SIMPLE HOMOTOPY TYPES.*

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1. Introduction. This is a sequel to two papers [1] entitled "Combinatorial Homotopy," Parts (I) and (II). It deals with what I have previously called the "nucleus," but which will now be called the *simple homotopy type* of a complex. It is closely related to parts of [1] and [3] but the treatment is so different that we shall start again from the beginning.

Let $\{K\}$ be the class of all (cell) complexes,[2] as defined in CH (I), which are of the same homotopy type as a given complex K. Let $K' \equiv K$ (i. e. $K' \varepsilon \{K\}$) and let $\bar{\phi}: K \equiv K'$ be the class of maps which are homotopic to a given homotopy equivalence, $\phi: K \equiv K'$. If $\phi': K' \equiv K''$, we define $\bar{\phi}'\bar{\phi}$ by

$$\bar{\phi}'\bar{\phi} = \overline{\phi'\phi}: K \equiv K''.$$

It is easily verified that the classes $\bar{\phi}$, with this multiplication, form a groupoid,[3] G, whose unit elements are the classes $\bar{1}: K' \equiv K'$, for every $K' \varepsilon \{K\}$, where $1: K' \rightarrow K'$ is the identical map. Our plan is to analyse this groupoid in algebraic terms.

First consider the group, $G_K \subset G$, which consists of the classes $\bar{\phi}: K \equiv K$. We define an additive Abelian group, $\mathbf{T}$, which depends only on $\pi_1(K)$. The group $\mathbf{T}$ admits G_K as a group of operators and we shall define a crossed homomorphism $\tau: G_K \rightarrow \mathbf{T}$. We call $\tau(g)$ the torsion of a given element $g \varepsilon G_K$. If $\phi: K \equiv K'$, where $K' \neq K$, we define a class of elements $\tau(\phi) \subset \mathbf{T}$, which we call the *torsion* of ϕ. We describe ϕ as a *simple* (*homotopy*) *equivalence* if, and only if, $\tau(\phi) = 0$. We say that K and K' are of the same *simple homotopy type*, and shall write $K \equiv K' \, (\Sigma)$, if, and only if, there is a simple equivalence $\phi: K \equiv K'$. It will follow from the

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[1] *Bulletin of the American Mathematical Society*, vol. 55 (1949), pp. 213-45 and 453-96. These papers will be referred to as CH (I) and CH (II).

[2] Until the final section we assume that any given complex is finite and connected. We also assume that the points in our complexes are taken from some aggregate, σ, which is given in advance. The power of σ shall exceed that of the continuum, so that it is not exhausted by any one (finite) complex, and the points in Hilbert space shall be included in σ.

[3] See [6], p. 132.

definition of $\tau(\phi)$ that $K \equiv K'$ (Σ) is an equivalence relation. We then prove that $K \equiv K'$ (Σ) if, and only if, K can be transformed into K' by a "formal deformation," which is defined in much the same way as in [1]. Thus the elementary transformations, or "moves," do not appear in the definition of simple equivalence but in a theorem which is analogous to Tietze's theorem [4] on discrete groups. Similarly it is proved that two complexes are of the same n-type if, and only if, they can be interchanged by elementary transformations of the sort used in [1] to define the "n-group."

It was proved in [3] that the Reidemeister-Franz torsion,[5] when defined, is an invariant of the simple homotopy type. Using this fact, examples were given of complexes, which are of the same homotopy type but not of the same simple homotopy type. However, if $\mathrm{T} = 0$, then $K \equiv K'$ (Σ) if $K \equiv K'$. It will be obvious that this is so if $\pi_1(K) = 1$. It follows from Theorems 14, 15 in [11] that $\mathrm{T} = 0$ if $\pi_1(K)$ is of order 2, 3, 4 or cyclic infinite.

It is an open question whether or not the simple homotopy type is a topological invariant. However we shall prove that it is a combinatorial invariant in the following sense. If K' is a sub-division of K, then the identical map $K \to K'$ is a simple equivalence.[6] Any differentiable manifold has a "preferred" class of triangulations,[7] any two of which are combinatorially equivalent in the sense of Newman. Also any analytic variety has a preferred class of triangulations,[7] any two of which have a common sub-division. Therefore the simple homotopy type has an invariant status in differential and alegbraic geometry and in the study of analytic varieties.

2. The group T. Let R be a ring with a unit element 1. Eventually R will be the group ring [8] of $\pi_1(K)$ but here we only assume that, if A is a free R-module of (finite) rank n, then any free R-module, which is isomorphic [9] to A, also has rank n. This condition is equivalent to the

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[4] See [7], p. 46.

[5] See [8], [9] and p. 1209 of [3]. In Section **12** below it is shown, in the case of Lens space, how this is related to our torsion. See also [10].

[6] This may turn out to be a wider definition, even for simplicial complexes, than the one based on Newman's "moves," or on recti-linear sub-divisions (see [12], [13]). For example, we do not enquire whether or not the vertex scheme of a given "curvilinear" triangulation of an n-simplex is a formal n-element, as defined by Newman.

[7] See [2] and [14].

[8] By the group ring of a group, Γ, we shall always mean the integral group ring, in which the additive group is the ordinary free Abelian group, which is freely generated by the elements of Γ.

[9] A module will always mean a free R-module and, unless the contrary is stated, a homomorphism will always mean an operator homomorphism.

condition that every regular R-matrix (i. e. one with elements in R and a 2-sided inverse) is square. Hence it is satisfied if there is a homomorphism, other than $R \to 0$, of R into a division ring, D. For such a homomorphism carries a regular R-matrix into a regular D-matrix, which is necessarily square. If R is the group ring of a group, Γ, then $\Gamma \to 1$ defines a homomorphism of R into the rational field. Therefore the condition of rank invariance is satisfied.

Let M be the module, of infinite rank, whose elements are the infinite sequences $(r_1, r_2, \cdots)$ $(r_i \varepsilon R)$, in which all but a finite number of $r_1, r_2, \cdots$ are zero. The elements in R will operate on M from the left.[10] Thus an operator, $r \varepsilon R$, transforms m into rm, where

$$m = (r_1, r_2, \cdots), \qquad rm = (rr_1, rr_2, \cdots).$$

Let $m_i \varepsilon M$ be the basis element which is given by $r_i = 1$, $r_j = 0$ if $j \neq i$. Let $M^n \subset M$ be the module generated[11] by $(m_1, \cdots, m_n)$ and M_n the one generated by $(m_{n+1}, m_{n+2}, \cdots)$, where $n \geq 0$ and $M^0 = 0$. Then M is the direct sum $M = M^n + M_n$ and a given element in M is in M^n for some value of n. We shall describe an endomorphism, $f: M \to M$, as *admissible* if, and only if, $fm_i = m_i$ for all sufficiently large values of i. If $f, g: M \to M$ are admissible endomorphisms[12] so, obviously, is $fg: M \to M$ and if $f: M \to M$ is an admissible automorphism so is f^{-1}. Therefore the admissible automorphims form a group, $\mathfrak{A}$.

Let $f: M \to M$ be an (admissible) endomorphism and let $fm_j = m_j$ if $j > p$. Let n_i be such that $fm_i \varepsilon M^{n_i}$ $(i = 1, \cdots, p)$ and let $n \geq \operatorname{Max}(n_i, p)$. Then $fm_i \varepsilon M^n$ for $i = 1, \cdots, n$ and $fm_j = m_j$ if $j > n$. Therefore $fM^n \subset M^n$, $fm = m$ if $m \varepsilon M_n$. We shall write $f = (f)^n: M \to M$, and f^n will denote the endomorphism, $f^n: M^n \to M^n$, which is induced by f. That is to say, $f^n m = fm$ if $m \varepsilon M^n$. Notice that $(f)^n = (f)^q$ if $q > n$. Therefore, if $f_i: M \to M$ is any finite set of endomorphisms, we may take $f_i = (f_i)^n$, for any value of n which is sufficiently large to be the same for each i. Notice also that any endomorphism $f': M^n \to M^n$ can be extended to a unique endomorphism, $(f)^n: M \to M$, such that $f^n = f'$. Obviously f^n is an automorphism if, and only if, $f \varepsilon \mathfrak{A}$.

Let $f = (f)^n$ be given by

$$fm_i = \sum_{j=1}^{\infty} f_{ij} m_j \qquad (f_{ij} \varepsilon R). \tag{2.1}$$

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[10] This has the disadvantage indicated by (2.3) below. But the convention $m \to mr$ would be inconvenient in the geometrical application.

[11] I. e. generated with the help of the operators in R.

[12] Unless the contrary is stated it is to be assumed that any given endomorphism of M is admissible.

Then the matrix $\mathbf{f} = [f_{ij}]$ is of the form

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}^n & 0 \\ 0 & \mathbf{1}_\infty \end{bmatrix} \tag{2.2}$$

where $\mathbf{f}^n$ is the matrix of $f^n : M^n \to M^n$ and $\mathbf{1}_\infty$ is the infinite unit matrix. Let $g : M \to M$ be given by

$$gm_i = \sum_j g_{ij} m_j.$$

Since $fr = rf$, where $r \varepsilon R$ is any operator, we have

$$fgm_i = \sum_j g_{ij} fm_j = \sum_{j,k} g_{ij} f_{jk} m_k. \tag{2.3}$$

Therefore $fg : M \to M$ corresponds to the matrix $\mathbf{gf}$.

Let $g : M \to M$ be given by

$$gm_i = m_i + rm_j, \qquad gm_k = m_k \qquad (j, k \neq i; r \varepsilon R). \tag{2.4}$$

Then g has an inverse, which is given by (2. 4), with r replaced by $-r$. It is therefore an (admissible) automorphism. Let $\Sigma_1 \subset \mathfrak{A}$ be the group generated by all such automorphisms, for all values of i, j, r.

Let A and B be the modules generated by disjoint sub-sets, $m_{i_1}, \cdots, m_{i_p}$ and $m_{j_1}, \cdots, m_{j_q}$, of the basis elements $m_1, m_2, \cdots,$. Let $h : A \to B$ be an arbitrary homomorphism and let $g : M \to M$ be given by

$$g(a + b) = a + (ha + b), \qquad gm_l = m_l, \tag{2.5}$$

where $a \varepsilon A$, $b \varepsilon B$ and $l \neq i_\rho$ or j_σ. Then g is the resultant of the homomorphisms

$$m_{i_\rho} \to m_{i_\rho} + h_{\rho\sigma} m_{j_\sigma}, \qquad m_k \to m_k \qquad (k \neq i_\rho),$$

where $hm_{i_\rho} = h_{\rho 1} m_{j_1} + \cdots + h_{\rho q} m_{j_q}$. These are of the form (2. 4), whence $g \varepsilon \Sigma_1$.

THEOREM 1. *Σ_1 is an invariant sub-group of $\mathfrak{A}$ and $\mathfrak{A}/\Sigma_1$ is Abelian.*

Let $f, f' \subset \mathfrak{A}$. We shall write $f \equiv f'$ if, and only if, $f = gf'g'$, where $g, g' \subset \Sigma_1$. This is obviously an equivalence relation. Assume that $ff' \equiv f'f$ for every pair $f, f' \subset \mathfrak{A}$. Let $g \varepsilon \Sigma_1$, and let $f' = gf^{-1}$. Then

$$fgf^{-1} \equiv gf^{-1}f = g.$$

Therefore $fgf^{-1} \varepsilon \Sigma_1$, whence Σ_1 is invariant in $\mathfrak{A}$. Also $\mathfrak{A}/\Sigma_1$ is Abelian since $ff' \equiv f'f$ for every pair $f, f' \subset \mathfrak{A}$.

We proceed to prove that $ff' \equiv f'f$. Let $A = M^p$, let B be the module generated by $m_{p+1}, \cdots, m_{2p}$ and let g be given by (2.5). Then $g = (g)^{2p}$ and

$$g^{2p} = \begin{bmatrix} \mathbf{1}_p & \boldsymbol{h} \\ 0 & \mathbf{1}_p \end{bmatrix},$$

where $\boldsymbol{h} = [h_{\rho\sigma}]$ and $\mathbf{1}_p$ is the unit matrix of order p. Let $f = (f)^{2p} \varepsilon \mathfrak{A}$ and let

$$\boldsymbol{f}^{2p} = \begin{bmatrix} \boldsymbol{f}_{11} & \boldsymbol{f}_{12} \\ \boldsymbol{f}_{21} & \boldsymbol{f}_{22} \end{bmatrix},$$

where $\boldsymbol{f}_{11}$, $\boldsymbol{f}_{22}$ are square matrices of order p. Let $\boldsymbol{f}'^{2p}$ and $\boldsymbol{f}'_{\lambda\mu}$ $(\lambda, \mu = 1, 2)$ be similarly defined in terms of $f' = (f')^{2p}$. We shall write $\boldsymbol{f}^{2p} \equiv \boldsymbol{f}'^{2p}$ if, and only if, $f \equiv f'$. Then

$$\boldsymbol{f}^{2p} \equiv \boldsymbol{f}^{2p}\boldsymbol{g}^{2p} = \begin{bmatrix} \boldsymbol{f}_{11} & \boldsymbol{f}_{11}\boldsymbol{h} + \boldsymbol{f}_{12} \\ \boldsymbol{f}_{21} & \boldsymbol{f}_{21}\boldsymbol{h} + \boldsymbol{f}_{22} \end{bmatrix}. \tag{2.6}$$

Similarly a right hand multiple of the second column may be added to the first. Also $\boldsymbol{f}^{2p} \equiv \boldsymbol{g}^{2p}\boldsymbol{f}^{2p}$, and $\boldsymbol{g}^{2p}\boldsymbol{f}^{2p}$ is obtained from $\boldsymbol{f}^{2p}$ by a similar operation on the rows.

Let $f, f' \subset \mathfrak{A}$ be given and let p be so large that

$$f = (f)^{p} = (f)^{2p}, \qquad f' = (f')^{p} = (f')^{2p}.$$

Let $\boldsymbol{r} = \boldsymbol{f}^{p}$, $\boldsymbol{r}' = \boldsymbol{f}'^{p}$. Then $\boldsymbol{r}$, $\boldsymbol{r}'$ are regular matrices. Therefore, beginning with (2.6), with $\boldsymbol{h} = \boldsymbol{r}^{-1}$ and $\boldsymbol{f}$ replaced by $\boldsymbol{f}'\boldsymbol{f}$, we have

$$\begin{aligned} \boldsymbol{f}'^{2p}\boldsymbol{f}^{2p} &= \begin{bmatrix} \boldsymbol{r}'\boldsymbol{r} & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{r}'\boldsymbol{r} & \boldsymbol{r}' \\ 0 & \mathbf{1}_p \end{bmatrix} \equiv \begin{bmatrix} 0 & \boldsymbol{r}' \\ -\boldsymbol{r} & \mathbf{1}_p \end{bmatrix} \\ &\equiv \begin{bmatrix} 0 & \boldsymbol{r}' \\ -\boldsymbol{r} & 0 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{r} & \boldsymbol{r}' \\ -\boldsymbol{r} & 0 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{r} & \boldsymbol{r}' \\ 0 & \boldsymbol{r}' \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{r} & 0 \\ 0 & \boldsymbol{r}' \end{bmatrix}. \end{aligned}$$

Similarly

$$\begin{bmatrix} 0 & \boldsymbol{r}' \\ -\boldsymbol{r} & 0 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{r}' & 0 \\ 0 & \boldsymbol{r} \end{bmatrix} \equiv \boldsymbol{f}^{2p}\boldsymbol{f}'^{2p}.$$

Therefore $\boldsymbol{f}\boldsymbol{f}' \equiv \boldsymbol{f}'\boldsymbol{f}$ and the theorem is proved.

Since $\mathfrak{A}/\Sigma_1$ is Abelian it follows that $\mathfrak{A}^c \subset \Sigma_1$, where $\mathfrak{A}^c$ is the commutator sub-group of $\mathfrak{A}$. Therefore we have the corollary:

COROLLARY. *If $\Sigma \subset \mathfrak{A}$ is any sub-group, which contains Σ_1, then Σ is invariant and $\mathfrak{A}/\Sigma$ is Abelian.*

The totality of automorphisms $(f)^n \varepsilon \mathfrak{A}$, for a fixed value of n, is obviously a sub-group, $(\mathfrak{A})^n \subset \mathfrak{A}$. It follows from Theorem 1 that

$(\Sigma_1)^n = \Sigma_1 \cap (\mathfrak{A})^n$ is an invariant sub-group [13] of $(\mathfrak{A})^n$ and that $(\mathfrak{A})^n/(\Sigma_1)^n$ is Abelian. Let $\mathfrak{A}^n$ be the group of (operator) automorphisms, $f^n: M^n \to M^n$, and let $\phi: (\mathfrak{A})^n \to \mathfrak{A}^n$ be given by $\phi(f)^n = f^n$. Then ϕ is obviously an isomorphism.[14] It follows from the invariance of $(\Sigma_1)^n$ in $(\mathfrak{A})^n$ that $\Sigma_1{}^n = \phi(\Sigma_1)^n$ is invariant in $\mathfrak{A}^n$ and that $\Sigma_1{}^n$ is independent of the particular isomorphism $\phi: (\mathfrak{A})^n \approx \mathfrak{A}^n$. Also $\mathfrak{A}^n/\Sigma_1{}^n$ is Abelian.

Let Λ be a sub-group of the multiplicative group of regular elements in R (that is, elements with two-sided inverses), which contains both ± 1. Let $g: M \to M$ be given by

$$(2.7) \qquad gm_i = \lambda m_i + rm_j, \qquad gm_k = m_k \qquad (j, k \neq i),$$

where $\lambda \in \Lambda$, $r \in R$. Then $g \in \mathfrak{A}$ and g^{-1} is given by (2.7) with λ, r replaced by λ^{-1}, $-\lambda^{-1}r$. Let Σ_Λ be the sub-group of $\mathfrak{A}$, which is generated by all automorphisms of the form (2.7), for every choice of i, j, λ and r. Clearly $\Sigma_1 \subset \Sigma_\Lambda$. Therefore Σ_Λ is invariant and $\mathrm{T} = \mathfrak{A}/\Sigma_\Lambda$ is Abelian. We shall keep Λ fixed and shall write Σ, T for Σ_Λ, T_Λ. The elements of Σ will be called *simple automorphisms*. We shall write T additively and $\tau(f) \in \mathrm{T}$ will denote the co-set containing a given $f \in \mathfrak{A}$.

Our "torsion" will be defined in terms of T. An element of torsion will correspond to an isomorphism of one module, of finite rank, onto another. In order to classify such isomorphisms in term of T we need a standard class, which have "zero torsion." We therefore proceed to define a class of "basic modules" in M, which are related by a standard class of automorphisms, called permutations.

By a *basic module*, $A \subset M$, we shall mean the one generated by $m_{i_1}, \cdots, m_{i_p}$, for any (distinct) values of $i_1, \cdots, i_p$. We shall call $(m_{i_1}, \cdots, m_{i_p})$ *the* basis of A. We allow $p = 0$, in which case the set $(m_{i_1}, \cdots, m_{i_p})$ is empty and $A = M^0$. Let $p \geq 0$ and let M_A be the module generated by the remaining basic elements, $m_j \neq m_{i_\rho}$, of M. Then M is the direct sum $M = A + M_A$. Let B be a basic module and let $(m_{j_1}, \cdots, m_{j_q})$ be its basis. We shall only allow ourselves to form the direct sum $A + B = B + A$, if $A \cap B = 0$. In this case $A + B$ will be the basic module, whose basis is

$$(m_{i_1}, \cdots, m_{i_p}, m_{j_1}, \cdots, m_{j_q}),$$

[13] The example II, on p. 1233 of [3] shows that $(\Sigma_1)^n$ may be a larger group than the one which is generated by transformations of the form (2.4), with $i, j \leq n$. I see no reason to suppose that the latter is necessarily an invariant sub-group of $(a)^n$.

[14] An isomorphism, without qualification, will always mean an isomorphism onto.

not the set of all pairs (a, b), with $a \varepsilon A$, $b \varepsilon B$. Let C be a given basic module. Then $C = A + B$ will always mean that A, B are basic modules, with disjoint bases, of which C is the direct sum.

Let $i_1, \cdots, i_n$ be any permutation of $1, \cdots, n$, for any $n \geq 1$. Let $P: M \to M$ be the automorphism, which is given by

$$Pm_j = m_{i_j}, \qquad Pm_k = m_k \qquad (j = 1, \cdots, n; k > n).$$

We shall call P a *permutation*. It follows from (2. 7), with $\lambda, r = \pm 1$, that the transformations

$$(m_i, m_j) \to (-m_i + m_j, m_j) \to (-m_i + m_j, m_i) \to (m_j, m_i)$$

determine simple automorphisms. Therefore $P \varepsilon \Sigma$. Let A, B be basic modules of the same rank and let n be so large that the bases of A, B are both contained in M^n. Then there is obviously a permutation, $P = (P)^n$, such that $PA = B$. The totality of permutations is obviously a sub-group of $\mathcal{A}$.

Let $\alpha: A \approx A'$, where A, A' are basic modules. Since A and A' have the same rank, accoording to our condition on R, there is a permutation, P, such that $PA' = A$. Let $f: M \to M$ be given by

$$(2.8) \qquad f(a + m) = P\alpha a + m \qquad (a \varepsilon A, m \varepsilon M_A),$$

and let $\tau(\alpha) = \tau(f)$. Let P' be any other permutation such that $P'A' = A$ and let f' be defined by (2. 8), with P replaced by P'. Since $P'P^{-1}A = A$ the permutation $P'P^{-1}$ permutes the basis elements of A among themselves. Therefore $P'': M \to M$, given by

$$P''(a + m) = P'P^{-1}a + m \qquad (m \varepsilon M_A),$$

is a permutation. Since $P\alpha A = A$, it follows from (2. 8) that $f' = P''f$. Therefore

$$\tau(f') = \tau(P'') + \tau(f) = \tau(f).$$

Therefore $\tau(\alpha)$ does not depend on the choice of P. We shall describe α as a *simple isomorphism*, and shall write $\alpha: A \approx A'$ (Σ), if, and only if, $\tau(\alpha) = 0$. It follows from (2. 8) that $\tau(\alpha) = 0$ if $\alpha = 1: A \approx A$. In particular $\tau(\alpha) = 0$ if $A = A' = M^0$.

Let α, P, f, mean the same as in (2. 8), let $\alpha': A' \approx A''$ and let P' be a permutation such that $P'A'' = A'$. Then $\tau(\alpha') = \tau(f')$, $\tau(\alpha'\alpha) = \tau(f'')$, where f' and f'' are given by (2. 8) with α, P replaced by α', P' and by $\alpha'\alpha$, PP'. Clearly $P^{-1}M_A = M_{A'}$. Therefore

$$Pf'P^{-1}f(a+m) = Pf'P^{-1}(P\alpha a + m) = Pf'(\alpha a + P^{-1}m)$$
$$= P(P'\alpha'\alpha a + P^{-1}m) = PP'\alpha'\alpha a + m = f''(a+m).$$

Therefore

$$\tau(\alpha'\alpha) = \tau(Pf'P^{-1}f) = \tau(f') + \tau(f) = \tau(\alpha') + \tau(\alpha).$$

Since $\tau(1) = 0$ it follows that $\tau(\alpha^{-1}) = -\tau(\alpha)$.

Let $\alpha: A \approx A'$ be the isomorphism induced by a permutation, $P': M \to M$, such that $P'A = A'$. Then f, given by (2.8), is a permutation. Therefore $\tau(\alpha) = 0$.

Let A, B and A', B' be two pairs of basic modules such that $A \cap B = A' \cap B' = 0$. Let $\gamma: A + B \to A' + B'$ be a homomorphism such that $\gamma B \subset B'$. Then $\gamma(a+b) = \alpha a + (ha + \beta b)$, where $\alpha a \varepsilon A', ha, \beta b \subset B'$. It is easily verified that $\alpha: A \to A'$, $h: A \to B'$, $\beta: B \to B'$ are homomorphisms.

THEOREM 2. *If either:*

(i) *γ is an isomorphism*[14] *and either α is an isomorphism into or β is onto, or if*

(ii) *α, β are isomorphisms, then α, β, γ are all isomorphisms and*

$$\tau(\gamma) = \tau(\alpha) + \tau(\beta).$$

Let $\gamma: A + B \approx A' + B'$. If $\beta b = 0$, then $\gamma b = \beta b = 0$, whence $b = 0$. Therefore β is an isomorphism into. Let $a' \varepsilon A'$ be given. Then $\alpha a + (ha + \beta b) = \gamma(a+b) = a'$ for some $a \varepsilon A$, $b \varepsilon B$. Since $ha + \beta b \varepsilon B'$ we have $\alpha a = a'$. Therefore α is onto. Let $\alpha: A \approx A'$ and let $b' \varepsilon B'$ be given. Then $\alpha a + (ha + \beta b) = b'$ for some $a \varepsilon A$, $b \varepsilon B$. Since $\alpha a \varepsilon A'$ we have $\alpha a = 0$. Therefore $a = 0$, $ha = 0$ and $\beta b = b'$. Therefore β is onto. Let $\beta: B \approx B'$ and let $\alpha a = 0$. Then $\gamma(a - \beta^{-1}ha) = \alpha a + (ha - ha) = 0$. Therefore $a - \beta^{-1}ha = 0$. Since $\beta^{-1}ha \varepsilon B$ it follows that $a = 0$. Therefore $\alpha: A \approx A'$. Thus α, β, γ are isomorphisms if (i) is satisfied.

Let α, β be isomorphisms. Then $\gamma = \gamma^*\delta$, where $\gamma^*: A + B \to A' + B'$, $\delta: A + B \to A + B$ are given by $\gamma^*(a+b) = \alpha a + \beta b$, $\delta(a+b) = a + (\beta^{-1}ha + b)$. Obviously γ^*, δ are isomorphisms and so therefore is γ. Moreover $g: M \to M$ is of the form (2.5), where

$$g(a+b+m) = \delta(a+b) + m \qquad (m \varepsilon M_{A+B}).$$

Let P be a permutation such that $PA' = A$, $PB' = B$. Let $f = f_\alpha$ be defined by (2.8) and let f_β, f_γ, f_{γ^*} be similarly defined in terms of β, γ, γ^* and the same permutation P. Then $f_\alpha b = b$, $f_\beta a = a$ and

$$f_\gamma(a+b) = P\alpha a + P(ha + \beta b) = f_\alpha + f_\beta(\beta^{-1}ha + b) = f_\alpha f_\beta g(a+b).$$

Since $g \in \Sigma$ it follows that

$$\tau(\gamma) = \tau(f_\gamma) = \tau(f_\alpha) + \tau(f_\beta) + \tau(g) = \tau(\alpha) + \tau(\beta)$$

and the proof is complete.

COROLLARY. *If any two of α, β, γ are simple isomorphisms, so is the third.*

Let $\theta: R \approx R$ be an automorphism of R and let $s_\theta: M \to M$ be the transformation which is given by

$$s_\theta(r_1, r_2, \cdots) = (\theta r_1, \theta r_2, \cdots). \tag{2.9}$$

Obviously $s_{\theta^{-1}} = s_\theta^{-1}$ and $s_\theta s_\phi = s_{\theta\phi}$, where $\phi: R \approx R$. Also $s_\theta(rm) = (\theta r)s_\theta m$, where $r \in R$, $m \in M$. Hence it follows that, if $f: M \to M$ is an (operator) endomorphism, then $(s_\theta f s_\phi)rm = (\theta\phi r)(s_\theta f s_\phi)m$. Therefore $f^\theta r = rf^\theta$ where $f^\theta = s_\theta f s_\theta^{-1}$. Since $s_\theta m_i = m_i$ it follows that $f^\theta \in \mathcal{A}$ if $f \in \mathcal{A}$. Let $g: M \to M$ be given by (2.7). Since $s_\theta m = m_i$ we have $g^\theta m_i = (\theta\lambda)m_i + m_j$, $g^\theta m_k = m_k$. Therefore g^θ is also of the form (2.7) if $\theta\lambda \in \Lambda$. Clearly $(g_1 g_2)^\theta = g_1^\theta g_2^\theta$ and it follows that $f^\theta \in \Sigma$ if $f \in \Sigma$, provided $\theta\Lambda \subset \Lambda$.

We shall describe $\theta: R \approx R$ as a Λ-*automorphism* if, and only if, $\theta\Lambda = \Lambda$. The totality of Λ-automorphisms is obviously a group Θ. Since $f^\theta \in \Sigma$ if $f \in \Sigma$ and $\theta \in \Theta$ it follows that T admits Θ as a group of operators, according to the rule

$$\theta\tau(f) = \tau(f^\theta). \tag{2.10}$$

Let $x \in R$ be any regular element, not necessarily an element of Λ, and let $\theta_x r = xrx^{-1}$. I say that

$$\theta_x \tau = \tau \tag{2.11}$$

for each $\tau \in \mathrm{T}$. For let $f \in \mathcal{A}$ be given by (2.1). Then

$$f^{\theta_x} m_i = s_{\theta_x} \Sigma_j f_{ij} m_j = \Sigma_j (x f_{ij} x^{-1}) m_j.$$

Let $f = (f)^n$ and let $g_x = (g_x)^n: M \to M$ be given by

$$g_x(r_1, \cdots, r_n, r_{n+1}, \cdots) = (r_1 x, \cdots, r_n x, r_{n+1}, \cdots).$$

Then $g_x m_i = x m_i$ if $i \leq n$. Since $fxm = xfm$,

$$f g_x m_i = x f m_i = \sum_{j=1}^{n} x f_{ij} m_j = \sum_{j=1}^{n} (x f_{ij} x^{-1}) x m_j = g_x f^{\theta_x} m_i \quad (i = 1, \cdots, n).$$

Therefore $f^{\theta_x} = g_x^{-1} f g_x$ and $\tau(f^{\theta_x}) = -\tau(g_x) + \tau(f) + \tau(g_x) = \tau(f)$, which proves (2.11).

Let $f \varepsilon \mathfrak{A}$ and let $\boldsymbol{f}$ and $\boldsymbol{f}^n$ mean the same as in (2.2). Let g be given by (2.7) and let $\boldsymbol{g}$ be its matrix. Then $\boldsymbol{gf}$ is obtained from $\boldsymbol{f}$ by the following operations

(2.12) a) *multiplying a row from the left by an element* $\lambda \varepsilon \Lambda$,

b) *adding a left multiple of one row to another, the multiplier being an arbitrary element* $r \varepsilon R$.

Therefore $f \varepsilon \Sigma$ if, and only if, $\boldsymbol{f} \to \mathbf{1}_\infty$ by a finite sequence of such transformations. Let $\boldsymbol{f} \to \mathbf{1}_\infty$ by such a sequence, $\sigma_1, \cdots, \sigma_p$, and let $f = (f)^n$. Then there is a $k \geq 0$ such that no row of $\boldsymbol{f}$, after the $(n+k)$-th is involved in any of $\sigma_1, \cdots, \sigma_p$. Therefore $\sigma_1, \cdots, \sigma_p$ transform[13] $\boldsymbol{f}^{n+k}$ into $\mathbf{1}_{n+k}$, where

$$\boldsymbol{f}^{n+k} = \begin{bmatrix} \boldsymbol{f}^n & 0 \\ 0 & \mathbf{1}_k \end{bmatrix}$$

Let R be the group ring of a group Γ and let Λ consist of the elements $\pm \gamma$, where $\gamma \varepsilon \Gamma$. If Γ is Abelian, the determinant, $|\boldsymbol{f}^n|$, of $\boldsymbol{f}^n$ can be calculated in the ordinary way. Obviously $|\boldsymbol{f}^n|$ is unaltered by (2.12b) or by an "expansion," $\boldsymbol{f}^n \to \boldsymbol{f}^{n+k}$, and a transformation of the form (2.12a) changes $|\boldsymbol{f}^n|$ into $\pm \gamma |\boldsymbol{f}^n|$. Therefore $\pm |\boldsymbol{f}^n| \varepsilon \Gamma$ if $f \varepsilon \Sigma$. Let Γ be cyclic of order 5 and let $\gamma \neq 1$. Then $(1 - \gamma - \gamma^4)(1 - \gamma^2 - \gamma^3) = 1$. Therefore $f: M \to M$, given by $f(r_1, r_2, r_3, \cdots) = \{r_1(1 - \gamma - \gamma^4), r_2, r_3, \cdots\}$ is in $\mathfrak{A}$, but not in Σ. Therefore $\mathrm{T} \neq 0$. On the other hand it follows from the theory of integral, unimodular matrices, in case $\Gamma = 1$, and from Theorems 14, 15 in [11], that $\mathrm{T} = 0$ if Γ is of order 1, 2, 3, 4 or is cyclic infinite.

We continue, until Section **9**, without the assumption that R is a group ring.

3. Chain systems. By a *chain system*, $C = \{C_n\}$, we shall mean a family of basic modules, $C_n \subset M$, together with a boundary operator, $\partial = \{\partial_n\}$, which is a family of (operator) homomorphisms, $\partial_n : C_n \to C_{n-1}$, such that $\partial_n \partial_{n+1} = 0$. For the sake of completeness we define $\partial_0 C_0 = C_{-1} = 0$. Each C_n, being a basic module, is of finite rank. We do not require C_0 to be of rank 1, as we did in section **8** of CH(II). For example, we allow $C_0 = 0$. We assume that $C_n = 0$ for all sufficiently large values of n. If $C_n = 0$ when $n > N \geq 0$, but $C_N \neq 0$, we write $N = \dim C$. We write $C = 0$, and dim $C = -1$, if $C_n = 0$ for every $n \geq 0$. We insist that $C_p \cap C_q = 0$ if $p \neq q$ and C shall be the set-theoretic union of the groups $C_0, C_1, \cdots$. Thus $c \varepsilon C$ means that $c \varepsilon C_n$ and $c + c'$ is only defined if $c, c' \subset C_n$, for some $n \geq 0$. Also ∂ is a map, $\partial : C \to C$, of the set C into itself.

Until Section **9** we shall only consider chain mappings,[15] $f: C \to C'$, of C into a chain system, $C' = \{C'_n\}$, such that each $f: C_n \to C'_n$ is an operator homomorphism. That is to say, in the terminology of CH (II), f is associated with the identical isomorphism, $R \to R$. Thus [16] $\partial f = f \partial$, $fr = rf$, where $r \varepsilon R$ is an operator. Also $f \simeq g: C \to C'$ will mean that

$$g_n - f_n = \partial_{n+1}\eta_{n+1} + \eta_n \partial_n \qquad (n \geqq 0), \tag{3.1}$$

where $\eta = \{\eta_n\}$ is a chain deformation operator, and $f: C \equiv . C'$ will mean that there is a chain mapping, $f': C' \to C$, such that $f'f \simeq 1$, $ff' \simeq 1$ in the sense of (3.1). We shall call $f: C \to C'$ a simple isomorphism, and shall write $f: C \approx C'$ (Σ), if, and only if, it is a chain mapping such that $f_n: C_n \approx C'_n$ (Σ), for each $n \geq 0$.

Let B, C be given chain systems and let $B_n = B'_n + B''_n$, $C_n = C'_n + C''_n$, where, according to our convention, B'_n, B''_n, C'_n, C''_n are basic modules. Let $f: B \to C$ be a chain mapping such that

$$f_n(b' + b'') = f'_n b' + (g_n b' + f''_n b'') \qquad (b' \varepsilon B'_n, b'' \varepsilon B''_n),$$

for each $n \geq 0$, where $f'_n b' \varepsilon C'_n$, $g_n b', f''_n b'' \subset C''_n$.

Lemma 1. *If any two of* $\{f_n\}$, $\{f'_n\}$, $\{f''_n\}$ *are families of simple isomorphisms, so is the third.*

This follows immediately from the corollary to Theorem 2.

Let $C_n = C'_n + C''_n$ and let $\partial C'_n \subset C'_{n-1}$ for each $n \geqq 0$. Let $C' = \{C'_n\}$ and let $\partial': C' \to C'$ be defined by $\partial' c' = \partial c'$. Then $\partial'\partial' c' = \partial\partial c' = 0$. Under these, and only these conditions, we shall describe C', with the boundary operator ∂', as a *sub-system* of C. If also $\partial C''_n \subset C''_{n-1}$ $(n \geq 0)$, so that $\partial(c' + c'') = \partial' c' + \partial'' c''$ $(c' \varepsilon C'_n, c'' \varepsilon C''_n)$, then $C'' = \{C''_n\}$, with boundary operator ∂'', is also a sub-system. In this case we shall call C the *direct sum*, $C = C' + C'' = C'' + C'$, of C' and C''. Let C', C'' be given, disjoint,[17] chain systems. Then the direct sum, $C' + C''$, will be the system which consists of the groups $C'_n + C''_n$, with $\partial(c' + c'') = \partial' c' + \partial'' c''$. Similarly we define the direct sum of any finite set of disjoint chain systems.

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[15] At this stage we do not impose any restriction such as $f_0 m_i = \lambda m_j$ on f_0, where m_i is a basis element of C_0. For example, $C \to 0$ is a chain mapping.

[16] We shall often use ∂ to denote the boundary operator in each of two or more systems, $C, C', C'', \cdots$, which occur in the same context. On other occasions we shall use $\partial, \partial', \partial'', \cdots$ to denote the boundary operators in $C, C, C'', \cdots$.

[17] We describe two or more basic modules, or chain systems, as disjoint if, and only if, $0 \varepsilon M$ is their only common element.

Let C' be a sub-system of C and let $C_n = C'_n + C''_n$. Let $j_n : C_n \to C''_n$ and $\partial''_n : C''_n \to C''_{n-1}$ be defined by

$$(3.2) \qquad j_n(c' + c'') = c'', \qquad \partial''_n c'' = j_{n-1}\partial_n c''.$$

Then $\partial''_n j_n = j_{n-1}\partial_n$, since $\partial C' \subset C'$ and $j_{n-1}C'_{n-1} = 0$. Therefore

$$\partial''_n \partial''_{n+1} j_{n+1} = \partial_n j_n \partial_{n+1} = j_{n-1}\partial_n\partial_{n+1} = 0,$$

whence $\partial''_n\partial''_{n+1} = 0$. Therefore $C'' = \{C''_n\}$, with $\partial'' = \{\partial''_n\}$ as boundary operator, is a chain system. We call it the *residue system*, mod C′, and write $C'' = C - C'$. Notice, however, that an element in C'' is an element in the basic module C''_n, for some $n \geq 0$, not a residue class of elements in C. Notice also that $j = \{j_n\}$ is a chain mapping, $j : C \to C''$; also that $c - jc \in C'$, whence

$$(3.3) \qquad \partial c'' - \partial'' c'' = \partial c'' - j\partial c'' \in C'.$$

Let B', C' be sub-systems of chain systems B, C and let $B'' = B - B'$, $C'' = C - C'$. Let $f : B \to C$ be a chain mapping such that $fB' \subset C'$. Then $f' : B' \to C'$, given by $f'b' = fb'$, is obviously a chain mapping. Let

$$f''_n = j_n f_n : B''_n \to C''_n.$$

Then $f''j = jf$, where $j : B \to B''$ is defined in the same way as $j : C \to C''$. Since $\partial''j = j\partial$ we have

$$f''\partial''j = jf\partial = j\partial f = \partial'' f'' j,$$

where ∂ operates on B, C and ∂'' on B'', C''. Therefore f'' is a chain mapping. We shall call $f' : B' \to C'$, $f'' : B'' \to C''$ the chain mappings *induced* by f. It follows from Lemma 1 that, if any two of f, f', f'' are simple isomorphisms, so is the third.

Let A be a common sub-system of B and C. Then we shall describe a chain mapping, $f : B \to C$, as *rel.* A if, and only if, $fa = a$ for each $a \in A$. We shall say that $f \simeq g : B \to C$, rel. A, if, and only if, $g - f = \partial\eta + \eta\partial$, where $\eta : B \to C$ is a deformation operator such that $\eta A = 0$.

Let $Z_n(C) = \partial_n^{-1}(0)$ and let

$$H_n(C) = Z_n(C) - \partial_{n+1}C_{n+1} \qquad (n \geq 0).$$

A chain mapping, $f : B \to C$, obviously induces a family of homomorphisms

$$f_* : H_n(B) \to H_n(C).$$

Let C' be a sub-system of C, let $i : C' \to C$ be the identical map, which is obviously a chain mapping, and let $j: C \to C''$ mean the same as before. Let $z'' \varepsilon Z_n(C'')$. Then it follows from (3.3) that $\partial z'' \varepsilon Z_{n-1}(C')$. Therefore ∂ induces a family of homomorphisms $d_*: H_n(C'') \to H_{n-1}(C')$, where $H_{-1}(C') = 0$. It is known[18] that the sequence of homomorphisms,

$$(3.4) \qquad \cdots \underset{d_*}{\to} H_n(C') \underset{i_*}{\to} H_n(C) \underset{j_*}{\to} H_n(C'') \underset{d_*}{\to} \cdots \underset{d_*}{\to} H_{-1}(C'),$$

is exact, meaning that the kernel of each homomorphism is the image group of its predecessor. We prove that $d_* H_n(C'') = i_*^{-1}(0)$. Let $\bar{z} \varepsilon H_n(X)$ $(X = C, C'$ or $C'')$ be the residue class containing a given element $z \varepsilon Z_n(X)$. Let $z'' \varepsilon Z_n(C'')$. Then $\partial z'' = i\partial z'' \varepsilon Z_{n-1}(C)$, and

$$i_* d_* \bar{z}'' = i_* \overline{\partial z''} = \overline{i\partial z''} = \overline{\partial z''} = 0.$$

Therefore $d_* H_n(C'') \subset i_*^{-1}(0)$. Conversely, let $i_* \bar{z} = 0$, where $z' \varepsilon Z_{n-1}(C')$. This means that $iz' \varepsilon \partial C_n$, or that $z' = \partial c = \partial(c' + z'')$ $(c' \varepsilon C'_n, z'' \varepsilon C''_n)$. Therefore, writing $z' - \partial c' = z'_1$, we have

$$\bar{z}' = \bar{z}'_1 = \overline{\partial z''} = d_* \bar{z}'',$$

whence $d_* H_n(C') = i_*^{-1}(0)$. It follows from similar arguments that $i_* H_n(C') = j_*^{-1}(0)$ and that $j_* H_n(C) = d_*^{-1}(0)$.

4. Deformation retracts. Let $C' \subset C$ be a sub-system and let $i: C \to C$ be the identical chain mapping. A chain mapping, $k: C \to C'$, will be called a *retraction* if, and only if, $ki = 1$. We shall call C' a *deformation retract* (D. R.) of C if, and only if, there is a retraction, $k: C \to C'$, such that $ik \simeq 1$, rel. C'. Let $ik \simeq 1$, rel. C', and let $k': C \to C'$ be any other retraction. Then $ik' \simeq ik'ik = ik \simeq 1$, rel. C'.

THEOREM 3. *A sub-system $C' \subset C$ is a D. R. of C if, and only if, $H_n(C - C') = 0$ for every $n \geq 0$.*

Let C' be a D. R. of C and let $k: C \to C'$ be a retraction. Then

$$(4.1) \qquad 1 - ik = \partial\eta + \eta\partial,$$

where $\eta: C \to C$ is a deformation operator such that $\eta C' = 0$. Let $C'' = C - C'$ and let $z'' \varepsilon Z_n(C'')$. Then $\partial z'' \varepsilon C'$, whence $\eta\partial z'' = 0$. Clearly $ji = 0$, $jz'' = z''$, where $j: C \to C''$ is given by (3.2). Therefore

$$z'' = j(1 - ik)z'' = j(\partial\eta + \eta\partial)z'' = j\partial\eta z'' = \partial j\eta z''.$$

Therefore $H_n(C'') = 0$ $(n \geq 0)$.

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[18] See Theorem 3.3 in [15].

Conversely, let $H_n(C'') = 0$ for every $n \geq 0$. Assume that there are homomorphisms,

$$k_r : C_r \to C'_r, \ \eta_{r+1} : C_r \to C_{r+1} \qquad (r = -1, \cdots, n-1),$$

such that $k_r i_r = 1$, $\partial_r k_r = k_{r-1}\partial_r$ and

$$(4.2)_r \qquad i_r k_r - 1 = \partial_{r+1}\eta_{r+1} + \eta_r \partial_r,$$

these conditions being vacuous if $n = 0$. Let $m''_1, \cdots, m''_p$ $(m''_\lambda = m_i)$ be the basis of C''_n and let

$$(4.3) \qquad (1 + \eta_n \partial_n) m''_\lambda = c'_\lambda + c''_\lambda \qquad (c'_\lambda \in C'_n, \ c''_\lambda = C''_n).$$

It follows from $(4.2)_{n-1}$ that

$$\partial_n(1 + \eta_n\partial_n) = (1 + \partial_n\eta_n)\partial_n = (i_{n-1}k_{n-1} - \eta_{n-1}\partial_{n-1})\partial_n = i_{n-1}k_{n-1}\partial_n.$$

Therefore $\partial_n(c'_\lambda + c''_\lambda) = i_{n-1}k_{n-1}\partial_n m''_\lambda = k_{n-1}\partial_n m''_\lambda$. Therefore $\partial''_n c''_\lambda = 0$. Since $H_n(C'') = 0$ we have $c''_\lambda = \partial''_{n+1} a''_\lambda$, for some $a''_\lambda \in C''_{n+1}$. Let $a'_\lambda = c''_\lambda - \partial_{n+1} a''_\lambda \in C'_n$. Then it follows from (4.3) that

$$(4.4) \qquad (1 + \eta_n\partial_n) m''_\lambda = c'_\lambda + a'_\lambda + \partial_{n+1} a''_\lambda.$$

Let $k_n : C_n \to C'_n$ and $\eta_{n+1} : C_n \to C_{n+1}$ be the operator homomorphisms defined by $k_n c' = c'$, $\eta_{n+1} c' = 0$ and

$$k_n m''_\lambda = c'_\lambda + a'_\lambda, \qquad \eta_{n+1} m''_\lambda = - a''_\lambda.$$

Then $(4.2)_n$ follows from (4.4). Also

$$\partial_n k_n c' = \partial_n c' = k_{n-1}\partial_n c'$$

$$\partial_n k_n m''_\lambda = \partial_n(c'_\lambda + a'_\lambda) = \partial_n(c'_\lambda + c''_\lambda) = k_{n-1}\partial_n m''_\lambda.$$

Therefore, starting with $k_{-1} = \eta_{-1} = 0$, the theorem follows by induction on n.

COROLLARY 1. *$C \equiv 0$ if, and only if, $H_n(C) = 0$ for every $n \geq 0$.*

COROLLARY. 2. *C' is a D. R. of C if, and only if, $C - C' \equiv 0$.*

LEMMA 2. *If C' is a D. R. of C then $C \approx C' + C''$ (Σ), rel. C', where $C'' = C - C'$.*

Let $C^* = C' + C''$. Then $C^*_n = C_n$ and $\partial^* : C^* \to C^*$ is given by $\partial^*(c' + c'') = \partial' c' + \partial'' c''$. Let i, k, η mean the same as in (4.1) and let $f : C^* \to C$ be given by

$$f(c' + c'') = (c' - kc'') + c'' = c' + (\partial\eta + \eta\partial)c''.$$

Then $\partial fc' = \partial c' = f\partial^* c'$ and

$$\partial fc'' = \partial(\partial\eta + \eta\partial)c'' = \partial\eta\partial c'',$$
$$f\partial^* c'' = (\partial\eta + \eta\partial)\partial'' c'' = (\partial\eta + \eta\partial)(\partial c'' + c'_1) \qquad (c'_1 \varepsilon C').$$

Since $\partial C' \subset C'$ and $\eta C' = 0$ it follows that $f\partial^* c'' = \partial\eta\partial c'' = \partial fc''$. Therefore f is a chain mapping. It follows from Lemma 1 that $f: C^* \approx C\ (\Sigma)$ and the lemma is proved.

5. Simple equivalence. We shall describe a chain system, B, as *elementary* if, and only if, $B_n = 0$ when $n \neq r - 1,\ r$, for some $r \geq 1$, and $\partial_r : B_r \approx B_{r-1}\ (\Sigma)$. This being so, it is obvious that $H_n(B) = 0$ for every $n \geq 0$. Therefore $B \equiv 0$, by Theorem 3, Corollary 1. We shall describe B as *collapsible* if, and only if, it is the direct sum of a finite set of elementary systems. Clearly $B \equiv 0$ if B is collapsible. It is obvious that B' is collapsible if $B' \approx B\ (\Sigma)$, where B is collapsible; also that, if B, B' are disjoint and collapsible, then $B + B'$ is collapsible; also that the direct sum of a set of r-dimensional elementary systems is itself elementary.

Let $B_n = A_n + Z_n$, let $\partial'_n : A_n \approx Z_{n-1}\ (\Sigma)$ and let $\partial_n : B_n \to B_{n-1}$ be given by $\partial_n a = \partial'_n a$, $\partial_n Z_n = 0$, $(n = 1, 2, \cdots)$. Then $B = \{B_n\}$, with $\partial = \{\partial_n\}$ as boundary operator, is the direct sum of the elementary systems $(\cdots, 0, A_n, Z_{n-1}, 0, \cdots)$. Therefore B is collapsible and any collapsible system is obviously of this form.

We shall say that C, C' are in the same *simple equivalence class*, and shall write $C \equiv C'\ (\Sigma)$ if, and only if, there are collapsible systems, B, B' such that

$$f: B + C \approx B' + C'\ (\Sigma). \tag{5.1}$$

This being so, it follows from Theorem 3, Corollary 2, that C, C' are D. R.'s of $B + C$, $B' + C'$. Let

$$i: C \to B + C, \qquad i': C' \to B' + C'$$
$$k: B + C \to C, \qquad k': B' + C' \to C'$$

be the identity maps and any retractions. Let

$$g \simeq k'fi : C \to C' \tag{5.2}$$

and let $g' \simeq kf^{-1}i' : C' \to C$. Then

$$g'g \simeq kf^{-1}i'k'fi \simeq kf^{-1}fi = 1. \tag{5.3}$$

Similarly $gg' \simeq 1$. Therefore $g: C \equiv C'$. We shall describe a chain mapping, $g: C \to C'$, as a *simple equivalence* and shall write $g: C \equiv C'\ (\Sigma)$, if, and only if, it is related by (5. 2) to some simple isomorphism of the type (5. 1). It follows from (5. 3) that, if $g: C \equiv C'\ (\Sigma)$ and if $g'': C' \to C$ is such that $gg'' \simeq 1$, then g'' is a simple equivalence. Obviously $g: C \equiv C'\ (\Sigma)$ if $g: C \approx C'\ (\Sigma)$.

The relation $C \equiv C'\ (\Sigma)$ is obviously reflexive and symmetric. We proceed to prove that it is transitive; also that, if $g: C \equiv C'\ (\Sigma)$ and $g': C' \equiv C''\ (\Sigma)$, then $g'g: C \equiv C''\ (\Sigma)$. Let C, C' be related by (5. 1), let $f^*: B^* + C' \approx B'' + C''\ (\Sigma)$, where B^*, B'' are collapsible, and assume that

$$(5.4) \qquad B' \cap B^* = B^* \cap (B + C) = B' \cap (B'' + C'') = 0.$$

Then $B + B^* + C \approx B' + B^* + C' \approx B' + B'' + C''\ (\Sigma)$, whence $C \equiv C''\ (\Sigma)$. If (5. 4) are not satisfied we apply a permutation, $P_n: B'_n \to A'_n$, to each module B'_n, thus transforming B' into a new system, A', such that $P: B' \approx A'\ (\Sigma)$, where $P = \{P_n\}$, and $A' \cap C = 0$. Let

$$h: B' + C' \approx A' + C'\ (\Sigma)$$

be given by $h(b' + c') = Pb' + c'$. Then

$$(5.5) \qquad hf: B + C \approx A' + C'\ (\Sigma).$$

Similarly we can replace B^* by $A^* = P^*B^*$. We can choose the basic modules A'_n, A^*_n in such a way that (5. 4) are satisfied when B', B^* are replaced by A', A^*. Therefore $C \equiv C''\ (\Sigma)$, and it follows that $C \equiv C'\ (\Sigma)$ is an equivalence relation.

Let $g: C \to C'$ be given by (5. 2) and let h mean the same as in (5. 5). Then $g \simeq (k'h^{-1})(hf)i: C \to C'$ and $k'h^{-1}: A' + C' \to C'$ is obviously a retraction. Also $k'h^{-1}$ can be extended to a retraction, $A' + A^* + C' \to C'$, by mapping A^* on zero. Therefore, if $g: C \equiv C'\ (\Sigma)$ and $g': C' \equiv C''\ (\Sigma)$ we lose no generality in assuming that g satisfies (5. 2) and that

$$f': B' + C' \approx B'' + C''\ (\Sigma), \qquad g \simeq k''f'i': C' \to C'',$$

where $k'': B'' + C'' \to C''$ is a retraction. This being so,

$$f'f: B + C \approx B'' + C''\ (\Sigma)$$

and

$$g'g \simeq k''f'i'k'fi \simeq k''f'fi: C \to C''.$$

Therefore $g'g: C \equiv C''\ (\Sigma)$.

A non-zero element, which is common to two chain systems, will be called an *accidental intersection*, unless it is in a common sub-system, which is explicitly mentioned in the context. Accidental intersections between any finite set of systems, $C, \cdots$, can always be eliminated, as in the paragraph containing (5.5), by replacing $C, \cdots$, by a set of chain systems, $PC, \cdots$, where $P_n: C_n \to (PC)_n$, is a suitable set of permutations. When the context requires it, we shall always assume that this has already been done.

Let $C \equiv C' \ (\Sigma)$, $C^* \equiv C'^* \ (\Sigma)$. Then $B + C \approx B' + C' \ (\Sigma)$, $B^* + C^* \approx B'^* + C'^* \ (\Sigma)$, where B, B' etc. are collapsible. Therefore, in the absence of accidental intersections, it follows from Lemma 1, in Section **3**, that

$$B + B^* + C + C^* \approx B' + B'^* + C' + C'^* \ (\Sigma),$$

whence

$$C + C^* \equiv C' + C'^* \ (\Sigma). \tag{5.6}$$

Let A be a common sub-system of C and C'. We shall write $C \equiv C'' \ (\Sigma)$, rel. A, if, and only if,

$$f: B + C \approx B' + C' \ (\Sigma), \text{rel. } A, \tag{5.7}$$

where B, B' are collapsible.

THEOREM 4. a) *If $C \equiv C' \ (\Sigma)$, rel. A, then $C - A \equiv C' - A' \ (\Sigma)$.*

b) *If $C - A \equiv 0 \ (\Sigma)$, then $C \equiv A \ (\Sigma)$, rel. A.*

Let C, C' be related by (5.7). Since f induces the identity, $A \to A \ (\Sigma)$, it follows from Lemma 1 that

$$f': (B + C) - A \approx (B' + C') - A \ (\Sigma),$$

where f' is the chain mapping induced by f. Obviously

$$(B + C) - A = B + (C - A), \qquad (B' + C') - A = B' + (C' - A),$$

which proves (a).

Let $C'' \equiv 0 \ (\Sigma)$, where $C'' = C - A$. Then $B + C'' \equiv B' \ (\Sigma)$, where B, B' are collapsible. Since $C'' \equiv 0$ it follows from Theorem 3, Corollary 2, that A is a D. R. of C. Therefore $C \approx A + C'' \ (\Sigma)$, rel. A, by Lemma 2. Therefore

$$B + C \approx B + C'' + A \approx B' + A \ (\Sigma), \text{rel. } A.$$

Therefore $C \equiv A \ (\Sigma)$, rel. A, and the theorem is proved.

By a (p, q)-system, C, we shall mean a chain system such that $C_n = 0$ if $n < p$ or if $n > q$ $(p \leq q)$.

LEMMA[19] 3. *If $C \equiv C'$ (Σ), where C, C' are (p, q)-systems, there are collapsible (p, q)-systems, B, B', such that $B + C \approx B' + C'$ (Σ).*

Let $f: A + C \approx A' + C'$ (Σ), where A, A' are collapsible. Let $n = \dim A > q$. Then $n = \dim(A + C) = \dim(A' + C')$. It follows from the definition of a collapsible system that

$$A = B^1 + \cdots + B^m, \tag{5.8}$$

where each B^i is an elementary system. Let E be the direct sum of all the n-dimensional summands, B^i, and let D be the direct sum of the others. Let D', $E' \subset A'$ be similarly defined. Then $E_n = (A + C)_n$, $E'_n = (A' + C')_n$, and $\partial_n : E_n \approx E_{n-1}$ (Σ), $\partial'_n : E'_n \approx E'_{n-1}$ (Σ) since E, E' are elementary systems. Also $f_n : E_n \approx E'_n$ (Σ). Therefore

$$fE_{n-1} = f\partial E_n = \partial' f E_n = \partial' E'_n = E'_{n-1}$$

and

$$f_{n-1} = \partial' f_n \partial_n^{-1} : E_{n-1} \approx E'_{n-1} \ (\Sigma).$$

Therefore $f: A + C \approx A' + C'$ (Σ) induces a simple isomorphism $E \approx E'$ (Σ). Since $A + C - E = D + C$, $A' + C' - E' = D' + C'$, it follows from Lemma 1 that

$$D + C \approx D' + C' \ (\Sigma). \tag{5.9}$$

Now let $A_r \neq 0$ for some $r < p$ and let s be the least value of r with this property. Since $C'_s = 0$ and $A + C \approx A' + C'$ (Σ) it follows that s has the same property in A'. Let E now denote the direct sum of the $(s + 1)$-dimensional summands in (5.8) and let D be the direct sum of the others. Let D', $E' \subset A'$ be similarly defined. Then $D_s = D'_s = 0$, by the minimal property of s. Therefore

$$(D + C)_s = (D' + C')_s = 0. \tag{5.10}$$

Since E, E' are elementary systems we have

$$\partial : E_{s+1} \approx E_s \ (\Sigma), \qquad \partial' : E'_{s+1} \approx E'_s \ (\Sigma). \tag{5.11}$$

Let $c \in C_{s+1}$, $d \in D_{s+1}$ and let $f(d + c) = e' + d' + c'$. Since $\partial'(d + c') = 0$, in consequence of (5.10), we have $\partial' e' = \partial'(e' + d' + c') = \partial' f(d + c) = f\partial(d + c) = 0$. Therefore it follows from (5.11) that $e' = 0$. Therefore $f(D + C) \subset D' + C'$. Let $h: E \rightarrow E'$ be the chain mapping induced by f. Then it follows from (5.10) that $h_s = f_s : E_s \approx E'_s$ (Σ) and also, since $he - fe \in (D' + C')_{s+1}$ if $e \in E_{s+1}$, that

[19] Cf. Theorem 1 on p. 1202 of [3].

$$\partial' h_{s+1} e = \partial' f_{s+1} e = f_s \partial e = h_s \partial e.$$

Therefore

$$h_{s+1} = \partial_{E'}^{-1} h_s \partial_E : E_{s+1} \approx E'_{s+1} \, (\Sigma),$$

where $\partial_E = \partial \mid E_{s+1}$, $\partial_{E'} = \partial' \mid E'_{s+1}$. Therefore $h : E \approx E' \, (\Sigma)$ and (5.9) again follows from Lemma 1. Lemma 3 now follows by induction on m in (5.8).

We are now approximately half way through the algebraic preliminaries. The simple homotopy equivalences will be defined as those which induce simple chain equivalences, in a sense explained in Section **10** below. But we have still to relate chain equivalences to the group T, which is defined in Section **2** above. The first step in this is to associate an element, $\tau(C) \varepsilon \mathrm{T}$, with each system, C, such that $C \equiv 0$. We shall do this by transforming C into an $(m, m+1)$-system, C^m, in which $\partial_{m+1} : C^m_{m+1} \approx C^m_m$, and defining $\tau(C) = (-1)^m \tau(\partial_{m+1})$. In Section **8** below we define a chain system called the "mapping cylinder," C^*, of a given chain equivalence $f : C \equiv C'$. This contains C as a sub-system and $C^* - C \equiv 0$. We define $\tau(f) = \tau(C^* - C)$. We shall also need to consider the effect of a Λ-automorphism, $\theta : R \approx R$, operating on T, because the chain mapping induced by a homotopy equivalence, $\phi : K \equiv K'$, is "associated" with an isomorphism $\pi_1(K) \approx \pi_1(K')$, namely the one induced by ϕ.

6. Null-equivalent systems. Let $C \equiv 0$. Then $k \simeq 1 : C \to C$, where $kC = 0$. Therefore there is a chain deformation operator, $\eta : C \to C$, such that

$$(6.1) \qquad \partial\eta + \eta\partial = 1.$$

Let $\delta = \eta\partial\eta$. That is to say, $\delta = \{\delta_n\}$, where $\delta_n = \eta_n \partial_n \eta_n : C_{n-1} \to C_n$. Therefore δ is also a chain deformation operator. It follows from (6.1) that $\partial\eta\partial = (1 - \eta\partial)\partial = \partial$. Therefore $\partial\delta + \delta\partial = \partial\eta\partial\eta + \eta\partial\eta\partial = \partial\eta + \eta\partial = 1$. Also $\partial\eta\eta = (1 - \eta\partial)\eta = \eta(1 - \partial\eta) = \eta\eta\partial$. Therefore $\delta\delta = \eta\partial\eta\eta\partial\eta = \eta\eta\eta\partial\partial\eta = 0$. Thus

$$(6.2) \qquad \partial\delta + \delta\partial = 1, \qquad \delta\delta = 0.$$

Let $P_1, P_2 : M \to M$ be permutations and let B be the elementary system in which

$$B_0 = 0, \quad B_i = P_i C_0, \quad B_n = 0 \qquad (i = 1, 2; n > 2)$$

and $\partial P_2 c_0 = P_1 c_0$ ($c_0 \varepsilon C_0$). Let $C' = B + C$. Then $C'_n = C_n$ if $n \neq 1$ or 2 and

$$\begin{cases} C'_1 = P_1C_0 + C_1, & C'_2 = P_2C_0 + C_0 \\ \partial'_1(P_1c_0 + c_1) = \partial_1 c_1, & \partial'_2(P_2c_0 + c_2) = P_1c_0 + \partial_2 c_2, \end{cases}$$

where $c_i \varepsilon C_i$. Let C^* be the system which consists of the same groups, $C^*_n = C'_n$, with $\partial^*_n = \partial_n$ if $n > 2$ and

$$(6.3) \qquad \partial^*_1(P_1c_0 + c_1) = c_0, \qquad \partial^*_2(P_2c_0 + c_2) = \delta_1 c_0 + \partial_2 c_2.$$

Let $f: C' \to C^*$ be given by $f_n = 1$ if $n \neq 1$ and $f_1(P_1c_0 + c_1) = P_1\partial_1 c_1 + (\delta_1 c_0 + c_1)$. Then

$$\partial^*_1 f_1(P_1c_0 + c_1) = \partial_1 c_1 = f_0\partial'_1(P_1c_0 + c_1)$$

$$\partial^*_2 f_2(P_2c_0 + c_2) = \delta_1 c_0 + \partial_2 c_2 = P_1\partial_1\partial_2 c_2 + \delta_1 c_0 + \partial_2 c_2$$

$$= f_1(P_1c_0 + \partial_2 c_2) = f_1\partial'_2(P_2c_0 + c_2)$$

and $\partial^*_n f_n - f_{n-1}\partial'_n = \partial_n - \partial_n = 0$ if $n > 2$. Therefore f is a chain mapping. Let $g_1, h_1: C'_1 \to C'_1$ be given by

$$g_1(P_1c_0 + c_1) = P_1(c_0 + \partial_1 c_1) + c_1$$

$$h_1(P_1c_0 + c_1) = -P_1c_0 + (\delta_1 c_0 + c_1).$$

Since $\partial_1\delta_1 = \partial_1\delta + \delta_0\partial_0 = 1$ we have

$$g_1h_1(P_1c_0 + c_1) = g_1\{-P_1c_0 + (\delta_1 c_0 + c_1)\}$$

$$= P_1(-c_0 + c_0 + \partial_1 c_1) + (\delta_1 c_0 + c_1) = f_1(P_1c_0 + c_1).$$

Therefore $f_1 = g_1h_1$. It follows from Theorem 2 in Section **2** that $g_1, h_1: C'_1 \approx C'_1\ (\Sigma)$ and hence that $f: C \approx C'\ (\Sigma)$.

It follows from (6.3) that $C^* = B' + C^1$, where $B'_n = 0$ if $n > 1$,

$$B'_0 = C_0, B'_1 = P_1C_0, (\partial^* \mid B'_1) = P_1^{-1}: B'_1 \approx B'_0\ (\Sigma)$$

and

$$(6.4) \qquad C^1_0 = 0,\ C^1_1 = C_1,\ C^1_2 = P_2C_0 + C_2,\ C^1_n = C_n \qquad (n > 2)$$

with $\partial^1: C^1 \to C^1$ given by $\partial^1_n = \partial_n$ if $n > 2$ and

$$(6.5) \qquad \partial^1_2(P_2c_0 + c_2) = \delta_1 c_0 + \partial_2 c_2.$$

Let $m \geq 1$ and assume that there is a system, C^m, such that $C \equiv C^m\ (\Sigma)$ and

$$C^m_0 = \cdots = C^m_{m-1} = 0, \qquad C^m_{m+p+1} = C_{m+p+1} \qquad (p > 0).$$

Then it follows from the above argument, with C^m_{m+n} playing the part of C_n, that $C^m \equiv C^{m+1}\ (\Sigma)$, where C^{m+1} satisfies the same conditions as C^m, with m

replaced by $m+1$. It follows by induction on m that there is such a system for each value of m. Moreover $C^m \equiv C \equiv 0$. Therefore equations analogous to (6.2) are satisfied in C^m.

Let $N = \dim C$ and let $m \geq N - 1$. Then $C^m{}_n = 0$ unless $n = m$ or $m+1$. Therefore (6.2) reduces to

$$\text{(6.6)} \qquad \begin{cases} \partial_{m+1}\delta_{m+1} = 1 : C^m{}_m \to C^m{}_m \\ \delta_{m+1}\partial_{m+1} = 1 : C^m{}_{m+1} \to C^m{}_{m+1}. \end{cases}$$

Therefore $\partial_{m+1} : C^m{}_{m+1} \approx C^m{}_m$ and $\delta_{m+1} = \partial^{-1}{}_{m+1}$. We define the *torsion*, $\tau(C)$, of C as

$$\text{(6.7)} \qquad \tau(C) = (-1)^m \tau(\partial_{m+1}) = (-1)^{m+1}\tau(\delta_{m+1}).$$

In (6.4) and (6.5) let C, C^1 be replaced by C^m, C^{m+1}, with $C^m{}_n = 0$ if $n > m+1$. Then (6.4), (6.5) become

$$C^{m+1}{}_{m+1} = C^m{}_{m+1}, \qquad C^m{}_{m+2} = PC^m{}_m,$$

where P is a permutation, and

$$\partial_{m+2} = \delta_{m+1}P^{-1} : C^{m+1}{}_{m+2} \to C^{m+1}{}_{m+1}.$$

Since $\tau(P) = 0$ it follows from (6.7) that $\tau(C) = (-1)^{m+1}\tau(\delta_{m+1}) = (-1)^{m+1}\tau(\partial_{m+2})$. Therefore $\tau(C)$ does not depend on the choice of $m \geq N-1$. However $\tau(C)$ appears to depend on the particular choice of δ in (6.2) and on the construction for C^m. The following theorem shows that it does not.

THEOREM 5. *$\tau(C)$ depends only on C. Also $\tau(C) = \tau(C')$, if and only if, $C \equiv C'$ (Σ), given that $C \equiv C' \equiv 0$.*

Let $C \equiv C'$ (Σ), where $C \equiv 0$, and let C^m, C'^m be any given $(m, m+1)$-systems such that $C^m \equiv C$ (Σ), $C'^m \equiv C'$ (Σ). Let $\tau(C)$ be defined by (6.7), where C^m is now this given system, and let $\tau(C')$ be similarly defined in terms of C'^m. In particular we may have $C = C'$. Therefore, when we have proved that $\tau(C) = \tau(C')$, it will follow that $\tau(C)$ depends only on C and also that $\tau(C) = \tau(C')$ if $C \equiv C'$ (Σ).

By Lemma 3 $f : B + C^m \approx B' + C'^m$ (Σ), where B, B' are collapsible, and hence elementary $(m, m+1)$-systems. It follows from Theorem 3, Corollary 2, that $B + C^m \equiv B' + C'^m \equiv 0$. Therefore it follows from relations analogous to (6.6) that

$$\partial : (B + C^m)_{m+1} \approx (B + C^m)_m$$

$$\partial' : (B' + C'^m)_{m+1} \approx (B' + C'^m)_m.$$

Moreover $\partial = f_m^{-1}\partial' f_{m+1}$ since $f\partial = \partial' f$. Since f_m, f_{m+1} are simple isomorphisms it follows that $\tau(\partial) = \tau(\partial')$. Also

$$\begin{cases} \partial(b + c) = \partial_B b + \partial_{m+1} c & (b \in B_{m+1},\ c \in C^m{}_{m+1}) \\ \partial'(b' + c') = \partial_{B'} b' + \partial'_{m+1} c', \end{cases}$$

where $\partial_B = \partial \mid B_{m+1}$, $\partial_{B'} = \partial' \mid B'_{m+1}$. By the definition of an elementary system, $\tau(\partial_B) = \tau(\partial_{B'}) = 0$ and it follows from Theorem 2, in Section **2**, that $\tau(\partial_{m+1}) = \tau(\partial) = \tau(\partial') = \tau(\partial'_{m+1})$. Therefore $\tau(C) = \tau(C')$.

Conversely let $\tau(C) = \tau(C')$ and let $C^m \equiv C$ (Σ), $C'^m \equiv C'$ (Σ), where C^m, C'^m are $(m, m+1)$-systems. Let $C^m{}_m$ be of rank p and $C'^m{}_m$ of rank p'. If $p \neq p'$, say $p < p'$, we replace C^m by $B + C^m$, where B is an elementary $(m, m+1)$-system, such that B_m is of rank $p' - p$. Therefore we assume that $p = p'$. Moreover, after applying suitable permutations to $C^m{}_m$, $C^m{}_{m+1}$ we assume that $C^m{}_m = C'^m{}_m$, $C^m{}_{m+1} = C'^m{}_{m+1}$. Then $g = \partial'_{m+1}\partial^{-1}{}_{m+1} : C^m{}_m \approx C^m{}_m$ and $(-1)^m \tau(g) = \tau(C') - \tau(C) = 0$. Therefore $g : C^m{}_m \approx C^m{}_m$ (Σ). Let $f : C^m \to C'^m$ be given by $f_m = g$, $f_{m+1} = 1$. Then $\partial'_{m+1} f_{m+1} \equiv \partial'_{m+1} = f_m \partial_{m+1}$. Therefore $f : C^m \approx C'^m$ (Σ) and the theorem is proved.

Obviously $\tau(0) = 0$. Therefore we have the corollary:

Corollary. $C \equiv 0$ (Σ) *if, and only if,* $\tau(C) = 0$.

Let C' be a sub-system of C, and let $C'' = C - C'$.

Theorem 6. *If any two of* C, C', C'' *are chain equivalent to* 0, *so is the third and* $\tau(C) = \tau(C') + \tau(C'')$.

Let X, Y, Z denote C, C', C'' in any order and let $X \equiv Y \equiv 0$. Then $H_n(X) = 0$, $H_n(Y) = 0$ for every $n \geq 0$, according to Theorem 3, Corollary 1. Therefore it follows from the exactness of the sequence (3.4) that $H_n(Z) = 0$, for every $n \geq 0$, and hence that $Z \equiv 0$.

Let $C \equiv C' \equiv C'' \equiv 0$. Then $C \approx C' + C''$ (Σ), according to Theorem 3, Corollary 2, and Lemma 2, and it follows from Theorem 5 that $\tau(C) = \tau(C' + C'')$. For a sufficiently large value of m we have $C' \equiv A'$ (Σ), $C'' = A''$ (Σ), where A', A'' are $(m, m+1)$-systems. Therefore it follows from (5.6) that $C' + C'' \equiv A' + A''$ (Σ) and from Theorem 5 that $\tau(C) = \tau(A' + A'')$. Let $\partial' : A'_{m+1} \to A_m$, $\partial'' : A''_{m+1} \to A''_m$, be the boundary homomorphisms, which are isomorphisms since $A' \equiv A'' \equiv 0$. Then it follows from Theorems 2 and 5 that

$$\tau(A' + A'') = (-1)^m \{\tau(\partial') + \tau(\partial'')\} = \tau(A') + \tau(A'') = \tau(C') + \tau(C'')$$

and the theorem is proved.

For purposes of calculation we exhibit the structure of the system, C^m, which is defined by reiterating the construction $C \rightarrow C^1$, leading to (6.4). To begin with we do not require $m + 1 \geqq \dim C$. Let $m = 2k$ or $2k + 1$ $(k \geqq 0)$ and let $D_0, D_1 \subset M$ be the basic modules

$$(6.8) \qquad \begin{cases} D_0 = C_0 + C_1 + \cdots + C_r \\ D_1 = C_1 + C_3 + \cdots + C_{2k+1}, \end{cases}$$

where $r = m$ if $m = 2k$, $r = m + 1$ if $m = 2k + 1$. Let $D = D_0 + D_1 = C_0 + C_1 + \cdots + C_{m+1}$ and let $\partial : D \rightarrow D$ be the homomorphism which is determined by $\partial : C \rightarrow C$. Let $\delta' : D \rightarrow D$ be the homomorphism which is given by

$$\begin{aligned} \delta' c &= \delta c \qquad \text{if } c \in C_s \qquad (s < m + 1) \\ &= 0 \qquad \text{if } c \in C_{m+1}. \end{aligned}$$

Then $\delta'\delta' = 0$. Also it follows from (6.2) that

$$(6.9) \qquad \begin{aligned} (\partial\delta' + \delta'\partial) c &= c \qquad \text{if } c \in C_s \qquad (s < m + 1) \\ &= \delta\partial c \qquad \text{if } c \in C_{m+1}. \end{aligned}$$

Also $\partial D_i \subset D_j$, $\delta' D_i \subset D_j$ $(i \neq j; i, j = 0, 1)$. Let

$$(6.10) \qquad \Delta = \partial + \delta' : (D, D_0, D_1) \rightarrow (D, D_1, D_0)$$

and let $\Delta_i : D_i \rightarrow D_j$ be the homomorphism which is induced by Δ. Then

$$(6.11) \qquad \Delta_i \Delta_j d = \Delta_i \Delta d = \Delta\Delta d \qquad (d \in D_j).$$

Since $\partial\partial = \delta'\delta' = 0$ it follows from (6.9) that

$$(6.12) \qquad \begin{aligned} \Delta\Delta c &= (\partial + \delta')(\partial + \delta') c \\ &= c \qquad \text{if } c \in C_s \ (s < m + 1); \ = \delta\partial c \qquad \text{if } c \in C_{m+1}. \end{aligned}$$

Also $\delta'\partial C_{m+2} \subset \delta' C_{m+1} = 0$, whence $\Delta\partial C_{m+2} = 0$.

Let $i = 0$, $j = 1$ if $m = 2k$ and let $i = 1$, $j = 0$ if $m = 2k + 1$. Then $C_{m+1} \subset D_j$ and it follows from (6.11) and (6.12) that

$$(6.13) \qquad \begin{aligned} &\Delta_j \Delta_i = 1 \\ &\Delta_i \Delta_j c = c \qquad \text{if } c \in C_s \ (s < m + 1); \ = \delta\partial c \qquad \text{if } c \in C_{m+1}. \end{aligned}$$

Let C^m, with boundary operator ∂^m, be the chain system, which is given by [20] $C^m{}_n = 0$ if $n < m$,

$$(6.14) \qquad \begin{cases} C^m{}_m = D_i = \cdots + C_{m-2} + C_m \\ C^m{}_{m+1} = D_j = \cdots + C_{m-1} + C_{m+1} \\ \partial^m{}_{m+1} = \Delta_j, \end{cases}$$

[20] Cf. (5) on p. 205 of [10].

and $C^m{}_n = C_n$, $\partial^m{}_n = \partial_n$ if $n > m+1$. Since $\Delta\partial C_{m+2} = 0$ it follows that $\partial^m\partial^m = 0$. Let δ^m be the deformation operator, which is given by

$$(6.15) \qquad \begin{aligned} &\delta^m{}_{m+1} = \Delta_i \\ &\delta^m{}_{m+2}c = 0 \qquad \text{if } c \,\varepsilon\, C_s \ (s < m+1)\,; = \delta_{m+2}c \qquad \text{if } c \,\varepsilon\, C_{m+1} \end{aligned}$$

and $\delta^m{}_n = \delta_n$ if $n > m+2$. Then $\delta^m{}_{n+1}\delta^m{}_n = 0$ if $n > m+1$ and

$$\delta^m{}_{m+2}\delta^m{}_{m+1}(\cdot\cdot\cdot + c_m) = \delta_{m+2}\delta_{m+1}c_m = 0,$$

where $c_m \,\varepsilon\, C_m$. Therefore $\delta^m\delta^m = 0$. Also it follows from (6.13) that

$$\partial^m{}_{m+1}\delta^m{}_{m+1} = 1, \qquad \partial^m{}_{m+2}\delta^m{}_{m+2} + \delta^m{}_{m+1}\partial^m{}_{m+1} = 1,$$

whence $\partial^m\delta^m + \delta^m\partial^m = 1$.

Let C^{m+1} be the system, with boundary operator ∂^{m+1}, which is obtained from C^m by the construction, $C \to C^1$, leading to (6.4), with $C^m{}_{m+n}$ playing the part of C_n and with P_2C_0 replaced by C_0 and P_2 by 1 in (6.4), (6.5). Then

$$C^{m+1}{}_{m+1} = C^m{}_{m+1} = \cdot\cdot\cdot + C_{m-1} + C_{m+1}$$
$$C^{m+1}{}_{m+2} = C^m{}_m + C^m{}_{m+2} = \cdot\cdot\cdot + C_m + C_{m+2}$$

and (6.5) becomes

$$\partial^{m+1}{}_{m+2}(c^m + c_{m+2}) = \partial^m{}_{m+1}c^m + \partial_{m+2}c_{m+2} = \Delta_i c^m + \partial_{m+2}c_{m+2} = \Delta^*(c^m + c_{m+2}),$$

where $c^m \,\varepsilon\, C^m{}_m$, $c_{m+2} \,\varepsilon\, C_{m+2}$ and Δ^* is defined by (6.10), with m replaced by $m+1$. Therefore C^{m+1} is defined in the same way as C^m and we define δ^{m+1} by (6.15), with m replaced by $m+1$. Starting with $C^0 = C$, it follows by induction on m that the construction $C \to C^1$, reiterated m times, leads to C^m. Therefore $C \equiv C^m$ (Σ). We now take $m \geq \dim C - 1$, thus giving an explicit definition of ∂_{m+1}, δ_{m+1} in (6.7).

Let $R' \subset R$ be a sub-ring, which is the image of R in a homomorphism, $\phi : R \to R'$, such that $\phi r' = r'$ if $r' \,\varepsilon\, R'$. Let $1 \,\varepsilon\, R'$ and let $\Lambda' = \phi\Lambda$. It is easily verified that Λ' is a (multiplicative) group . Let $M' \subset M$ be the sub-module, which consists of all the elements $(r'_1, r'_2, \cdot\cdot\cdot)$, where $r'_i \,\varepsilon\, R'$, and let R' be such that every admissible automorphism, $M' \to M'$, is Λ-simple. That is to say, every matrix of the form (2.2), with elements in R', can be transformed into the unit matrix by a finite sequence of the transformations (2.12), with $\lambda \,\varepsilon\, \Lambda$, $r \,\varepsilon\, R$.

Let $\psi : M \to M'$ be given by $\psi(r_1, r_2, \cdot\cdot\cdot) = (\phi r_1, \phi r_2, \cdot\cdot\cdot)$. Then $\psi m_i = m_i$ and $\psi(rm) = (\phi r)\psi m$. Let $f : M \approx M$ be an admissible automorphism, which is given by $fm_i = \Sigma_j f_{ij}m_j$, $(f_{ij} \,\varepsilon\, R)$. Then $\psi f m_i = \Sigma_j(\phi f_{ij})m_j$ and it follows that $f_{ij} \,\varepsilon\, R'$ if (and only if) $\psi f = f\psi$. Therefore $f \,\varepsilon\, \Sigma$ if $\psi f = f\psi$.

Let C be a chain system, with boundary operator ∂, and let $\psi: M \rightarrow M'$ mean the same as before. Then $\psi C_r \subset C_r$, since $\psi m_i = m_i$, and $\partial\psi$, $\psi\partial$ are two families of homomorphisms $\partial\psi, \psi\partial: C_n \rightarrow C_{n-1}$.

LEMMA 4. *If $C \equiv 0$ and if $\partial\psi = \psi\partial$, then $C \equiv 0$ (Σ).*

Let $C \equiv 0$ and let $\partial\psi = \psi\partial$. Let $\eta: C \rightarrow C$ be a deformation operator, which satisfies (6.1), and let $\xi: C \rightarrow C$ be the deformation operator determined by $\xi m_i = \psi\eta m_i$, $\xi r = r\xi$. Since $\partial\psi = \psi\partial$ and $\psi m_i = m_i$ we have $(\partial\xi + \xi\partial) m_i = \psi(\partial\eta + \eta\partial) m_i = m_i$. Therefore $\partial\xi + \xi\partial = 1$. Also $\psi\xi = \xi$, since $\psi\psi = \psi$. Therefore $\xi\psi m_i = \xi m_i = \psi\xi m_i$, whence $\xi\psi = \psi\xi$. Moreover $\delta = \xi\partial\xi$ is a deformation operator such that $\delta\psi = \psi\delta$, which satisfies (6.2). It follows from (6.5) and induction on m, or from the explicit formulae (6.14), that ∂_{m+1}, in (6.6), may be constructed so as to commute with ψ. Therefore ∂_{m+1} is a simple isomorphism. Therefore $\tau(C) = 0$ and the lemma follows from the corollary to Theorem 5.

Let R be the group ring of a group Γ, let Λ consist of the elements $\pm\gamma$ ($\gamma \in \Gamma$) and let R' consist of the integral multiples of $1 \in \Gamma$. Let $\phi: R \rightarrow R'$ be given by $\phi\Gamma = 1$. Then we have the corollary:

COROLLARY. *Let $C \equiv 0$, let $(m^n{}_1, \cdots, m^n{}_{p_n})$ be the basis of C_n $(m^n{}_i = m_{j_i})$ and let*

$$\partial m^n{}_i = \Sigma_j d^n{}_{ij} m^{n-1}{}_j \qquad (n = 1, 2, \cdots),$$

where $d^n{}_{ij}$ are integers. Then $C \equiv 0$ (Σ).

7. Conjugate systems.

Let C be a given chain system with boundary operator ∂. Let $\theta: R \approx R$ be a given automorphism and let $s_\theta: M \rightarrow M$ mean the same as in (2.9). We shall also use s_θ to denote the semi-linear transformation $s': C_r \rightarrow C_r$, which is given by[21] $s'c = s_\theta c$ for each $c \in C_r$ $(r = 0, 1, \cdots)$. Let

$$\partial^\theta = s_\theta \partial s_\theta^{-1}: C_n \rightarrow C_{n-1} \qquad (n = 1, 2, \cdots).$$

Then $\partial^\theta\partial^\theta = s_\theta\partial\partial s_\theta^{-1} = 0$. Obviously $\partial^\theta r = r\partial^\theta$ for each operator $r \in R$. Therefore ∂^θ is a boundary operator. Let C^θ be the chain system, which consists of the modules C_n with the boundary operator ∂^θ. We shall describe C^θ as *conjugate* to C.

Let $f: C \rightarrow C'$ be a chain mapping, let $f^\theta{}_n = s_\theta f_n s_\theta^{-1}: C_n \rightarrow C'_n$, and let $f^\theta = \{f^\theta{}_n\}$. Obviously $f^\theta r = rf^\theta$ and $f^\theta\partial^\theta = s_\theta f\partial s_\theta^{-1} = s_\theta\partial f s_\theta^{-1} = \partial^\theta f^\theta$. There-

[21] $s_\theta C_r = C_r$ since $s_\theta m_i = m_i$ and C_r is a basic module in M.

fore $f^\theta: C^\theta \to C'^\theta$ is a chain mapping. On transforming the relevant equations by s_θ we see that, if $f: C \equiv C'$, then

$$f^\theta: C^\theta \equiv C'^\theta. \tag{7.1}$$

Let $C \equiv 0$ and let ∂, δ satisfy (6.2). Then ∂^θ and $\delta^\theta = s_\theta \delta s_\theta^{-1}$ obviously satisfy (6.2). It follows from (6.5) and induction on m, or from the explicit formulae (6.14), that the construction for C^m, with ∂, δ replaced by $\partial^\theta, \delta^\theta$, leads to the conjugate system $(C^m)^\theta$. Therefore

$$\tau(C^\theta) = (-1)^m \tau(\partial^\theta{}_{m+1}). \tag{7.2}$$

Let $\theta\Lambda = \Lambda$. Then it follows from (7.2) that

$$\tau(C^\theta) = \theta\tau(C), \tag{7.3}$$

where $\theta: \mathrm{T} \to \mathrm{T}$ is defined by (2.10).

8. Mapping cylinders. Let C, C' be disjoint chain systems, with boundary operators ∂, ∂', and let $f: C \to C'$ be a chain mapping. Let αC_{n-1} be the image of C_{n-1} in a simple isomorphism $\alpha: C_{n-} \approx \alpha C_{n-1}$, which is induced by a permutation $P_{n-1}: M \to M$. Then αC_{n-1} is a basic module. Let $C^*{}_n$ be the direct sum $C^*{}_n = C'_n + C_n + \alpha C_{n-1}$. Let $\partial: C^*{}_n \to C^*{}_{n-1}$ be defined by

$$(8.1) \quad \begin{cases} \text{a)} \quad \partial^* c = \partial c, \quad \partial^* c' = \partial' c' & (c \,\varepsilon\, C_n,\ c' \,\varepsilon\, C'_n) \\ \text{b)} \quad \partial^* \alpha c = (f - 1 - \alpha\partial) c & (c \,\varepsilon\, C_{n-1}). \end{cases}$$

We shall write (8.1b) as $\partial^*\alpha = f - 1 - \alpha\partial$, using 1, f as abbreviations for i, $i'f$, where $i: C_n \to C^*{}_n$, $i': C'_n \to C^*{}_n$ are the identities. Obviously $\partial^*\partial^*(C'_n + C_n) = 0$ and $\partial^*\partial^*\alpha = \partial' f - \partial - \partial^*\alpha\partial = (f-1)\partial - (f - 1 - \alpha\partial)\partial = 0$. Therefore $C^* = \{C^*{}_n\}$, with ∂^* as boundary operator, is a chain system. We shall call it the *mapping cylinder* of f. Clearly C, C' are subsystems of C.

LEMMA 5. *$C^* - C'$ is collapsible.*

Let $C'' = C^* - C'$. Then $C''_n = C_n + \alpha C_{n-1}$ and

$$\partial''(c_2 + \alpha c_1) = \partial c_2 - (1 + \alpha\partial) c_1 = (\partial c_2 - c_1) - \alpha\partial c_1, \tag{8.2}$$

where $c_2 \,\varepsilon\, C_n$, $c_1 \,\varepsilon\, C_{n-1}$. Let $\partial^0: C'' \to C''$ be given by

$$\partial^0 c = 0, \qquad \partial^0 \alpha c = c.$$

Then $\partial^0\partial^0 = 0$ and it follows that $\{C''_n\}$, with ∂^0 as boundary operator, is a chain system C^0, which is obviously collapsible. Let $g: C'' \to C^0$ be given by

$g(c_2 + \alpha c_1) = c_2 + \alpha(\partial c_2 - c_1)$. Then $g\partial''(c_2 + \alpha c_1) = g(\partial c_2 - c_1 - \alpha\partial c_1) = \partial c_2 - c_1 + \alpha(-\partial c_1 + \partial c_1) = \partial^0 g(c_2 + \alpha c_1)$. Therefore it follows from Lemma 1, Section **3**, that $g: C'' \approx C^0$ (Σ), and Lemma 5 is proved.

It follows from Lemma 5 and Theorem 4 that $C^* \equiv C'$ (Σ), rel. C'. Therefore C' is a D. R. of C^* and any retraction $k': C^* \to C'$ is a simple equivalence. Let k' be given by $k'c = fc$, $k'c' = c'$, $k'\alpha c = 0$ $(c \varepsilon C, c' \varepsilon C')$. Then it follows from (8.1) that $k'\partial^* = \partial' k'$. Therefore k' is a chain mapping, which is a retraction, since $k'c' = c'$. Also $k'c = fc = fic$. Therefore we have the corollary.

COROLLARY. *$k': C^* \equiv C'$ (Σ), rel. C', and $f = k'i$.*

Let $C(f) = C^* - C$. Then $C_n(f) = C'_n + \alpha C_{n-1}$, and it follows from (8.1) that the boundary operator, $\partial^f: C(f) \to C(f)$, is given by

$$\partial^f c' = \partial' c', \qquad \partial^f \alpha = f - \alpha\partial. \tag{8.3}$$

LEMMA 6. *$C(f) \equiv 0$ if, and only if, $f: C \equiv C'$.*

Let $C(f) \equiv 0$. Then C is a D. R. of C^*, according to Theorem 3, Corollary 2. Therefore $i: C \equiv C^*$ and, by the corollary to Lemma 5, $k': C^* \equiv C'$. Therefore $f: C \equiv C'$.

Conversely, let $f: C \equiv C'$. Then

$$f_* = k'_* i_*: H_n(C) \approx H_n(C') \qquad (n = 0, 1, \cdots)$$

and it follows that $i_*: H_n(C) \approx H_n(C^*)$. Therefore it follows from the exactness of the sequence

$$H_n(C) \to H_n(C^*) \to H_n\{C(f)\} \to H_{n-1}(C) \to H_{n-1}(C^*)$$

that $H_n\{C(f)\} = 0$ for every[22] $n \geq 0$. Therefore $C(f) \equiv 0$ by Theorem 3, Corollary 1.

LEMMA 7. *If $f \simeq g: C \to C'$, then $C(f) \approx C(g)$ (Σ).*

Let $g - f = \partial\eta + \eta\partial$, where $\eta: C \to C'$ is a deformation operator, and let βC_n and $\beta: C_n \approx \beta C_n$ be the analogues, in $C(g)$, of αC_n and α. Let $h: C(f) \to C(g)$ be given by $h(c' + \alpha c) = (c' - \eta c) + \beta c$, which we write as $hc' = c'$, $h\alpha = \beta - \eta$. Then $\partial h = h\partial$ in C' and since $hf = f$ we have[23]

$$\partial h\alpha = \partial(\beta - \eta) = g - \beta\partial - (g - f - \eta\partial) = f - (\beta - \eta)\partial = hf - h\alpha\partial = h\partial\alpha.$$

Therefore it follows from Lemma 1 that $h: C(f) \approx C(g)$ (Σ) and the lemma is proved.

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[22] Cf. Section **3** in [5].

[23] We now use ∂ to denote the boundary operator in all our systems.

Let $f: C \equiv C'$. Then it follows from Lemma 6 that $C(f) \equiv 0$. We define $\tau(f) = \tau\{C(f)\}$ and call $\tau(f)$ the *torsion* of f. It follows from Lemma 7 that $\tau(f)$ depends only on the chain-homotopy class containing f and we shall also call it the *torsion* of this class.

Let $f: C \to C'$, $f': C' \to C''$ be any chain mappings.

LEMMA 8. *There is a chain system D, containing $C(f)$ as a sub-system, such that*[24] $D - C(f) = C(f')$ *and* $D \equiv C(f'f)$ (Σ).

Let $\alpha' C'_n \subset C(f')$ and $\alpha'' C_n \subset C(f'f)$ be the analogues of αC_n. Let C'^* be the mapping cylinder of f' and let D be the direct sum $D = C'^* + C(f)$ with the "united sub-system" C'. That is to say $D_n = C''_n + C'_n + \alpha' C'_{n-1} + \alpha C_{n-1}$ and $\partial: D \to D$ is determined by $\partial: C'^* \to C'^*$ and $\partial: C(f) \to C(f)$, which coincide in C'. Thus $\partial\alpha' = f' - 1 - \alpha'\partial$, $\partial\alpha = f - \alpha\partial$. Moreover $C(f)$ is a sub-system of D and obviously $D - C(f) = C(f')$.

Let D' be the direct sum $D' = C'^* + C(f'f)$, with the united sub-system C''. Then $D'_n = C''_n + C'_n + \alpha' C'_{n-1} + \alpha'' C_{n-1}$ and $\partial\alpha'' = f'f - \alpha''\partial$.

Let $g: D \to D'$ be given by

$$g(c'^* + \alpha c) = (c'^* - \alpha' f c) + \alpha'' c \qquad (c \,\varepsilon\, C,\ c'^* \,\varepsilon\, C'^*),$$

which we write as $gc'^* = c'^*$, $g\alpha = \alpha'' - \alpha' f$. Then $\partial g = g\partial$ in C'^*. Since $gfc = fc$ we have

$$\begin{aligned}\partial g\alpha = \partial\alpha'' - \partial\alpha' f &= f'f - \alpha''\partial - (f' - 1 - \alpha'\partial)f = f - \alpha''\partial + \alpha'\partial f \\ &= f - (\alpha'' - \alpha' f)\partial = g(f - \alpha\partial) = g\partial\alpha.\end{aligned}$$

Therefore $g: D \approx D'$ (Σ). Clearly $D' - C(f'f) = C'^* - C''$ and it follows from Lemma 5 and Theorem 4(b) that $C(f'f) \equiv D' \approx D$ (Σ). This completes the proof.

Let $f: C \equiv C'$, $f': C' \equiv C''$.

THEOREM 7. $\tau(f'f) = \tau(f') + \tau(f)$.

Let D mean the same as in Lemma 8. Then it follows from Lemma 8 and Theorems 5, 6 that $\tau\{C(f'f)\} = \tau(D) = \tau\{C(f')\} + \tau\{C(f)\}$, and the theorem is proved.

LEMMA 9. *If* $f: C \approx C'$ (Σ) *then* $\tau(f) = 0$.

Let $f: C \approx C'$ (Σ) and let C^* be the mapping cylinder of f. Let $g: C(f) \to C^* - C'$ be given by $g(c' + \alpha c) = f^{-1}c' - \alpha c$. Then $\partial g c' = g\partial c'$ and it follows from (8.2) that

[24] I. e. $D - C(f) = C(f')$ if the permutations $\alpha': C'_n \to \alpha' C'_n \subset C(f')$ are suitably chosen.

$$\partial g\alpha = 1 + \alpha\partial = f^{-1}f + \alpha\partial = g(f - \alpha\partial) = g\partial\alpha.$$

Therefore $g: C(f) \approx C^* - C'$ (Σ) and the lemma follows from Lemma 5 and the corollary to Theorem 5.

Let $f: C \equiv C'$ and let us discard the (implicit) condition that $C \cap C' = 0$. Let $h: A \approx C$ (Σ), where A is a chain system such that $A \cap C' = 0$. Then $fh: A \equiv C'$ and we define $\tau(f)$ by $\tau(f) = \tau(fh)$. Let (h', A) be any other pair such that $h': A \approx C$ (Σ), $A' \cap C' = 0$. Let $h'': A'' \approx C$ (Σ), where A'' is disjoint from A, A', C'. Then $fh: A \equiv C'$, $h^{-1}h'': A'' \approx A$ (Σ), $fh'': A'' \equiv C'$. Therefore it follows from Theorem 7 and Lemma 9 that $\tau(fh'') = \tau(fh) + \tau(h^{-1}h'') = \tau(fh)$. Similarly $\tau(fh'') = \tau(fh')$. Therefore $\tau(f)$ is independent of the choice of h, A.

Let $f \simeq g: C \equiv C'$, where $C \cap C' \neq 0$. Then $fh \cong gh: A \equiv C'$, and in consequence of Lemma 7 we have:

THEOREM 8. *If $f \simeq g: C \equiv C'$, then $\tau(f) = \tau(g)$.*

Let $f: C \equiv C'$, $f': C' \equiv C''$. Let $h: A \approx C$ (Σ), $h': A' \approx C'$ (Σ), where A. A' are disjoint from C', C'' and from each other. Then $f'h': A' \equiv C''$, $h'^{-1}fh: A \equiv A'$, $f'fh: A \equiv C''$, and $h'^{-1}: C' \approx A'$ (Σ), $fh: A \equiv C'$. Therefore it follows from Theorem 7, and Lemma 9 that

$$\begin{aligned}\tau(f'f) = \tau(f'fh) = \tau(f'h' \cdot h'^{-1}fh) &= \tau(f'h') + \tau(h'^{-1}fh) \\ &= \tau(f'h') - \tau(h') + \tau(fh) = \tau(f') + \tau(f).\end{aligned}$$

Therefore Theorem 7 is valid, even when C, C', C'' are not disjoint from each other.

Similarly Lemma 9 is valid, even if $C \cap C' \neq 0$.

THEOREM 9. *Given $g: C \equiv C'$, then $\tau(g) = 0$ if, and only if, $g: C \equiv C'$ (Σ).*

It follows from Theorem 7 and Lemma 9 that we may assume $C \cap C' = 0$. This being so, let $\tau(g) = 0$ and let C^* be the mapping cylinder of g. Since $\tau\{C(g)\} = \tau(g) = 0$ it follows from Theorem 5 that $C(g) \equiv 0$ (Σ). Therefore it follows from Theorem 4(b) that $C^* \equiv C$ (Σ), rel. C, whence $i: C \equiv C^*$ (Σ). Therefore it follows from Corollary to Lemma 5 that $g: C \equiv C'$ (Σ).

Conversely, let $g: C \equiv C'$ (Σ). Then $f: B + C \approx B' + C'$ (Σ), $g \simeq k'fi$, where B, B' are collapsible systems, $i: C \to B + C$ is the identity and $k': B' + C' \to C'$ is a retraction. Assume that $\tau(i) = \tau(i') = 0$, where $i': C' \to B' + C'$ is the identity. Then $\tau(k') = 0$, since $\tau(k') + \tau(i') = \tau(k'i') = \tau(1) = 0$. Also $\tau(f) = 0$, according to Lemma 9, and it follows from Theorems 7, 8 that $\tau(g) = 0$.

It remains to prove that $\tau(i) = \tau(i') = 0$. Let $h : A \approx C\ (\Sigma)$, where $A \cap (B + C) = 0$. Since $ihA = C$ it follows from (8.3) that $C(ih) = B + C(h)$, and from Theorems 5, 6 and Lemma 6 that $\tau\{C(ih)\} = \tau(B) + \tau\{C(h)\} = 0$. Therefore $\tau(i) = \tau(ih) = 0$ and similarly $\tau(i') = 0$. This completes the proof.

Let A, A' be sub-systems of C, C' and let $h : C \rightarrow C'$ be a chain mapping such that $hA \subset A'$. Let $B = C - A$, $B' = C' - A'$ and let $f : A \rightarrow A'$, $g : B \rightarrow B'$, be the chain mappings induced by h.

THEOREM 10. *If any two of f, g, h are chain equivalences so is the third, and $\tau(h) = \tau(f) + \tau(g)$.*

Assuming that $C \cap C' = 0$ we have

$$C_n(h) = C'_n + \alpha C_{n-1} = A'_n + B'_n + \alpha A_{n-1} + \alpha B_{n-1} = C_n(f) + C_n(g).$$

Let $D = C(h) - C(f)$. Then $D_n = C_n(g)$ and I say that $D = C(g)$. For let ∂_X denote the boundary operator in X, where X stands for any of the systems $C, C', \cdots$. Since $A' \subset C(f)$ we have [25]

$$\partial_D b' \equiv \partial_{C(h)} b' = \partial_{C'} b' \equiv \partial_{B'} b' = \partial_{C(g)} b', \qquad \text{mod. } C(f),$$

where $b' \varepsilon B'$. Since $\alpha A \subset C(f)$ we have $\alpha c_1 \equiv \alpha c_2$, mod. $C(f)$, if $c_1 \equiv c_2$, mod. A, where $c_1, c_2 \subset C$. Therefore

$$\partial_D \alpha b \equiv \partial_{C(h)} \alpha b = hb - \alpha \partial_C b \equiv gb - \alpha \partial_B b = \partial_{C(g)} b, \qquad \text{mod. } C(f),$$

where $b \,\varepsilon\, \mathrm{B}$. Therefore $\partial_D d \equiv \partial_{C(g)} d$, mod. $C(f)$, for any $d \,\varepsilon\, D$. But $D \cap C(f) = 0$. Therefore $\partial_D = \partial_{C(g)}$, whence $D = C(g)$. The theorem now follows from Lemma 6 and Theorem 6.

In consequence of Theorem 6 we have:

COROLLARY. *If any two of f, g, h are simple equivalences, so is the third.*

Using the same notation as in Theorem 10, let $A' = A = C \cap C'$ and let $h : C \rightarrow C'$ be rel. A. Then we define the *mapping cylinder*, C^*, of h in the same way as when $A = 0$, except that $C^*_n = C'_n + B_n + \alpha B_{n-1} = B'_n + C_n + \alpha B_{n-1}$ where $\alpha : B_n \approx \alpha B_n\ (\Sigma)$ is induced by a permutation, $P_n : M \rightarrow M$, and ∂^* is given by (8.1), with $c \,\varepsilon\, B$ and $\partial = \partial_B$ in (8.1b). Obviously $C^* - C = C(g)$. Therefore it follows from Theorem 10, with $f = 1$, and from Lemma 6, that $h : C \equiv C'$ if, and only if, $C^* - C \equiv 0$. It

[25] If Y is a sub-system of X, then $x \equiv x'$, mod $Y\ (x, x' \subset X)$ means that $x - x' \,\varepsilon\, Y$. Clearly $\partial_z x \equiv \partial_x x$, mod Y, where $Z = X - Y$.

follows from the corollary to Theorem 10 and Theorem 9 that h is a simple equivalence if, and only if,

$$C^* - C \equiv 0\ (\Sigma). \tag{8.4}$$

Let $f: C \equiv C'$ and let $\theta: R \approx R$ be any Λ-automorphism. Then $f^\theta: C^\theta \equiv C'^\theta$, according to (7.1). Since α, in (8.1), is the isomorphism induced by a permutation it follows that $\alpha s_\theta = s_\theta \alpha$. Therefore it follows from (8.3) that $C(f^\theta) = C(f)^\theta$ and from (7.3) that

$$\tau(f^\theta) = \theta\tau(f). \tag{8.5}$$

Let $f: C \equiv C'$. In order to calculate $\tau(f)$ we need to know a deformation operator, $\xi: C(f) \to C(f)$, such that $\partial\xi + \xi\partial = 1$. Let f and $f': C' \to C$ be related by $f'f - 1 = \partial\eta + \eta\partial$, $ff' - 1 = \partial\eta' + \eta'\partial$, where $\eta: C \to C$, $\eta': C' \to C'$ are deformation operators. Let

$$\mu = f\eta - \eta' f: C \to C'. \tag{8.6}$$

Then

$$\begin{aligned} \partial\mu + \mu\partial &= \partial f\eta - \partial\eta' f + f\eta\partial - \eta' f\partial = f(\partial\eta + \eta\partial) - (\partial\eta' + \eta'\partial)f \\ &= f(f'f - 1) - (ff' - 1)f = 0. \end{aligned}$$

Hence it follows by a straightforward calculation that $\partial\xi + \xi\partial = 1$, where $\xi: C(f) \to C(f)$ is given by

$$\begin{cases} \xi\alpha = \alpha\eta - \mu\eta \\ \xi \mid C' = \alpha f' - \mu f' - \eta'. \end{cases} \tag{8.7}$$

9. The groupoid $\mathfrak{G}$. Let R be the integral group ring of a group Γ. We need to consider chain mappings which do not necessarily commute with the operators $r \in R$. By a *chain mapping*, $f: C \to C'$, *associated with an automorphism*, $\theta: R \approx R$, we shall mean a family of homomorphisms, $f_n: C_n \to C'_n$, such that $f\partial = \partial f$ and $fr = (\theta r)f$. We now insist that $C_0 \neq 0$ and that, if m_i is a basis element of C_0, then $f_0 m_i$ shall be a basis element of C'_0. This ensures that f is associated with only one automorphism θ (unlike $C \to 0$, for example). If $f': C' \to C''$ is associated with $\theta': R \approx R$, then $f'f: C \to C''$ is obviously associated with $\theta'\theta$. Let $x \in R$ be any regular element. Then $\partial x = x\partial$ and $x(rc) = (xrx^{-1})xc$. Therefore $x: C \to C$, given by $c \to xc$, is a chain mapping associated with the inner automorphism θ_x. We shall confine ourselves to chain mappings associated with those automorphisms of R, which are determined by automorphisms of Γ, and we shall use the same symbol to denote $\theta: \Gamma \approx \Gamma$ and the corresponding automorphism of R.

We define chain homotopy and chain equivalence as in CH(II), with Γ playing the part of $\bar{\rho}_1$. Thus $f \simeq g: C \to C'$ means that

$$(9.1) \qquad \gamma g - f = \partial\eta + \eta\partial,$$

where $\gamma \varepsilon \Gamma$ and $\eta: C \to C'$ is a chain deformation operator associated with the same automorphism, θ, as f.. As in CH(II) it follows that g is associated with $\theta\gamma^{-1}\theta$. We shall write $f \cong g$ if, and only if, f, g are related by (9.1), with $\gamma = 1$. As in the ordinary theory of homotopy or chain homotopy, $f \simeq g$ implies $fh \simeq gh$, $h'f \simeq h'g$, where h, h' are any chain mappings of the form $h: C^0 \to C$, $h': C' \to C''$.

We say that $f: C \to C'$ is a *chain equivalence* and write $f: C \equiv C'$, if, and only if there is a chain mapping, $g: C' \to C$, such that $gf \simeq 1$, $fg \simeq 1$. Let $\gamma, \gamma' \subset \Gamma$ be such that $\gamma gf \cong 1$, $\gamma' fg \cong 1$. Then $f'f \cong 1$, where $f' = \gamma g$. On transforming $\gamma' fg \cong 1$ by γ' we have $fg\gamma' \cong 1$. Therefore $ff' \cong ff'fg\gamma' \cong fg\gamma' \cong 1$.

Let $f: C \equiv C'$, where f is associated with $1: R \approx R$. That is to say, $fr = rf$. Let $f': C' \to C$ be such that $f'f \cong 1$, $ff' \cong 1$. Since $f'f$ is associated with only one $\theta: R \approx R$ and since $fr = rf$ it follows that $f'r = rf'$. Therefore $f: C \equiv C'$, in the sense of Lemma 6.

Let $f: C \equiv C'$ be associated with $\theta: R \approx R$ and let $\theta_1, \theta_2: R \approx R$ be given. Using the same notation as in Section 7 we have

$$\partial'^{\theta_1} s_{\theta_1} f s_{\theta_2}{}^{-1} = s_{\theta_1} \partial' f s_{\theta_2}{}^{-1} = s_{\theta_1} f \partial s_{\theta_2}{}^{-1} = s_{\theta_1} f s_{\theta_2}{}^{-1} \partial^{\theta_2}.$$

Therefore

$$(9.2) \qquad s_{\theta_1} f s_{\theta_2}{}^{-1}: C^{\theta_2} \to C'^{\theta_1}$$

is a chain mapping, which is obviously associated with $\theta_1\theta\theta_2{}^{-1}$. Let $f': C' \to C$ be such that $f'f \cong 1$, $ff' \cong 1$. On transforming $f'f \cong 1$ by s_θ we have $s_\theta f' f s_\theta{}^{-1} \cong 1: C^\theta \to C^\theta$. Also $fs_\theta{}^{-1} \cdot s_\theta f' \cong 1: C' \to C'$. Therefore it follows from (9.2), with $\theta_1 = 1$, $\theta_2 = \theta$, that $fs_\theta{}^{-1}: C^\theta \equiv C'$. Moreover $(fs_\theta{}^{-1})r = r(fs_\theta{}^{-1})$. We define $\tau(f)$ by

$$(9.3) \qquad \tau(f) = \tau(fs_\theta{}^{-1})$$

and call it the *torsion* of f. Let $\theta_1, \theta_2: R \approx R$ be arbitrary. Then $s_{\theta_1} f s_{\theta_2}{}^{-1}$ is associated with $\theta_1\theta\theta_2{}^{-1}$ and it follows from (9.3) and (8.5) that

$$(9.4) \qquad \tau(s_{\theta_1} f s_{\theta_2}{}^{-1}) = \tau(s_{\theta_1} f s_{\theta_2}{}^{-1} \cdot s_{\theta_2} s_\theta{}^{-1} s_{\theta_1}{}^{-1}) = \tau(s_{\theta_1} \cdot fs_\theta{}^{-1} \cdot s_{\theta_1}{}^{-1}) = \theta_1 \tau(f).$$

Let $f' : C' \equiv C''$ and let f' be associated with $\theta' : R \approx R$. Then $f'f : C \equiv C''$ is associated with $\theta'\theta$, and it follows from Theorem 7 and (9.4) that

$$\tau(f'f) = \tau(f'fs_\theta^{-1}s_{\theta'}^{-1}) = \tau(f's_{\theta'}^{-1} \cdot s_{\theta'}fs_\theta^{-1}\, s_{\theta'}^{-1}) \tag{9.5}$$
$$= \tau(f') + \tau(s_{\theta'} \cdot fs_\theta^{-1} \cdot s_{\theta'}^{-1}) = \tau(f') + \theta'\tau(f).$$

Let $f \cong g : C \equiv C'$ and let f, g be related by (9.1). Then $\gamma g, f$ and η are all associated with the same automorphism, θ, and

$$\gamma g s_\theta^{-1} - fs_\theta^{-1} = \partial'\eta s_\theta^{-1} + \eta s_\theta^{-1} s_\theta \partial s_\theta^{-1} = \partial'\xi + \xi\partial^\theta,$$

where $\xi = \eta s_\theta^{-1}$. Therefore $fs_\theta^{-1} \cong \gamma g s_\theta^{-1}$ and it follows from Theorem 8 that $\tau(f) = \tau(\gamma g)$. Also it follows from (9.5) and (2.11) that $\tau(\gamma g) = \tau(\gamma) + \theta_\gamma \tau(g) = \tau(\gamma) + \tau(g)$, where $\gamma : C' \to C'$ is the chain mapping $c' \to \gamma c'$, which is obviously a chain equivalence. Since $\gamma s_{\theta\gamma}^{-1} m_i = \gamma m_i$, where m_i is any basis element of C'_n, it follows from (2.7) that $\gamma s_{\theta\gamma}^{-1} : C'^{\theta\gamma} \approx C'\ (\Sigma)$, whence $\tau(\gamma) = 0$. Therefore $f \cong g$ implies

$$\tau(f) = \tau(g). \tag{9.6}$$

Let $\mathfrak{G}$ be the totality of chain homotopy classes of equivalences between all the chain systems, which are equivalent to a given one. Let $f : C \equiv C'$, $f' : C' \equiv C''$ be such equivalences and $\bar{f}$, $\bar{f}'$ the corresponding chain homotopy classes. We define $\bar{f}'\bar{f} = \overline{f'f}$. It may be verified that, when multiplication is thus defined, $\mathfrak{G}$ is a groupoid. Let f be associated with θ. Then we define $f : \mathrm{T} \to \mathrm{T}$ by $f\tau = \theta\tau$. Let $f \cong g$. Then g is associated with $\theta_\gamma^{-1}\theta$, for some $\gamma \in \Gamma$, and it follows from (2.11) that $g\tau = \theta_\gamma^{-1}\theta\tau = \theta\tau = f\tau$. Therefore a single-valued map $\bar{f} : \mathrm{T} \to \mathrm{T}$ is defined by $\bar{f}\tau = f\tau$. Obviously $1 \cdot \tau = \tau$ if 1 is any identical map, $C \to C$. Since $f'f$, if it exists, is associated with $\theta'\theta$, where f, f' are associated with θ, θ', it follows that $\bar{f}'(\bar{f}\tau) = (\bar{f}'\bar{f})\tau$. Therefore we say that T admits $\mathfrak{G}$ as a *groupoid of operators.*

It follows from (9.6) that a single-valued map $\tau : \mathfrak{G} \to \mathrm{T}$ is defined by $\tau(\bar{f}) = \tau(f)$ and from (9.5) that

$$\tau(\bar{g}\bar{f}) = \tau(\bar{g}) + \bar{g}\tau(\bar{f}). \tag{9.7}$$

Therefore τ is what, by a natural extension of the language of group theory, we call a *crossed homomorphism* of $\mathfrak{G}$ into T. We call $\tau(\bar{f})$ the *torsion* of $\bar{f}$. Given C it is easy to construct a chain system $C' \equiv C$ and an equivalence, $f : C \equiv C'$, such that $\tau(f)$ is a given element $\tau_0 \in \mathrm{T}$. For example, let $d : A \approx B$ be an isomorphism such that $\tau(d) = \tau_0$, where A, B are basic modules, which are disjoint from C and from each other Let $m > \dim C$ and let C' be the system which consists of C, with its own boundary operator, and

$C'_m = B$, $C'_{m+1} = A$, with $\partial'_{m+1} = d$. Then it follows from Theorem 10 that $\tau(i) = \tau_0$, where i is the identical map $C \to C'$.

10. Homotopy types of complexes. Let K be a given complex and let a 0-cell $e^0 \varepsilon K^0$ be taken as base-point for $\pi_1(K)$. Let $\tilde{K}$ be the universal covering complex of K, in which the points are classes of paths joining e^0 to points in K. Let [26] $C(\tilde{K})$ be defined in the same way as $C(K)$ in Section **12** of CH(II) and let $(c^n_1, \cdots, c^n_{p_n})$ be a natural basis for $C_n(\tilde{K}) = H_n(\tilde{K}^n, \tilde{K}^{n-1})$. Let R, Γ, M mean the same as before, with $\gamma : \pi_1(K) \approx \Gamma$. Let $R(K)$ be the group ring of $\pi_1(K)$. Let $C_n(K) \subset M$ be a basic module of rank p_n $(n = 0, 1, \cdots)$, such that $C_i(K) \cap C_j(K) = 0$ if $i \neq j$. Let $(m^n_1, \cdots, m^n_{p_n})$ be the basis of $C_n(K)$ and let $k_n : C_n(\tilde{K}) \approx C_n(K)$ be defined by

$$(10.1) \qquad k_n(r_1 c_{n1} + \cdots + r_{p_n} c^n_{p_n}) = (\gamma r_1) m^n_1 + \cdots + (\gamma r_{p_n}) m^n_{p_n},$$

where $r_i \varepsilon R(K)$. Let

$$(10.2) \qquad \partial_n = k_{n-1} \partial'_n k_n^{-1} : C_n(K) \to C_{n-1}(K),$$

where ∂' is the boundary operator in $C(\tilde{K})$. Obviously $\partial\partial = 0$ and $\partial r = r\partial$ $(r \varepsilon R)$. Therefore $C(K) = \{C_n(K)\}$, with $\partial = \{\partial_n\}$ as boundary operator, is a chain system and $k = \{k_n\} : C(\tilde{K}) \approx C(K)$ is an isomorphism associated with γ.

The arbitrariness in the definition of $C(K)$ consists of

a) the choice of the base point e^0,

(10.3) b) the choice of $\gamma : \pi_1(K) \approx \Gamma$,

c) the choices of the bases $\{c^n_i\}$ and of the basic modules $C_n(K)$.

Let another 0-cell $e_1^0 \varepsilon K^0$ be taken as base point and let $\tilde{K}_1$ and $C(\tilde{K}_1)$ be the corresponding universal covering complex and chain system. Let

$$\alpha : \pi_1(K, e_1^0) \approx \pi_1(K, e^0), \qquad \phi : \tilde{K}_1 \approx \tilde{K}$$

be the isomorphisms [27] determined by a path $(I, 0, 1) \to (K, e^0, e_1^0)$. Let $h : C(\tilde{K}_1) \approx C(\tilde{K})$ be the isomorphism induced by ϕ. Then h is obviously associated with α, and the same system, $C(K)$, is defined by

[26] Here we reserve the symbol $C(K)$ for a system in which $C_n(K)$ is a basic module in M, as defined by (10.1), (10.2) below. We no longer insist that the elementary chain $c^0 \varepsilon C_0(K)$, associated with a e-cell which covers e^0, shall be associated with the base point in K.

[27] We use the symbol $\approx$ to denote the relation of isomorphism between complexes as well as between groups and chain systems.

$$\gamma\alpha : \pi_1(K, e_1{}^0) \approx \Gamma, \qquad kh : C(\tilde{K}_1) \approx C(K),$$

as by α and k. The effect of choosing a different path, $(I, 0, 1) \to (K, e^0, e_1{}^0)$, is to replace α, h by $\theta_x\alpha$, xh, where $x \varepsilon \pi_1(K, e^0)$. Since

$$x(r_1c^n{}_1 + \cdots + r_{p_n}c^n{}_{p_n}) = (\theta_x r_1)xc^n{}_1 + \cdots + (\theta_x r_{p_n})xc^n{}_{p_n}$$

it follows that the resulting alterations, $\gamma\alpha \to \gamma\theta_x\alpha$ and $kh \to kxh$, are included in (b) and (c).

Let γ be replaced by $\gamma' : \pi_1(K) \approx \Gamma$ and let $\theta = \gamma'\gamma^{-1} : \Gamma \approx \Gamma$. Then it follows from (10.1) and (10.2) that k_n is replaced by $s_\theta k_n$ and ∂ by $\partial^\theta = s_\theta \partial s_\theta{}^{-1}$. Therefore $C(K)$ is replaced by the conjugate system $C^\theta(K)$.

Any other natural basis for $C_n(\tilde{K})$ is of the form $(\pm x_1c^n{}_1, \cdots, \pm x_{p_n}c^n{}_{p_n})$, where $x_i \varepsilon \pi_1(K)$. Any other basic module of rank p_n is of the form PC_n, where $P : M \to M$ is a permutation. Therefore a change in (c) leads to a new system $C'(K) \approx C(K)$ (Σ).

Therefore $C(K)$ is determined up to a transformation, $C(K) \to C'(K)$, which is the resultant of a semi-linear transformation, $C(K) \to C^\theta(K)$, followed by a simple isomorphism $C^\theta(K) \approx C'(K)$ (Σ).

Let $K' \equiv K$ and let $k' : C(\tilde{K}) \approx C(K')$ be defined in the same way as $k : C(\tilde{K}) \approx C(K)$, in terms of an isomorphism $\gamma' : \pi_1(K') \approx \Gamma$. Let $g : C(\tilde{K}) \equiv C(\tilde{K}')$ and $\alpha : \pi_1(K) \approx \pi_1(K')$ be the chain equivalence and the isomorphism induced by a homotopy equivalence [28] $\phi : K \equiv K'$. Let

$$(10.4) \qquad f = k'gk^{-1} : C(K) \to C(K'), \qquad \theta = \gamma'\alpha\gamma^{-1} : \Gamma \approx \Gamma.$$

Then it may be verified that f is a chain equivalence associated with θ. We describe it as the chain equivalence *induced* by ϕ and we define $\tau(\phi) = \tau(f)$. Let $g^* : C(\tilde{K}) \equiv C(\tilde{K}')$ be the chain equivalence induced by a homotopic map $\phi^* \simeq \phi$ and let $f^* = k'g^*k^{-1}$. Then $g^* \simeq g$ and it follows that $f^* \simeq f$. Therefore $\tau(\phi^*) = \tau(\phi)$. Hence, and by the two preceding paragraphs, $\tau(\phi)$ depends only on the homotopy class, $\bar{\phi}$, of maps $K \to K'$, which contains ϕ, on γ, γ' and on the choice of base points [29] in K, K'. We define $\tau(\bar{\phi}) = \tau(\phi)$. Let γ and γ' be replaced by $\gamma_1 : \pi_1(K) \approx \Gamma$ and $\gamma'_1 : \pi_1(K') \approx \Gamma$. Let $\theta = \gamma_1\gamma^{-1}$, $\theta' = \gamma'_1\gamma'^{-1}$. Then k, k' are replaced by $s_\theta k$, $s_{\theta'}k'$ and f by $s_{\theta'}fs_\theta{}^{-1}$. Therefore it follows from (9.4) that $\tau(\bar{\phi})$ is replaced by $\theta'\tau(\bar{\phi})$.

Let us write $\tau \equiv \tau'$, where $\tau, \tau \subset \mathrm{T}$, if, and only if, $\tau' = \theta\tau$, where

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[28] All our maps and homotopies of complexes will be cellular and it is always to be understood that a given map, $K \to K'$, carries e^0 into e'^0, where $e^0 \varepsilon K'^0$ are the base points.

[29] Actually $\tau(\phi)$ does not depend on the choice of base points since $\theta_\gamma\tau = \tau$ for any $\gamma \varepsilon \Gamma$, $\tau \varepsilon \mathrm{T}$.

$\theta:\Gamma\approx\Gamma$. Obviously $\tau\equiv\tau'$ is an equivalence relation and we shall describe the corresponding equivalence classes as *θ-classes.* It follows from the preceding paragraph that the θ-class, $\bar{\tau}(\bar{\phi})$, which contains $\tau(\bar{\phi})$, is uniquely determined by the homotopy class $\bar{\phi}$. We call it the *torsion* of $\bar{\phi}$, or of any map $\phi\in\bar{\phi}$.

We shall describe $\phi:K\equiv K'$ as a *simple* (*homotopy*) *equivalence,* and shall write $\phi:K\equiv K'\ (\Sigma)$, if, and only if, $\tau(\phi)=0$. We shall say that K,K' are of the same *simple homotopy type,* and shall write $K\equiv K'\ (\Sigma)$, if, and only if, there is a simple homotopy equivalence $\phi:K\equiv K'\ (\Sigma)$.

Let $\phi':K'\equiv K''$ and let $C(K'')\approx C(K'')$ be defined in the same way as $C(K)$ and $C(K')$, in terms of an isomorphism $\gamma'':\pi_1(K'')\approx\Gamma$. Let ϕ' be associated with $\alpha':\pi_1(K')\approx\pi_1(K'')$ and let $\theta'=\gamma''\alpha'\gamma'^{-1}:\Gamma\approx\Gamma$. Then it follows from (10.4) and (9.5) that

$$(10.5)\qquad\qquad \tau(\phi'\phi)=\tau(\phi')+\theta'\tau(\phi).$$

Therefore, if ϕ,ϕ' are simple equivalences, so is $\phi'\phi$. Obviously $\tau(\psi)=0$ if $\psi\simeq 1:K\rightarrow K$. Therefore, taking $K''=K$ and $\phi'\phi\simeq 1$, it follows from (10.5) that a homotopy inverse of a simple homotopy equivalence is itself a simple homotopy equivalence. Therefore $K\equiv K'\ (\Sigma)$ is an equivalence relation.

Let G_K be the aggregate of homotopy classes, $\bar{\phi},\bar{\psi},\cdots$, of homotopy equivalences, $\phi,\psi,\cdots$, of K into itself. Let $\bar{\psi}\bar{\phi}=\overline{\psi\phi}$. Then G_K, with this multiplication, is obviously a group. Let $e^0\in K^0$ and $k:C(K)\approx C(K)$ be fixed and let $\mathfrak{G}_K$ be the sub-group of the groupoid $\mathfrak{G}$, which consists of the chain homotopy classes of chain equivalences $C(K)\equiv C(K)$. Let $f_\phi=f$, where f is given by (10.4), with $K'=K$, $\gamma'=\gamma$, $k'=k$, and let $\bar{f}_\phi\in\mathfrak{G}$ be the class which contains f_ϕ. Then $\bar{\phi}\rightarrow\bar{f}_\phi$ is obviously a homomorphism of G_K into $\mathfrak{G}_K$. Let $\tau_K:G_K\rightarrow\mathrm{T}$ be the map which is given by $\tau_K(\bar{\phi})=\tau(\bar{\phi})$. It is the resultant of $\bar{\phi}\rightarrow\bar{f}_\phi$, followed by the crossed homomorphism $\bar{f}\rightarrow\tau(\bar{f})$. Therefore τ_K is a crossed homomorphism, in which G_K operates on T according to the rule $\bar{\phi}\tau=\bar{f}_\phi\tau$.

Let us take $\Gamma=\pi_1(K,p_0)$, where $p_0\in K$, and let $\gamma:\pi_1(K,e^0)\approx\Gamma$ be the isomorphism determined by a path in K, which joins p_0 to e^0. Then the degree of arbitrariness in γ is that it may be replaced by $\theta\gamma$, where $\theta:\Gamma\approx\Gamma$ is an inner automorphism. In this case f_ϕ is replaced by $f_\phi{}^\theta$ and τ_K by $\theta\tau_K:G_K\rightarrow\mathrm{T}$. But $\theta\tau=\tau$, according to (2.11), whence $\theta\tau_K=\tau_K$. Therefore τ_K is uniquely determined by K when $\Gamma=\pi_1(K,p_0)$. For the reasons given in discussing (10.3), τ_K is independent of the choice of e^0.

Let K_0 be a connected sub-complex of K, which contains e^0 and is such that $i:\pi_1(K_0)\approx\pi_1(K)$, where i is the injection homomorphism. Then $\tilde{K}_0=\boldsymbol{p}^{-1}K_0$ may be taken as the universal covering complex of K_0, where $\boldsymbol{p}:\tilde{K}\to K$ is the covering map. Let $C_n(\tilde{K}_0)\subset C_n(\tilde{K})$ be the sub-module consisting of the n-chains carried by $\tilde{K}_0$ and let k_n mean the same as in (10.1). A natural basis for $C_n(\tilde{K}_0)$ is part of a natural basis for $C_n(\tilde{K})$ and it follows that $C(K_0)$, with

$$(10.6)\qquad C_n(K_0)=k_nC_n(\tilde{K}_0),$$

is a sub-system of $C(K)$. Let $U=K-K_0$ and let us denote the residue system $C(K)-C(K_0)$ by $C(U)=C(K)-C(K_0)$. Let $\tilde{U}=\boldsymbol{p}^{-1}U$ and let $C_n(\tilde{U})\subset C_n(\tilde{K})$ be the sub-module consisting of the n-chains carried by $\tilde{U}$. Then obviously $C_n(U)=k_nC_n(\tilde{U})$.

When dealing with such a pair of complexes K and $K_0\subset K$ we shall always assume that $C(K_0)$ is imbedded in $C(K)$ in the way described above.

Let $K_0\subset K$, $L_0\subset L$ be sub-complexes of given complexes K, L. Let $\phi:(K,K_0)\to(L,L_0)$ be a map such that $\phi\mid K-K_0$ is an isomorphism onto $L-L_0$ and let $\phi_0:K_0\to L_0$ be the map which is induced by ϕ.

THEOREM [30] 11. *If ϕ_0 is a simple equivalence, so is ϕ.*

Let $h:C(K)\to C(L)$ and $f:C(K_0)\to C(L_0)$ be the chain mappings which are induced by ϕ and ϕ_0. Then it is obvious that $hC(K_0)\subset C(L_0)$ and that f is the chain mapping induced by h. Since $\phi\mid K-K_0$ is an isomorphism onto $L-L_0$ it is also obvious that $g:C(K-K_0)\approx C(L-L_0)\ (\Sigma)$, where g is the chain mapping induced by h. Therefore the Theorem follows from Theorem 10.

As an application of Theorem 11 let $\phi_0:K_0\equiv L_0$, where L_0 consists of a single 0-cell, let L be formed from K by shrinking K_0 into the point L_0 and let $\phi:K\to L$ be the "identification map." Since $\pi_1(L_0)=1$ it follows that ϕ_0 is a simple equivalence and so therefore is ϕ. In particular we can take $K_0\subset K^1$ to be a tree containing K^0. Then L^0 consists of the single 0-cell L_0.

11. Combinatorial invariance.

In this section we prove:

THEOREM 12. *If K' is a sub-division of K the identical map, $i:K\to K'$, is a simple equivalence.*

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[30] Cf. Theorem 12 in Section **8** of (I). Obviously $\phi_0: K_0\equiv L_0(\Sigma)$ if $K_0=e^0$, $L_0=e'^0$, where $e^0\in K^0$, $e'^0\in L^0$ are the base points. In this case the theorem states that $\phi: K\equiv L(\Sigma)$ if $\phi: K\approx L$.

Let P be a given complex and $Q \subset P$ a sub-complex, which is a D. R. of P.

LEMMA 10. *If every circuit in $P - Q$ is contractible to a point in P, then the identical map, $Q \to P$, is a simple equivalence.*

We first prove the theorem, assuming the truth of the lemma. Let $\phi' : K' \approx L$, where L is a new complex, which does not meet K. Let $\phi = \phi' i : K \to L$. Then $i = \phi'^{-1}\phi$ and $\phi'^{-1} : L \equiv K' \ (\Sigma)$ according to Theorem 11. Therefore it is sufficient to prove that $\phi : K \equiv L \ (\Sigma)$. Let P be the mapping cylinder of ϕ. We regard P as $K \times I$, with $(x, 0) = x \ (x \varepsilon K)$ and $K \times 1$ sub-divided to form L. Let e_0 be a principal cell (i. e. one which is an open sub-set) of K and let $K_1 = K - e_0$. Proceeding by induction we define a sequence of sub-complexes

$$K = K_0, K_1, \cdots, K_n = K^{-1},$$

such that $K_{\lambda+1} = K_\lambda - e_\lambda$, where e_λ is a principal cell of K_λ. Let $P_\lambda = K \cup (K_\lambda \times I)$. Then $P_{\lambda+1}$ is a D. R.[31] of P_λ and $P_\lambda - P_{\lambda+1}$ is the point-set $e_\lambda \times (0, 1\rangle$, where $(0, 1\rangle$ is the half open interval $0 < t \leq 1$. Therefore $\pi_1(P_\lambda - P_{\lambda+1}) = 1$ and it follows from .Lemma 10 that $i_\lambda : P_{\lambda+1} \equiv P_\lambda \ (\Sigma)$, where i_λ is the identical map. Therefore

$$j = i_0 \cdots i_{n-1} : P_n = K \equiv P \ (\Sigma).$$

Similarly $k : L \equiv P \ (\Sigma)$, where k is the identical map. Let $\psi : P \to L$ be given by $\psi(x, t) = \phi x$. Then $\psi : P \equiv L \ (\Sigma)$, since ψ is a homotopy inverse of k. Therefore $\phi = \psi j : K \equiv L \ (\Sigma)$ and Theorem 12 is proved.

It remains to prove Lemma 10. Since Q is a D. R. of P it is easily proved that the chain system $C(Q)$ is a D. R. of $C(P)$ and that a retraction $\psi : P \to Q$ induces a retraction $k : C(P) \to C(Q)$. We have to prove that k is a simple equivalence and this will follow from Theorem 4, Section **5**, when we have proved that $C(U) \equiv 0 \ (\Sigma)$, where $U = P - Q$.

Let $U_1, \cdots, U_m$ be the components of U, which are finite in number since U is the union of a finite number of (connected) cells. Let $\tilde{P}$ be the universal covering complex of P and let $\boldsymbol{p} : P \to \tilde{P}$ be the covering map. Let $\tilde{U}_\lambda$ be any component of $\boldsymbol{p}^{-1} U_\lambda$ and let $U^* = \tilde{U}_1 \cup \cdots \cup \tilde{U}_m$. Since P and $\tilde{P}$ are locally connected, U_λ and $\tilde{U}_\lambda$ are open sets. Let $e^n{}_1, \cdots, e^n{}_{q_n}$ be the n-cells in U. It follows from the condition on the circuits in U, which is satisfied a fortiori by the circuits in U_λ, that $\boldsymbol{p} \mid \tilde{U}_\lambda$ is a homeomorphism onto U_λ. Therefore $\boldsymbol{p} \mid U^*$ is a homeomorphism onto U. Therefore

[31] See Theorem 1. 4(ii) in [16].

U^* contains precisely one, $\tilde{e}^n_i$, of the cells in $\tilde{P}$, which cover e^n_i. Let $c^n_i \varepsilon C_n(\tilde{P})$ be the element which is represented by a characteristic map for $\tilde{e}^n_i$. Then $(c^n_1, \cdots, c^n_{q_n})$ is a basis for $C_n(\tilde{U})$, which is part of a natural basis for $C_n(\tilde{P})$. Moreover $C_n(U^*)$, which consists of the n-chains carried by U^*, is the ordinary free Abelian group, which is freely generated by $c^n_1, \cdots, c^n_{q_n}$, without the help of the operators in $\pi_1(P)$.

Since each component of $\tilde{U}$ is open it follows that no cell in $\tilde{U} - U^*$ meets the closure of $\tilde{e}^n_i$. Therefore

$$\text{(11.1)} \qquad \partial c^n_i = \sum_{j=1}^{q_{n-1}} d_{ij}{}^n c_j{}^{n-1} + c'_i{}^{n-1},$$

where $d_{ij}{}^n$ are integers and $c'_i{}^{n-1} \varepsilon C_n(\tilde{Q})$ $(\tilde{Q} = \mathbf{p}^{-1}Q)$. Let $m_i{}^n = k_n c_i{}^n$, where k_n means the same as in (10.1). Then it follows from (11.1) that $\partial : C(U) \to C(U)$ is given by $\partial m_i{}^n = \sum_{j=1}^{q_{n-1}} d_{ij}{}^n m_j{}^{n-1}$. Therefore $C(U) \equiv 0\ (\Sigma)$, by the corollary to Lemma 4, in Section **6**. This proves Lemma 10.

12. Lens spaces. By way of an example let A, B be the chain systems determined by lens spaces of types (m, p), (m, q), where m is the order of their fundamental groups and $q \equiv k^2 p(m)$. That is to say, A, B play the part of $C(K)$ in Section **10** and $\pi_1(P)$, $\pi_1(Q)$ in Section **15** of CH(II) are both replaced by Γ. The generators ξ, η in CH(II) are replaced by a generator $\gamma \varepsilon \Gamma$ and we denote the integer r by h, to avoid confusion with $r \varepsilon R$. Otherwise the notations will be the same as in CH(II). Thus $\partial : A \to A$ and $\partial : B \to B$ are given by

$$\text{(12.1)} \qquad \begin{cases} \partial a_1 = (\gamma - 1) a_0, & \partial a_2 = \sigma_m(\gamma) a, \qquad \partial a_3 = (\gamma^p - 1) a_2 \\ \partial b_1 = (\gamma - 1) b_0, & \partial b_2 = \sigma_m(\gamma) b_1, \qquad \partial b_3 = (\gamma^q - 1) b_2, \end{cases}$$

where $\sigma_t(\gamma^s) = 1 + \gamma^s + \cdots + \gamma^{(t-1)s}$.

The Reidemeister-Franz torsion in A and B is r_A and r_B, where

$$\text{(12.2)} \qquad r_A = (\gamma - 1)(\gamma^p - 1), \qquad r_B = (\gamma - 1)(\gamma^q - 1).$$

Let $\theta : \Gamma \approx \Gamma$ be given by $\theta\gamma = \gamma^k$ and let $u : A \to B$, $v : B \to A$ be the chain mappings, associated with θ and with θ^{-1}, which are given by $ua_n = b_n$, $vb_n = a_n$ if $n = 0$ or 3 and

$$\text{(12.3)} \qquad \begin{cases} ua_1 = \sigma_k(\gamma) b_1, & ua_2 = \sigma_k(\gamma^{kp}) b_2 \\ vb_1 = \sigma_l(\gamma) a_1, & vb_2 = \sigma_l(\gamma^{lq}) a_2, \end{cases}$$

where $kl = 1 + mh$. As shown in CH(II), $vu - 1 = \partial\eta + \eta\partial$, $uv - 1$

$= \partial\eta + \eta\partial$, where $\eta a_n = \eta b_n = 0$ if $n \neq 1$ and $\eta a_1 = h a_2$, $\eta b_1 = h b_2$. Notice that $\eta\eta = 0$ and $\eta s_\theta = s_\theta \eta$, since a_n, b_n are generators, m_{i_n}, m_{j_n}, of M.

It follows from (12. 1) that

$$(12.\ 4) \qquad \partial^\theta a_1 = (\gamma^k - 1) a_0, \qquad \partial^\theta a_2 = \sigma_m(\gamma) a_1, \qquad \partial^\theta a_3 = (\gamma^{kp} - 1) a_2,$$

and from (12. 3) that

$$(12.\ 5) \qquad s_\theta v b_1 = \sigma_l(\gamma^k) a_1, \qquad s_\theta v b_2 = \sigma_l(\gamma^q) a_2,$$

since $kl \equiv 1(m)$. We proceed to calculate $\tau(u) = \tau\{C(us_\theta^{-1})\}$, where $C(f)$ means the same as in Section **8**. Let $C = C(us_\theta^{-1})$ and let $a'_n = \alpha a_n$, where α means the same as in (8. 2). Then (b_n, a'_{n-1}) is a basis for C_n $(a'_{-1} = b_4 = 0)$. Since $us_\theta^{-1} a_n = u a_n$ it follows from (8. 2), (12. 3) and (12. 4) that $\partial : C \to C$ is given by

$$\begin{aligned} \partial b_1 &= (\gamma - 1) b_0, & \partial a'_0 &= b_0 \\ \partial b_2 &= \sigma_m(\gamma) b_1, & \partial a'_1 &= \sigma_k(\gamma) b_1 - (\gamma^k - 1) a'_0 \\ \partial b_3 &= (\gamma^q - 1) b_2, & \partial a'_2 &= \sigma_k(\gamma^{kp}) b_2 - \sigma_m(\gamma) a'_1 \\ & & \partial a'_3 &= b_3 - (\gamma^{kp} - 1) a'_2. \end{aligned}$$

It is easily verified that $\mu = 0$, where μ is given by (8. 6), and similarly that $s_\theta v \eta = \eta s_\theta v$. Since $\eta\eta = 0$ a straightforward caluculation shows that $\delta\delta = 0$, where δ is given by (8. 7), with $\xi = \delta$. It follows from (8. 7) and (12. 5) that $\delta b_0 = a'_0$ and

$$\begin{aligned} \delta b_1 &= - h b_2 + \sigma_l(\gamma^k) a'_1, & \delta a'_0 &= 0 \\ \delta b_2 &= \sigma_l(\gamma^q) a'_2, & \delta a'_1 &= h a'_2 \\ \delta b_3 &= a'_3, & \delta a'_2 &= 0. \end{aligned}$$

Let $D_0 = C_0 + C_2 + C_4$, $D_1 = C_1 + C_3$, as in (6. 8) with $m = \dim C - 1 = 3$. Then D_0, D_1 have (b_0, a'_1, b_2, a'_3), (a'_0, b_1, a'_2, b_3) as bases and $\Delta_0 = \partial + \delta : D_0 \to D_1$ is given by

$$\begin{aligned} \Delta_0 b_0 &= a'_0 \\ \Delta_0 a'_1 &= -(\gamma^k - 1) a'_0 + \sigma_k(\gamma) b_1 + h a'_2 \\ \Delta_0 b_2 &= \sigma_m(\gamma) b_1 + \sigma_l(\gamma^q) a'_2 \\ \Delta_0 a'_3 &= -(\gamma^{kp} - 1) a'_2 + b_3. \end{aligned}$$

Let $f_0 : D_0 \approx D_0$, $f_1 : D_1 \approx D_1$ be the simple automorphisms which are given by

$$\begin{aligned} f_0 a'_1 &= a'_1 + (\gamma^k - 1) b_0, & f_0 c &= c & (c &= b_0, b_2, a'_3) \\ f_1 b_3 &= b_3 + (\gamma^{kp} - 1) a'_2, & f_1 c &= c & (c &= a'_0, b_1, a'_2). \end{aligned}$$

Then $\Delta'_0 = f_1\Delta_0 f_0 : D_0 \to D_1$ is given by $\Delta'_0 b_0 = a'_0$, $\Delta'_0 a'_3 = b_3$ and

$$\Delta'_0 a'_1 = \sigma_k(\gamma) b_1 + h a'_2$$
$$\Delta'_0 b_2 = \sigma_m(\gamma) b_1 + \sigma_l(\gamma^q) a'_2.$$

Let i, j be any integers and let $\rho = (i, j)$. If $i \leq j$ we have, writing $\sigma_s = \sigma_s(\gamma)$ $(\sigma_0 = 0)$,

$$\sigma_j - \gamma^{j-i}\sigma_i = \sigma_{j-i}.$$

Therefore it follows by induction on $i + j$ that the matrix $[\sigma_i, \sigma_j]$ can be reduced to $[\sigma_\rho, 0]$ by a sequence of transformations of the form

$$[\sigma_i, \sigma_j] \to [\sigma_i, \sigma_j - \gamma^s\sigma_i] \text{ or } [\sigma_i - \gamma^s\sigma_j, \sigma_j]$$

followed, if necessary, by $[0, \sigma_\rho] \to [\sigma_\rho, 0]$. Since $(k, m) = 1$ it follows that

$$\begin{bmatrix} \sigma_k(\gamma) & h \\ \sigma_m(\gamma) & \sigma_l(\gamma^q) \end{bmatrix} \to \begin{bmatrix} 1 & r \\ 0 & t \end{bmatrix} \qquad (r, t \subset R)$$

by such elementary transformations of the rows. Since these transformations alter the determinant by a factor ± 1, at most, we have $t = \pm d$, where

$$d = \sigma_k(\gamma)\sigma_l(\gamma^q) - h\sigma_m(\gamma).$$

I say that

$$(12.6) \qquad d r_B = \theta r_A,$$

where r_A, r_B are given by (12.2). For let χ be the homomorphism of R into the complex field, which is given by $\chi r = \omega$, where $\omega^m = 1$. Then $\chi(dr_B) = \chi(\theta r_A) = 0$ if $\omega = 1$, and if $\omega \neq 1$ we have

$$\chi(dr_B) = [(\omega^k - 1)(\omega^{lq} - 1)]/[(\omega - 1)(\omega^q - 1)](\omega - 1)(\omega^q - 1)$$
$$= (\omega^k - 1)(\omega^{lq} - 1) = \chi(\theta r_A).$$

Therefore $\chi(dr_B - r_A) = 0$ and (12.6) follows from the orthogonality relations between the group characters. Therefore $\tau(u)$ is essentially the same as the inverse of the element π in Lemma 5 on p. 1209 of [3].

13. Formal deformations. As in CH(II) let I^n be the n-cube in Hilbert space, which is given by $0 \leq t_1, \cdots, t_n \leq 1$, $t_i = 0$ if $i > n$, with $I^0 = (0, 0, \cdots)$. Let

$$E_0^{n-1} = \partial I^n - (I^{n-1} - \partial I^{n-1}) \qquad (n \geq 1).$$

Let e^n be a principal cell of a complex K and let e^{n-1} be a principal cell of $K - e^n$ $(n \geq 1)$. We shall describe the transformation $K \to K_1 = K - e^n - e^{n-1}$ as an *elementary contraction* if, and only if, e^n has a characteristic map, $f: I^n \to \bar{e}^n$, such that $fE_0^{n-1} \subset K_1$ and $f \mid I^{n-1}$ is a characteristic map for e^{n-1}. The inverse, $K_1 \to K$, of an elementary contraction, $K \to K_1$, will be called an *elementary expansion.* An elementary expansion may also be defined as follows. Let E^n be an n-element, which is disjoint from a given complex, K, and let E^{n-1} be a hemisphere [32] of ∂E^n. Let $e^n = E^n - \partial E^n$, $e^{n-1} = \partial E^n - E^{n-1}$ and let $f: (E^{n-1}, \partial E^{n-1}) \to (K^{n-1}, K^{n-2})$ be an arbitrary map. Let $K_1 = K \cup e^{n-1} \cup e^n$ be the complex formed by identifying each point $p \varepsilon E^{n-1}$ with $fp \varepsilon K^{n-1}$. Then $K \to K_1$ is obviously an elementary expansion.

Either an elementary expansion or an elementary contraction will be called an *elementary deformation.* The resultant, $K_0 \to K_r$, of a finite sequence of elementary deformations,

$$K_i \to K_{i+1} \qquad (i = 0, \cdots, r-1), \tag{13.1}$$

will be called a *formal deformation.* We also include the identical transformation, $K \to K$, of any complex, among the formal deformations. We shall denote a formal deformation by $D: K_0 \to K_r$, and $K_r = DK_0$ will mean that K_r is obtained from K_0 by a formal deformation D. If each $K_i \to K_{i+1}$ is an elementary expansion (contraction) then $K_0 \to K_r$ will be called an *expansion* (*contraction*) and we shall say that K_0 *expands* (*contracts*) into K_r. If $D: K_0 \to K_r$ is the resultant of the sequence (13.1), then the resultant of the sequence $K_{i+1} \to K_i$ is the formal deformation, $D^{-1}: K_r \to K_0$, *inverse* to D.

Let $D: K_0 \to K_1$ be an elementary contraction. Obviously E_0^{n-1} is a D. R. of I^n. Therefore [33] K_1 is a D. R. of K_0 and $[D]$ will denote the homotopy class of maps, $K_0 \to K_1$, which contains a retraction. If $D: K_0 \to K_1$ is an elementary expansion then $[D]$ will denote the homotopy class of maps, $K_0 \to K_1$, which contains the identity. Let $D = D_{r-1} \cdots D_0: K_0 \to K_r$, where D_i stands for (13.1). We define $[D]$ by $[D] = [D_{r-1}] \cdots [D_0]$. If $1: K \to K$ is the identical formal deformation then $[1]$ will denote the homotopy class of the identical map $K \to K$. It is obvious that, if $D: K \to K'$ and $D': K' \to K''$ are formal deformations, then $K'' = D'DK$ and

$$[D'D] = [D'][D]. \tag{13.2}$$

Also it is easily verified that $[D^{-1}][D] = [1]$.

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[32] I. e. the image of E_0^{n-1} in some homeomorphism $\partial I^n \to \partial E^n$.

[33] See Lemma 2 in Section 4 of [4].

Let L be a sub-complex of K, which may be empty. By a formal deformation, $D: K \to K'$, rel. L, we shall mean the resultant of a sequence of elementary deformations, none of which removes a cell of L. By $K' = DK$, rel. L, we shall mean that K' is the image of K in such a formal deformation, D. If $K' = DK$, rel. L, then $[D]$ obviously contains at least one map $\phi_0: K \to K'$, rel. L, where rel. L means that $\phi_0 y = y$ if $y \varepsilon L$. We restrict $[D]$ to maps, $\phi: K \to K'$, rel. L, such that $\phi \simeq \phi_0$, rel. L.

We shall describe a map $\phi: K \to K'$, rel. L, as a *restricted equivalence*, rel. L, if, and only if, $K' = DK$, rel. L, and $\phi \varepsilon [D]$. We shall write $K \equiv K' (\Sigma)$, rel. L, if, and only if, there is a simple equivalence $\phi: K \equiv K' (\Sigma)$, which is rel. L.

THEOREM 13. *$K' = DK$, rel. L, if, and only if, $K \equiv K' (\Sigma)$, rel. L, in which case the restricted equivalences, $K \to K'$, rel. L, are the same as the simple equivalences, $K \to K'$, rel. L.*

In order to prove this we shall need some lemmas, which are proved in the following section.

14. Lemmas on formal deformations. Let K, K' be complexes, with a common sub-complex, L, which may be empty, and let $(K - L) \cap K' = 0$. Let $\phi: K \to K'$ be a map rel. L. By the *mapping cylinder* of ϕ we mean the complex, P, which is formed from[34] $K \times I$ by identifying $(x, 0)$ with x, $(x, 1)$ with ϕx and $y \times I$ with y, for each point $x \varepsilon K$ and $y \varepsilon L$. Let

(14. 1) $$\phi_r: \pi_r(K) \approx \pi_r(L) \qquad (r = 1, \cdots, m),$$

where ϕ_r is the homomorphism induced by ϕ. Then the argument[35] used in the case $L = 0$ shows that

(14. 2) $$i_r: \pi_r(K) \approx \pi_r(P) \qquad (r = 1, \cdots, m),$$

where i_r is the injection. Therefore it follows from the exactness of the sequence

$$\pi_r(K) \to \pi_r(P) \to \pi_r(P, K) \to \pi_{r-1}(K) \to \pi_{r-1}(P),$$

that $\pi_r(P, K) = 0$ for $r = 1, \cdots, m$, where $\pi_1(P, K) = 0$ means that i_1 is onto $\pi_1(P)$.

[34] After replacing $K \times I$ by a homeomorph, if necessary, we assume that it has no point in common with K or K'.

[35] See Section **3** of [5].

LEMMA 11. *P contracts into K'.*

Let $K = K_0 \cup e^n$, where e^n is a principal cell in $K - L$. Then $P = P_0 \cup e^n \cup e^{n+1}$, where $e^{n+1} = e^n \times (0, 1)$ and P_0 is the mapping cylinder of $\phi \mid K_0 : K_0 \to K'$. Let $f : I^n \to \bar{e}^n$ be a characteristic map for e^n. Then $g : I^{n+1} \to \bar{e}^{n+1}$, given by $g(t_1, \cdots, t_n, t) = \{f(t_1, \cdots, t_n), t\}$, is obviously a characteristic map for e^{n+1} and $gE_0{}^n \subset P_0$ and $gx = fx$ if $x \varepsilon I^n$. Therefore P contracts into P_0. Therefore the lemma follows by induction on the number of cells in $K - L$.

Let $\psi : P \to K'$ be given by

$$\psi \mid K' = 1, \qquad \psi(x, t) = \phi x \qquad\qquad (x \varepsilon K).$$

Since $\psi \mid K' = 1$ and since any two retractions $P \to K'$ are homotopic to each other, we have $\psi \varepsilon [D]$, where $D : P \to K'$ is any contraction. Also $\phi = \psi i$, where $i : K \to P$ is the identical map. If $K = D_1 P$, rel. K, then $i \varepsilon [D_1{}^{-1}]$. Hence, and from (13.2), we have the corollary:

COROLLARY. *If $K = D_1 P$, rel. K, then $\phi \varepsilon [DD_1{}^{-1}]$.*

Let K, K' and L be as in Lemma 11, except that $K - L$ and $K' - L$ may now have points in common. Let $\phi : (K, L) \approx (K', L)$, rel. L.

LEMMA 12. *ϕ is a restricted equivalence, rel. L.*

First let $K \cap K' = L$ and let P be the mapping cylinder of ϕ. Then P may also be regarded as the mapping cylinder of ϕ^{-1} and the Lemma follows from Lemma 11 and its corollary.

If $K' - L$ meets K we replace the points in $K' - L$ by new ones, thus forming a complex K'', such that $\phi' : K'' \approx K'$, rel. L, and $K \cap K'' = K' \cap K'' = L$. By what we have already proved, ϕ' and $\phi'^{-1}\phi : K \approx K''$ are restricted equivalences, rel. L. Therefore it follows from (13.2) that ϕ is a restricted equivalence, rel. L, and the lemma is proved.

Let K_0, K_1 be complexes with a common sub-complex, K, and let $K_i = K \cup e_i{}^n$ $(i = 0, 1; n \geq 1)$. Let $f_i : I^n \to e_i{}^n$ be a characteristic map for $e_i{}^n$ in K_i.

LEMMA 13. *If $f_0 \mid \partial I^n \simeq f_1 \mid \partial I^n$ in K, then $K_1 = DK_0$, rel. K.*

First assume that $e_0{}^n \cap e_1{}^n = 0$ and unite K_0, K_1 in the complex $K^* = K_0 \cup K_1$. Let $g_t : \partial I^n \to K$ be a homotopy of $g_0 = f_0 \mid \partial I^n$ into $g_1 = f_1 \mid \partial I^n$ and let $f : \partial I^{n+1} \to K^*$ be given by

$$f(t_1,\cdots,t_n,i)=f_i(t_1,\cdots,t_n) \qquad \{i=0,1;\ (t_1,\cdots,t_n)\,\varepsilon\, I^n\}$$
$$f(s_1,\cdots,s_n,t)=g_t(s_1,\cdots,s_n) \qquad \{t\,\varepsilon\, I;\ (s_1,\cdots,s_n)\,\varepsilon\,\partial I^n\}.$$

We attach a new cell, e^{n+1}, to K^* by means of the map [36] f, thus forming a complex, $L=K^*\cup e^{n+1}$, in which e^{n+1} has a characteristic map, $h: I^{n+1}\to\bar{e}^{n+1}$ such that $h\mid\partial I^{n+1}=f$. Since $h(x,0)=f_0x$ $(x\,\varepsilon\, I^n)$ and [37] $hE_0^n\subset L-e_0^n$ it follows that $L\to K_1$ is an elementary contraction. Similarly $L\to K_0$ is an elementary contraction. Therefore $K_0\to L\to K_1$ is a formal deformation, rel. K.

If $e_0^n\cap e_1^n\neq 0$ we attach a new cell, e'^n_0, to K, by means of the map $f_0\mid\partial I^n$, taking care that $e'^n_i\cap e_i^n=0$ $(i=0,1)$. Then $K_0\to K\cup e'_0\to K_1$ is a formal deformation, rel. K, and the lemma is proved.

Let P be a given complex, let $P_0\subset P$ be a sub-complex and let $k_n(P-P_0)$ denote the number of n-cells in $P-P_0$. Let K be a sub-complex of P_0 and let $D_0: P_0\to Q_0$ be a formal deformation, rel. K. We shall describe a formal deformation, $D: P\to Q$, rel. K, as an *extension* of D_0 if, and only if, Q_0 is a sub-complex of Q and

$$k_n(Q-Q_0)=k_n(P-P_0) \qquad (n=0,1,\cdots).$$

LEMMA 14. *$D_0: P_0\to Q_0$ has an extension $D: P\to Q$.*

Let $D: P\to Q$ be an extension of D_0 and let $D': Q\to Q'$, rel. K, be an extension of a formal deformation $D'_0: Q_0\to Q'_0$. Then $D'D: P\to Q'$ is obviously an extension of $D'_0D_0: P_0\to Q'_0$. Let $P_1\subset P$ be a sub-complex, which contains P_0. Let $D_1: P_1\to Q_1$, rel. K, be an extension of D_0 and let $D: P\to Q$ be an extension of D_1. Then D is obviously an extension of D_0. Therefore the Lemma will follow by a double induction on the number of elementary deformations in D_0 and on the number of cells in $P-P_0$ when we have proved it in case D_0 is an elementary deformation and $P-P_0$ is a single cell.

Let $P=P_0\cup e^n$ and let D_0 be an elementary expansion $D_0: P_0\to Q_0=P_0\cup e^{p-1}\cup e^p$. If e^n has a point in common with $e^{p-1}\cup e^p$ we apply a preliminary formal deformation, $P\to P'$, rel. P_0, as in Lemma 13, so as to replace e^n by a cell which is disjoint from $e^{p-1}\cup e^p$. Then P' and Q_0 may be united in a complex

$$Q=P'\cup e^{p-1}\cup e^p$$

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[36] That is to say, we attach an $(n+1)$-element, E^{n+1} $(e^{n+1}=E^{n+1}-\partial E^{n+1})$, to K^* by means of the map fh': $\partial E^{n+1}\to K^*$, where h': $\partial E^{n+1}\to\partial I^{n+1}$ is a homomorphism.

[37] We recall that $E_0^{n+1}=\partial I^{n+1}-(I^n-\partial I^n)$.

and $P \rightarrow P' \rightarrow Q$ is an extension of D_0.

Let D_0 be an elementary contraction,

$$D_0 : P_0 \rightarrow Q_0 = P_0 - e^p - e^{p-1},$$

and let $f : I^n \rightarrow \bar{e}^n$ be a characteristic map for e^n. Since Q_0 is a D. R. of P_0 there is a map, $f' : \partial I^n \rightarrow Q_0$, which is homotopic, in P_0, to $f \mid \partial I^n$. We attach a new cell, e'^n, to P_0 by means of the map f', thus forming a complex $P' = P_0 \cup e'^n$. Then $P' = D'P$, rel. P_0, by Lemma 13. Since $\partial e'^n \subset Q_0$ it follows that $P' \rightarrow Q = P' - e^p - e^{p-1}$ is an elementary contraction and $P \rightarrow P' \rightarrow Q$ is an extension of D_0. This proves the lemma.

Let K be a (connected) sub-complex of P such that $\pi_n(P, K) = 0$ $(n = 1, \cdots, r)$.

LEMMA 15. *There is a formal deformation $D : P \rightarrow Q$, rel. K, such that $k_n(Q - K) = 0$ if $n \leq r$ and $k_n(Q - K) = k_n(P - K)$ if $n > r + 2$.*

Let $0 \leq p \leq r$ and assume that, if $p > 0$, then $k_n(P - K) = 0$ for $n = 0, \cdots, p - 1$. For the sake of clarity we consider the case $p = 0$ separately. Let $p = 0$ and let $e_1^0, \cdots, e_k^0$ be the 0-cells in $P - K$. Since P is connected there is a map

$$g_i : (E_0^1, I^0, E_0^0) \rightarrow (P^1, e_i^0, K^0).$$

Let $E_1^2, \cdots, E_k^2$ be a set of 2-elements, which are disjoint from P and from each other. Let $h_i : \partial E_i^2 \rightarrow \partial I^2$ be a homeomorphism and let

$$e_i^2 = E_i^2 - \partial E_i^2, \qquad e_i^1 = h_i^{-1}(I^1 - \partial I^1) = h_i^{-1}(\partial I^2 - E_0^1).$$

We attach E_i^2 to P by means of the map $g_i h_i : h_i^{-1} E_0^1 \rightarrow P^1$, thus forming a complex $P^* = P \cup \bigcup_i (e_i^1 \cup e_i^2)$. Then $P \rightarrow P^*$ is an expansion. The complex $K^* = K \cup \bigcup_i (e_i^0 \cup e_i^1)$ contracts into K. By Lemma 14 there is an extension, $P^* \rightarrow M_0$, rel. K, of the contraction $K^* \rightarrow K$. Then

$$k_0(M_0 - K) = k_0(P^* - K^*) = 0$$
$$k_n(M_0 - K) = k_n(P^* - K^*) = k_n(P - K) \qquad (n > 2).$$

Thus we have eliminated the 0-cells from $P - K$ at the expense of introducing $k_0(P - K)$ new 2-cells.

Now let $p > 0$, let $e_1^p, \cdots, e_k^p$ be the p-cells in $P - K$ and let $f_i : I^p \rightarrow \bar{e}_i^p$ be a characteristic map for e_i^p. Since $k_{p-1}(P - K) = 0$, whence

$P^{p-1} = K^{p-1}$, we have $f_i \partial I^p \subset K$. Since $\pi_p(P, K) = 0$ and since I^p, $E_0{}^p$ are two hemispheres of $\partial E_0{}^{p+1} = \partial I^{p+1}$, the map f_i can be extended to a map,

$$(14.3) \qquad g_i : (E_0{}^{p+1}, E_0{}^p) \to (P^{p+1}, K^p).$$

It now follows, in exactly the same way as when $p = 0$, that there is a complex $M_p = D_p P$, rel. K, such that $k_n(M_p - K) = 0$ if $n \leq p$ and $k_n(M_p - K) = k_n(P - K)$ if $n > p + 2$. Therefore the lemma follows by induction on p.

15. Proof of Theorem 13. Let $D : K \to K_1 = K \cup e^{n-1} \cup e^n$, $(n \geq 1)$ be an elementary expansion. Then $i \,\varepsilon\, [D]$, where $i : K \to K_1$ is the identity. Also K is a D. R. of K_1 and $K_1 - K$ is simply connected. Therefore it follows from Lemma 10 that $i : K \to K_1$ is a simple equivalence. Since $k : K_1 \to K$ is a homotopy inverse of i if $k \,\varepsilon\, [D^{-1}]$, it follows that k is also a simple equivalence. Therefore $\phi : K \to DK$ is a simple equivalence, rel. L, if $\phi \,\varepsilon\, [D]$, where D is any elementary deformation, rel L. Therefore it follows from an inductive argument that $\phi : K \equiv K'$ (Σ), rel. L, if $K' = DK$, rel. L, and $\phi \,\varepsilon\, [D]$.

Conversely, let $\phi : K \equiv K'$ (Σ), rel. L. Then it follows from Theorem 11 and Lemma 12 that we may, without loss of generality, replace K' by K'', where $K'' \approx K'$, rel. L. Therefore we assume that $K \cap K' = L$ and also that $e^0 = e'^0 \,\varepsilon\, L$, where $e^0 \,\varepsilon\, K^0$, $e'^0 \,\varepsilon\, K'^0$ are the base points, thus excluding the case $L = 0$. Let P, with base point e^0, be the mapping cylinder of ϕ. Since $\phi : K \equiv K'$ the relations (14.1), (14.2) hold for every $n \geq 1$. Also K' is a D. R. of P. Therefore $i'_1 : \pi_1(K') \approx \pi_1(P)$, where i'_1 is the injection. Therefore $C(K)$, $C(K')$ are sub-systems of $C(P)$, according to the convention (10.6). Let $j : C(L) \to C(P)$ be the chain mapping induced by the identical map $L \to P$. Then $A = jC(L) = C(K) \cap C(K')$. Let $h : C(K) \to C(K')$ be the chain mapping induced by ϕ. Then h is obviously rel. A, since ϕ is rel. L. It may be verified[38] that $C(P)$ is the mapping cylinder of h, as defined in the paragraph containing (8.4). Since, by hypothesis, h is a simple equivalence we have

$$(15.1) \qquad C(P - K) \equiv 0 \ (\Sigma).$$

according to (8.4).

[38] See Section **14** of CH(II), with $\gamma c_\lambda{}^n = 0$ if $c_\lambda{}^n$ corresponds to a cell in $L \subset K$ (not the L in CH(II)).

It follows from (14.1) that $\pi_r(P, K) = 0$ for every $r \geq 1$. Let

(15.2) $q = \operatorname{Max}\{\dim(P - K), 3\} = \operatorname{Max}\{\dim(K - L) + 1, \dim(K' - L), 3\}$.

Then it follows from Lemma 15 that there is a complex $Q = D_1 P$, rel. K, such that

$$Q = K \cup e_1^{q-1} \cup \cdots \cup e_s^{q-1} \cup e_1^q \cup \cdots \cup e_t^q. \tag{15.3}$$

By the first part of the Theorem, with K, K', L replaced by P, Q, K, we have $C(Q) \equiv C(P) (\Sigma)$, rel. $C(K)$. Therefore it follows from Theorem 4(a) and (15.1) that $C'' = C(Q - K) \equiv C(P - K) \equiv 0 \; (\Sigma)$, whence $\partial''_q : C''_q \approx C''_{q-1} \; (\Sigma)$, by the corollary to Theorem 5. Therefore $s = t$. Let

$$\partial''_q m_i^q = \sum_{j=1}^{s} d_{ij} m_j^{q-1} \qquad (d_{ij} \in R),$$

where $(m_1^r, \cdots, m_s^r)$ is the basis of C''_r $(r = q - 1, q)$. Since $\tau(\partial''_q) = 0$ the matrix $\boldsymbol{d} = [d_{ij}]$ can be annihilated by an expansion

$$\boldsymbol{d} \to \begin{bmatrix} \boldsymbol{d} & 0 \\ 0 & \mathbf{1}_k \end{bmatrix},$$

followed by a sequence of elementary transformations of the form (2.12), followed by a contraction $\mathbf{1}_{s+k} \to \mathbf{1}_0$, where $\mathbf{1}_0$ is the empty matrix. Since $q - 1 \geq 2$ it follows from arguments on pp. 289, 290 of [1], with minor alterations,[39] that the transformation $\boldsymbol{d} \to \mathbf{1}_0$ can be "copied geometrically" by a formal deformation $Q \to K$, rel. K. Therefore $K = D_2 P$, rel. K, and the Theorem follows from the corollary to Lemma 11.

Let us describe r as the order of an elementary expansion $K \to K_1 = K \cup e^{r-1} \cup e^r$, and also of its inverse, $K_1 \to K$. Let $\phi : K \equiv K' \; (\Sigma)$, rel. L and let $K^p, K'^p \subset L$, for some $p \geq -1$. Then the following addendum to Theorem 13 is implicit in the proofs of Lemmas 11-15 and of Theorem 13.

ADDENDUM. *$K' = DK$, rel. L, and $\phi \in [D]$, where D is the resultant of elementary deformations, whose orders lie between $p + 2$ and $q + 1$ inclusive, where q is given by* (15.2).

This addendum has the following application. It follows from Theorems 11 and 13 that, by means of a formal deformation, we can reduce a given complex to one which has a given point, e^0, as its only 0-cell. It is some-

[39] Let $r(\alpha_1, \cdots, \alpha_k)$ mean the same as in Theorem 19 of [1], with $K^* = Q^{q-1}$ and $k = s$. Then $m_j^{q-1} \to \alpha_j$ determines an isomorphism $C''_{q-1} \approx r(\alpha_1, \cdots, \alpha_k)$.

times convenient to restrict ourselves to a class of complexes, all of which have the same point, e^0, as their only 0-cell. Let K_0, K_r be two such complexes and let $K_r = DK_0$. Then it follows from Theorem 13 and its addendum, with $p = 0$, that $[D] = [D']$, where $D': K_0 \to K_r$ is the resultant of elementary deformation, $K_i \to K_{i+1}$, whose orders exceed 1. Therefore $K_j{}^0 = e^0$ for each $j = 0, \cdots, r$.

16. *n*-types. By a *cluster of n-spheres, attached to a space X, at a point* $x_0 \varepsilon X$, we shall mean a set of n-spheres, $\{S_i{}^n\}$, such that $X \cap S_i{}^n = x_0$ and $S_i{}^n - x_0$ does not meet $S_j{}^n$ if $i \neq j$. If X is a complex and $x_0 \varepsilon X^{n-1}$, then $X \cup \{S_i{}^n\}$ is the complex $X \cup \{e_i{}^n\}$, where $e_i{}^n = S_i{}^n - x_0$. At this stage we assume that, X being a finite complex, the number of n-spheres in a cluster attached to X is finite.

Let K^n, L^n be complexes of at most n dimensions $(n > 1)$ and let $\phi: K^n \to L^n$ be an $(n-1)$-homotopy equivalence, as defined in Section 2 of CH(I).

THEOREM 14. *There is a simple equivalence,*

$$\psi: K^n \cup \{S_{1i}{}^n\} \equiv L^n \cup \{S_{2j}{}^n\} \quad (\Sigma),$$

such that $\psi x = \phi x$ *if* $x \varepsilon K^{n-1}$, *where* $\{S_{1i}{}^n\}$, $\{S_{2j}{}^n\}$ *are clusters of n-spheres attached to* K^{n-1}, L^{n-1}.

Assuming that $K^n \cap L^n = 0$, let P be the mapping cylinder of ϕ. Then P^n is the union of K^n, L^n and the cells $e^r \times (0, 1)$, where $e^r \varepsilon K^{n-1}$. We attach a cluster of n-spheres,

$$\{S_{2\rho}{}^n\} = S_{21}{}^n \cup \cdots \cup S_{2k}{}^n,$$

to a 0-cell $e'^0 \varepsilon L^0$, where k is to be determined later. Using Lemma 13, we transfer these over P^n to a 0-cell of K^n, so that they become a cluster, $\{S_\rho{}^n\}$, attached to K^{n-1}. Since $\phi: K^n \equiv_{n-1} L^n$ it follows that (14.1) and (14.2) are satisfied with $m = n - 1$. Therefore

(16.1)
a) $\pi_r(P^n, K^n) = 0$ *if* $r = 1, \cdots, n-1$,
b) *the injection,* $\pi_{n-1}(K^n) \to \pi_{n-1}(P^n)$, *is an isomorphism (onto).*

These conditions are obviously satisfied by K^* and P^*, where

$$K^* = K^n \cup \{S_\rho\}, \qquad P^* = P^n \cup \{S_\rho{}^n\}.$$

Therefore it follows from (16. 1a) and Lemma 15 that there is a formal deformation

$$D_1 : P^* \to Q = K^* \cup e_1^{n-1} \cup \cdots \cup e_a^{n-1} \cup e_1^n \cup \cdots \cup e_b^n, \quad \text{rel. } K^*.$$

Now let $k = a$. On considering the effect of a simple elementary deformation, rel. K^*, it follows inductively that (16. 1) are also satisfied by K^* and Q. Let

$$g_\rho : (E_0^n, I^{n-1}) \to (Q, \bar{e}_\rho^{n-1}) \qquad (\rho = 1, \cdots, a)$$

mean the same as in (14. 3). Since $g_\rho \mid \partial I^n = g_\rho \mid \partial E_0^n$ is homotopic in Q to a constant map, it follows from Lemma 13 that there is a formal deformation $Q \to Q'$, rel. K^n, which replaces each $S_\rho^n - e^0$ by a cell, $e_{b+\rho}^n$, with a characteristic map $g'_\rho : I^n \to \bar{e}_{b+\rho}^n$, such that $g'_\rho \mid \partial I^n = g_\rho \mid \partial I^n$. Therefore it follows, as in the proof of Lemma 15, that there is a formal deformation

$$D_2 : Q' \to Q'' = K^n \cup e'^n_1 \cup \cdots \cup e'^n_b, \quad \text{rel. } K^n.$$

Let $h : I^n \to \bar{e}'^n_i$ be a characteristic map for e'^n_i. Then it follows from (16. 1b) that $h_i \mid \partial I^n$ is homotopic in K^n to a constant map. Therefore it follows from Lemma 13 that there is a formal deformation

$$D_3 : Q'' \to K^n \cup S_{11}^n \cup \cdots \cup S_{1b}^n, \quad \text{rel. } K^n,$$

where $\{S_{1i}^n\}$ is a cluster of n-spheres attached to K^{n-1}. On reversing these constructions we have

$$P^n \cup S_{21}^n \cup \cdots \cup S_{2k}^n = D(K^n \cup S_{11}^n \cup \cdots \cup S_{1b}^n), \quad \text{rel. } K^n.$$

Let $P_0^n \subset P^n$ be the mapping cylinder of $\phi \mid K^{n-1}$. Then P^n is the union of P_0^n and the n-cells in K^n. Let $f : I^n \to \bar{e}^n$ be a characteristic map for an n-cell $e^n \varepsilon K^n$. Then $f \mid \partial I^n$ is obviously homotopic in P_0^n to $\phi(f \mid \partial I^n) : \partial I^n \to L^{n-1}$. Since the latter can be extended to $\phi f : I^n \to L^n$ it follows that $f \mid \partial I^n$ is homotopic in P_0^n to a constant map. Therefore

$$P^*_0 = P_0^n \cup S_{21}^n \cup \cdots \cup S_{2l}^n = D'(P^n \cup S_{21}^n \cup \cdots \cup S_{2k}^n), \quad \text{rel. } P_0^n,$$

where $l \geq k$ and $S_{2,k+1}, \cdots, S_{2l}^n$ are n-spheres, attached to $e'^0 \varepsilon L^n$, which correspond to the n-cells in K^n. It follows from Lemma 11 that

$$L^* = L \cup S_{21}^n \cup \cdots \cup S_{2l}^n = D^* P_0^*,$$

where D^* is a contraction. Let $\psi^* : P_0^* \to L^*$ be given by $\psi^* \mid L^* = 1$,

$\psi^*(x, t) = \phi x$ $(x \varepsilon K^{n-1})$. Then $\psi^* \varepsilon [D^*]$ and the conditions of the theorem are satisfied by a map $\psi \varepsilon [D^*D'D]$. This completes the proof.

It follows from this theorem that any two complexes of the same n-type can be interchanged by a finite sequence of elementary deformations and transformations of the form

$$(16.2) \qquad K \to L = K - e^r, \qquad L \to K = L \cup e^r \qquad (r > n),$$

where e^r is a principal cell of K. For the transformations $K \to K^n \to K^n \cup E^{n+1} \to K^n \cup S^n$ are the resultants of such sequences, where $E^{n+1} = e^0 \cup e^n \cup e^{n+1}$, $S^n = \partial E^{n+1} = e^0 \cup e^n$, and $K^n \cap E^{n+1} = e^0 \varepsilon K^0$. Conversely a formal deformation preserves the homotopy type, and hence the n-type of a complex. Also $K^n = L^n$, whence K, L are of the same n-type, if they are related by (16.2). Thus the n-type may be defined in terms of formal deformations and elementary transformations of the form (16.2).

It follows from Lemma 2 in Section **9** of CH(I) and from Lemma 13 and Theorem 12 above that, if K is any complex, there is a simplicial complex $K^* = DK$. Moreover, Sections **14-16** may be interpreted as referring to formal deformations of the kind considered in [3]. Therefore the class of simplicial complexes, which, when treated as cell-complexes, are of the same simple homotopy type, or n-type, as a given one, K, is the same as the "nucleus," or "n-group," of K, as defined in [1].

17. Homotopy systems.[40] We proceed to the simple equivalence theory of homotopy systems. In this section we confine ourselves to systems, ρ, such that $\dim \rho < \infty$ and each group ρ_n has a finite basis.

We modify the definition of a homotopy system, ρ, by associating a class of *preferred bases* with each ρ_n. Let $(a_1, \cdots, a_p)$ be a preferred basis for ρ_n. Then $(a'_1, \cdots, a'_p)$ shall be a preferred basis for ρ_n if, and only if, $a'_i = a_{j_i}^{\pm 1}$, in case $n = 1$, or $a'_i = \pm x_i a_{j_i}$ if $n > 1$, where $x_i \varepsilon \rho_1$ and $j_1, \cdots, j_p$ is a permutation of $1, \cdots, p$. We shall only admit that $f: \rho \approx \rho'$ if f carries a preferred basis for each ρ_n into a preferred basis for ρ'_n. The preferred bases for $\rho(K)$, where K is a complex, shall be the natural bases, as defined in Section **5** of CH(II). If K^0 is a single 0-cell, then the natural bases for $\rho_1(K)$ are uniquely defined. In general they depend on the choice of a tree $T \subset K^1$, which contains K^0. In this case $\rho(K)$ is the homotopy

[40] The main purpose of this section is to prove Theorem 17, which was announced in Section **7** of CH(II).

system of the pair, (K, T). However we shall continue to write it as $\rho(K)$. A complex, K, will be called a (geometrical) realization of a given system, ρ, if, and only if, $\rho \approx \rho(K)$, subject to our condition concerning preferred bases. The process of realizing ρ by K will consist of defining a particular $f: \rho \approx \rho(K)$. Having (implicitly) done this, we shall use ρ and $a \varepsilon \rho$ to denote $\rho(K)$ and fa. By a *basis* for ρ_n we shall always mean a preferred basis.

Let C and $h: \rho \rightarrow C$ be defined as in Section **8** of CH(II). Then C shall be a chain system of the kind introduced in Section **2** above, R being the group ring of $\bar{\rho}_1 = \rho_1/d\rho_2$. We insist that, if $(a_1, \cdots, a_p)$ is a (preferred) basis for ρ_n and if $(m'_1, \cdots, m'_p)$ is the basis of C_n, then $ha_i = \pm \bar{x}_i m'_{j_i}$, where $\bar{x}_i \varepsilon \bar{\rho}_1$ and $\bar{x}_i = 1$ if $n = 1$.

We are going to define a sub-system, ρ', of a homotopy system ρ. This is not quite so simple as in the case of chain systems, for the following reason. Let $\rho'_1 \subset \rho_1$ be the sub-group generated by part of a basis for ρ_1. Let $\rho'_2 \subset \rho_2$ be the sub-group generated by ρ'_1, operating on a set of elements, $(a'_1, \cdots, a'_k)$, in a basis for ρ_2. Let $d\rho'_2 \subset \rho'_1$ and let $d': \rho'_2 \rightarrow \rho'_1$ be the homomorphism induced by $d: \rho_2 \rightarrow \rho_1$. Then ρ'_2 is not necessarily a free crossed (ρ'_1, d')-module. For example, let ρ_1 have a single free generator, x, and ρ_2 a pair of basis elements, a, b, such that $da = x$, $db = 1$. Let $\rho'_1 = \rho_1$ and let ρ'_2 be generated by ρ_1, operating on b. Since $db = 1$ we have $a + b = b + a$, whence $xb - b = a + b - a - b = 0$. Therefore ρ'_2 is not a free ρ_1-module.

Let $\bar{\rho}'_1 = \rho'_1/d\bar{\rho}'_2$ and let $i_*: \bar{\rho}'_1 \rightarrow \bar{\rho}_1$ be the homorphism induced by the identical map $i_1: \rho'_1 \rightarrow \rho_1$.

LEMMA 16. *Let* $i_*^{-1}(1) = 1$. *Then* ρ'_2 *is a free crossed* (ρ'_1, d')*-module, having* $(a'_1, \cdots, a'_k)$ *as a basis.*

Let ρ''_2 be the free crossed (ρ'_1, d'')-module, which is defined in terms of the symbolic generators (x', α_i) and the map $\alpha_i \rightarrow da'_i$ $(i = 1, \cdots, k;$ $x' \varepsilon \rho'_1)$. Obviously $d''\rho''_2 = d\rho'_2$. Let $a''_i \varepsilon \rho''_2$ be the basis element which corresponds to the generator $(1, \alpha_i)$. It follows from Lemma 2 in Section **2** of CH(II) that an operator homomorphism, $i_2: \rho''_2 \rightarrow \rho_2$, associated with $i_1: \rho'_1 \rightarrow \rho_1$, is defined by $i_2 a''_i = a'_i$. Obviously $i_2\rho''_2 = \rho'_2$ and the lemma will follow when we have proved that $i_2^{-1}(0) = 0$.

Let $C_2 = h\rho_2$, $C''_2 = h''\rho''_2$ be ρ_2, ρ''_2 made Abelian and let $j: C''_2 \rightarrow C_2$ be the homomorphism induced by i_2. Since $a'_i = i_2 a''_i$ and $jh'' = hi_2$ it follows that $(jh''a''_1, \cdots, jh''a''_k)$ is part of a basis for C_2. Since $d''\rho''_2 = d\rho'_2$ and $i_*^{-1}(1) = 1$ it follows that $j^{-1}(0) = 0$. Let $a'' \varepsilon i_2^{-1}(0)$. Then

$i_1 d''a'' = di_2 a'' = 1$, $jh''a'' = hi_2 a'' = 0$. Therefore $d''a'' = 1$, $h''a'' = 0$ and it follows from Lemma 1 in CH(II) that $a'' = 0$. Therefore $i_2^{-1}(0) = 0$ and the lemma is proved.

Let ρ'_1, ρ'_2 satisfy the conditions of Lemma 16. Let $\rho'_p \subset \rho_p$ $(p = 3, 4, \cdots)$ be the sub-group which is generated by $\bar{\rho}'_1$, operating on part of a basis for ρ_p, and let $d\rho'_p \subset \rho'_{p-1}$. Let $d' : \rho'_p \to \rho'_{p-1}$ be the homomorphism induced by $d : \rho_p \to \rho_{p-1}$. Then $\rho' = \{\rho'_p\}$, with d' as boundary operator, is a homotopy system, which we describe as a *sub-system* of ρ, on the understanding that a (preferred) basis for ρ'_n $(n \geqq 1)$ is part of a basis for ρ_n.

Let ρ be a given homotopy system, let $Z_n(\rho) = d_n^{-1}(0)$ and let [41]

$$G_1(\rho) = \bar{\rho}_1, \qquad G_n(\rho) = Z_n(\rho) - d_{n+1}\rho_{n+1} \qquad (n > 1).$$

A homomorphism, $f : \rho \to \rho'$, obviously induces a family of homomorphisms $f_* : G_n(\rho) \to G_n(\rho')$ $(n = 1, 2, \cdots)$. It may be verified in the same way as in ordinary homology theory that $f_* : G_n(\rho) \approx G_n(\rho')$ if $f : \rho \equiv \rho'$. The converse is proved below.

Let $\rho' \subset \rho$ be a sub-system and let $Z_n(\rho, \rho') = d_n^{-1}\rho'_{n-1}$ $(n > 1)$. Let $a \,\varepsilon\, Z_2(\rho, \rho')$, $a' \,\varepsilon\, \rho'_2$. Then $a + a' - a = (da)a' \,\varepsilon\, \rho'_2$, since $da \,\varepsilon\, \rho'_1$. Therefore ρ'_2 is an invariant sub-group of $Z_2(\rho, \rho')$. So therefore is the direct sum $\rho'_2 + d\rho_3$, since $d\rho_3 \subset Z_2(\rho)$, which is in the centre of ρ_2. Let

$$G_n(\rho, \rho') = Z_n(\rho, \rho') - (\rho'_n + d\rho_{n+1}) \qquad (n > 1).$$

Let

$$(17.1) \qquad G_n(\rho') \underset{i_*}{\to} G_n(\rho) \underset{j_*}{\to} G_n(\rho, \rho') \underset{d_*}{\to} G_{n-1}(\rho') \underset{i_*}{\to} \cdots \underset{i_*}{\to} G_1(\rho)$$

be the homomorphisms, which are induced by $i : \rho' \to \rho$, the identical map $Z_n(\rho) \to Z_n(\rho, \rho')$ and by $d \mid Z_n(\rho, \rho')$. Then it may be verified, as in ordinary homology theory, that the sequence (17.1) is exact.

Let $f : \rho \to \rho'$ be a homomorphism of ρ into a system ρ', with boundary operator d'. Let $f_* : \bar{\rho}_1 \approx \bar{\rho}'_1$. We proceed to define a system, ρ^*, which we shall call the *mapping cylinder* of f. We realize the systems $\rho^2 = (\rho_1, \rho_2)$ and ρ'^2 by complexes $K = K^2$ and $K' = K'^2$, such that $K^0 = K'^0 = e^0 = K \cap K'$. By Theorem 4 in CH(II), $f : \rho^2 \to \rho'^2$ can be realized by a map $\phi : K \to K'$. Let P be the mapping cylinder of ϕ, with $e^0 \times I$ shrunk into the point e^0. Then $P^0 = e^0$. We define $\rho^*_n = \rho_n(P) = \rho_n(P^2)$ $(n = 1, 2)$.

Since ϕ induces $f_* : \bar{\rho}_1 \approx \bar{\rho}'$ and since K' is a D. R. of P, it follows that

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[41] If $\rho = \rho(K)$ then $G_1(\rho) \approx \pi_1(K)$, $G_2(\rho) \approx \pi_2(K)$, $G_n(\rho) \approx H_n(K)$ if $n > 2$.

$i_*:\bar{\rho}_1 \approx \bar{\rho}^*{}_1$, $i'_*:\bar{\rho}'_1 \approx \bar{\rho}^*{}_1$, where i_*, i'_* are the injections. Therefore it follows from the proof of Lemma 16 that the injections

$$(17.2) \qquad i:\rho^2 \to \rho(P^2), \qquad i':\rho'^2 \to \rho(P^2)$$

are isomorphisms (into).

Let $\delta:\rho^2 \to \rho(P)$ be the deformation operator determined by the homotopy $\delta_t: K \to P$, which is given by $\delta_t p = (p, t)$ $(p \varepsilon K)$. Then

$$(17.3) \qquad d^*{}_n\delta_n = f_{n-1} - 1 - \delta_{n-1}d_n \qquad (\delta_1 d_1 = 0),$$

where $n = 2, 3$ and $\rho^*{}_1$ is written additively. Let $n \geq 3$ and let $\delta_n\rho_{n-1}$ be a free $\bar{\rho}^*{}_1$-module, which is the image of ρ_{n-1} in an operator homomorphism, δ_n, whose kernel is the commutator sub-group of ρ_{n-1}. Thus $\delta_n:\rho_{n-1} \approx \delta_n\rho_{n-1}$ if $n > 3$. We take $\delta_3\delta_2 \subset \rho_3(P)$ and δ_3 shall mean the same as before. Let $\rho^*{}_n$ be the direct sum $\rho^*{}_n = \rho'_n + \rho_n + \delta_n\rho_{n-1}$ $(n \geqq 3)$. We imbed ρ_n, ρ'_n in $\rho^*{}_n$ by means of $i:\rho_n \to \rho^*{}_n$, $i':\rho'_n \to \rho^*{}_n$, where i, i' mean the same as in (17.2) if $n = 1, 2$, and $ia = (0, a, 0)$, $i'a' = (a', 0, 0)$ if $n > 2$. We define $d^*{}_n:\rho^*{}_n \to \rho^*{}_{n-1}$ by $d^*{}_n a = d_n a$, $d^*{}_n a' = d'_n a'$ and by (17.3), with $n \geq 2$. If $\{a_i{}^n\}$ and $\{a'_j{}^n\}$ are bases for ρ_n and ρ'_n, then the union of $\{a_i{}^n\}$, $\{a'_j{}^n\}$ and $\{\delta_n a_k{}^{n-1}\}$ shall be a (preferred) basis for $\rho^*{}_n$. It follows from an argument in Section **8** above that $d^*{}_n d^*{}_{n+1} = 0$ if $n \geq 3$. Also $d^*d^* = 0$ in $\rho(P)$. Therefore $d^*{}_n d^*{}_{n+1} = 0$ for every $n > 1$. Clearly $d^*{}_n$ is an operator homomorphism and it follows that $\rho^* = \{\rho^*{}_n\}$, with $d^* = \{d^*{}_n\}$ as boundary operator, is a homotopy system. We call it the *mapping cylinder* of $f:\rho \to \rho'$.

Let $i':\rho' \to \rho^*$ be the identical map and let $k':\rho^* \to \rho'$ be given by

$$k'a = fa, \qquad k'a' = a', \qquad k'\delta a = 0 \qquad (a \varepsilon \rho, a' \varepsilon \rho').$$

Then $k'i' = 1$ and it is easily verified that $d'k' = k'd^*$ and that $i'k' - 1 = d^*\delta^* + \delta^*d^*$, where $\delta^*a = \delta a$, $\delta^*\rho' = \delta^*\delta\rho = 0$. Therefore $k':\rho^* \equiv \rho'$ and $k'_*: G_n(\rho^*) \approx G_n(\rho')$. Clearly $f = k'i$, where $i:\rho \to \rho^*$ is the identical map, whence $f_* = k'_*i_*$. Therefore, if each f_* is an isomorphism (onto), so is i_*. In this case it follows from the exactness of (17.1) that

$$(17.4) \qquad G_n(\rho^*, \rho) = 0 \qquad (n \geq 1),$$

where $G_1(\rho^*, \rho) = 0$ means that $\rho^*{}_1 = i_*\rho_1$.

Let $r \geqq 2$ and let $(a_0{}^n, a_1{}^n, \cdots, a_{p_n}{}^n)$ be a preferred basis for ρ_n $(n = r - 1, r)$. If $r = 2$ let $\rho'_1 \subset \rho_1$ be the sub-group generated by $(a_1{}^1, \cdots, a_{p_1}{}^1)$ and if $r > 2$ let $\rho'_1 = \rho_1$. If $n > 1$ $(n = r - 1, r)$ let

$\rho'_n \subset \rho_n$ be the sub-group which is generated by ρ'_1 operating on $(a_1^n, \cdots, a_{p_n})$. In any case let

$$da_0^r = a_0^{r-1} - a'_0, \qquad da_i^r \in \rho_{r-1} \qquad\qquad (i = 1, \cdots, p_r),$$

where $a'_0 \in \rho'_{r-1}$ and ρ_1 is written additively if $r = 2$. Then $d\rho'_r \subset \rho'_{r-1}$ and the conditions of Lemma 16 are satisfied[42] by ρ'_1, ρ'_2. Therefore $\rho' = \{\rho'_n\}$, with $\rho'_n = \rho_n$ if $n \neq r-1$ or r, is a sub-system of ρ. Let $i : \rho' \to \rho$ be the identical map and let $k_n : \rho_n \to \rho'_n$ the operator homomorphism, which is given by $k_n \mid \rho'_n = 1$ and $k_r a_0^r = 0$, $k_{r-1} a_0^{r-1} = a'_0$. Then it is easily verified that $kd = d'k$, whence $k : \rho \to \rho'$ is a homomorphism. We have $ki = 1$ and $ik - 1 = d\xi + \xi d$, where $\xi : \rho \to \rho$ is the deformation operator given by

$$\xi\rho' = 0, \qquad \xi a_0^r = 0, \qquad \xi a_0^{r-1} = - a_0^r.$$

Therefore $i : \rho' \equiv \rho$ and $k : \rho \equiv \rho'$. Notice that, if $f : \rho \to \rho'$ is any homomorphism such that $fi \simeq 1$, then $f \simeq fik \simeq k$.

We shall describe a homomorphism $f : \rho^0 \to \rho^1$ as an *elementary equivalence* if, and only if, ρ^0, ρ^1 are related to each other in the same way as ρ, ρ' in the preceding paragraph, and $f \cong i$ or $f \cong k$, according as $\rho^0 \subset \rho^1$ or $\rho^1 \subset \rho^0$. We shall describe a homomorphism $f : \rho \to \rho^*$ as a *simple equivalence*, $f : \rho \equiv \rho_0\ (\Sigma)$, if, and only if, it is the resultant of a finite sequence of isomorphisms and elementary equivalences.

Let C, C' be chain systems associated with given homotopy systems ρ, ρ^1. Let $g : C \to C'$ be the chain mapping induced by a homomorphism $f : \rho \to \rho'$.

THEOREM 15. *$f : \rho \equiv \rho'(\Sigma)$ if, and only if, $g : C \equiv C'\ (\Sigma)$.*

This follows from the lemmas in Section **14** and the proof of Theorem 13, restated in terms of homotopy ssytems.

Let ρ be a homotopy system and σ a free $\bar{\rho}_1$-module, with a finite basis $(b_1, \cdots, b_q)$. Let $\rho_n^0 = \rho_n + \sigma$, $\rho_p^0 = \rho_p$ $(p \neq n)$, for a given value of $n \geq 2$. Let $d^0 : \rho_r^0 \to \rho_{r-1}^0$ be defined by $d^0 \mid \rho = d$, $d^0\sigma = 0$. Then[43] $\rho^0 = \{\rho_r^0\}$, with d^0 as boundary operator, is a homotopy system. We say that $\rho \to \rho^0$ is the result of *attaching a cluster* of n-cycles to ρ. If $\{a_i\}$ is a preferred basis for ρ_n, then $\{a_i, b_j\}$ shall be a preferred basis for ρ_n^0 and the preferred bases for ρ_p $(p \neq n)$ shall be the same in ρ^0 as in ρ.

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[42] The homomorphism $k_1 : \rho_1 \to \rho'_1$, defined below, induces $i_*^{-1} : \rho_1 \approx \rho'_1$.

[43] If $n = 2$ then ρ_n^0 is a free crossed module since $d^0\sigma = 1$.

Let $\dim \rho$, $\dim \rho' \leq n$ $(n \geq 2)$ and let $f: \rho \to \rho'$ be a homomorphism such that

$$(17.5) \qquad f_*: G_r(\rho) \approx G_r(\rho') \qquad (r = 1, \cdots, n-1).$$

Then we have the following generalization of Tietze's theorem.

THEOREM 16. *There is a simple equivalence,* $f^0: \rho^0 \equiv \rho'^0$ (Σ), *such that* $f^0 a = fa$ *if* $a \in \rho_r$ $(r < n)$, *where* ρ^0, ρ'^0 *are formed by attaching clusters of n-cycles to* ρ, ρ'.

Since $f_*: \bar{\rho}_1 \approx \bar{\rho}'_1$ we can construct the mapping cylinder, ρ^*, of f. Then the theorem follows from the proof of Theorem 14, translated into algebraic terms.

The following corollary may be deduced from Theorem 16, or proved directly with the help of (17.4).

COROLLARY. *If* $\dim \rho$, $\dim \rho' \leq n-1$, *then* (17.5) *implies* $f: \rho \equiv \rho'$.

THEOREM 17. *If* $f: \rho(K) \equiv \rho'$, *where* K *is a complex, then* ρ' *can be realized by a complex,* K', *and in such a way*[44] *that* f *has a realization* $\phi: K \equiv K'$.

This follows from Theorem 16 and an argument which is essentially the same as the proof of Theorem 9 on p. 1228 of [3].

18. Infinite complexes. Let K_1 be a CW-complex, as defined in CH(I), which may be infinite. Let $K_0 \subset K_1$ be a sub-complex such that $K_1 = K_0 \cup \bigcup_a (e_a^{n-1} \cup e_a^n)$, where $\{e_a^{n-1}, e_a^n\}$ is an indexed aggregate of cells such that $e_a^{n-1} \cup e_a^n$ is an open subset of K_1 and $K_0 \to K_0 \cup e_a^{n-1} \cup e_a^n$ is an elementary expansion, for each α. Then $K_0 \to K_1$ will be called a *composite expansion* and $K_1 \to K_0$ a *composite contraction.* It follows from the argument used in the finite case, and (I), in Section **5** of CH(I), that K_0 is a D. R. of K_1. By a *formal deformation,* $D: K \to L$, we shall mean the resultant of a finite sequence of composite expansions and contractions. We restrict ourselves to complexes of finite dimensionality. Then the proofs of the lemmas in Section **14** and of Theorem 14 apply to infinite complexes, after a few trivial alterations.

We also admit homotopy systems of finite dimensionality, in which the

[44] In general ρ' can also be realized by a complex, K', in such a way that f has no realization $K \to K'$.

groups may have infinite bases. We define a simple equivalence, $f: \rho \equiv \rho'$, where ρ, ρ' are two such systems, by analogy with a formal deformation $D: K \rightarrow L$. Then Theorems 16, 17 can be extended without difficulty to systems in which the bases may be infinite.

It remains to be seen whether or not the purely algebraic theory developed in Section **2-9** can be extended to systems of modules with infinite bases, in such a way as to yield a generalization of Theorem 13 to infinite complexes.

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