

On the construction and topological invariance of the Pontryagin classes

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Abstract We use sheaves and algebraic L -theory to construct the rational Pontryagin classes of fiber bundles with fiber \mathbb{R}^n . This amounts to an alternative proof of Novikov’s theorem on the topological invariance of the rational Pontryagin classes of vector bundles. Transversality arguments and torus tricks are avoided.

Keywords L -theory · Characteristic classes · Sheaves · Poincaré duality

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1 Introduction

The “topological invariance of the rational Pontryagin classes” was originally the statement that for a *homeomorphism* of smooth manifolds, $f: N \rightarrow N'$, the induced map

$$f^*: H^*(N'; \mathbb{Q}) \rightarrow H^*(N; \mathbb{Q})$$

takes the Pontryagin classes $p_i(TN') \in H^{4i}(N'; \mathbb{Q})$ of the tangent bundle of N' to the Pontryagin classes $p_i(TN)$ of the tangent bundle of N . The topological invariance was proved by Novikov [21], some 45 years ago. This breakthrough result and its torus-related method of proof have stimulated many subsequent developments in topological manifolds, notably the formulation of the Novikov conjecture, the Kirby–Siebenmann structure theory and the Chapman–Ferry–Quinn et al. controlled topology.

The topological invariance of the rational Pontryagin classes has been reproved several times [11, 25, 28]. In this paper we give yet another proof, using sheaf-theoretic ideas. We

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associate to a topological manifold M a kind of “tautological” co-sheaf on M of symmetric Poincaré chain complexes, in such a way that the cobordism class is a topological invariant by construction. We then produce excision and homotopy invariance theorems for the cobordism groups of such co-sheaves, and use the Hirzebruch signature theorem to extract the rational Pontryagin classes of M from the cobordism class of the tautological co-sheaf on M .

For the historical context, see the Appendix.

Definition 1.1 Write $\text{TOP}(n)$ for the space of homeomorphisms from \mathbb{R}^n to \mathbb{R}^n and $\text{PL}(n)$ for the space (geometric realization of a simplicial set) of PL-homeomorphisms from \mathbb{R}^n to \mathbb{R}^n . Let $\text{TOP} = \bigcup_n \text{TOP}(n)$ and $\text{PL} = \bigcup_n \text{PL}(n)$.

The topological invariance can also be formulated in terms of the classifying spaces, as the statement that the homomorphism $H^*(B\text{TOP}; \mathbb{Q}) \rightarrow H^*(BO; \mathbb{Q})$ induced by the inclusion $O \rightarrow \text{TOP}$ is onto.¹ Here

$$H^*(BO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \dots]$$

where $p_i \in H^{4i}(BO; \mathbb{Q})$ is the i -th “universal” rational Pontryagin class. We note also that the inclusion $B\text{SO} \rightarrow BO$ induces an isomorphism in rational cohomology, and the inclusion $B\text{STOP} \rightarrow B\text{TOP}$ induces a surjection in rational cohomology by a simple transfer argument. Therefore it is enough to establish surjectivity of $H^*(B\text{STOP}; \mathbb{Q}) \rightarrow H^*(B\text{SO}; \mathbb{Q})$.

Hirzebruch’s signature theorem [14] expressed the signature of a closed smooth oriented $4i$ -dimensional manifold M as the evaluation on the fundamental class $[M] \in H_{4i}(M)$ of the \mathcal{L} -genus

$$\mathcal{L}(TM) \in H^{4*}(M; \mathbb{Q}),$$

that is

$$\text{signature}(M) = \langle \mathcal{L}(TM), [M] \rangle \in \mathbb{Z} \subset \mathbb{Q}.$$

It follows that, for any closed smooth oriented n -dimensional manifold N and a closed $4i$ -dimensional framed submanifold $M \subset N \times \mathbb{R}^k$ (where *framed* refers to a trivialized normal bundle),

$$\text{signature}(M) = \langle \mathcal{L}(TN), [M] \rangle \in \mathbb{Z} \subset \mathbb{Q}.$$

By Serre’s finiteness theorem for homotopy groups,

$$H^{n-4i}(N; \mathbb{Q}) \cong \varinjlim_k [\Sigma^k N_+, S^{n-4i+k}] \otimes \mathbb{Q}$$

with $N_+ = N \cup \{\text{pt.}\}$, and by Pontryagin-Thom theory $[\Sigma^k N_+, S^{n-4i+k}]$ can be identified with the bordism group of closed framed $4i$ -dimensional submanifolds $M \subset N \times \mathbb{R}^k$. It is thus possible to identify the component of $\mathcal{L}(TN)$ in

$$H^{4i}(N; \mathbb{Q}) \cong \text{hom}(H^{n-4i}(N; \mathbb{Q}), \mathbb{Q})$$

with the linear map

$$H^{n-4i}(N; \mathbb{Q}) \longrightarrow \mathbb{Q}; M \mapsto \text{signature}(M).$$

¹ The point is that topological n -manifolds have topological tangent bundles with structure group $\text{TOP}(n)$ which are sufficiently natural under homeomorphisms. See [17].

The assumption that N be closed can be discarded if we use cohomology with compact supports where appropriate. Then we identify the component of $\mathcal{L}(TN)$ in

$$H^{4i}(N; \mathbb{Q}) \cong \text{hom}(H_c^{n-4i}(N; \mathbb{Q}), \mathbb{Q})$$

with the linear map

$$H_c^{n-4i}(N; \mathbb{Q}) \rightarrow \mathbb{Q}; M \mapsto \text{signature}(M).$$

Now we can choose N in such a way that the classifying map $\kappa : N \rightarrow BSO(n)$ for the tangent bundle is highly connected, say $(4i + 1)$ -connected. Then $\kappa^* : H^{4i}(BSO(n); \mathbb{Q}) \rightarrow H^{4i}(N; \mathbb{Q})$ takes \mathcal{L} of the universal oriented n -dimensional vector bundle to $\mathcal{L}(TN)$, and so the above description of $\mathcal{L}(TN)$ in terms of signatures can be taken as a *definition* of the universal $\mathcal{L} \in H^{4i}(BSO(n); \mathbb{Q})$, or even $\mathcal{L} \in H^{4i}(BSO; \mathbb{Q})$.

Since this definition relies almost exclusively on transversality arguments, which carry over to the PL setting, we can deduce immediately, as Thom [31] did, that Hirzebruch’s \mathcal{L} -genus extends to

$$\mathcal{L} \in H^{4*}(BSPL; \mathbb{Q}).$$

As the rational Pontryagin classes are polynomials (with rational coefficients) in the components of \mathcal{L} , the PL invariance of the rational Pontryagin classes follows from this rather straightforward argument. It was also clear that the *topological* invariance of the rational Pontryagin classes would follow from an appropriate transversality theorem in the setting of topological manifolds. However, Novikov’s proof did not exactly deduce the topological invariance of the Pontryagin classes from a topological transversality statement. Instead, he proved that signatures of the submanifolds were homeomorphism invariants by showing that for a homeomorphism $f : N \rightarrow N'$ of closed *smooth* oriented n -dimensional manifolds N, N' and a closed framed $4i$ -dimensional submanifold $M' \subset N' \times \mathbb{R}^k$ it is possible to make the proper map $f \times \text{id} : N \times \mathbb{R}^k \rightarrow N' \times \mathbb{R}^k$ transverse regular at M' , keeping it proper, and the smooth transverse image $M \subset N \times \mathbb{R}^k$ has

$$\text{signature}(M) = \text{signature}(M') \in \mathbb{Z}.$$

This was done using non-simply-connected methods. Subsequently, it was found that the ideas in Novikov’s proof could be extended and combined with non-simply-connected surgery theory to prove transversality for topological (non-smooth, non-PL) manifolds. Details on that can be found in [16]. It is now known that $H^*(BTOP; \mathbb{Q}) \cong H^*(BO; \mathbb{Q})$.

The collection of the signatures of framed $4i$ -dimensional submanifolds $M \subset N \times \mathbb{R}^k$ of a topological n -dimensional manifold N was generalized in Ranicki [24] to a fundamental class $[N]_{\mathbf{L}} \in H_n(N; \mathbf{L}^\bullet)$ with coefficients in a spectrum \mathbf{L}^\bullet of symmetric forms over \mathbb{Z} . However, [24] made some use of topological transversality. This paper will remedy this by expressing $[N]_{\mathbf{L}}$ as the cobordism class of the tautological co-sheaf on N , using the local Poincaré duality properties of N .

In any case the “modern” point of view in the matter of the topological invariance of rational Pontryagin classes is that it merits a treatment separate from transversality discussions. Topological manifolds ought to have (tangential) rational Pontryagin classes because they satisfy a local form of Poincaré duality. More precisely, if N is a topological n -manifold, then for any open set $U \subset N$ we have a Poincaré duality isomorphism between the homology of U and the cohomology of U with compact supports. The task is then to use this refined form of Poincaré duality to make invariants in the homology or cohomology of N . Whether or not these invariants can be expressed as characteristic classes of the topological tangent bundle of N becomes a question of minor importance.

Remark In this paper we make heavy use of homotopy direct and homotopy inverse limits of diagrams of chain complexes and chain maps. They can be defined like homotopy direct and homotopy inverse limits of diagrams of spaces. That is to say, the Bousfield-Kan formulae [3] for homotopy direct and homotopy inverse limits of diagrams of spaces can easily be adapted to diagrams of chain complexes: products (of spaces) should be replaced by tensor products (of chain complexes), and where standard simplices appear they should, as a rule, be replaced by their cellular chain complexes.

We often rely on [8, Sect. 9], a collection of conversion and comparison theorems for homotopy direct and homotopy inverse limits.

2 Duality and L -theory: generalities

In the easiest setting, we start with an additive category \mathcal{A} and the category \mathcal{C} of all chain complexes in it, graded over \mathbb{Z} and bounded from above and below. We assume given a functor

$$(C, D) \mapsto C \boxtimes D$$

from $\mathcal{C} \times \mathcal{C}$ to chain complexes of abelian groups. This is subject to bilinearity and symmetry conditions:

- for fixed D in \mathcal{C} , the functor $C \mapsto C \boxtimes D$ takes contractible objects to contractible objects, and takes homotopy cocartesian (= homotopy pushout) squares to homotopy cocartesian squares ;
- there is a binatural isomorphism $\tau : C \boxtimes D \rightarrow D \boxtimes C$ satisfying $\tau^2 = \text{id}$.

We assume that every object C in \mathcal{C} has a “dual” (w.r.t. \boxtimes). This means that the functor

$$D \mapsto H_0(C \boxtimes D)$$

is co-representable in the homotopy category \mathcal{HC} , so that there exists C^{-*} in \mathcal{C} and a natural isomorphism

$$H_0(C \boxtimes D) \cong [C^{-*}, D]$$

where the square brackets denote chain homotopy classes of maps. Let’s note that in this case the n -fold suspension $\Sigma^n C^{-*}$ co-represents the functor $C \mapsto H_n(C \boxtimes D)$ in \mathcal{HC} .

Example 2.1 \mathcal{A} is the category of f.g. free left modules over $\mathbb{Z}[\pi]$, for a fixed group π . For C and D in \mathcal{C} we let

$$C \boxtimes D = C^t \otimes_{\mathbb{Z}[\pi]} D,$$

using the standard involution $\sum a_g g \mapsto \sum a_g g^{-1}$ on $\mathbb{Z}[\pi]$ to turn C into a chain complex of right $\mathbb{Z}[\pi]$ -modules C^t . Then the dual C^{-*} of any C in \mathcal{C} exists and can be defined explicitly as the chain complex of right module homomorphisms $\text{hom}_{\mathbb{Z}[\pi]}(C^t, \mathbb{Z}[\pi])$ on which $\mathbb{Z}\pi$ acts by left multiplication:

$$(rf)(c) := r(f(c))$$

for $r \in \mathbb{Z}[\pi]$, $c \in C^t$ and $f \in \text{hom}_{\mathbb{Z}[\pi]}(C^t, \mathbb{Z}[\pi])$.

Definition 2.2 An n -dimensional symmetric algebraic Poincaré complex in \mathcal{C} consists of an object C in \mathcal{C} and an n -dimensional cycle φ in $(C \boxtimes C)^{h\mathbb{Z}/2}$ whose image in $H_n(C \boxtimes C)$ is nondegenerate (i.e., adjoint to a homotopy equivalence $\Sigma^n C^{-*} \rightarrow C$).

Remark 2.3 $(C \boxtimes C)^{h\mathbb{Z}/2} = \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C \boxtimes C)$ where W is your favorite projective resolution of the trivial module \mathbb{Z} over the ring $\mathbb{Z}[\mathbb{Z}/2]$.

Example: Suppose that \mathcal{A} is the category of f.g. free abelian groups. Let C be the cellular chain complex of a space X which is the realization of a f.g. simplicial set, and also a Poincaré duality space of formal dimension n . Then you can use an Eilenberg-Zilber diagonal

$$W \otimes C \longrightarrow C \otimes C$$

(respecting $\mathbb{Z}/2$ -actions) and evaluate the adjoint $C \rightarrow (C \otimes C)^{h\mathbb{Z}/2}$ on a fundamental cycle to get a nondegenerate n -cycle $\varphi \in (C \otimes C)^{h\mathbb{Z}/2}$.

Definition 2.4 An $(n + 1)$ -dimensional symmetric algebraic Poincaré pair in \mathcal{C} consists of a morphism $f : C \rightarrow D$ in \mathcal{C} , an n -dimensional cycle φ in $(C \boxtimes C)^{h\mathbb{Z}/2}$ and an $(n + 1)$ -dimensional chain ψ in $(D \boxtimes D)^{h\mathbb{Z}/2}$ such that $(f \boxtimes f)(\varphi) = \partial\psi$ and

- the image of φ in $H_n(C \boxtimes C)$ is nondegenerate,
- the image of ψ in $H_{n+1}(D \boxtimes D/C)$ is nondegenerate, where D/C is shorthand for the algebraic mapping cone of $f : C \rightarrow D$.

The *boundary* of the SAP pair $(C \rightarrow D, \psi, \varphi)$ is (C, φ) , an n -dimensional SAPC.

With these definitions, it is straightforward to design bordism groups $L^n(\mathcal{C})$ of n -dimensional SAPCs in \mathcal{C} . Elements of $L^n(\mathcal{C})$ are represented by n -dimensional SAPCs in \mathcal{C} . We say that two such representatives, (C, φ) and (C', φ') , are *bordant* if $(C \oplus C', \varphi \oplus -\varphi')$ is the boundary of an SAP pair of formal dimension $n + 1$.

Furthermore there exist generalizations of the definitions of SAPC and SAP pair (going in the direction of m -ads) which lead automatically to the construction of a spectrum $\mathbf{L}^\bullet(\mathcal{C})$ such that $\pi_n \mathbf{L}^\bullet(\mathcal{C}) \cong L^n(\mathcal{C})$. These activities come under the heading *symmetric L-theory of \mathcal{C}* . This is an idea going back to Mishchenko [20]. It was pointed out in [23] that there is an analogue of Mishchenko’s setup, the *quadratic L-theory of \mathcal{C}* , where “homotopy fixed points” of $\mathbb{Z}/2$ -actions are replaced by “homotopy orbits”. For example:

Definition 2.5 An n -dimensional quadratic algebraic Poincaré complex in \mathcal{C} consists of an object C in \mathcal{C} and an n -dimensional cycle φ in $(C \boxtimes C)_{h\mathbb{Z}/2}$ whose image in $H_n(C \boxtimes C)$ (under the transfer) is nondegenerate.

Remark 2.6 $(C \boxtimes C)_{h\mathbb{Z}/2} = W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (C \boxtimes C)$ where W is that resolution, as above.

The bordism groups of QAPCs are denoted $L_n(\mathcal{C})$ and the corresponding spectrum is $\mathbf{L}_\bullet(\mathcal{C})$. There is a (norm-induced) comparison map $\mathbf{L}_\bullet(\mathcal{C}) \rightarrow \mathbf{L}^\bullet(\mathcal{C})$. On the algebraic side, a key difference between $L_n(\mathcal{C})$ and $L^n(\mathcal{C})$ is that elements of $L_n(\mathcal{C})$ can always be represented by “short” chain complexes (concentrated in degree k if $n = 2k$, and in degrees k and $k + 1$ if $n = 2k + 1$), while that is typically not the case for elements of $L^n(\mathcal{C})$. On the geometric side, $L_n(\mathcal{C})$ also has the immense advantage of being directly relevant to differential topology as a surgery obstruction group (Wall [33]), for the right choice(s) of \mathcal{C} . But the comparison maps $L_n(\mathcal{C}) \rightarrow L^n(\mathcal{C})$ are always isomorphisms away from the prime 2, and since we are interested in rational questions, there is no need to make a very careful distinction between symmetric and quadratic L -theory here.

Example 2.7 If \mathcal{A} is the category of f.g. free abelian groups, \mathcal{C} the corresponding chain complex category, then

$$L_n(\mathcal{C}) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod{4} \\ 0 & n \equiv 1 \pmod{4} \\ \mathbb{Z}/2 & n \equiv 2 \pmod{4} \\ 0 & n \equiv 3 \pmod{4} \end{cases}$$

If \mathcal{A} is the category of f.d. vector spaces over \mathbb{Q} , and \mathcal{C} the corresponding chain complex category, then

$$L_n(\mathcal{C}) \cong L^n(\mathcal{C}) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty & n \equiv 0 \pmod{4} \\ 0 & n \equiv 1, 2, 3 \pmod{4} \end{cases}$$

where $(\dots)^\infty$ denotes a countably infinite direct sum.

Remark 2.8 We need a mild generalization of the setup above. Again we start with an additive category \mathcal{A} . Write $\mathcal{B}(\mathcal{A})$ for the category of all chain complexes of \mathcal{A} -objects, bounded from below (but not necessarily from above). We suppose that a full subcategory \mathcal{K} of $\mathcal{B}(\mathcal{A})$ has been specified, closed under suspension, desuspension, homotopy equivalences, direct sums and mapping cone constructions, so that the homotopy category \mathcal{HK} is a triangulated subcategory of $\mathcal{HB}(\mathcal{A})$. We assume given a functor

$$(C, D) \mapsto C \boxtimes D$$

from $\mathcal{K} \times \mathcal{K}$ to chain complexes of abelian groups. This is subject to the usual bilinearity and symmetry conditions:

- for fixed D in \mathcal{K} , the functor $C \mapsto C \boxtimes D$ takes contractible objects to contractible ones and preserves homotopy cocartesian (= homotopy pushout) squares;
- there is a binatural isomorphism $\tau : C \boxtimes D \rightarrow D \boxtimes C$ satisfying $\tau^2 = \text{id}$.

We assume that every object C in \mathcal{K} has a “dual” (w.r.t. \boxtimes). This means that the functor $D \mapsto H_0(C \boxtimes D)$ on \mathcal{HK} is co-representable. From these data we construct L -theory spectra $\mathbf{L}_\bullet(\mathcal{K})$ and $\mathbf{L}^\bullet(\mathcal{K})$ as before. (Some forward “hints”: Our choice of additive category \mathcal{A} is fixed from Definition 3.1 onwards, and for \mathcal{K} we take the category \mathcal{D}' defined in Sect. 4.)

3 Chain complexes in a local setting

Let X be a locally compact, Hausdorff and separable space. Let $\mathcal{O}(X)$ be the poset of open subsets of X . We introduce an additive category $\mathcal{A} = \mathcal{A}_X$ whose objects are free abelian groups (typically not finitely generated) equipped with a system of subgroups indexed by $\mathcal{O}(X)$.

Definition 3.1 An object of \mathcal{A} is a free abelian group F with a basis S , together with subgroups $F(U) \subset F$, for $U \in \mathcal{O}(X)$, such that the following conditions are satisfied.

- $F(\emptyset) = 0$ and $F(X) = F$.
- Each subgroup $F(U)$ is generated by a subset of S .
- For $U, V \in \mathcal{O}(X)$ we have $F(U \cap V) = F(U) \cap F(V)$.

A morphism in \mathcal{A} from F_0 to F_1 is a group homomorphism $F_0 \rightarrow F_1$ which, for every $U \in \mathcal{O}(X)$, takes $F_0(U)$ to $F_1(U)$.

Example 3.2 For $i \geq 0$, the i -th chain group $C_i = C_i(X)$ of the singular chain complex of X has a preferred structure of an object of \mathcal{A} . The preferred basis is the set $S_i = S_i(X)$ of singular i -simplices in X . For open U in X , let $C_i(U) \subset C_i(X)$ be the subgroup generated by the singular simplices with image contained in U . The boundary operator from $C_i(X)$ to $C_{i-1}(X)$ is an example of a morphism in \mathcal{A} .

Definition 3.3 We write $\mathcal{B}(\mathcal{A})$ for the category of chain complexes in \mathcal{A} , graded over \mathbb{Z} and bounded from below.

Next we list some conditions which we might impose on objects in $\mathcal{B}(\mathcal{A})$, to define a subcategory in which we can successfully do L -theory. The kind of object that we are most interested in is described in the following example.

Example 3.4 Take a map $f : Y \rightarrow X$, where Y is a compact ENR (euclidean neighborhood retract). Let $C(f)$ be the object of $\mathcal{B}(\mathcal{A})$ defined as follows: $C(f)(X)$ is the singular chain complex of Y , with the standard graded basis, and $C(f)(U) \subset C(f)(X)$ for $U \in \mathcal{O}(X)$ is the subcomplex generated by the singular simplices of Y whose image is in $f^{-1}(U)$.

We start by listing some of the obvious but remarkable properties which we see in this Example 3.4. Let C be any object of $\mathcal{B}(\mathcal{A})$.

Definition 3.5 We say that C satisfies the *sheaf type condition* if, for any subset \mathcal{W} of $\mathcal{O}(X)$, the inclusion

$$\sum_{V \in \mathcal{W}} C(V) \longrightarrow C\left(\bigcup_{V \in \mathcal{W}} V\right)$$

is a homotopy equivalence.²

We say that C satisfies *finiteness condition* (i) if the following holds. There exists an integer $a \geq 0$ such that

for open sets $V_1 \subset V_2$ in X such that the closure of V_1 is contained in V_2 , the inclusion $C(V_1) \rightarrow C(V_2)$ factors up to chain homotopy through a chain complex D of f.g. free abelian groups, with $D_i = 0$ if $|i| > a$.

We say that C satisfies *finiteness condition* (ii) if

there exists a compact K in X such that $C(U)$ depends only on $U \cap K$. (Then we say that C is *supported in K* .)

Lemma 3.6 *Example 3.4 satisfies the sheaf type condition and the two finiteness conditions.*

Proof The sheaf type condition is well known from the standard proofs of excision in singular homology. A crystal clear reference for this is [7, III.7.3]. For finiteness condition (i), choose a finite simplicial complex Z and a retraction $r : Z \rightarrow Y$ (with right inverse $j : Y \rightarrow Z$). Replacing the triangulation of Z by a finer one if necessary, one can find a finite simplicial subcomplex Z' of Z containing $j(f^{-1}U)$ and such that $r(Z') \subset f^{-1}(V)$. Then the inclusion $f^{-1}U \rightarrow f^{-1}(V)$ factors through Z' . The singular chain complex of Z' is homotopy equivalent to the cellular chain complex of Z' , a chain complex of f.g. free abelian groups which is zero in degrees < 0 and in degrees $> \dim(Z)$. Finiteness condition (ii) is satisfied with $K = f(Y)$. □

² The sum sign is for an internal sum taken in the chain complex $C(X)$, not an abstract direct sum.

Definition 3.7 Let $\mathcal{C} \subset \mathcal{B}(\mathcal{A})$ consist of the objects which satisfy the sheaf type conditions and the two finiteness conditions.

In the following proposition, we write \mathcal{C}_X and \mathcal{C}_Y etc. rather than \mathcal{C} , to emphasize the dependence on a space such as X or Y .

Definition 3.8 A map $f: X \rightarrow Y$ induces a “pushforward” functor $f_*: \mathcal{C}_X \rightarrow \mathcal{C}_Y$, defined by $f_*C(U) = C(f^{-1}(U))$ for C in \mathcal{C}_X and open $U \subset Y$.

Returning to the shorter notation (\mathcal{C} for \mathcal{C}_X), we spell out two elementary consequences of the sheaf condition.

Lemma 3.9 Let C be an object of \mathcal{C} and let \mathcal{W} be a finite subset of $\mathcal{O}(X)$. If \mathcal{W} is closed under unions (i.e., for any $V, W \in \mathcal{W}$ the union $V \cup W$ is in \mathcal{W}) then the inclusion-induced map

$$C\left(\bigcap_{V \in \mathcal{W}} V\right) \longrightarrow \operatorname{holim}_{V \in \mathcal{W}} C(V)$$

is a homotopy equivalence.

Proof Choose $V_1, \dots, V_k \in \mathcal{W}$ such that every element of \mathcal{W} is a union of some (at least one) of the V_i . For nonempty $S \subset \{1, \dots, k\}$ let $V_S = \bigcup_{i \in S} V_i$. There is an inclusion

$$\operatorname{holim}_{V \in \mathcal{W}} C(V) \longrightarrow \operatorname{holim}_{\text{nonempty } S \subset \{1, \dots, k\}} C(V_S)$$

and we show first that this is a homotopy equivalence. It is induced by a map f of posets. On the right-hand side, we have the poset of nonempty subsets of $\{1, \dots, k\}$ partially ordered by reverse inclusion, and on the left-hand side we have \mathcal{W} itself, partially ordered by inclusion. The map f is given by $S \mapsto V_S$. Under these circumstances it is enough to show that f has a (right) adjoint g . But this is clear: for $V \in \mathcal{W}$ let $g(V) = \{i \mid V_i \subset V\}$.

Now it remains to show that the inclusion-induced map

$$C\left(\bigcap_{i=1}^k V_i\right) \longrightarrow \operatorname{holim}_{\text{nonempty } S \subset \{1, \dots, k\}} C(V_S)$$

is a homotopy equivalence. We show this without any restrictive assumptions on V_1, \dots, V_k .

The map fits into a commutative square

$$\begin{CD} C\left(\bigcap_{i=1}^{k-1} V_i \cap V_k\right) = C\left(\bigcap_{i=1}^k V_i\right) @>>> \operatorname{holim}_{\emptyset \neq S \subset \{1, \dots, k\}} C(V_S) \\ @VVV @VVV \\ \operatorname{holim}_{\emptyset \neq S \subset \{1, \dots, k-1\}} C(V_S \cap V_k) @>>> \operatorname{holim}_{\emptyset \neq S \subset \{1, \dots, k-1\}} \operatorname{holim}(C(V_S) \rightarrow C(V_{S \cup k}) \leftarrow C(V_k)). \end{CD}$$

The left-hand vertical arrow in the square is a homotopy equivalence by inductive assumption. The lower horizontal arrow is induced by homotopy equivalences

$$C(V_S \cap V_k) \longrightarrow \operatorname{holim}(C(V_S) \rightarrow C(V_{S \cup k}) \leftarrow C(V_k))$$

and is therefore also a homotopy equivalence. The right-hand vertical arrow is contravariantly induced by a map of posets,

$$(S, T) \mapsto T$$

where S is a nonempty subset of $\{1, \dots, k - 1\}$ and $T = S$ or $T = S \cup k$ or $T = k$. Hence it is enough, by [8, 9.7], to verify that the appropriate categorical “fibers” of this map of posets have contractible classifying spaces. For fixed T' , a nonempty subset of $\{1, \dots, k\}$, the appropriate fiber is the poset of all (S, T) as above with $T \subset T'$. It is easy to verify that the classifying space is contractible. \square

Lemma 3.10 *Let C be an object of \mathcal{C} and let \mathcal{W} be a subset of $\mathcal{O}(X)$. If \mathcal{W} is closed under intersections (i.e., for any $V, W \in \mathcal{W}$ the intersection $V \cap W$ is in \mathcal{W}) then the inclusion-induced map*

$$\text{hocolim}_{V \in \mathcal{W}} C(V) \longrightarrow C\left(\bigcup_{V \in \mathcal{W}} V\right)$$

is a homotopy equivalence.

Proof In the case where \mathcal{W} is finite, the proof is analogous to that of Lemma 3.9. The general case follows from the case where \mathcal{W} is finite by an obvious direct limit argument. \square

4 A zoo of subcategories

The category $\mathcal{C} = C_X$ defined in Sect. 3 should be regarded as a provisional work environment. It has two shortcomings.

- A morphism $f : C \rightarrow D$ in \mathcal{C} which induces homotopy equivalences $C(U) \rightarrow D(U)$ for every open $U \subset X$ need not be a chain homotopy equivalence in \mathcal{C} .

We can fix that rather easily, and will do so in this section, by defining *free* objects in \mathcal{C} and showing that all objects in \mathcal{C} have free resolutions. This leads to a decomposition of \mathcal{C} into full subcategories \mathcal{C}' and \mathcal{C}'' , where \mathcal{C}' contains the free objects and \mathcal{C}'' contains the objects which we regard as *weakly equivalent to 0*.

- Given the decomposition of \mathcal{C} into \mathcal{C}' and \mathcal{C}'' , we are able to set up a good duality theory either in \mathcal{C}' or, less formally, in \mathcal{C} modulo \mathcal{C}'' . The resulting quadratic L -theory spectrum is still a functor of X , because $\mathcal{A}, \mathcal{C}, \mathcal{C}', \mathcal{C}''$ depend on X . For this functor we are able to prove homotopy invariance, but not excision.

We solve this problem not by adding further conditions to the list in Definition 3.5, but instead by defining a full subcategory \mathcal{D} of \mathcal{C} in terms of generators. To be more precise, \mathcal{D} is generated by all objects which are weakly equivalent to 0 and all the examples of 3.4 obtained from singular simplices $f : \Delta^k \rightarrow X$, using the processes of extension, suspension and desuspension. The good duality theory in \mathcal{C}' or \mathcal{C} modulo \mathcal{C}'' restricts to a good duality theory in \mathcal{D}' or \mathcal{D} modulo \mathcal{D}'' , where $\mathcal{D}' = \mathcal{D} \cap \mathcal{C}'$ and $\mathcal{D}'' = \mathcal{D} \cap \mathcal{C}''$. The corresponding L -theory functor $X \mapsto \mathbf{L}^\bullet(\mathcal{D}_X)$ satisfies homotopy invariance and excision.

Unfortunately it is not clear that *all* the objects of \mathcal{C} obtained as in Example 3.4 belong to \mathcal{D} . They do however belong to $r\mathcal{D}$, the idempotent completion of \mathcal{D} within \mathcal{C} . We are not able to prove excision for the functor $X \mapsto \mathbf{L}^\bullet(r\mathcal{D}_X)$, but we do have a long exact “Rothenberg”

sequence showing that the inclusion $\mathbf{L}^\bullet(\mathcal{D}_X) \rightarrow \mathbf{L}^\bullet(r\mathcal{D}_X)$ is a homotopy equivalence away from the prime 2.

$$\begin{array}{ccccccc}
 & & \mathcal{C}' & \longleftarrow & r\mathcal{D}' & \longleftarrow & \mathcal{D}' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B}(A) & \longleftarrow & \mathcal{C} & \longleftarrow & r\mathcal{D} & \longleftarrow & \mathcal{D} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathcal{C}'' & \longleftarrow & r\mathcal{D}'' & \longleftarrow & \mathcal{D}''
 \end{array}$$

Definition 4.1 A nonempty open subset U of X determines a functor F on $\mathcal{O}(X)$ by

$$F(V) = \begin{cases} \mathbb{Z} & \text{if } U \subset V \\ 0 & \text{otherwise.} \end{cases}$$

With the obvious basis for $F(X)$, this becomes an object of \mathcal{A} which we call *free on one generator attached to U* . Any direct sum of such objects is called *free*.

Example 4.2 The singular chain group $C_i(X)$ of X , with additional structure as in Example 3.2, is typically *not* free in the sense of 4.1.

Definition 4.3 Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory consisting of the objects which are free in every dimension. Let $\mathcal{C}'' \subset \mathcal{C}$ be the full subcategory consisting of the objects C for which $C(U)$ is contractible, for all $U \in \mathcal{O}(X)$.

Definition 4.4 A morphism $f: C \rightarrow D$ in \mathcal{C} is a *weak equivalence* if its mapping cone belongs to \mathcal{C}'' .

Lemma 4.5 Every morphism in \mathcal{C} from an object of \mathcal{C}' to an object of \mathcal{C}'' is nullhomotopic.

Proof The nullhomotopy can be constructed by induction over skeleta, using the following “projective” property of free objects in \mathcal{A} . Let a diagram

$$\begin{array}{ccc}
 & & B_0 \\
 & & \downarrow f \\
 A & \longrightarrow & B_1
 \end{array}$$

in \mathcal{A} be given where f is strongly onto (that is, the induced map $B_0(U) \rightarrow B_1(U)$ is onto for every U) and A is free. Then there exists $g: A \rightarrow B_0$ making the diagram commutative. □

Lemma 4.6 For every object D of \mathcal{C} , there exists an object C of \mathcal{C}' and a morphism $C \rightarrow D$ which is a weak equivalence.

Proof The morphism $C \rightarrow D$ can be constructed inductively using the fact that, for every B in \mathcal{A} , there exists a free A in \mathcal{A} and a morphism $A \rightarrow B$ which is strongly onto. □

The next lemma means that objects of \mathcal{C}' are “cofibrant”:

Lemma 4.7 *Let $f : C \rightarrow D$ and $g : E \rightarrow D$ be morphisms in \mathcal{C} . Suppose that C is in \mathcal{C}' and g is a weak equivalence. Then there exists a morphism $f^\sharp : C \rightarrow E$ such that gf^\sharp is homotopic to f .*

Proof By Lemma 4.5, the composition of f with the inclusion of D in the mapping cone of g is nullhomotopic. Choosing a nullhomotopy and unravelling that gives $f^\sharp : C \rightarrow E$ and a homotopy from gf^\sharp to f . □

Definition 4.8 We write \mathcal{HC}' for the homotopy category of \mathcal{C}' . By all the above, this is equivalent to the category $\mathcal{C}/\mathcal{C}''$ obtained from \mathcal{C} by making invertible all morphisms whose mapping cone belongs to \mathcal{C}'' .

In the following lemma, we write \mathcal{C}_X and \mathcal{C}_Y etc. instead of \mathcal{C} to emphasize the dependence of \mathcal{C} on a space such as X or Y .

Lemma 4.9 *The “pushforward” functor $f_* : \mathcal{C}_X \rightarrow \mathcal{C}_Y$ determined by $f : X \rightarrow Y$ restricts to a functor $\mathcal{C}''_X \rightarrow \mathcal{C}''_Y$. If f is an open embedding, it also restricts to a functor $\mathcal{C}'_X \rightarrow \mathcal{C}'_Y$.* □

This completes our discussion of freeness and weak equivalences in \mathcal{C} . We now turn to the concept of *decomposability*, which is related to excision.

Definition 4.10 Let \mathcal{D} be the smallest full subcategory of $\mathcal{C} = \mathcal{C}_X$ with the following properties.

- All objects of \mathcal{C} obtained from maps $\Delta^k \rightarrow X$ (where $k \geq 0$) by the method of Example 3.4 belong to \mathcal{D} .
- If $C \rightarrow D \rightarrow E$ is a short exact sequence in \mathcal{C} and two of the three objects C, D, E belong to \mathcal{D} , then the third belongs to \mathcal{D} .
- $\mathcal{D} \supset \mathcal{C}''$, that is, all weakly contractible objects in \mathcal{C} belong to \mathcal{D} .

When we say that an object of \mathcal{C} is *decomposable*, we mean that it belongs to \mathcal{D} .

Definition 4.11 $\mathcal{D}' := \mathcal{D} \cap \mathcal{C}'$ and $\mathcal{D}'' := \mathcal{D} \cap \mathcal{C}''$.

Lemma 4.12 *Let C in \mathcal{C} be obtained from a map $Y \rightarrow X$ as in Example 3.4, where Y is a compact ENR. Then C is a retract in \mathcal{C} of an object of \mathcal{D} .*

Proof Choose a retraction $r : Z \rightarrow Y$ where Z is a finite simplicial complex (with right inverse $j : Y \rightarrow Z$, say). Then $fr : Z \rightarrow X$ determines an object of \mathcal{C} as in Example 3.4, and this is clearly in \mathcal{D} . The object of \mathcal{C} determined by f is a retract of the object of \mathcal{D} determined by fr . □

Lemma 4.13 *The rule $X \mapsto \mathcal{D}_X$ is a covariant functor.*

Lemma 4.14 *Let V and W be open subsets of X such that the closure of V in X is contained in W . Given D in \mathcal{D}_X , there are C in \mathcal{D}_W and a morphism $g : C \rightarrow D$ in \mathcal{D}_X such that $g_U : C(U) \rightarrow D(U)$ is a homotopy equivalence for any open $U \subset V$.*

Proof If that claim is true for a particular D and all V, W as in the statement, then we say that D has property P . It is enough to verify the following.

- (i) Every object obtained from a map $\Delta^k \rightarrow X$ by the method of Example 3.4 has property P .

- (ii) If $a: D \rightarrow E$ is a weak equivalence in \mathcal{D}_X and if one of D, E has property P , then the other has property P .
- (iii) If $f: D \rightarrow E$ is any morphism in \mathcal{D}_X , and both D and E have property P , then the mapping cone of f has property P .

The proof of (i) is straightforward using barycentric subdivisions. Also, one direction of (ii) is trivial: if D has property P , then E has property P . For the converse, suppose that E has property P . For fixed V and W , choose $g: F \rightarrow E$ with F in \mathcal{D}_W such that $F(U) \rightarrow E(U)$ is a homotopy equivalence for all $U \subset V$. Without loss of generality, F is free in every dimension. (Otherwise use Lemma 4.6.) Then F belongs to \mathcal{D}'_X also. By Lemma 4.7, there exists a morphism $h: F \rightarrow D$ such that the composition $ah: F \rightarrow E$ is homotopic to g . Then h is a morphism which solves our problem.

For the proof of (iii), we fix V and W and choose an open W_0 such that $V \subset W_0 \subset W$, and the closure of V in X is contained in W_0 while the closure of W_0 in X is contained in W . Then we choose C in \mathcal{C}_W and $C \rightarrow E$ in \mathcal{C}_X inducing homotopy equivalences $C(U) \rightarrow E(U)$ for all open $U \subset W_0$. We also choose B in \mathcal{C}_{W_0} and $B \rightarrow D$ inducing homotopy equivalences $B(U) \rightarrow D(U)$ for all open $U \subset V$. Without loss of generality, B is free in every dimension. Hence there exists $B \rightarrow C$ making the diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ D & \longrightarrow & E \end{array}$$

commutative up to homotopy. Any choice of such a homotopy determines a map from the mapping cone of $B \rightarrow C$ to the mapping cone of $D \rightarrow E$ which solves our problem. In particular the mapping cone of $B \rightarrow C$ belongs to \mathcal{C}_W . □

Corollary 4.15 *Let $X = Y \cup Z$ where Y, Z are open in X . Then for any D in \mathcal{D}_X , there exists a morphism $C \rightarrow D$ in \mathcal{D}_X such that C is in \mathcal{D}_Y and the mapping cone of $C \rightarrow D$ is weakly equivalent to an object of \mathcal{D}_Z .*

Proof Choose an open neighborhood V of $X \setminus Z$ in X such that the closure of V (in X) is contained in Y . Apply Lemma 4.14 with this V and $W = Y$. □

5 Duality in a local setting

Definition 5.1 For an object C in \mathcal{C} and open subsets $U, V \subset X$ with $V \subset U$, let $C(U, V)$ be the chain complex $C(U)/C(V)$ (of free abelian groups).

Definition 5.2 For objects C and D of \mathcal{C} , let

$$C \boxtimes D = \operatorname{holim}_{\substack{U \subset X \text{ open,} \\ K_1, K_2 \subset X \text{ closed} \\ K_1 \cap K_2 \subset U}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2).$$

The local Poincaré duality properties of topological manifolds do not directly suggest the above definition of $C \boxtimes D$, but rather an asymmetric definition, as follows.

Definition 5.3 For objects C and D of \mathcal{D} , let

$$C \boxtimes^? D = \operatorname{holim}_{\substack{U \subset X \text{ open} \\ K \subset U \text{ closed}}} C(U, U \setminus K) \otimes_{\mathbb{Z}} D(U).$$

Remark 5.4 Both $C \boxtimes D$ and $C \boxtimes^? D$ are contractible if either C or D are in \mathcal{D}'' .

Lemma 5.5 *The specialization map $C \boxtimes D \rightarrow C \boxtimes^? D$ (obtained by specializing to $K_2 = X$ in the formula for $C \boxtimes D$) induces an isomorphism in homology.*

Proof We begin with an informal argument. Let ξ be an n -cycle in $C \boxtimes D$. For $K_1, K_2 \subset X$ closed, $U \subset X$ open and $K_1 \cap K_2$ contained in U , abbreviate

$$F(U, K_1, K_2) = C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2).$$

Choose $K_1^+ \subset U$ closed, $K_1^- \subset X$ closed so that $K_1 = K_1^+ \cup K_1^-$ and $K_2 \cap K_1^- = \emptyset$. Then we have a commutative diagram

$$\begin{array}{ccccc} F(U, K_1^+, K_2) & \longrightarrow & F(U, K_1^+ \cap K_1^-, K_2) & \longleftarrow & F(U, K_1^-, K_2) \\ \uparrow = & & \uparrow = & & \uparrow \\ F(U, K_1^+, K_2) & \longrightarrow & F(U, K_1^+ \cap K_1^-, K_2) & \longleftarrow & F(U \setminus K_2, K_1^-, K_2) = 0 \\ \downarrow = & & \downarrow = & & \downarrow \\ F(U, K_1^+, K_2) & \longrightarrow & F(U, K_1^+ \cap K_1^-, K_2) & \longleftarrow & F(U \setminus K_2, K_1^+ \cap K_1^-, K_2) = 0 \\ \uparrow & & \uparrow & & \uparrow \\ F(U, K_1^+, X) & \longrightarrow & F(U, K_1^+ \cap K_1^-, X) & \longleftarrow & F(U \setminus K_2, K_1^+ \cap K_1^-, X) \end{array}$$

By the sheaf properties, the coordinate of ξ in $F(U, K_1, K_2)$ is sufficiently determined by the projection of ξ to the homotopy inverse limit of the top row of the diagram. By diagram chasing, this is sufficiently determined by the projection of ξ to the homotopy inverse limit of the bottom row. But that information is stored in the image of ξ in $C \boxtimes^? D$.

The argument can be formalized as follows. For fixed open U , closed K_1 and K_2 with $K_1 \cap K_2$ contained in U , we consider the poset \mathcal{P} of “decompositions” $K_1 = K_1^+ \cup K_1^-$ where $K_1^+ \subset U$ and K_1^- are closed, $K_2 \cap K_1^- = \emptyset$. The ordering is such that $(J_1^+, J_1^-) \leq (K_1^+, K_1^-)$ if and only if $J_1^+ \subset K_1^+$ and $J_1^- \subset K_1^-$. We need to know that $B\mathcal{P}$ is contractible. To see this let \mathcal{Q} be the poset of all closed neighborhoods of $K_1 \cap K_2$ in $K_1 \cap U$. Then $(K_1^+, K_1^-) \mapsto K_1^+$ is a functor $v: \mathcal{P} \rightarrow \mathcal{Q}$. Fixing some $J \in \mathcal{Q}$, let \mathcal{P}_J be the poset of all $(K_1^+, K_1^-) \in \mathcal{P}$ with $K_1^+ \subset J$. This contains as a terminal sub-poset the set of all (K_1^+, K_1^-) with $K_1^+ = J$, and the latter is clearly (anti-)directed. Hence $B\mathcal{P}_J$ is contractible. This verifies the hypotheses in Quillen’s theorem A for the functor v , so that $Bv: B\mathcal{P} \rightarrow B\mathcal{Q}$ is a homotopy equivalence. But \mathcal{Q} is again directed, so $B\mathcal{Q}$ is contractible. Therefore $B\mathcal{P}$ is contractible. Now we can write

$$\begin{aligned} C \boxtimes D &= \operatorname{holim}_{U, K_1, K_2} F(U, K_1, K_2) \\ &\simeq \operatorname{holim}_{U, K_1, K_2} \operatorname{holim}_{K_1^+, K_1^-} F(U, K_1, K_2) \\ &\cong \operatorname{holim}_{U, K_1^+, K_1^-, K_2} F(U, K_1^+ \cup K_1^-, K_2) \\ &\simeq \operatorname{holim}_{U, K_1^+, K_1^-, K_2} \operatorname{holim} [F(U, K_1^+, K_2) \rightarrow F(U, K_1^+ \cap K_1^-, K_2) \leftarrow F(U, K_1^-, K_2)]. \end{aligned}$$

(The usual conventions apply: K_1, K_2 closed in X , with $K_1 \cap K_2$ contained in the open set U , and $K_1 = K_1^- \cup K_1^+$ is a decomposition of the type we have just discussed.) Using that, we

obtain from the rectangular twelve term diagram above a map $g : H_*(C \boxtimes^? D) \rightarrow H_*(C \boxtimes D)$. Indeed an n -cycle in $C \boxtimes^? D$ determines an n -cycle in

$$\operatorname{holim}_{U, K_1^+, K_1^-, K_2} \operatorname{holim} [F(U, K_1^+, X) \rightarrow F(U, K_1^+ \cap K_1^-, X) \leftarrow F(U \setminus K_2, K_1^+ \cap K_1^-, X)]$$

and we use the vertical arrows in the twelve-term diagram to obtain an n -cycle in

$$\operatorname{holim}_{U, K_1^+, K_1^-, K_2} \operatorname{holim} [F(U, K_1^+, K_2) \rightarrow F(U, K_1^+ \cap K_1^-, K_1) \leftarrow F(U, K_1^-, K_2)]$$

whose homology class is well defined. By construction, $g : H_*(C \boxtimes^? D) \rightarrow H_*(C \boxtimes D)$ is left inverse to the projection-induced map $H_*(C \boxtimes D) \rightarrow H_*(C \boxtimes^? D)$. But it is clearly also right inverse (specialize to $K_2 = X$ and then $K_1^- = \emptyset$ in the diagrams above). \square

Example 5.6 Let X be a compact ENR. For open U in X , let $C(U)$ be the singular chain complex of U , regarded as a subcomplex of C . By Lemma 3.6, applied to the identity $X \rightarrow X$, this functor $U \mapsto C(U)$ satisfies the sheaf type condition and the two finiteness conditions. We construct a “canonical” map

$$\nabla : C(X) \longrightarrow C \boxtimes C .$$

To start with we have the following diagram:

$$C(X) \longrightarrow \operatorname{holim}_{U, K} C(X, X \setminus K) \xleftarrow{\cong} \operatorname{holim}_{U, K} C(U, U \setminus K)$$

where U and K are open in X and closed in X , respectively, with $K \subset U$. The first arrow is induced by the quotient maps $C(X) \rightarrow C(X, X \setminus K)$ and the second map is induced by the inclusions $C(U, U \setminus K) \rightarrow C(X, X \setminus K)$ which are chain homotopy equivalences by the sheaf property. Inversion of the second arrow in the diagram gives us a map

$$\gamma : C(X) \longrightarrow \operatorname{holim}_{U, K} C(U, U \setminus K)$$

well defined up to contractible choice (in particular, well defined up to chain homotopy).

Next we make use of a certain chain map

$$\zeta : \operatorname{holim}_{U, K} C(U, U \setminus K) \longrightarrow C \boxtimes C .$$

This is determined by the compositions

$$C(U, U \setminus (K_1 \cap K_2)) \longrightarrow \operatorname{sing. chain cx. of } (U \times U, U \times U \setminus (K_1 \times U \cup U \times K_2)) \downarrow C(U, U \setminus K_1) \otimes C(U, U \setminus K_2)$$

(for closed $K_1, K_2 \subset X$ with intersection $K_1 \cap K_2 \subset U$), where the first arrow is induced by the diagonal map and the second arrow is an Eilenberg-Zilber map. Now we define

$$\nabla = \zeta \gamma : C(X) \longrightarrow C \boxtimes C .$$

This is a refinement of the standard Eilenberg-Zilber-Alexander-Whitney diagonal chain map $C(X) \rightarrow C(X) \otimes C(X)$, which we can recover by composing with the projection (alias specialization) from $C \boxtimes C$ to $C(X) \otimes C(X)$.

If X is an oriented closed topological n -manifold and $\omega \in C(X)$ is an n -cycle representing a fundamental class for X , then $\nabla(\omega)$ is an n -cycle in $C \boxtimes C$ which, as we shall see in

Proposition 5.8 below, is “nondegenerate”. This reflects the fact that not only X , but also each open subset of X satisfies a form of Poincaré duality.

Example 5.7 Let (X, Y) be a pair of compact ENRs. For open U in X , let $C(U)$ be the singular chain complex of U and let $D(U)$ be the singular chain complex of $U \cap Y$. By Lemma 3.6, both C and D satisfy the sheaf type condition and the two finiteness conditions. A straightforward generalization of the previous example gives

$$\nabla : C(X)/D(X) \longrightarrow (C \boxtimes C)/(D \boxtimes D) .$$

If X is an oriented compact topological n -manifold with boundary Y , and $\omega \in C(X)/D(X)$ is an n -cycle representing a fundamental class for the pair (X, Y) , then $\nabla(\omega)$ is an n -cycle in $(C \boxtimes C)/(D \boxtimes D)$ whose image in $C \boxtimes (C/D)$ is nondegenerate (Proposition 5.8 below).

Proposition 5.8 *Let C and D be objects of \mathcal{C} . Let $[\varphi] \in H_n(C \boxtimes D)$ and suppose that, for every open $U \subset X$ and every $j \in \mathbb{Z}$, the map*

$$\operatorname{colim}_{\text{cpct } K \subset U} H^{n-j}C(U, U \setminus K) \longrightarrow H_jD(U) ,$$

slant product with the coordinate of φ in $C(U, U \setminus K) \otimes D(U)$, is an isomorphism. Then $[\varphi]$ is nondegenerate in the following sense: for any E in \mathcal{D} , the map $f \mapsto f_[\varphi]$ is an isomorphism from $[D, E]$, the morphism set in $\mathcal{C}/\mathcal{C}' \cong \mathcal{H}\mathcal{C}'$, to $H_n(C \boxtimes E)$.*

Proof The idea is very simple. Given $[\psi] \in H_n(C \boxtimes E)$, the slant product with $[\psi]$ gives us homomorphisms

$$\operatorname{colim}_{\text{closed } K \subset U} H^{n-j}C(U, U \setminus K) \longrightarrow H_jE(U)$$

for $j \in \mathbb{Z}$. By our assumption on $[\varphi]$, we have $\operatorname{colim}_K H^{n-j}C(U, U \setminus K) \cong H_jD(U)$ and so we get homomorphisms

$$H_jD(U) \longrightarrow H_jE(U)$$

for all $j \in \mathbb{Z}$, naturally in U . It remains “only” to construct a morphism $f_\psi : D \rightarrow E$ in \mathcal{D} or in $\mathcal{H}\mathcal{D}/\mathcal{H}\mathcal{D}' \cong \mathcal{H}\mathcal{D}'$ inducing these homomorphisms of homology groups.

By Lemma 5.5 or otherwise, we may represent the class $[\psi]$ by an n -cycle

$$\psi \in C \boxtimes^? E = \operatorname{holim}_{\substack{U \subset X \text{ open} \\ K \subset U \text{ closed}}} C(U, U \setminus K) \otimes_{\mathbb{Z}} E(U).$$

We shall show first of all that this n -cycle determines a chain map

$$\psi^{\text{ad}} : \operatorname{hocolim}_{K \subset U} C(X, X \setminus K)^{n-*} \longrightarrow \operatorname{holim}_{V \supset U} E(V)$$

natural in the variable U . In fact the source of this map is clearly a covariant functor of the variable U , by *extension of the indexing poset*. The target is homotopy equivalent to $E(U)$, and we regard it as a covariant functor of U by *restriction of the indexing poset*. More precisely, if $U_1 \subset U_2$, then the poset of open subsets of X containing U_2 is contained in the subset of open subsets of X containing U_1 . Corresponding to that inclusion of posets we have a projection map

$$\operatorname{holim}_{V \supset U_1} E(V) \longrightarrow \operatorname{holim}_{V \supset U_2} E(V)$$

and that is what we use.

To give a precise description of ψ^{ad} now, we fix a string $K_0 \subset K_1 \subset \dots \subset K_r$ of compact subsets of X , and a string of open subsets $U_s \subset U_{s-1} \subset \dots \subset U_0$ with $K_r \subset U_s$. Let $\ell\Delta^r$ and $\ell\Delta^s$ be the cellular chain complexes of Δ^r and Δ^s (the ℓ is for *linearization*). We can define ψ^{ad} by associating to every choice of two such strings a chain map

$$\ell\Delta^r \otimes_{\mathbb{Z}} C(X, X \setminus K_0)^{n-*} \longrightarrow \text{hom}_{\mathbb{Z}}(\ell\Delta^s, E(U_0)),$$

or equivalently, a chain map

$$\ell\Delta^s \otimes_{\mathbb{Z}} \ell\Delta^r \longrightarrow C(X, X \setminus K_0) \otimes_{\mathbb{Z}} E(U_0)$$

of degree n . (As the input strings vary, these chain maps are subject to obvious compatibility conditions.) But in fact the coordinate of $\psi \in C \boxtimes^? E$ corresponding to the strings (K_0, \dots, K_r) and (U_s, \dots, U_0) is a chain map

$$\ell\Delta^s \otimes \ell\Delta^r \longrightarrow C(U_0, U_0 \setminus K_0) \otimes E(U_0)$$

of degree n . We need only compose with the inclusion-induced maps

$$E(U_0) \otimes D(U, U \setminus K_r) \longrightarrow E(U_0) \otimes D(X, X \setminus K_0)$$

to get the data we need.

Now abbreviate

$$\begin{aligned} PC^{(n-*)}(U) &= \text{hocolim}_{K \subset U} C(X, X \setminus K)^{n-*}, \\ PE(U) &= \text{holim}_{V \supset U} E(V), \\ PD(U) &= \text{holim}_{V \supset U} D(V) \end{aligned}$$

(writing P for *provisional*). Then we have a homotopy commutative diagram of natural transformations of functors on $\mathcal{O}(X)$,

$$\begin{array}{ccccc} D & \longleftarrow & C^{(n-*)} & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ PD & \xleftarrow{\varphi^{\text{ad}}} & PC^{(n-*)} & \xrightarrow{\psi^{\text{ad}}} & PE \end{array}$$

as follows. The maps $E \rightarrow PE$ and $D \rightarrow PD$ are obvious (diagonal) constructions. We note that $E(U) \rightarrow PE(U)$ and $D(U) \rightarrow PD(U)$ are homology equivalences for all U . The object $C^{(n-*)}$ is chosen in C' and the map from it to $PC^{(n-*)}$ induces homology equivalences $C^{(n-*)}(U) \rightarrow PC^{(n-*)}(U)$ for all U by construction. (This uses Lemma 4.6.) The arrows in the top row are then constructed to make the diagram homotopy commutative. (This uses Lemma 4.7.) The horizontal arrows in the left-hand square of the diagram are homology equivalences (for every choice of input U) and consequently the arrow from $C^{(n-*)}$ to D is an isomorphism in C/C'' . We choose an inverse for it and compose with $C^{(n-*)} \rightarrow E$ to obtain the desired element $[f_\psi] \in [D, E]$. Finally, it is just a matter of inspection to see that the rule $[\psi] \mapsto [f_\psi]$ is inverse to the map from $[D, E]$ to $H_n(C \boxtimes E)$ given by slant product with $[\varphi]$. □

Example 5.9 Recall that the natural Eilenberg–Zilber map

$$\text{sg ch cx of } (Y \times Z) \longrightarrow (\text{sg ch cx of } Y) \otimes (\text{sg ch cx of } Z)$$

(for arbitrary spaces Y and Z) admits a refinement to a natural equivariant chain map

$$W \otimes (\text{sg ch cx of } (Y \times Z)) \longrightarrow (\text{sg ch cx of } Y) \otimes (\text{sg ch cx of } Z)$$

where W is a free resolution of \mathbb{Z} as a trivial module over the group ring $\mathbb{Z}[\mathbb{Z}/2]$. Here *equivariance* means that

$$\begin{array}{ccc} W \otimes (\text{sg ch cx of } Y \times Z) & \longrightarrow & (\text{sg ch cx of } Y) \otimes (\text{sg ch cx of } Z) \\ T \otimes \text{perm.} \downarrow \cong & & \text{perm.} \downarrow \cong \\ W \otimes (\text{sg ch cx of } Z \times Y) & \longrightarrow & (\text{sg ch cx of } Z) \otimes (\text{sg ch cx of } Y) \end{array}$$

commutes for all Y and Z , where T is the generator of $\mathbb{Z}/2$ acting on W and “perm.” stands for a permutation of the factors Y and Z .

Applying this in the situation of Example 5.6, we deduce immediately that

$$\nabla : C(X) \longrightarrow C \boxtimes C$$

has a refinement to

$$\nabla^{h\mathbb{Z}/2} : C(X) \longrightarrow (C \boxtimes C)^{h\mathbb{Z}/2}.$$

Suppose now that X is a closed topological manifold, and $\omega \in C(X)$ is a fundamental cycle. Then with the nondegeneracy property which we have already established, we can say informally that $(C, \nabla^{h\mathbb{Z}/2}(\omega))$ is an n -dimensional SAPC in \mathcal{C} (slightly against the rules, since we have not established that every object in \mathcal{C} has a dual).

Similarly, in the situation and notation of Example 5.7, we obtain $\nabla^{h\mathbb{Z}/2}(\omega)$, an n -cycle in $(C \boxtimes C)^{h\mathbb{Z}/2} / (D \boxtimes D)^{h\mathbb{Z}/2}$. With the nondegeneracy property which we have already established, this allows us to say informally that $((C, D), \nabla^{h\mathbb{Z}/2}(\omega))$ is an SAP pair.

There are “economy” versions of $C \boxtimes D$ other than $C \boxtimes^? D$. Suppose that \mathcal{Q} is a collection of closed subsets of X with the following properties.

- \mathcal{Q} is closed under finite intersections (i.e., for Q_1 and Q_2 in \mathcal{Q} , we have $Q_1 \cap Q_2 \in \mathcal{Q}$);
- for every compact subset K of X and open $U \subset X$ containing K , there exist $r \geq 0$ and $Q_1, \dots, Q_r \in \mathcal{Q}$ such that

$$K \subset \bigcup_{i=1}^r Q_i \subset U.$$

For C and D in \mathcal{D} we define

$$C \boxtimes^{??} D = \text{holim}_{\substack{U \in \mathcal{O}(X) \\ K_1, K_2 \in \mathcal{Q} \\ K_1 \cap K_2 \subset U}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2).$$

This depends on \mathcal{Q} , not just on C and D , but in practice it will be clear what \mathcal{Q} is.

Lemma 5.10 *The specialization map $C \boxtimes D \longrightarrow C \boxtimes^{??} D$ is a chain homotopy equivalence.*

Proof Step 1: We assume that X is compact. Let \mathcal{Q}' be the collection of all subsets of X which are finite unions of subsets $K \in \mathcal{Q}$. The specialization map $C \boxtimes D \longrightarrow C \boxtimes^{??} D$ is a composition of two specialization maps

$$\begin{array}{c}
 \text{holim}_{\substack{U \in \mathcal{O}(X) \\ K_1, K_2 \text{ closed} \\ K_1 \cap K_2 \subset U}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2) \\
 \downarrow f \\
 \text{holim}_{\substack{U \in \mathcal{O}(X) \\ K_1, K_2 \in \mathcal{Q}' \\ K_1 \cap K_2 \subset U}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2) \\
 \downarrow g \\
 \text{holim}_{\substack{U \in \mathcal{O}(X) \\ K_1, K_2 \in \mathcal{Q} \\ K_1 \cap K_2 \subset U}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2).
 \end{array}$$

We are going to show that both of these are homotopy equivalences. For the specialization map f , it suffices to observe that, by our assumptions on \mathcal{Q} , the triples (U, K_1, K_2) with $K_1, K_2 \in \mathcal{Q}'$ and $K_1 \cap K_2 \subset U$ form an *initial* sub-poset in the poset of all triples (U, K_1, K_2) with K_1, K_2 closed and $K_1 \cap K_2 \subset U$. This refers to the usual ordering,

$$(U, K_1, K_2) \leq (V, L_1, L_2) \iff U \subset V \text{ and } K_1 \supset L_1, K_2 \supset L_2.$$

For the specialization map g , it suffices by [8, 9.7] to show that for open $U \subset X$ and L_1, L_2 in \mathcal{Q}' with $L_1 \cap L_2 \subset U$, the canonical map

$$C(U, U \setminus L_1) \otimes_{\mathbb{Z}} D(U, U \setminus L_2) \longrightarrow \text{holim}_{\substack{K_1, K_2 \in \mathcal{Q} \\ K_1 \subset L_1, K_2 \subset L_2}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2)$$

is a chain homotopy equivalence. But this is true by Lemma 3.9. (*Some details:* The target of this map can also be described, up to homotopy equivalence, as a double homotopy limit

$$\text{holim}_{\mathcal{R}} \text{holim}_{\substack{K_1, K_2 \in \mathcal{R} \\ K_1 \subset L_1, K_2 \subset L_2}} C(U, U \setminus K_1) \otimes D(U, U \setminus K_2)$$

where \mathcal{R} runs through the *finite* subsets of \mathcal{Q} which are “large enough”. By *large enough* we mean that there exist $K_{11}, K_{12}, \dots, K_{1r}$ and $K_{21}, K_{22}, \dots, K_{2s}$ in \mathcal{R} such that

$$\bigcup_{i=1}^r K_{1i} = L_1, \quad \bigcup_{j=1}^s K_{2j} = L_2.$$

For fixed \mathcal{R} , we have an Alexander-Whitney type homotopy equivalence

$$\begin{aligned}
 & \text{holim}_{\substack{K_1, K_2 \in \mathcal{R} \\ K_1 \subset L_1, K_2 \subset L_2}} C(U, U \setminus K_1) \otimes D(U, U \setminus K_2) \\
 & \simeq \left(\text{holim}_{\substack{K_1 \in \mathcal{R} \\ K_1 \subset L_1}} C(U, U \setminus K_1) \right) \otimes \left(\text{holim}_{\substack{K_2 \in \mathcal{R} \\ K_2 \subset L_2}} D(U, U \setminus K_2) \right)
 \end{aligned}$$

natural in \mathcal{R} . By Lemma 3.9 and because \mathcal{R} is large enough, the projections

$$\begin{aligned} C(U, U \setminus L_1) &\longrightarrow \operatorname{holim}_{\substack{K_1 \in \mathcal{R} \\ K_1 \subset L_1}} C(U, U \setminus K_1), \\ D(U, U \setminus L_2) &\longrightarrow \operatorname{holim}_{\substack{K_2 \in \mathcal{R} \\ K_2 \subset L_2}} D(U, U \setminus K_2) \end{aligned}$$

are homotopy equivalences. Putting these facts together, we see that

$$\operatorname{holim}_{\substack{K_1, K_2 \in \mathcal{Q} \\ K_1 \subset L_1, K_2 \subset L_2}} C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2)$$

is homotopy equivalent to the homotopy inverse limit of a constant functor,

$$\mathcal{R} \mapsto C(U, U \setminus K_1) \otimes_{\mathbb{Z}} D(U, U \setminus K_2).$$

Since the poset of all \mathcal{R} is directed, the homotopy inverse limit is homotopy equivalent to the unique value of that functor.)

Step 2: X is arbitrary (but still locally compact Hausdorff and separable). Choose a compact $Y \subset X$ which belongs to \mathcal{Q} and is a neighborhood for the support of C and for the support of D . Let

$$\mathcal{Q}^Y = \{Q \in \mathcal{Q} \mid Q \subset Y\},$$

a collection of compact subsets of Y which is closed under finite intersections. For open $U \subset Y$ put

$$C^Y(U) = C(U \cup (X \setminus Y)), \quad D^Y(U) = D(U \cup (X \setminus Y)).$$

Now we have a commutative diagram of specialization maps

$$\begin{array}{ccc} C \boxtimes D & \longrightarrow & C \boxtimes^{??} D \\ \downarrow & & \downarrow \\ C^Y \boxtimes D^Y & \longrightarrow & C^Y \boxtimes^{??} D^Y \end{array}$$

using \mathcal{Q}^Y to define the lower row. By step 1, the lower horizontal arrow is a homotopy equivalence. It is therefore enough to show that the two vertical arrows are homotopy equivalences. This follows easily from the fact that the inclusion of posets $\iota : \mathcal{W} \rightarrow \mathcal{O}(X)$ has a left adjoint, where $\mathcal{O}(X)$ consists of all open subsets of X and \mathcal{W} consists of all open subsets of X containing $X \setminus Y$. The left adjoint is given by $U \mapsto \lambda(U) = U \cup (X \setminus Y)$. Note also that the inclusion-induced maps $C(U) \cong C(\lambda(U))$ and $D(U) \cong D(\lambda(U))$ are isomorphisms. Thus, λ induces maps $C^Y \boxtimes D^Y \rightarrow C \boxtimes D$ and $C^Y \boxtimes^{??} D^Y \rightarrow C \boxtimes^{??} D$ which are homotopy inverses for the vertical arrows in our square. The homotopies are induced by natural transformations, the “unit” and the “counit” of the adjunction of ι and λ . □

6 Products

Let X and Y be locally compact Hausdorff and separable spaces.

Definition 6.1 For C in C_X and D in C_Y , the *tensor product* $C \otimes D$ of C and D is the ordinary tensor product of chain complexes $C \otimes_{\mathbb{Z}} D$, with the system of subcomplexes defined by

$$(C \otimes D)(W) = \sum_{\substack{U, V \\ U \times V \subset W}} C(U) \otimes_{\mathbb{Z}} D(V)$$

for $W \in \mathcal{O}(X \times Y)$.

We are aiming to show that $C \otimes D$ is in $C_{X \times Y}$. This is surprisingly hard. We begin with two lemmas.

Lemma 6.2 For $i \in \{1, 2, \dots, k\}$ let U_i be open in X and let V_i be open in Y . For nonempty $S \subset \{1, 2, \dots, k\}$ put $U_S = \bigcap_{\lambda \in S} U_\lambda$ and $V_S = \bigcap_{\lambda \in S} V_\lambda$. The following map (induced by obvious inclusions) is a homotopy equivalence:

$$\operatorname{hocolim}_{\emptyset \neq S \subset \{1, 2, \dots, k\}} C(U_S) \otimes_{\mathbb{Z}} D(V_S) \longrightarrow \sum_{i=1}^k C(U_i) \times D(V_i),$$

where the sum $\sum_{i=1}^k$ is taken inside $C(X) \otimes_{\mathbb{Z}} D(Y)$.

Proof We proceed by induction on k . The square of inclusion maps

$$\begin{array}{ccc} \sum_{i=1}^{k-1} C(U_i \cap U_k) \otimes D(V_i \cap V_k) & \longrightarrow & C(U_k) \otimes D(V_k) \\ \downarrow & & \downarrow \\ \sum_{i=1}^{k-1} C(U_i) \otimes D(V_i) & \longrightarrow & \sum_{i=1}^k C(U_i) \otimes D(V_i) \end{array} \quad (*)$$

is a homotopy pushout square. Indeed, it is a pushout square in which the horizontal arrows (in fact, also the vertical arrows) are cofibrations, i.e., split injective as maps of graded abelian groups. The pushout property can be verified in terms of bases: each of the four terms in the square is the graded free abelian group generated by a certain graded set.

Next, for nonempty $S \in \{1, \dots, k - 1\}$, write $E(S) = C(U_S) \times D(V_S)$. Then it is clear that

$$\begin{array}{ccc} \operatorname{hocolim}_S E(S \cup k) & \longrightarrow & \operatorname{hocolim}_S E(k) \\ \downarrow & & \downarrow \\ \operatorname{hocolim}_S E(S) & \longrightarrow & \operatorname{hocolim}_S \operatorname{hocolim}(E(S) \leftarrow E(S \cup k) \rightarrow E(k)), \end{array} \quad (**)$$

where S refers to nonempty subsets of $\{1, \dots, k - 1\}$, commutes up to a preferred homotopy h and, as such, is a homotopy pushout square. The square $(**)$ maps to $(*)$ by a forgetful map which also takes the homotopy h to zero. By inductive hypothesis, three of the four arrows which constitute this map $(**)$ \rightarrow $(*)$ between squares are homotopy equivalences, and therefore all are homotopy equivalences. Now it only remains to show that the canonical map

$$\begin{array}{c} \operatorname{hocolim}_{\emptyset \neq S \subset \{1, 2, \dots, k-1\}} \operatorname{hocolim}(E(S) \leftarrow E(S \cup k) \rightarrow E(k)) \\ \downarrow \\ \operatorname{hocolim}_{\emptyset \neq S \subset \{1, 2, \dots, k\}} E(S) \end{array}$$

is a homotopy equivalence. We have dealt with this kind of task before, in the proof of Lemma 3.9 and Lemma 3.10, and it can be dealt with in the same way here. \square

Lemma 6.3 *In the situation of Lemma 6.2, let W be the union of the sets $U_i \times V_i$ for $i = 1, \dots, k$. Then the inclusion*

$$\sum_{i=1}^k C(U_i) \otimes_{\mathbb{Z}} D(V_i) \longrightarrow (C \otimes D)(W)$$

is a homotopy equivalence.

Proof Step 1: We assume to begin with that W itself has the form $U \times V$ for some U open in X and some V open in Y . It is easy to construct finite open coverings $\{U'_\lambda \mid \lambda \in \Lambda\}$ of U , and $\{V'_\gamma \mid \gamma \in \Gamma\}$ of V , such that every open set $U_i \times V_i$ is a union of (some) of the sets $U'_\lambda \times V'_\mu$. Now the composition of inclusions

$$\begin{array}{c} \sum_{(\lambda, \mu)} C(U'_\lambda) \otimes D(V'_\mu) \\ \downarrow \\ \sum_i C(U_i) \otimes D(V_i) \\ \downarrow \\ C(U \times V) \end{array}$$

is a homotopy equivalence, because the source can be written as

$$\left(\sum_{\lambda} C(U'_\lambda) \right) \otimes_{\mathbb{Z}} \left(\sum_{\mu} C(V'_\mu) \right)$$

and we are assuming the sheaf type condition for C and D . Therefore it remains only to show that the first of these inclusions admits a homotopy left inverse. Using Lemma 6.2, we may replace $\sum_i C(U_i) \otimes D(V_i)$ by

$$\text{hocolim}_{\emptyset \neq T \subset \{1, \dots, k\}} C(U_T) \otimes D(V_T).$$

In that expression, $C(U_T)$ can be replaced by the sum of the $C(U'_R)$ for $U'_R \subset U_T$, where R is a nonempty subset of Λ . Similarly $D(V_T)$ can be replaced by the sum of the $D(V'_S)$ for $V'_S \subset V_T$, where S is a nonempty subset of Γ . (Here we use the sheaf type conditions for C and D again.) After these modifications, we have an obvious projection map from that homotopy colimit to

$$\sum_{(\lambda, \mu)} C(U'_\lambda) \otimes D(V'_\mu).$$

This provides the required left homotopy inverse.

Now we look at the general case. Let $W = \bigcup_{i=1}^k U_i \times V_i$ as given. Let $U_{k+1} \subset X$ and $V_{k+1} \subset Y$ be open and suppose $U_{k+1} \subset V_{k+1} \subset W$. It is enough to show that the inclusion

$$\sum_{i=1}^k C(U_i) \otimes D(V_i) \longrightarrow \sum_{i=1}^{k+1} C(U_i) \otimes D(V_i)$$

is a homotopy equivalence. (For then we can repeat the process by adding on as many terms $C(U') \otimes D(V')$ with $U' \times V' \subset W$ as we like, and thereby approximate $(C \otimes D)(W)$.) There is a pushout square

$$\begin{CD} \sum_{i=1}^k C(U_i \cap U_{k+1}) \otimes D(V_i \cap V_{k+1}) @>>> C(U_{k+1}) \otimes D(V_{k+1}) \\ @VVV @VVV \\ \sum_{i=1}^k C(U_i) \otimes D(V_i) @>>> \sum_{i=1}^{k+1} C(U_i) \otimes D(V_i) \end{CD}$$

in which the horizontal arrows are cofibrations. It follows that the square is a homotopy pushout square. By step 1, the upper horizontal arrow is a homotopy equivalence. Therefore the lower horizontal arrow is a homotopy equivalence. \square

Lemma 6.4 *If C belongs to \mathcal{C}_X and D belongs to \mathcal{C}_Y , then $C \otimes D$ belongs to $\mathcal{C}_{X \times Y}$.*

Proof For the sheaf condition, suppose that $W \subset X \times Y$ is open and $W = \bigcup W_\alpha$. Then W is the union of all open sets $U_i \times V_i$ which are contained in some W_α , and therefore the inclusion

$$\sum_{\alpha} (C \otimes D)(W_\alpha) = \sum_i C(U_i) \otimes D(V_i) \longrightarrow (C \otimes D)(W)$$

is a homotopy equivalence by Lemma 6.3 and passage to the direct limit.

For finiteness condition (ii), suppose that C has support in a compact subset K of X and D has support in a compact subset L of Y . Then it is clear that $C \otimes D$ has support in $K \times L \subset X \times Y$.

This leaves finiteness condition (i) to be established. We recall what it requires. We have to find an integer $c \geq 0$ such that, for open $W \subset W'$ in $X \times Y$, where W has compact closure in W' , the inclusion $(C \otimes D)(W) \rightarrow (C \otimes D)(W')$ factors through a chain complex of f.g. free abelian groups whose i -th chain group is zero whenever $|i| > c$. Let $a, b \in \mathbb{Z}$ be the corresponding integers for C and D . Let us provisionally say that an open set W in $X \times Y$ is *good* if

W has compact closure in X and, for every open W'' in X containing the closure \bar{W} of W , there exists another open W' with compact closure in X such that

$$W \subset \bar{W} \subset W' \subset \bar{W}' \subset W$$

and the inclusion map $(C \otimes D)(W') \rightarrow (C \otimes D)(W'')$ factors through a bounded chain complex of f.g. free abelian groups.

Then it is easy to verify:

- any W of the form $W = U \times V$, with compact closure in $X \times Y$, is good;
- if W_1, W_2 and $W_1 \cap W_2$ are good open subsets of $X \times Y$, then $W_1 \cup W_2$ is good.

It follows that if W is any finite union of subsets of the form $U \times V$, where U and V are open in X and Y , respectively, with compact closures, then W is good. While that does not prove all we need, it will now be sufficient to show that, for any W open in $X \times Y$, the chain complex $(C \otimes D)(W)$ is homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-ab$ and $(a + 2)(b + 2) + 1$. To that end, we note that for any open U in X , the chain complex $C(U)$ is homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-a$ and $a + 1$, as it is homotopy equivalent to a sequential homotopy colimit of chain complexes concentrated in degrees between $-a$ and a . Consequently, for open U', U'' in X with $U' \subset U''$, the chain complex

$C(U', U'') = C(U')/C(U'')$ is homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-a$ and $a + 2$. Similar remarks apply to D in place of C . If W is open in $X \times Y$ and is a finite union of subsets of the form $U_\alpha \times V_\alpha$, then we may replace $(C \otimes D)(W)$ by the homotopy equivalent subcomplex

$$\sum_{\alpha} C(U_\alpha) \otimes D(V_\alpha).$$

This admits a finite filtration by subcomplexes such that the subquotients of the filtration have the form

$$C(U', U'') \otimes_{\mathbb{Z}} D(V', V'')$$

for some open U', U'' in X and V', V'' in Y , with $U' \subset U''$ and $V' \subset V''$. It is therefore homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-ab$ and $(a + 2)(b + 2)$. Finally, an arbitrary open W in $X \times Y$ can be written as a monotone union of subsets W_i where each W_i is a finite union of open subsets of the form $U_\alpha \times V_\alpha$. Since each $(C \otimes D)(W_i)$ is homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-ab$ and $(a + 2)(b + 2)$, it follows that $(C \otimes D)(W)$ is homotopy equivalent to a chain complex of free abelian groups concentrated in degrees between $-ab$ and $(a + 2)(b + 2) + 1$. □

Corollary 6.5 *If C belongs to \mathcal{D}_X and D belongs to \mathcal{D}_Y , then $C \otimes D$ belongs to $\mathcal{D}_{X \times Y}$.*

Proof It is clear from the definition that, for fixed C , the functor $D \mapsto C \otimes D$ from \mathcal{C}_X to $\mathcal{C}_{X \times Y}$ respects weak equivalences and short exact sequences. The same is true if we fix D and allow C to vary. Therefore it is enough to prove the claim in the special case where C and D are obtained from maps $f: \Delta^k \rightarrow X$ and $g: \Delta^\ell \rightarrow Y$, so that $C(U)$ and $D(V)$ are the singular chain complexes of $f^{-1}(U)$ and $g^{-1}(V)$, respectively, for U open in X and V open in Y . Let E in $\mathcal{C}_{X \times Y}$ be the object obtained from $f \times g: \Delta^k \times \Delta^\ell \rightarrow X \times Y$, so that $E(W)$ is the singular chain complex of $(f \times g)^{-1}(W)$, for W open in $X \times W$. It is easy to show that E belongs to $\mathcal{D}_{X \times Y}$. There is an easy Eilenberg-Zilber type map

$$C \otimes D \longrightarrow E$$

in $\mathcal{C}_{X \times Y}$. For open W in $X \times Y$ of the form $W = U \times V$ with U open in X and V open in Y , this specializes to a map of chain complexes

$$(C \otimes D)(W) \longrightarrow E(W)$$

which is a chain homotopy equivalence by the very Eilenberg-Zilber theorem. It follows then from the sheaf properties that $(C \otimes D) \rightarrow E(W)$ is always a chain homotopy equivalence, for arbitrary open W in $X \times Y$. Consequently $C \otimes D$ belongs to $\mathcal{D}_{X \times Y}$. □

Proposition 6.6 *Let C and E be objects of \mathcal{C}_X . Let D and F be objects of \mathcal{C}_Y . Then the following specialization map is a homotopy equivalence (and a “fibration”, i.e., split surjective as a map of graded abelian groups):*

$$\begin{aligned}
 (C \otimes D) \boxtimes (E \otimes F) &= \operatorname{holim}_{W, P_1, P_2} (C \otimes D)(W, W \setminus P_1) \otimes (E \otimes F)(W, W \setminus P_2) \\
 &\quad \downarrow \\
 &\operatorname{holim}_{\substack{W=U \times V \\ P_1=J_1 \times K_1 \\ P_2=J_2 \times K_2}} (C \otimes D)(W, W \setminus P_1) \otimes (E \otimes F)(W, W \setminus P_2).
 \end{aligned}$$

Proof As in the proof of Lemma 5.10, there is no loss of generality in assuming that X and Y are both compact. In that case we may replace the target of our specialization map by

$$\operatorname{holim}_{\substack{W \\ P_1=J_1 \times K_1 \\ P_2=J_2 \times K_2}} (C \otimes D)(W, W \setminus P_1) \otimes (E \otimes F)(W, W \setminus P_2),$$

dropping the condition $W = U \times V$. (This makes no difference to the homotopy type because, in the poset of triples (W, P_1, P_2) where $P_1 = J_1 \times K_1$ and $P_2 = J_2 \times K_2$, those triples which have $W = U \times V$ for some U and V form an “initial” sub-poset.) Now our map has the form

$$(C \otimes D) \boxtimes (E \otimes F) \longrightarrow (C \otimes D) \boxtimes^{??} (E \otimes F)$$

as in Lemma 5.10, and it is a homotopy equivalence by that same lemma. □

Definition 6.7 We construct a map

$$(C \boxtimes D) \otimes (E \boxtimes F) \longrightarrow (C \otimes E) \boxtimes (D \otimes F)$$

by composing the following:

$$\begin{aligned}
 &(C \boxtimes D) \otimes (E \boxtimes F) \\
 &\quad \downarrow \\
 &\operatorname{holim}_{\substack{W=U \times V \\ P_1=J_1 \times K_1 \\ P_2=J_2 \times K_2}} (C \otimes D)(W, W \setminus P_1) \otimes (E \otimes F)(W, W \setminus P_2) \\
 &\quad \downarrow \\
 &(C \otimes E) \boxtimes (D \otimes F)
 \end{aligned}$$

The second arrow is a right inverse for the chain map in Proposition 6.6. The first arrow is induced by isomorphisms

$$\begin{aligned}
 & (C(U, U \setminus J_1) \otimes D(U, U \setminus J_2)) \otimes (E(V, V \setminus K_1) \otimes F(V, V \setminus K_2)) \\
 & \quad \downarrow \cong \\
 & ((C \otimes E)(W, W \setminus P_1)) \otimes ((D \otimes F)(W, W \setminus P_2))
 \end{aligned}$$

where $W = U \times V$ and $P_i = J_i \times K_i$ for $i = 1, 2$. We describe the composite map informally as $\varphi \otimes \psi \mapsto \varphi \tilde{\otimes} \psi$, where φ and ψ are chains in $C \boxtimes D$ and $E \boxtimes F$, respectively.

Definition 6.8 Assume $C = D$ and $E = F$ in Definition 6.7. We construct a map

$$(C \boxtimes C)^{h\mathbb{Z}/2} \otimes (E \boxtimes E)^{h\mathbb{Z}/2} \longrightarrow ((C \otimes E) \boxtimes (C \otimes E))^{h\mathbb{Z}/2}$$

by composing the following:

$$\begin{aligned}
 & (C \boxtimes C)^{h\mathbb{Z}/2} \otimes (E \boxtimes E)^{h\mathbb{Z}/2} \\
 & \quad \downarrow \\
 & \left(\begin{array}{c} \text{holim} \\ W=U \times V \\ P_1=J_1 \times K_1 \\ P_2=J_2 \times K_2 \end{array} (C \otimes E)(W, W \setminus P_1) \otimes (C \otimes E)(W, W \setminus P_2) \right)^{h\mathbb{Z}/2} \\
 & \quad \downarrow \\
 & ((C \otimes E) \boxtimes (C \otimes E))^{h\mathbb{Z}/2}
 \end{aligned}$$

(details as in Definition 6.7, with superscripts $h\mathbb{Z}/2$ added on where appropriate). We describe the composite map informally as $\varphi \otimes \psi \mapsto \varphi \tilde{\otimes} \psi$, where φ and ψ are chains in $(C \boxtimes C)^{h\mathbb{Z}/2}$ and $(E \boxtimes E)^{h\mathbb{Z}/2}$, respectively.

Example 6.9 In particular, suppose that (C, φ) and (E, ψ) are “symmetric objects” (not necessarily Poincaré) of dimensions m and n respectively, in \mathcal{C}_X and \mathcal{C}_Y respectively, so that φ is an m -cycle in $(C \boxtimes C)^{h\mathbb{Z}/2}$ and ψ is an n -cycle in $(D \boxtimes D)^{h\mathbb{Z}/2}$. Then we have

$$(C \otimes D, \varphi \tilde{\otimes} \psi),$$

a symmetric object of dimension $m + n$ in $\mathcal{C}_{X \times Y}$.

Example 6.10 Let X and Y be compact ENRs. Let C be the functor taking an open $U \subset X$ to the singular chain complex of U . Let E be the functor taking an open $V \subset Y$ to the singular chain complex of V . Let F be the functor taking an open $W \subset X \times Y$ to the singular chain complex of W . By all the above, we have the following diagram

$$\begin{array}{ccc}
 C(X) \otimes E(Y) & \xrightarrow{\nabla \otimes \nabla} & (C \boxtimes C)^{h\mathbb{Z}/2} \otimes (E \boxtimes E)^{h\mathbb{Z}/2} \\
 \downarrow \simeq & & \downarrow \text{dotted} \\
 F(X \times Y) & \xrightarrow{\nabla} & (F \boxtimes F)^{h\mathbb{Z}/2} .
 \end{array}$$

which is commutative up to a chain homotopy. (We could be more precise about that by specifying a “contractible choice” of such chain homotopies.) The dotted arrow is given as in Definition 6.8 by $\varphi \otimes \psi \mapsto \varphi \otimes \psi$. We leave it to the reader to establish the homotopy commutativity.

7 Duality and decomposability

We turn to a discussion of duality, first in \mathcal{C} , then in \mathcal{D} and then in $r\mathcal{D}$. Let C and D be objects of \mathcal{C} which admit duals $C^{(-*)}$ and $D^{(-*)}$. (In other words, the functors $E \mapsto H_0(C \boxtimes E)$ and $E \mapsto H_0(D \boxtimes E)$ on $\mathcal{H}\mathcal{D}/\mathcal{H}\mathcal{D}'$ are co-representable.) Fix nondegenerate 0-cycles

$$\varphi \in C \boxtimes C^{(-*)} , \quad \psi \in D \boxtimes D^{(-*)} .$$

Let $f : C \rightarrow D$ be any morphism. We can assume that $D^{(-*)}$ and $C^{(-*)}$ are in \mathcal{C}' . Choose a morphism $g : D^{(-*)} \rightarrow C^{(-*)}$ such that $g_*(\psi) \in D \boxtimes C^{(-*)}$ is homologous to $f_*(\varphi)$. Then choose a 1-chain $\zeta \in D \boxtimes C^{(-*)}$ such that $d\zeta = g_*(\psi) - f_*(\varphi)$. Now for every E in \mathcal{C} the square

$$\begin{array}{ccc}
 \text{hom}(C^{(-*)}, E) & \xrightarrow{g_*} & \text{hom}(D^{(-*)}, E) \\
 \text{slant with } \varphi \downarrow & & \downarrow \text{slant with } \psi \\
 C \boxtimes E & \xrightarrow{f_*} & D \boxtimes E
 \end{array}$$

is homotopy commutative (slant with ζ provides a homotopy), and the vertical arrows are homology equivalences. Writing $M(g)$ and $M(f)$ for the mapping cone of g and f respectively, we have homotopy cofiber sequences

$$\begin{array}{ccccc}
 \text{hom}(M(g), E) & \longrightarrow & \text{hom}(C^{(-*)}, E) & \xrightarrow{g_*} & \text{hom}(D^{(-*)}, E) , \\
 & & C \boxtimes E & \xrightarrow{f_*} & D \boxtimes E \longrightarrow M(f) \boxtimes E .
 \end{array}$$

Our homotopy commutative square therefore implies that the functor $E \mapsto H_0(M(f) \boxtimes E)$ is again co-representable, with representing object $\Sigma^{-1}M(g)$. In particular the identity class in $H_0 \text{hom}(M(g), M(g))$ corresponds to some nondegenerate class $[\lambda] \in H_1(M(f) \boxtimes M(g))$. This construction of $[\lambda]$ shows also that, for every open $U \in \mathcal{O}(X)$ and closed $K \subset X$ contained in U , we have a commutative diagram

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \uparrow & & \uparrow \\
 H^{j+1}M(f)(U, U \setminus K) & \xrightarrow{\text{slant with } \lambda} & H_j M(g)(U) \\
 \uparrow & & \uparrow \\
 H^j C(U, U \setminus K) & \xrightarrow{\text{slant with } \varphi} & H_j C^{(-*)}(U) \\
 \uparrow f^* & & \uparrow g^* \\
 H^j D(U, U \setminus K) & \xrightarrow{\text{slant with } \psi} & H_j D^{(-*)}(U) \\
 \uparrow & & \uparrow \\
 H^j M(f)(U, U \setminus K) & \xrightarrow{\text{slant with } \lambda} & H_{j+1} M(g)(U) \\
 \uparrow & & \uparrow \\
 \vdots & & \vdots
 \end{array}$$

with exact columns. This leads us to the following conclusion.

Lemma 7.1 *Objects of \mathcal{D} have duals which are again in \mathcal{D} . For E and F in \mathcal{D} , an element $[\lambda] \in H_0(E \boxtimes F)$ is nondegenerate if and only if, for all open U in X , the slant product with $[\lambda]$ is an isomorphism*

$$\text{colim}_K H^j E(U, U \setminus K) \longrightarrow F(U)$$

(where K runs through the closed subsets of X which are contained in U).

Proof By the preceding discussion, it is enough to show that an object E of \mathcal{D} constructed as in Example 3.4 from a map $f : \Delta^k \rightarrow X$ has a dual F in \mathcal{C} , with nondegenerate $[\lambda] \in H_0(E \boxtimes F)$ say, that F is again decomposable, and that the slant product with λ is an isomorphism

$$\text{colim}_K H^j E(U, U \setminus K) \longrightarrow F(U)$$

for all $U \in \mathcal{O}(X)$. By Example 5.7 and Proposition 5.8, all that is true. □

Corollary 7.2 *Objects of $r\mathcal{D}$ have duals which are again in $r\mathcal{D}$. For E and F in $r\mathcal{D}$, an element $[\lambda] \in H_0(E \boxtimes F)$ is nondegenerate if and only if, for all open U in X , the slant product with $[\lambda]$ is an isomorphism*

$$\text{colim}_K H^j E(U, U \setminus K) \longrightarrow F(U).$$

Proof Let E be an object in $r\mathcal{D}$. We can assume that E is in $r\mathcal{D}'$ and that E admits a “complement” E' , also in $r\mathcal{D}'$, so that $E'' = E \oplus E'$ belongs to \mathcal{D} . Now E'' admits a dual, say F'' in \mathcal{D}' , coupled to E'' by means of

$$[\lambda''] \in H_0(E'' \boxtimes F'').$$

The retraction map $q : E'' \rightarrow E$ (via E) has a dual $p : F'' \rightarrow F$, so that $(q \otimes \text{id})_*[\lambda''] = (\text{id} \otimes p)_*[\lambda''] \in H_0(E \boxtimes F)$. As q is idempotent, p is idempotent up to homotopy. We

can now produce a splitting $F'' \simeq F \oplus F'$. Namely, for open U in X we let $F(U)$ be the homotopy colimit of

$$F''(U) \xrightarrow{p} F''(U) \xrightarrow{p} F''(U) \xrightarrow{p} \dots$$

and we let $F'(U)$ be the homotopy colimit of

$$F''(U) \xrightarrow{\text{id}-p} F''(U) \xrightarrow{\text{id}-p} F''(U) \xrightarrow{\text{id}-p} \dots$$

Then it is clear that F and F' belong to $r\mathcal{D}'$ and

$$H_0(E'' \boxtimes F'') \cong H_0(E \boxtimes F) \oplus H_0(E' \boxtimes F') \oplus H_0(E' \boxtimes F) \oplus H_0(E \boxtimes F').$$

The equation $(q \otimes \text{id})_*[\lambda''] = (\text{id} \otimes p)_*[\lambda'']$ shows that $[\lambda'']$ lives in

$$H_0(E \boxtimes F) \oplus H_0(E' \boxtimes F').$$

Write $[\lambda''] = [\lambda] \oplus [\lambda']$ with $[\lambda] \in H_0(E \boxtimes F)$ and $[\lambda'] \in H_0(E' \boxtimes F')$. It is straightforward to show that $[\lambda] \in H_0(E \boxtimes F)$ is nondegenerate and that the slant product with $[\lambda]$ is an isomorphism

$$\text{colim}_K H^j E(U, U \setminus K) \longrightarrow F(U),$$

because $[\lambda'']$ has the analogous properties. The universal property of $[\lambda]$ now implies that, for arbitrary F^\sharp in $r\mathcal{D}$ with $[\lambda^\sharp] \in H_0(E \boxtimes F^\sharp)$, the element $[\lambda^\sharp]$ is nondegenerate if and only if the slant product with it is an isomorphism

$$\text{colim}_K H^j E(U, U \setminus K) \longrightarrow F^\sharp(U).$$

□

Lemma 7.3 *The rule $X \mapsto \mathcal{D}_X$ is a covariant functor, preserving duality.*

Proof Let $f: X \rightarrow Y$ be a map (between locally compact separable Hausdorff spaces). It is clear that, for C in \mathcal{D}_X , we have f_*C in \mathcal{D}_Y . For C and D in \mathcal{D}_X , there is a specialization map

$$C \boxtimes D \longrightarrow f_*C \boxtimes f_*D.$$

We need to show that this takes nondegenerate classes in $H_0(C \boxtimes D)$ to nondegenerate classes in $H_0(f_*C \boxtimes f_*D)$. In the case where X and Y are both compact, this follows from the nondegeneracy criterion given in Lemma 7.1. (For U open in Y , every compact subset of $f^{-1}(U)$ is contained in some $f^{-1}(K)$ for compact $K \subset U$.) Finally, because of finiteness condition (ii) in Definition 3.5, it is easy to reduce to a situation where X and Y are both compact. □

Corollary 7.4 *The rule $X \mapsto \mathcal{D}_X$ is a covariant functor, preserving duality.*

Proposition 7.5 *The tensor product $\mathcal{D}_X \times \mathcal{D}_Y \longrightarrow \mathcal{D}_{X \times Y}$ is compatible with duality.*

Proof Let C, E be objects of \mathcal{D}_X and let D, F be objects of \mathcal{D}_Y . Let $[\lambda] \in H_0(C \boxtimes E)$ and $[\mu] \in H_0(D \boxtimes F)$. Then we have $[\lambda \bar{\otimes} \mu] \in H_0((C \otimes D) \boxtimes (E \otimes F))$. What we have to show is that if $[\lambda]$ and $[\mu]$ are nondegenerate, then $[\lambda \bar{\otimes} \mu]$ is nondegenerate. This is a statement about the three triangulated categories $\mathcal{H}\mathcal{D}'_X, \mathcal{H}\mathcal{D}'_Y$ and $\mathcal{H}\mathcal{D}'_{X \times Y}$. For each of the three triangulated categories we know that duality preserves exact triangles. We also know that the \boxtimes product

preserves exact triangles when one input variable is fixed. Hence, using repeated five lemma arguments, we can easily reduce the claim about the nondegeneracy of $[\lambda \bar{\otimes} \mu]$ to the special case where C and D are among the standard generators of \mathcal{D}_X and \mathcal{D}_Y , respectively. That is, C is weakly equivalent to the object of \mathcal{D}_X obtained from some map $f: \Delta^k \rightarrow X$ by the method of Example 3.4, and D is weakly equivalent to the object of \mathcal{D}_X obtained from some map $g: \Delta^\ell \rightarrow Y$ by the same method. (It seems better to say “weakly equivalent” rather than “equal” because we might want to apply the resolution procedure of Lemma 4.6 to obtain objects in \mathcal{D}'_X and \mathcal{D}'_Y , respectively.) In that case, we also have a clear idea what E and F are, and what $[\lambda]$ and $[\mu]$ are. Namely, E is (up to desuspensions) the quotient of C by its “boundary” (the object obtained from $f|_{\partial \Delta^k}$ by the method of Example 3.4), and F is (up to desuspensions) the quotient of D by its “boundary” (the object obtained from $g|_{\partial \Delta^\ell}$ by the method of Example 3.4). Also, $[\lambda]$ can be described as the class of $\nabla(\omega_k)$ where ω_k is a relative fundamental cycle for the manifold-with-boundary Δ^k , and $[\mu]$ can be described as the class of $\nabla(\omega_\ell)$ where ω_ℓ is a relative fundamental cycle for the manifold-with-boundary Δ^ℓ .

Next, $C \otimes D$ can be identified with the object obtained from

$$f \times g : \Delta^k \times \Delta^\ell \longrightarrow X \times Y$$

by the method of Example 3.4. Also $E \otimes F$ can be identified (up to desuspensions) with the quotient of $C \otimes D$ by its “boundary”, which is the object obtained from $f \times g$ restricted to $\partial(\Delta^k \times \Delta^\ell)$ by the method of Example 3.4. Now Example 6.10 implies that $[\lambda \bar{\otimes} \mu]$ can be described as the class of $\nabla(\omega_k \times \omega_\ell)$. Since $\omega_k \times \omega_\ell$ is a relative fundamental cycle for $\Delta^k \times \Delta^\ell$, this implies (with Examples 5.6, 5.7 and Proposition 5.8) that $[\lambda \bar{\otimes} \mu]$ is indeed nondegenerate. □

8 The excisive signature

Lemma 8.1 *The functor $X \mapsto \mathbf{L}^\bullet(\mathcal{D}_X)$ is homotopy invariant.*

Proof It is enough to show that the maps $X \rightarrow X \times [0, 1]$ given by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$ induce the same homomorphisms

$$\pi_* \mathbf{L}^\bullet(\mathcal{D}_X) \longrightarrow \mathbf{L}^\bullet(\mathcal{D}_{X \times [0,1]}).$$

That is easily done by using the 1-dimensional manifold with boundary $[0, 1]$ and the corresponding SAPC in $\mathcal{C}_{[0,1]}$, and tensor product with that, to produce appropriate bordisms. □

Theorem 8.2 *The functor $X \mapsto \mathbf{L}^\bullet(\mathcal{D}_X)$ is excisive. In detail:*

- (i) *For open U, V subset X with $U \cup V = X$, the commutative square of inclusion-induced maps*

$$\begin{CD} \mathbf{L}^\bullet(\mathcal{D}_{U \cap V}) @>>> \mathbf{L}^\bullet(\mathcal{D}_U) \\ @VVV @VVV \\ \mathbf{L}^\bullet(\mathcal{D}_V) @>>> \mathbf{L}^\bullet(\mathcal{D}_X). \end{CD}$$

is homotopy (co)cartesian ;

(ii) For a finite or countably infinite disjoint union $X = \coprod X_\alpha$, the inclusions $X_\alpha \rightarrow X$ induce a (weak) homotopy equivalence

$$\bigvee_{\alpha} \mathbf{L}^\bullet(\mathcal{D}_{X_\alpha}) \longrightarrow \mathbf{L}^\bullet(\mathcal{D}_X).$$

Proof Excision property (ii) is a straightforward consequence of finiteness property (ii). With Corollary 4.15, the proof of excision property (i) can be given using a mechanism which is very nicely abstracted in a paper by Vogel [32, 1.18, 6.1]. \square

Theorem 8.3 *The relative homotopy groups of the inclusion $\mathbf{L}^\bullet(\mathcal{D}_X) \rightarrow \mathbf{L}^\bullet(r\mathcal{D}_X)$ are vector spaces over $\mathbb{Z}/2$.*

Proof Let $K_0(\mathcal{D})$ be the Grothendieck group of \mathcal{D} (with one generator $[C]$ for each object C , a relation $[C] \sim [D]$ if C and D are weakly equivalent, and a relation $[C] - [D] + [E] = 0$ for every short exact sequence $C \rightarrow D \rightarrow E$ in \mathcal{D}). Define the Grothendieck group of $r\mathcal{D}$ similarly. Let

$$\tilde{K}_0$$

be the cokernel of the inclusion-induced map $K_0(\mathcal{D}) \rightarrow K_0(r\mathcal{D})$. The group $\mathbb{Z}/2$ acts on this by means of (degree 0) duality. The long exact sequence of homotopy groups of the inclusion map $\mathbf{L}^\bullet(\mathcal{D}_X) \rightarrow \mathbf{L}^\bullet(r\mathcal{D}_X)$ can be described as a ‘‘Rothenberg’’ sequence:

$$\cdots \rightarrow L^n(\mathcal{D}_X) \rightarrow L_n(r\mathcal{D}_X) \rightarrow \hat{H}^n(\mathbb{Z}/2; \tilde{K}_0) \rightarrow L^{n-1}(\mathcal{D}_X) \rightarrow L^{n-1}(r\mathcal{D}_X) \rightarrow \cdots$$

where \hat{H}^* denotes Tate cohomology. \square

Remark 8.4 Lemma 8.1 and Theorems 8.2 and 8.3 have analogues for quadratic L -theory which can be proved in the same way.

If our locally compact Hausdorff separable space X is an ENR, then

$$\mathbf{L}^\bullet(\mathcal{D}_X) \simeq X_+ \wedge \mathbf{L}^\bullet(\mathcal{D}_{\text{pt}}) = X_+ \wedge \mathbf{L}^\bullet(\mathbb{Z}).$$

This follows from homotopy invariance and the two excision properties by the standard arguments going back to Eilenberg and Steenrod. Here $\mathbf{L}^\bullet(\mathbb{Z})$ is the symmetric L -theory spectrum of the ring \mathbb{Z} (with the trivial involution).

Remark 8.5 If X is the polyhedron of a simplicial complex $\mathbf{L}^\bullet(\mathcal{D}_X)$ has the homotopy type of the spectrum $\mathbf{L}^\bullet(\mathbb{Z}, X)$ constructed in [24, §10] from the (\mathbb{Z}, X) -category of [26] endowed with a chain duality. See also [18, 27].

If X is a compact oriented topological n -manifold with boundary ∂X , then the identity map $X \rightarrow X$ determines by Examples 3.4 and 5.7 an n -dimensional SAP pair in $r\mathcal{D}_X$ (with boundary in $r\mathcal{D}_{\partial X}$) which in turn determines an element in

$$\pi_n(\mathbf{L}^\bullet(r\mathcal{D}_X), \mathbf{L}^\bullet(r\mathcal{D}_{\partial X})) \cong_{\mathbb{Z}[1/2]} \pi_n(\mathbf{L}^\bullet(\mathcal{D}_X), \mathbf{L}^\bullet(\mathcal{D}_{\partial X})) \cong H_n(X, \partial X; \mathbf{L}^\bullet(\mathbb{Z})).$$

Definition 8.6 This element in $\pi_n(\mathbf{L}^\bullet(r\mathcal{D}_X), \mathbf{L}^\bullet(r\mathcal{D}_{\partial X}))$ is the excisive signature of $(X, \partial X)$.

Remark 8.7 If X is triangulable, then we can regard the excisive signature of $(X, \partial X)$ as an element of $\pi_n(\mathbf{L}^\bullet(\mathcal{D}_X), \mathbf{L}^\bullet(\mathcal{D}_{\partial X}))$. In fact the excisive signature of a compact topological manifold X with boundary, not necessarily triangulable, can always be regarded as an element of

$$\pi_n(\mathbf{L}^\bullet(\mathcal{D}_X), \mathbf{L}^\bullet(\mathcal{D}_{\partial X})).$$

This follows easily from the fact that $X \times I^n$ for sufficiently large n admits a handle decomposition. Unfortunately the proof of that fact (existence of handle decomposition) given e.g. in [16] is hard and uses ideas which are quite closely related to Novikov’s original proof of the topological invariance of Pontryagin classes. For this reason we do not wish to use the “handle decomposition” argument. We have already avoided it by introducing $r\mathcal{D}_X$ and proving Theorem 8.3.

Remark 8.8 Let $f : Y \rightarrow X$ be a degree 1 normal map of closed n -dimensional topological manifolds. By Examples 3.4 and 5.7, the map $f : Y \rightarrow X$ and the identity map $\text{id} : X \rightarrow X$ determine two n -dimensional SAP objects $(C(f), \varphi)$ and $(C(\text{id}), \psi)$ in $r\mathcal{D}_X$ (and even in \mathcal{D}_X , by the previous remark). The map f induces a chain map $C(f) \rightarrow C(\text{id})$ which respects the symmetric structures, so that there is a splitting up to weak equivalence in $r\mathcal{D}_X$ or \mathcal{D}_X ,

$$(C(f), \varphi) \simeq (C(\text{id}), \psi) \oplus (K, \zeta).$$

We expect that the nondegenerate symmetric structure ζ on K has a canonical refinement to a (nondegenerate) quadratic structure, determined by the bundle data which come with the normal map f .

9 The Poincaré dual of the excisive signature

There is a rational homotopy equivalence

$$\mathbf{L}^\bullet(\mathbb{Z}) \simeq_{\mathbb{Q}} \bigvee_{i \geq 0} S^{4i} \wedge \mathbf{H}\mathbb{Q},$$

unique up to homotopy. For a compact oriented topological n -manifold X with boundary, the Poincaré dual of the “rationalized” excisive signature of $(X, \partial X)$ is therefore a class in

$$\bigoplus_{i \geq 0} H^{4i}(X; \mathbb{Q}).$$

We shall show that it is a characteristic class associated with the topological tangent bundle of X , a bundle with structure group $\text{TOP}(n)$.

Lemma 9.1 *The suspension isomorphism*

$$H_0(\text{pt}; \mathbf{L}^\bullet(\mathbb{Z})) \longrightarrow H_1(I, \partial I; \mathbf{L}^\bullet(\mathbb{Z}))$$

takes the unit 1 to the excisive signature of $(I, \partial I)$.

Proposition 9.2 *Let X be a compact oriented topological n -manifold X with boundary and let $Y = X \times [0, 1]$, so that $Y/\partial Y \cong \Sigma(X/\partial X)$. The suspension isomorphism*

$$H_n(X, \partial X; \mathbf{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Z}[1/2] \longrightarrow H_{n+1}(Y, \partial Y; \mathbf{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Z}[1/2]$$

takes the excisive signature of $(X, \partial X)$ to the excisive signature of $(Y, \partial Y)$.

Proof This follows from the previous lemma and the product formula in Example 6.10. \square

Proposition 9.3 *Let X be a compact oriented topological n -manifold X with boundary, $Y \subset X$ a compact codimension zero submanifold with locally flat boundary, $Y \cap \partial X = \emptyset$. Then, under the homomorphism*

$$H_n(X, \partial X; \mathbf{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Z}[1/2] \longrightarrow H_n(Y, \partial Y; \mathbf{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Z}[1/2]$$

induced by the quotient map $X/\partial X \rightarrow Y/\partial Y$, the excisive signature of $(X, \partial X)$ maps to the excisive signature of $(Y, \partial Y)$.

Proof This is a consequence of the naturality of ∇ in Example 5.6. □

Proposition 9.4 *The Poincaré dual of the (rationalized) excisive signature of a compact oriented manifold with boundary is a characteristic class Λ for euclidean bundles, evaluated on the tangent (micro)bundle of the manifold. (The characteristic class Λ is defined for euclidean bundles on compact ENRs, and is invariant under stabilisation, i.e., replacing a euclidean bundle $E \rightarrow Y$ by $E \times \mathbb{R} \rightarrow Y$.)*

Remark 9.5 While the construction of Λ as such is elementary, we use a technical fact from geometric topology to show that $\Lambda(TY)$ is Poincaré dual to $\sigma(Y, \partial Y)$ in the case where Y is a compact n -manifold. This fact is the existence of stable normal bundles for embeddings of topological manifolds [13].

Proof of Proposition 9.4 Let Y be a finite simplicial complex and let $E \rightarrow Y$ be a bundle on Y with fibers homeomorphic to \mathbb{R}^k . We would like to find a compact topological n -manifold X for some n , and a homotopy equivalence $f : Y \rightarrow X$ such that f^*TX is isomorphic to $E \times \mathbb{R}^{n-k} \rightarrow Y$, a stabilized version of $E \rightarrow Y$. Assuming that a sufficiently canonical choice of such an X and f can be made, we may then *define* the characteristic class associated with E on Y to be f^* of the Poincaré dual of the excisive signature of $(X, \partial X)$.

For the first step of this program, we choose an embedding $Y \rightarrow \mathbb{R}^\ell$ which is linear on each simplex of Y . Let Y_r be a regular neighborhood of Y in \mathbb{R}^ℓ . Choose an extension of E to a euclidean bundle $E_r \rightarrow Y_r$. Let $X \rightarrow Y_r$ be the bundle of $(k + 1)$ -disks on Y_r obtained from $E_r \rightarrow Y_r$ by fiberwise one-point-compactification, followed by fiberwise join with a point. Then X is a compact oriented manifold of dimension $\ell + k + 1$. Let $f : Y \rightarrow X$ be the composition of the inclusion $Y \rightarrow Y_r$ with any section of $X \rightarrow Y_r$. It is clear that f^*TX is identified with $E \times \mathbb{R}^{\ell+1} \rightarrow Y$.

We now define, as promised,

$$\Lambda(E \rightarrow Y) = f^*(u_X) \in \bigoplus_{i \geq 0} H^{4i}(Y; \mathbb{Q})$$

where $u_X \in \bigoplus_{i \geq 0} H^{4i}(X; \mathbb{Q})$ is the Poincaré dual of the rationalized excisive signature $\sigma(X, \partial X)$, in other words

$$u_X \cap [X, \partial X] = \sigma(X, \partial X).$$

From Proposition 9.2 we deduce that this is well defined, i.e., independent of the choice of an ℓ and an embedding $Y \rightarrow \mathbb{R}^\ell$. (More precisely Proposition 9.2 gives us the permission to make ℓ as large as we like, and for large ℓ any two embeddings $Y \rightarrow \mathbb{R}^\ell$ are isotopic.) From Proposition 9.3, we deduce that Λ is a characteristic class. Namely, suppose that we have euclidean bundles $E \rightarrow Y$ and $E' \rightarrow Y'$ and a simplicial map $g : Y \rightarrow Y'$ such that $g^*E' \cong E$. Taking ℓ large, we can choose embeddings $Y \rightarrow \mathbb{R}^\ell$ and $Y' \rightarrow \mathbb{R}^\ell$, with regular neighborhoods Y_r and Y'_r , in such a way that there is a codimension zero embedding $g_r : Y_r \rightarrow Y'_r$ making the following diagram homotopy commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\text{inclusion}} & Y_r \\ \downarrow g & & \downarrow g_r \\ Y' & \xrightarrow{\text{inclusion}} & Y'_r \end{array}$$

Now Proposition 9.3 can be applied to the embedding g_r and gives the desired conclusion, that $g^* \Lambda(E') = \Lambda(E)$ in $\bigoplus_{i \geq 0} H^{4i}(Y; \mathbb{Q})$.

Finally we can mechanically extend the definition of Λ to obtain a characteristic class defined for euclidean bundles on compact ENRs. Indeed let Y be a compact ENR; then Y is a retract of some finite simplicial complex Y_1 . Hence any euclidean bundle on Y extends to one on Y_1 . We can evaluate the characteristic class Λ there, and pull back to the cohomology of Y . To show that this is well defined, use the following: if Y is a retract of a finite simplicial complex Y_1 , and also a retract of a finite simplicial complex Y_2 , then the union of Y_1 and Y_2 along Y is again an ENR. See [15].

This is not the end of the proof, because we still have to show that

$$\Lambda(TY) \cap (Y, \partial Y) = \sigma(Y, \partial Y) \in \bigoplus_{i \geq 0} H_{n-4i}(Y, \partial Y; \mathbb{Q})$$

holds in the case where Y is a compact n -manifold with boundary. To establish this, we choose first of all a locally flat embedding $Y \rightarrow \mathbb{R}^\ell$ for some ℓ . This can be done by the method of [12, Chap. 1, Theorem 3.4]. In view of this we write $Y \subset \mathbb{R}^\ell$. Increasing ℓ if necessary, we may also assume [13] that Y has a normal microbundle in \mathbb{R}^ℓ , and by [17] we may also assume that it has a normal bundle $N \rightarrow Y$ in \mathbb{R}^ℓ . Choose a neighborhood Y_r of Y in \mathbb{R}^ℓ such that Y is a retract of Y_r and Y_r is a codimension zero PL submanifold of \mathbb{R}^ℓ . According to our definition of Λ , we now have to extend the euclidean bundle $TY \rightarrow Y$ to a euclidean bundle $E_r \rightarrow Y_r$ (which is easy). Then we should replace $E_r \rightarrow Y_r$ by $E_r \times \mathbb{R} \rightarrow Y_r$, which completes to a disk bundle $X \rightarrow Y_r$, etc.; we then have to find $\sigma(X, \partial X)$ and pass to Poincaré duals.— Altogether we now have an embedding $Y \rightarrow X$ by composing

$$Y \xrightarrow{\text{incl.}} Y_r \xrightarrow{\text{incl.}} E_r \times \mathbb{R} \xrightarrow{\text{incl.}} X$$

where the second arrow is any section of the bundle projection $E_r \times \mathbb{R} \rightarrow Y_r$. To complete the proof, it suffices to show that Y has a trivial normal disk bundle X' in X , and to apply Proposition 9.3 to the inclusion $X' \rightarrow X$. Here we note that the existence of a trivial normal bundle (with fibers $\cong \mathbb{R}^{\ell+1}$) implies the existence of a trivial normal disk bundle. But it is clear that Y has a normal bundle in X , identified with $N \times_Y (TY \times \mathbb{R})$, and this is clearly trivial since already $N \times_Y TY$ is trivial. \square

Proposition 9.6 *On vector bundles, the characteristic class Λ agrees with Hirzebruch’s total \mathcal{L} -class.*

Proof Let Y be a closed oriented topological manifold of dimension $4i$. By construction, the scalar product $\langle \Lambda_{4i}(TY), [Y] \rangle$ is equal to the image of $\sigma(Y)$ under the specialization (alias assembly) map

$$H_{4i}(Y; \mathbf{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Q} \rightarrow \pi_{4i} \mathbf{L}^\bullet(\mathbb{Z}) \otimes \mathbb{Q}.$$

In other words it is equal to the signature of Y . This holds in particular when Y is smooth. As this property characterizes the \mathcal{L} -class on vector bundles, we have $\Lambda = \mathcal{L}$ on vector bundles. \square

Appendix: the historical context

The referee of the paper recommended the provision of the historical context. Since the referee’s report did this so admirably, we simply append the relevant part of the report.

Besides Novikov's original proof of the topological invariance of the rational Pontryagin classes, the work of Kirby and Siebenmann [16] gives a geometric approach based on transversality. Different proofs have been given by Gromov [11] and by Sullivan and Teleman [30]. The latter employs Sullivan's construction of a Lipschitz structure on topological manifolds. Moreover, Pedersen, Roe and Weinberger [22] have given an indirect proof, using bounded topology, of the topological invariance of the K -homology class of the signature operator, which also follows from the Sullivan-Teleman proof, but avoids the use of Lipschitz structures. Madsen and Rothenberg [19] have given an equivariant generalization to odd order groups using a transversality approach. This was subsequently extended by Rothenberg and Weinberger [29] and Cappell, Shaneson and Weinberger [4] to all proper Lie group actions.

As for self dual sheaves, they arise explicitly in the work of Goresky and MacPherson [10]; and the Witt groups of self-dual sheaves and associated characteristic classes have been used in the work of Cappell and Shaneson [5, 6] for applications to non-locally flat embeddings and the understanding of singular hypersurfaces, and to give formulae for characteristic classes of manifolds when one has a singular fibration between them. The papers of [37] and of Woolf [35], and the book of Banagl [1] also shed light on the structure of self-dual sheaves; and self-dual sheaves are studied in homotopy stratified settings in Friedman [9]. See Brasselet et al. [2] for an application of self-dual sheaves in algebraic geometry. Finally, other applications of self-dual sheaves in topology appear in the work of Cappell, Shaneson and Weinberger [4] as well as a discussion of their topological invariance, which in general settings there draws on work of Yamasaki [36] (and also draws on work of Weiss [34]).

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