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THE OCHANINE k-INVARIANT IS A BROWN–KERVAIRE INVARIANT

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0. INTRODUCTION

FOR $(8m + 2)$-dimensional closed *Spin*-manifolds, one can define on the one side the finite set of Brown–Kervaire invariants, and on the other side Ochanine's invariant k. Both are $\mathbb{Z}/2$-valued invariants of *Spin*-bordism, where the first are defined cohomologically as the Arf-invariant of certain quadratic refinements of the intersection form on $H^{4m+1}(M; \mathbb{Z}/2)$, and the second can be defined as a KO-characteristic number which by the real family index theorem has an analytic interpretation as the mod 2 index of a twisted Dirac-operator. Ochanine showed that these invariants agree on the class of *Spin*-manifolds, for which all Stiefel–Whitney numbers containing an odd-dimensional Stiefel–Whitney class vanish. On the other hand, it is not difficult to construct two different Brown–Kervaire invariants in dimension 34.

We show here (Theorem 7.1) that Ochanine's invariant is in fact a Brown–Kervaire invariant; in particular, it vanishes if $H^{4m+1}(M; \mathbb{Z}/2) = 0$, and is an invariant of the *Spin*-homotopy type (Corollary 7.3). This result is in analogy to the Hirzebruch signature theorem and can be considered as a $\mathbb{Z}/2$-valued cohomological index theorem for the above operator. The proof uses the integral elliptic homology of Kreck and Stolz (which in particular characterizes invariants with a multiplicativity property in $\mathbb{H}P^2$-bundles) and the theory of Kristensen about secondary cohomology operations (which gives a Cartan formula necessary for the computation of certain secondary operations in $\mathbb{H}P^2$-bundles).

1. BROWN–KERVAIRE INVARIANTS

In [3], Brown generalized the Kervaire invariant of framed manifolds to the bordism theory Ω^ξ_* associated to a fibration $\xi: B \to BO$:

$$K: Q^\xi_{2n} \times \Omega^\xi_{2n} \to \mathbb{Z}/8.$$

Here Q^ξ_{2n} denotes the *set of parameters*

$$Q^\xi_{2n} := \{h \in Hom(\pi_{2n}(M\xi \wedge K_n), \mathbb{Z}/4) \mid h(\lambda_\xi) = 2\}$$

where $K_n := K(\mathbb{Z}/2, n)$, and where the stable map $\lambda_\xi: S^{2n} \to S^0 \wedge K_n \to M\xi \wedge K_n$ is induced by the non-trivial map in $\pi^{st}_{2n}(K_n) = \mathbb{Z}/2$. The set Q^ξ_{2n} has a transitive and effective action of $H^n B$ (cohomology always with $\mathbb{Z}/2$-coefficients) and is non-empty iff the Wu class $v_{n+1}(\xi)$ vanishes. Brown defined $K_h(M^{2n})$, which we call the *Brown–Kervaire invariant* of M^{2n} *with parameter* h, as the $\mathbb{Z}/8$-valued Arf-invariant of the $\mathbb{Z}/4$-valued quadratic form

$$q_h: H^n M^{2n} \to \{M^{2n}, K_n\} \xrightarrow{\cong} [S^{2n}, M\nu \wedge K_n] \xrightarrow{\nu_* \wedge 1} [S^{2n}, M\xi \wedge K_n] \xrightarrow{h} \mathbb{Z}/4$$

where the first map is stabilization, the second is S-duality, and the third is given by the ξ-structure $v: M^{2n} \to B$ on M^{2n}. The map q_h is a quadratic refinement of the $\mathbb{Z}/2$-intersection pairing $H^n M^{2n} \times H^n M^{2n} \to \mathbb{Z}/2$. If the Wu class $v_n(\xi)$ vanishes, the pairing is even and q_h takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$; in this case $K_h(M^{2n})$ is the ordinary $\mathbb{Z}/2$-valued Arf-invariant of the quadratic form q_h.

By definition, the map K_h is linear in the second variable, but not in the first variable where one has

$$K_{h+x}(M^{2n}) - K_h(M^{2n}) = i_4^8(q_h(v^*x)[M^{2n}])$$

for $x \in H^n B$ and $i_4^8: \mathbb{Z}/4 \subsetneq \mathbb{Z}/8$ the natural inclusion. This shows that for M, x, y with $v^*(xy)[M] \neq 0$, at least one of the three Brown–Kervaire invariants K_{h+x}, K_{h+y} and K_{h+x+y} is different from K_h because of

$$q_h(v^*(x+y))[M] - q_h(v^*x)[M] - q_h(v^*y)[M] = i_2^4(v^*(xy)[M] \neq 0.$$

2. SPIN MANIFOLDS

For *Spin*-bordism, $\xi: BSpin \to BO$, the total Wu class has the form

$$v(\xi) = 1 + v_4 + v_8 + v_{12} + \cdots$$

because of $\chi(Sq^{2k+1}) = \chi(Sq^{2k})Sq^1$ and $\chi(Sq^{4k+2}) = \chi(Sq^{4k})Sq^2 + Sq^1\chi(Sq^{4k})Sq^1$, showing that the condition $v_{n+1}(\xi) = 0$ is satisfied for $n = 2m$ and $n = 4m + 1$.

In the first case, the Pontragin square $\wp$ gives a canonical $h_\wp \in Q_{4m}^{Spin}$ and then $K_\wp$ is equal to the signature mod 8 by a theorem of Morita [13].

The second case is more complicated; here one also has $v_n(\xi) = 0$ and gets thus $\mathbb{Z}/2$-valued invariants. The parameter set Q_{8m+2}^{Spin} grows according to $|Q_{8m+2}^{Spin}| = |H^{4m+1} BSpin|$ and $H^*BSpin = \mathbb{Z}/2[w_k | k \neq 1, 2^s + 1]$. This gives for the first values $|Q_2^{Spin}| = |Q_{10}^{Spin}| = |Q_{18}^{Spin}| = 1$, $|Q_{26}^{Spin}| = 4$ and $|Q_{34}^{Spin}| = 16$. On the other side, Ochanine proved in [14] that in dimensions 2, 10, 18 and 26, the Brown–Kervaire invariants agree with his invariant k (defined in the next section). Actually, he proved this for the larger class of invariants $K: \Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$ with the properties

(1) $K(M^{8m} \times \bar{S}^1 \times \bar{S}^1) = sign(M^{8m}) \bmod 2$

(2) $H^{4m+1} M^{8m+2} = 0 \Rightarrow K(M^{8m+2}) = 0$

where $\bar{S}^1$ denotes the circle with the non-trivial *Spin*-structure. Ochanine called these invariants *generalized Kervaire invariants* because for Brown–Kervaire invariants, (2) is obviously satisfied and (1) follows from the results in [4].

In [14], there is also an example of two different generalized Kervaire invariants in dimension 34: Let M^{10} and M^{24} be closed *Spin*-manifolds with the only (tangential) non-zero Stiefel–Whitney numbers

$$w_6 w_4[M^{10}],\ w_8^2 w_4^2[M^{24}],\ w_7^2 w_6 w_4[M^{24}],\ w_6^4[M^{24}],\ w_6^2 w_4^2[M^{24}],\ w_4^6[M^{24}]$$

(M^{10} and M^{24} exist by [1, 12]). Define $M^{34} := M^{10} \times M^{24}$ and let $K: \Omega_{34}^{Spin} \to \mathbb{Z}/2$ be a generalized Kervaire invariant; then $K + w_{12} w_8 w_7^2[\ ]$ is also a generalized Kervaire invariant, with $w_{12} w_8 w_7^2[M^{34}] \neq 0$.

A modification of this example shows also the existence of two different Brown–Kervaire invariants in dimension 34: Let $x := w_{13} w_4$ and $y := w_{10} w_7$; then we also have that $v^*(xy)[M^{34}] \neq 0$, which implies that at least one of K_{h+x}, K_{h+y} and K_{h+x+y} is different from K_h for every $h \in Q_{34}^{Spin}$.

We come now to the construction of certain Brown–Kervaire invariants by unstable secondary cohomology operations [3]

$$\phi: ker(\alpha) \to coker(\beta)$$

with range of definition and indeterminacy given by

$$\alpha := (Sq^{4m}, Sq^{4m}Sq^1): H^{4m+1}X \to H^{8m+1}X \oplus H^{8m+2}X,$$

$$\beta := Sq^2 + Sq^1: H^{8m}X \oplus H^{8m+1}X \to H^{8m+2}X.$$

These operations are associated to the decomposition

$$Sq^{4m+2} = Sq^2Sq^{4m} + Sq^1Sq^{4m}Sq^1$$

and are quadratic refinements of the cup pairing modulo the indeterminacy [5]. We call them *Brown–Peterson secondary cohomology operations*. Two of them differ by a primary operation $\gamma: H^{4m+1}X \to H^{8m+2}X$ which lies in the stable range and is thus given by an element $\gamma \in A^{4m+1}$ in the Steenrod algebra. If X is a 1-connected *Spin*-manifold M^{2n}, we have $ker(\alpha) = H^nM^{2n}$, $coker(\beta) = H^{2n}M^{2n} = \mathbb{Z}/2$, and get therefore a Brown–Kervaire invariant K_ϕ by $q_h := \phi$, which we call a *Brown–Peterson–Kervaire invariant*. If $v_\gamma \in H^{4m+1}M^{8m+2}$ denotes the generalized Wu class of $\gamma = \phi - \phi'$, we have

$$K_\phi(M^{8m+2}) - K_{\phi'}(M^{8m+2}) = \phi(v_\gamma)[M^{8m+2}].$$

3. OCHANINE'S INVARIANT

In [14], Ochanine defined an invariant

$$k: \Omega^{Spin}_{8m+2} \to \mathbb{Z}/2$$

by $k(M^{8m+2}) := sign(W^{8m+4})/8 \bmod 2$, where $\partial W^{8m+4} = M^{8m+2} \times \bar{S}^1$. Such a *Spin*-manifold W^{8m+4} exists by [1] and has signature divisible by 8, and k is well-defined because of Novikov additivity and Ochanine's signature theorem.

Ochanine gave in [15] another construction of k in terms of KO-characteristic numbers. We recall the coefficients of the KO-theory

$$KO_* = \frac{\mathbb{Z}[\eta, \omega, \mu, \mu^{-1}]}{2\eta = \eta^3 = \eta\omega = 0, \omega^2 = 4\mu}$$

where $\eta \in KO_1$, $\omega \in KO_4$ and $\mu \in KO_8$ are given by the Hopf bundles (viewed as real vector bundles) over the real, quaternion, and Cayley projective lines $\mathbb{R}P^1 = S^1$, $\mathbb{H}P^1 = S^4$, and $\mathbb{O}P^1 = S^8$. Now define for a real vector bundle $E \to X$

$$\Theta_q(E) := \bigotimes_{n \geq 1} (\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)) \quad \in KO^0(X)[[q]]$$

where $\Lambda_u(E) = \sum_{u \geq 0} u^k \Lambda^k(E)$ and $S_u(E) = \sum_{u \geq 0} u^k S^k(E)$ are the total exterior, respectively, symmetrical powers of E. For the trivial line bundle we have, in particular,

$$\theta(q) := \Theta_q(1) = \prod_{n \geq 1} \frac{1 - q^{2n-1}}{1 - q^{2n}}$$

which we also view as an element in $\mathbb{Z}[[x]]$. Because of $\Theta_q(E \oplus F) = \Theta_q(E)\Theta_q(F)$ we can extend Θ_q to

$$\Theta_q: KO^0(X) \to KO^0(X)[[q]].$$

For a n-dimensional closed *Spin*-manifold M^n, Ochanine defined

$$\beta(M^n) := \langle \Theta_q(TM^n - n), [M^n]_{KO} \rangle = \theta(q)^{-n} \langle \Theta_q(TM^n), [M^n]_{KO} \rangle \quad \in KO_n[[q]]$$

where $[M^n]_{KO} \in KO_n(M^n)$ denotes the Atiyah–Bott–Shapiro orientation of M^n and $\langle , \rangle : KO^m(X) \otimes KO_n(X) \to KO_{n-m}$ the Kronecker pairing. This gives a multiplicative *Spin*-bordism invariant, the *Ochanine elliptic genus*

$$\beta : \Omega_n^{Spin} \to KO_n[[q]].$$

The coefficient of q^0 is obviously given by $\langle 1, [M^n]_{KO} \rangle \in KO_n$, which is the definition of the *Atiyah α-invariant* $\alpha : \Omega_n^{Spin} \to KO_n$.

Ochanine proves in [15] that the Pontrjagin character of β gives the q-expansion at the cusp ∞ of the universal elliptic genus $\phi : \Omega_*^{SO} \to \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ where $\delta = \phi(\mathbb{C}P^2)$, $\varepsilon = \phi(\mathbb{H}P^2)$. Thus, β extends ϕ for *Spin*-manifolds to the dimensions $8m + 1$ and $8m + 2$. Furthermore, β takes values in the ring of modular forms over KO_*, more precisely:

THEOREM 3.1 (Ochanine [15]). *The image of β is the subring generated by η, $\omega\delta_0$, $\mu\delta_0$ and $\mu\varepsilon$ of the ring $M^{\Gamma_0(2)}(KO_*) = KO_*[\delta_0, \varepsilon]/\eta(\delta_0 - 1) \subset KO_*[[q]]$ of modular forms over KO_*, where $\delta_0 \in \mathbb{Z}[[q]]$ and $\varepsilon \in \mathbb{Z}[[q]]$ are given by*

$$\delta_0 = -8\delta = 1 + 24 \sum_{n \geq 1} \left(\sum_{d|n,\, d \text{ odd}} d \right) q^n, \qquad \varepsilon = \sum_{n \geq 1} \left(\sum_{d|n,\, n/d \text{ odd}} d^3 \right) q^n.$$

In particular, one has for a $(8m + 2)$-dimensional closed *Spin*-manifold M^{8m+2}

$$\beta(M^{8m+2}) = (a_0(M^{8m+2}) + a_1(M^{8m+2})\bar{\varepsilon} + \cdots + a_m(M^{8m+2})\bar{\varepsilon}^m)\eta^2\mu^m$$

with homomorphisms $a_i : \Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$, where

$$\bar{\varepsilon} = \sum_{k \geq 0} q^{(2k+1)^2} = q + q^9 + q^{25} + \cdots \quad \in \mathbb{Z}/2[[q]]$$

is the reduction of ε modulo two. Obviously the lowest coefficient a_0 is again given by the Atiyah α-invariant.

THEOREM 3.2 (Ochanine [15]). *The highest coefficient a_m is equal to the Ochanine invariant k.*

This gives an expression of k in terms of KO-characteristic numbers: Let $q(\varepsilon) \in \mathbb{Z}[[\varepsilon]]$ be any formal power series whose $\mathbb{Z}/2$-reduction is the inverse power series of $\bar{\varepsilon} \in \mathbb{Z}/2[[q]]$, then a_m is the coefficient of $\bar{\varepsilon}^m$ in the polynomial $\beta_{q(\varepsilon)} \in KO_{8m+2}[\bar{\varepsilon}]$ which is obtained from the formal power series $\beta \in KO_{8m+2}[[q]]$ by inserting $q(\varepsilon)$ for q. We get (suppressing $\eta^2\mu^m$)

$$a_m = \frac{1}{2\pi i} \oint \frac{d\varepsilon}{\varepsilon^{m+1}} \beta_{q(\varepsilon)} = \frac{1}{2\pi i} \oint \frac{dq}{q^{m+1}} \left(\frac{\varepsilon}{q} \right)^{-m} \beta_q \bmod 2$$

because $q d\varepsilon/dq = \varepsilon \bmod 2$. Thus, $k = a_m$ is the coefficient of q^m in $f(q)^{-8m} \cdot \beta \in KO_{8m+2}[[q]]$ where

$$f(q) := \sum_{n \geq 1} q^{\binom{n}{2}} = 1 + q + q^3 + q^6 + \cdots$$

since $\varepsilon/q = f(q^8) \bmod 2 = f(q)^8 \bmod 2$. Together with the real family index theorem in dimension $8m+2$, this shows that k has an analytical interpretation as the mod 2 index of a twisted Dirac operator:

THEOREM 3.3 (Atiyah and Singer [2]). *Let M be an $(8m+2)$-dimensional closed Spin-manifold and $E \in KO^0(X)$. Define $e \in \mathbb{Z}/2$ by $\langle E, [M]_{KO}\rangle = e\eta^2\mu^m \in KO_{8m+2}$, then*

$$e = dim_{\mathbb{C}}\, ker(D_E) \bmod 2,$$

where D_E is the Dirac operator of M twisted by the virtual bundle E.

COROLLARY 3.4. *The Ochanine invariant of an $(8m+2)$-dimensional closed Spin-manifold M is equal to the mod 2 index of the Dirac operator twisted by the virtual bundle E_m, which is the coefficient of q^m in $f(q)^{-8m}\theta(q)^{-8m-2}\Theta_q(TM) \in KO^0(M)[[q]]$,*

$$k(M) = dim_{\mathbb{C}}\, ker(D_{E_m}) \bmod 2.$$

4. INTEGRAL ELLIPTIC HOMOLOGY

This theory of Kreck and Stolz [7] is a refinement at the prime two of the elliptic homology theory of Landweber, Ravenel and Stong and has a very geometric definition in terms of $\mathbb{H}P^2$-bundles. Here we need only the coefficients of this theory. We consider $\mathbb{H}P^2$-bundles $p: N^{k+8} \to M^k$ where the structure group is $PSp(3)$ acting in the canonical way on $\mathbb{H}P^2 = PSp(3)/P(Sp(2)\times Sp(1))$ and N^{k+8}, M^k are closed *Spin*-manifolds. Such bundles are the pullback of the universal $\mathbb{H}P^2$-bundle $E := BP(Sp(2)\times Sp(1)) \to BPSp(3)$ by a classifying map $f: M^k \to BPSp(3)$. On the level of bordism one has homomorphisms

$$\Psi: \Omega_k^{Spin}(BPSp(3)) \to \Omega_{k+8}^{Spin}, \quad [M,f] \mapsto [N = f^*E]$$
$$\pi: \Omega_k^{Spin}(BPSp(3)) \to \Omega_k^{Spin}, \quad [M,f] \mapsto [M]$$

and we set

$T_* := \operatorname{im}\Psi = \{\text{total spaces of } \mathbb{H}P^2\text{-bundles}\} \subset \Omega_*^{Spin}$
$\tilde{T}_* := \Psi(ker\,\pi) = \{\text{total spaces of } \mathbb{H}P^2\text{-bundles with zero-bordant base}\} \subset \Omega_*^{Spin}$.

There are the following results, where α and β are the invariants of Atiyah and Ochanine.

THEOREM 4.1 (Stolz [16]). *$T_* = ker\,\alpha$.*

THEOREM 4.2 (Kreck and Stolz [7]). *$\tilde{T}_* = ker\,\beta$.*

If we set $ell_* := \Omega_*^{Spin}/\tilde{T}_*$, then we have:

THEOREM 4.3 (Kreck and Stolz [7]). *Let $s := [\bar{S}^1]$, $k := [K^4]$, $b := [B^8]$ and $h := [\mathbb{H}P^2]$ be the Spin-bordism classes of the non-trivial circle, the Kummer-surface, the Bott-manifold and the quaternion plane. Then*

$$ell_* = \frac{\mathbb{Z}[s,k,b,h]}{2s = s^3 = sk = 0,\ k^2 = 4b + 256h}.$$

Remarks. Theorem 4.1 was the key step in Stolz' proof [16] of the Gromov–Lawson conjecture. By $ell_* \cong \operatorname{im}\beta$, we get from Theorem 4.3, just Ochanine's Theorem 3.1 about $\operatorname{im}\beta$.

COROLLARY 4.4. *Let* $K:\Omega^{Spin}_{8m+10}\to\mathbb{Z}/2$ *and* $K':\Omega^{Spin}_{8m+2}\to\mathbb{Z}/2$ *be homomorphisms with*
(i) $K(N^{8m+10})=K'(M^{8m+2})$ *for any* $\mathbb{H}P^2$*-bundle* $p:N^{8m+10}\to M^{8m+2}$,
(ii) $K(N^{8m+8}\times\bar{S}^1\times\bar{S}^1)=sign(N^{8m+8})\bmod 2$.
Then K *is equal to the Ochanine invariant,* $K=k$.

Proof. The condition (i) is equivalent to $K\circ\Psi=K'\circ\pi$, thus K factors over ell_{8m+10} which is a $\mathbb{Z}/2$-vector space with basis $(s^2h^ib^j)_{i+j=m+1}$. By (ii), we have $K(s^2h^ib^j)=sign(h^ib^j)\bmod 2=k(s^2h^ib^j)$. ∎

5. SECONDARY OPERATIONS AND $\mathbb{H}P^2$-BUNDLES

In order to apply the results of the previous section, we want to compute Brown–Peterson–Kervaire invariants of $\mathbb{H}P^2$-bundles. Let $p:N^{8m+10}\to M^{8m+2}$ be a $\mathbb{H}P^2$-bundle of *Spin*-manifolds classified by $f:M^{8m+2}\to BPSp(3)$. Since $BPSp(3)$ is 1-connected, we can by surgery assume that M^{8m+2} is 1-connected without changing the bordism class $[M^{8m+2},f]\in\Omega^{Spin}_{8m+2}(BPSp(3))$. Then N^{8m+10} is also 1-connected. By the Leray–Hirsch theorem we have

$$H^{4m+5}N^{8m+10}=p^*H^{4m+5}M^{8m+2}\oplus xp^*H^{4m+1}M^{8m+2}\oplus x^2p^*H^{4m-3}M^{8m+2}$$

where $x\in H^4N^{8m+10}$ is the pullback of the universal Leray–Hirsch generator in $H^4BP(Sp(2)\times Sp(1))$ belonging to the universal $\mathbb{H}P^2$-bundle $BP(Sp(2)\times Sp(1))\to BPSp(3)$.

Now assume that $\phi:(ker\,\alpha)^{4m+5}\to(coker\,\beta)^{8m+10}$ is a Brown–Peterson operation giving a Brown–Peterson–Kervaire invariant in dimension $8m+10$. For N^{8m+10} we have $ker\,\alpha=H^{4m+5}N^{8m+10}$ and $coker\,\beta=H^{8m+10}N^{8m+10}=\mathbb{Z}/2$, and for $a=p^*a'\in p^*H^{4m+5}M^{8m+2}$ we get

$$\phi(a)=p^*\phi(a')\in p^*H^{8m+10}M^{8m+2}=0.$$

On the two other summands $xp^*H^{4m+1}M^{8m+2}$ and $x^2p^*H^{4m-3}M^{8m+2}$ the operation ϕ does not vanish in general, but the following algebraic lemma tells us that we can apply Corollary 4.4 if we would have $\phi(xp^*a')=x^2p^*\phi'(a')$ on the middle summand, with $\phi':(ker\,\alpha)^{4m+1}\to(coker\,\beta)^{8m+2}$, another appropriate Brown–Peterson operation.

LEMMA 5.1. *Let* V *be a finite-dimensional* $\mathbb{Z}/2$*-vector space and* $q:V\to\mathbb{Z}/2$ *be a non-degenerate quadratic form which vanishes on a sub-Lagrangian* V_-. *Define* $V_0:=V^{\perp}_-/V_-$ *and* $q_0:V_0\to\mathbb{Z}/2$ *by* $q_0(v+V_-):=q(v)$. *Then* q_0 *is a non-degenerate quadratic form on* V_0 *and*

$$\mathrm{Arf}(q)=\mathrm{Arf}(q_0).$$

Proof. We use the definition of the Arf-invariant as the "democratic invariant":

$$\mathrm{Arf}(q)=sgn\sum_{v\in V}(-1)^{q(v)}.$$

It is easy to see that q_0 is well-defined and non-degenerate. We set $V_+:=V/V^{\perp}_-$, then the pairing $V_-\times V_+\to\mathbb{Z}/2$ is also well-defined and non-degenerate. If we choose $\mathbb{Z}/2$-linear splittings $V_0\subsetneq V^{\perp}_-$ and $V_+\subsetneq V$ we have $V=V_-\oplus V_0\oplus V_+$ and

$$\begin{aligned}\mathrm{Arf}(q)&=sgn\sum_{v_-\in V_-,v_0\in V_0,v_+\in V_+}(-1)^{q(v_-+v_0+v_+)}\\&=sgn\sum_{v_0\in V_0}(-1)^{q(v_0)}\left(\sum_{v_+\in V_+}(-1)^{q(v_+)+v_0v_+}\left(\sum_{v_-\in V_-}(-1)^{v_-v_+}\right)\right)\end{aligned}$$

because of $q(v_- + v_0 + v_+) = q(v_0) + q(v_+) + v_-v_+ + v_0v_+$. But $\sum_{v_- \in V_-} (-1)^{v_-v_+}$ is 0 for $v_+ \neq 0$ and $|V_-|$ for $v_+ = 0$, which gives

$$\operatorname{Arf}(q) = sgn \sum_{v_0 \in V_0} (-1)^{q(v_0)} ((-1)^{q(0)+0} \cdot |V_-|)$$

$$= sgn \sum_{v_0 \in V_0} (-1)^{q(v_0)} = \operatorname{Arf}(q_0). \qquad \blacksquare$$

Remark. This lemma can also be proved by constructing a "good" symplectic basis, but the proof here generalizes also to the $\mathbb{Z}/8$-valued Arf-invariant.

6. A PRODUCT FORMULA OF KRISTENSEN

The previous section shows that we need a product formula for the unstable secondary cohomology operation ϕ applied to $x \cdot y$ with $x \in H^4 N^{8m+10}$ and $y := p^*a$, $a \in H^{4m+1} M^{8m+2}$. Sum and product formulas for secondary cohomology operations of this type were obtained by Kristensen in a series of papers [8–10]; see in particular [11] for a short survey on his product formula.

Kristensen worked in the category of simplicial sets which is no restriction because its homotopy theory is equivalent to the homotopy theory of topological spaces, and used cochain operations to represent secondary cohomology operations. A *cochain operation* $a = (a_k)_{k \in \mathbb{N}}$ of degree $n \in \mathbb{N}$ is a series of natural transformations $a_k : C^k(\ ) \to C^{k+n}(\ )$ of the normalized cochain functor for simplicial sets (coefficients are always $\mathbb{Z}/2$). The a_k need neither to be linear nor to commute with the coboundary $\delta : C^k(\ ) \to C^{k+1}(\ )$. Kristensen defined a differential Δ in the graded $\mathbb{Z}/2$-vector space $\mathcal{O}^*$ of these operations by $(\Delta a)_k := \delta a_k + a_{k+1}\delta$ (here, $a(0) = 0$ follows by naturality from the vanishing of the *normalized* cochains of the simplicial point) and showed that:

THEOREM 6.1 (Kristensen [8]). *Let $a \in \mathcal{O}^n$ with $\Delta a = 0$ and define a cohomology operation $[a]$ of degree n in each dimension k by $[a]([x]) := [a_k(x)]$ for all $x \in C^k X$ with $\delta x = 0$, then $[a]$ is well-defined and stable. This gives an isomorphism*

$$H(\mathcal{O}^*, \Delta) = A^*.$$

This isomorphism is also compatible with composition, but in contrast to the Steenrod algebra A^*, the cochain operations $\mathcal{O}^*$ do not built an algebra because in general its elements consist of non-linear mappings and the composition is thus not right distributive. For example, by using a system of cup-i products one defines cochain operations $sq^i \in \mathcal{O}^i$ as

$$(sq^i)_k(x) := x \cup_{k-i} x + x \cup_{k-i+1} \delta x, \quad x \in C^k X$$

which give the Steenrod squares $Sq^i = [sq^i]$. While the Sq^i are linear they are induced from quadratic maps sq^i.

Kristensen proved also a r-variable version of the above theorem

$$H(\mathcal{O}^{*(r)}, \Delta^{(r)}) = \bigoplus_r A^*$$

where a cochain operation a of degree n in r variables is a series of natural transformations a_k: $C^k(\ ) \times \cdots \times C^k(\ ) \to C^{k+n}(\ )$ and the differential $\Delta^{(r)}$ is defined by $(\Delta^{(r)}a)(x_1, \ldots, x_r) :=$

$\delta a(x_1, \dots, x_r) + a(\delta x_1, \dots, \delta x_r)$. As an application, for each $a \in \mathcal{O}^n$ with $\Delta a = 0$ there exists a r-variable cochain operation $d_a \in \mathcal{O}^{n-1(r)}$ with

$$(\Delta^{(r)} d_a)(x_1, \dots, x_r) = a\left(\sum_{i=1}^{r} x_i\right) - \sum_{i=1}^{r} a(x_i)$$

because the left-hand side measures the deviation of a from linearity which vanishes in $\oplus_r A^n$ since $[a] \in A^n$ is linear.

We come now to the representation of secondary cohomology operations by cochain operations, see [8]. We start with a relation $\sum_{i=1}^{s} \alpha_i \beta_i = \gamma$ of degree n in the Steenrod algebra, where $\alpha_i \in A^{n_i}$, $\beta_i \in A^{m_i}$ with $n_i + m_i = n$ for $i = 1, \dots, s$, and $\gamma \in A^n$. If we write α_i, β_i and γ as sums of admissible monomials in the Sq^k, then the corresponding expressions with Sq^k replaced by sq^k are representing cochain operations a_i, b_i and c. The cochain operation $r := \sum_{i=1}^{s} a_i b_i + c \in \mathcal{O}^n$ has the property $\Delta r = 0$ and $[r] = 0$, thus there exists a cochain operation $R \in \mathcal{O}^{n-1}$ with $\Delta R = r$. Now, let $[x] \in H^k X$ be in the kernel of all the β_i, and $k < excess(\gamma)$. Since $[b_i(x)] = 0$ there are $w_i \in C^{k+m_i-1} X$ with $\delta w_i = b_i(x)$, and furthermore $c(x) = 0$ by the definition of the excess and of the sq^i. Consider

$$\phi(x) := R(x) + \sum_{i=1}^{s} a_i(w_i)$$

then $\phi(x) \in C^{k+n-1} X$ and a short computation gives $\delta\phi(x) = 0$. Kristensen shows that choosing other w_i' with $\delta w_i' = b_i(x)$ or another $x' \in [x]$ changes the cohomology class $[\phi(x)] \in H^{k+n-1} X$ by elements in $\sum_{i=1}^{s} \mathrm{im}(\alpha_i : H^{k+m_i-1} X \to H^{k+n-1} X)$. Thus, for $k < excess(\gamma)$ we have defined a secondary cohomology operation

$$\phi : ker\left(H^k X \xrightarrow{(\beta_1, \dots, \beta_s)} \bigoplus_{i=1}^{s} H^{k+m_i} X\right) \to coker\left(\bigoplus_{i=1}^{s} H^{k+m_i-1} X \xrightarrow{(\alpha_1 + \cdots + \alpha_s)} H^{k+n-1} X\right),$$

which is stable if γ vanishes. Furthermore, a different choice of R' with $\Delta R' = r$ is given by $e := R' - R \in \mathcal{O}^n$ with $\Delta e = 0$, and then one has $\phi' - \phi = [e] \in A^{n-1}$ for the corresponding secondary cohomology operations. In the following we say that ϕ is associated with the "relation"

$$\rho := \sum_{i=1}^{s} \alpha_i \otimes \beta_i \in A^* \otimes A^*$$

and is defined in dimensions $k < excess(\mu(\rho))$, where $\mu : A^* \otimes A^* \to A^*$ denotes the product in the Steenrod algebra. These operations are equivalent to those constructed in the topological category from $\sum_{i=1}^{s} \alpha_i \beta_i = \gamma$. We remark that in the case of $|x| < excess(\beta_i)$ for all $i = 1, \dots, s$ we have $b_i(x) = 0$ and can thus make the canonical choice $w_i = 0$. But also r vanishes then in this dimension and R can be chosen with $R(x) = 0$ (one can easily see this in the topological category by choosing the zero map between the appropriate Eilenberg–MacLane spaces as a representative of β_i). In particular, one has then $\phi([x]) = 0$.

Now we want to compute a product formula for the operation ϕ. The product formula for a stable primary cohomology operation is given by the coproduct $\psi : A^* \to A^* \otimes A^*$ in the Steenrod algebra, and for relations we have the coproduct

$$\psi^{(2)} := (1 \otimes t \otimes 1)(\psi \otimes \psi) : A^* \otimes A^* \to A^* \otimes A^* \otimes A^* \otimes A^*$$

with the Hopf algebra property $(\mu \otimes \mu)\psi^{(2)} = \psi\mu$. Suppose now that we have

$$\psi^{(2)}\rho = \sum_{n \in N} \rho_n' \otimes \varepsilon_n'' + \sum_{m \in M} \varepsilon_m' \otimes \rho_m''$$

with ρ'_n, ρ''_m, ε'_m, $\varepsilon''_n \in A^* \otimes A^*$, where we regard the ρ'_n, ρ''_m as relations. This decomposition is designed for the case that $0 = \beta_i([x][y]) = \sum_{j \in B_i} \beta'_{ij}([x])\beta''_{ij}([y])$ holds true because in each summand at least one factor is zero, which Kristensen calls the *complementary case*; his method works only under this condition (see [9, 10]).

A first conjecture would be that (on the common domain of definition and modulo the total indeterminacy) one has then $\phi([x][y]) = \sum_{n \in N} \phi'_n([x]) \delta''_n([y]) + \sum_{m \in M} \delta'_m([x])\phi''_m([y])$ with secondary cohomology operations ϕ'_n, ϕ''_m associated with ρ'_n, ρ''_m, and $\delta''_n := \mu(\varepsilon''_n)$, $\delta'_m := \mu(\varepsilon'_m)$. But the situation is a little more complicated because a relation gives in general more than one secondary operation (which differ by stable primary operations), so this equation can be only true if one adds at the left-hand side $\varepsilon([x] \otimes [y])$ with a certain primary cohomology operation $\varepsilon \in A^* \otimes A^*$, whose computation was the main problem in Kristensen's product formula for ϕ.

For the computation of $\phi([x][y]) = [R(xy) + \sum_{i=1}^{s} b_i(w_i)]$ we need two parts: Firstly, an expansion of $R(xy)$ (with xy meaning the cup product of cochains), and secondly, cochains w_i with $\delta w_i = b_i(xy)$ which are given in terms of the complementarity condition. The second problem leads to *cochain operations* $\mathscr{Q}^*$ *of the second kind*; these are series $G = (G_{i,j})_{i,j \in \mathbb{N}}$ of natural transformations $G_{i,j}: C^i(\ ) \times C^j(\ ) \to C^{i+j+n}(\ )$, and one has a differential $\nabla: \mathscr{Q}^* \to \mathscr{Q}^{*+1}$ by $(\nabla G)(x, y) := \delta G(x, y) + G(\delta x, y) + G(x, \delta y)$. Kristensen proves in [9] that

$$H(\mathscr{Q}^*, \nabla) = A^* \otimes A^*.$$

As an application of this theorem, let $\alpha \in A^n$ and the terms in the coproduct $\psi\alpha = \sum \alpha'_k \otimes \alpha''_k$ be represented by cochain operations a, a'_k and a''_k. Then there exists a cochain transformation $T_a \in \mathscr{Q}^{n-1}$ of the second kind measuring the deviation of the Cartan formula on the cochain level,

$$\nabla T_a(x, y) = a(xy) + \sum a'_k(x)a''_k(y) + d_a(\delta xy, x\delta y) + |x| d_a(x\delta y, x\delta y).$$

For the proof, one computes that ∇ of the right-hand side is zero (in order to get this, one has to include the linearity defects d_a) and that it represents $\alpha([x][y]) + \sum \alpha'_k([x])\alpha''_k([y]) = 0$. Now we can construct our w_i with $\delta w_i = b_i(xy)$ as

$$w_i := \sum_{j \in B'_i} w'_{ij} b''_{ij}(y) + \sum_{j \in B''_i} b'_{ij}(x) w''_{ij} + T_{b_i}(x, y)$$

where $\delta w'_{ij} = b'_{ij}(x)$ for $j \in B'_i$, $\delta w''_{ij} = b''_{ij}(y)$ for $j \in B''_i$, and $B_i = B'_i \sqcup B''_i$. Attacking the first problem, Kristensen defines the following cochain operation $A \in \mathscr{Q}^{n-1}$ of the second kind:

$$A(x, y) := R(xy) + \sum_{n \in N} R'_n(x) d''_n(y) + \sum_{m \in M} d'_m(x) R''_m(y) + T_r(x, y) + D_R(x, y).$$

Here $\Delta R = r$, $\Delta R'_n = r'_n$ and $\Delta R''_m = r''_m$ are representing cochain operations in $\mathscr{O}^*$ for the relations ρ, ρ'_n, $\rho''_m \in A^* \otimes A^*$ and d''_n, d'_m represent δ''_n, $\delta'_m \in A^*$. The cochain operation $T_r \in \mathscr{Q}^{n-1}$ is constructed as T_a above by the property

$$\nabla T_r(x, y) = r(xy) + \sum_{n \in N} r'_n(x) d''_m(y) + \sum_{m \in M} d'_m(x) r''_m(y) + d_r(\delta xy, x\delta y) + |x| d_r(x\delta y, x\delta y)$$

and similar $D_R \in \mathscr{Q}^{n-1}$ is constructed by

$$\nabla D_R(x, y) = R(\delta xy + x\delta y) + R(\delta xy) + R(x\delta y) + d_r(\delta xy, x\delta y) + |x| d_r(x\delta y, x\delta y).$$

While the operation T_r measures the Cartan defect of the relation r on the cochain level, the operation D_R is included to give $\nabla A = 0$ (D_R can be chosen to vanish on cocycles and is therefore neglected in [11]). Thus, we get a primary operation

$$\varepsilon := [A] \in A^* \otimes A^*$$

and Kristensen proved:

THEOREM 6.2 (Kristensen [9]). *Under the complementarity assumptions on the cohomology classes* $[x]$, $[y]$ *and* $\psi^{(2)}\rho = \sum_{n\in N} \rho'_n \otimes \varepsilon''_n + \sum_{m\in M} \varepsilon'_m \otimes \rho''_m$, *we have (on the common domain of definition and modulo the total indeterminacy)*

$$\phi([x][y]) = \sum_{n\in N} \phi'_n([x])\delta''_n([y]) + \sum_{m\in M} \delta'_m([x])\phi''_m([y]) + \varepsilon([x]\otimes[y])$$

with secondary cohomology operations ϕ, ϕ'_n, ϕ''_m *associated with the relations* ρ, ρ'_n, ρ''_m; *with* $\delta''_n := \mu(\varepsilon''_n)$, $\delta'_m := \mu(\varepsilon'_m)$, *and with* $\varepsilon \in A^* \otimes A^*$ *constructed as above.*

In the application of this formula, one has the problem that the term ε is not effectively computed by the other data. This problem was later solved by Kristensen, see [10, 11]. In particular, he gave an explicit formula for the following triple series of relations, which are linear combinations of the Adem relations:

$$\rho^k_{ab} := \sum_{j\in\mathbb{Z}} \left(\binom{b-1-j}{k+b-a-2j} + \binom{b-1-j}{j+b-a}\right) Sq^{k-j} \otimes Sq^j, \quad k, a, b \in \mathbb{Z}.$$

Here we use the conventions $Sq^k = 0$ for $k < 0$ and $\binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$ for $k \geqslant 0$, $\binom{n}{k} = 0$ for $k < 0$. One has a decomposition

$$\psi^{(2)}\rho^k_{ab} = \sum_{i,j\in\mathbb{Z}} \rho^{k-i-j}_{a-2i,b-i} \otimes (Sq^j \otimes Sq^i) + \sum_{i,j\in\mathbb{Z}} (Sq^j \otimes Sq^i) \otimes \rho^{k-i-j}_{a-j,b-i}.$$

THEOREM 6.3 (Kristensen [10]). *There exists an essentially unique choice of cochain operations* R^k_{ab} *for the relations* ρ^k_{ab} *such that the primary term* ε *in the product formula for the associated secondary cohomology operations is given by*

$$\varepsilon^k_{ab} = (Sq^1 \otimes (Sq^2Sq^1 + Sq^3))$$
$$\psi\left(\sum_{j\in\mathbb{Z}} \left(\binom{b-1-j}{k+b-a-2j} + \binom{b-1-j}{j+b-a}\right)(Sq^{k-j-3}Sq^{j-2} + Sq^{k-j-2}Sq^{j-3})\right).$$

We only mention that the proof uses:

- The Eilenberg–MacLane complex $K(\mathbb{Z}/2, 1)$, which is a simplicial $\mathbb{Z}/2$-vector space with zero differential in its normalized cochain complex, and has thus the only non-zero cochain $u^n \in C^nK(\mathbb{Z}/2, 1)$ in each dimension where u is the fundamental cocycle. Then one can relate the action of $A^k_{ab} \in \mathcal{Q}^{k-1}$ on (u^n, u^m) to the action of R^k_{ab} and T^k_{ab}.
- The cobar resolution $\bar{A}^{\otimes n}$ of the Steenrod algebra, which has homology $\Lambda(Q_0, Q_1, \ldots)$ where in particular $Q_0 = Sq^1$ and $Q_1 = Sq^2Sq^1 + Sq^3$. Then A^k_{ab} gives an element in $\bar{A}^{\otimes 2}$ whose boundary in $\bar{A}^{\otimes 3}$ can again be expressed in terms of A^k_{ab}.
- Special systems of cochain operations for the T_r with good combinatorial properties, whose existence was proved in [10]. See also the appendix of [11].

7. THE MAIN THEOREM

We prove now the main result.

THEOREM 7.1. *In each dimension* $8m+2$, *there exists a Brown–Peterson–Kervaire invariant* K_ϕ *which is equal to Ochanine's invariant* k.

Proof. By the corollary in Section 4, we have to show that for each $m \in \mathbb{N}$, there exists a Brown–Peterson–Kervaire invariant $K_{\phi_{m+1}}$ in dimension $8m+10$ and an invariant K' in dimension $8m+2$ which satisfy property (ii) (actually we will show that K' is also a Brown–Peterson–Kervaire invariant K_{ϕ_m}). By the lemma in Section 5, we have for each $\mathbb{H}P^2$-bundle $p: N^{8m+10} \to M^{8m+2}$ that $K_{\phi_{m+1}}(N^{8m+10}) = \mathrm{Arf}(q_0)$ where $q_0: H^{4m+1}M^{8m+2} \to \mathbb{Z}/2$ denotes $q_0(y') := \phi_{m+1}(x \cdot p^*y')[N^{8m+10}]$. We compute now the product formula for $\phi_{m+1}(xy)$, $y := p^*y'$, by Kristensen's theory (Section 6), where ϕ_{m+1} is associated with

$$\rho_{m+1} := Sq^2 \otimes Sq^{4m+4} + Sq^1 \otimes Sq^{4m+4}Sq^1.$$

We first have to check the complementarity conditions for xy. The summands in

$$Sq^{4m+4}(xy) = \sum_{i=0..4m+4} Sq^i x \cdot Sq^{4m+4-i} y$$

and

$$Sq^{4m+4}Sq^1(xy) = \sum_{i=0..4m+4} Sq^i Sq^1 x \cdot Sq^{4m+4-i} y + \sum_{i=0..4m+4} Sq^i x \cdot Sq^{4m+4-i} Sq^1 y$$

are all zero, in detail:

- For $i = 0, 1, 2$ in the first and the second sums, and $i = 0, 1$ in the third sum, because then the dimension $|y| = 4m+1$ is smaller than the excess of the operation acting on y.
- For $i = 5, \ldots, 4m+4$ in the first and the third sums, and $i = 6, \ldots, 4m+4$ in the second sum, because then the dimension $|x| = 4$ is smaller than the excess of the operation acting on x.
- For $i = 3, 4$ in the first sum because then $Sq^{4m+1}y = p^*Sq^1(Sq^{4m}y') = 0$ since M^{8m+2} is *Spin*, respectively $Sq^{4m}y = p^*Sq^{4m}y' = 0$ since M^{8m+2} is 1-connected.
- For $i = 3, 4, 5$ in the second sum because then $Sq^i Sq^1 x = 0$; here we use that x is the pullback of the universal Leray–Hirsch generator x^{univ} which satisfies $Sq^1 x^{\mathrm{univ}} = 0$ (see [16]).
- For $i = 2, 3, 4$ in the third sum because then $Sq^{4m+2}Sq^1 y \in p^*H^{8m+4}M^{8m+2} = 0$ and $Sq^{4m+1}Sq^1 y \in p^*H^{8m+3}M^{8m+2} = 0$, respectively, $Sq^{4m}Sq^1 y = p^*Sq^{4m}Sq^1 y' = 0$ since M^{8m+2} is *Spin*.

According to these facts we choose our splitting of

$$\psi^{(2)}\rho = \sum_{\substack{i=0,\ldots,4m+4 \\ j=0,1,2}} \sigma^1_{ji} + \sum_{\substack{i=0,\ldots,4m+4 \\ j=0,1}} \sigma^2_{ji} + \sum_{\substack{i=0,\ldots,4m+4 \\ j=0,1}} \sigma^3_{ji}$$

with

$$\sigma^1_{ji} := (Sq^j \otimes Sq^i) \otimes (Sq^{2-j} \otimes Sq^{4m+4-i})$$

$$\sigma^2_{ji} := (Sq^j \otimes Sq^i Sq^1) \otimes (Sq^{1-j} \otimes Sq^{4m+4-i})$$

$$\sigma^3_{ji} := (Sq^j \otimes Sq^i) \otimes (Sq^{1-j} \otimes Sq^{4m+4-i}Sq^1)$$

in the following way:

$$\psi^{(2)}\rho = \left(\sum_{\substack{i,j \\ i \neq 3,4}} \sigma_{ji}^1 + \sum_{\substack{i,j \\ i \neq 3,4,5}} \sigma_{ji}^2 + \sum_{\substack{i,j \\ i \neq 2,3,4}} \sigma_{ji}^3 \right) + \left(\sum_{\substack{i,j \\ i = 3,4}} \sigma_{ji}^1 + \sum_{\substack{i,j \\ i = 3,4,5}} \sigma_{ji}^2 + \sum_{\substack{i,j \\ i = 2,3,4}} \sigma_{ji}^3 \right)$$

where we denote the first bracket by Σ_1 and the second by Σ_2. Now the summands of Σ_1, which we consider as $\rho' \otimes \varepsilon''$ or $\varepsilon' \otimes \rho''$ according to that x or y gives the "reason" for being zero, contribute all with $\phi'(x)\delta''(y) = 0$ or $\delta'(x)\phi''(y) = 0$ to the sum formula for $\phi(xy)$. This holds because the kernel condition for ϕ', ϕ'' is satisfied by the fact that the excess is larger than the dimension (giving $w_k' = 0$ and $R' = 0$ as natural choices for ϕ'; and analogously for ϕ''). We say that Σ_1 consists of *trivial terms*. In contrast to this, the 18 summands in Σ_2 do not vanish by this reason; we call them *critical terms*. We show now that in our situation 16 of these terms vanish, with the remaining two terms giving exactly $x^2 \cdot p^*\phi_m(y)$.

We remark that in the case where the secondary operation of the one side of a term is defined and the primary operation of the other side of the term vanishes, the whole term (including its undeterminacy) vanishes. This applies to the terms σ_{13}^1, σ_{23}^1, σ_{14}^1, σ_{24}^1, σ_{13}^3, σ_{14}^3, which we view as $\varepsilon' \otimes \rho''$, and to σ_{03}^2, σ_{13}^2, σ_{04}^2, σ_{14}^2, σ_{05}^2, which we view as $\rho' \otimes \varepsilon''$. Furthermore, if a term $\varepsilon' \otimes \rho''$ has the property that the degree of the relation satisfies $|\rho''| > 4m + 2$, one gets $\phi''(y) \subset p^*\phi''(y') = 0$ because $\phi''(y') \subset H^{4m+|\rho''|}M^{8m+2} = 0$. This applies to σ_{03}^1, σ_{02}^3, σ_{12}^3 and σ_{03}^3. Then remain the terms σ_{15}^2, σ_{04}^1 and σ_{04}^3. We consider first

$$\sigma_{15}^2 = (Sq^1 \otimes Sq^5 Sq^1) \otimes (Sq^0 \otimes Sq^{4m-1}) =: \rho' \otimes \varepsilon''.$$

The associated secondary operation ϕ' has the property $Sq^1\phi' = 0$, thus we get for $z \in \phi'(x)$ that $z \cdot Sq^{4m-1}y = z \cdot Sq^1 Sq^{4m-2}y = Sq^1(z \cdot Sq^{4m-2}y) = 0$ showing that the term $\phi'(x)\delta''(y)$ (including its undeterminacy) vanishes. The sum of the last two terms can be factorized as

$$\sigma_{04}^1 + \sigma_{04}^3 = (Sq^0 \otimes Sq^4) \otimes (Sq^2 \otimes Sq^{4m} + Sq^1 \otimes Sq^{4m} Sq^1) = (1 \otimes Sq^4) \otimes \rho_m.$$

With the product formula 6.3 of Kristensen, we have proved

$$\phi_{m+1}(xy) = Sq^4(x)\phi_m(y) + \varepsilon(x \otimes y) = x^2 \cdot p^*\phi_m(y') + \varepsilon(x \otimes y).$$

Now, we have to compute the primary term ε, which comes from the cochain operation A of Section 6, where r and R are given by $r_{m+1} := sq^{4m+6} + sq^2 sq^{4m+4} + sq^1 sq^{4m+4} sq^1$ and $\Delta R_{m+1} = r_{m+1}$. We note that the Kristensen relations $\rho_{2b,b}^{a+b}$ of Section 6 are nothing but the Adem relations written as

$$\rho_{2b,b}^{a+b} = Sq^a \otimes Sq^b + \sum_{j \in \mathbb{Z}} \tbinom{b-1-j}{a-2j} Sq^{a+b-j} \otimes Sq^j$$

and the corresponding primary terms $\varepsilon_{2b,b}^{a+b}$ are ($Q_0 := Sq^1$, $Q_1 := Sq^2 Sq^1 + Sq^3$)

$$\varepsilon_{2b,b}^{a+b} = (Q_0 \otimes Q_1)$$
$$\cdot \psi \left(Sq^{a-3} Sq^{b-2} + Sq^{a-2} Sq^{b-3} + \sum_{j \in \mathbb{Z}} \tbinom{b-1-j}{a-2j} (Sq^{a+b-j-3} Sq^{j-2} + Sq^{a+b-j-2} Sq^{j-3}) \right).$$

We need in particular

$$\dot{\rho}_n := \rho_{4n,2n}^{2n+1} = Sq^1 \otimes Sq^{2n} + Sq^{2n+1} \otimes 1$$

$$\dot{\varepsilon}_n := \varepsilon_{4n,2n}^{2n+1} = 0$$

$$\ddot{\rho}_m := \rho^{4m+2}_{8m,4m} = Sq^2 \otimes Sq^{4m} + Sq^{4m+2} \otimes 1 + Sq^{4m+1} \otimes Sq^1$$

$$\ddot{\varepsilon}_m := \varepsilon^{4m+2}_{8m,4m} = \textstyle\sum_{i=0..4m-3} Q_0 Sq^i \otimes Q_1 Sq^{4m-3-i}$$

and decompose r_{m+1} as $\ddot{r}_{m+1} + \dot{r}_{2m+2} sq^1$, which shows that we can choose R_{m+1} as the linear combination $\ddot{R}_{m+1} + \dot{R}_{2m+2} sq^1$ with the special system of Kristensen's cochain operations. Furthermore, the cochain operations measuring the Cartan defect of r_{m+1} satisfy $T_{\ddot{r}_{m+1} + \dot{r}_{2m+2} sq^1} = T_{\ddot{r}_{2m+1}} + T_{\dot{r}_{2m+2} sq^1}$ and for cocycles x, y, $[T_{\dot{r}_{2m+2} sq^1}(x, y)] = [\dot{r}_{2m+2} T_{sq^1}(x, y) + T_{\dot{r}_{2m+2}}(sq^1 x, y) + T_{\dot{r}_{2m+2}}(x, sq^1 y)]$ (see [11]). Looking now at the definition of A one gets for cocycles x, y that $\varepsilon([x] \otimes [y]) = [\ddot{A}_{m+1}(x, y) + \dot{A}_{2m+2}(sq^1 x, y) + \dot{A}_{2m+2}(x, sq^1 y)]$ which shows that for the cohomology classes x and y

$$\varepsilon(x \otimes y) = \ddot{\varepsilon}_{m+1}(x \otimes y) = \sum_{i=0..4m+1} Q_0 Sq^i x \cdot Q_1 Sq^{4m+1-i} y.$$

Applied to our case, the only term which can give a contribution has to contain the factor $Sq^4 x$ since we are in the top dimension, but this term does not show up in the sum because of $Q_0 Sq^i = 0$ for i odd. Summarizing our computation we have shown that

$$\phi_{m+1}(xy) = x^2 p^* \phi_m(y')$$

where the secondary operations ϕ_{m+1} and ϕ_m are constructed by using Kristensen's special system of cochain operations. Now the proof is finished since

$$\begin{aligned} K_{m+1}(N^{8m+10}) &= \mathrm{Arf}(\phi_{m+1}) = \mathrm{Arf}(y' \mapsto \phi_{m+1}(xp^*y')[N^{8m+10}]) \\ &= \mathrm{Arf}(y' \mapsto \phi_m(y')[M^{8m+2}]) = K_m(M^{8m+2}). \end{aligned}$$ ■

COROLLARY 7.12. *Ochanine's invariant k vanishes for $H^{4m+1}M^{8m+2} = 0$.*

Ochanine showed in [15] that an orientation-preserving homotopy equivalence between two closed oriented manifolds with $w_2 = 0$ gives a natural bijection between the both sets of *Spin*-structures on the two manifolds. In particular, one defines a *Spin-homotopy equivalence* between two *Spin*-manifolds as an orientation-preserving homotopy equivalence which maps the *Spin*-structure of the one to that of the other. Furthermore, Ochanine showed that generalized Kervaire invariants are invariants of the *Spin*-homotopy type.

COROLLARY 7.3. *Ochanine's invariant k is an invariant of the Spin-homotopy type.*

In [15], Ochanine defined $\kappa : \Omega^{Spin}_* \to KO_* \otimes \mathbb{Z}/2$ by

$$\kappa(M^n) := \begin{cases} sign(M^n)\mu^m & \otimes 1 \quad \text{for } n = 8m \\ k(M^n \times \bar{S}^1)\eta\mu^m & \otimes 1 \quad \text{for } n = 8m+1 \\ k(M^n)\eta^2\mu^m & \otimes 1 \quad \text{for } n = 8m+2 \\ \frac{1}{16} sign(M^n)\omega\mu^m & \otimes 1 \quad \text{for } n = 8m+4 \\ 0 & \qquad \text{otherwise} \end{cases}$$

and showed that κ is a ring homomorphism; this summarizes the multiplicative properties of k. Now, the signature is an invariant of the oriented homotopy type, and the definition of *Spin*-homotopy equivalence is compatible with products.

Corollary 7.4. *κ is an invariant of the Spin-homotopy type.*

In contrast to this, the Atiyah α-invariant (and thus also the Ochanine β-invariant) is *not* an invariant of the *Spin*-homotopy type, because it detects some exotic spheres in dimension 9 which have clearly the *Spin*-homotopy type of the standard sphere. With the result of Kahn [6] for oriented manifolds in mind, saying that the rational multiples of the signature are the only rational characteristic numbers which are invariants of the oriented homotopy type, we end with an open problem.

Problem. *Besides the multiples of the signature and the Ochanine k-invariant,*

$$c \cdot sign(M^{4m}), \qquad k(M^{8m+1} \times \bar{S}^1), \qquad \text{and} \qquad k(M^{8m+2}),$$

are there other KO-characteristic numbers of Spin-manifolds which are invariants of the Spin-homotopy type?

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