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AN INTERPRETATION OF G. WHITEHEAD'S GENERALIZATION OF H. HOPF'S INVARIANT

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In the present paper[1], the generalized H. Hopf's invariant $H: \pi_{d+n+1}(S_{n+1}) \to \pi_{d+n+1}(S_{2n+1})$, due to G. Whitehead [10], is given a new definition which has some similarity with the original H. Hopf's definition [5]. The invariant $H(f)$ appears as depending in particular on the position in S_{d+n+1} of the inverse images M_d, M'_d by f: $S_{d+n+1} \to S_{n+1}$ of two regular values q, q' in S_{n+1}.

Arnold Shapiro has defined the linking coefficient of two spheres S_p, S_q imbedded (without common point) in E_{m+1} for $p + q > m$.[2] In § 5, the notion of linking coefficient is extended to $(p + q - m)$-connected π-manifolds M_p, M'_q in E_{m+1}. It is an element of the stable homotopy group $\pi_{r+N}(S_N)$, where $r = p + q - m$ (π-manifold = manifold which can be imbedded in some euclidean space with a trivial normal bundle).

In the definition of H given in § 3, M_d and M'_d are π-manifolds but need not be $(d - n)$-connected. As a consequence, H will in general also depend on the fields of normal vectors over M and M'. Therefore, it cannot be considered strictly as a linking coefficient which should be uniquely determined by the position in space of the two manifolds. It is an open question whether the method can be used to define the linking coefficient of (non-necessarily $(p + q - m)$-connected) π-manifolds M_p, M'_q in E_{m+1} by going over to the quotient of $\pi_{p+q-m+N}(S_N)$ by some suitable subgroup.

As an application of the new definition of H, it is proved that any regular imbedding (without self-intersection) of the d-sphere into euclidean $(d + n)$-space induces over S_d the trivial normal bundle provided that $2n > d + 1$. A partial result in this direction was announced in [6].

1. Notations

E_{m+1} will denote euclidean $(m + 1)$-space (space of infinite sequence of real numbers t_i, $i=0, 1, 2, \cdots$, such that $t_M=0$ for $M>m$). $E_m \subset E_{m+1}$. The

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[2] Unpublished. The author has attended a talk by Arnold Shapiro at the Harvard-M.I.T. Colloquium entitled *On higher linking coefficients*. Some of the results stated without proof by Arnold Shapiro during his talk are used in the present paper. The proofs given here are my own.

m-sphere S_m is the subspace of E_{m+1} given by $\sum_i (t_i)^2 = 1$. Thus $S_m \subset S_{m+1}$. The normal to E_m in E_{m+1} will be the vector $\mathbf{t}_m = (\delta_{1m}, \cdots, \delta_{im}, \cdots)$. The sphere S_m is oriented by the ordered set of vectors $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_m$ tangent at $(1, 0, \cdots)$. We have $\dot{B}_{m+1} = S_m$, where B_{m+1} is the ball in E_{m+1} characterized by $\sum_i (t_i)^2 \leqq 1$, with the orientation induced by $B_{m+1} \subset E_{m+1}$.

We set the mapping $s : E_{m+1} \to S_{m+1}$ to be the one given by the formula

$$(1.1) \qquad s(t_0, t_1, \cdots, t_m) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t_0}{1 + t^2}, \cdots, \frac{2t_m}{1 + t^2} \right),$$

where t^2 stands for $\sum_i (t_i)^2$. Notice that s is orientation preserving.

For the reader's convenience and in order to fix orientation conventions, we recall the following well-known construction due to L. Pontrjagin, B. Eckmann and R. Thom [7], [3] and [8]:

V_r being a closed p-dimensional submanifold of E_{r+N}(C^2-imbedded) with a field of normal N-frames[3] $\mathbf{F}_N = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$, a point $u \in E_{r+N}$ in a *tubular* neighborhood U of V of radius ρ lies in the normal plane to V at some well defined point $x \in V$. Let $u_1, u_2, \cdots, u_N$ be the coordinates of u relative to the vectors $\mathbf{v}_1(x), \mathbf{v}_2(x), \cdots, \mathbf{v}_N(x)$; $\sum_i (u_i)^2 \leqq \rho^2$. To $(V_r ; \mathbf{F}_N)$ is associated a map $\gamma : S_{r+N} \to S_N$ defined by

$$(1.2) \qquad \begin{cases} \gamma(su) = (1 - 2y^2, 2y_1(1 - y^2)^{1/2}, \cdots, 2y_N(1 - y^2)^{1/2}) & \text{for } u \in U, \\ \gamma(S - sU) = (-1, 0, \cdots, 0), \end{cases}$$

where $y_i = u_i/\rho$ and $y^2 = \sum_i (y_i)^2$.

REMARK 1.3. If the field $\mathbf{F}_N$ orthogonal to V_r in E_{r+N} is replaced by $\mathbf{F}'_N$ obtained by letting a non-singular constant N by N matrix A act on the vectors of $\mathbf{F}_N$, then the maps γ and γ' attached to $(V; \mathbf{F})$ and $(V; \mathbf{F}')$ respectively are related by $\gamma' = (\sigma i_N) \circ \gamma$, where $\sigma = \text{sign} \det(A)$.[4]

By an obvious generalization of the above argument one associates with a submanifold V_r of the manifold X_{r+N} together with a field $\mathbf{F}_N$ of N-frames orthogonal to V_r in X_{r+N} a mapping $X_{r+N} \to S_N$(X_{r+N} being assumed to carry some Riemannian metric). Here V_r is not necessarily closed. However, one should require $\dot{V}_r \subset \dot{X}_{r+N}$. It is also convenient to require orthogonality of the tangent plane $T_x(V_r)$ and $T_x(\dot{X}_{r+N})$ at $x \in \dot{V}_r$.

Recall that any homotopy class of mappings $X_{r+N} \to S_N$, where X_{r+N} is a C^∞-differentiable manifold can be represented by a map associated with some submanifold $V_r \subset X_{r+N}$ and field of N-frames orthogonal to V_r in X_{r+N}. Indeed, any such class contains a map f: $X_{r+N} \to S_N$ of class C^{r+N}

[3] In this paper *N-frame* means ordered set of N linearly independent vectors.

[4] Recall that $(ki_N) \circ \gamma = k\gamma + (1/2)k(k-1)\,[i_N, i_N] \circ H_0(\gamma)$, see P. J. Hilton, *On the homotopy groups of the union of spheres*, J. London Math. Soc., 30 (1955), 154-172. In particular, if $\gamma = E\lambda$, $H_0(\gamma) = 0$ and $(ki_N) \circ \gamma = k\gamma$. This is the case if $r < N - 1$.

and it is well known that (under these differentiability conditions) the set of regular values of f is an everywhere dense open subset of S_N (see [8, Théorème I.3]). Let $q \in S_N$ be a regular value of f. The inverse image $f^{-1}(q)$ is an r-dimensional submanifold V_r, C^{r+N}-imbedded in X_{r+N}. Furthermore, f induces a linear mapping F_x[5] of the tangent plane to X at x into the tangent plane to S_N at $f(x)$. The fact that q is a regular value of f means by definition that F_x is *onto* for every $x \in V_r$. The tangent plane to V_r at x is (as a vector space) the kernel of F_x and its orthogonal complement $L_N(x)$ (in the tangent plane to X at x) is mapped isomorphically onto the tangent plane to S_N at q. It follows that a fixed N-frame $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_N$ tangent to S_N at q and inducing the positive orientation of S_N admits for every $x \in V_r$ a unique inverse image $\mathbf{v}_1(x), \mathbf{v}_2(x), \cdots, \mathbf{v}_N(x)$ by F_x in $L_N(x)$. When x runs over V_r, we obtain a field $\mathbf{F}_N = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$ of N-frames orthogonal to V_r in X_{r+N}. The mapping $X_{r+N} \rightarrow S_N$ associated with V_r and $\mathbf{F}_N$ is homotopic to the mapping f we started from.

Notice that if $\gamma : S_{r+N} \rightarrow S_N$ is the map associated with $V_r \subset E_{r+N}$ and $F_N = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$, then $a = (1, 0, \cdots) \in S_N$ is a regular value of γ and the derived map Γ_x at $x \in V_r$ sends $\mathbf{v}_i(x)$ into $\mathbf{t}_i$, $i = 1, 2, \cdots, N$.

Since most of the maps occurring in this paper will be described in terms of manifolds and fields with which they are associated by the above procedure, we proceed to a description of the various needed homotopy operations in these terms.

1.4. *The suspension homomorphism* $E: \pi_{p+q}(S_q) \rightarrow \pi_{p+q+1}(S_{q+1})$ can be defined as follows (see [3], page 22):

Using $E_{p+q} \subset E_{p+q+1}$ and adjoining to the vectors of $\mathbf{F}_q = \{\mathbf{v}_1, \cdots, \mathbf{v}_q\}$ at each $x \in V_p$ the vector $\mathbf{t}_{p+q}$ orthogonal to E_{p+q} in E_{p+q+1} (eventually multiplied by a *positive* continuous real-valued function of x), we obtain V_p as a submanifold of E_{p+q+1} with a normal field of $(q+1)$-frames $\mathbf{F}_{q+1} = \{\mathbf{v}_1, \cdots, \mathbf{v}_q, \mathbf{t}_{p+q}\}$. The attached map $S_{p+q+1} \rightarrow S_{q+1}$ is the suspension of the map $S_{p+q} \rightarrow S_q$ attached to $(V_p\,;\, \mathbf{F}_q)$. There is agreement in sign with the usual E defined by suspension on the *last* coordinate (or alternately by $E(f) = f * i_0$ as in [10], § 3 V, page 206).

1.5. *The Hopf construction* associates with a mapping $\varphi : S_p \times S_q \rightarrow S_m$ a mapping $G\varphi : S_{p+q+1} \rightarrow S_{m+1}$ (see [5] and [10], § 3, VI, page 208). We shall make use of the following definition: Replace φ by a C^{p+q}-differentiable approximation which we denote again by φ. Let $b \in S_m$ be a regular value of φ; denote by $V_r (r = p + q - m)$ the inverse image $\varphi^{-1}(b)$ and by $\mathbf{F}_m = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ the field of m-frames orthogonal to V_r in $S_p \times S_q$ induced by φ and a fixed frame $\mathbf{u}_1, \cdots, \mathbf{u}_m$ at b giving the positive orienta-

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[5] F_x will be called the derivative of f at x.

tion of S_m. In order to obtain V_r as a submanifold of E_{p+q+1}, we imbed $S_p \times S_q$ into E_{p+q+1} by the mapping $i: S_p \times S_q \to E_{p+q+1}$ given by the formula

$$(1.6)\qquad i(x, y) = (x_0(\varepsilon y_0 + 1), \cdots, x_q(\varepsilon y_0 + 1), \varepsilon y_1, \cdots, \varepsilon y_q)\ ,$$

where x and y stand for $(x_0, x_1, \cdots, x_p) \in S_p$ and $(y_0, y_1, \cdots, y_q) \in S_q$ respectively ($\varepsilon < 1$). Notice that $i(S_p \times S_q)$ is the boundary of a region in E_{p+q+1} homeomorphic to $S_p \times B_{q+1}$. Denote by $\mathbf{w}(x, y)$, for every $(x, y) \in S_p \times S_q$, the unit vector orthogonal to $S_p \times S_q$ and pointing outwards this region.

Let $G'\varphi$ be the homotopy class of the map $S_{p+q+1} \to S_{m+1}$ associated with V_r (as a submanifold of E_{p+q+1} by the injection i) with the field consisting of the vectors of $\mathbf{F}$ followed by the vector $\mathbf{w}$.

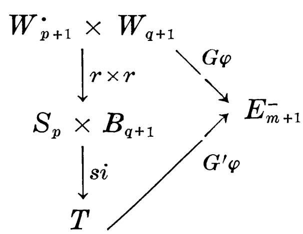
CLAIM. $G\varphi = (-1)^p G'\varphi$, *where G is defined as in* [10, § 3, VI, p. 208].

PROOF. Denote by W_{N+1} the cube in E_{N+1} defined by $-1 \leqq t_i \leqq +1$, $i = 0, 1, \cdots, N$. We have $S_{p+q+1} = rW^{\cdot}_{p+q+2}$, where r denotes radial projection $r: W_{N+1} \to B_{N+1}$ as given by the formula

$$(1.7)\qquad r(\mathbf{t}) = a\mathbf{t} \quad (a = |\mathbf{t}| / \|\mathbf{t}\|,\ |\mathbf{t}| = \max|t_i|,\ \|\mathbf{t}\| = (\textstyle\sum_i t_i^2)^{1/2})\ .$$

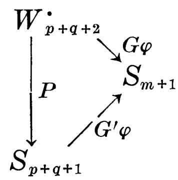
Notice that r is orientation preserving.

We have $W^{\cdot}_{p+q+2} = W^{\cdot}_{p+1} \times W_{q+1} + (-1)^{p+1} W_{p+1} \times W^{\cdot}_{q+1}$ and $G\varphi \mid W^{\cdot}_{p+1} \times W^{\cdot}_{q+1} = \varphi \mid rW^{\cdot}_{p+1} \times rW^{\cdot}_{q+1}$. The following diagram (in which $T = si(S_p \times B_{q+1})$) is homotopy commutative relative to $(W^{\cdot}_{p+1} \times W^{\cdot}_{q+1}, S_m)$:



Indeed, $G'\varphi \mid T^{\cdot} \simeq \varphi = \varphi \mid rW^{\cdot}_{p+1} \times rW^{\cdot}_{q+1} = G\varphi \mid W^{\cdot}_{p+1} \times W^{\cdot}_{q+1}$. Now any two extensions $\Phi_0, \Phi_1; (X, A) \to (E^-_{m+1}, S_m)$ of homotopic maps $\varphi_0, \varphi_1: A \to S_m$ are homotopic relative (A, S_m).[6] Any extension P: $W^{\cdot}_{p+q+2} \to S_{p+q+2}$ of $si(r \times r): W^{\cdot}_{p+1} \times W_{q+1} \to T \subset S_{p+q+1}$, such that $P(W_{p+1} \times W^{\cdot}_{q+1}) \subset S_{p+q+1} - T$ has degree $(-1)^p$ (because i has "local" degree $(-1)^p$ and s, $r \times r$ are orientation preserving). We thus have the following homotopy commutative diagram

[6] i.e., if $\varphi_t: A \to S_m$ exists, then $\Phi_t: (X, A) \to (E^-_{m+1}, S_m)$ with $\Phi_t | A = \varphi_t$ exists for any choice of Φ_0, Φ_1 extending φ_0, φ_1 respectively.

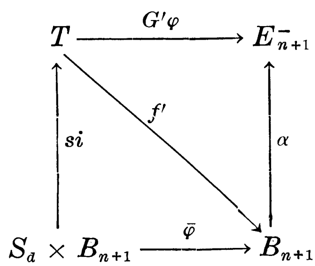


from which $G\varphi = (-1)^p G'\varphi$ follows.

1.8. *The Hopf homomorphism* $J: \pi_d(\mathrm{SO}(n+1)) \to \pi_{d+n+1}(S_{n+1})$. Let $S_d \subset E_{d+1} \subset E_{d+n+1}$ and regard the set of vectors $\mathbf{x}, \mathbf{t}_{d+1}, \cdots, \mathbf{t}_{d+n}$ as an orthonormal basis in the $(n+1)$-plane orthogonal to S_d in E_{d+n+1} at $x \in S_d$. Let $M: S_d \to \mathrm{SO}(n+1)$ be a mapping representing some element $\mu \in \pi_d(\mathrm{SO}(n+1))$ and let $\mathbf{v}_0(x), \mathbf{v}_1(x), \cdots, \mathbf{v}_n(x)$ be the row vectors of $M(x)$ relative to the basis $\mathbf{x}, \mathbf{t}_{d+1}, \cdots, \mathbf{t}_{d+n}$; i.e., $\mathbf{v}_i(x) = a_{i0}(x)\mathbf{x} + \sum_{j=1}^{n} a_{ij}(x)\mathbf{t}_{d+j}$, where $a_{ij}(x)$, $i = 0, \cdots, n$, $j = 0, 1, \cdots, n$ are the entries in the matrix $M(x)$. By construction (1.2), there is a map $f': S_{d+n+1} \to S_{n+1}$ associated with S_d and $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n$. Define $J'\mu$ = homotopy class of f' (it is clear that the homotopy class of f' depends only on the homotopy class of M).

CLAIM. $(-1)^d J'\mu = ((-1)^n i_{n+1}) \circ J\mu$, *where* $J\mu$ *is as in* [10, § 5, p.214].

PROOF. We compare the homotopy class of f' with $G'\varphi = (-1)^d G\varphi = (-1)^d J\mu$, where $\varphi: S_d \times S_n \to S_n$ is the map defined by $\varphi(x, y) = M(x)y$. Notice that $G'\varphi(T) \subset E^-_{n+1}$, where $T = si(S_d \times B_{n+1})$. Denote by $\bar{\varphi}: S_d \times B_{n+1} \to B_{n+1}$ the obvious extension of φ, defined again by $\bar{\varphi}(x, y) = M(x)y$. The following diagram, where α is given by $\alpha(y_0, \cdots, y_n) = (y_0, \cdots, y_n, -(1-y^2)^{1/2})$ is homotopy commutative relative to the boundaries.



The maps $f' \circ si$ and $\bar{\varphi}$ are equal. $G'\varphi \circ si \simeq \alpha \circ \bar{\varphi}$ rel. $(S_d \times S_n, S_n)$ because their restrictions to $S_d \times S_n$ are homotopic. It follows that the upper triangle of the diagram is homotopy commutative (relative boundaries) and since α has degree $(-1)^n$, we have $G'\varphi \simeq ((-1)^n i_{n+1}) \circ f'$.

1.9. *The join*. Let $\alpha: S_p \to S_m$, $\beta: S_q \to S_n$ be mappings. Identifying S_N, B_{N+1} with $W^{\cdot}_{N+1}$, W_{N+1} respectively under radial projection r (see (1.7)), define $\lambda : W^{\bullet}_{p+q+2} \to W^{\bullet}_{m+n+2}$ by the formula

$$(1.10) \qquad \lambda(x, y) = \begin{cases} (\alpha(x), \|y\| \beta(y/\|y\|)) & \text{if } (x, y) \in W^{\bullet}_{p+1} \times W_{q+1} \\ (\|x\| \alpha(x/\|x\|), \beta(y)) & \text{if } (x, y) \in W_{p+1} \times W^{\bullet}_{q+1} \end{cases}$$

($W^{\bullet}_{p+q+2} = W^{\bullet}_{p+1} \times W_{q+1} \cup W_{p+1} \times W^{\bullet}_{q+1}$). The mapping λ is known to be homotopic to the join $\alpha * \beta$ (see [2, Lemma 2.2]).

We construct a map $\pi(\alpha, \beta) : S_{p+q} \to S_{m+n}$ as follows: Let $(W_{p-m}; \mathbf{F}_m)$, $(W_{q-n}; \mathbf{F}_n)$ be manifolds and fields with which α and β respectively are associated (up to homotopy; $W_{p-m} \subset E_p$, $W_{q-n} \subset E_q$). Consider $W_{p-m} \times W_{q-n}$ as a submanifold of $E_p \times E_q = E_{p+q}$ with the obvious field $\mathbf{F}_m \times \mathbf{F}_n$. This induces the map $\pi(\alpha, \beta)$.

LEMMA 1.11. $E\pi(\alpha, \beta) \simeq (-1)^{q+m} \alpha * \beta$.

PROOF. $E\pi(\alpha, \beta)$ is associated with $i(W_{p-m} \times W_{q-n})$ and the field consisting of the images by i of the vectors of $\mathbf{F}_m \times \mathbf{F}_n \times (-1)^{p+q}\mathbf{w}$. In the definition of λ, $S_p \times B_{q+1}$ and $S_m \times B_{n+1}$ are mapped with degree $+1$ into S_{p+q+1}, S_{m+n+1} respectively. Since $i : S_p \times B_{q+1} \to E_{p+q+1}$ has local degree $(-1)^p$ and s is orientation preserving, we have $(-1)^{p+q} E\pi(\alpha, \beta) \cong (-1)^p([(-1)^m i_{m+n+1}](a * \beta))$, from which 1.11 follows.

2. Generalized Hopf's construction

Let $f : M_p \to E_{p+u}$ and $f' : M'_q \to E_{q+v}$ be regular C^2-imbeddings of the closed manifolds M_p, M'_q into euclidean spaces and assume the existence of fields $\mathbf{F}_u$, $\mathbf{F}'_v$ of u- and v-frames orthogonal to M_p and M'_q in E_{p+u} and E_{q+v} respectively.

LEMMA 2.1. *With any homotopy class of mappings $\varphi : M_p \times M'_q \to S_m$ there is associated a homotopy class $G(\varphi, f, \mathbf{F}, f', \mathbf{F}') \in \pi_{p+q+u+v}(S_{m+u+v})$.*

The class G is obtained as follows: Take φ to be differentiable of class C^{p+q} and let U_φ be a spherical[7] open subset of S_m in which φ takes on only regular values. Choosing $b \in U_\varphi$ and $\mathbf{F}_m(b)$ to be a fixed m-frame tangent to S_m at b and inducing the positive orientation of S_m, we obtain a submanifold $V_r = \varphi^{-1}(b) \subset M_p \times M'_q (r = p + q - m)$ together with a field of m-frames $\mathbf{F}_m$ orthogonal to V_r in $M_p \times M'_q$ (with the metric induced by the imbedding $f \times f' : M_p \times M'_q \to E_{p+u+q+v}$). The vectors of $\mathbf{F}_m$ at $x \in V_r$ are the inverse images of the vectors of $\mathbf{F}_m(b)$ by the derivative Φ_x of φ at x. Regarding V_r as a submanifold of $E_{p+u+q+v}$ (using $f \times f' \mid V_r$) carrying an orthogonal field of $(m + u + v)$-frames consisting of the vectors of $\mathbf{F}_u$ followed in order by the vectors of $\mathbf{F}'_v$ and those of $\mathbf{F}_m$, we obtain a

[7] U_φ consists of the points whose spherical distance to some point of S_m is smaller than some $\varepsilon > 0$.

mapping $\gamma\varphi : S_{p+q+u+v} \to S_{m+u+v}$ by Construction 1.2.

CLAIM. *The homotopy class of $\gamma\varphi$ depends only on the homotopy class of φ (and on f, f', $\mathbf{F}_u$, $\mathbf{F}'_v$).*

PROOF. Obviously the homotopy class of $\gamma\varphi$ does not depend on the choice of $b \in U_\varphi$ and by Remark 1.3, it does not depend on the choice of $\mathbf{F}_m(b)$. Let $\varphi_0, \varphi_1 : M_p \times M'_q \to S_m$ be C^{p+q}-maps taking on regular values *in a given* spherical open subset $U \subset S_m$. Assume $\varphi_0 \simeq \varphi_1$. Let $\psi: M_p \times M'_q \times I \to S_m$ be a homotopy. By Lemma IV. 5 of [8], ψ may be chosen to be differentiable of class C^{p+q}. There exists then by Théorème 1.3 of [8] an open spherical subset U_ψ of U consisting of regular values of the three maps $\psi, \varphi_0, \varphi_1$. Choose $b \in U_\psi$ and let $W_{r+1} = \psi^{-1}(b)$. It is easily seen that the boundary of W consists of the manifolds $V^0 = \varphi_0^{-1}(b)$ and $V^1 = \varphi_1^{-1}(b)$, i.e. $W^\cdot = \pm(V^1 - V^0)$. Assuming, without loss of generality,[8] that the derivatives of ψ with respect to t are zero for $t = 0, 1$, the fields induced by φ_0 and φ_1 over V^0 and V^1 coincide with the restriction over $W^\cdot$ of the field given by ψ over W (the fields induced by $\psi, \varphi_0, \varphi_1$ are supposed to be the inverse images of the same m-frame $\mathbf{F}_m(b)$). Regarding W as a submanifold of $E_{p+u} \times E_{q+v} \times I$ in the obvious way, we obtain (by straightforward generalization of construction 1.2 to manifolds with boundaries) a mapping $S_{p+q+u+v} \times I \to S_{m+u+v}$ which is a homotopy of $\gamma\varphi_0$ to $\gamma\varphi_1$.

From this follows:

(1) The homotopy class of $\gamma\varphi$ does not depend on the choice of U_φ. If $\gamma(\varphi, U_\varphi)$ and $\gamma(\varphi, U'_\varphi)$ are the mappings $S_{p+q+u+v} \to S_{m+u+v}$ obtained from two different choices, then set $\varphi_0 = r \circ \varphi$, $\varphi_1 = r' \circ \varphi$, where $r, r' : S_m \to S_m$ are diffeomorphisms of degree $+1$, such that $r(U_\varphi) = U$, $r'(U'_\varphi) = U$. We have $\varphi_0 \simeq \varphi_1$ and $\gamma(\varphi, U_\varphi) = \gamma\varphi_0 \simeq r\varphi_1 = \gamma(\varphi, U'_\varphi)$.

(2) If φ, φ' are any two homotopic C^{p+q}-maps $M_p \times M'_q \to S_m$, there is a spherical open subset U' of U in which φ and φ' both admit only regular values. The above claim follows by replacing φ, φ' by $\varphi_0 = r \circ \varphi$ and $\varphi_1 = r \circ \varphi'$, where $r : S_m \to S_m$ is again a diffeomorphism of degree $+1$, such that $r(U') = U$.

The question: How far does $G(\varphi, f, \mathbf{F}_u, f', \mathbf{F}'_v)$ depend on $f: M_p \to E_{p+u}$, $f' : M'_q \to E_{q+v}$ and on the fields $\mathbf{F}_u$, $\mathbf{F}'_v$? will be only partly answered by the following two lemmas.

LEMMA 2.2. *If M_p and M'_q are $(p+q-m)$-connected, then $G(\varphi, f, \mathbf{F}_u, f' \mathbf{F}'_v)$ does not depend on the choice of the fields $\mathbf{F}_u$, $\mathbf{F}'_v$, at least up to left composition with $\pm\, i_{m+u+v}$,* where i_{m+u+v}: $S_{m+u+v} \to S_{m+u+v}$ *is the identity mapping.*

It would be easy to set up orientation conventions to avoid the possible

[8] If it were not so, replace ψ: $M_p \times M'_q \times I \to S_m$ by ψ^* which takes at point (x, x', t) the value $\psi(x, x', (1\text{-}\cos \pi t)/2)$.

factor $\pm i_{m+u+v}$. However, in the sequel, it is cumbersome to prevent the appearance of manifolds and fields with the "wrong" orientation (whatever convention is chosen).

PROOF OF LEMMA 2.2. Let $\bar{\mathbf{F}}_u$, $\bar{\mathbf{F}}'_v$ be an alternate choice of fields of u- resp. v-frames orthogonal to M_p and M'_q in E_{p+u}, E_{q+v}. They define together with $\mathbf{F}_u$, $\mathbf{F}'_v$ mappings $\omega: M_p \to \mathbf{O}(u)$, $\omega' : M'_q \to \mathbf{O}(v)$ given by $\omega(x) = (\mathbf{v}_i(x)\cdot\bar{\mathbf{v}}_j(x))$, $i, j=1, \cdots, u$, where $\mathbf{v}_i(x)$, $\bar{\mathbf{v}}_i(x)$ are the vectors of $\mathbf{F}_u$, $\bar{\mathbf{F}}_u$ at x (we assume as we obviously may that the $\mathbf{v}_i(x)$ as well as the $\bar{\mathbf{v}}_j(x)$ are mutually orthogonal unit vectors; $\mathbf{O}(N)$ denotes the orthogonal group in N variables). Define $\omega \times \omega' : M_p \times M'_q \to \mathbf{O}(u + v)$ by

$$\omega \times \omega'(x, x') = \begin{pmatrix} \omega(x) & 0 \\ 0 & \omega'(x') \end{pmatrix}.$$

Since $M_p \times M'_q$ is $(p + q - m)$-connected, the inclusion $V_{p+q-m} \subset M_p \times M'_q$ is homotopic to zero. Therefore, $\omega \times \omega' \mid V_r$ is homotopic to zero. Hence, there exists a continuous family $\mathbf{F}_{u+v}(t)$, $0 \leqq t \leqq 1$, of fields over V_r of $(u + v)$-frames orthogonal to $M_p \times M'_q$ in $E_{p+q+u+v}$, such that $\mathbf{F}_{u+v}(0) = \mathbf{F}_u \times \mathbf{F}'_v \mid V_r$ and $\mathbf{F}_{u+v}(1) = A.\bar{\mathbf{F}}_u \times \bar{\mathbf{F}}'_v \mid V_r$, where A is some constant $(u + v)$ by $(u + v)$ matrix. The family $\gamma_t\varphi$ of mappings $S_{p+q+u+v} \to S_{m+u+v}$ associated with V_r and the field $\mathbf{F}_{u+v}(t) \times \mathbf{F}_m$ provide a homotopy of $\gamma(\varphi, f, \mathbf{F}_u, f', \mathbf{F}'_v)$ to $[\sigma i_{m+u+v}] \circ \gamma(\varphi, f, \bar{\mathbf{F}}_u, f', \bar{\mathbf{F}}'_v)$, where $\sigma =$ sign det $A(\mathbf{F}_m$ is the field over V_r, tangent to $M_p \times M'_q$ which is induced by φ from some "positive" m-frame at the point $\varphi(V_r) \in S_m)$.

LEMMA 2.3. *If $p + q + 1 < u + v + 2m$ and M_p, M'_q are $(p + q - m)$-connected, then $G(\varphi, f, \mathbf{F}_u, f', \mathbf{F}'_v)$ does not depend either on the imbeddings f, f' (again up to left composition with $\pm i_{m+u+v}$).*

This follows from the fact that under condition $p + q + 1 < u+v+2m$ (in other words $p + q + u + v \geqq 2r + 2$) two imbeddings $f \times f'$ and $\bar{f} \times \bar{f}'$ of V_r into $E_{p+q+u+v}$ are regularly isotopic [14, Theorem 6]. By the covering homotopy theorem one can carry along the fields during the deformation and then Lemma 2.2 yields the result.

3. A homotopy invariant

Let $f: S_{d+n+1} \to S_{n+1}$ be a C^{d+n+1}-differentiable mapping and let U, U' be disjoint spherical open subsets of S_{n+1} consisting of regular values of f. Take points q, q' in U, U' respectively. Denote by M_d, M'_d the inverse images by f of q, q' and let $\mathbf{F}_{n+1} = \{\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_n\}$, $\mathbf{F}'_{n+1} = \{\mathbf{u}'_0, \mathbf{u}'_1, \cdots, \mathbf{u}'_n\}$ be the fields of $(n + 1)$-frames orthogonal to M_d, M'_d in S_{d+n+1} obtained by taking the inverse images by the derivatives of f of fixed $(n + 1)$-frames tangent to S_{n+1} at q, q' and inducing the positive orientation of S_{n+1}. Assume that q, q' have been chosen such that $a^*=(-1, 0, \cdots) \in S_{d+n+1}$ does not belong to $M_d \cup M'_d$ and identify $S_{d+n+1}-a^*$ with E_{d+n+1} under stereographic projection s. Consider the map $\varphi : M_d \times M'_d \to S_{d+n}$ given by

$$\varphi(x, y') = (y' - x)/\| y' - x \| \qquad (sx \in M_d, sy' \in M'_d), \tag{3.1}$$

where points in E_{d+n+1} are treated as (radius) vectors. The mapping φ is well defined since $M_d \cap M'_d = 0$. Let $h(f)$ be obtained from φ by the generalized Hopf's construction: $h(f) = G(\varphi, M_d, \mathbf{F}_{n+1}, M'_d, \mathbf{F}'_{n+1})$. It will be proved in the next section that $h(f)$ does not depend on the arbitrary choices of q, q', etc.$\cdots$, but depends actually only upon the homotopy class of the map $f: S_{d+n+1} \to S_{n+1}$. The invariant h will be seen to provide a homomorphism of $\pi_{d+n+1}(S_{n+1})$ into $\pi_{2d+2n+2}(S_{d+3n+2})$ (or alternately into the stable group $\pi_{d-n+N}(S_N)$) which coincides up to sign and stable suspension with the generalization by G. Whitehead of H. Hopf's invariant (whenever this last is defined, i.e. for $d<2n-1$). I do not know whether for $d \geqq 2n-1$ h coincides with the suspension of some of Hilton's generalizations of Hopf's invariant.

4. Proof of the homotopy invariance

U, U' being connected sets of regular values of $f: S_{d+n+1} \to S_{n+1}$, different choices of q, q' within the fixed U, U' amount to changing $\varphi : M_d \times M'_d \to S_{d+n}$ within its homotopy class. By Lemma 2.1, this does not change $h(f)$. Thus $h(f)$ has now been proved to be well defined if a C^{d+n+1}-map f: $S_{d+n+1} \to S_{n+1}$ *together with* $U, U' \in S_{n+1}$ consisting of regular values of f are given. We prove below that $h(f)$ does not depend on U, U' either and depend only on the homotopy class of f. The proof is based on the following:

LEMMA 4.1 *Let U, U' be given disjoint open spherical subsets of S_{n+1} and let f_0, f_1 be differentiable mappings $S_{d+n+1} \to S_{n+1}$ for which U, U' consist of regular values. If $f_0 \simeq f_1$, then $h(f_0) = h(f_1)$.* (W*e assume* $n > 0$.)

By an argument similar to the one used in the proof of Lemma 2.1, this lemma implies that $h(f)$ does not depend on U, U'. (If U_1, U'_1 is another pair of disjoint spherical open subsets of S_{n+1} consider a diffeomorphism $r: S_{n+1} \to S_{n+1}$ of degree $+1$, such that $r(U_1) = U$ and $r(U'_1) = U'$. Then set $f_0 = f$ and $f_1 = r \circ f$ and apply above lemma.) Similarly, Lemma 4.1 implies the homotopy invariance of $h(f)$.

PROOF OF LEMMA 4.1. Let f be a differentiable homotopy of f_0 to f_1, i.e. a C^{d+n+2}-differentiable mapping $S_{d+n+1} \times I \to S_{n+1}$ ($I=$ unit interval $0 \leqq t \leqq 1$), such that $f(x, 0) = f_0(x), f(x, 1) = f_1(x)$. We may again assume that the partial derivatives of f with respect to t are zero for $t=0, 1$ (see footnote 8). The set of regular values of f being an open everywhere dense subset of S_{n+1}, there exist spherical neighborhoods $\bar{U} \subset U$ and $\bar{U}' \subset U'$ consisting of regular values of f, f_0, f_1. Assume $q \in \bar{U}, q' \in \bar{U}'$ and $q, q' \notin f(a^* \times I)$; recall we assumed $n > 0$. The sets $X = f^{-1}(q)$ and $X' = f^{-1}(q')$ are $(d+1)$-dimensional disjoint manifolds with boundary, imbedded in $E_{d+n+1} \times I$ by $s^{-1} \times$ ident. : $(S_{d+n+1} - a^*) \times I \to E_{d+n+1} \times 1$ (see (1.1)). Let $M_i = f_i^{-1}(q)$, $M'_i = f_i^{-1}(q')$. Since $f \mid S_{d+n+1} \times \{i\} = f_i$, we have $M_i = X \cap (S_{d+n+1} \times \{i\})$ and similarly $M'_i = X' \cap (S_{d+n+1} \times \{i\})$. It is easily seen that $\dot{X} = \pm (M_1 - M_0)$, $\dot{X}' = \pm(M'_1 - M'_0)$. Denote by φ_i the mapping $\varphi_i : M_i \times M'_i \to S_{d+n}$ defined by formula (3.1) and, regarding $E_{d+n+1} \times I$ as subset of E_{d+n+2} under $(t_0, \cdots, t_{d+n}) \times t \to (t_0, \cdots, t_{d+n}, t)$ and treating points of $E_{d+n+1} \times I$ as vectors in E_{d+n+2}, define the map $\varphi : X \times X' \to S_{d+n+1}$ by the formula

$$\varphi(x, x') = (x' - x)/\| x' - x \| . \tag{4.2}$$

We have $\varphi_i = \varphi \mid M_i \times M'_i$ $(S_{d+n} \subset S_{d+n+1})$.

Let $U_\varphi \subset S_{d+n+1}$ be an open set consisting of regular values of φ, having non-void intersection $\bar{U}_\varphi = U_\varphi \cap S_{d+n}$ with S_{d+n} and such that $\bar{U}_\varphi$ consists of regular values of both φ_0 and φ_1. Take $b \in \bar{U}$ and denote by W the $(d-n+1)$-dimensional manifold $\varphi^{-1}(b) \subset X \times X'$. Let the $(d+n+1)$-frame $\mathbf{w}_1, \cdots, \mathbf{w}_{d+n}, \mathbf{t}$ consist of the vectors of a fixed $(d+n)$-frame $\mathbf{w}_1, \cdots, \mathbf{w}_{d+n}$ tangent to S_{d+n} at b, inducing the positive orientation of S_{d+n}, together with $\mathbf{t} = \mathbf{t}_{d+n+1} = (0, \cdots, 0, 1) \in E_{d+n+2}$. Let Φ_p be the derivative of φ at $p \in \varphi^{-1}(b)$ (Φ_p maps linearly the space T_p tangent to $X \times X'$ at $p \in \varphi^{-1}(b)$ *onto* the tangent plane to S_{d+n+1} at b). Denote by $\mathbf{v}_1(p), \cdots, \mathbf{v}_{d+n}(p), \mathbf{v}_{d+n+1}(p)$ the uniquely determined vectors of N_p (the orthogonal complement in T_p of the space tangent to W_{d-n+1} at p), such that

$$\Phi_p(\mathbf{v}_j(p)) = \mathbf{w}_j \quad \text{for } j = 1, \cdots, d+n+1 \ (\mathbf{w}_{d+n+1} = \mathbf{t}). \tag{4.3}$$

Thereby, $X \times X'$ carries the metric induced by the imbedding

$$X \times X' \subset E \times I \times E' \times I' \qquad (E = E' = E_{d+n+1}).$$

Notice that W is contained in the subset $\Delta \subset E \times I \times E' \times I'$ consisting of those points $(\mathbf{u}, t, \mathbf{u}', t')$ for which $t = t'$. Indeed, if $\varphi(x, x') = b$, $x \in X$, $x' \in X'$, we see from $\varphi(x, x') = [\mathbf{u}' - \mathbf{u} + (t' - t)\mathbf{t}]/(\| \mathbf{u}' - \mathbf{u} \|^2 + |t' - t|^2)^{1/2}$ and $\mathbf{b} \cdot \mathbf{t} = 0$, that $t = t'(x = (\mathbf{u}, t), x' = (\mathbf{u}', t'))$. The boundary $W^{\cdot}$ of W consists of two manifolds V_i, $i = 0, 1$, satisfying $V_i \subset \Delta_i$, where Δ_i is the set of those $(\mathbf{u}, t, \mathbf{u}', t')$ for which $t = t' = i$. Therefore $V_i = W \cap (M_i \times M'_i) =$

$\varphi_i^{-1}(b)$. It is convenient to introduce the injection $\Psi : E \times I \times E' \times I' \to E_{2d+2n+3} \times J$ given by $\Psi(\mathbf{u}, t, \mathbf{u}', t') = (\mathbf{u}, \mathbf{u}', (t' - t)/\sqrt{2}) \times (t + t')/\sqrt{2}$, where $J = [0, \sqrt{2}]$. Then ΨW is a submanifold of $E_{2d+2n+2} \times J$ and the boundary of ΨW consists of $\Psi V_i, i=0,1$, such that $\Psi V_i \subset E_{2d+2n+2} \times \{i\sqrt{2}\}$. Now, W carried in $E \times I \times E' \times I'$ a field of normal $(d+3n+3)$-frames consisting of the vectors of $\mathbf{F}_{n+1}$ followed in order by those of $\mathbf{F}'_{n+1}$ and by $\mathbf{v}_1, \cdots, \mathbf{v}_{d+n}, \mathbf{v}_{d+n+1}$. Since $\mathbf{t}' - \mathbf{t}$ is orthogonal to W and, as a simple computation shows, $\Phi_p(\mathbf{t}'-\mathbf{t}) = 2\mathbf{t}/\| \mathbf{u}' - \mathbf{u} \|$ for every $p = (\mathbf{u}, t, \mathbf{u}', t) \in W$, it follows that $\mathbf{v}_{d+n+1}(p)$ has for every $p \in W$ the constant direction $\mathbf{t}' - \mathbf{t}$. In other words, the images by Ψ of $\mathbf{F}_{n+1}, \mathbf{F}'_{n+1}, \mathbf{v}_1, \cdots, \mathbf{v}_{d+n}$ provide a field of $(d + 3n + 2)$-frames orthogonal to ΨW in $E_{2d+2n+2} \times J$. There is associated with ΨW and the just mentioned field a mapping $\lambda : S_{2d+2n+2} \times J \to S_{d+3n+3}$. Identifying $E \times \{i\} \times E' \times \{i\}$ with $E_{2d+2n+2}$ under $(\mathbf{u}, i, \mathbf{u}', i) \leftrightarrow (\mathbf{u}, \mathbf{u}')$, the mapping $\Psi \mid E \times \{i\} \times E' \times \{i\}$ is the inclusion $E_{2d+2n+2} \subset E_{2d+2n+3}$. Therefore, $\lambda_i = \lambda \mid S_{2d+2n+2} \times \{i\sqrt{2}\}$ is associated with V_i (as a submanifold of $E_{2d+2n+2}$), together with the field $\mathbf{F}_{n+1}, \mathbf{F}'_{n+1}, \mathbf{v}_1, \cdots, \mathbf{v}_{d+n}$. However, $\gamma\varphi_i$ is associated with the same manifold and field. Therefore, $\lambda_i \simeq \gamma\varphi_i$, and since $\lambda_0 \simeq \lambda_1$, it follows $h(f_0) = h(f_1)$, which proves Lemma 4.1.

5. Higher linking coefficients

Let $f : S_p \to E_{m+1}$ and $f' : S_q \to E_{m+1}$ be regular C^2-imbeddings of the p- and q-dimensional spheres into euclidean $(m + 1)$-space, such that $\Sigma_p = f(S_p)$ and $\Sigma'_q = f'(S_q)$ have no common point. Assume $m > \max(p, q) + 1$, then the spaces $X = S_{m+1} - s\,\Sigma_p$ and $X' = S_{m+1} - s\,\Sigma'_q$ ($s : E_{m+1} \to S_{m+1}$ is the stereographic projection of Section 1) are simply connected. By Alexander duality, their homology groups are isomorphic to the homology groups of the spheres S_{m-p} and S_{m-q} respectively (in every dimension and with any coefficients). We set the isomorphisms as follows: Let x^* be a point of Σ_p and let $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{m-p}$ be an orthonormal basis in the $(m - p + 1)$-plane orthogonal to Σ_p at x^*, such that $\mathbf{v}_0, \mathbf{v}_1, \cdots, \cdots, \mathbf{v}_{m-p}$ followed in order by the vectors of a "positive" p-frame tangent to Σ_p at x^* determine the positive orientation of E_{m+1}. Define $j : S_{m-p} \to X$ by

$$j(y_0, \cdots, y_{m-p}) = s(x^* + \varepsilon \textstyle\sum_{i=0}^{m-p} y_i \mathbf{v}_i) ,$$

and similarly $j' : S_{m-q} \to X'$ by interchanging p and q in the definition of j ($\varepsilon > 0$, smaller than the radii of tubular neighborhoods of Σ_p, Σ'_q). We assume for later convenience, that $a^* = (-1, 0, \cdots) \in S_{m+1}$ does not belong to $j(S_{m-p}) \cup j'(S_{m-q})$. With these conventions, we have

$$L(j(S_{m-p}), f(S_p)) = +1,\ L(j'(S_{m-q}), f'(S_q)) = +1 ,$$

where L denotes the linking coefficient (in the usual sense of [1, Kap. XI, § 1, 6]). The mappings j, j' induce isomorphisms $j_*: H_k(S_{m-p}) \to H_k(X)$ and $j'_*: H_k(S_{m-q}) \to H_k(X')$ for every k. By a theorem of J.H.C. Whitehead [12, Theorem 3, p. 216], j and j' also induce isomorphisms of the homotopy groups $j_*: \pi_k(S_{m-p}) \to \pi_k(X)$ and $j'_*: \pi_k(S_{m-q}) \to \pi_k(X')$ and there exist mappings $k: X \to S_{m-p}$ and $k': X' \to S_{m-q}$ which are both right and left homotopy inverses for j and j' respectively.

Let $\alpha \in \pi_p(S_{m-q})$ and $\alpha' \in \pi_q(S_{m-p})$ be represented by the mappings $k' \circ s \circ f$ and $k \circ s \circ f'$ respectively. Following Arnold Shapiro, α and α' can be regarded as the linking coefficients $L(fS, f'S_q)$ and $L(f'S_q, fS_p)$ respectively. In general however, for $p=q$, α and α' differ from each other not only in sign.

Let $\varphi: S_p \times S_q \to S_m$ be defined by the formula

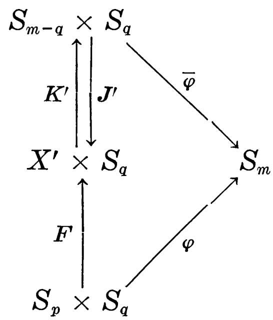
$$\varphi(x, y) = (f'y - fx)/||f'y - fx|| ,$$

and let $G\varphi \in \pi_{p+q+1}(S_{m+1})$ be the homotopy class of the mapping obtained from φ by Hopf's construction.

Lemma 5.1 $G\varphi = (-1)^{m+1}E^{q+1}\alpha = (-1)^{pq+p+q}E^{p+1}\alpha'$.

Proof. Consider the diagram

(5.2)



where $F = sf \times i_q$, $J' = j' \times i_q$, $K' = k' \times i_q$ ($i_q =$ identity mapping $S_q \to S_q$ and $\bar{\varphi}(x, y) = [f'y - s^{-1}j'x]/||f'y - s^{-1}j'x||$. The diagram is homotopy commutative, i.e. $\bar{\varphi} \circ K' \circ F \simeq \varphi$. This is seen by introducing $\bar{\bar{\varphi}}: (X' - a^*) \times S_q \to S_m$ defined by $\bar{\bar{\varphi}}(x', y) = [f'y - s^{-1}x'] / || f'y - s^{-1}x' ||$. Indeed, $\bar{\bar{\varphi}}(J'(x, y)) = \bar{\bar{\varphi}}(j'x, y) = [f'y - s^{-1}j'x] / ||f'y - s^{-1}j'x|| = \bar{\varphi}(x, y)$ and $\bar{\bar{\varphi}}(F(x, y)) = \bar{\bar{\varphi}}(fx, y) = [f'y - fx] / ||f'y - fx|| = \varphi(x, y)$. Therefore, $\bar{\varphi} \circ K' \circ F = \bar{\bar{\varphi}} \circ J' \circ K' \circ F \simeq \bar{\bar{\varphi}} \circ F = \varphi$.

The homotopy $\bar{\bar{\varphi}} \circ J' \circ K' \circ F \simeq \bar{\bar{\varphi}} \circ F$ follows from the fact that $j' \circ k' \circ f$ and f which map S_p into $X' - a^*$ are not only homotopic in X' but moreover in $X' - a^*$, since $p < m$.

By [1, Kap. XII, Anhang, p. 496], we have

$$\text{degree } \bar{\varphi} = (-1)^{m-q+1}L(j'(S_{m-q}), f'(S_q)) = (-1)^{m-q+1} .$$

Now, $K' \circ F = (k' \times i_q) \circ (sf \times i_q) = (k' \circ sf) \times i_q$ represents $\alpha \times i_q$. Introducing the map $f_\alpha : \dot{W}_{p+q+2} \rightarrow \dot{W}_{m+2}$ defined by

$$f_\alpha(x, y) = \begin{cases} (k'sfx, y) & \text{for } x \in \dot{W}_{p+1}, y \in W_{q+1} \\ (\| x \| \, k'sf(x/\| x \|), y) & \text{for } x \in W_{p+1}, y \in \dot{W}_{q+1}, \end{cases}$$

we have the homotopy commutative diagram

$$\begin{array}{ccc} S_{m+1} & & \\ \uparrow & \overset{G\bar{\varphi}}{\searrow} & \\ f_\alpha & & S_{m+1} \\ | & \underset{G\varphi}{\nearrow} & \\ S_{p+q+1} & & \end{array}$$

where $\dot{W}_{N+1}$ and S_N have been identified under radial projection (see Section 1). It is well known and easily verified that $f_\alpha \simeq +E^{q+1}k'sf \in E^{q+1}\alpha$. On the other hand, if $\bar{\varphi} : S_a \times S_b \rightarrow S_{a+b}$ has degree d, then $G\bar{\varphi} : S_{a+b+1} \rightarrow S_{a+b+1}$ has degree $(-1)^b d$. Thus in our case, $G\bar{\varphi}$ has degree $(-1)^q(-1)^{m-q+1} = (-1)^{m+1}$. It follows $G\varphi = (-1)^{m+1}E^{q+1}\alpha$.

By interchanging p, q, we obtain $G\varphi' = (-1)^{m+1}E^{p+1}\alpha'$, where $\varphi' : S_q \times S_p \rightarrow S_m$ is defined by $\varphi'(y, x) = [fx - f'y] / \| fx - f'y \|$. The diagram

$$\begin{array}{ccc} S_p \times S_q & \xrightarrow{\varphi} & S_m \\ \downarrow \lambda & & \downarrow \mu \\ S_q \times S_p & \xrightarrow{\varphi'} & S_m \end{array}$$

where $\lambda(x, y) = (y, x)$, $\mu(z) = -z$, is commutative. It induces a diagram

$$\begin{array}{ccc} S_{p+q+1} & \xrightarrow{G\varphi} & S_{m+1} \\ \downarrow \lambda' & & \downarrow \mu' \\ S_{p+q+1} & \xrightarrow{G\psi'} & S_{m+1} \end{array}$$

where λ' has degree $(-1)^{(p+1)(q+1)}$. Since $\mu' = -E\mu$ has degree $(-1)^m$, it follows $G\varphi = (-1)^{m+(p+1)(q+1)}G\varphi' = (-1)^{pq+p+q}E^{p+1}\alpha'$.

The definition of higher linking coefficients can be generalized as follows : Let M_p and M'_q be $(p + q - m)$-connected closed π-manifolds. Let $f : M_p \rightarrow E_{m+1}$ and $f' : M'_q \rightarrow E_{m+1}$ be continuous mappings of M_p, M'_q into euclidean $(m + 1)$-space, such that $f(M_p) \cap f'(M'_q) = 0$. Define

$\varphi : M_p \times M'_q \to S_m$ by $\varphi(x, x') = [f'x' - fx]/\|f'x' - fx\|$. Since M_p, M'_q (as π-manifolds) can be imbedded in some euclidean spaces with fields of orthogonal frames, they can be imbedded in E_{p+m+1} and E_{q+m+1} respectively with fields $\mathbf{F}_{m+1}$, $\mathbf{F}'_{m+1}$ of orthogonal $(m+1)$-frames ($m \geqq \max(p, q)$ is assumed). Let $f_1 : M_p \to E_{p+m+1}$, $f'_1 : M'_q \to E_{q+m+1}$ denote these imbeddings. The generalized Hopf's construction $G(\varphi, f_1, \mathbf{F}, f'_1, \mathbf{F}')$ provides an element in $\pi_{p+q+2m+2}(S_{3m+2})$, in other words in the stable group $\pi_{r+N}(S_N)$ $r = p+q-m$, which can be regarded as the linking coefficient $L(f(M_p), f'(M'_q))$ of $f(M_p)$ and $f'(M'_q)$ in E_{m+1}. The $(p+q-m)$-connectedness of M and M' guarantees (by Lemma 2.3) that $G(\varphi, f_1, \mathbf{F}, f'_1, \mathbf{F}')$ be independent of the arbitrary choices of f_1, f'_1, $\mathbf{F}$ and $\mathbf{F}'$.

Obviously $L(X, X')$, whenever defined, is a bilinear function.

6. The value of $h(f)$ for some special mappings f.

Let $p : S_d \longrightarrow E_{d+n}$ be a regular C^2-imbedding of the d-sphere into euclidean $(d+n)$-space. Since $E_{d+n} \subset E_{d+n+1}$, we can interpret p as being an imbedding into E_{d+n+1}. Assume that there exists over $\Sigma_d = p(S_d)$ in E_{d+n+1} a field $\mathbf{F}_{n+1}$ of $(n+1)$-frames orthogonal to Σ_d. Choose the vectors $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n$ of $\mathbf{F}_{n+1}$ such that if $\mathbf{v}_{n+1}(x), \cdots \mathbf{v}_{d+n}(x)$ is a tangent frame at some point $x \in \Sigma_d$ inducing the positive orientation of Σ_d then $\mathbf{v}_0(x), \cdots, \mathbf{v}_n(x), \mathbf{v}_{n+1}(x), \cdots, \mathbf{v}_{d+n}(x)$ in this order induce the positive orientation of E_{d+n+1}. The pair $(\Sigma_d; \mathbf{F}_{n+1})$ induces by construction (1.2) a mapping $f : S_{d+n+1} \to S_{n+1}$ for which we want to calculate $h(f)$.

Let ν be the homotopy class of the map $N : S_d \to S_n$ defined as follows: Denote by $\mathbf{t}$ ($= \mathbf{t}_{d+n}$ in the notations of Section 1) the constant vector normal to E_{d+n} in E_{d+n+1}. Then, $N(x) = (\mathbf{t}\cdot\mathbf{v}_0(x), \mathbf{t}\cdot\mathbf{v}_1(x), \cdots, \mathbf{t}\cdot\mathbf{v}_n(x))$, where $\mathbf{t}\cdot\mathbf{v}_i(x)$ denotes the scalar product.

LEMMA 6.1. *For the above mapping f, one has*

$$h(f) = (-1)^{(n+1)d+1}E^{d+2(n+1)}\nu .$$

Let $X = S_{d+n+1} - s\,\Sigma_d$ and let the map $p' : S_d \to X$ be given by

$$p'(x) = s[p(x) + \varepsilon v_0(x)] ,$$

where $\varepsilon > 0$ is smaller than the radius of a tubular neighborhood of Σ_d in E_{d+n+1}. Let α' be the homotopy class (in $\pi_d(X)$) of $p' : S_d \to X$. We first prove the

LEMMA 6.2. $j_*\nu = -\alpha'$, *where $j : S_n \to X$ is defined by*

$$j(y) = s[p(a) + \varepsilon \textstyle\sum_{k=1}^{n} y_k \mathbf{v}_k(a)] ,$$

a being some fixed point of S_d.

Notice that j is the same as in § 5. It induces an isomorphism $j_* : \pi_d(S_n) \to \pi_d(X)$.

PROOF. Let $\psi : S_d \times S_n \to X$ be defined by the formula

$$\psi(x, y) = s[p(x) + \varepsilon \sum_{k=0}^{n} y_k \mathbf{v}_k(x)] .$$

The mapping ψ induces a homomorphism $\psi_* : \pi_d(S_d \times S_n) \to \pi_d(X)$. We shall identify $\pi_d(S_d \times S_n)$ with $\pi_d(S_d) \oplus \pi_d(S_n)$ under $\sigma_* \oplus \tau_*$ induced by $\sigma(x) = (x, b)$ and $\tau(y) = (a, y)$. Since $\psi_*(1, 0)$ is represented by $p' : S_d \to X$, one has $\psi_*(1, 0) = \alpha'$. Similarly, because $\psi \mid a \times S_n = j$, we have $\psi_*(0, \lambda) = j_* \lambda$. The homomorphism ψ_* is therefore given by the formula

$$\psi_*(m, \lambda) = m\alpha' + j_* \lambda . \tag{6.3}$$

Consider the mapping $p'' : S_d \to X$ given by $p''(x) = s[p(x) + \varepsilon t]$. Obviously, $p'' \simeq 0$ in X (because $p''(S_d) \subset s\{x_{d+n} \geqq \varepsilon\} \subset X$ and $\{x_{d+n} \geqq \varepsilon\}$ has vanishing homotopy groups). On the other hand, we can express p'' by

$$p''(x) = s[p(x) + \varepsilon \sum_{k=0}^{n} (\mathbf{t} \cdot \mathbf{v}_k(x)) \mathbf{v}_k(x)]$$

as well. Therefore, $p''(x) = \psi(x, N(x))$. By (6.3), the mapping p'' (homotopic to zero) represents $\psi_*(1, \nu) = \alpha' + j_* \nu$. In other words, $j_* \nu = -\alpha'$.

PROOF OF LEMMA 6.1. Let $\varphi : S_d \times S_d \to S_{d+n}$ be the mapping defined by

$$\varphi(x_1, x_2) = [p(x_2) + \varepsilon \mathbf{v}_0(x_2) - p(x_1)] / \| p(x_2) + \varepsilon \mathbf{v}_0(x_2) - p(x_1) \| .$$

By Lemma 5.1, $G\varphi = (-1)^d E^{d+1} j_*^{-1} \alpha'$. Therefore, by the above lemma, $G\varphi = (-1)^{d+1} E^{d+1} \nu$. It remains to prove that $h(f) = (-1)^{nd} E^{2n+1} G\varphi$. We shall prove $h(f) = (-1)^{d(n+1)} E^{2n+1} G'\varphi$, from which the assertion follows by 1.5.

$h(f)$ is represented by $\gamma\varphi$, obtained from φ by generalized Hopf's construction. $\gamma\varphi$ is associated with the image by $f \times f'$ in $E_{2d+2n+2}$ of the manifold $V_{d-n} = \varphi^{-1}(b)$, where b is some regular value of φ in S_{d+n}. $E^{2n+1}G'\varphi$ is associated with $i(V_{d-n}) \subset E_{2d+1} \times E_{2n+1} = E_{2d+2n+2}$. By Lemma 2.3 in which $p = 2 = d$, $u = v = n + 1$, $m = d+n$ $(d+d+1 < 2(n + 1) + 2(d+n)$ and S_d is $(d - n)$-connected), we have $\gamma\varphi \simeq \pm E^{2n+1} G'\varphi$.

To obtain the sign, choose an orientation of V_r, for instance the one induced by a tangent r-frame $\mathbf{u}_1, \cdots, \mathbf{u}_r$ at some $x \in V_r$, such that $\mathbf{u}_1, \cdots, \mathbf{u}_r$ followed in order by the vectors of the $(d + n)$-frame $\mathbf{w}_1, \cdots, \mathbf{w}_{d+n}$ induced by φ at x induce the positive orientation of $S_d \times S_d$ ($\mathbf{w}_1, \cdots, \mathbf{w}_{d+n}$ is assumed to be induced by φ from a tangent frame to S_{d+n} at b inducing the positive orientation of S_{d+n}). Denote by $\mathbf{v}'_0, \cdots, \mathbf{v}'_n$ the vectors of the field $\mathbf{F}'_{n+1}$ over $p'(S_d)$ obtained by carrying along $\mathbf{F}_{n+1}$ (by the homotopy covering theorem). $\gamma\varphi$ is associated with

$$\mathbf{v}_0, \cdots, \mathbf{v}_n, \mathbf{v}'_0, \cdots, \mathbf{v}'_n, \mathbf{w}_1, \cdots, \mathbf{w}_{d+n} .$$

The image under $p \times p'$ of the vectors $\mathbf{u}_1, \cdots, \mathbf{u}_r$ followed by these vectors induce the orientation $(-1)^{(n+1)d}$ of $E_{2d+2n+2}$. The map $E^{2n+1}G'\varphi$ is associated with

$$\mathbf{w}_1, \cdots, \mathbf{w}_{d+n}, \mathbf{w}, \mathbf{t}_{2d+1}, \cdots, \mathbf{t}_{2d+2n+1} ,$$

where $\mathbf{w}$ is the exterior normal to $i(S_d \times B_{d+1})$ in E_{2d+1}. The images by i of $\mathbf{u}_1, \cdots, \mathbf{u}_r$ followed by the above vectors induce the positive orientation of $E_{2d+2n+2}$. Thus, $\gamma\varphi \simeq [(-1)^{(n+1)d} i_{d+3n+2}] \circ E^{2n+1}G'\varphi$. Therefore, $h(f) = (-1)^{(n+1)d} E^{2n+1}G'\varphi$.

REMARK 6.4. If the field $\mathbf{F}_{n+1}$ orthogonal to Σ_d in E_{d+n+1} (and satisfying the orientation convention) is replaced by $\bar{\mathbf{F}}_{n+1}$ obtained by letting the constant $(n+1)$ by $(n+1)$ matrix act on the vectors of $\mathbf{F}_{n+1}$, then ν is obviously changed into $\sigma\nu$, where $\sigma =$ sign det A. It is easily seen that $h(f)$ does not change (although f is replaced by $(\sigma i_{n+1}) \circ f$). Therefore, Formula 6.1 reads in general $h(f) = \sigma(-1)^{(n+1)d+1} E^{d+2(n+1)}\nu$.

This situation occurs in the following

LEMMA 6.5. *If $f \in J\mu$, where $\mu \in \pi_d(\mathbf{SO}(n+1))$, then $h(f) = (-1)^{n+1} E^{d+2(n+1)} \phi_* \mu$. ($\phi_* : \pi_d(\mathbf{SO}(n+1)) \to \pi_d(S_n)$ is induced by ϕ which maps $A = (a_{ij})$, $i, j = 0, \cdots, n$ into $(a_{0,n}, a_{1,n}, \cdots, a_{n,n}) \in S_n$).*

PROOF. By 1.7, it is sufficient to prove $h(f') = (-1)^{d+1} E^{d+2(n+1)} \phi_* \mu$, where f' representing $J'\mu$ is a map of the sort considered in Lemma 6.1 above. The corresponding ν is the class of the mapping $N : S_d \to S_n$, where $N(x) = (\mathbf{t} \cdot \mathbf{v}_0(x), \cdots, \mathbf{t} \cdot \mathbf{v}_n(x))$. Here, $\mathbf{v}_i(x) = a_{i0}(x)\mathbf{x} + \sum_{j=1}^{n} a_{ij}(x)\mathbf{t}_{d+j}$. Hence, $\mathbf{t} \cdot \mathbf{v}_i(x) = a_{i,n}$, i.e. $N = \phi \circ M(\mathbf{t} = \mathbf{t}_{d+n})$. Thus $\nu = \phi_* \mu$ and by 6.1, together with the above remark 6.4, $h(f') = (-1)^{dn}(-1)^{(n+1)d+1} E^{d+2(n+1)}\nu = (-1)^{d+1} E^{d+2(n+1)} \phi_* \mu$.

Next, we want to calculate $h(f)$ for a mapping f obtained by Hopf's construction. We need a preliminary lemma.

Let $\varphi : S_p \times S_q \to S_m$ be a C^{p+q}-map of type (α, β). Denote by $(V_r; \mathbf{F}_m)$, $r = p+q-m$, a manifold and field with which φ is associated up to homotopy ($V_r \subset S_p \times S_q$). Let $\pi_1 : V_r \to S_p$ and $\pi_2 : V_r \to S_q$ be the restrictions over V_r of the projections $S_p \times S_q \to S_p$ and $S_p \times S_q \to S_q$ respectively (projection mappings : $(x, y) \to x$, $(x, y) \to y$). The mappings π_1, π_2 are associated up to homotopy with some submanifolds W_{q-m}, W_{p-m} of V_r and fields $\mathbf{F}_p$, $\mathbf{F}_q$ respectively. Using $i : S_o \times S_q \to E_{p+q+1}$ of Section 1 (Formula (1.6)), we obtain W_{q-m}, W_{p-m} as submanifolds of E_{p+q+1} carrying the fields of

$(p+m+1)$-resp. $(q+m+1)$-frames $\mathbf{F}_m \times \mathbf{F}_p \times \mathbf{w}$, $\mathbf{F}_m \times \mathbf{F}_q \times \mathbf{w}$. Denote by $\beta' : S_{p+q+1} \to S_{p+m+1}$ and $\alpha' : S_{p+q+1} \to S_{q+m+1}$ the corresponding mappings. We have the

LEMMA 6.6. $\alpha' \simeq (-1)^{p+q} E^{q+1}\alpha$ and $\beta' \simeq (-1)^{p+q+pq} E^{p+1}\beta$.

(We use the same letter for a map and its homotopy class whenever no confusion can arise.) It follows, in particular, that the homotopy classes of α', β' are uniquely determined by those of α, β.

PROOF. Let us prove $\alpha' = (-1)^{p+q} E^{q+1}\alpha$. Let $c \in S_m$ be a regular value of both φ and $\varphi \mid S_p \times b = \alpha$. Take $V_r = \varphi^{-1}(c)$. It can be assumed that the derivatives $(\partial\varphi_i/\partial y_j)_{(x,b)}$ are all zero ($y_1, \cdots, y_q$ being local coordinates on $x \times S_q$ around (x, b) and φ_i giving φ in terms of local coordinates on S_m around $\varphi(x, b)$). If it is not so, we replace φ by φ^* which takes at $(x; y_1, \cdots, y_q)$ the value $\varphi(x; y_1 f(y^2), \cdots, y_q f(y^2))$, where $y^2 = \sum (y_i)^2$ and $f(t)$, defined for $0 \leqq t < \infty$, is a real valued monotone increasing C^m-differentiable function such that $f(0) = 0$ and $f(t) = 1$ for $t \geqq \rho^2 > 0$.[9] Here ρ is chosen so small that (x, y) is a regular point of φ for every $x \in V_r \cap (S_p \times b)$ and $y \in U_\rho(b)$. These conditions guarantee that c is also a regular value of both φ^* and $\varphi^* \mid S_p \times b$. We write again φ instead of φ^*.

The condition $(\partial\varphi_i/\partial y_j)_{(x,b)} = 0$ guarantees that the tangent plane to $x \times S_q$ at (x, b) is a subspace of the tangent plane to V_r at this point $(r \geqq q)$. Therefore, the orthogonal complement of $T_{(x,b)}(V_r)$ in $T_{(x,b)}(S_p \times S_q)$ is equal to the orthogonal complement of $T_{(x,b)}(W_{p-m})$ in $T_{(x,b)}(S_p \times b)$. It follows that *the restriction over* W_{p-m} of the field of m-frames (orthogonal to V_r in $S_p \times S_q$) induced by φ is equal to the field of m-frames (orthogonal to W_{p-m} in $S_p \times b$) induced by $\alpha = \varphi \mid S_p \times b$. Let us denote by $\mathbf{v}_1(x), \cdots, \mathbf{v}_m(x)$ the vectors of this field at $x \in W_{p-m}$. The inclusion $i : S_p \times S_q \to E_{p+q+1}$ yields the manifold $i(W_{p-m}) \subset E_{p+q+1}$ with which α' is associated. An obvious modification of 1.4 shows that $+E^{q+1}\alpha$ is associated with $i(W_{p-m})$ and $\mathbf{v}_1, \cdots, \mathbf{v}_m$ followed in order by the constant vectors $\mathbf{t}_{p+1}, \cdots, \mathbf{t}_{p+q}, (-1)^{p+q}\mathbf{w}$, where $\mathbf{w}$ is the exterior normal to $i(S_p \times B_{b+1})$ in E_{p+q+1}. Now, the field over $i(W_{p-m})$ in E_{p+q+1} with which α' is associated consists of $\mathbf{v}_1, \cdots, \mathbf{v}_m, \mathbf{u}_1, \cdots, \mathbf{u}_q, \mathbf{w}$, where $\mathbf{u}_1, \cdots, \mathbf{u}_q$ are vectors tangent to V_r and orthogonal to W_{p-m} and mapped onto a fixed q-frame by $\pi_2 : V_r \to S_q$ (or rather $\mathbf{u}_1, \cdots, \mathbf{u}_q$ are the I-images of such vectors, where I is the derivative of i). In other words, $\mathbf{u}_1, \cdots, \mathbf{u}_q$ is a fixed frame (independent of (x, b)) tangent to $i(x \times S_q)$ at $i(x, b)$ and inducing the positive orientation of $i(x \times S_q)$. We may as well take $\mathbf{u}_i = \mathbf{t}_i$, $i = 1, \cdots, q$.

[9] Take for instance $f(t) = 1 - (t - \rho^2)^{2N}/\rho^{4N}$ for $0 \leqq t \leqq \rho^2$ and $= 1$ for $\rho^2 < t$, where $2N > m$.

Thus the fields with which α' and E^{q+1} are associated differ only in the sign of the last vector. It follows $\alpha' = (-1)^{p+q}E^{q+1}\alpha$.

By interchanging p and q, it is clear that $\beta' = \pm E^{p+1}\beta$. The determination of the sign $\beta' = (-1)^{p+q+pq}E^{p+1}\beta$ is left to the reader.

LEMMA 6.7. *Let $f : S_{p+q+1} \to S_{m+1}$ be obtained by Hopf's construction from a map $\psi : S_p \times S_q \to S_m$ of type (α, β). Then*

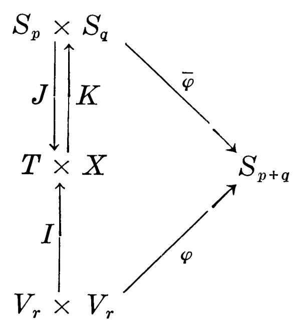
$$h(f) = (-1)^{q+1}E^{p+q+1}(\alpha * \beta) .$$

PROOF. Let $V_r = \psi^{-1}(c)$, where c is a regular value of ψ (assumed to be of class C^{p+q}). Define $i' : S_p \times S_q \to E_{p+q+1}$ by

$$i'(x, y) = i(x, y) + \eta w(x, y) , \tag{6.8}$$

where $0 < \eta < 1 - \varepsilon$ (thus i' is an *imbedding*). The set $si'(S_p \times S_q)$ disconnects $S_{p+q+1} = sE_{p+q+1} \cup a^*$ and we denote by X the closure of the region which does not contain $sT = si(S_p \times B_{q+1})$. It is easily seen that $X \simeq B_{p+1} \times S_q$ and $j = s \circ i' \mid a \times S_q : a \times S_q \to X$ induces an isomorphism $j_* : \pi_k(S_q) \to \pi_k(X)$ in every dimension (j_* coincides up to sign $(-1)^{pq}$ with the j_* of § 5). Let $k : X \to S_q$ be the map induced by the projection $B_{p+1} \times S_q \to S_q$ ($ksi' = p_2$, where $p_2(x, y) = y$). Consider the diagram

(6.9)



where $I(u, u') = (i(u), si'(u'))$, $J(x, y) = (i(x, 0), j(y))$, $K(i(x, y), z) = (x, kz)$ and $\bar{\varphi}$ is defined by $\bar{\varphi}(x, y) = [i'(y) - i(x)] / \| i'(y) - i(x) \|$.

The diagram is homotopy commutative (i.e. $\bar{\varphi} \circ K \circ I \simeq \varphi$) and

$$\text{degree } \bar{\varphi} = (-1)^{p+1}L(i(S_p), i'(S_q)) = (-1)^p .$$

Notice that $K \circ I$ is the map $\pi_1 \times \pi_2$, where $\pi_1 : V_r \to S_p$, $\pi_2 : V_r \to S_q$ are the restrictions to V_r of the projections $p_1, p_2 (p_1(x, y) = x, p_2(x, y) = y)$. Indeed,

$$K \circ I(u, u') = K(i(u),\ si(u')) = (p_1(u),\ ksi'(u')) = (p_1(u), p_2(u_{\prime})) .$$

We can re-write the diagram (6.9) as follows

$$
(6.10)\qquad
\begin{array}{ccc}
S_p \times S_q & & \\
\uparrow{\scriptstyle \pi_1 \times \pi_2} & \searrow{\scriptstyle \bar{\varphi}} & \\
 & & S_{p+q} \\
 & \nearrow{\scriptstyle \varphi} & \\
V_r \times V_r & &
\end{array}
$$

$h(f)$ is the class of the map $\gamma : S_{2(p+q+1)} \to S_{p+q+2(m+1)}$ obtained by generalized Hopf's construction $\gamma(\varphi, i \mid V_r, \mathbf{F}_m \times \mathbf{w}, i' \mid V_r, \mathbf{F}'_m \times \mathbf{w}')$. Since the homotopy class of γ depends on the *homotopy class* of φ (and not on φ itself), $h(f)$ is represented as well by $\gamma(\pi_1 \times \pi_2 \circ \bar{\varphi}, i \mid V_r, \mathbf{F}_m \times \mathbf{w}, i' \mid V_r, \mathbf{F}'_m \times \mathbf{w}')$. Denote by $\alpha' : S_{p+q+1} \to S_{q+m+1}$ and $\beta' : S_{p+q+1} \to S_{p+m+1}$ the mappings associated with $(i \mid W_{p-m} ; \mathbf{F}_m \times \mathbf{F}_q \times \mathbf{w})$ and $(i' \mid W_{q-m} ; \mathbf{F}'_m \times \mathbf{F}_p \times \mathbf{w}')$ respectively, where $(W_{p-m} \subset V_r; \mathbf{F}_q)$, $(W_{q-m} \subset V_r; \mathbf{F}_p)$ are manifolds and fields with which $\pi_2 : V_r \to S_q$ and $\pi_1 : V_r \to S_p$ are associated. Since $\pi(\beta', \alpha')$ defined in 1.9 is associated with $(i \mid W_{q-m} \times i' \mid W_{p-m}; \mathbf{F}_m \times \mathbf{F}_p \times \mathbf{w} \times \mathbf{F}'_m \times \mathbf{F}_q \times \mathbf{w}')$, we have $\gamma \simeq (-1)^{pm+q+p}\pi(\beta', \alpha')$. By Lemma 1.11, it follows: $E\gamma \simeq (-1)^{pm+q+p+p+q+1+p+m+1}\beta' * \alpha' = (-1)^{mp+m+p}\beta' * \alpha'$. Now by Lemma 6.6, $\alpha' = (-1)^{p+q}E^{q+1}\alpha$, $\beta' = (-1)^{p+q+pq}E^{p+1}\beta$. Therefore, $E\gamma \simeq (-1)^{mp+m+p+pq} E^{p+1}\beta * E^{q+1}\alpha$. By known formulae about the join (see [2, Formula (2.3) and Corollary 2.6]) it follows $E\gamma \simeq (-1)^{q+m}E^{p+q+2}\alpha * \beta$ which implies $h(f) = (-1)^{q+m}E^{p+q+1}(\alpha * \beta)$ since $h(f)$ lies in the stable range.

7. Identification of $h(f)$

THEOREM (7.1). *Let f be an element of $\pi_{d+n+1}(S_{n+1})$. Assume $d < 2n-1$, then $H(f)$, the generalized Hopf's invariant is defined* ([10], Section 5).
CLAIM : $h(f) = (-1)^n E^{d+n+1}H(f)$.

PROOF. Since $d < 2n - 1$, by Corollary 6.4 in [11] the element f of $\pi_{d+n+1}(S_{n+1})$ can be obtained by Hopf's construction from a map $\psi : S_d \times S_n \to S_n$ of type (α, i_n). By Lemma 6.7, $h(f) = E^{d+n+1}(\alpha * i_n)$. In [10], G. Whitehead has proved that $H(f) = (-1)^n(\alpha * i_n)$ (Theorem 5.1 with sign corrected as in [13, Formula (6.1)]). Hence the assertion.

8. Application to the normal bundle of a sphere in euclidean space

We first prove a lemma which is seemingly nowhere stated in the literature.

LEMMA 8.1. *Let $f : S_{d+n+1} \to S_{n+1}$ be a differentiable map such that for some regular value $c \in S_{n+1}$, the manifold $f^{-1}(c)$ is diffeomorphic to the sphere S_d. Assume $d<2n$, then the homotopy class of f belongs to the image of $J : \pi_d(\mathrm{SO}(n+1)) \to \pi_{d+n+1}(S_{n+1})$.*

PROOF. We prove the lemma for $E^k f$ by decreasing induction on $k = \cdots, 1, 0$. For large values of k, $E^k f \in \pi_{d+n+k+1}(S_{n+k+1})$ is in the image of J because for $2d+1<d+n+k+1$, the imbedding of S_d into euclidean $(d+n+k+1)$-space as inverse image of c by $E^k f$ is isotopic to the standard imbedding (see [14], Theorem 6, § 12). By the covering homotopy theorem, we can carry along the normal field during the deformation. This provides a homotopy between $E^k f$ and some $J\mu_k$, $\mu_k \in \pi_d(\mathrm{SO}(n+k+1))$. Let $E^k f = J\mu_k$, $k > 0$. We proceed to prove that $E^{k-1} f = J\mu_{k-1}$ for some $\mu_{k-1} \in \pi_d(\mathrm{SO}(n+k))$. Consider the diagram

$$\begin{array}{ccccccc}
 & & \pi_{d+n+k}(S_{n+k}) & \xrightarrow{E} & \pi_{d+n+k+1}(S_{n+k+1}) & & \\
 & \overset{P}{\nearrow} & \uparrow J & & \uparrow J & \overset{H'}{\searrow} & \\
\pi_{d+1}(S_{n+k}) & & & & & & \pi_d(S_{n+k}) \\
 & \underset{\partial}{\searrow} & & & & \underset{\phi_*}{\nearrow} & \\
 & & \pi_d(\mathrm{SO}(n+k)) & \xrightarrow{i_*} & \pi_d(\mathrm{SO}(n+k+1)) & &
\end{array}$$

(Notations of [4, § 4]). Since $H'E^k f = 0$ $(k > 0)$, we have $\phi_*\mu_k = 0$ by $H'J\mu_k = \phi_*\mu_k$ (see [4], Formula (4.3)). Thus by exactness of the homotopy sequence of $\mathrm{SO}(n+k+1)/\mathrm{SO}(n+k) = S_{n+k}$, we have $\mu_k = i_*\mu'_{k-1}$ for some $\mu'_{k-1} \in \pi_d(\mathrm{SO}(n+k))$. Since $Ej = -Ji_*$ ([13, Formula (2.1)]),

$$E(E^{k-1}f + J\mu'_{k-1}) = E^k f - Ji_*\mu'_{k-1} = 0 .$$

By exactness of the G. Whitehead's sequence (upper sequence of the diagram), there exists an element $\alpha \in \pi_{d+1}(S_{n+k})$ such that $P\alpha = E^{k-1}f + J\mu'_{k-1}$. Since $P = J\partial$ ([9, Theorem (3.2)]), it follows $E^{k-1} = J(\partial\alpha - \mu'_{k-1}) = J\mu_{k-1}$. setting $\mu_{k-1} = \partial\alpha - \mu'_{k-1}$.

The above argument is valid as long as the G. Whitehead's sequence exists and $H'J = \phi_*$. This is guaranteed (for every $k>0$) by the restriction $d < 2n$.

THEOREM 8.2. Let *$p : S_d \to E_{d+n}$ be a regular imbedding (without self-intersection) of the d-dimensional sphere into euclidean $(d+n)$-space. Assume $d < 2n - 1$, then the normal $\mathrm{SO}(n)$-bundle over S_d induced by p is trivial.*

PROOF. We proceed by decreasing induction on n. For large n $(n>d+1)$ the theorem is a trivial consequence of the fact that p is isotopic in E_{d+n} to the standard imbedding (the theorem was also known for $n=d$ and $n=d+1$, see [6]).

Let $p: S_d \rightarrow E_{d+n}, n \leqq d \leqq 2n-2$ be a regular imbedding and regard E_{d+n} as subspace of E_{d+n+1}. Assume by induction hypothesis, that the normal bundle over S_d induced by the imbedding into E_{d+n+1} is trivial and let $\mathbf{F}_{n+1}=\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be some field of $(n+1)$-frames orthogonal to $p(S_d)$ in E_{d+n+1}. Let $\mathbf{t}$ be the normal to E_{d+n} in E_{d+n+1} and let ν be the homotopy class of the map $N: S_d \rightarrow S_n$ given by $N(x)=(\mathbf{t}\cdot\mathbf{v}_0(x), \mathbf{t}\cdot\mathbf{v}_1(x), \cdots, \mathbf{t}\cdot\mathbf{v}_n(x))$. Denoting by $f: S_{d+n+1} \rightarrow S_{n+1}$ the map associated with $(p(S_d); \mathbf{F}_{n+1})$, we have by Lemma 6.1 and Theorem 7.1: $H(f)=(-1)^{(n+1)(d+1)}E^{n+1}\nu$. Since by Lemma 8.1, $f=J\mu$ for some $\mu \in \pi_d(\mathrm{SO}(n+1))$, we have $E^{n+1}\nu = \pm E^{n+1}H'J\mu = \pm E^{n+1}\phi_*\mu$. Now for $d<2n-1$, $E: \pi_d(S_n) \rightarrow \pi_{d+1}(S_{n+1})$ is an isomorphism and therefore $\nu=\phi_*\mu' (\mu'=\pm\mu)$. Geometrically, this means that $p(S_d)$ admits a field of normal n-frames in E_{d+n}.

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