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# SMOOTH HOMOLOGY SPHERES AND THEIR FUNDAMENTAL GROUPS

BY
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Let $M^n$ be a smooth homology $n$-sphere, i.e. a smooth $n$-dimensional manifold such that $H_*(M^n) \simeq H_*(S^n)$. The fundamental group $\pi$ of $M$ satisfies the following three conditions:

(1) $\pi$ has a finite presentation,

(2) $H_1(\pi)=0$,

(3) $H_2(\pi)=0$,

where $H_i(\pi)$ denotes the $i$th homology group of $\pi$ with coefficients in the trivial $Z\pi$-module $Z$. Properties (1) and (2) are trivial and (3) follows from the theorem of Hopf [2] which asserts that $H_2(\pi)=H_2(M)/\rho\pi_2(M)$, where $\rho$ denotes the Hurewicz homomorphism.

For $n>4$ we will prove the following converse

THEOREM 1. *Let $\pi$ be a group satisfying the conditions* (1), (2) *and* (3) *above, and let n be an integer greater than* 4. *Then, there exists a smooth manifold $M^n$ such that $H_*(M^n) \cong H_*(S^n)$ and $\pi_1(M) \cong \pi$.*

The proof is very similar to the proof used for the characterization of higher knot groups in [5]. Compare also the characterization by K. Varadarajan of those groups $\pi$ for which Moore spaces $M(\pi, 1)$ exist [9].

Not much seems to be known for $n \leqq 4$. If $M^3$ is a 3-dimensional smooth manifold with $H_*(M) \simeq H_*(S^3)$, then $\pi=\pi_1(M)$ possesses a presentation with an equal number of generators and relators. (Take a Morse function $f$ on $M$ with a single minimum and a single maximum. Then $f$ possesses an equal number of critical points of index 1 and 2.)

Also, under restriction to *finite* groups there is the following

THEOREM 2. *Let $M^3$ be a 3-dimensional manifold such that $H_*(M) \simeq H_*(S^3)$. Suppose that $\pi_1(M)$ is finite. Then, either $\pi_1(M)=\{1\}$ or else, $\pi_1(M)$ is isomorphic to the binary icosahedral group with presentation*

$$(x, y; x^2 = y^3 = (xy)^5).$$

This is implicitly well known: The hypotheses imply that $\pi=\pi_1(M)$ is a group of fix-point free transformations of a homotopy 3-sphere. Any such group belongs to a list established by Suzuki [8] and even to the shorter list of Milnor [7]. The

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binary icosahedral group is the only (nontrivial) group in Milnor's list satisfying $H_1\pi=0$.

REMARK. Curiously enough, it seems that the trivial group and the binary icosahedral group are the only available examples of finite groups $\pi$ satisfying $H_1\pi=0$ and having a presentation with an equal number of generators and relators.

For $n=4$, it follows from the proof below that every finitely presented group $\pi$ with an equal number of generators and relators and satisfying $H_1\pi=0$ is the fundamental group of some homology 4-sphere. It seems unlikely that these conditions should characterize the fundamental groups of homology 4-spheres, but I do not know of a counterexample.

1. **Proof of Theorem 1.** We start with a finite presentation of $\pi$:

$$(x_1, \ldots, x_\alpha; R_1, \ldots, R_\beta)$$

and the manifold

$$M_0 = (S^1 \times S^{n-1})\# \cdots \#(S^1 \times S^{n-1}),$$

connected sum of $\alpha$ copies of $S^1 \times S^{n-1}$. Choosing as usual a contractible open set $U$ in $M_0$ as "base point", the condition $n>2$ implies that $\pi_1(M, U)$ is free on $\alpha$ generators. After identification of free generators of $\pi_1(M, U)$ with $x_1, \ldots, x_\alpha$, the elements $R_1, \ldots, R_\beta$ of the free group on $x_1, \ldots, x_\alpha$ can be represented by disjoint differentiable imbeddings $\phi_i$: $S^1 \times D^{n-1} \to M_0$, $i=1, \ldots, \beta$. Moreover, these can be chosen so that the spherical modification $\chi(\phi_1, \ldots, \phi_\beta)$ can be framed (see [6]). The resulting manifold $M_1=\chi(M_0; \phi_1, \ldots, \phi_\beta)$ is stably parallelizable as was $M_0$ and $\pi_1(M_1)\cong\pi$ since $n>3$.

From the homology exact sequences of the pairs $(M_0, \bigcup_i \phi_i(S^1 \times D^{n-1}))$ and $(M_1, \bigcup_i \phi_i'(D^2 \times S^{n-2}))$, where $\phi_i'$ denotes the natural imbedding $D^2 \times S^{n-2} \to M_1$ with $\phi_i'|S^1 \times S^{n-2}=\phi_i|S^1 \times S^{n-2}$, one concludes that $H_i(M_1)=0$ for $i\neq 0, 2, n-2, n$, and that $H_2(M_1)$ is free abelian of rank $\gamma=\beta-\alpha$. (Observe that $\beta\geqq\alpha$ since $\pi$ abelianized is trivial.) Hence, there exist bases $\xi_1, \ldots, \xi_\gamma$ of $H_2(M_1)$ and $\eta_1, \ldots, \eta_\gamma$ of $H_{n-2}(M_1)$ respectively such that $\xi_i \cdot \eta_j=\delta_{ij}$, where $\xi_i \cdot \eta_j$ is the homology intersection number, and $\delta_{ij}$ is the Kronecker delta.

By the theorem of Hopf mentioned above, $H_2\pi=H_2(M_1)/\rho\pi_2(M_1)$, and so by condition (3) on $\pi$, the Hurewicz homomorphism $\rho$: $\pi_2(M_1) \to H_2(M_1)$ is surjective.

Since $n>4$, the classes $\xi_1, \ldots, \xi_\gamma$ can be represented by disjoint differentiable imbeddings $f_i$: $S^2 \to M_1$, $i=1, \ldots, \gamma$, and $M_1$ being stably parallelizable, these extend to disjoint imbeddings $\psi_i$: $S^2 \times D^{n-2} \to M_1$. It follows from the arguments in [6, §5] that the manifold $M=\chi(M_1; \psi_1, \ldots, \psi_\gamma)$ obtained by spherical modification is a homology sphere with $\pi_1(M)\cong\pi$.

2. Many of the homology spheres occurring in the literature are constructed as the boundary of a contractible manifold. We investigate this question whether, in general, homology spheres bound contractible manifolds.

The following well-known construction provides an example of a 3-dimensional homology sphere which does not bound a contractible manifold.

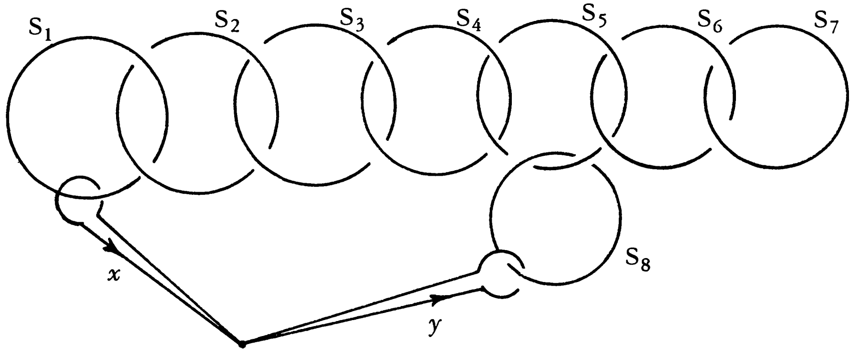
Let $W = D^4 + (\phi_1) + \cdots + (\phi_8)$ be the manifold obtained from the 4-disc by attaching eight handles of type 2 using unknotted imbeddings (with disjoint images) $\phi_i \colon S^1 \times D^2 \to S^3$, such that the matrix of linking numbers

$$L(\phi_i(S^1 \times x_0), \phi_j(S^1 \times x_1))$$

with $x_0 \neq x_1$, $x_0, x_1 \in D^2$, is:

$$L = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

This matrix is unimodular and has signature 8. The picture shows the images




$S_i = \phi_i(S^1 \times 0)$ in $R^3 = S^3 - \text{point}$. From the handle decomposition of $W$ one sees that $\pi_1 W = \{1\}$, and $H_2 W$ is free abelian on 8 generators. An easy calculation shows that $\pi_1(bW)$ is the group with presentation:

$$(x, y; x^5 = y^2 = (x^{-1}y)^3),$$

where $x$ and $y$ are the classes of the loops shown on the picture. Hence, $bW$ is a homology sphere. If $bW$ were the boundary of a contractible manifold $V$, one could form the closed 4-manifold $M = W \cup V$, and the above matrix $L$ would be the matrix of intersection numbers of $H_2M$. From $L$ one reads off the Stiefel-Whitney class $w_2M = 0$ and the signature $\sigma(M) = 8$, contradicting Rohlin's theorem which says that if $w_2M=0$ for a closed orientable 4-manifold $M$, then $\sigma(M)\equiv 0 \bmod 16$. (Cf. [4].)

In contrast, we have

THEOREM 3. *Every 4-dimensional homology sphere bounds a contractible manifold. If $M^n$ is a smooth oriented homology sphere with $n\geqq 5$, there exists a unique smooth homotopy sphere $\Sigma^n_M$ such that $M^n \# \Sigma^n$ bounds a contractible smooth manifold.*

This has been obtained independently by the Hsiang brothers [3].

Let first $M^4$ be a smooth 4-dimensional homology sphere which we think of as imbedded in $S^{k+4}$ with $k$ large. It is easy to see that $M$ has trivial normal bundle. Indeed, the only obstruction to trivializing the normal bundle is an element $\mathcal{O} \in H^4(M; \pi_3(SO_k))=Z$. It is known that $2\mathcal{O}= \pm p_1[M]$, where $p_1$ is the Pontryagin class. (Cf. e.g. [4].) The latter vanishes by the Thom-Hirzebruch formula $\frac{1}{3}p_1[M] =\sigma(M)=0$. Now, choosing a trivialization of the normal bundle of $M^4$ in $S^{k+4}$, the Thom construction provides an element in $\pi_{k+4}(S^k)$. Since $\pi_{k+4}(S^k)=0$, we see that $M^4$ is the boundary of a parallelizable manifold $W^5$. This manifold can be modified to get a new 5-manifold $V^5$ with $bV^5=bW^5=M^4$ such that $\pi_1V=\{1\}$ and $\pi_2V=0$. (Cf. [6, §5].) Clearly, $V$ is contractible.

Assume then that $n\geqq 5$.

We associate with the given smooth homology $n$-sphere $M^n$ some smooth homotopy sphere $\Sigma^n$. Observe that $M^n$ is stably parallelizable. Since

$$H^i(M; \pi_{i-1}(SO)) = 0 \qquad \text{for } i < n,$$

there is only one possible obstruction $\mathcal{O} \in H^n(M; \pi_{n-1}(SO))=\pi_{n-1}(SO)$ to producing a trivialization of the stable tangent bundle of $M$. It is known that $\mathcal{O}$ belongs to the kernel of the homomorphism $J\colon \pi_{n-1}(SO) \to \Pi_{n-1}$, where $\Pi_{n-1} =\pi_{n+k-1}(S^k)$, $k$ large. Thus $\mathcal{O}=0$ by the same arguments as in [6, §3].

We obtain $\Sigma^n$ from $M^n$ by framed surgery in dimensions 1 and 2. Let $x_1, \ldots, x_\alpha$ be a finite set of generators of $\pi_1(M, U)$, where $U$ is a contractible open "base set" in $M$. Let $\phi_1, \ldots, \phi_\alpha$ be smooth imbeddings of $S^1 \times D^{n-1}$ into $M^n$ with disjoint images representing $x_1, \ldots, x_\alpha$ respectively. We use $\phi_1, \ldots, \phi_\alpha$ to attach $\alpha$ handles of type 2 to $I\times M$ along $(1)\times M$, where $I=[0, 1]$. Let

$$V_0 = I\times M+(\phi_1)+\cdots+(\phi_\alpha)$$

be the resulting $(n+1)$-manifold. We may assume that the imbeddings $\phi_1, \ldots, \phi_\alpha$ have been chosen so that $V_0$ is parallelizable. The manifold $N=bV_0-(0)\times M$ will be called the right-hand boundary of $V_0$. It is easily checked that $\pi_1N=\pi_1V_0=\{1\}$, and $H_iN=0$ for $3\leqq i\leqq n-3$ (if $n\geqq 6$). The groups $H_2N$ and $H_{n-2}N$ are free abelian of rank $\alpha$ and the inclusion $N \subsetneq V_0$ induces an isomorphism $H_2N\cong H_2V_0$.

Let $\xi_1, \ldots, \xi_\alpha$ and $\eta_1, \ldots, \eta_\alpha$ be bases of $H_2N$ and $H_{n-2}N$ respectively such that $\xi_i\cdot\eta_j=\delta_{ij}$. Then, representing the classes $\xi_1, \ldots, \xi_\alpha$ by disjoint differentiable imbeddings $\psi_1, \ldots, \psi_\alpha$ of $S^2\times D^{n-2}$ into $N$ we construct

$$V_1 = I\times M+(\phi_1)+\cdots+(\phi_\alpha)+(\psi_1)+\cdots+(\psi_\alpha)$$

by attaching $\alpha$ handles of type 3 to $V_0$ along $N$ using $\psi_1, \ldots, \psi_\alpha$. Again this can be done so that $V_1$ is parallelizable. Now, the right-hand boundary $bV_1-(0)\times M$ is a smooth homotopy $n$-sphere $\Sigma^n$. We think of it as oriented so that the (oriented) boundary of $V_1$ is $-\Sigma^n-(0)\times M$, where the orientation of $V_1$ is given by the one of $I\times M$.

*Claim*: $M^n \# \Sigma^n$ bounds a contractible manifold. Indeed, let $t: (I\times D^n, bI\times D^n) \to (V_1, bV_1)$ be a smooth imbedding with $t(\text{int } I\times D^n)\subset \text{int } V_1$, then after rounding off corners we get a smooth contractible manifold $V=V_1-t(I\times \text{int } D^n)$ whose boundary is diffeomorphic to $M\#\Sigma$.

It remains to prove the uniqueness of $\Sigma$. Let $\Sigma_1$ be a homotopy $n$-sphere such that $M\#\Sigma_1$ is the boundary of a contractible manifold $W$. Using a connected sum of $V_1$ (as constructed above) and $I\times\Sigma_1$ along $t(I\times S^{n-1})$ and $I\times\sigma^{n-1}$, where $\sigma^{n-1}$ is the boundary of a smooth $n$-disc $\delta^n$ in $\Sigma_1$, one gets a manifold $W'$ whose homotopy type is $S^n$, and $bW'=(-\Sigma)\#\Sigma_1+(-M)\#\Sigma_1$. ($W'$ is obtained from the disjoint union

$$\{V_1-t(I\times(0))\}+\{I\times\Sigma_1-I\times(0)\}$$

under the identification of $t(x, ry)\in t(I\times D^n)$ with $(x, (1-r)y)\in I\times\delta^n$ for $x\in I$, $y\in S^{n-1}$ and $0<r<1$.)

Now, paste the manifold $W$ along the left boundary $((0)\times M)\#\Sigma_1$ of $W'$ by the identity diffeomorphism

$$bW = M\#\Sigma_1 \to ((0)\times M)\#\Sigma_1.$$

The resulting union $W\cup W'$ is a contractible manifold as follows easily using the van Kampen and Mayer-Vietoris theorems. The boundary of $W\cup W'$ is $(-\Sigma)\#\Sigma_1$. Therefore, $\Sigma$ and $\Sigma_1$ are $h$-cobordant, and thus diffeomorphic since we assume $\dim\Sigma=\dim\Sigma_1\geqq 5$.

COROLLARY. *Every combinatorial homology sphere $K^n$ of dimension $n\neq 3$ is the boundary of a contractible combinatorial manifold.*

By Hirsch's obstruction theory [1, Theorem 3.1], every combinatorial manifold $K^n$ admits a smoothness structure in the neighborhood of its 7-skeleton. We need a smoothness structure in a neighborhood $N$ of the 2-skeleton of $K^n$. Then $N$ is parallelizable and we can apply the above surgery arguments requiring the imbeddings to have their images in the smooth subset. We obtain a manifold $W^{n+1}$ such that $bW=\Sigma^n+M^n$, where $\Sigma^n$ is a combinatorial homotopy $n$-sphere and the inclusion $\Sigma\subset W$ is a homotopy equivalence. Since $n\geqq 5$, $\Sigma^n$ is PL-homeomorphic to $b\Delta^{n+1}$, and pasting $\Delta^{n+1}$ to $W^{n+1}$ by such a homeomorphism provides a contractible manifold $V^{n+1}$ with boundary $M^n$.

Thus, as a by-product, we have proved the probably well-known fact that every combinatorial homology sphere admits a smoothness structure.

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