The signature of ramified coverings*

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1.

Let X be a compact oriented differentiable manifold of dimension m without boundary on which the cyclic group G_n of order n acts by orientation preserving diffeomorphisms. The set Y of fixed points of this action is a differentiable submanifold of X not necessarily connected. The various connectedness components of Y can have different dimensions, they are not necessarily orientable.

We assume that all components of Y have codimension 2 and G_n operates freely on $X - Y$. Then X/G_n is an oriented manifold. The natural projection

$$\pi\colon X \longrightarrow X/G_n$$

maps Y bijectively onto a submanifold Y' of X/G_n. For any point $x \in Y$ we can introduce local coordinates (ξ, z) of X centered at x with

$$\xi \in \mathbf{R}^{m-2},\ z \in \mathbf{C}\ , \qquad\qquad Y \text{ given locally by } z = 0\ ,$$

and local coordinates (ξ', z') of X/G_n centered at $\pi(x)$ with

$$\xi' \in \mathbf{R}^{m-2},\ z' \in \mathbf{C}\ , \qquad\qquad Y' \text{ given locally by } z' = 0$$

such that $\pi\colon X \to X/G_n$ has the local description

$$z' = z^n\ , \qquad \xi' = \xi\ .$$

Under these circumstances we call X an n-fold ramified covering of X/G_n with branching locus Y'.

From now on suppose X has dimension $4k$. The signature of X is defined as follows. Consider over $H^{2k}(X, \mathbf{R})$ the quadratic form

$$(1) \qquad Q(\alpha, \beta) = (\alpha \cup \beta)[X] \qquad\qquad \text{for } \alpha, \beta \in H^{2k}(X, \mathbf{R})\ .$$

Q is a bilinear symmetric form over a real vector space and by

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definition

$$\operatorname{sign}(X) = p^+ - p^- ,$$

where p^+ is the number of positive entries, p^- the number of negative entries in a diagonalisation of Q. If, as before, G_n operates on X, then $H^{2k}(X, \mathbf{R})$ is a G_n-module. The action of G_n on $H^{2k}(X, \mathbf{R})$ preserves Q. We can decompose $H^{2k}(X, \mathbf{R})$ as follows

$$(2) \qquad H^{2k}(X, \mathbf{R}) = H_+ \oplus H_- ,$$

where H_+ and H_- are Q-orthogonal, and where Q is positive-definite on H_+ and negative-definite on H_- and

$$T(H_+) = H_+ , \quad T(H_-) = H_- \qquad \text{for } T \in G_n .$$

For any $T \in G_n$ we define

$$(3) \qquad \operatorname{sign}(X, T) = \operatorname{tr}(T \mid H_+) - \operatorname{tr}(T \mid H_-)$$

where tr denotes the trace. It is easy to show that sign (X, T) does not depend on the choice of H_+ and H_- (compare [4, p. 578]). For $T = 1$ we have

$$\operatorname{sign}(X, T) = \operatorname{sign}(X) .$$

We wish to relate sign (X) and sign (X/G_n). For this we need information on sign (X, T). This is furnished by the fixed point theorem of Atiyah-Bott-Singer [2], [3], [4], or rather by a special case of it. There is an elliptic operator on X of order 1 whose index is sign (X) (see [4, § 6]). The Atiyah-Singer index theorem applied to this operator gives sign (X) in terms of Pontrjagin numbers [7, Th. 8.2.2], the Atiyah-Bott-Singer fixed point theorem gives a formula for sign (X, T) involving the Pontrjagin classes of X and the normal bundle of Y as a vector bundle on which G_n acts. We shall give the precise formula for sign (X, T) later.

2.

The map $\pi: X \to X/G_n$ induces a ring homomorphism

$$\pi^*: H^*(X/G_n, \mathbf{R}) \longrightarrow H^*(X, \mathbf{R}) .$$

LEMMA. *π^* maps $H^*(X/G_n, \mathbf{R})$ bijectively on $H^*(X, \mathbf{R})^{G_n}$, the ring of elements of $H^*(X, \mathbf{R})$ invariant under the operations of G_n.*

This is a well-known fact true under much more general circumstances. (Compare A. Grothendieck, Tôhoku Math. J. 9 (1957), 119–221; Chap. V.)

We have for $\alpha, \beta \in H^{2k}(X/G_n, \mathbf{R})$

$$(\pi^*\alpha \cup \pi^*\beta)[X] = n(\alpha \cup \beta)[X/G_n] .$$

Therefore sign (X/G_n) equals the signature of the form Q of X (see (1)) when restricted to

$$H^{2k}(X, \mathbf{R})^{G_n} = H_+^{G_n} \oplus H_-^{G_n} \qquad \text{(see (2))} .$$

Thus

$$\text{(4)} \qquad \text{sign}\,(X/G_n) = \dim H_+^{G_n} - \dim H_-^{G_n} .$$

Using a well-known formula for the dimension of the invariant part of a representation of a finite group, we obtain from formulas (3) and (4)

$$\text{(5)} \qquad \text{sign}\,(X/G_n) = \frac{1}{n} \sum\nolimits_{T \in G_n} \text{sign}\,(X, T)$$

3.

Let us first consider the case $n = 2$. Then $G_2 = \{1, T\}$ where T is an orientation preserving involution of X. We have

$$\text{(6)} \qquad \text{sign}\,(X, T) = \text{sign}\,(Y \circ Y)$$

where $Y \circ Y$ is the oriented self-intersection cobordism class (see [8], [9], [4, Prop. 6.15]). Thus (5) becomes

$$\text{(7)} \qquad 2\,\text{sign}\,(X/G_2) = \text{sign}\,(X) + \text{sign}\,(Y \circ Y) .$$

Remarks. (i) The formula (6) is a consequence of the Atiyah-Bott-Singer fixed point theorem. Formula (6) holds for any orientation preserving differentiable involution, the fixed point set Y need not have codimension 2. The fixed theorem gives sign (X, T) in terms of characteristic numbers. Applying the signature theorem to $Y \circ Y$ eliminates the characteristic numbers to give a theorem which is trivial for $T =$ Identity and therefore is weaker than the version coming from the Atiyah-Bott-Singer fixed point theorem. Jänich and Ossa [10] have proved (6), for arbitrary codimension of Y, by elementary methods. It is not true for manifolds with boundary, even if one assumes $Y \cap \partial X = \varnothing$. The mistake gives rise to an interesting invariant for free involutions on $(4k - 1)$-manifolds as studied in [4, § 7], [8], [9]. Also (7) is not true for manifolds with boundary.

(ii) In (6) and (7) the fixed point set Y need not be orientable.

Example. If X is the complex projective plane canonically oriented and T complex conjugation with respect to homogeneous coordinates (z_0, z_1, z_2), then Y is the real projective plane and

$$\operatorname{sign}(Y \circ Y) = -1 .$$

X/G_2 is a rational homology 4-sphere which checks with (7).

(iii) If $\dim X = 4$ and Y is orientable, then $\operatorname{sign}(Y \circ Y)$ is the self-intersection number $Y \circ Y$ in X which equals $(Y' \circ Y')/2$ where $Y' \circ Y'$ is the self-intersection number of Y' in X/G_2 (compare (13)). Thus (7) can be written in this case as

$$(8) \qquad \operatorname{sign}(X) = 2 \operatorname{sign}(X/G_2) - \frac{1}{2}(Y' \circ Y') .$$

Formula (8) is true also if Y is non-orientable. Then $Y' \circ Y'$ has to be considered as oriented self-intersection cobordism class in X/G_2.

4.

We make the assumptions of § 1 and wish to give a formula for $\operatorname{sign}(X, T)$ with $T \in G_n$, but T different from the identity. Observe that Y is orientable if $n \geqq 3$. For $n = 2$ we assume Y orientable, the non-orientable case having been settled in § 3. We orient Y and the normal bundle ν of Y such that these orientations span the given orientation of X. Then ν has SO(2) as structural group and may therefore be regarded as a complex line bundle with U(1) as structural group. This can be done equivariantly with respect to G_n. Then the operation of T in ν determines a complex eigenvalue t with $t^n = 1$ and $t \neq 1$.

THEOREM. *Under the assumptions of* § 1 (*with* $\dim X = 4k$) *we have*

$$(9) \qquad \operatorname{sign}(X, T) = \operatorname{sign} \frac{(t+1) + (t-1)Y}{(t-1) + (t+1)Y} Y .$$

This formula is to be interpreted as follows: Develop

$$\frac{(t+1) + (t-1)y}{(t-1) + (t+1)y} y ,$$

where $t \in \mathbf{C}$ $(t \neq 1)$ and y an indeterminate, as a formal power series in y.

$$(10)\qquad \frac{(t+1)+(t-1)y}{(t-1)+(t+1)y}\,y = \frac{t+1}{t-1}\,y - \frac{4t}{(t-1)^2}\,y^2 + \cdots .$$

We construct a sequence of oriented submanifolds of X

$$\cdots \subset Y_3 \subset Y_2 \subset Y_1 = Y \subset X .$$

If Y_r is already constructed, then we make the embedding $i\colon Y_r \to X$ transversal to Y. Let j be a transversal map approximating i, then $Y_{r+1} = j^{-1}(Y)$. The orientations of Y_{r+1} and $j^*\nu$ span the orientation of Y_r. The oriented cobordism classes of the Y_r are independent of all other choices involved. The cobordism class of Y_r is denoted by

$$Y^r = Y \circ \cdots \circ Y \in \Omega^{4k-2r} .$$

If we replace in (10) the power y^r by Y^r, then we get an element of the cobordism algebra $\Omega^* \otimes \mathbf{C}$, where of course $Y^r = Y \circ \cdots \circ Y$ does not represent a power with respect to the multiplication in Ω^*. Recall that the signature is a ring homomorphism $\Omega^* \to \mathbf{Z}$ which vanishes by definition on Ω^m for $m \not\equiv 0 \pmod 4$. The right side of (9) is the signature of the element of $\Omega^* \otimes \mathbf{C}$ obtained by replacing y^r by Y^r. Thus (9) means (for $\dim X = 4k$)

$$\operatorname{sign}(X, T) = -\frac{4t}{(t-1)^2}\operatorname{sign}(Y \circ Y) + \cdots .$$

If Y is not connected, then the eigenvalue t has to be taken separately for each connectedness component of Y, and the right side of (9) represents a sum over the connectedness components. Changing the orientation of Y and simultaneously of the normal bundle ν has the effect that $Y^r \in \Omega^{4k-2r}$ is replaced by $(-1)^r Y^r$ and t by t^{-1}. Since

$$\frac{(t+1)+(t-1)y}{(t-1)+(t+1)y}\,y$$

remains unchanged under the substitution $t \to t^{-1}$, $y \to -y$ it does not matter which orientations we take as long as the orientations of Y and ν span the given orientation of X. Actually we have even more freedom with the orientations, since in (9) we have $\operatorname{sign}(Y^r) \neq 0$ only for r even. Formula (9) is a consequence of the Atiyah-Bott-Singer fixed point theorem, more precisely of the G-signature theorem [4, p. 582]. When applying it the invariance under the substitution $t \to t^{-1}$, $y \to -y$ has to observed because in [4] *loc cit.* the eigenvalue t is supposed to have a positive imaginary part.

When deducing (9) from the G-signature theorem we may assume $t \neq -1$. If $t = -1$, then T is an involution (which can happen only if n is even). But (9) reduces to (6) for $t = -1$. Without further explanation we write down the G-signature theorem of [4, p. 582] for our case and $t \neq -1$. We use precisely the notation of [4]. This gives for $t = e^{i\theta}$ $(0 < \theta < \pi)$

$$\operatorname{sign}(X, T) = 2^{2k-1}\left\{\mathfrak{L}(Y)\frac{1}{\tanh\dfrac{x+i\theta}{2}}\right\}[Y]$$

where $x \in H^2(X, \mathbf{Z})$ is the Poincaré dual of Y. Substitute in the expression in $\{\quad\}$ each $(2r)$-dimensional class α by $2^r\alpha$, then we get

$$\operatorname{sign}(X, T) = \left\{\widetilde{\mathfrak{L}}(Y)\frac{1}{\tanh\left(x + i\dfrac{\theta}{2}\right)}\right\}[Y]\,,$$

where $\widetilde{\mathfrak{L}}(Y) = \sum_{j=0}^{\infty} L_j(Y)$ is the total L-class of Y introduced in [7]. Since for $t = e^{i\theta}$

$$\frac{1}{\tanh\left(x + i\dfrac{\theta}{2}\right)} = \frac{(t+1) + (t-1)\tanh x}{(t-1) + (t+1)\tanh x}\,,$$

we get (9) by using the virtual indices or signatures of [7, § 9].

5.

Still making the assumptions of § 1 we calculate $\operatorname{sign}(X/G_n)$ using (5) and (9). Observe that (9) remains correct if T is the identity and $t = 1$, because then the right side of (9) reduces to $\operatorname{sign}(Y^\circ)$ and $Y^\circ = X$. There is the following identity between rational functions.

$$(11) \qquad \frac{1}{n}\sum_{t^n=1}\frac{(t+1)+(t-1)y}{(t-1)+(t+1)y} = \frac{(1+y)^n + (1-y)^n}{(1+y)^n - (1-y)^n}\,.$$

The sum is over all n^{th} roots of unity. By virtue of (11), the formulas (5) and (9) imply

THEOREM. *Let X be a compact oriented differentiable manifold of dimension $4k$ without boundary on which the cyclic group G_n of order n acts by orientation preserving diffeomorphisms. If all components of the fixed point set Y of this action have* *codimension 2 and if G_n acts freely on $X - Y$, then X/G_n is a compact oriented differentiable manifold with*

$$(12)\qquad \begin{cases} \operatorname{sign}(X/G_n) = \operatorname{sign} \dfrac{(1+Y)^n + (1-Y)^n}{(1+Y)^n - (1-Y)^n} Y \\ \qquad = \dfrac{1}{n} \operatorname{sign}(X) + \dfrac{n^2-1}{3n} \operatorname{sign}(Y \circ Y) + \cdots . \end{cases}$$

Suppose Y is orientable, consider the submanifold Y' of X/G_n ($Y = Y'$ under the projection $\pi\colon X \to X/G_n$). Orient Y and Y' in the same way and regard their normal bundles ν in X and ν' in X/G_n as complex line bundles. Then it follows easily that

$$(13)\qquad \nu' = \nu^n = \nu \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} \nu .$$

As we shall see later ν' and ν (as bundles over Y') can be extended to complex line bundles E' and E over X/G_n with

$$E' = E^n \qquad\qquad (n\text{-fold tensor product})$$

and such that the first Chern class

$$c_1(E') = x' \in H^2(X/G_n, \mathbf{Z})$$

is the Poincaré dual of Y' in X/G_n. Therefore $x' = nc_1(E)$. Represent $c_1(E)$ by an oriented submanifold U of X/G_n. Then we can deduce from (12) and (13) a formula for sign (X) in terms of the signatures of the oriented self-intersection cobordism classes $U^r = U \circ \cdots \circ U$ where the "self-intersection" takes place in X/G_n. Leaving the details to the reader we obtain

$$(14)\qquad \begin{aligned} \operatorname{sign} X &= \operatorname{sign} \frac{(1+U)^n - (1-U)^n}{(1+U)^n + (1-U)^n} \cdot \frac{1}{U} \\ &= n \operatorname{sign}(X/G_n) - \frac{n(n^2-1)}{3} \operatorname{sign}(U \circ U) + \cdots . \end{aligned}$$

If dim $X = 4$, then (14) can be written as

$$(15)\qquad \operatorname{sign} X = n \operatorname{sign}(X/G_n) - \frac{n^2-1}{3n}(Y' \circ Y')$$

where $Y' \circ Y'$ is the self-intersection number of the branching locus in X/G_n.

Remarks. (i) If Y is empty, then we have an unramified covering and (12) gives the well-known formula

$$\operatorname{sign}(X) = n \operatorname{sign}(X/G_n) .$$

This is not known in the *topological case*. Though formula (5) remains correct, the preceding formula cannot be deduced from it. Namely, for topological manifolds it is not known whether sign (X, T) vanishes for T without fixed points.

(ii) In the formulas (12), (14), and (15) the orientations of Y, U, and Y' respectively, do not play any role. Observe that in (12) and (14) the rational functions on the right side of the equations are even. Therefore only self-intersections Y^r etc. with r even occur. The orientation of Y^r does not change if one changes the orientation of Y.

6.

PROPOSITION. *Let M be a compact oriented differentiable manifold without boundary. Let Y' be an oriented differentiable submanifold of codimension 2, and $x' \in H^2(M, \mathbf{Z})$ the Poincaré dual of Y'. Suppose that x' is divisible by n in $H^2(M, \mathbf{Z})$. Then there exists a G_n-manifold X with $X/G_n = M$ and with branching locus Y' such that all the assumptions of § 1 are satisfied.*

For the proof we consider the complex line bundle E' with $c_1(E') = x'$. It has a differentiable section $s: M \to E'$ which vanishes on Y', is different from zero on $M - Y'$, and is transversal to $s_0(M)$ where s_0 is the zero section of E'. Since x' is divisible by n we can find a complex line bundle E with

$$E^n = E \otimes \cdots \otimes E = E' .$$

If $\rho: E \to E'$ denotes the map

$$\rho: v \longmapsto v \otimes \cdots \otimes v ,$$

then $X = \rho^{-1}s(M)$ satisfies all the assumptions of § 1.

Of course, X in the above proposition is in general not uniquely determined. Its signature is given by (14) where U is a submanifold representing $c_1(E)$.

If the assumptions of § 1 are satisfied for a given G_n-manifold X, then X is obtainable by the method of the preceding proposition. This can be seen as follows. Over $X/G_n - Y'$ we have a principal G_n-bundle, and over a tubular neighborhood $\mathcal{U}$ of Y' a principal $\mathbf{C}^*$-bundle coming from the normal bundle ν of Y in X. If we extend the structural group of the principal G_n-bundle to $\mathbf{C}^*$ we get a principal $\mathbf{C}^*$-bundle over $X/G_n - Y'$ which can be identified on $\mathcal{V} - Y'$ with the principal C^*-bundle of ν. Thus we get a principal C^*-bundle over X/G_n. We denote its associated complex line bundle by E. Then E^n is a complex line bundle E' with $c_1(E') = x'$, the Poincaré dual of Y'. As can be checked, X is obtained from E and E' as in the above proposition.

7.

The construction of **6** can be done in the complex analytic case. We need a minor modification. Let M be a compact complex manifold, and D a divisor on M given by meromorphic functions f_i defined in open sets U_i with $\bigcup U_i = M$ such that, on $U_i \cap U_j$, the function f_i/f_j has neither zeros nor poles (see [7, § 15.2]). We say, D has simple zeros and poles if and only if for a suitable open covering each f_i is a coordinate function z_1 in a local coordinate system $(z_1, z_2, \cdots, z_n)$ defined in U_i or is the inverse z_1^{-1} of such a coordinate function or is constant and different from zero and ∞. A divisor D has simple zero and poles if and only if $D = D_1 - D_2$ where D_1, D_2 are non-singular divisors [7, § 15.2, p. 115] with no common zeros, i.e., the complex submanifolds Y_1' and Y_2' of complex codimension 1 determined by D_1 and D_2 do not intersect.

We assume that

$$\{D\} = E \otimes \cdots \otimes E = E^n . \tag{16}$$

Here $\{D\}$ is the holomorphic complex line bundle defined by D and E is a holomorphic complex line bundle whose n^{th} power as a holomorphic line bundle is $\{D\}$. Any holomorphic line bundle L has an associated holomorphic bundle $\hat{L}$ with the complex projective line as fibre obtained by adding a point at infinity to each fibre of L.

The line bundle $\{D\}$ has transition functions

$$f_{ij} = f_i/f_j \colon U_i \cap U_j \longrightarrow C^*$$

and a meromorphic section given by the functions f_i. This meromorphic section defines a holomorphic section

$$s \colon M \longrightarrow \widehat{\{D\}} .$$

Because of (16) we have the "n^{th} power maps"

$$\rho \colon E \longrightarrow \{D\} , \qquad \hat{\rho} \colon \hat{E} \longrightarrow \widehat{\{D\}} .$$

Then $X = \hat{\rho}^{-1}s(M)$ is a compact complex manifold which is an n-fold holomorphic ramified covering of M with $Y' = Y_1' \cup Y_1'$. Here

Y' is determined by the zeros and poles of the divisor D. Since we are interested in the signature of X, the orientations given to the components of Y' do not matter (§ 5, *Remark* (ii)).

Formula (16) implies that the Chern class $c_1\{D\}$ is divisible by n. This divisibility is also sufficient for the existence of a holomorphic line bundle E with $E^n = \{D\}$. For the proof we use the exact sequence

$$H^1(M, \Omega) \xrightarrow{\exp} H^1(M, \mathbf{C}^*_\omega) \xrightarrow{c_1} H^2(M, \mathbf{Z}) \xrightarrow{j} H^2(M, \Omega)$$

where Ω is the sheaf of germs of holomorphic functions and $H^1(M, \mathbf{C}^*_\omega)$ the group of holomorphic line bundles under the tensor product [7, § 15.9]. If $c_1\{D\} = nx$, then x is in the kernel of j, therefore in the image of c_1. Thus $c_1\{D\} = c_1(E_1^n)$, where E_1 is a holomorphic line bundle, and $\{D\}E_1^{-n}$ is in the image of exp. In $H^1(M, \Omega)$ every element is divisible by n. Therefore $\{D\}E_1^{-n} = L^n$ where L is in the image of exp. q.e.d.

If M is an algebraic surface, then the ramified covering X is again algebraic. We have the following theorem.

THEOREM. *Let M be a non-singular algebraic surface and $D = D_1 - D_2$ a divisor on M where D_1, D_2 are non-singular curves which do not intersect. Suppose that the integral homology class of D is divisible by n. Then there exists an algebraic surface X which is a ramified covering of M along D (we have $M = X/G_n$ where the cyclic group G_n acts on X by holomorphic maps and freely outside the set of fixed points). For any such X we have*

$$\text{sign}(X) = n\,\text{sign}(M) - \frac{n^2-1}{3n} D \circ D\,, \qquad (\text{see (15)})\,. \tag{17}$$

Here $D \circ D$ is the self-intersection number of the homology class of the divisor D.

Remark. If in the preceding theorem $\text{sign}(M) = 0$ and $D \circ D < 0$ with $n \geq 2$, we get examples of algebraic surfaces with $\text{sign}(X) \geqq 2$. In fact, Atiyah [1] and Kodaira [11] have in this way constructed algebraic surfaces with arbitrary large signatures, as we shall recall in § 8. The existence of algebraic surfaces with $\text{sign}(X) \geqq 2$ contradicts an earlier conjecture in algebraic geometry. Borel [5] has proved the existence of discontinuous groups operating freely on bounded homogeneous symmetric domains and which have a compact orbit space. Also his result led to examples of algebraic surfaces with arbitrary large signatures (compare [7, § 22.3]).

8.

The formula for the signature of ramified coverings was motivated by papers of Atiyah [1] and Kodaira [11] who studied the signature of ramified coverings in some special cases which are of particular interest because they show that the signature of the total space of a differentiable fibre bundle need not be equal to the product of the signatures of base and fibre. This multiplicative property holds however if the fundamental group of the base operates trivially on the cohomology of the fibre [6]. The construction in §§ 6 and 7 occurs essentially in [1] except that Atiyah studies only double coverings. We report briefly on the family of algebraic surfaces studied by Kodaira [11]. The calculation of Pontrjagin classes occuring in [1] and [11] can be replaced by formula (17).

Let C_1 be a Riemann surface (algebraic curve) of genus $g_1 \geqq 1$. Since $H_1(C_1, \mathbf{Z}_2) \neq 0$ there exists a Riemann surface C_2 which is an unramified double covering of C_1. Let $\tau: C_2 \to C_2$ be the covering translation; τ is a free involution. The genus g_2 of C_2 is given by

$$2(2 - 2g_1) = 2 - 2g_2 , \qquad g_2 = 2g_1 - 1 .$$

The group $H_1(C_2, \mathbf{Z}_n)$ is a homomorphic image of $\pi_1(C_2)$. Let C_3 be the Riemann surface (algebraic curve) which is associated to the universal unramified covering of C_2 with $H_1(C_2, \mathbf{Z}_n)$ as fibre. The degree of this covering map $f: C_3 \to C_2$ is n^{2g_2}. The genus g_3 of C_3 is

$$g_3 = n^{2g_2}(g_2 - 1) + 1 .$$

In the cartesian product $C_3 \times C_2$ we consider the graph Γ_f of f. The self-intersection number of Γ_f in $C_3 \times C_2$ equals

$$\Gamma_f \circ \Gamma_f = \deg(f) \cdot (2 - 2g_2) = 4n^{4g_1-2} \cdot (1 - g_1) . \tag{18}$$

The class of Γ_f in $H_2(C_3 \times C_2, Z_n)$ is determined by $f^*: H^*(C_2, \mathbf{Z}_n) \to H^*(C_3, \mathbf{Z}_n)$. By the very construction f^* is 0 in dimensions 1 and 2. The same holds for $(\tau f)^* = f^* \tau^*$. In dimension 0, both f^* and $(\tau f)^*$ are the identity. Therefore Γ_f and $\Gamma_{\tau f}$ have the same homology class in $H_2(C_3 \times C_2, \mathbf{Z}_n)$. We can now apply the theorem of § 7 with $M = C_3 \times C_2$ and $D = \Gamma_f - \Gamma_{\tau f}$. Since

$$\operatorname{sign}(M) = \operatorname{sign}(C_3) \operatorname{sign}(C_2) = 0$$

and

$$D \circ D = 2\Gamma_f \circ \Gamma_f = -8n^{4g_1-2}(g_1 - 1) ,$$

we have

THEOREM. *For any $g_1 \geqq 1$ and $n \geqq 2$ there exists an algebraic surface $X(g_1, n)$ with*

$$\text{sign}\, X(n, g_1) = 8\frac{n^2 - 1}{3}(g_1 - 1)n^{4g_1-3} .$$

The Kodaira algebraic surface $X(g_1, n)$ is fibered differentiably over C_3 with algebraic curves $C_2'(x)$ as fibres which are n-fold ramified coverings of C_2 with the two branching points $f(x)$ and $\tau f(x)$ for $x \in C_3$. The fibres therefore have genus $ng_2 = n(2g_1 - 1)$. Thus $X(g_1, n)$ is a 4-dimensional manifold fibered differentiable over a 2-dimensional manifold C_3 of genus $n^{4g_1-2}(2g_1 - 2) + 1$ with a 2-dimensional manifold C_2' of genus $n(2g_1 - 1)$ as fibre. For $g_1 \geqq 2$ we have sign $X(g_1, n) \neq 0$ whereas sign (C_2') sign $(C_3) = 0 \cdot 0 = 0$. Thus the signature does not behave multiplicatively in this differentiable fibre bundle. Observe that the signature is defined to be 0 for manifolds of a dimension not divisible by 4. A differentiable bundle with total space, base, and fibre of dimensions divisible by 4 is given by the cartesian product of $X(g_1, n) = X$ with itself. Then

$$\text{sign}\, X^2 = (\text{sign}\, X)^2 \neq 0 ,$$

whereas the base C_3^2 and the fibre $(C_2')^2$ have vanishing signatures.

$X(g_1, n)$ gives for $g_1 \geqq 2$ a family of Riemann surfaces $C_2'(x)$ $(x \in C_3)$ which is locally not trivial. The complex structure of $C_2'(x)$ varies with x (see [1] and [11]). As Atiyah [1] shows, this phenomenon is closely related to the non-multiplicativity of the signature.

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