ON FAMILIES OF SELF ADJOINT OPERATORS

by

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§1. Introduction

In this paper, we shall discuss the behaviour of the eigenvalues of a family of Dirac operators, and in particular, calculate the local contributions to the spectral flow in terms of easily obtained algebraical data.

We start with a compact oriented manifold M, fibred over the circle S^1, with map π:

$$\begin{array}{c} M \\ \downarrow \pi \\ S^1 \end{array} .$$

This induces a map on homotopy:

$$\pi_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$$

and we may therefore construct the infinite cyclic covering $\tilde{M}$ of M, the sheets of which are labelled by elements of $\mathbb{Z}$. So, now $\tilde{M}$ is fibred over $\mathbb{R}$

Let N denote the fibre; we shall assume throughout this paper that N is even dimensional, say $\dim N = 2n$; so that $\dim M = 2n + 1$.

Each representation of $\pi_1(S^1) = \mathbb{Z}$, and its corresponding character χ, gives rise to a local coefficient system χ_θ on $\tilde{M}$, and thus to a self-adjoint operator:

$$D_\chi : \Omega^{\text{even}}(M;\chi) \to \Omega^{\text{even}}(M;\chi)$$

produced from the Dirac-type operator on M, associated with χ (namely $\pm *d \pm d^*$: see definition 2.1 for the precise form).

The characters of representations of $\mathbb{Z}$ are given by maps:

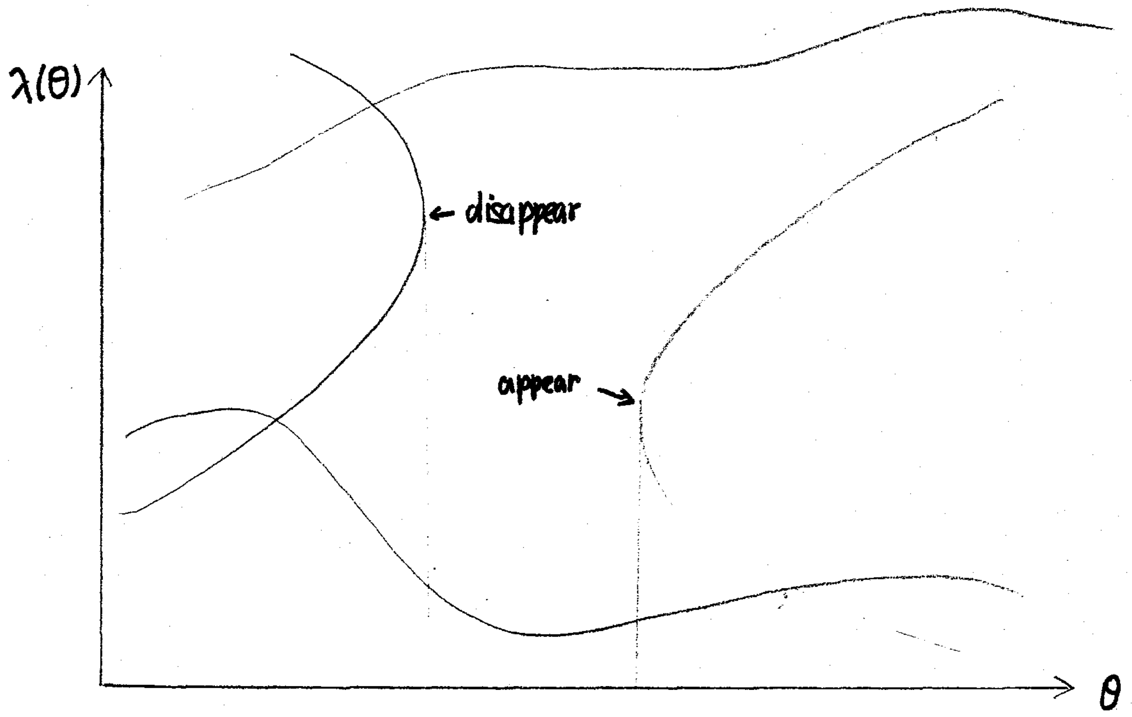
$$\mathbb{Z} \to \mathbb{C}^*$$
$$n \to z^n$$

for any $z \in \mathbb{C}^*$. However, unitary representations require that $|z| = 1$, and thus form a one-parameter family specified by θ s.t. $e^{i\theta} = z$. These characters are denoted χ_θ. When χ is the identity, the above operator D_χ (i.e. D_{χ_θ}) is that discussed in [1]. We will abbreviate D_{χ_θ} to D_θ.

So, we have obtained a one-parameter family D_θ of self-adjoint operators on $\Omega^{even}(M;\chi_\theta)$, parametrised by $\theta \in S^1$. Any such family has an associated natural invariant, namely the spectral flow which we shall now define. In §8, we shall show how D_θ can be conjugated to operators all acting on the same space, and varying linearly with θ (see Lemma 8.1). Thus the associated eigenvalues, which must all be real since the operators are self-adjoint, will also depend real analytically on θ (see proposition 8.2). Intuitively, one understands this last statement as follows. As θ moves around S^1, the eigenvalues will change smoothly. They cannot 'suddenly appear or disappear', since then the number of eigenvalues for a fixed θ would change abruptly, which it cannot do, due to the self-adjointness (see diagram below). The rigourous proof of this statement involves Puiseaux expansions (see §8, Proposition 8.2)

Since $\chi_{2\pi}$ and χ_0 are identical, thus the eigenvalues of D_θ must come into coincidence for θ's differing by multiples of 2π. The spectral flow is defined to be:

$n_+ - n_-$



where n_+ is the number of eigenvalues crossing $\lambda = 0$ from -ve to +ve and n_- is the number of eigenvalues crossing $\lambda = 0$ from +ve to -ve, counting multiplicies of eigenvalues, as θ flows from 0 to 2π. (See the diagram below).

Since M is fibred over S^1 with fibre N, there is a map:

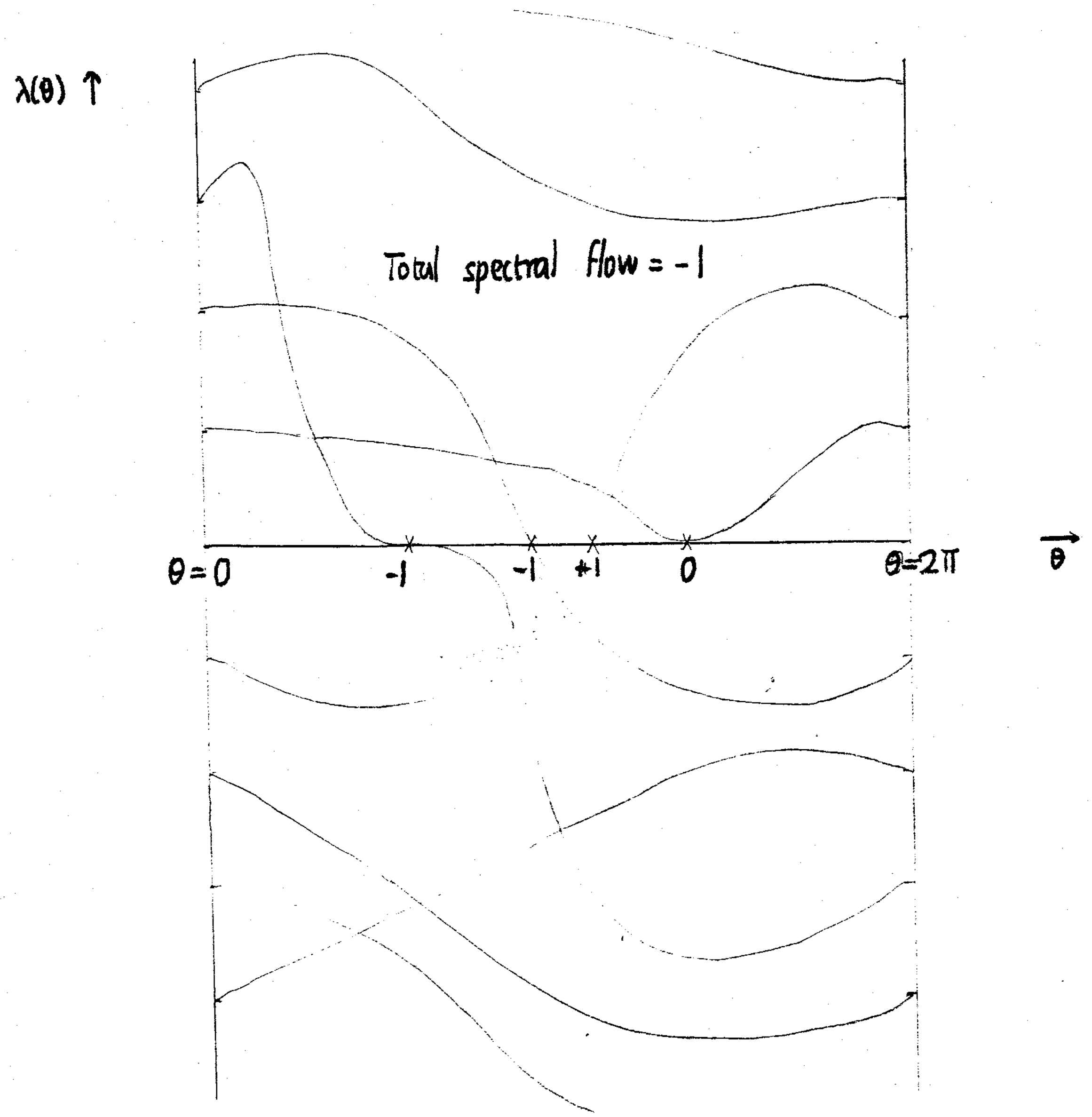
$$N \to N$$

corresponding to each element of $\pi_1(S^1) = \mathbb{Z}$. These are the <u>monodromy actions</u>. In particular, we refer to the mapping corresponding to $1 \in \mathbb{Z}$ as the <u>monodromy map</u>:

$$T : N \to N .$$

This map induces a mapping of the cohomology $H^*(N)$:

$$T^* : H^*(N) \to H^*(N) .$$



Our aim is to relate this monodromy map T^* to the eigenvalue variation of the Dirac operators $\{D_\theta\}$. More precisely, we shall find the <u>multiplicities</u> of intersection of the graphs of eigenvalues of the D_θ against θ with the θ-axis; and the associated <u>crossing numbers</u> at such intersections, in terms of the monodromy. We shall obtain a local version of the global result that:

(spectral flow of eigenvalues of D_θ as θ varies round S^1)
= (signature of monodromy induced on middle cohomology)

which in itself is an easy consequence of the Atiyah-Singer index theorem (see §2, Theorem 2.2).

More generally, we are trying to link the local behaviour of the eigenvalues of D_θ with the monodromy on cohomology. This local behaviour involves properties such as:

(i) θ's where an eigenvalue vanishes;

(ii) number of eigenvalues vanishing at that value of θ;

(iii) the order of contact of such an eigenvalue with the θ-axis;

(iv) the crossing numbers of such eigenvalues.

The monodromy contains information such as:

(i) the number of Jordan blocks, their sizes and corresponding eigenvalues;

(ii) the signatures of the Jordan blocks (see §11, Theorem 11.5).

To obtain a link we split the problem into three stages:

$$H(N) \leftrightarrow H_c(\tilde{M}) \leftrightarrow H_a(\tilde{M}) \leftrightarrow \text{Coker } D .$$

The first stage connects N and $\tilde{M}$, and is discussed in §§3, 4. See Theorems 3.1 and 4.6. The second stage is the connection between the 'analytic' cohomology of $\tilde{M}$, and the 'algebraic', i.e. compactly supported, cohomology of $\tilde{M}$. This is discussed in §5, see Theorem 5.3. Finally, in §6 we connect H_a and the cokernel of D (considered as a module over the analytic functions on S^1), see Theorem 6.2.

The Hermitian form on $H^n(N)$ (middle dimensional cohomology) which we use is the standard one; that is:

$$(\alpha,\beta) = \int_N \alpha \wedge \overline{\beta} .$$

This we relate to a certain Hermitian form on Coker D, see §7, Theorem 7.1.

This Hermitian form we relate to the eigenvalue structure of D_θ. The eigenvalue structure of D_θ is found using perturbation theory in §8. In §9, we apply the theory of normal forms, from Appendix II, so as to produce the connection between local D_θ eigenvalue structure and the Hermitian form on Coker D, and hence with the Hermitian form on H(N). This gives our main Theorem 9.1.

§2. The global result

In this section, we wish to show that the spectral flow of D_θ $(\theta \in S^1)$ is equal to the signature of the natural form:

$$(\alpha,\beta) = \int_N \alpha \wedge \bar{\beta} \tag{2.1}$$

on the middle cohomology $H^n(N)$ of N. The signature is usually called the signature of N, sign(N).

For each $\theta \in S^1$, we have an operator D_θ defined:

$$\Omega^{even}(M;\chi_\theta) \to \Omega^{even}(M;\chi_\theta)$$

such that $D_\theta^2 = dd^* + d^*d$ where d^* denotes the adjoint of d with respect to the Hermitian inner product on p-forms:

$$\Omega^p(M;\chi_\theta) \times \Omega^p(M;\chi_\theta) \to \mathbb{R}$$

$$(\alpha,\beta) \to \int_M \alpha \wedge *\bar{\beta} . \tag{2.2}$$

Let q be the dimension of M, i.e. $2n+1$. Then we define a map:

$$\tau : \Omega^p \to \Omega^{q-p}$$

$$\omega \to (-1)^{\frac{1}{2}p(p-1)+\frac{1}{4}q(q-1)} (*\omega) \tag{2.3}$$

where $*$ denotes the Hodge star operator on Ω^p. Then:

$$\tau^2 = (-1)^{\frac{1}{2}(q-p)(q-p-1)+\frac{1}{4}q(q-1)} (-1)^{\frac{1}{2}p(p-1)+\frac{1}{4}q(q-1)} . *^2$$

$$= (-1)^{\frac{1}{2}[q^2-q(2p+1)+2p^2] + \frac{1}{2}q(q-1)} . *^2$$

$$= (-1)^{q^2-q-qp+p^2} . (-1)^{p(q-p)}$$

$$= (-1)^{q^2-q} = 1$$

and

$$\tau d\tau = (-1)^{\frac{1}{2}(p^2-p)+\frac{1}{4}q(q-1)} (-1)^{\frac{1}{2}(q-p)(q-p+1)+\frac{1}{4}q(q-1)} (*d*) \text{ on } \Omega^p$$

since if $\delta \in \Omega^p$, then $d\tau\omega \in \Omega^{q-p+1}$

$$= (-1)^{\frac{1}{2}[q^2-q(2p-1)+2(p^2-p)]+\frac{1}{2}q(q-1)} (*d*)$$

$$= (-1)^{q^2-qp+p^2-p} (*d*)$$

$$= (-1)^{q(q-p)} (*d*) \text{ as } p^2 - p \text{ is even .}$$

However, if $x \in \Omega^{p-1}$, $y \in \Omega^p$,

then $<dx,y> = \int_M dx \wedge (*y)$

$$= \int_M d(x \wedge *y) - (-1)^{\partial x} x \wedge d(*y)$$

$$= (-1)^p <x,(*^{-1}d*)y> \text{ since } \int_M d\omega = 0 \text{ by Stokes' Th.}$$

where $<,>$ denotes the inner product (2.2). That is:

$$d^* = (-1)^p (*^{-1}d*)$$

$$= (-1)^p (-1)^{(q-p+1)(p-1)} (*d*) \text{ since if } \omega \in \Omega^p \text{ then } d*\omega \in \Omega^{q-p+1}$$

$$= (-1)^{q(p-1)-1} (*d*)$$

$$= (-1)^{q(p-1)-1} (*d*) \text{ since } q^2 - q \text{ is even}$$

$$= -\tau d\tau, \text{ by above .}$$

Thus $dd^* + d^*d = -d\tau d\tau - \tau d\tau d$

$$= (d\tau - \tau d)^2 \text{ since } (d\tau)(\tau d) = d^2 = 0 \text{ as } \tau^2 = 1 \text{ and } (\tau d)(d\tau) = \tau d^2\tau = 0 .$$

Hence we define:

<u>Definition 2.1.</u> $D_\theta = (d\tau - \tau d)$

On $\Omega^p, D_\theta = (-1)^{\frac{1}{2}p(p-1)+\frac{1}{4}q(q-1)} d* - (-1)^{\frac{1}{2}p(p+1)+\frac{1}{4}q(q-1)} *d$

$$= (-1)^{\frac{1}{2}p(p-1)+\frac{1}{4}q(q-1)} (d* + (-1)^{p+1} *d) .$$

When we use maps $\Omega^{even} \to \Omega^{even}$, q is even, and thus D_θ is $\pm(d* - *d)$. Now, we have ensured that $D_\theta^2 = \Delta$, the Laplacian.

We now proceed to prove:

<u>Theorem 2.2.</u> <u>The spectral flow of the eigenvalues of</u> D_θ <u>around</u> S^1 <u>equals the signature of</u> N.

The proof that the spectral flow is defined in this case, i.e. that the different branches of eigenvalues can be separated out, can be found in §8 (proposition 8.2). To prove theorem 2.2, we can construct a new space:

$$X = M \times [0,1]$$

and define on it an operator:

$$\mathcal{D} = D_t + \partial/\partial t .$$

We map the parameter $t \in [0,1]$ to $e^{2\pi i t}$ on S^1, and so for each t, there is an associated self-adjoint operator D_t, that is $D_{e^{2\pi i t}}$.

Now X has two boundaries:

$$S_0 = M \times \{0\}$$

$$S_1 = M \times \{1\} .$$

Each of these boundaries gives a space $L^2(S)$, which we can split:

$$L^2(S_t) = H_t^+ \oplus H_t^- \quad (t = 0 \text{ or } 1)$$

where H_t^+ is specified by a positive spectrum for D_t, and H_t^- is specified by a negative spectrum for D_t. The boundary conditions used at the two ends S_0, S_1, are that:

$$f \in H_0^- \quad \text{near} \quad t = 0$$

$$f \in H_1^+ \quad \text{near} \quad t = 1 .$$

We prove theorem 2.2 in two parts:

Lemma 2.3. Index$(\mathcal{D}(X'))$ = (spectral flow of D_θ) .

Lemma 2.4. Index $(\mathcal{D}(X'))$ = sign(N) .

Here $X' = M \times S^1$.

Proof of Lemma 2.3. The Atiyah-Patodi-Singer formula [1], for the index of $\mathcal{D}$, when applied to the cylinder:

$$X_t = M \times [0,t]$$

gives:

$$2 \text{ index}(\mathcal{D}(X_t)) = \eta_t - \eta_0 + \int_{X_t} L$$

where $\eta_t(s) = \Sigma(\text{sgn } \lambda)|\lambda|^{-s}$ is summed over eigenvalues of D_t and η_t denotes the limiting value of $\eta_t(s)$ as $s \downarrow 1$. The integral on the right-hand side varies continuously with t - this is the only information we require about the behaviour of this term. The factor '2' comes in, since the $\mathcal{D}$ defined here acts on the space of even-dimensional forms only, whereas that in [1] acts on the space of all forms.

However, index$(\mathcal{D}(X_t))$ is an integer, and thus can jump by integer amounts as t varies continuously from 0 to 1.

The integral is a continuous function of t, and thus, if $t = t_0$ is a value at which $\operatorname{index}(\mathcal{D}(X_t))$ jumps, then:

$$2\Delta_{t_0}(\operatorname{index}(\mathcal{D}(X_t))) = \Delta_{t_0}(\eta_t)$$

where Δ_{t_0} denotes the change as t passes through t_0.

Since $\eta(s) = \Sigma(\operatorname{sgn}\lambda)|\lambda|^{-s}$

$$\Delta_{t_0}(\eta_t) = 2 \quad (\text{crossing number for eigenvalues of } D_t, \text{ at } t = t_0).$$

This is because as we pass through $t = t_0$, only those terms in $\eta(s)$ corresponding to eigenvalues of D_t crossing zero change, and they change by:

$$\pm 2|\lambda|^{-s} \quad (\text{sign is the crossing number}).$$

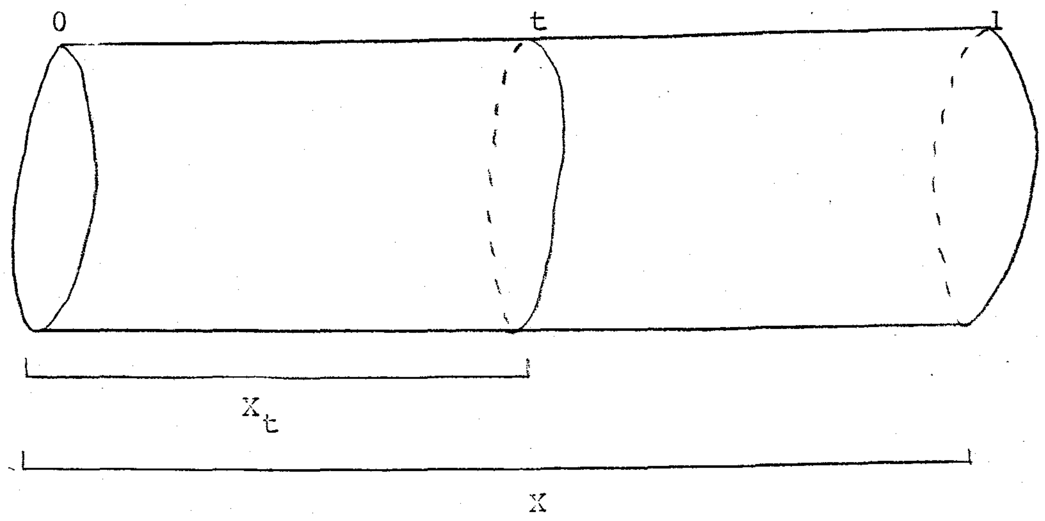
As $s \to 0$, this just gives twice the crossing number.

Thus $2\Delta_{t_0}(\operatorname{index}\mathcal{D}(X_t)) = 2 \times (\text{crossing number at } t_0)$.

Hence the total jump in index is:

$$\begin{aligned}\operatorname{index}(\mathcal{D}(X_1)) - \operatorname{index}(\mathcal{D}(X_0)) &= \sum_{\substack{\text{jump points} \\ t_0}} \Delta_{t_0}(\operatorname{index}\mathcal{D}(X_t)) \\ &= \sum_{\substack{\text{jump points} \\ t_0}} (\text{crossing number at } t_0) \\ &= (\text{spectral flow of eigenvalues of } D_t).\end{aligned}$$

However, $X_1 = X$ and $X_0 = M \times \{0\}$. So, the left hand side is just $\operatorname{index}(\mathcal{D}(X))$.



Hence index($\mathcal{D}$(X)) = (spectral flow of D_t eigenvalues).

However, D_0 and D_1 are identical, and so $\eta_0 = \eta_1$.

Thus $2 \text{ index}(\mathcal{D}(X)) = \eta_1 - \eta_0 + \int_X L$ by Atiyah-Patodi-Singer formula

$$= \int_X L$$

$$= 2 \text{ index}(\mathcal{D}(X'))$$

where $X' = M \times S^1$ is a manifold without boundary.

So index($\mathcal{D}$(X')) = (spectral flow of D_t eigenvalues).

Q.E.D.

Proof of Lemma 2.4. To connect the index of $\mathcal{D}$ with the signature of N, we use the main Atiyah-Singer index theorem [2].

Since M is fibred over S^1 with fibre N, $X' = M \times S^1$ is fibered over $S^1 \times S^1 = T^2$ with fibre N:

$$\begin{array}{ccc} M & \to & X' \\ \downarrow N & & \downarrow N \\ S' & \to & T^2 \end{array} .$$

Let D_0 be the signature operator on X. Then the operator $\mathcal{D} = D_t + \partial/\partial t$ on X' can be written as

$$\mathcal{D} = D_N \otimes D_{T^2} .$$

For, suppose we consider a form on X. This can be written as a function of θ and u, where u is the S^1 variable in M. So, we start with a form:

$$\omega(\theta,u)$$

which has:

$$\omega(\theta,u + 2\pi) = e^{i\theta}\omega(\theta,u)$$

$$\omega(\theta+2\pi,u) = \omega(\theta,u) .$$

When we untwist,

$$\omega(\theta,u) = e^{i\theta u}\phi(\theta,u)$$

say. Thus, ϕ is now periodic with periods 2π.

However, $\mathcal{D}$ as defined on ϕ acts as:

$$\phi(\theta,u) \to e^{-i\theta u}\ \mathcal{D}(e^{i\theta u}\ \phi(\theta,u))$$

$$= e^{-i\theta u}[D_\theta + \partial/\partial\theta](e^{i\theta u}\phi(\theta,u))$$

$$= e^{-i\theta u}[e^{i\theta u}\partial\phi/\partial\theta + iue^{i\theta u}\phi(\theta,u)$$

$$+ e^{i\theta u}D_\theta\phi(\theta,u) + B_\phi(\theta,u) \times \theta e^{i\theta u}]$$

$$= (D + \partial/\partial\theta)\phi + (iu + \theta B)\phi$$

where B is the symbol of D_θ.

So, the symbol of $\mathcal{D}$ is $\theta B + iu$. It is thus D_N tensored by D_{T^2}, where we twist by a line bundle L .

Thus $\text{index}(\mathcal{D}) = \text{index}(D_N \otimes D_{T^2})$

$$= \text{index}_{T^2}(D_{T^2} \otimes \text{index}(D_N))$$

where $\text{index}(D_N) \in K(T^2)$.

Since L describes the twist, thus:

$$(\text{index } D_N) = (\text{sign } N) \otimes L$$

i.e. $(\text{index } \mathcal{D}) = (\text{sign } N) \text{ index}_{T^2}(D_{T^2} \otimes L)$.

However, D_{T^2} is just $\bar{\partial}$. This is due to the 'iu' term in the above symbol, which just means $\bar{\partial}$ on T^2. The first Chern class of L is 1, due to the $e^{iu\theta}$ twisting above.

So $\text{index}_{T^2}(D_{T^2} \otimes L) = \text{index}_{T^2}(\bar{\partial} \otimes L)$

$$= 1 - g + c_1(L)$$

$$= 1$$

since the genus of a torus is 1.

Thus $\text{index } \mathcal{D} = (\text{sign } N) \text{ index}_{T^2}(D_{T^2} \otimes L)$

$$= (\text{sign } N).$$

Q.E.D.

Putting Lemmas 2.3 and 2.4 together gives Theorem 2.2.

§3. The connection between $H(N)$ and $H_c(\tilde{M})$

Let $H_c(\tilde{M})$ denote the cohomology of $\tilde{M}$ with compact support. We represent $\tilde{M}$ as $N \times \mathbb{R}$. Define a map:

$$H^p(N) \to H_c^{p+1}(\tilde{M})$$

by using $\omega \to \chi_{[0,1]}(t)\ (\omega \wedge dt)$ for $\omega \in \Omega^p(N)$ where $\chi_{[0,1]}(t)$ denotes the characteristic function of the interval $[0,1]$ in t (we often abbreviate this to $\chi_{[0,1]}$). The above defines a map on the level of ~~forms~~ distributions which can be extended to a map on cohomology. For, if:

$$[\omega] = 0 \quad \text{in} \quad H^p(N)$$

then $\omega = d\phi$. Thus:

$$\chi_{[0,1]}\ \omega \wedge dt = d(\chi_{[0,1]}\phi \wedge dt)$$

which corresponds to zero cohomology class in $H_c^{p+1}(\tilde{M})$.

Furthermore, if $[\chi_{[0,1]}\ \omega \wedge dt] = 0$ in $H_c^{p+1}(\tilde{M})$ then $\chi_{[0,1]}\ \omega \wedge dt = dx$ some $x \in \Omega_c^p(\tilde{M})$. Let us write x as:

$$x = y + z \wedge dt$$

some forms y, z on $\tilde{M}$, not involving dt, of degrees $p, p-1$ respectively. Then:

$$dx = dy + dz \wedge dt + dt \wedge \partial y/\partial t$$

and so
$$\left.\begin{aligned} \chi_{[0,1]}\omega &= dz + (-1)^p\ \partial y/\partial t \\ 0 &= dy \end{aligned}\right\} \qquad \begin{matrix}\text{(i)}\\ \text{(ii)}\end{matrix}$$

Since x is compactly supported, we may integrate (i) over a sufficiently large interval containing $[0,1]$ so that $[y] = 0$. Hence ω is a closed form, and $[\omega] = 0$ in $H^p(N)$. Thus the above map is injective.

Since $\tilde{M}$ is fibred over $\mathbb{R}$ with fibre N, the space $\tilde{M}$ is contractible to N, and so the cohomologies of $\tilde{M}$ and N are identical. Thus the above map is bijective:

$$H^p(N) \cong H_c^{p+1}(\tilde{M})$$

(The map is linear, and $H^p(N)$ is finite dimensional as N is compact).

The map preserves the overall integral:

$$\int_{\tilde{M}} (\omega \wedge dt)\chi_{[0,1]} = \int_N \int_{\mathbb{R}} \chi_{[0,1]}\, \omega \wedge dt$$
$$= \int_N \omega \quad ,$$

by definition. However,

$$H_c^{p+1}(\tilde{M}) \cong H^{p+1}(\tilde{\Omega}_c)$$

where $\tilde{\Omega}_c$ denotes the space of all forms on $\tilde{M}$ with compact support.

The shift operator $\tilde{T}$ on $\tilde{M}$ is defined by:

$$\tilde{M} \to \tilde{M}$$

$$(x,t) \to (x,t+1)$$

for all $x \in N$, $t \in \mathbb{R}$. This induces an operator, which we also call $\tilde{T}$, on the forms on $\tilde{M}$, and thus also on the cohomology of $\tilde{M}$. So, the ring:

$$\tilde{A} = C[\tilde{T},\tilde{T}^{-1}]$$

acts on the cohomology spaces $H_c^{p+1}(\tilde{M})$, $H^{p+1}(\tilde{\Omega}_c)$, and thus these cohomologies can be viewed as $\tilde{A}$-modules.

There is also an action on $H^p(N)$ given by the monodromy T on N. So $H^p(N)$ can be viewed as an A-module, where $A = C[T,T^{-1}]$. It can also be viewed as an $\tilde{A}$-module: $\tilde{\Omega}_c$ is free, whereas H is not free.

Suppose $\omega \in H^p(N)$. The image of $T\omega$ in $H_c^{p+1}(\tilde{M})$ is:

$$\chi_{[0,1]}(T\omega \wedge dt) \qquad (T^* \text{ has been abbreviated to } T) .$$

At a point (x,t) $(x \in N, t \in \mathbb{R})$,

$$[\chi_{[0,1]}\, T\omega \wedge dt](x,t) = ((T\omega)(x) \wedge dt)\chi_{[0,1]}$$

$$(\tilde{T}[\chi_{[0,1]}\omega \wedge dt])(x,t) = ((\omega \wedge dt)\chi_{[0,1]})(x,t-1)$$

$$= ((T\omega \wedge dt)\chi_{[1,2]})(x,t) .$$

Thus, since $((T\omega)(x) \wedge dt)\chi_{[0,1]}$ and $(T\omega \wedge dt)\chi_{[1,2]}$ are in the same cohomology class, the image of $T\omega \in H^p(N)$ in $H_c^{p+1}(\tilde{M})$ is the image under $\tilde{T}$ of the form in $H_c^{p+1}(\tilde{M})$ corresponding to $\omega \in H^p(N)$. Hence:

$$\begin{array}{ccc} H^p(N) & \xrightarrow{T} & H^p(N) \\ \downarrow & & \downarrow \\ H_c^{p+1}(M) & \xrightarrow{\tilde{T}} & H_c^{p+1}(M) \end{array}$$

is a commuting square.

Note that in order to make T, $\tilde{T}$ correspond, we have chosen T to be the inverse of the 'natural' map, that is, it is the monodromy on N (or induced on $\Omega^p(N)$ or $H^p(N)$) when t is reduced from 1 to 0.

The maps:

$$H^p(N) \to H_c^{p+1}(\tilde{M}) \cong H^{p+1}(\tilde{\Omega}_c)$$

are bijections, and preserve the $\tilde{A}$-module structure. So, we have obtained:

<u>Theorem 3.1.</u> <u>There exists a bijection:</u>

$$H^p(N) \to H^{p+1}(\tilde{\Omega}_c)$$

<u>and this map transforms the A-module structure of</u> $H^p(N)$ <u>into the $\tilde{A}$-module structure of</u> $H^{p+1}(\tilde{\Omega}_c)$.

The map we chose above is by no means unique. The cleanest way is to use:

$$\omega \to \delta(\omega \wedge dt)$$

where δ is the delta function at $t = 0$. This uses distributions, but creates problems in the next section. This map is the 'spiky' extreme. On the other extreme, we may use:

$$\omega \to \beta(t)(\omega \wedge dt)$$

where β is a C^∞ function of integral 1. However, the map we used is in between these two extremes - it is used so as to simplify the reasoning in §4.

§4. The Fourier Transform

The space, $\tilde{\Omega}_c$, consists of forms on $\tilde{M}$ with compact support. For any form $\omega \in \tilde{\Omega}_c$, define its Fourier transform $\hat{\omega}$, a form on $\tilde{M}$ dependent on ζ by:

Definition 4.1. $\hat{\omega}(\zeta) = \sum_{k \in \mathbb{Z}} (\tilde{T}^{-k}\omega)\zeta^k.$

The form $\hat{\omega}$ has coefficients in the space of Laurent series in ζ. Restricting $\hat{\omega}$ to a form on $N \times [0,1]$, we see that it has only a finite number of Fourier components. This is because:

$$(\tilde{T}^{-k}\omega)|_{N\times[0,1]} = \tilde{T}^{-k}(\omega|_{N\times[k,k+1]})$$

which vanishes for all $k \in \mathbb{Z}$, except a finite number, since ω is compactly supported.

Note also that:

$$\begin{aligned}
\hat{\omega}(\zeta)|_{N\times[\alpha,\alpha+1]} &= \sum_{k\in\mathbb{Z}} (\tilde{T}^{-k}\omega)|_{N\times[\alpha,\alpha+1]}\ \zeta^k \\
&= \sum_{k\in\mathbb{Z}} \tilde{T}^{-k}(\omega|_{N\times[k+\alpha,k+\alpha+1]})\zeta^k \\
&= \tilde{T}^{\alpha} \sum_{k\in\mathbb{Z}} \tilde{T}^{-(k+\alpha)}(\omega|_{N\times[k+\alpha,k+\alpha+1]})\zeta^k \\
&= \tilde{T}^{\alpha}\ \zeta^{-\alpha} \sum_{\ell\in\mathbb{Z}} \tilde{T}^{-\ell}(\omega|_{N\times[\ell,\ell+1]})\zeta^{\ell} \quad \text{where } \ell = k+\alpha \\
&= \tilde{T}^{\alpha}\ \zeta^{-\alpha} \sum_{\ell\in\mathbb{Z}} (\tilde{T}^{-\ell}\omega)|_{N\times[0,1]}\ \zeta^{\ell} \\
&= \tilde{T}^{\alpha}\ \zeta^{-\alpha}(\hat{\omega}(\zeta)|_{N\times[0,1]})
\end{aligned}$$

and hence $\hat{\omega}(\zeta)$ is determined uniquely by its value on $N \times [0,1]$: the rest is essentially just shifted.

Thus $\hat{\omega}(\zeta)$ can be considered as a form on M, since:

$$\hat{\omega}(\zeta)|_{N\times[1,2]} = 1/\zeta\ \tilde{T}(\hat{\omega}(\zeta)|_{N\times[0,1]})$$

So $\hat{\omega}(\zeta)$ is a form on M which twists by $1/\zeta$, what we would call an element of:

$$\Omega(M;\chi_{-\theta})$$

where $\zeta = e^{i\theta}$.

<u>Lemma 4.2</u>. $\int_M \hat{\omega}(1) = \int_{\tilde{M}} \omega$.

<u>Proof</u>: Now
$$\int_M \hat{\omega}(1) = \int_{N\times[0,1]} \sum_{k\in\mathbb{Z}} (\tilde{T}^{-k}\omega)$$
$$= \sum_{k\in\mathbb{Z}} \int_{N\times[0,1]} \tilde{T}^{-k}(\omega|_{N\times[k,k+1]})$$
$$= \sum_{k\in\mathbb{Z}} \int_{N\times[k,k+1]} \omega \quad \text{since } \tilde{T} \text{ preserves integrals}$$
$$= \int_{\tilde{M}} \omega \quad \text{since } \tilde{M} \text{ is fibred over } \mathbb{R}, \text{ fibre } N$$

Q.E.D.

We define an inner product on $H^n(N)$:

<u>Definition 4.3</u>. $(\alpha,\beta) = \int_N \alpha \wedge \overline{\beta}$

This inner product is indefinite, and its signature is, by definition, sign (N). By theorem 3.1, we have a bijection from $H^n(N)$ to $H^{n+1}(\tilde{\Omega}_c)$, and thus we can induce an inner product on $H^{n+1}(\tilde{\Omega}_c)$.

By the above, any form $\omega \in \tilde{\Omega}_c$ can be mapped to its Fourier transform a form $\hat{\omega}$ on M, which depends on ζ. Since N is compact, $H^n(N)$ is finite dimensional, and thus so is $H^{n+1}(\tilde{\Omega}_c)$. We shall abbreviate $H^{n+1}(\tilde{\Omega}_c)$ to H_c. The action of $\tilde{T}$ on $\tilde{\Omega}_c$ has a characteristic polynomial, and so

~~$\forall \omega \in \tilde{\Omega}_c$~~ $\forall x \in H_c$,

$$p(\tilde{T})[\omega] = 0 \quad \text{in} \quad H_c$$

some polynomial p.

The map $a \to \hat{a}$ gives:

$$\tilde{T}a \to \sum_{k\in\mathbb{Z}} (\tilde{T}^{-k}(\tilde{T}a))\zeta^k$$

$$= \sum_{k\in\mathbb{Z}} (\tilde{T}^{-(k-1)}a)\zeta^k$$

$$= \sum_{k\in\mathbb{Z}} (\tilde{T}^{-k}a)\zeta^{k+1} = \zeta\hat{a} .$$

So, if $[\omega] \in H_c$, then

$$p(\tilde{T})\omega = dx$$

some $x \in \tilde{\Omega}_c$ and polynomial p. Thus, taking Fourier transforms,

$$p(\zeta)\hat{\omega} = d\hat{x} .$$

We define an inner product on $\tilde{\Omega}_c$ by:

<u>Definition 4.4</u>. <u>If</u> $a,b \in \tilde{\Omega}_c$, <u>define</u>:

$$(a,b) = \operatorname*{Res}_{\zeta=0,\infty}\left[\frac{1}{p(\zeta)}\left(\int_{\tilde{M}}\left(\sum_{k\in\mathbb{Z}} T^k x\cdot\zeta^{-k}\right) \wedge \overline{b}\right)\frac{d\zeta}{\zeta}\right] .$$

<u>When</u> $x,y \in H_c$, <u>define</u>

$$\langle x,y\rangle = \int_{\tilde{M}} \hat{x} \wedge \overline{y} .$$

Thus $(a,b) = \operatorname*{Res}_{\zeta=0,\infty} \frac{\langle x,b\rangle}{p(\zeta)}\frac{d\zeta}{\zeta}$.

Here, $\langle\, ,\, \rangle$ is a pairing on H_c, with values in the ring of Laurent series in ζ. The symbols $\operatorname*{Res}_{\zeta=0,\infty}$ and $\operatorname*{Res}_{|\zeta|=1}$ (which we shall use later) denote a sum of residues over all poles in the list of ζ-values: $\zeta = 0,\infty$ in the first case, and $|\zeta| = 1$ in the second case.

<u>Proposition 4.5.</u> <u>The inner product</u> $(\ ,\ )$ <u>on</u> $\tilde{\Omega}_C$ <u>induces a well-defined inner product on</u> H_C .

<u>Proof</u>: We must show that:

(i) if $p(\tilde{T})a = dx'$, we get the same result for (a,b) on interchange of x, x';

(ii) if $p(\tilde{T})q(\tilde{T})a = d(q(\tilde{T})x)$, we get the same result for (a,b) if we replace p and x, by $p\cdot q$ and $q(\tilde{T})x$;

(iii) (a,b) is independent of the choice of p and x;

(iv) if $b = db_1$ $(a,b) = 0$;

(v) if $a = da_1$ $(a,b) = 0$;

(vi) $(a,b) = (-1)^{(\deg a)(\deg b)}\,\overline{(b,a)}$;

(i) Now $\operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle x,b\rangle}{p(\zeta)}\frac{d\zeta}{\zeta}\right| - \operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle x',b\rangle}{p(\zeta)}\frac{d\zeta}{\zeta}\right|$

$$= \operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle x-x',b\rangle}{p(\zeta)}\frac{d\zeta}{\zeta}\right|$$

$$= \operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle x-x',q(\tilde{T})b\rangle}{p(\zeta)\bar{q}(\zeta^{-1})}\frac{d\zeta}{\zeta}\right| \quad \text{(using (b) below)}$$

$$= \operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle x-x',dy\rangle}{p(\zeta)\bar{q}(\zeta^{-1})}\frac{d\zeta}{\zeta}\right|$$

$$= \pm \operatorname*{Res}_{\zeta=0,\infty}\left|\frac{\langle d(x-x'),y\rangle}{p(\zeta)\bar{q}(\zeta^{-1})}\frac{d\zeta}{\zeta}\right| \quad \text{(using (d) below)}$$

$$= 0 \qquad \text{since } d(x-x') = 0$$

i.e. (a,b) is unchanged when we replace x by x' .

(ii) Now $\operatorname*{Res}_{\zeta=0,\infty}\left(\frac{\langle q(\tilde{T})x,b\rangle}{p(\zeta)q(\zeta)}\frac{d\zeta}{\zeta}\right) = \operatorname*{Res}_{\zeta=0,\infty}\left(\frac{\langle x,b\rangle}{p(\zeta)}\frac{d\zeta}{\zeta}\right)$ (using (a) below)

and so we get the same result for (a,b) using:

$q(\tilde{T})x$, $p(\zeta)q(\zeta)$

as we do using $x, p(\zeta)$.

(iii) Suppose that:

$$p(\tilde{T})a = dx$$

$$q(\tilde{T})a = dx' \ .$$

We wish to show that replacing p, x by q, x' leaves the inner product (a,b) unchanged. Let r be the h.c.f. of p,q. Then r can be expressed as $ps + qt$ some polynomials s.t. Thus,

$$[(a,b) \text{ using } (r, t(\tilde{T})x' + s(\tilde{T})x)]$$

$$= [(a,b) \text{ using } (p, (\tfrac{p}{r})(\tilde{T})(t(\tilde{T})x' + s(\tilde{T})x))] \text{ by (ii)}$$

$$= [(a,b) \text{ using } (p,x)], \text{ by (i)}$$

and similarly l.h.s. equals (a,b) calculated using q,y. Hence the values of (a,b) calculated using p, x and using q, y are equal.

(iv) Suppose that $b = db_1$, $dx = p(\tilde{T})a$. Then

$$(a,b) = \operatorname*{Res}_{\zeta=0,\infty} \left(\frac{\langle x,b\rangle}{p(\zeta)} \frac{d\zeta}{\zeta}\right)$$

$$= \operatorname*{Res}_{\zeta=0,\infty} \left(\frac{\langle x,db_1\rangle}{p(\zeta)} \frac{d\zeta}{\zeta}\right)$$

$$= \pm \operatorname*{Res}_{\zeta=0,\infty} \left(\frac{\langle dx,b_1\rangle}{p(\zeta)} \frac{d\zeta}{\zeta}\right) \quad \text{(using (d) below)}$$

$$= \pm \operatorname*{Res}_{\zeta=0,\infty} \left(\frac{\langle p(T)a,b_1\rangle}{p(\zeta)} \frac{d\zeta}{\zeta}\right) .$$

However, using (a) below, this r.h.s. becomes:

$$\pm \operatorname*{Res}_{\zeta=0,\infty} \left\{ \langle a,b_1\rangle \frac{d\zeta}{\zeta} \right\} = 0$$

since $\langle a,b_1\rangle$ is a Laurent polynomial.

(v) Now, if $a = da_1$, we can use $x = a_1$, $p = 1$ to get:

$$(a,b) = \operatorname*{Res}_{\zeta=0,\infty} \left\{ \langle a_1,b\rangle \frac{d\zeta}{\zeta} \right\}$$

$= 0$ since $\langle a_1,b\rangle$ is a Laurent polynomial .

(vi) Finally, $(a,b) = \operatorname*{Res}_{\zeta=0,} \left\{ \frac{\langle x\ ,b\rangle}{p(\zeta)} \cdot \frac{d\zeta}{\zeta} \right\}$

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ \frac{\langle x,q(\tilde{T})b\rangle}{p(\zeta)\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\} \quad \text{(using (b) below)}$$

where $p(\tilde{T})a = dx$, $q(\tilde{T})b = dy$

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ \frac{\langle x,dy\rangle}{p(\zeta)\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\}$$

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ (-1)^{\deg x+1} \frac{\langle dx,y\rangle}{p(\zeta)\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\}$$

(using (d) below)

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ (-1)^{\deg x+1} \frac{\langle p(\tilde{T})a,y\rangle}{p(\zeta)\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\}$$

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ (-1)^{\deg x+1} \frac{\langle a,y\rangle}{\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\}$$

(using (a) below)

$$= \operatorname*{Res}_{\zeta=0,\infty} \left\{ (-1)^{\deg x+1} (-1)^{(\deg a)(\deg y)} \frac{\overline{\langle y,a\rangle(\bar{\zeta}^{-1})}}{\bar{q}(\zeta^{-1})} \frac{d\zeta}{\zeta} \right\}$$

$$= \operatorname*{Res}_{z=\infty,0} \left\{ (-1)^{(\deg a)(\deg b-1)} \left| \overline{\frac{\langle y,a\rangle}{q(z)}} \right| \frac{dz}{z} \right\}$$

where $z = \zeta^{-1}$

$$= (-1)^{(\deg a)(\deg b)}\, \overline{(b,a)}$$

as required.

In this proof, various properties of $<,>$ have been used:

(a) $<p(\tilde{T})x,y> = p(\zeta)<x,y>$.

(b) $<x,p(\tilde{T})y> = \bar{p}(\zeta^{-1})<x,y>$.

(c) $\overline{<x,y>(\zeta)} = <y,x>(\bar{\zeta}^{-1})$.

(d) $<dx,y> = (-1)^{\partial x+1}<x,dy>$.

Since $<,>$ defined in definition 4.4 is a sum:

$$\sum_{k\in\mathbb{Z}} \zeta^k \left\{ \int_{\tilde{M}} (\tilde{T}^{-k}x) \hat{,} \bar{y}) \right\}$$

and T, d commute, thus (d) holds. As the Fourier transform of $P(T)x$ is $p(\zeta)\hat{x}$, thus (a) holds. Clearly (b) follows from (a) and (c):

$$\begin{aligned} <x,p(\tilde{T})y>(\zeta) &= \overline{<p(\tilde{T})y,x>(\bar{\zeta}^{-1})} && \text{by (c)} \\ &= \overline{p(\bar{\zeta}^{-1})<y,x>(\bar{\zeta}^{-1})} && \text{by (a)} \\ &= \bar{p}(\zeta^{-1})<x,y> && \text{by (c)} . \end{aligned}$$

So, finally, we must prove (c). However,

$$\begin{aligned} \overline{<x,y>(\zeta)} &= \overline{\sum_{k\in\mathbb{Z}} \zeta^k \left\{ \int_{\tilde{M}} (\tilde{T}^{-k}x) \wedge \bar{y} \right\}} \\ &= \sum_{k\in\mathbb{Z}} \bar{\zeta}^k \int_{\tilde{M}} (\tilde{T}^{-k}\bar{x} \wedge y) \end{aligned}$$

$$= \sum_{k\in\mathbb{Z}} \bar{\zeta}^{k} \int_{\tilde{M}} (y \wedge \tilde{T}^{-k} \bar{x})$$

(since $\tilde{M}$ is odd dimensional, $a \wedge b = b \wedge a$ whenever a,b have complementary dimensions).

$$= \sum_{k\in\mathbb{Z}} \bar{\zeta}^{k} \int_{\tilde{M}} (\tilde{T}^{k} y \wedge \bar{x})$$

since $\tilde{T}$ preserves integrals

$$= \langle y,x\rangle (\overline{\zeta^{-1}})$$

as required.

Q.E.D.

We shall now show that this inner product on H_c is identified with that on $H(N)$ under the map in Theorem 3.1.

<u>Theorem 4.6.</u> <u>The map of Theorem 3.1 transforms the inner product of definition 4.3 to that of definition 4.4.</u>

<u>Proof</u>. Suppose $\alpha,\beta \in H^n(N)$ correspond to $a,b \in \tilde{\Omega}_c^{n+1}$. Then:

$$p(\tilde{T})a = dx$$

for some $x \in \tilde{\Omega}_c^n$. Thus:

$$(a,b) = \operatorname*{Res}_{\zeta=0,\infty} \int_{\tilde{M}} \frac{\hat{x}\wedge\bar{b}}{p(\zeta)} \frac{d\zeta}{\zeta}$$

$$= \operatorname*{Res}_{\zeta=0,\infty} \sum_{k\in\mathbb{Z}} \left(\int_{\tilde{M}} \frac{\tilde{T}^k x\wedge\bar{b}}{p(\zeta)} \zeta^{-k} \frac{d\zeta}{\zeta}\right)$$

$$= \operatorname*{Res}_{\zeta=0,\infty} \sum_{k\geq 0} \left(\int_{\tilde{M}} \tilde{T}^k x\wedge\bar{b}\right) \frac{1}{\zeta^k p(\zeta)} \frac{d\zeta}{\zeta}$$

$$+ \operatorname*{Res}_{\zeta=0,\infty} \sum_{k>0} \left(\int_{\tilde{M}} \tilde{T}^{-k} x\wedge b\right) \frac{\zeta^k}{p(\zeta)} \frac{d\zeta}{\zeta} \qquad (*)$$

The polynomial p may be assumed, without loss of generality to be of the form:

$$p(z) = a_0 + a_1 z + \ldots + a_p z^p$$

with $a_0 \neq 0$. For, if $p(z) = z^{\ell} q(z)$ some $\ell > 0$ with $q(z)$ of the above form, then:

$$\tilde{T}^{\ell} q(\tilde{T}) a = dx$$

i.e. $q(\tilde{T}) a = d(\tilde{T}^{-\ell} x)$

and so we can replace p by q, so long as x is replaced by $\tilde{T}^{-\ell} x$.

Hence $1/p(\zeta)$ can be written as a series in ζ with first term $1/a_0$, near $\zeta = 0$. The residues at 0 in the second term for (a,b) above $(*)$, thus all vanish. To investigate the residues at $\zeta = \infty$, put $z = 1/\zeta$. Then:

$$\tilde{T}^k x \wedge \bar{b} \frac{1}{\zeta^k p(\zeta)} = (\tilde{T}^k x \wedge \bar{b}) \frac{z^k}{p(1/z)} .$$

However,

$$z^k/p(1/z) = z^k/(a_o + a_1/z + \ldots + a_p/z^p)$$

$$= z^{k+p}/(a_p + a_{p-1} z + \ldots + a_0 z^p)$$

and thus the residues at $\zeta = \infty$ in the first term in $(*)$ above, all vanish, except possibly for $k = 0$. Thus

$$(a,b) = \operatorname*{Res}_{\zeta=0} \sum_{k\geq 0} \left\{ \int_{\tilde{M}} \tilde{T}^k x \wedge \bar{b}) \right\} \frac{1}{\zeta^k p(\zeta)} \frac{d\zeta}{\zeta}$$

$$+ \operatorname*{Res}_{z=0} \sum_{k\geq 0} \left\{ \int_{\tilde{M}} \tilde{T}^{-k} x \wedge \bar{b}) \right\} \frac{1}{z^k p(1/z)} \frac{dz}{z}$$

$$= \sum_{k\geq 0} \left\{ \int_{\tilde{M}} \tilde{T}^k x \wedge \bar{b} \right\} \quad (\text{coeff of } \zeta^k \text{ in } 1/p(\zeta))$$

$$+ \sum_{k\geq 0} \left\{ \int_{\tilde{M}} T^{-k} x \wedge \bar{b} \right\} \quad (\text{coeff of } \zeta^k \text{ in } 1/p(1/\zeta)) \qquad (i))$$

However, $p(\tilde{T})a = dx$ and since a,b are the images of $\alpha,\beta \in H^n(N)$ in H_c, thus:

$$\left.\begin{aligned} a &= \chi_{[0,1]}\ \alpha \wedge dt \\ b &= \chi_{[0,1]}\ \beta \wedge dt \end{aligned}\right\} \quad \text{(see §3)} .$$

Thus $\displaystyle\int_{\tilde{M}} (\tilde{T}^k x \wedge \bar{b}) = \int_0^1 \int_N (\tilde{T}^k x \wedge \bar{\beta} \wedge dt)$

$$= \int_N T^k \left\{\int_{-k}^{-k+1} z\, dt\right\} \wedge \bar{\beta}$$

where z is that part of x involving no dt's .

Let $x = z + y \wedge dt$ where y,z are forms not involving dt. Then $dx = dz \pm \partial z/\partial t \wedge dt + dy \wedge dt$. Since:

$$\begin{aligned} dx &= p(\tilde{T})a \\ &= p(\tilde{T})(\chi_{[0,1]}\ \alpha) \\ &= p_\mu(T^\mu \alpha) \quad \text{on that part of } \tilde{M} \text{ where } \mu < t < \mu + 1 \end{aligned}$$

$\Rightarrow \quad dz = 0$

and $dy \pm \dfrac{\partial z}{\partial t} = p_\mu(T^\mu \alpha)$ on that part of $\tilde{M}$ where $\mu < t < \mu + 1$.

A solution of this equation is thus:

$$z(x,t) = p_0\alpha + \ldots + p_{\mu-1}T^{\mu-1}\alpha + (t-\mu)p_\mu T^\mu \alpha \quad \text{if} \quad \mu < t < \mu + 1$$

Therefore,

$$\int_k^{k+1} z dt = \begin{cases} 0 & \text{if} \quad k \le -1 \\ p_0\alpha + \ldots + p_{k-1}T^{k-1}\alpha + \tfrac{1}{2}p_k T^k \alpha & \text{if} \quad k \ge 0 \end{cases} .$$

From (i), we thus obtain:

$$(a,b) = \sum_{k\geq 0} \left[\int_N T^k \left\{\int_{-k}^{-k+1} zdt\right\} \wedge \overline{\beta}\right] \text{(coeff. of } \zeta^k \text{ in } \frac{1}{p(\zeta)})$$

$$+ \sum_{k\geq 0} \left[\int_N T^{-k} \left\{\int_{k}^{k+1} adt\right\} \wedge \overline{\beta}\right] \text{(coeff. of } \zeta^k \text{ in } \frac{1}{p(\frac{1}{\zeta})})$$

$$= \int_N (\tfrac{1}{2}\, p_0\alpha) \wedge \overline{\beta} \quad \text{(coeff. of } \zeta^0 \text{ in } \frac{1}{p(\zeta)})$$

$$+ \sum_{k\geq 0} \left[\int_N [T^{-k}(p_0\alpha+\ldots+p_{k-1}T^{k-1}\alpha + \tfrac{1}{2}p_kT^k\alpha)] \wedge \overline{\beta} \quad \text{(coeff of } \zeta^k \text{ in } \frac{1}{p(\frac{1}{\zeta})})\right]$$

$$= \tfrac{1}{2} \int_N \alpha\wedge\overline{\beta} + \tfrac{1}{2} \sum_{k\geq 0} \left(\int_N p_k\alpha\wedge\overline{\beta}\right) \text{(Coeff. of } \zeta^k \text{ in } \frac{1}{p(\frac{1}{\zeta})})$$

$$+ \sum_{k>0} \sum_{s=0}^{k-1} \int_N p_s(T^{s-k}\alpha)\ \overline{\beta}) \text{(Coeff. of } \zeta^k \text{ in } \frac{1}{p(\frac{1}{\zeta})}) .$$

$$= \tfrac{1}{2} \int_N \alpha\wedge\overline{\beta} + \tfrac{1}{2} \sum_{k\geq 0} \left\{\int_N p_k\alpha\wedge\overline{\beta}\right\} \text{(Coeff. of } \zeta^0 \text{ in } \frac{1}{\zeta^k p(\frac{1}{\zeta})})$$

$$+ \sum_{s=0}^{\infty} \sum_{k=s+1}^{\infty} \left\{\int_N p_s(T^{-(k-s)}\right\} \wedge\overline{\beta} \text{ (Coeff. of } \zeta^{k-s} \text{ in } \frac{1}{\zeta^s p(\frac{1}{\zeta})})$$

$$= \tfrac{1}{2} \int_N \alpha\wedge\overline{\beta} + \tfrac{1}{2} \left\{\int_N \alpha\wedge\overline{\beta}\right\} [\text{coeff. of } \zeta^0 \text{ in } \sum_{k\geq 0} \frac{p_k}{\zeta^k p(\frac{1}{\zeta})}]$$

$$+ \sum_{s=0}^{\infty} \sum_{\ell=1}^{\infty} \left\{\int_N (T^{-\ell}\alpha) \wedge\overline{\beta}\right\} [\text{coeff. of } \zeta^\ell \text{ in } \frac{p_s}{\zeta^s p(\frac{1}{\zeta})}]$$

putting $\ell = k - s$

$$= \tfrac{1}{2} \int_N \alpha\wedge\overline{\beta} + \tfrac{1}{2} \left\{\int_N \alpha\wedge\overline{\beta}\right\} \text{ (Coeff of } \zeta^0 \text{ in } 1)$$

$$+ \sum_{\ell=1}^{\infty} \left\{\int_N (T^{-\ell}\alpha)\wedge\overline{\beta}\right\} [\text{Coeff. of } \zeta^\ell \text{ in } 1]$$

$$= \int_N \alpha\wedge\overline{\beta}$$

since last term vanishes. Q.E.D.

§5. The relation between H_a and H_c

Let Ω_a consist of all forms $\omega \in \Omega^*(\tilde{M})$ such that

$$\sum_{n=-\infty}^{\infty} (\tilde{T}^n \omega)\zeta^{-n}$$

converges pointwise in ζ when ζ is on the unit circle $|\zeta| = 1$, and is C^∞ convergent on compact subsets of $\tilde{M}$. Let H_a denote the homology of the Ω_a complex. As noted in §3, there are many possible maps $H(N) \to H_c$, all of which are equivalent. For the purpose of this section, all forms are C^∞. This is achieved by 'rounding the corners' of $\chi_{[0,1]}\alpha \wedge dt$ in the map of §3.

Define A' to be the ring of analytic functions on S^1. Then Ω_a is an A'-module, the action of A' on Ω_a being defined by $\tilde{T}$.

By Theorem 10.11 (see §10), we obtain:

<u>Proposition 5.1</u>. $H_a = H_c \otimes_{\tilde{A}} A'$

Now $H_c = H(\tilde{\Omega}_c)$ is the homology of forms with compact support on $\tilde{M}$. The Fourier transform (Definition 4.1) transforms $\omega \in \tilde{\Omega}_c$ to $\hat{\omega}(\zeta) \in \Omega(M;\chi_\theta)$ where $\zeta = e^{-i\theta}$.

Thus H_c can be written as a direct sum of parts, each corresponding to a particular ζ:

$$H_c = \bigoplus_{\zeta} H_c^{(\zeta)} \qquad (\zeta \in \mathbb{C}^*)$$

where $H_c^{(\zeta)}$ is the homology associated with $\Omega(M;\chi)$. So H_a can be identified with that part of H_c corresponding to $\zeta \in S^1$.

By Proposition 5.1,

$$H_a = \bigoplus_{|\zeta|=1} H_c^{(\zeta)} .$$

The inner product on H_c, as defined in Definition 4.4 is:

$$(a,b) = \operatorname*{Res}_{\zeta=0,\infty} \int_{\tilde{M}} \frac{\hat{x}\wedge\bar{b}}{p(\zeta)} \frac{d\zeta}{\zeta} \qquad \text{for } a,b \in H_c$$

where $p(T)a = dx$ some polynomial p. If p has all its roots on S^1, then:

$$(a,b) = -\operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} \frac{\hat{x}\wedge\bar{b}}{p(\zeta)} \frac{d\zeta}{\zeta}$$

by the residue theorem, since the only roots of p are on the unit circle.

So, the inner product defined on Ω_a, and thus on H_a, which is induced by the inner product defined on Ω_c (or H_c) is given by:

<u>Definition 5.2.</u> <u>If</u> $a,b \in H_a$, <u>define</u>:

$$(a,b)_{H_a} = -\operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} \frac{\hat{x}\wedge\bar{b}}{p(\zeta)} \frac{d\zeta}{\zeta} .$$

The inner product on H_c is thus a direct sum of inner products on the individual $H_c^{(\zeta)}$. The inner product on H_a is, when H_a is identified as:

$$\bigoplus_{\zeta\in S^1} (H_c^{(\zeta)}$$

precisely the restriction of the inner product on H_c, and is thus also a direct sum.

We thus obtain:

<u>Theorem 5.3</u>

(i) H_c <u>can be written as a direct sum of spaces</u> $H_c^{(\zeta)}$, <u>and then</u> H_a <u>is isomorphic to the direct sum of those</u> $H_c^{(\zeta)}$ <u>corresponding to</u> $\zeta \in S^1$.

(ii) <u>The inner product on</u> H_c <u>decomposes into inner products on</u> $H_c^{(\zeta)}$, <u>and that on</u> H_a <u>is the sum of those parts of the</u> H_c <u>inner product on</u> $H_c^{(\zeta)}$ <u>spaces where</u> $|\zeta| = 1$.

Note that inner products can be defined between spaces of forms like Ω^*, which do not have an associated unique dimension - that is, between forms which are linear combinations of forms of different dimensions. This is done using linearity, and 'incompatible' dimensions give rise to a zero result. For example, the inner product defined on H_c^* (Definition 4.4) is only 'compatible' between forms a, b of degrees s.t.

$$\deg(\hat{x}) + \deg(b) = 2n + 1$$

i.e. $\deg(a) + \deg(b) = 2(n + 1)$.

For any other pair we obtain a zero result.

§6. The connection between Coker D and H_a

The operators D_θ act on $\Omega^{even}(M;\chi_\theta)$ for $\theta \in S^1$. We can consider the family D_θ to consist of operators dependent analytically on a variable $\zeta = e^{i\theta} \in S^1$. Thus, we can consider its image to be a subspace of $\Omega^{even}(M;\chi_\theta)$ defined as $\mathrm{Im}(D_\theta)$, considered as formally dependent on ζ.

So (Coker D) refers to the space of forms in $\Omega^{even}(M;\chi_\theta)$ which can be written as analytic functions of $\zeta = e^{i\theta}$, up to an equivalence relation under which:

$$\omega_1 \sim \omega_2$$

iff $(\omega_1 - \omega_2) = D\phi$ some ϕ in this space of 'analytic' forms on M, in $\Omega^{even}(M;\chi_\theta)$. So, we are considering D as a map:

$$V \to V$$

where V is the space of all forms ω which depend analytically on ζ, and for each $\zeta = e^{i\theta}, \omega(\zeta) \in \Omega^{even}(M;\chi_\theta)$. This map takes $\omega(\zeta)$ to $(D_\theta \omega(\zeta))$.

Define $\Omega_m = \{(\omega/p(\tilde{T}))$ where $\omega \in \Omega_a$ and p is a non-zero polynomial withzeros only on $|z| = 1\}/\sim$

where $\sim$ is the equivalence relation:

$$\omega/p(\tilde{T}) \sim \omega'/p'(\tilde{T})$$

iff $p'(\tilde{T})\omega = p(\tilde{T})\omega'$.

In this section we investigate the connection between Coker D and H_a, and in the next section, we shall discuss their inner products.

<u>Lemma 6.1.</u> <u>If</u> $\omega \in \Omega_a$, <u>then there exist unique</u> $x_1, x_2 \in \Omega_m$ <u>such that</u>

$$\omega = x_1 + x_2$$

<u>and</u> $p(\tilde{T})x_1 = dy_1$

$$p(\tilde{T})x_2 = d^*y_2$$

<u>some polynomial</u> p <u>whose roots are all on</u> $|z| = 1$, <u>and some</u> $y_1, y_2 \in \Omega_a$.

That is, $x_1 = d(\text{element of } \Omega_m)$ and $x_2 = d^*(\text{element of } \Omega_m)$.

<u>Proof</u> : Let $\tilde{\omega}(\theta) = \sum_{n=-\infty}^{\infty} (e^{-in\theta}\tilde{T}^n\omega)$

Then $\tilde{T}\tilde{\omega}(\theta) = e^{i\theta}\cdot\tilde{\omega}(\theta)$, and so $\tilde{\omega}(\theta) \in \Omega(M;\chi_\theta)$. Suppose M_0 is a fundamental region for the action of $\tilde{T}$ on $\tilde{M}$, i.e. of the form $N \times [0,1]$.

Then $\tilde{\omega}(\theta)|_{M_0} = \sum_{n=-\infty}^{\infty} (e^{-in\theta}\tilde{T}^n(\omega|_{\tilde{T}^{-n}M_0}))$

$$\Rightarrow \int_0^{2\pi} e^{in\theta}\tilde{\omega}(\theta)|_{M_0} d\theta = 2\Pi\cdot\tilde{T}^n(\omega|_{\tilde{T}^{-n}M_0})$$

and so $\omega|_{\tilde{T}^n M_0} = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\tilde{T}^n(\tilde{\omega}(\theta)|_{M_0})d\theta$.

However, M is compact, and so the Hodge theorem for the coefficient system χ_θ provides, for generic θ, a Green's function G_ζ, so that:

$$G_\zeta D_\zeta = D_\zeta G_\zeta = 1$$

where D_ζ is our Dirac-type operator. The points at which such a G_ζ does not exist are those with $\ker(D_\zeta) \neq 0$. There are finitely many such ζ's, namely the eigenvalues of T.

We reduce the situation to a finite-dimensional one by introducing a cut-off, so that we only examine those parts of the space V spanned by eigenvectors of D corresponding to eigenvalues λ s.t. $|\lambda| \le c$ where c is a suitable cut-off. The value of c is chosen so as to include all those eigenvalues crossing zero at some θ.

Then, for $\zeta \notin \{\zeta_j\}$ where ζ_j are the exceptional ζ's, we have

$$\tilde{\omega}(\theta) = (\tau d + d\tau)(G_\zeta \tilde{\omega})$$

$$= d\tilde{\omega}_1 + d^*\tilde{\omega}_2$$

where $\tilde{\omega}_1 = \tilde{\omega}_2 = \tau G_\zeta \tilde{\omega}$. However, $\tilde{\omega}_1$, $\tilde{\omega}_2$ are analytic functions of ζ away from the ζ_j. Their behaviour near ζ_j's may have poles. Since the situation is finite-dimensional, thus these poles are all of finite order. Let

$$p(\zeta) = \prod_j (\zeta - \zeta_j)^{n_j}$$

where n_j is the order of the pole at ζ_j. Then:

$$p(\zeta)\tilde{\omega}_i \qquad (i = 1,2)$$

have no poles. Define:

$$y_i|_{(\tilde{T}^n M_0)} = \frac{1}{2\pi}\int_0^{2\pi} p(e^{i\theta})\tilde{T}^n(\tilde{\omega}_i(\theta))e^{-in\theta}d\theta \quad (i = 1,2) .$$

Then, the Fourier transform of y_i is $p(e^{i\theta})\tilde{\omega}_i(\theta)$, i.e. $y_i = p(\tilde{T})\omega_i$.

Thus $(p(\tilde{T})\omega) = p(\zeta)\tilde{\omega}(\theta)$

$$= d(p(\zeta)\tilde{\omega}_1(\theta)) + d^*(p(\zeta)\tilde{\omega}_2(\theta))$$

$$= d\tilde{y}_1 + d^*\tilde{y}_2 .$$

Hence $p(\tilde{T}) = dy_1 + d^*y_2$.

Moreover, $\omega = x_1 + x_2$ where:

$$\left.\begin{aligned} x_1 &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{T}^n d\tilde{\omega}_1(\theta)e^{-in\theta}d\theta \\ x_2 &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{T}^n d^*\tilde{\omega}_2(\theta)e^{-in\theta}d\theta \end{aligned}\right\} \quad \text{on } \tilde{T}^n M_0$$

since $\omega = \frac{1}{2\pi}\int_0^{2\pi} \tilde{T}^n(d\tilde{\omega}_1(\theta) + d^*\tilde{\omega}_2(\theta))e^{-in\theta}d\theta$ on $\tilde{T}^n M_0$.

So $x_1, x_2 \in \Omega_a$, while $y_1, y_2 \in \Omega_a$ and p is a polynomial all of whose roots are on the unit circle.

Q.E.D.

Using this lemma, we now show the equivalence of H_a and Coker D, exhibiting the isomorphism.

Theorem 6.2. $H_a \cong$ Coker D

Proof : Define a map θ from H_a to Coker D as follows. Any element of H_a can be represented as $[x+y]$ where:

$[x] \in H_a^{even}$

$[y] \in H_a^{odd}$.

We define the map as:

$$\theta : H_a \to \text{Coker } D$$

$$[x+y] \to [x+\tau y].$$

Well-defined in Coker D

(i) Since $[x+y] \in H_a$, there exists a polynomial $p(\tilde{T})$ such that

$$p(\tilde{T})(x+y) = d(x'+y')$$

some $x' \in \Omega_a^{odd}$, $y' \in \Omega_a^{even}$.

$$\left.\begin{array}{ll}\text{Thus} & p(\tilde{T})x = dx' \\ \text{and} & p(\tilde{T})y = dy'\end{array}\right\} .$$

By Lemma 6.1, $\exists\, x_j, y_j \in \Omega_a$ $(j = 1,2)$ s.t.

$$x' = x_1 + x_2$$

$$y' = y_1 + y_2$$

and $q(\tilde{T})x_1, q(\tilde{T})y_1 \in \text{Im } d$

$q(\tilde{T})x_2, q(\tilde{T})y_2 \in \text{Im } d^*$, say d^*z_2, d^*z_2'

some polynomial $q(\tilde{T})$ in $\tilde{T}$, whose roots all lie on $|z| = 1$.

Hence $\theta([x+y]) = [x + \tau y]$

$$\begin{aligned}\text{and}\quad p(\tilde{T})q(\tilde{T})\theta([x+y]) &= q(\tilde{T})[dx'+\tau dy'] \\ &= q(\tilde{T})[dx_2+\tau dy_2] \\ &= [dd^*z_2+\tau dd^*z_2'] \\ &= [(d\tau + \tau d)(d\tau z_2+\tau d\tau z_2')] \\ &= 0\end{aligned}$$

since $D = d\tau + \tau d$ and $d^* = \tau d\tau$ (see §2, Definition 2.1).

Thus $\theta([x+y]) \in \text{Coker } D$.

(ii) We next show that if $\omega = x + y$, $x \in \Omega_a^{\text{even}}(M)$, $y \in \Omega_a^{\text{odd}}(M)$ vanishes in H_a, then:

$$\theta([\omega]) = 0$$

in Coker D. So, suppose that:

$$x = dx', \quad x' \in \Omega_a^{odd}(M)$$

$$y = dy', \quad y' \in \Omega_a^{even}(M) \ .$$

Then $\theta([\omega]) = dx' + \tau dy'$. Apply Lemma 6.1. to x', y':

$$\left.\begin{aligned} x' &= x_1 + x_2 \\ y' &= y_1 + y_2 \end{aligned}\right\}$$

where $q(\tilde{T})x_1, \ q(\tilde{T})y_1 \in \text{Im } d$

$q(\tilde{T})x_2, \ q(\tilde{T})y_2 \in \text{Im } d^*$, say d^*z_2, d^*z_2'

some polynomial $q(\tilde{T})$. Thus:

$$q(\tilde{T})\theta([\omega]) = [dd^*z_2 + \tau dd^*z_2']$$

$$= [(d\tau+\tau d)(d\tau z_2 + d^*z_2')]$$

i.e. $$\theta([\omega]) = [(d\tau+\tau d)(\tau x_2+y_2)]$$

$$= 0 \text{ in Coker } D$$

as required.

<u>Injectivity</u>

Suppose $[x+\tau y] = \dot{\theta}([x+y])$ vanishes in Coker D. That is, $x+\tau y = D\sigma$ some $\sigma \in \Omega_a^{ev}$. Then since $x,y \in H_a$.

$$\left.\begin{aligned} p(\tilde{T})x &= dx' \\ p(\tilde{T})y &= dy' \end{aligned}\right\} \qquad (*)$$

some $x' \in \Omega_a^{odd}$, $y' \in \Omega_a^{even}$ and some polynomial p whose roots are all on $|z| = 1$.

Thus $p(\tilde{T})(\tau d\sigma + d\tau\sigma) = p(\tilde{T})(x + \tau y)$

$$= dx' + \tau dy' \ .$$

By uniqueness of the representation in Lemma 6.1,

$$\left.\begin{aligned} p(\tilde{T})d\tau\sigma &= dx' \\ p(\tilde{T})d\sigma &= dy' \end{aligned}\right\}$$

and comparing with (*) above, $x = d\tau\sigma$, $y = d\sigma$. Thus $x + y = d\tau\sigma + d\sigma$, and is thus a closed form. So, it vanishes in that, hence proving injectivity.

Surjectivity

Suppose that:

$$[\omega] \in \text{Coker } D \ .$$

Then, for some polynomial p with all its roots on the unit circle,

$$p(\tilde{T})\omega = D\eta$$

some $\eta \in \Omega_a$.

Hence $\omega = D\phi$, some ϕ with $p(\tilde{T})\phi = \eta$ (this ϕ is an element of Ω_m). Thus:

$$\omega = (d\tau + \tau d)\phi$$

$$= d(\tau\phi) + \tau(d\phi)$$

$$\Rightarrow [\omega] = \theta([d(\tau\phi + \phi)])$$

$$\in \text{Im}\,\theta$$

since $d(\tau\phi + \phi) \in H_a$. This is because:

$$p(\tilde{T})d(\tau\phi + \phi) = d(\tau\eta + \eta)$$

$$= 0 \quad \text{since we are in } H_a \ .$$

Hence θ is surjective.

Q.E.D.

There is therefore a map from a subset of $H(N)$ to Coker D which is a bijection. This is given by the composition of the maps so far outlined.

§7. Calculation of the quadratic form on Coker D

In the last section, we described a map:

$$\theta : H_a \to \text{Coker } D$$

$$[x + y] \to [x + \tau y]$$

for $x \in \Omega_a^{\text{even}}$, $y \in \Omega_a^{\text{odd}}$. The quadratic form on H_a (Definition 5.2) was:

$$(a,b) = - \operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} \frac{\sum_k (\tilde{T}^k x \wedge \bar{b}) \zeta^{-k}}{p(\zeta)} \frac{d\zeta}{\zeta} \qquad (*)$$

where $p(\tilde{T})a = dx$, some polynomial p, all of whose roots are on the unit circle.

Let $\Delta = dd^* + d^*d$ be the Laplacian on $\Omega^{\text{even}}(M;\chi_\theta)$. Let G_Δ be the Green's operator for Δ. For generic θ, i.e. $e^{i\theta} \neq \zeta_j$ for the finite number of special points ζ_j, Δ is invertible, and so:

$$\Delta G_\Delta = G_\Delta \Delta = 1.$$

Define $G_d = d^* G_\Delta$. Then for $a,b \in H_a$, let p be a polynomial with roots only on $|z| = 1$ such that:

$$p(\tilde{T})a = dx .$$

The pairing:

$$\langle\alpha,\beta\rangle = \sum_k \int_{\tilde{M}} \tilde{T}^k \alpha \wedge \bar{\beta} \cdot \zeta^{-k}$$

$$= \int_{\tilde{M}} \hat{\alpha} \wedge \bar{\beta} \qquad \text{(see Definition 4.4)}$$

on Ω_a^* extends to a pairing on Ω_m^*. However,

$$d(G_d a) = dd^* G_\Delta a$$

$$= (dd^* + d^*d) G_\Delta a \quad \text{since} \quad da = 0$$

$$= a \ .$$

Thus, if $\widehat{(G_d a)}$ has no poles, then:

$$(a,b) = - \operatorname*{Res}_{|\zeta|=1} \sum_k \int_{\tilde{M}} \zeta^{-k} \tilde{T}^k \widehat{(G_d a)} \wedge \bar{b}$$

using (*) with x replaced by $G_d a$ and p replaced by 1.
So:

$$(a,b) = - \operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} \widehat{(G_d a)} \wedge \bar{b}$$

$$= - \operatorname*{Res}_{|\zeta|=1} \langle G_d a, b \rangle \ .$$

If, however, $\widehat{(G_d a)}$ has poles of ζ_j, then we can find a polynomial p whose roots lie on the unit circle, (and which are the ζ_j), such that:

$$p(\tilde{T})(G_d a) = \omega$$

is in H_a, as its Fourier transform $p(\zeta)\widehat{(G_d a)}$ has no poles.
Then, we obtain:

$$(a,b) = - \operatorname*{Res}_{|\zeta|=1} \frac{1}{p(\zeta)} \sum_k \int_{\tilde{M}} \zeta^{-k} \tilde{T}^k \omega \wedge \bar{b}$$

$$= - \operatorname*{Res}_{|\zeta|=1} \frac{1}{p(\zeta)} \langle \omega, b \rangle \ .$$

By defintion, when $\langle , \rangle$ is extended from Ω_a to Ω_m,

$$\langle \omega, b \rangle = p(\zeta) \quad \langle G_d a, b \rangle$$

and so the inner product on H_a can be written:

$$(a,b) = - \operatorname*{Res}_{|\zeta|=1} \langle G_d a, b \rangle \ .$$

Theorem 7.1. Under the isomorphism of Theorem 6.2 between H_a and Coker D, the inner product on H_a (see Definition 5.2) corresponds to the inner product:

$$(a,b) = -\operatorname*{Res}_{|\zeta|=1} \langle(\tau G_d + G_d\tau)a, \tau b\rangle$$

on Coker D.

Note that here again, $\langle,\rangle$ is the extended inner product.

Proof: Suppose that $x = x_1 + x_2$ and $y = y_1 + y_2$ are elements of Ω_a. Thus:

$$x_1, y_1 \in \Omega_a^{even}$$

$$x_2, y_2 \in \Omega_a^{odd}$$

and $\left.\begin{aligned} p(T)x_i &= dx_i' \\ p(T)y_i &= dy_i' \end{aligned}\right\}$ for $i = 1,2$

some polynomial p whose roots are all on the unit circle, and some $x_i', y_i' \in \Omega_a^*$.

Thus,

$$(x,y)_{H_a} = -\operatorname*{Res}_{|\zeta|=1} \langle G_d(x_1+x_2), (y_1+y_2)\rangle \quad \text{by above}$$

$$= -\operatorname*{Res}_{|\zeta|=1} [\langle G_d x_1, y_1\rangle + \langle G_d x_2, y_2\rangle]$$

since $G_d x_1 \in \Omega_a^{odd}$, $G_d x_2 \in \Omega_a^{even}$

$y_1 \in \Omega_a^{even}$, $y_2 \in \Omega_a^{odd}$

and so the cross-terms have incompatible dimensions: for $\langle a,b\rangle$, compatibility means that $\partial a + \partial b = 2n + 1$ (i.e. dimensions have opposite parities).

However, $\theta([x]) = [x_1 + \tau x_2]$

and $\theta([y]) = [y_1 + \tau y_2]$

$$\Rightarrow \langle(\tau G_d + G_d\tau)(x_1 + \tau x_2), \tau(y_1 + \tau y_2)\rangle$$

$$= \langle \tau G_d x_1, \tau y_1\rangle + \langle \tau G_d \tau x_2, \tau y_1\rangle + \langle G_d \tau x_1, \tau y_1\rangle + \langle G_d x_2, \tau y_1\rangle$$

$$+ \langle \tau G_d x_1, y_2\rangle + \langle \tau G_d \tau x_2, y_2\rangle + \langle G_d \tau x_1, y_2\rangle + \langle G_d x_2, y_2\rangle \quad .$$

Since $G_d = d^* G_\Delta = \tau d \tau G_\Delta$

$\Rightarrow \quad \tau G_d = d\tau G_\Delta$.

Thus

$$\left.\begin{aligned} \langle \tau G_\Delta x_1, y_1\rangle &= \langle d\tau G_\Delta x_1, y_2\rangle = 0 \\ \langle \tau G_\Delta \tau x_2, y_2\rangle &= \langle d G_\Delta x_2, y_2\rangle = 0 \end{aligned}\right\} \quad \text{as} \quad dy_2 = 0,\ \langle d\alpha, \beta\rangle = \pm\langle \alpha, d\beta\rangle$$

$$\left.\begin{aligned} \langle \tau G_\Delta \tau x_1, y_2\rangle &= \langle \tau d G_\Delta x_1, y_2\rangle = 0 \\ \langle G_\Delta \tau x_1, \tau y_1\rangle &= \langle \tau d G_\Delta x_1, \tau y_1\rangle = 0 \end{aligned}\right\} \quad \text{as} \quad dG_\Delta x_1 = 0$$

$$\langle \tau G_\Delta \tau x_2, \tau y_1\rangle = \langle d G_\Delta x_2, \tau y_1\rangle = 0 \quad \text{as} \quad dG_\Delta x_2 = 0$$

$$\begin{aligned} \langle G_\Delta x_2, \tau y_1\rangle &= \langle \tau d \tau G_\Delta x_2, \tau y_1\rangle \\ &= \pm\langle d\tau G_\Delta x_1, y_1\rangle \\ &= 0 \qquad\qquad \text{as} \quad dy_1 = 0 \ . \end{aligned}$$

Hence $\langle(\tau G_d + G_d\tau)(x_1 + \tau x_2),\ \tau(y_1 + \tau y_2)\rangle$

$$= \langle G_d x_2, y_2\rangle + \langle \tau G_d x_1, \tau y_1\rangle \quad .$$

However, $\langle \tau a, \tau b \rangle = \sum_{k\in\mathbb{Z}} \int_{\tilde{M}} (\tilde{T}^k \tau a \wedge \tau b)\zeta^{-k}$

$$= \sum_{k\in\mathbb{Z}} \int_{\tilde{M}} (-1)^{\partial a.\partial b} (\tilde{T}^k a \wedge b)\zeta^{-k}$$

since $\tau a \wedge \tau b = (-1)^{\partial a \cdot \partial b}\, a \wedge b$

$= \langle a,b \rangle$, since $p(2n+1-p)$ is always even .

So $\langle (\tau G_d + G_d\tau)(x_1 + \tau x_2),\ \tau(y_1 + \tau y_2)\rangle$

$$= \langle G_d x_1, y_1\rangle + \langle G_d x_2, y_2\rangle$$

and hence:

$$(x,y)_{H_a} = -\operatorname*{Res}_{|\zeta|=1} \langle (\tau G_d + G_d\tau)\theta(x), \tau\theta(y)\rangle .$$

Q.E.D.

Finally, note that $\tau G_d + G_d\tau = G_D$. For,

$\tau\Delta = \tau(dd^* + d^*d)$

$= \tau(d\tau d\tau + \tau d\tau d)$ as $d^* = \tau d\tau$

$= \tau d\tau d\tau + d\tau d$ as $\tau^2 = 1$

$= (\tau d\tau d + d\tau d\tau)\tau$

$= \Delta\tau$

i.e. τ, Δ commute, and hence τ, G_Δ commute. Thus:

$(\tau d+d\tau)(\tau G_d+G_d\tau) = \tau d\tau G_d + \tau dG_d\tau + dG_d + d\tau G_d\tau$

$= \tau d\tau d^* G_\Delta + \tau dd^* G_\Delta\tau + dd^* G_\Delta + d\tau d^* G_\Delta\tau$

$= \tau dd^*\tau G_\Delta + dd^* G_\Delta$

since $\tau d\tau d^* = d^{*2} = 0$, $d\tau d^* = \tau d^{*2} = 0$

and G_Δ, τ commute

$$= (d^*d + dd^*)G_\Delta$$

$$= \Delta G_\Delta .$$

Hence $\tau G_d + G_d \tau$ is the Green's operator of $\tau d + d\tau = D$. We denote it by G_D. Thus the inner product corresponding to definition 5.2. on H_a is given by:

<u>Corollary 7.2</u>. $(a,b)_{\text{Coker } D} = - \operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} \widehat{(G_D a)} \wedge \tau \bar{b}$

§8. Perturbation theory

Consider D_θ as a family of operators on the spaces:

$$\Omega^{even}(M;\chi_\theta) .$$

Since, for each θ, the space on which D_θ acts is different, it is difficult to compare their eigenvalues, or do any kind of perturbation theory. So, we must either:

(a) conjugate D_θ, so as to obtain operators depending on θ, which act on a common space, and then apply the standard perturbation theory;

or

(b) consider D_θ as the restriction of a fixed operator $\mathcal{D}$ (not to be confused with the operator of §2) on $\Omega^{even}(M)$ to a subspace:

$$V_\theta = \Omega^{even}(M;\chi_\theta) \subseteq V = \Omega^{even}(\tilde{M})$$

and then V_θ can be defined as the subset of V satisfying a certain 'algebraic' condition - algebraic in $e^{i\theta}$. This approach is essentially the one used in the last two sections.

Since $\Omega^{even}(M;\chi_\theta)$ involves a twist of $e^{i\theta}$ as we go around S^1, we can define a map:

$$\alpha_\theta : \Omega^{even}(M;\chi_0) \to \Omega^{even}(M;\chi_\theta)$$

$$\omega \to e^{-i\theta u}\omega$$

where u is the function:

$$u(x) = \frac{1}{2\pi i} \log(\pi(x)) \qquad (x \in M)$$

and π is the projection of M onto S^1, considered as a subset of $\mathbb{C}^*$.

The conjugated operator D_θ is:

$$D_\theta : \Omega^{even}(M;\chi_0) \to \Omega^{even}(M;\chi_0)$$

and
$$D_\theta\omega = (\alpha_\theta \circ \mathcal{D} \circ \alpha_\theta^{-1})\omega$$
$$= e^{-i\theta u}\mathcal{D}(e^{i\theta u}\omega)$$
$$= e^{-i\theta u}(e^{i\theta u}\mathcal{D}\omega + \mathcal{D}(e^{i\theta u})\omega)$$
$$= (D_0 + \theta B)\omega$$

where B is a constant operator; that is, it does not depend upon θ. Thus we have:

Lemma 8.1. The operators D_θ can be conjugated so as all act on the same space $\Omega^{even}(M;\chi_0)$, and they become:

$$D_0 + \theta B$$

some operator B independent of θ.

These new operators all act on the same space. All their eigenvalues are real, since the original $\mathcal{D}$ is self-adjoint and thus has real eigenvalues. Only a finite number of eigenvalues, thought of as functions of θ, cross the θ-axis, as θ varies. Let P be the orthogonal projection of $\Omega^{even}(M;\chi_0)$ onto the space N:

$$N = \langle N_\theta : D_\theta \text{ has a zero eigenvalue}\rangle$$

where N_θ is the null space of D_θ. For each θ, N_θ is finite-dimensional, and there are only a finite number of possible θ's which give non-trival N_θ. Hence N must be finite-dimensional.

So $\{PD_\theta P\}$ is a one-parameter family of operators which depend on θ in a linear manner, and have the same local eigenvalue properties as $\{D_\theta\}$. They do, however, act on a finite-dimensional space, unlike the $\{D_\theta\}$.

We now consider the situation in which we wish to determine the local behaviour of eigenvalues of a family of (finite-dimensional) Hermitian matrices:

$$A + \theta B \ .$$

Their characteristic equations are given by:

$$f(\theta,\lambda) = 0$$

for some polynomial f on λ,θ, whose roots for λ are real for all values of θ. The local behaviour of such a root can be written in a Puiseaux expansion as a series in $(\theta - \theta_0)^{1/m}$ some m. Let its leading term be:

$$\lambda \sim A(\theta - \theta_0)^{r/m}$$

say, some constant A. Since the number of real roots is constant (all roots are real), $m = 1$, and so we can describe the eigenvalues locally, and therefore globally, by analytic functions of θ. Hence:

<u>Proposition 8.2.</u> <u>The eigenvalues of</u> D_θ <u>are real analytic functions of</u> $\theta \in S^1$.

We will now use perturbation theory to find this leading term in the local behaviour near a value $\theta = \theta_0$ of θ at which the eigenvalue vanishes. Without loss of generality,

$\theta = 0$, and the order of contact is m say. So, we suppose that the local behaviours of the eigenvalue and corresponding eigenvector are given by:

$$\lambda = \lambda^{(m)}\theta^m + \lambda^{(m+1)}\theta^{m+1} + \ldots$$

$$\underline{v} = \underline{v}^{(0)} + \theta\underline{v}^{(1)} + \ldots$$

Let $\{\underline{v}_i\}$ be the eigenvectors of A, with corresponding eigenvalues λ_i. Then:

$$(A+\theta B)(\underline{v}^{(0)}+\theta\underline{v}^{(1)}+\ldots) = (\theta^{(m)}\lambda^{(m)}+\ldots)(\underline{v}^{(0)}+\theta\underline{v}^{(1)}+\ldots)$$

with, say, $\underline{v}^{(0)} = \underline{v}_i$ where $\lambda_i = 0$. We assume $\{\underline{v}_i\}$ to be an orthonormal system, which it is possible to choose since A is self-adjoint.

Equating coefficients of θ^r, we obtain:

$$A\underline{v}^{(r)} + B\underline{v}^{(r-1)} = 0 \qquad \forall r < m \qquad \text{(i)}$$

$$A\underline{v}^{(m)} + B\underline{v}^{(m-1)} = \lambda^{(m)}\underline{v}^{(0)} \qquad (r = m) \qquad \text{(ii)} .$$

<u>Lemma 8.3.</u> <u>In the above situation:</u>

(a) $$<\underline{v}_i|B|\underline{v}^{(r)}> = \begin{cases} 0 & \underline{\text{if}} \quad r < m-1 \\ \lambda^{(m)} & \underline{\text{if}} \quad r = m-1 . \end{cases}$$

(b) $X\underline{v}^{(r)} = \underline{v}^{(r-1)}$ <u>where</u> $X = -B^{-1}A$, <u>for</u> $r < m$.

(c) $\underline{v}^{(r)} = -GB\underline{v}^{(r-1)} + \underline{\alpha}^{(r)}$ <u>some</u> $\underline{\alpha}^{(r)}$ <u>in the null space of</u> A, <u>for</u> $1 \le r < m-1$.

<u>Proof</u>

(a) Now from (i), (ii)

$$<\underline{v}_i|A|\underline{v}^{(r)}> + <\underline{v}_i|B|\underline{v}^{(r-1)}> = \begin{cases} 0 & \text{if} \quad r < m \\ \lambda^{(m)} & \text{if} \quad r = m . \end{cases}$$

However, $\underline{Av}_i = \underline{0}$, and $\underline{A}$ is self-adjoint. Thus:

$$<\underline{v}_i|B|\underline{v}^{(r-1)}> = \begin{cases} 0 & \text{if } r < m \\ \lambda^{(m)} & \text{if } r = m \end{cases}$$

and so

$$<\underline{v}_i|B|\underline{v}^{(r)}> = \begin{cases} 0 & \text{if } r < m-1 \\ \lambda^{(m)} & \text{if } r = m-1 . \end{cases}$$

(b) Since, from (i),

$$B\underline{v}^{(r-1)} + A\underline{v}^{(r)} = 0 \qquad \text{for } r < m$$

$$\Rightarrow \qquad \underline{v}^{(r-1)} = -B^{-1}A\underline{v}^{(r)}$$
$$= X\underline{v}^{(r)}$$

where $X = -B^{-1}A$.

(c) From (i),

$$<\underline{v}_j|A|\underline{v}^{(r)}> + <v_j|B|\underline{v}^{(r-1)}> = 0$$

$$\Rightarrow \quad \lambda_j<\underline{v}_j|\underline{v}^{(r)}> + <v_j|B|\underline{v}^{(r-1)}> = 0$$

and so $<\underline{v}_j|\underline{v}^{(r)}> = -1/\lambda_j \; <v_j|B|\underline{v}^{(r-1)}>$ if $\lambda_j \neq 0$.

Thus $\underline{v}^{(r)} = -GB\underline{v}^{(r-1)} + \underline{\alpha}^{(r)}$ some $\underline{\alpha}^{(r)} \in N(A)$.

Q.E.D.

§9. Comparison of eigenvalue and inner product structures on Coker D.

Consider a single eigenvalue crossing at $\theta = \theta_0$. W.l.o.g., $\theta_0 = 0$, and so:

$$D_\theta = A + \theta B$$

where $A = D_0$. However,

$$\begin{aligned} D_\theta \omega &= e^{-i\theta u}\mathcal{D}(e^{i\theta u}\omega) \\ &= e^{-i\theta u}(\tau d + d\tau)(e^{i\theta u}\omega) \\ &= e^{-i\theta u}[e^{i\theta u}\tau(d\omega) + \tau(i\theta e^{i\theta u}du \wedge \omega) \\ &\quad + e^{i\theta u}d(\tau\omega) + (i\theta e^{i\theta u}du) \wedge \tau\omega] \\ &= (\tau d + d\tau)\omega + i\theta(\tau(du \wedge \omega) + du \wedge \tau\omega) \end{aligned}$$

i.e. $B = i(\tau(du \wedge \omega) + du \wedge \tau\omega)$.

Thus,

$$\begin{aligned} \tfrac{1}{i}\mathcal{D}B\omega &= (d\tau + \tau d)(\tau(du \wedge \omega) + du \wedge \tau\omega) \\ &= d(du\wedge\omega) + d(\tau(du\wedge\tau\omega)) + \tau d(\tau(du\wedge\omega)) + \tau d(du\wedge\tau\omega) \\ &= -du \wedge d\omega + \tau d^*(du\wedge\tau\omega) + d^*(du\wedge\omega) - \tau(du\wedge d\tau\omega) \\ &= -du \wedge d\omega + \tau(d^*du\wedge\tau\omega) - \tau(du\wedge d^*\tau\omega) \\ &\quad + d^*du \wedge \omega - du \wedge d^*\omega - \tau(du\wedge d\tau\omega) \\ &= -du \wedge (d+d^*)\omega - \tau(du\wedge(d\tau+d^*\tau)\omega) \\ &\quad + d^*du \wedge \omega + \tau(d^*du\wedge\tau\omega) . \end{aligned}$$

However, $d + d^* = \tau D$ and $d^*du = 0$

$$\Rightarrow \quad \frac{1}{i}DB\omega = - du \wedge \tau D\omega - \tau(du \wedge D\omega)$$

$$= - \frac{1}{i}BD\omega \ .$$

So B,D anticommute, and thus B,G anticommute. Also

$$-B^2\omega = (\tau(du\wedge) + du\wedge\tau)(\tau(du\wedge\omega) + du\wedge\tau\omega)$$

$$= \tau(du\wedge\tau(du\wedge\omega)) + \tau(du\wedge du\wedge\tau\omega)$$

$$+ du \wedge \tau\tau(du\wedge\omega) + du \wedge \tau(du\wedge\tau\omega)$$

$$= \tau(du\wedge\tau(du\wedge\omega)) + du \wedge \tau(du\wedge\tau\omega) \quad \text{as} \quad \tau^2 = 1 \ .$$

However, if ω is in the basis, then:

(i) when ω contains 'du', the first term vanishes;

(ii) when ω does not contain 'du', the second term vanishes.

It is easy to check the signs, and one finds that the right-hand side B always ω, on even forms. Thus:

$$B^2 = - I \ .$$

Note that B is the symbol of the Dirac-type operators D_θ.

In Lemma 8.3, $X = -B^{-1}A$, and so its adjoint with respect to the inner product $<x,\tau y>$, the natural positive definite inner product, is:

$$X^* = -AB^{-1}$$

since A, B are self-adjoint. Thus:

$$X^*G = -AB^{-1}G$$

$$= AGB^{-1} \qquad (B^{-1}, G \text{ anti-commute})$$

$$= GAB^{-1} \qquad (G = G_A)$$

$$= -GB^{-1}A \qquad (B^{-1}, G \text{ anti-commute})$$

$$= GX .$$

By Corollary 7.2, if $\langle a, b\rangle' = \operatorname*{Res}_{|\zeta|=1} \int_{\tilde{M}} (\hat{a} \wedge \bar{b})$, then

$$(a,Xb)_{\text{Coker } D} = -\langle G_D a, Xb\rangle'$$

$$= -\langle X^* G_D a, b\rangle'$$

$$= -\langle G_D X_a, b\rangle', \quad \text{by above}$$

$$= (Xa,b)_{\text{Coker } D} .$$

So X is self-adjoint, with respect to this inner product; and thus iX is skew-adjoint with respect to $(\ ,\ )_{\text{Coker } D}$.

However, if $\underline{e}_\mu = i^\mu \underline{v}^{(\mu)}$, $Y = -iX$, then:

(i) Y is skew-adjoint w.r.t. $(\ ,\ )_{\text{Coker } D}$.

(ii) $Y\underline{e}_\mu = \underline{e}_{\mu-1}$ if $1 \le \mu \le m-1$ and $Y\underline{e}_1 = \underline{0}$.

As we will show in §11 (see Theorem 11.5), the signature of the inner product on this space is:

$$\begin{cases} 0 & \text{if } m \text{ is even} \\ \text{the sign of } \alpha(-1)^{\frac{m+1}{2}} & \text{if } m \text{ is odd} \end{cases}$$

where $\alpha = (\underline{e}_{m-1}, \underline{e}_0)$.

This is because Coker D is spanned by:

$$\underline{v}_i, \theta\underline{v}_i, \ldots \theta^{m-1}\underline{v}_i .$$

However, in Coker D, Im D is identified with zero. Since $D = A + \theta B$, thus, multiplication by θ is identified with $(B^{-1}A)$, and thus $\underline{v}_i, \theta\underline{v}_i, \ldots, \theta^{m-1}\underline{v}_i$ are identified with:

$$\underline{v}, (-B^{-1}A)\underline{v}, \ldots, (-B^{-1}A)^{m-1}\underline{v}$$

i.e. $\underline{v}_0, \ldots, \underline{v}^{(m-1)}$ using $\underline{v} = \underline{v}^{(m-1)}$. This gives the part of Coker D associated with $\lambda_i(\theta)$ crossing zero at $\theta = 0$.

However,

$$(\underline{v}^{(0)}, \underline{v}^{(r)}) = \langle \underline{v}^{(0)} | G | \underline{v}^{(r)} \rangle$$

$$= \langle \underline{v}^{(0)} | B | (GB)\underline{v}^{(r)} \rangle$$

since $BGB = -BBG = G$

$$= -\langle \underline{v}^{(0)} | B | \underline{v}^{(r+1)} \rangle$$

by Lemma 8.3(c)

and $\langle \underline{v}_j | B | \underline{v}^{(0)} \rangle = 0$ whenever $\lambda_j = 0$

from Lemma 8.3(a) .

So $$(\underline{v}^{(0)}, \underline{v}^{(r)}) = \begin{cases} 0 & \text{for } r+1 < m \\ -\lambda^{(m)} . & \text{for } r+1 = m \end{cases}$$

$$\Rightarrow \qquad \alpha = (\underline{e}_{m-1}, \underline{e}_0)$$

$$= (-i)^{m-1} (\underline{v}^{(m-1)}, \underline{v}^{(0)})$$

$$= (-1)^{m-1/2} (-\lambda^{(m)})$$

$$= (-1)^{m+1/2} \lambda^{(m)} .$$

So, the signature of that part of Coker D corresponding to this crossing is:

$$\begin{cases} 0 & \text{if } m \text{ is even} \\ \operatorname{sgn}(\lambda^{(m)}) & \text{if } m \text{ is odd} \end{cases} .$$

That is, it is the crossing number of the eigenvalue at $\theta = 0$.

Since we have obtained maps, which composed give:

$H^n(N) \to \text{Coker } D$

thus, for ζ on the unit circle (i.e. $e^{i\theta}$), the signature of the inner product on the subset of $H^n(N)$ corresponding to the eigenvalue ζ of T, must equal that of the corresponding subset of Coker D. By above, this is the local component of the spectral flow for that one eigenvalue crossing.

Those $\zeta \in \mathbb{C}^*$ with $|\zeta| \neq 1$ come in pairs, as eigenvalues of T on $H^n(N)$, by Theorem 11.5 (§11). These components of $H^n(N)$ are annihilated when we map to Coker D. However, the total signature on a pair of such spaces, corresponding to $\zeta, \bar{\zeta}^{-1}$ vanishes (see §11, Theorem 11.5), and hence the total spectral flow is just sign N, the signature of our indefinite inner product on $H^n(N)$.

Theorem 9.1. *Suppose* M *is a compact oriented smooth manifold fibred over* S, *with fibre* N; *and let* $\tilde{M}$ *be its infinite cyclic covering. Let* D_θ *denote the Dirac-type operator on* $\Omega^{even}(M,\chi_\theta)$, *the space of forms on* M *with local coefficient system twisted by* $e^{i\theta}$. *Let* T *denote the induced monodromy action on the homology* $H(N)$ *of* N. *Then:*

(i) *There exists a mapping*

$H(N) \to \text{Coker } D$

which is an isomorphism when restricted to the parts of $H(N)$ *corresponding to eigenvalues of* T *on the unit circle.*

(ii) Jordan blocks of T on $H(N)$ correspond to eigenvalues of D_θ crossing zero, and the crossing number at $\theta = \theta_0$ of an eigenvalue is the signature of the Jordan block corresponding to the eigenvalue $e^{i\theta_0}$ of T.

(iii) Those Jordan blocks corresponding to eigenvalues of T not on the unit circle, come in pairs, and their signatures cancel out, so not contributing to the total.

Thus summing the results in (ii) and noting (iii), we obtain the global result of Theorem 2.2.

The argument here can mostly be generalised to the case where $\pi : M \to S^1$ is just a map, and not a fibration. In that case, §6's results remain unaltered, although Lemma 6.1 must be changed to include an extra term x_3 with $dx_3 = d^*x_3 = 0$. We must then consider only the torsion parts of H_a, H_c, Coker D, whereas in our case, the whole spaces were torsion. The final result is, however, exactly the same, with the above provisos.

§10. Appendix I: Analytic and algebraic homologies

Define Ω_c to be the subset of $\Omega(\tilde{M})$ consisting of forms with compact support. Let H_c be the corresponding homology. Let:

A = ring of 'algebraic' functions (Laurent polynomials)

A' = ring of 'analytic' functions on S^1 .

We shall show that:

$$H_a = H_c \otimes_A A'$$

where H_a is the A'-module homology, and H_c is the A-module homology (obtained using the Fourier transform, see §4). The proof is along the lines of Serre [3]. See also Lang [4].

Definition 10.1. *We say that* B *is A-flat if whenever the sequence of A-modules*:

$$E \to F \to G$$

is exact, the sequence:

$$E \otimes_A B \to F \otimes_A B \to G \otimes_A B$$

is also exact.

Definition 10.2. *We say that* (A,B) *is flat if* B/A *is A-flat as an A-module*.

So, we must show that (A,A') is a flat pair. From this it follows that:

$$H_a = H_c \otimes_A A' .$$

Define $Tor_i^A(B,E) = (i^{th}$ homology of $F \otimes E)$, where F is the complex which resolves B, as a free resolution of A-modules:

$$\underbrace{\cdots \to F_2 \to F_1 \to F_0}_{F} \to B \to 0 .$$

Then $Tor_1^A(B,E) = 0$ for all A-modules E if B is A-flat.

<u>Theorem 10.3</u>. (A,B) <u>is flat iff</u> B <u>is A-flat and the map</u> $E \to E \otimes_A B$, <u>for any A-module E is injective</u>.

<u>Proof</u>:

(i) Suppose that B is A-flat, and:

$$E \to E \otimes_A B$$

is injective for all A-modules E. Since:

$$0 \to A \to B \to B/A \to 0$$

is exact, thus:

$$Tor_1^A(A,E) \to Tor_1^A(B,E) \to Tor_1^A(B/A,E) \to A \otimes_A E \to B \otimes_A E$$

is exact. However, trivially:

$$Tor_1^A(A,E) = 0$$

$$A \otimes_A E = E$$

and thus:

$$0 \to Tor_1^A(B,E) \to Tor_1^A(B/A,E) \to E \to B \otimes_A E$$

is exact.

Our assumption is that the map:

$$E \to E \otimes_A B$$

is exact, and thus $\mathrm{Tor}_1^A(B/A,E)$ maps into 0 in E. This gives an exact sequence:

$$0 \to \mathrm{Tor}_1^A(B,E) \to \mathrm{Tor}_1^A(B/A,E) \to 0 .$$

Hence, we must have:

$$\mathrm{Tor}_1^A(B,E) \simeq \mathrm{Tor}_1^A(B/A,E) .$$

Since B is A-flat, the l.h.s. is trivial, and thus so is the r.h.s.. So B/A is A-flat, and hence (A,B) is flat.

Q.E.D.

Conversely, if (A,B) is flat, then B/A is A-flat, and thus:

$$\mathrm{Tor}_1^A(B/A,E) = 0 .$$

Hence, since:

$$0 \to \mathrm{Tor}_1^A(B,E) \to \mathrm{Tor}_1^A(B/A,E) \to E \to E \otimes_A B$$

is exact, so:

$$0 \to \mathrm{Tor}_1^A(B,E) \to 0 \to E \to E \otimes_A B$$

is exact. Hence:

(a) $\mathrm{Tor}_1^A(B,E) = 0$ i.e. B is A-flat, since it is true for all A-modules E;

(b) $E \to E \otimes_A B$ is injective.

Q.E.D.

Theorem 10.4. *If* $A \subseteq B \subseteq C$ *are three rings, and* (A,C), (B,C) *are flat pairs, then* (A,B) *is a flat pair*.

Proof: Suppose $0 \to E \to F$ is exact. We wish to show that:

$$E \otimes_A B \to F \otimes_A B$$

is injective. So suppose:

$$0 \to N \to E \otimes_A B \to F \otimes_A B$$

is exact, where N is the kernel of the map:

$$E \otimes_A B \to F \otimes_A B .$$

Then, since C is B-flat, thus:

$$0 \to N \otimes_B C \to (E \otimes_A B) \otimes_B C \to (F \otimes_A B) \otimes_B C$$

is exact. However,

$$(E \otimes_A B) \otimes_B C \cong E \otimes_A C .$$

Thus $0 \to N \otimes_B C \to E \otimes_A C \to F \otimes_A C$ is exact. Since C is A-flat, thus $E \otimes_A C \to F \otimes_A C$ is injective, and so $N \otimes_B C = 0$.

But, (B,C) is a flat pair, and so by Theorem 10.3, the map:

$$N \to N \otimes_B C$$

is injective. So $N = 0$, and thus B is A-flat.

Also, the composite map:

$$E \to E \otimes_A B \to (E \otimes_A B) \otimes_B C \cong E \otimes_A C$$

is injective, since (A,C) is flat (by theorem 10.3). Thus $E \to E \otimes_A B$ is injective. Thus, by Theorem 10.3, (A,B) is flat.

Q.E.D.

Theorem 10.5. Suppose E is an A-module of finite type, and A is a local Noetherian ring. If $E = mE$, then $E = 0$ (for any maximal ideal m).

Proof: Suppose $E \neq 0$ is generated by $e_1,\ldots,e_n$. Choose n to be the least number of basis elements. Since:

$$e_n \in E = mE$$

$$e_n = \Sigma(x_i e_i) \quad \text{some} \quad x_i \in m$$

Hence $(1-x_n)e_n = x_1e_1 + \ldots + x_{n-1}e_{n-1}$, and so:

$$e_n \in \langle e_1,\ldots,e_{n-1}\rangle$$

since $(1-x_n)$ is invertible in A. Thus E is generated by $e_1,\ldots,e_{n-1}$, a contradiction. So $E = 0$.

Q.E.D.

Lemma 10.6. Suppose A is a Noetherian ring, and E is an A-module of finite type, with an m-fibration, E_n, that is, $mE_n \subseteq E_{n+1}$ $\forall n$. Then:

$$E_S = \bigoplus_{n\in N\cup\{0\}} (E_n) \quad \text{is finite over} \quad S = \bigoplus_{n\in N\cup\{0\}} (m^n)$$

iff the filtration of E is m-stable (that is, $mE_n = E_{n+1}$ for sufficiently large n).

Proof: Let $F_n = \bigoplus_{r=0}^{n} (E_r)$

and $$G_n = E_n \oplus \ldots \oplus E_n \oplus mE_n \oplus m^2 E_n \oplus \ldots .$$

Then G_n is an S-submodule of E_S, and is given to be finite over E_S, since F_n is finite over A. However, $G_n - G_{n+1}$ and $V_b(G_n) = E_s$.

Since S is Noetherian, thus E_S is finite over S

iff $E_S = G_N$ some N

iff $E_{n+k} = m^k E_N$ $\forall k \geq 0$

iff the fibration of E is m-stable.

Q.E.D.

Lemma 10.7. If E is an A-module of finite type with stable m-filtration and F is a submodule, $F_n = F \cap E_n$, then F_n is a stable m-filtration of F.

Proof: Now $m(F \cap E_n) \subseteq mF \cap mE_n \subseteq F \cap E_{n+1}$. So F_n is an m-fibration of F, and,

$$F_S = \bigoplus_{n=0}^{\infty} (F_n) \subseteq E_S$$

is finite over S, as S is Noetherian. So, by Lemma 10.6, F is m-stable.

Q.E.D.

Theorem 10.8. If E is an A-module of finite type, then:

(a) The topology induced on a sub-module $F \subseteq E$ by the m-adic topology on E, coincides with the m-adic topology on F.

(b) Any sub-module of E is closed in the m-adic topology on E.

Proof:

(a) Applying Lemma 10.7 to $E_n = m^n E$, we see that there exists s such that:

$$F_n = m^{n-s} F_s \quad \forall n \geq s .$$

That is, $(F \cap m^n E) = m^{n-s}(F \cap m^s E)$ $\forall n \geq S$ (i).

This result is usually known as the Artin-Rees lemma.

Thus, since $m^n E$ generate the open neighbourhoods of 0 in E, thus $(F \cap m^n E)$ are the induced basis open neighbourhoods of 0 and F. From (i), this basis gives an m-adic topology on F. Hence any open set in the induced topology on F is open in the m-adic topology on F, as required.

(b) Let F be the intersection of all open sets containing 0. Then $F = mF$, since we are using the m-adic topology on E. By theorem 10.5, $F = 0$. Thus if $p \neq 0$, then $p \notin F$, and so there exists an open set $U \ni 0$ s.t. $p \notin U$. So E is separable.

Suppose $E' \subseteq E$ is a submodule. Then, E/E' is separable by the above argument. Hence E' is closed in E.

Q.E.D.

<u>Theorem 10.9.</u> <u>The map</u> $\varepsilon : E \otimes_A \hat{A} \to \hat{E}$ <u>is bijective if</u> E <u>is an A-module of finite type.</u>

<u>Proof</u>: Suppose that $0 \to R \to L \to E \to 0$ is an exact sequence of A-modules, where L is a free module of finite type. Since A is Noetherian, R is also of finite type. Theorem 10.8 shows that the m-adic topology on R is induced from the m-adic topology on L.

Clearly E is a quotient of L, since the map $L \to E$ is surjective. Hence:

$$0 \to \hat{R} \to \hat{L} \to \hat{E} \to 0$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccc}
R \otimes_A \hat{A} & \xrightarrow{\alpha} & L \otimes_A \hat{A} & \xrightarrow{\beta} & E \otimes_A \hat{A} & \longrightarrow & 0 \\
\downarrow \varepsilon'' & & \downarrow \varepsilon' & & \downarrow \varepsilon & & \\
\hat{R} & \xrightarrow{\alpha'} & \hat{L} & \xrightarrow{\beta'} & \hat{E} & \longrightarrow & 0 .
\end{array}$$

Since L is a free module, thus ε' is bijective. The two rows above are already known to be exact, and thus ε is surjective. For, if $x' \in \hat{E}$, then as β' is surjective:

$$x' = \beta'(y')$$

some $y' \in \hat{L}$. But ε' is bijective, i.e.:

$$y' = \varepsilon'(y)$$

some $y \in L \otimes_A \hat{A}$, and so:

$$\begin{aligned} x' &= \beta'(\varepsilon'(y)) \\ &= \varepsilon(\beta(y)) \qquad \text{(commutative square)} . \end{aligned}$$

Thus ε is surjective.

Since this holds for all finitely generated modules E, thus ε'' applied to R is surjective. Suppose that:

$$\varepsilon(x) = 0$$

some $x \in E \otimes_A \hat{A}$. Then $\exists\, y \in L \otimes_A \hat{A}$ s.t.

$$\beta(y) = x .$$

So
$$\begin{aligned} 0 &= \varepsilon(x) \\ &= \varepsilon(\beta(y)) \\ &= \beta'(\varepsilon'(y)) \qquad \text{(commutative square)} \end{aligned}$$

$\Rightarrow \quad \varepsilon'(y) \in \ker(\beta') = \mathrm{Im}(\alpha')$.

Thus $\varepsilon'(y) = \alpha'(z')$ some $z' \in \hat{R}$

$\Rightarrow$ $z' = \varepsilon''(z)$ some $z \in R \otimes_A \hat{A}$

since ε'' is surjective.

Thus, $\varepsilon'(y) = \alpha'(\varepsilon''(z))$

$= \varepsilon'(\alpha(z))$

$\Rightarrow$ $y = \alpha(z)$, as ε' is injective.

So $x = \beta(y)$

$= \beta(\alpha(z)) = 0$

and thus ε is injective.

Hence ε is bijective.

Q.E.D.

__Theorem 10.10__. $(A,\hat{A})$ __is flat__.

__Proof__: Now $\hat{A}$ is A-flat. By Theorem 10.3, we need only show that whenever E is an A-module of finite type:

$$E \to E \otimes_A \hat{A}$$

is injective.

By Theorem 10.9, ε is injective. Since E is finitely generated, $E \to \hat{E}$ is injective. Thus $(A,\hat{A})$ is flat.

Q.E.D.

__Theorem 10.11__. __Suppose__ $A \subseteq B$ __are two rings with__ $\hat{A} = \hat{B}$. __Then__ (A,B) __is a flat pair__.

__Proof__: By Theorem 10.10, $(A,\hat{A})$, $(B,\hat{B})$ are flat pairs. Thus by Theorem 10.4 applied to $A \subseteq B \subseteq C$ where $C = \hat{A} = \hat{B}$, we see that (A,B) is a flat pair. Q.E.D.

To apply this, put:

A = ring of Laurent polynomials

A' = ring of analytic functions on S^1 .

Choose $\zeta_0 \in S^1$. Then the local completions of A, A' are identical, namely formal power series about ζ_0. Thus the local rings A_{ζ_0}, A'_{ζ_0} form a flat pair by Theorem 10.11.

However, $H_{A'}$, H_A are both expressible as direct sums of local spaces $H_{A'}^{(\zeta)}$, $H_A^{(\zeta)}$ (see §5). The above theorem shows that locally these spaces are the same:

$$H_{A'}^{(\zeta)} = H_A^{(\zeta)} \otimes_A A' .$$

Summing over all $\zeta \in S^1$ gives:

$$H_{A'} = \left(\bigoplus_{\zeta \in S^1} H_A^{(\zeta)} \right) \otimes_A A'$$

i.e. $H_{A'} = H_A \otimes_A A'$.

§11. Appendix II : Normal forms in the orthogonal group

Suppose T is an orthogonal transformation for a certain indefinite form. We shall now find the normal form for such transformations up to conjugacy - that is, we find parameters which specify uniquely the conjugacy class of an element of U(p,q).

<u>Lemma 11.1.</u> *If* $\underline{A} \in U(p,q)$, *define for each eigenvalue* λ *of* $\underline{A}$, *a subspace of* $\mathbb{C}^{p+q}$:

$$V_\lambda = \ker(\underline{A} - \lambda\underline{I})^{p+q}$$

(the λ*-Jordan block of* $\underline{A}$*). Then* $V_\lambda \perp V_\mu$ *whenever* $\mu \neq \bar{\lambda}^{-1}$.

<u>Proof</u>. Suppose $\underline{v} \in V_\lambda$, $\underline{w} \in V_\mu$. Then $\exists\, m,n \in N$ such that

$$(\underline{A} - \lambda\underline{I})^m\underline{v} = \underline{0}$$

$$(\underline{A} - \mu\underline{I})^n\underline{w} = \underline{0} .$$

Define $\alpha_{k,\ell} = \langle(\underline{A} - \lambda\underline{I})^k\underline{v}, (\underline{A} - \mu\underline{I})^\ell\underline{w}\rangle$. Then we are given that:

$$\alpha_{k,\ell} = 0$$

whenever $k \geq m$ or $\ell \geq n$.

However,

$$\begin{aligned}\alpha_{k+1,\ell+1} &= \langle(\underline{A}-\lambda\underline{I})(\underline{A}-\lambda\underline{I})^k\underline{v}, (\underline{A}-\mu\underline{I})(\underline{A}-\mu\underline{I})^\ell\underline{w}\rangle \\ &= \langle(\underline{A}-\lambda\underline{I})^k\underline{v}, (\underline{A}-\mu\underline{I})^\ell\underline{w}\rangle(1+\lambda\bar{\mu}) \\ &\quad -\lambda\langle(\underline{A}-\lambda\underline{I})^k\underline{v},\underline{A}(\underline{A}-\mu\underline{I})^\ell\underline{w}\rangle \\ &\quad -\bar{\mu}\langle\underline{A}(\underline{A}-\lambda\underline{I})^k\underline{v}, (\underline{A}-\mu\underline{I})^\ell\underline{w}\rangle\end{aligned}$$

since $\underline{A}$ preserves the inner product

$$= (1-\lambda\bar{\mu})<(\underline{A}-\lambda\underline{I})^k\underline{v},\ (\underline{A}-\mu\underline{I})^\ell\underline{w}>$$
$$-\lambda<(\underline{A}-\lambda\underline{I})^k\underline{v},(\underline{A}-\mu\underline{I})^{\ell+1}\underline{w}>$$
$$-\bar{\mu}<(\underline{A}-\lambda\underline{I})^{k+1}\underline{v},\ (\underline{A}-\mu\underline{I})^\ell\underline{w}>$$
$$= (1-\lambda\bar{\mu})\alpha_{k,\ell} - \lambda\alpha_{k,\ell+1} - \bar{\mu}\alpha_{k+1,\ell}\ .$$

Thus, if $\mu \neq \bar{\lambda}^{-1}$, then $\lambda\bar{\mu} - 1 \neq 0$, and so:

$$\alpha_{k,\ell} = (1-\lambda\bar{\mu})^{-1}[\alpha_{k+1,\ell+1} + \lambda\alpha_{k,\ell+1} + \bar{\mu}\alpha_{k+1,\ell}]\ .$$

We apply this repeatedly, starting with $k = m - 1$, $\ell = n - 1$ and then decreasing ℓ to 0; and then carrying on with $k = m - 2$, $\ell = n - 1, \ldots, 0$ etc. We then obtain:

$$\alpha_{k,\ell} = 0 \quad \forall\ k,\ell \geq 0$$

$$\Rightarrow \quad \alpha_{0,0} = 0$$

and so $<\underline{v},\underline{w}> = 0$.

Hence $V_\lambda \perp V_\mu$.

Q.E.D.

We can thus split up the space into orthogonal pieces:

$V_\lambda \oplus V_{\bar{\lambda}}^{-1}$	for $\lvert\lambda\rvert \neq 1$	(both null)
V_λ	for $\lvert\lambda\rvert = 1$	.

<u>Lemma 11.2</u>. <u>If</u> $|\lambda| \neq 1$, <u>then the action of</u> A <u>on</u> $V_\lambda \oplus V_{\bar{\lambda}}^{-1}$ <u>is completely specified by its action,</u> B, <u>on</u> V_λ; <u>and then</u> $(B-\lambda I)$ <u>is nilpotent, the action on</u> $V_{\bar{\lambda}}^{-1}$ <u>being that of</u> $(B^T)^{-1}$.

<u>Proof</u>: Now V_λ, $V_{\bar{\lambda}}^{-1}$ are null spaces. Thus A's action on $V_\lambda \oplus V_{\bar{\lambda}}^{-1}$ is specified by the actions:

$$B = A|V_\lambda$$

$$B' = A|V_{\bar{\lambda}^{-1}} .$$

But A is orthogonal, and so:

$$\langle A\underline{v},\underline{v}'\rangle = \langle \underline{v},\underline{A}^{-1}\underline{v}'\rangle \quad \text{for } \underline{v} \in V_\lambda, \ \underline{v}' \in V_{\bar{\lambda}^{-1}}$$

i.e. $\langle B\underline{v},\underline{v}'\rangle = \langle \underline{v},\underline{B}'^{-1}\underline{v}'\rangle$.

So $\underline{B}^\dagger = \underline{B}'^{-1}$ i.e. $\underline{B}' = \underline{B}^{T-1}$. Since $\underline{B} = \underline{A}|_{V_\lambda}$, thus $(\underline{B}-\lambda\underline{I})$ must, by definition of V_λ, be nilpotent.

Q.E.D.

The rest of $\underline{A}$'s action is given by its action on V_λ's for which $|\lambda| = 1$. Suppose $|\lambda_0| = 1$. Then, map $\underline{A}$, an element of the Lie group, to say:

$$X = (\underline{A}-\lambda_0\underline{I})/ (\underline{A}+\lambda_0\underline{I})$$

an element of the Lie algebra. Then X is nilpotent, and skew-symmetric relative to the quadratic form.

Lemma 11.3. There exists a decomposition into orthogonal indecomposable Jordan blocks for X.

Proof: Now X is nilpotent, and so $\exists$ basis $e_{i,j}$ s.t.

$$\ker(X^k) = \langle e_{i,j} | i \le k\rangle$$

and $$X(e_{i,j}) = \begin{cases} e_{i-1,j} & (i \ge 1) \\ 0 & (i = 1) . \end{cases}$$

Define n_j to be the number of $e_{i,j}$ for each j: w.l.o.g. a decreasing sequence.

Since X is skew-adjoint with respect to the quadratic form:

$$\langle Xe_{i,j}, e_{k,\ell}\rangle = -\langle e_{i,j}, Xe_{k,\ell}\rangle$$

$$\Rightarrow \quad \langle e_{i-1,j},\ {}_{k,\ell}\rangle = -\langle e_{i,j}, e_{k-1,\ell}\rangle$$

$$\Rightarrow \quad \langle e_{i,j}, e_{k,\ell}\rangle = \begin{cases} -\langle e_{i+1,j}, e_{k-1,\ell}\rangle & \text{for } k > 1 \\ 0 & \text{for } k = 1 . \end{cases}$$

We now change basis so that the blocks $\langle e_{i,j}\rangle_i$ for each j are mutually orthogonal. Inductively, we need only show how to add combinations of $e_{j,1}$'s to $e_{i,2}$'s so as to make

$$e_{i,2} \perp e_{j,1} \qquad \forall i,j .$$

So, put $e'_{i,2} = e_{i,2} + \sum_{j\le i} (\alpha_{i-j+1} e_{j,1})$ some α_k's .

Then $Xe'_{i,2} = e_{i-1,2} + \sum_{j\le 1} (\alpha_{i-j+1} e_{j-1,1})$

$$= e'_{i-1,2}$$

and $\langle e'_{i,2}, e_{k,1}\rangle = \langle e_{i,2}, e_{k,1}\rangle + \sum_{j\le i} (\langle e_{j,1}, e_{k,1}\rangle \alpha_{i-j+1})$.

We require to find suitable $\alpha_1,..,\alpha_{n_2}$ s.t. this vanishes for $i = 1,2,...,n_2$, and $k = n_1$ (since the other k's follow using the skew-adjointness of X). However, the equations:

$$\sum_{j\le i} \langle e_{j,1}, e_{n_1,1}\rangle \alpha_{i-j+1} = -\langle e_{i,2}, e_{n_1,1}\rangle$$

have a triangular form, and therefore have a solution, since the diagonal terms are all $\langle e_{1,1}, e_{n_1,1}\rangle \neq 0$.

For, if $\langle e_{1,1}, e_{n_1,1}\rangle = 0$ then the matrix $\langle e_{i,1}, e_{j,1}\rangle$ would look like:

$$\begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 0 & & * & \end{pmatrix}$$

and thus there are $[n_1/2]$ null orthogonal vectors, and another vector orthogonal to all of these, while being linearly independent of them all. This is impossible in an n_1-dimensional space. Thus, such a basis exists.

Q.E.D.

In the basis, the only non-zero inner products are between elements with the same j. So we can now suppose X acts indecomposably on one such block say of size n.

<u>Lemma 11.4.</u> <u>In the above situation, the signature of the inner product can only be:</u>

$$\begin{cases} 0 & \text{if } n \text{ is even} \\ \pm 1 & \text{if } n \text{ is odd} \end{cases}$$

<u>and given this signature, there is a unique conjugacy class in the unitary group.</u>

<u>Proof</u>: Suppose e_i is a basis such that $X_{e_i} = e_{i-1}$ $(i = 2,\ldots,n)$ and $X_{e_1} = 0$. Since X is skew-adjoint,

$$\begin{aligned} \langle e_i, e_j \rangle &= \langle Xe_{i+1}, e_j \rangle \\ &= -\langle e_{i+1}, Xe_j \rangle \\ &= -\langle e_{i+1}, e_{j-1} \rangle \end{aligned}$$

for $i < n$, $j > 1$; and :

$$\langle e_1, e_j \rangle = \langle e_1, Xe_{j+1} \rangle = 0 \quad \text{for } j < n .$$

Thus the matrix $\langle \underline{e}_i, \underline{e}_j \rangle$ looks like

$$\begin{pmatrix} & & & & * \\ & 0 & & \cdot^{\cdot^{\cdot}} & \vdots \\ & & \cdot^{\cdot^{\cdot}} & & \vdots \\ & \cdot^{\cdot^{\cdot}} & & & \vdots \\ * & \cdots & \cdots & \cdots & * \end{pmatrix}$$

with $\langle \underline{e}_i, \underline{e}_j \rangle = \begin{cases} \alpha_{i+j-n}(-1)^{n-j} & \text{some } \alpha\text{'s} \quad (i+j>n) \\ 0 & (i+j \leq n) \end{cases}$.

However, a change of basis:

$$f_i = \sum_{j=1}^{i} (a_{i-j} e_j)$$

gives

$$Xf_i = \sum_{j=1}^{i} (a_{i-j} e_{j-1})$$

$$= \sum_{j=1}^{i-1} (a_{i-j-1} e_j)$$

$$= f_{i-1} .$$

Under this basis change $\alpha_i \to \beta_i$ where:

$$\beta_i = \langle f_i, f_n \rangle$$

$$= \langle \sum_{j=1}^{i} a_{i-j} e_j , \sum_{j=1}^{n} a_{n-j} e_j \rangle$$

$$= \sum_{j=1}^{i} \sum_{k=n+1-j}^{n} (a_{i-j} \bar{a}_{n-k} (-1)^{n-k} \alpha_{j+k-n})$$

$$= \sum_{j=1}^{i} \sum_{k=0}^{j-1} (a_{i-j} \bar{a}_k (-1)^k \alpha_{j-k})$$

$$= \sum_{k=0}^{i-1} \sum_{j=k+1}^{i} (a_{i-j} \bar{a}_k (-1)^k \alpha_{j-k})$$

$$= \sum_{k=0}^{i-1} \sum_{j=1}^{i-k} (\alpha_j (-1)^k \bar{a}_k a_{i-j-k})$$

$$= \sum_{j=1}^{i} \alpha_j \left[\sum_{k=0}^{i-j} \bar{a}_k a_{i-j-k} (-1)^k \right] .$$

Using $a_i = 0$, except for $a_0 = 1$, $a_r = a$ we get:

$$\beta_i = \begin{cases} \alpha_i + (a + (-1)^r \bar{a}) \alpha_{i-r} & (i > r) \\ \alpha_i & (i \leq r) . \end{cases}$$

Thus we can reduce the situation to $\beta_i = 0 \ \forall i > 0$, by choosing suitable a's, recursively. For,

$$a + (-1)^r \bar{a} \in \begin{cases} \mathbb{R} & \text{for } r \text{ even} \\ i\mathbb{R} & \text{for } r \text{ odd} . \end{cases}$$

However, α_i/α_{i-r} is real for r even and pure imaginary for odd r. Thus we can match up, and obtain:

$$\beta_i = 0 \quad \forall i > 0 .$$

We now have inner porduct matrix:

$$\begin{pmatrix} 0 & \cdots & 0 & \alpha \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ (-1)^{n-1}\alpha & 0 & \cdots & 0 \end{pmatrix}$$

where $\bar{\alpha} = (-1)^{n-1} \alpha$.

Hence, there are $[\frac{n}{2}]$ mutually orthogonal, linearly independent null vectors. The dimension of a null space is, however, at most $\min(p,q)$ where p,q are the dimensions of the positive and negative spaces respectively.

Thus $p,q \geq [\frac{n}{2}]$ while $p + q = n$. Hence:

(i) if n is even, $p = q = n/2$ (zero signature),

(ii) if n is odd, $\begin{cases} p = \frac{1}{2}(n+1), & q = \frac{1}{2}(n-1) \quad \text{(signature } +1) \\ \text{or} \quad p = \frac{1}{2}(n-1), & q = \frac{1}{2}(n+1) \quad \text{(signature } -1) \end{cases}$.

<u>Case (i)</u>. <u>n even</u>

Here $\bar{\alpha} = -\alpha$, and so α is pure imaginary. Using a map $e_i \to \lambda e_i$,

$$\langle e_i, e_j\rangle \to |\lambda|^2 \langle e_i, e_j\rangle$$

i.e. $\alpha \to |\lambda|^2\alpha$, and so w.l.o.g. $\alpha = \pm i$. Conjugation transforms $\alpha = +i$ into $\alpha = -i$, and so these cases are essentially isomorphic.

<u>Case (ii)</u> <u>n odd</u>

Hence $\bar{\alpha} = \alpha$, and so α is real. Suitable scaling will give $\alpha = \pm 1$. If $n = 2k + 1$, then put:

$$\left.\begin{aligned} a_i &= \tfrac{1}{2}\sqrt{2}\,(e_i + e_{2k+2-i}) \\ b_i &= \tfrac{1}{2}\sqrt{2}\,(e_i - e_{2k+2-i}) \end{aligned}\right\} \quad 1 \leq i \leq k .$$

Then $\langle a_i, a_j\rangle = \frac{1}{2}(\langle e_i, e_{2k+2-j}\rangle + \langle e_j, e_{2k+2-i}\rangle)$

$$= 0 \text{ unless } i = j$$

and similarly, $\langle b_i, b_j\rangle = 0$ unless $i = j$, and for all i,j, $\langle a_i, b_j\rangle = \langle b_i, a_j\rangle = 0$. Thus $\{a_i, b_i\}$ form an orthogonal set for $1 \leq i \leq k$, all orthogonal to e_{k+1}, with:

$$\langle a_i, a_i\rangle = \tfrac{1}{2}(\langle e_i, e_{2k+2-i}\rangle + \langle e_{2k+2-i}, e_i\rangle)$$

$$= (-1)^{i-1}\alpha$$

$$\langle b_i, b_i\rangle = -\tfrac{1}{2}(\langle e_i, e_{2k+2i}\rangle - \langle e_{2k+2-i}, e_i\rangle)$$

$$= (-1)^i\alpha$$

$$\langle e_{k+1}, e_{k+1}\rangle = (-1)^k\alpha$$

giving signature ± 1 according as $(-1)^k\alpha$ is +ve or −ve. So α is determined by the signature.

Existence

Finally, we constructively show the existence of a basis $\underline{e}_i$ with inner product as defined above.

Case (i) n odd

By above, we need only find a_i, b_i $(1 \le i \le k)$, e_{k+1} mutually orthogonal, with:

$$\langle a_i, a_i\rangle = (-1)^{i-1}\alpha$$

$$\langle b_i, b_i\rangle = (-1)^i\alpha$$

$$\langle e_{k+1}, e_{k+1}\rangle = (-1)^k\alpha$$

This is evidently possible, just so long as α, and the signature of the space correspond.

Case (ii) n even

Suppose $n = 2k$. Then α is pure imaginary, say $\alpha = i$.

$$\left.\begin{array}{ll}\text{Put} & a_i = 1/\sqrt{2}\,(e_i + ie_{2k+1-i}) \\ \text{and} & b_i = 1/\sqrt{2}\,(e_i - ie_{2k+1-i})\end{array}\right\} \quad i = 1,2,\ldots,k$$

Then $\{a_i, b_i\}$ form an orthogonal basis, and:

$$\langle a_i, a_i\rangle = (i/2)[\langle e_{2k+1-i}, e_i\rangle - \langle e_i, e_{2k+1-i}\rangle]$$

$$= (i/2)(2i)(-1)^i = (-1)^{i+1}$$

and $\langle b_i, b_i\rangle = (i/2)[\langle e_i, e_{2k+1-i}\rangle - \langle e_{2k+1-i}, e_i\rangle]$

$$= (i/2)(-2i)(-1)^i = (-1)^i .$$

Clearly, such a_i's, b_i's exist so long as the signature is zero. Hence, we find the e_i's by inverting the above relations giving a_i, b_i in terms of the e_i .

Q.E.D.

So we note the following:

Theorem 11.5

(i) If A is an endomorphism of a complex vector space V, let $V = \oplus(V_\lambda)$ with $V_\lambda = \oplus(V_{\lambda,j})$ be a Jordan decomposition into indecomposable A-modules with $(A-\lambda I)$ nilpotent of order $n(\lambda,j)$ on $V_{\lambda,j}$. The eigenvalues λ and the $n(\lambda,j)$ determine the conjugacy class of A in $GL(V)$.

(ii) If A preserves a non-degenerate Hermitian (indefinite) form on V, then the above decomposition can be chosen s.t.

$$V_{\lambda,j} \qquad \text{for } |\lambda| = 1$$

$$W_\lambda = V_\lambda \oplus V_{\bar{\lambda}^{-1}} \qquad \text{for } |\lambda| \neq 1$$

are all mutually orthogonal.

Then, the Hermitian form on each $V_{\lambda,j}$ for $|\lambda| = 1$ has signature:

$$\begin{cases} 0 & \text{if } n(\lambda,j) \text{ is even} \\ \pm 1 & \text{if } n(\lambda,j) \text{ is odd} \end{cases}$$

The conjugacy class of A *in the (indefinite) unitary group is determined by the Jordan block data* (λ's *and* $n(\lambda,j)$'s) *together with the signature* (± 1) *for each* $V_{\lambda,j}$ *where* $|\lambda| = 1$, *and* $n(\lambda,j)$ *is odd*. *All such data occur, so long as* $n(\lambda,j) = n(\bar{\lambda}^{-1},j)$.

References

[1] M.F. Atiyah, V.K. Patodi & I.M. Singer 'Spectral Asymmetry and Riemannian Geometry', Bull. London Math. Soc. 5 (1973) 229-234.

[2] M.F. Atiyah & I.M. Singer 'The index of elliptic operators III', Ann. of Math. 87 (1968) 546-604.

[3] J-P. Serre 'Géométrie Algébrique et Geometrie Analytique' Annales de l'institut Fourier 6 (1955-6) 1-42.

[4] S. Lang 'Algebra', Addison Wesley (1984) Ch. VI, §§5-7.