Differentiable Manifolds Which Are Homotopy Spheres

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§1. Introduction

This paper will study the problem of classifying differentiable n-manifolds which are homotopy spheres, under the relation of J-equivalence. (See the "dictionary" below.) It is shown that the equivalence classes form an abelian group which is denoted by Θ^n. The only groups Θ^n which I have been able to determine completely are the following:

$$\Theta^1 = \Theta^2 = 0, \quad \Theta^5 = 0, \quad \Theta^7 = Z_{28}, \quad \Theta^{11} = Z_{992} .$$

However partial information is obtained in many other cases. For example (according to 3.7, 5.8 and 6.9):

Theorem. For $k > 1$ the group Θ^{4k-1} is finite but non-trivial.

Section 2 of this paper will study a sum operation for connected manifolds of the same dimension. Section 3 defines an invariant λ' for certain (4k-1)-manifolds. Section 4 contains examples of homotopy spheres for which the invariant λ' takes on all possible values.

Section 5 describes a construction for simplifying manifolds, which was communicated to the author by R. Thom. Using this construction it is shown that the invariant $\lambda'(M)$ determines the J-equivalence class of M uniquely. A corresponding result for dimensions of the form $4k + 1$ is stated without proof. Section 6 studies the following question: Is every homotopy sphere the boundary of a π-manifold?

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Section 7 contains further discussion and a list of unsolved problems. Operations of "pasting together" manifolds and "straightening angles" are described in an appendix.

Dictionary of terms used. The word manifold will mean a compact, oriented, differentiable manifold, with or without boundaries. (The phrase "topological manifold" will be used in case the differentiable structure has not yet been specified.) The symbol $-M$ will be used for the manifold M with orientation reserved.

Two unbounded manifolds M_1, M_2 of the same dimension are J-equivalent if there exists a manifold W such that

1) the boundary ∂W is the disjoint union of M_1 and $-M_2$, and

2) both M_1 and M_2 are deformation retracts of W.

Thus J-equivalent manifolds belong to the same cobordism class and to the same homotopy type. This concept is due to Thom [3]. It is not known whether J-equivalent manifolds are necessarily diffeomorphic.

By a homotopy sphere we mean a (differentiable) manifold without boundary which has the homotopy type of a sphere. Similarly a homology sphere M must be unbounded and satisfy $H_*(M) \approx H_*(S^n)$. Here H_* denotes homology with integer coefficients, and S^n denotes the unit sphere in Euclidean space R^{n+1}. The notation D^{n+1} will be used for the disk bounded by S^n.

1.1 Lemma. Let $M^n = \partial W^{n+1}$ where M^n is simply connected and W^{n+1} is contractible. Then M^n is J-equivalent to S^n.

Proof. Choose an imbedding of D^{n+1} in the interior of W^{n+1}. Then $(W^{n+1}\text{-interior } (D^{n+1}))$ has* boundary equal to the disjoint union

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*Here the symbol - stands for set theoretic subtraction.

of M^n and S^n. It is not difficult to see that both boundaries are deformation retracts of W^{n+1}-interior (D^{n+1}).

A π-manifold W^n is characterized by the following property. If W^n is imbedded in a high dimensional Euclidean space R^{n+q}, then the normal bundle ν^q is trivial. This concept is due to J. H. C. Whitehead [2]. If W is a π-manifold, then clearly ∂W is also a π-manifold.

W^n will be called almost parallelizable if there exists a finite subset F so that W^n-F is parallelizable.

1.2 Lemma (J.H.C.Whitehead) Every parallelizable manifold is a π-manifold. Every π-manifold is almost parallelizable.

Proof. A field of tangent n-frames on $W^n \subset R^{n+q}$ induces a map f from W^n to the Stiefel manifold $V_{n+q,n}$. Note that f is covered by a bundle map from ν^q to a corresponding SO_q-bundle over $V_{n+q,n}$. But the space $V_{n+q,n}$ is (q-1)-connected. (See Steenrod [1] §25.6.) For $q > n$ this implies that f is homotopic to a constant; hence that ν^q is trivial.

Similarly a field of normal q-frames on W^n induces $f: W^n \rightarrow V_{n+q,q}$. Since $V_{n+q,q}$ is (n-1)-connected, the only obstruction to contracting f lies in

$$H^n(W^n; \pi_n(V_{n+q,q})).$$

But this cohomology group can be killed by removing a finite number of points from W^n.

A similar argument shows the following.

1.3 Lemma. If every component of W^n has a non-vacuous boundary, then the three concepts: parallelizable, π-manifold, and almost parallelizable, are equivalent.

The J-homomorphism of H. Hopf and G. Whitehead will be denoted by

$$J_n: \pi_n(SO_q) \longrightarrow \pi_{n+q}(S^q) .$$

(For a definition see Kervaire [4] §1.8. Caution: this homomorphism has nothing to do with J-equivalence.) It will always be assumed that q is large. This homomorphism will play a fundamental role in what follows.

§2. The connected sum of manifolds

Let M_1, M_2 be connected differentiable manifolds of the same dimension n. The sum $M_1 \# M_2$ is obtained by removing an n-cell from each, and then pasting the resulting boundaries together. There are three difficulties with this:

1) The pasting must be done in such a way that $M_1 \# M_2$ has an orientation compatible with that of both M_1 and M_2.

2) Even allowing for orientation, not every diffeomorphism between the boundaries will give rise to the same composite manifold. (According to Milnor [1] it is possible to paste together the boundaries of two 7-cells, obtaining a manifold which is not diffeomorphic to S^7.)

3) It is necessary to show that the result does not depend on which n-cell is chosen.

Definition. Choose an orientation preserving imbedding $h_1: R^n \to M_1$ and an orientation reversing imbedding $h_2: R^n \to M_2$. Let $M_1 \# M_2$ be obtained from the disjoint union of $M_1 - h_1(0)$ and $M_2 - h_2(0)$ by identifying $h_1(x)$ with $h_2(x/\|x\|^2)$ for each $x \neq 0$ in R^n.

Remark. It would be sufficient to specify $h_1(x)$ and $h_2(x)$ for $\|x\| < 1 + \varepsilon$ in order to construct this manifold $M_1 \# M_2$. In fact by removing all $h_i(x)$ with $\|x\| \leq 1/(1 + \varepsilon)$ from each M_i, and then identifying $h_1(x)$ with $h_2(x/\|x\|^2)$ for $1+\varepsilon > x > 1/(1+\varepsilon)$, we obtain the identical manifold $M_1 \# M_2$.

The following will be proved in a paper by J. Cerf.

2.1 Theorem of Cerf. Let M be a connected n-manifold. Given two orientation preserving imbeddings $f, f' : D^n \longrightarrow$ (interior M), there exists a diffeomorphism $g : M \longrightarrow M$ which satisfies $gf = f'$.

2.2 Corollary. The sum $M_1 \# M_2$ is well defined up to orientation preserving diffeomorphism.

Proof of the corollary. The only choice which occurred in the definition was the choice of imbeddings h_1, h_2. Given other imbeddings h_1', h_2', there exist diffeomorphisms g_i of M_i so that

$$g_i h_i(x) = h_i'(x) \quad \text{for} \quad \|x\| \leqq 1 + \varepsilon .$$

These g_i give rise to a diffeomorphism $g : M_1 \# M_2 \longrightarrow (M_1 \# M_2)'$; which completes the proof.

2.3 Lemma. Suppose that the unbounded manifolds M_1, M_2 are J-equivalent to M_1' and M_2' respectively. Then the sum $M_1 \# M_2$ is J-equivalent to $M_1' \# M_2'$.

Proof. If the dimension n is $\leqq 2$, then the assertion is clear. Hence we may assume that $n \geqq 3$. Choose manifolds W_i so that ∂W_i is the disjoint union of the deformation retracts M_i and $-M_i'$. Choose a differentiable arc a_i from $p_i \varepsilon M_i$ to $p_i' \varepsilon M_i'$ in W_i, so that the interior of a_i lies in the interior of W_i. We will see that the inclusion map

$$j : M_i - p_i \longrightarrow W_i - a_i$$

is a homotopy equivalence.

Since the codimension n of p_i in M_i is $\geqq 3$, the homomorphisms $\pi_1(M_i - p_i) \longrightarrow \pi_1(M_i)$, $\pi_1(W_i - a_i) \longrightarrow \pi_1(W_i)$ are isomorphisms. Hence

$$j_*: \pi_1(M_i - p_i) \longrightarrow \pi_1(W_i - a_i)$$

is an isomorphism.

Let $\hat{M}_i \subset \hat{W}_i$ denote the universal covering spaces, and let $\hat{p}_i \subset \hat{a}_i$ denote the inverse images of p_i, a_i. The inclusion

$$(\hat{M}_i, \hat{M}_i - \hat{p}_i) \longrightarrow (\hat{W}_i, \hat{W}_i - \hat{a}_i)$$

gives rise to a homomorphism between exact sequences of homology groups. Using the Five Lemma it follows that

$$\hat{j}_*: H_k(\hat{M}_i - \hat{p}_i) \longrightarrow H_k(\hat{W}_i - \hat{a}_i)$$

is an isomorphism for all k. Therefore j is a homotopy equivalence. (Compare J.H.C.Whitehead [3].)

Choose tubular neighborhoods N_i of a_i, and let W be a manifold obtained from $W_1 - N_1$ and $W_2 - N_2$ by pasting together the boundaries in such a way that ∂W is the disjoint union of $M_1 \# M_2$ and $-(M_1' \# M_2')$. Since the inclusions

$$M_i - (M_i \cap N_i) \longrightarrow W_i - N_i$$

are homotopy equivalences, it follows easily that the inclusion

$$M_1 \# M_2 \longrightarrow W$$

is a homotopy equivalence. A corresponding argument takes care of the inclusion $(M_1' \# M_2') \longrightarrow W$. This completes the proof of 2.3.

It is clear that the operation # is associative and commutative, providing that we do not distinguish between diffeomorphic manifolds. Furthermore the sphere acts as a zero element: $M \# S^n \approx M$.

2.4 Lemma. Suppose that M is a homotopy n-sphere. Then $M \# (-M)$ is J-equivalent to S^n .

Proof. Let U denote the interior of a disk $D^n \subset M$. Consider the topological manifold $(M-U) \times [0,1]$. This is differentiable, except along the "angles" $\partial U \times [0]$ and $\partial U \times [1]$. Let W be a differentiable manifold obtained from $(M-U) \times [0,1]$ by straightening these angles. (See the Appendix.) Then W is a contractible manifold with boundary $M \# (-M)$. Together with 1.1 this completes the proof.

Now combining 2.3 and 2.4 this proves:

2.5 Theorem. The set of all J-equivalence classes of homotopy n-spheres forms an abelian group under the operation $\#$.

This group will be denoted by Θ^n. It is clear that $\Theta^1 = 0$. Since Munkres [1] has shown that a 2-manifold has an essentially unique differentiable structure, it follows that $\Theta^2 = 0$.

[Two subgroups of Θ^n will also be studied. $\Theta^n(\pi)$ will denote the subgroup formed by all π-manifolds in Θ^n, and $\Theta^n(\partial \pi)$ will denote the subgroup formed by all boundaries of π-manifolds.]

§3. The invariant $\lambda'(M^{4k-1})$

Let M be a (4k-1)-manifold which is (1) a homology sphere, and (2) the boundary of some π-manifold W . The intersection number of two homology class α, β of W will be denoted by $\langle\alpha, \beta\rangle$. Let $I(W)$ denote the index of the quadratic form

$$\alpha \longrightarrow \langle \alpha, \alpha \rangle ,$$

where α varies over the Betti group $H_{2k}(W)/(\text{torsion})$. Integer coefficients are to be understood.

Define I_k as the greatest common divisor of $I(M)$ where M ranges over all almost parallelizable manifolds of dimension $4k$ which have no boundary. This number has been studied by Kervaire and Milnor [1]. (See 3.7.)

3.1 Lemma. The residue class of $I(W)$ modulo I_k is an invariant of the boundary M.

Proof. If M is the boundary of two parallelizable manifolds W_1 and W_2, let N be the unbounded 4k-manifold obtained from W_1 and $-W_2$ by pasting together the common boundary. Clearly

$$I(N) = I(W_1) - I(W_2) .$$

Let p be a point of M. Then the complement $N-p$ is parallelizable. In fact $N-p$ is the union of parallelizable manifolds W_1-p and W_2-p, having an intersection $M-p$ which is acyclic. Given a field of 4k-frames on W_1-p and on W_2-p, it is possible to deform one of the two so that they coincide along $M-p$. Therefore N is almost parallelizable; and

$$I(N) \equiv 0 \pmod{I_k} .$$

This completes the proof.

Not every residue class can occur:

3.2 Lemma. The index $I(W)$ of an almost parallelizable manifold is always divisible by 8; providing that ∂W is a homology sphere.

Proof. First observe that the intersection number $\langle \alpha, \alpha \rangle$ is always an even integer. This is the homology translation of the statement that

$$Sq^{2k} : H^{2k}(W, \partial W ; Z) \longrightarrow H^{4k}(W, \partial W ; Z_2)$$

is zero. If Sq^{2k} were not zero then the formulae of Wu (see Wu [1], Kervaire [2]) would imply that W had a non-trivial Stiefel-Whitney class in dimension $\leq 2k$.

Since ∂W is a homology sphere it follows by Poincare duality that the matrix of intersection numbers has determinant ± 1. But a quadratic form with determinant ± 1 which takes on only even values must have index divisible by 8. (Compare Milnor [4].) This completes the proof.

Definition. The residue class of $\frac{1}{8} I(W)$ modulo $\frac{1}{8} I_k$ will be denoted by $\lambda'(M)$.

3.3 Lemma. The properties of being (1) a homotopy n-sphere, and (2) the boundary of a π-manifold, are invariant under J-equivalence; and are preserved by the sum operation $\#$.

Hence the manifolds which have these properties give rise to a subgroup of Θ^n.

Definition. This subgroup will be denoted by $\Theta^n(\partial \pi)$.

3.4 Lemma. The invariant $\lambda'(M)$ depends only on the J-equivalence class of M. Furthermore

$$\lambda'(M_1 \# M_2) = \lambda'(M_1) + \lambda'(M_2).$$

The proofs of 3.3 and 3.4 are straightforward. Hence λ' gives rise to a homomorphism

$$\Lambda' : \Theta^{4k-1}(\partial \pi) \longrightarrow Z_{\frac{1}{8} I_k}$$

It will be proved in Sections 4, 5 that Λ' is an isomorphism, at least for $k > 1$.

The principal difficulty with the invariant λ' is that it is extremely difficult to compute. For example it would be very interesting to evaluate λ' for the topological spheres which are constructed in Milnor [1, 5] and Shimada [1]. The invariant λ which is defined in these papers is somewhat weaker, but much easier to compute.

The numbers $\frac{1}{8} I_k$ can be described as follows. Let B_k denote the k-th Bernoulli number:

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \ldots, \quad B_6 = 691/2730, \ldots \ .$$

Define j_k as the order of the image

$$J_{4k-1}(SO_q) \subset \pi_{q+4k-1}(S^q) \qquad \text{for large } q \ .$$

Define a_k to be 2 if k is odd and 1 if k is even. Then according to Kervaire and Milnor [1]:

3.5 Lemma. I_k is equal to

$$2^{2k-1}(2^{2k-1}-1) \ B_k \ j_k \ a_k / k \ .$$

The only unknown quantity here is the integer j_k .

3.6 Lemma. j_k is a multiple of the denominator of $B_k/4k$.

Proof. For k even, this is proved in Kervaire and Milnor [1]. For k odd this follows from the arguments of that paper, together with the following:

Theorem of Hirzebruch (Not yet published.) If the unbounded manifold M^{4k} has Stiefel-Whitney class w_2 equal to zero, and if k is odd, then the $\hat{A}$-genus $\hat{A}[M^{4k}]$ is an even integer.

On the other hand an upper bound for j_k is given by the order of the largest cyclic subgroup of $\pi_{q+4k-1}(S^q)$. The p-primary component of $\pi_{q+4k-1}(S^q)$ is known for $k < p^2(p-1)/2$ for any prime p. (See Toda [1,2].) The full group is known (to me) only for $k = 1, 2, 3$. It turns out that the upper bound for the p-primary factor of j_k is exactly equal to the lower bound in each known case.

Combining the preceding information, we have:

3.7 Lemma. The number $\frac{1}{8} I_k$ is equal to $a_k 2^{2k-2}(2^{2k-1}-1)$ (numerator B_k/k), multiplied by an integer whose prime factors p satisfy $p^2(p-1) \leq 2k$. In particular

$$\frac{1}{8} I_1 = 2 , \quad \frac{1}{8} I_2 = 28 , \quad \frac{1}{8} I_3 = 992, \quad \frac{1}{8} I_4 \text{ equals } 8128 \text{ times}$$

a power of 2.

§4. Construction of (4k-1)-manifolds

The following is perhaps the simplest example of a symmetric matrix with determinant ± 1, with only even elements on the diagonal, and with index different from zero. (Compare Milnor [4].)

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

We will construct a manifold W^{4k} which has the above intersection matrix.

Let T be a tubular neighborhood of the diagonal in $S^{2k} \times S^{2k}$: say the set of all pairs (x,y) with distance $d(x,y) \leq \varepsilon$. Thus T is a 4k-manifold having the homotopy tupe of S^{2k}. The intersection number of the fundamental 2k-cycle with itself is $+2$.

Let $\alpha : S^{2k} \longrightarrow S^{2k}$ be the "twelve hour rotation" which leaves the north pole p fixed, and satisfies $\alpha(x) = -x$ for x on the equator. Let $T' = (1 \times \alpha)\, T$ be the set of pairs (x,y) with $d(x,\alpha y) \leq \varepsilon$. Then T and T' intersect only in a small neighborhood of the pair (p,p), and a small neighborhood of the pair $(-p,-p)$.

The universal covering space of $T \cup T'$ consists of infinitely many disjoint copies of T and infinitely many disjoint copies of T'. Numbering these copies T_i and T_i', we may assume that each T_i' intersects only T_i and T_{i+1}.

Define W_1 as the subset

$$T_1 \cup T_1' \cup T_2 \cup T_2' \cup T_3 \cup T_3' \cup T_4 \cup T_4'$$

of this universal covering space. Thus W_1 is a topological 4k-manifold, having the homotopy type of a union of eight 2k-spheres with a single point in common. Choosing an appropriate basis for $H_{2k}(W_1)$, the intersection matrix is as follows:

$$\begin{pmatrix}
2 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
 & 1 & 2 & 1 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & 1 \\
 & & & & & & 1 & 2
\end{pmatrix}$$

To correct this intersection matrix it is necessary to introduce an intersection between T_1' and T_3, so as to obtain an intersection number -1. Choose a rotation of $S^{2k} \times S^{2k}$ which carries a region of T' near the "equator" onto a region of T near the "equator", so as to obtain an intersection number of

-1 . Matching the corresponding regions of T_1' and T_3 , we obtain a topological manifold W_2, with the required intersection matrix.

This manifold W_2 is differentiable except along eight "angles" which have been introduced in the boundary. Let W_3 be a differentiable manifold obtained by straightening these angles. (See the appendix.)

Unfortunately the transition from W_1 to W_2 changed the homotopy type. In fact the fundamental group $\pi_1(W_2) = \pi_1(W_3)$ is infinite cyclic. Next we will kill this fundamental group. A generator can be represented by a simple closed differentiable curve C lying on the boundary of W_3 .

Choose an imbedding $h: S^1 \times D^{4k-2} \longrightarrow \partial W_3$ which carries $S^1 \times 0$ onto the given curve C . Let W_4 be the space obtained from the disjoint union

$$W_3 \cup D^2 \times D^{4k-2}$$

by identifying $S^1 \times D^{4k-2}$ with its image under h . Then W_4 is simply connected. In fact W_4 has the same homotopy type as W_1; but the same intersection matrix as W_2 or W_3 .

This space W_4 is a differentiable manifold, except along the "angle" corresponding to $S^1 \times S^{4k-3}$. Let W_0 be a differentiable manifold obtained by "straightening" this angle.

4.1 <u>Theorem.</u> W_0 is a parallelizable 4k-manifold with boundary M_0 which is a homology (4k-1)-sphere. In fact for $k > 1$, M_0 is a homotopy sphere. The index $I(W_0)$ equals +8 .

Thus the invariant $\lambda'(M_0)$ is defined and equal to +1 .

4.2 <u>Corollary.</u> The homomorphism Λ' from $\Theta^{4k-1}(\partial\pi)$ to the cyclic group of order $\frac{1}{8} I_k$ is onto, providing that $k > 1$.