Graham Higman's Thesis "Units in Group Rings"

Robert Sandling
The University
Manchester M13 9PL.

The extent to which Higman's D. Phil. thesis [Hig 2] differs from his published paper [Hig 1] is not widely known. There have been few references to it: a mention in [Coh], an acknowledgement in [Jac 1], a remark by Higman himself [Hig 3]. It is not listed in the standard directory (ASLIB) of British theses.

Higman's paper, entitled "The Units of Group Rings", is widely known. It is important to algebraists for its definitive results on units in integral group rings of finite abelian groups and its characterisation of finite groups which have only trivial units. It is looked to by a broad spectrum of pure mathematicians as one of the papers establishing K-theory because of its connection with the work of Higman's supervisor, J.H.C. Whitehead, on what is now called Whitehead torsion.

The thesis represents the work of another year or more, the paper having been submitted in February 1939, the thesis deposited in the library in December 1940. Most of the paper is included in the thesis but in a completed or altered form; for example, the results on trivial units are derived in the context of the theory of orders. While further classes of infinite groups having only trivial units are introduced, the connection with Whitehead's original motivation in simple homotopy theory is no longer even hinted at. The proofs of what is now referred to as the triviality of the Whitehead group of an infinite cyclic group and of certain finite abelian groups are omitted.

There is more in the thesis, however, than in the paper, much of it still of great interest. It raises many questions still unanswered. More surprisingly, it supplies answers to many questions raised and answered in the intervening years by researchers unaware of Higman's results. Some of these will be apparent in the following summary of the notable acheivements of the thesis.

The thesis marks an early stage in the study of integral group rings as orders and of maximal orders in group rings; it also displays an early use of algebraic methods in the investigation of units in orders. It provides early examples of integral representations, in particular for certain metacyclic groups (namely, the affine group over the field of p elements). These representations have the feature, subsequently seen to be important in other cases, of reducing to triangular form modulo p.

A more complete characterisation is given of group rings having only trivial units. Much is said about non-trivial units. Illustrations are given to exemplify various points: non-conjugacy to trivial units; possibilities for the orders of units of finite order. In the integral case, it is shown that a non-trivial unit of finite order has no central element in its support. There are also results on the order and exponent of a group of normalised units of finite order.

The isomorphism problem for integral group rings of finite groups makes its appearance in two forms: whether the group ring determines the group; whether every subgroup of normalised units of finite order is isomorphic to a subgroup of the group. A positive solution for the second is given for metabelian nilpotent groups and for the subgroups of the metacyclic groups mentioned earlier. His proof for metabelian nilpotent groups contains the main elements of the eventual proof [Whitc.] of the general metabelian case.

It offers an early use of the homomorphism induced on their group rings by the projection of a group onto a factor group. Ideals such as the kernel of this homomorphism are manipulated with facility and the basis for such results as the isomorphism between G/G' and I/I^2 laid down.

Problems appearing in the paper in one form or another receive further discussion and more precise formulation. The existence of one-sided inverses is queried, as is the existence of zero divisors and non-trivial units in group rings of torsion-free groups. For the study of the latter, the one and two unique product conditions are formulated.

Although there are instances in all this material in which Higman did not see a point in as full a fashion as it is presently understood, it is remarkable how complete his insight was into the topics he included and how confident was his treatment of them.

This report provides a detailed summary of the thesis. The Abstract, most of the theorems and certain proofs are quoted in full. There are several reasons for attempting to bring this thesis to a wider audience. It is refreshing to observe how one of the first explorers on the ground saw the terrain, and surprising in hindsight to note how correctly he saw it: raising many of the main questions, spotting many of the main theorems, developing many of the main techniques. For a number of problems, it displays yet another independent researcher reaching the same conclusions. This may indicate the extent of what is true in a given area and so offer a guide to present researches.

It is a challenging exercise in mathematical bibliography to seek out the reappearances (at least in published form) of material in the thesis. Paradoxically I began this exercise before I was aware of Higman's thesis in an effort to correct inadequate attributions I had made in an ad hoc survey of the isomorphism problem at the previous Oberwolfach Tagung (February 1977) in this series organised by Klaus Roggenkamp. I had, in particular, overlooked much of the work of Berman and his collaborators (I have now translated many of their articles and can supply copies). Although no doubt still incomplete, my findings concerning the material in Higman's thesis are included here. Only occasionally do I discuss later developments. There are presently available several comprehensive monographs and surveys on group rings

[Zal-Mih; Bov; Den; Pas 4; Seh 2; Pol; Far]. These indicate the current status of most of the problems Higman considered.

A typewritten manuscript copy of the thesis is kept in the Radcliffe Science Library, Parks Road, Oxford, OX1 3QP. It is unbound and is noted as being "Deposited 20-12-40". It carries the reference number "MS.D.Phil.d.387". Photocopies are available on application to the library. In the United Kingdom, it is available on inter-library loan.

The manuscript consists of:

Title page.

Abstract (pp. 1-6)

1. Introduction and summary of results (pp. 1-5)
2. Definitions and Preliminary Considerations (pp. 6-10)
3. The Group Algebra of a Finite Group (pp. 11-20)
4. Integral Group Rings of Finite Groups (pp. 21-32)
5. Units of Finite Order in Integral Group Rings (pp. 33-51)
6. An Example (pp. 52-62)
7. Group Rings of Groups without Elements of Finite Order (pp. 63-78)

Acknowledgement: I wish to thank Professor Higman for his permission to include excerpts from his thesis in this article.

Abstract (pp. 1-6)

In the Abstract, Higman describes his results. His interests and emphases are clearly expressed in it and for this reason I include it in its entirety.

It also defines his notation. In my commentary I use a form of notation more common at present. Having parallel notations is clumsy at times, mainly for rings of coefficients. Higman uses K for a general ring or field of coefficients; I have used R for a ring and K for a field. He uses C for a ring of algebraic integers; I have again used R with K being the corresponding quotient field (Higman uses k). My notation for the augmentation ideal of the group ring RG is I(G), with I(H)RG denoting the ideal of RG generated by I(H), H a normal subgroup of G; Higman uses λ_H for this.

"UNITS IN GROUP RINGS

Abstract of Thesis

If G is any group, written multiplicatively, and K is any ring, then the finite formal sums

$$k_1g_1 + k_2g_2 + \ldots + k_rg_r\ , \ k_i \in K,\ g_i \in G,\ (i = 1,\ldots,r)\ ,$$

form a ring, when addition and multiplication is defined in the obvious way. This ring we call the group ring of G over K, and we write it as R(G,K). More precisely, R(G,K) may be defined as a ring which is a linear set over K, and has a basis which is a multiplicative group isomorphic to G. If K has the unity element

1, and if g_0 is the identity of G, then $1.g_0$ is a unity element of R(G,K). We shall identify, when confusion cannot arise thereby, elements k of K with the elements $k.g_0$ of R(G,K). The object of this thesis is to establish certain theorems on the units of R(G,K).

An element e_1 in a ring with unity element 1 will be called a left unit if there exists also an element e_2 such that $e_1e_2 = 1$; and e_2 will then be called a right unit, and a right inverse of e_1. If e_1 is also a right unit, then it has a uniquely defined inverse which we write as e_1^{-1}, and e_1 is then called a unit, simply. We shall deal always with rings R(G,K) in which the coefficient ring K has no right units which are not left units. Whether it is even so possible for R(G,K) to have right units which are not left units, I do not know. Certainly it is not in the most important cases, - for instance, if K can be embedded in a field, and G is of finite order. Any group ring has units of the form e.g, where e is a unit in K, and g is an element of G, for e.g has the inverse $e^{-1}.g^{-1}$. Such a unit we shall call trivial. One of the questions that naturally arises is, for what group rings these are the only units.

This thesis is chiefly concerned with the case in which G is a group of finite order, and K is the ring C of integers in an algebraic field k. We shall then speak of R(G,C) as an integral group ring. R(G,C) is then an order, but not in general a maximal order, of the linear associative algebra R(G,k). Accordingly, after an introductory section, and one on definitions, we discuss in Section 3 the group algebra R(G,k). This section is an exposition of well-known facts concerning the decomposition of the semi-simple algebra R(G,k) into simple components, and is based on the classical theory of representations of finite groups in algebraic fields, as developed by I. Schur.

In Section 4 we pass on to the consideration of integral group rings, and more particularly, to a determination of what such rings have only trivial units. The result is, that R(G,C), where C is the integer ring of the field k, has only trivial units in the following four cases, and these only:

(i) G is Abelian, and the orders of its elements all divide two; k is the rational field or an imaginary quadratic extension of it;

(ii) G is Abelian, and the orders of its elements all divide six; k is the rational field, or the extension of it by a complex cube root of unity;

(iii) G is Abelian, and the orders of its elements all divide four; k is the rational field, or the extension of it by a complex fourth root of unity (that is, by $i = \sqrt{-1}$);

(iv) G is the direct product of a quaternion group and an Abelian group the orders of whose elements all divide two; k is the rational field.

To establish this theorem, we prove first a theorem which is a particular case of a result due to O. Schilling, namely, that if k is of finite degree over the rational field the unit group of R(G,C) has a finite index in the unit group of any maximal order of $R(G,k)$ containing it. Secondly, we show that the group of trivial units in $R(G,C)$ is not contained as a proper sub-group of finite index in any group of units of $R(G,C)$. From these two theorems together, it follows that $R(G,C)$ has only trivial units if and only if a maximal order of $R(G,k)$ containing it has a finite unit group. This imposes severe restrictions on the possible simple components of $R(G,k)$, which give rise to our main result.

In the following section, we turn our attention to the units of finite order in $R(G,C)$. We find it convenient to treat normalised units only, - that is to say, units in which the sum of the coefficients is unity. This involves no essential loss, since any unit of finite order is the product of a normalised unit of finite order and a root of unity in C; and any group of units of finite order is isomorphic to the direct product of the corresponding group of normalised units and a group of roots of unity in C. In terms of normalised units, a theorem from section 4 becomes: The group G is not contained as a proper sub-group of finite index in any group of normalised units of $R(G,C)$. In particular this implies that a unit of finite order in the centrum of $R(G,C)$ must be trivial. If G is Abelian, therefore, all the units of finite order in $R(G,C)$ are trivial. This is of course no longer true if G is not Abelian. In fact, a unit of finite order in $R(G,C)$ need not necessarily even be conjugate to a trivial unit. We are able, however, to prove two results on units of finite order. The first states that the elements of any group of normalised units of finite order in $R(G,C)$ are linearly independent, and even linearly independent modulo m, for any integer m, and that therefore the order of such a group cannot exceed the order of G. The second states that the prime factors of the order of a normalised unit of $R(G,C)$ divide the order of G. If we add the assumption that G is soluble, we can show that the order of the unit divides the order of G. Lastly, we show for the very special class of groups whose lower central series terminates in the identity and whose second derived group consists of the identity, that a group of normalised units of finite order in $R(G,C)$ is isomorphic to a subgroup of G. This implies that $R(G,C)$ is not isomorphic to any other group ring $R(H,C)$ unless G is isomorphic to H. These last theorems are proved by methods different from those of the rest of the section, and our chief tools are the two-sided ideals λ_H, where H is a self conjugate subgroup of G, generated by the elements $h-1$, for all h in H. It should be added, that throughout this section the coefficient ring C is an arbitrary ring of algebraic integers, and may be taken to be ring of all algebraic integers.

In section 6, we apply the theorems we have proved to the detailed investigation of the ring $R(G,C)$ where G is generated by two elements a, b subject to the relations:-

$$a^p = b^{p-1} = 1 , \qquad b^{-1}ab = a^r$$

where p is an odd prime, and r a primitive number modulo p. Here, too, we show that a group of normalised units in $R(G,C)$ is isomorphic to a subgroup of G.

Finally, in section 7, we consider group rings of groups without elements of finite order. Naturally, the theorems proved are of an entirely different character. Notably, they do not depend at all on the coefficient ring K, provided that it has no zero divisors. We show, in fact, that if G satisfy the condition that every subgroup generated by a finite number of elements of G has a homomorphism on the free cyclic group, then if K has no zero divisors neither has R(G,K), and the units of R(G,K) are all trivial. The condition is satisfied by free groups and by free Abelian groups; and generally, by the direct product and the free product of any two groups that satisfy it.

As we have said, section 3 is a repetition of well-known facts; and the first theorems of section 4 are a particular case of a theorem due to Schilling. The rest of the thesis is original, though some of it has been published previously."

1. Introduction and summary of results (pp. 1-5)

The introductory section is much the same as the abstract, although there are differences in details included and excluded. More attempt is made here to place the work in context:

"The study belongs more particularly to the Theory of Groups, because from the algebraic point of view there is nothing distinctive about the particular non-maximal order R(G,C) - indeed it is quite possible for two distinct groups G, G^1 to have isomorphic algebras, even over the rational field, though their group rings over C are not isomorphic." (p.2).

"The group algebra of a finite group is, of course, well known, and affords one of the most convenient examples of a semi-simple algebra...Considerably less attention has hitherto been paid to integral group rings, or to the group rings of infinite groups." (p.4).

Reference is made to work of Reidemeister [Rei] and Franz [Fra 1,2] as dealing with "particular problems in connection with integral group rings" (p.4) but of "no relevance to the problems with which we are concerned" (p.5). It is interesting to note here that the facts about trivial units for cyclic groups were known to Franz [Fra 3]. He cites as "more relevant ... the work on units in maximal orders in semi-simple algebras" (p.5) in [Schi.] and [Eic]. Although he refers to his Theorem 4 as a particular case of a theorem of Schilling, he finds no overlap in the rest of his work on group rings as orders because "in a maximal order of a general semi-simple algebra there is clearly no analogue of a trivial unit" (p.5). Lastly he thanks his "successive supervisors, Mr J.H.C. Whitehead and Mr P. Hall, for many suggestions" (p.5).

2. Definitions and Preliminary Considerations (pp. 6-10)

Aside from setting out basic definitions, this section repeats the question (still unsettled; see [Den]) of whether right units are left units in group rings, a question raised in his paper. He gives an example of an algebra for which they are not, and remarks "there seems nothing to suggest that this is impossible in a group ring" (p.9).

He also sets down the basic relationships between the group ring of a group and that of a subgroup or factor group. In particular, he shows that, if H is a subgroup of G, then an element of RH is a unit in RG if and only if it is a unit in RH. He views the augmentation homomorphism as a particular case of the natural homomorphism from a group ring onto the group ring of a factor group. He introduces normalised units and observes that "the whole group of units of R(G,K) is the direct product of the group of normalised units and the unit group of K" (p.10).

These notions, as with that of the kernel I(H)RG of the homomorphism from RG to RG/H (p.43), are now taken for granted. In the 1930's they were not so familiar and facility in their use developed slowly and as much in connection with topology and cohomology [Rha, Hop, Fre, Lyn, Fox] as with algebra. Algebraists encountered these notions principally in connection with dimension subgroups, and developed methods based on identities in elements of the augmentation ideal (also used by Higman) rather than on structural relationships among ideals.

3. The Group Algebra of a Finite Group (pp. 11-20)

In this section the representation theory of a finite group over a field K which is an algebraic extension of the rationals, is reviewed for subsequent use. It is done from two points of view, that of the theory of algebras for which his references are [Wae] and [Alb], and that of "the classical theory" (p.12), the reference here being [Spe]. He finds the classical theory, with its explicit formulae connecting the basis of group elements of KG with the basis of primitive central idempotents, as more "in keeping with the nature of the subject" (p.12) because, if R denotes the integers of K, RG is "only defined in terms of the special basis (of group elements)" (p.12).

The result he reports in Theorem 1 is the general version of the result about the decomposition of KG given in his paper for the case in which G is abelian (this specialisation is here Theorem 2). Formulae in the matrix entries of the group elements in the irreducible representations are given in detail and play a large role in section 4.

The section concludes with remarks on isomorphisms of groups and group rings. He cites as an "obvious case" (p.19) of non-isomorphic groups having isomorphic group rings, that of two abelian groups of the same order over any field containing roots of unity of that order [Per-Wal]. He goes on to give an example over the rational field, "and therefore" (p.19) over any field of characteristic 0, that of the two non-abelian groups of order p^3, p an odd prime [Ber 3, Pas 1]. Another early example is given in an exercise in [Bou, p.117], that of the two non-abelian groups of order 8 over fields of characteristic not equal to 2 in which -1 is a square.

He remarks that, in both of his examples, the corresponding integral group rings are not isomorphic as shown by results of section 5. In [Col], there is a mention of this for the second example, and, in [Coh-Liv 1], a full proof.

He finishes the section with a formulation of the isomorphism problem for group rings: "Whether it is possible for two non-isomorphic groups to have isomorphic integral group rings I do not know; but the results of section 5 suggest that it is unlikely" (p.20). This tentative conjecture is repeated later in a stronger form. Subsequent work on the problem usually traces its origin to the problem of determining all groups H for which KG is isomorphic to KH for a given field K, a problem which was "proposed by R.M. Thrall at the Michigan Algebra Conference in the summer of 1947" [Per-Wal, p.420]. It took many years for the problem to be formulated as it is known today. Major impetus towards its study was its inclusion in the survey [Bra] and the text [Cur-Rei, p.262].

4. Integral Group Rings of Finite Groups (pp. 21-32).

One of the main results of his published paper is improved in this section. He classifies those group rings RG which have only trivial units with respect to R as well as with respect to G. While the result is little different, the method is very much so. The exposition is entirely in the context of orders, a concept not present in the paper. The fact that RG is an order in KG does not seem to have been the commonplace then that it is now; for example, expository treatments of orders did not cite this as a standard example. Thus the first item in the section is a proof that RG is an order in KG, the main argument of which shows that each of the elements of RG satisfies a monic polynomial equation with coefficients in R.

In the introductory section he discussed the relevance of the theory of orders to this special case and pointed out its distinguishing features. In this section he exploits such features and is able to prove statements about these orders in a wholly algebraic manner; this is in strong contrast with the analytic and geometric style of the theory for more general orders, particularly in the articles of Eichler and Schilling which he cites.

Having noted that RG is not generally a maximal order in KG ("the element $\frac{1}{n}\sum_{i=1}^{n} g^{(i)}$ is an idempotent of the centrum of R(G,k) and is therefore in every maximal order of R(G,k), but it is not in R(G,C)" (p.21) — the $g^{(i)}$ are the n elements of the group G), he then gives a theorem which measures how far removed from being maximal it is:

"Theorem 3. If k is a finite extension of the rational field, and O is any order of R(G,k) containing R(G,C), then R(G,C) contains the set n O of all multiples of elements in O by the order n of G."(p.22).

In the proof he assumes that the irreducible representations are such "that in each representation all the elements of O are represented by matrices whose elements are integers"(p.22). He then needs only remark that, from the classical formulae reviewed in the previous section, each of the standard basis elements E_{ij} of the matrix algebras in the representations corresponds to an element of KG having coefficients which are integers times $1/n$. For the initial assumption he refers to [Spe, Ch. 14, § 65] which, however, applies only to group elements and so to RG, not O (Speiser's methods are those of group matrices and determinants). He goes on to give his own proof:

"We may suppose Γ to be a representation in some finite algebraic field k' containing k. Let O' be the order in $R(G,k')$ consisting of all elements $\sum_i c_i O_i$ where c_i are integers in k', and O_i are elements in O. Let $e_1,\dots e_s$ be the unit vectors of the vector space in which the matrices of $\Gamma(G)$ act, and consider the vectors

$$Xe_1 = x_{11}e_1 + x_{12}e_2 + \dots + x_{1s}e_s, \tag{1}$$

for all matrices X representing elements of O'. Call a vector (1) of length r if $x_{1t} = 0$ for $t > r$ and consider the set I_r of all r-th coefficients x_{1r} of vectors of length r. For each r, I_r is fractional ideal in k' distinct from the zero ideal. For I_r plainly contains the difference of any two of its elements, and the product of any one of them by an integer in k'. Moreover we can find a number m_r such that any element of $m_r I_r$ is an integer. If $r = 1$, we have $Xe_1 = x_{11}e_1$ so that $x - x_{11}$ is a factor of the characteristic function of X. Since O' is an order, this function has integral coefficients, and therefore x_{11} is an integer, so that we may take $m_1 = 1$. Now since Γ is irreducible, we can, by Burnside's Theorem[2] choose an element E_r in $R(G,C')$, and therefore in O', such that its image in Γ is $m_r E_{1r}$, for some number m_r, where $E_{1r}\, e_r = e_1$, $E_{1r}\, e_k = 0$, $k \neq r$.
Then by (1) we have

$$m_r E_{1r} \cdot Xe_1 = m_r x_{ir} e_1,$$

so that $m_r x_{1r}$ must be integral, as required. Thus I_r is an ideal in k'. If it were the zero ideal, there would be less than s linearly independent vectors in (1), contrary to the hypothesis that Γ is irreducible. Since k' is a finite extension of the rational field, we can find an extension k'' of k', such that the ideals I_r^1 in k'', having the elements of I_r as basis, are principal ideals $(\alpha_r)^1$. Extending the order O' again to O'' in $R(G,k'')$ and supposing X in (1) to be chosen now from the matrices representing elements of O'', we have that there are vectors $f_1, f_2,\dots,f_s$ in the set (1) such that

$$f_r = f_{r1}e_1 + \dots + f_{r\,r-1}e_{r-1} + \alpha_r e_r .$$

Then every vector of (1) can be written as $\sum_r y_r f_r$ with integral coefficients y_r. Choosing $f_1, f_2,\dots,f_s$ as a new basis for the vector space, we obtain an equivalent representation Γ' with the desired properties.

(2) Cf. van der Waerden, op.cit. Ch. XVII, § 121.

(1) See Hilbert, Jahresberichte der Deutschen Mathematiker-Vereinigung, 4, p. 224, (1897)." (pp. 23-24).

With the groundwork completed, he can readily establish the theorem, Theorem 5, which he needs for the classification of group rings with all units trivial. Theorems 4 and 6 are ancillary to it, the latter being proved by a lemma which is restated as Theorem 10 in the subsequent section.

"Theorem 4. If C is the integer ring of a finite extension k of the rational field, then the unit group of R(G,C) has a finite index in the unit group of any order O of R(G,k) containing R(G,C)."(pp. 24-25).

"Theorem 5. A necessary and sufficient condition that all the units of R(G,C) be trivial, is that a maximal order of R(G,k) containing R(G,C) have a unit group of finite order."(p.26).

"Theorem 6. The group of trivial units in R(G,C) is not contained as a proper subgroup of finite index in any group of units of R(G,C)."(p.26).

Having transformed the question into one about maximal orders, he can apply known facts about orders in full matrix rings, fields and division algebras to narrow down the candidates to those of the four cases (as listed in the Abstract). The fourth case requires extra discussion as in the paper. The section ends (there being no Theorem 7) with the statement of the main theorem and its extension to the case of torsion groups:

"Theorem 8. The group ring R(G,C) of a finite group G over a ring of algebraic integers C has only trivial units if and only if G and k, the quotient field of C, have one of the forms I, II, III, IV, listed above." (p. 31).

"Theorem 9. The statement of Theorem 8 remains true if for the words "a finite group G" we substitute "a group G all of whose elements have finite order."."(p.32). For G abelian, Theorem 8 appears in [Coh-Liv 2].

5. Units of Finite Order in Integral Group Rings (pp. 33-51)

This section opens with a stronger form of the isomorphism problem. It and the next are concentrated on providing examples to support this "plausible theorem" (p.33). He states it as follows:

"The theorems that we prove are all partial cases of the plausible theorem:- A group of units of finite order in R(G,C) is isomorphic to a group of trivial units" (p.33).

The "starting point"(p.33) for the results here, a lemma in the previous section, is repeated as Theorem 10:

"Theorem 10. The coefficient of the identity of G in a non-trivial unit of finite order in G is zero." (p.33).

It is a theorem usually attributed to Berman [Ber 1] (see [Tak] also). Berman was the next investigator to study units in group rings systematically and so to re-derive results already discovered by Higman. There has been a similar ignorance of the extent of Berman's contributions, partly due to the relative inaccessibility of his publications and to insufficient reviewing. A case in point is his theorem that the integral group ring determines the class sums of the group. This and other Soviet contributions to group ring problems are described in [Seh 2] (see also [Bov] and [Zal-Mih]).

Higman's proof is the same as that which is now considered standard:

"Let E be a unit of finite order in $R(G,C)$, in which the coefficient of the identity is the non-zero integer a. Let $\bar{E}$ be the image of E in the regular representation of G. We have trace $\bar{E} = na$, where n is the order of G. The characteristic roots of $\bar{E}$ are roots of unity, $w_1, \ldots, w_n$, and therefore

(1) $$|\text{trace } \bar{E}| = |w_1 + w_2 + \ldots + w_n| \leqslant |w_1| + |w_2| + \ldots + |w_n| = n$$

and the same is true for all the conjugates of trace $\bar{E}$. Therefore $|N(a)| \leqslant 1$, where $N(a)$ stands for the norm of a, and is therefore a non-zero rational integer. Thus equality must hold here, and therefore also in (1). That is to say

$$|w_1 + w_2 + \ldots + w_n| = |w_1| + |w_2| + \ldots + |w_n|,$$

which can only happen if $w_1 = \ldots = w_n$. Then, however, the matrix $\bar{E}$, being of finite order, must be scalar, so that $E = w.1$, and therefore E is trivial, which proves the Lemma." (p.27).

The next two theorems follow readily from Theorem 10. The first, a result in [Coh-Liv 2], is a "quantitative" (p.33) theorem:

"Theorem 11. The elements of a group of normalised units of finite order in $R(G,C)$ are linearly independent. The order of such a group is at most equal to the order of G." (p.33).

The second is "qualitative" (p.35):

"Theorem 12. The prime factors of the order of a group of normalised units of finite order in $R(G,C)$ divide the order of G." (p.35).

For G soluble, this is improved to:

"Theorem 13. If the group G is soluble and of order n, then the order of any normalised unit of finite order in $R(G,C)$ divides n." (p.37).

The proof, as he records in a Corollary (p.42), shows rather more, namely that the order divides the product of the exponents of the quotients of any series of normal subgroups of G having abelian quotients. The proofs of Theorem 12 and 13 involve matrix and algebraic number calculations as well as an induction argument in the latter. When rediscovered, these results were in a more precise and general form. In [Ber 1], Theorem 13 is established for any finite group. In [Coh-Liv 2], it is shown that the exponent of a group of normalised units divides the exponent of G; by symmetry, the exponent of a group basis equals that of G, a result also in [Pas 1].

Specific non-trivial units have always been difficult to find. The most familiar, a unit of infinite order in the group ring of the cyclic group of order 5, was one of the first to be recorded; it is attributed to Mordell in that part of Whitehead's paper [White., p.284] which makes reference to Higman's yet to be published calculations for certain cyclic groups. Many units of finite order are given in the thesis. To emphasise the significance of the ring of coefficients being one of algebraic integers, he provides a unit of order 3 in the rational group algebra of the quaternion group, namely, $1 - \frac{1}{4}(1 - xyz)(3 + x + y + z)$ where $G = \langle x, y : x^2 = y^2 = z^2 = xyz\rangle$ (p.36). He considers it "a corollary to Theorem 10" (p.36) that a non-trivial unit of finite order has no central elements in its support [Ber 2], and that therefore there are no examples of non-trivial units of finite order to be found in abelian groups, the result in his paper. Berman had stated a stronger form of this corollary in [Ber 1] and used it to draw this same conclusion; note, however, that the results of [Ber 1] are for the ring of rational integral coefficients while many of those in [Ber 2] are stated for rings of algebraic integers.

He gives an infinite family of units of order 2 for the non-abelian group of order 6 (namely, $y + k(x - x^2)(1 + y)$, where k is a rational integer and $G = \langle x, y : x^3 = y^2 = (xy)^2 = 1\rangle$ (p.36)). At the time, Taussky also worked on units in group rings [Tau 2] and discovered, among other things, various units for this group although she had no occasion to publish them before [Tau 1]. The two units of order 2 which she gives correspond to Higman's units with $k = 1$ and $k = -1$ for a suitable choice of generators. Her method of group matrices, and, in particular, circulants in the case of a cyclic group, reproves Higman's theorem on units for abelian groups in the cyclic case and provides explicit non-trivial units when they exist [New-Tau]. As reported in [Tau 1], Iwasawa constructed many units for the non-abelian group of order 6. His constructions were methodical and yielded Higman's units and many more, for example, $k(x - x^2) + ((1 + k^2) + k^2x - 2k^2x^2)y$. He also noted that his methods gave units for any dihedral group of order at least 6, namely, $\alpha + (1 \pm \alpha)y$ where $\alpha = k(x - x^{-1})$. (My thanks to Professors Taussky-Todd and Iwasawa for their assistance.)

Higman included these examples in the non-abelian group of order 6 to show that units of finite order need not be conjugate to trivial units. This he established by examining their images in an integral representation of the group and by observing that, for odd k, the matrix representing the unit above could not be similar to any of the matrices representing group involutions. It is reported in [Ber 4] that this point was demonstrated in [Ros]; her example is presumably that for the dihedral group of order 8 published in [Ber-Ros] (my thanks to G. Karpilovsky for the details of this reference). According to [Hug-Pea], Zassenhaus also raised this question and it served as one of the motivations for [Hug-Pea]; Hughes and Pearson answer it with a unit which is one of Higman's. Recently much interest has focussed on the question of conjugacy in the larger rational group algebra and of conjugacy of group bases; these are questions that Higman does not mention.

The second half of the section turns to the proof that, for a nilpotent metabelian group, a group of units of finite order is isomorphic to a group of trivial units. His methods are "rather different methods from those used in proving the previous theorem" (p.43). They involve manipulation of the ideals $\lambda_H = I(H)RG$ where H is a normal subgroup of G. Before proving the theorem, Theorem 14, he gathers together some facts about these ideals:

"Lemma 1. If $x_1,\ldots,x_r$, of orders $n_1,\ldots,n_r$ are a basis of the Abelian group G, then

$$\mathbf{x} = \mu_1(x_1 - 1) + \mu_2(x_2 - 1) + \ldots + \mu_r(x_r - 1) \equiv 0\ (\lambda_G^2)\quad (\mu_\alpha \text{ integers})$$

if and only if μ_α is divisible by n_α, $\alpha = 1,\ldots,r$." (p.43).

"Lemma 2. If ϕ is an element of $R(H,C)$, where H is a self-conjugate subgroup of G, then $\phi \equiv 0\ (\lambda_G\lambda_H)$ if and only if $\phi \equiv 0\ (\lambda_H^2)$." (p.44).

"Corollary to Lemmas 1 and 2. Let G be a group, H a self conjugate subgroup of G. Then if g is an element of G

(i) $g \equiv 1 \mod \lambda_G^2$ if and only if g is in G';

(ii) $g \equiv 1 \mod \lambda_G\lambda_H$ if and only if g is in H';

(iii) $g \equiv 1 \mod (\lambda_G\lambda_H, \lambda_G\lambda_H)$ if and only if g is in (G,H)." (p.46).

Part (iii), in which $(\lambda_G\lambda_H, \lambda_H\lambda_G)$ means $I(G)I(H) + I(H)I(G)$ and (G,H) denotes the subgroup generated by the commutators of elements of G with those of H, is included to accommodate the fact "that multiplication of the ideals λ_H is not in general commutative" (p.45).

Lemma 1 is presently understood as asserting an isomorphism between G/G' and $I(G)/I(G)^2$, G' being the commutator subgroup of an arbitrary group G. This appears in [Hop, pp. 49-50], [Fre, p.291] and [Coh], where it is proved in a basis-free manner. (My thanks to M. Oganjuren for the Hopf reference). He also gives the formula

$$"\ x_1^{a_1}x_2^{a_2}\ldots x_r^{a_r} \equiv 1 + a_1(x_1 - 1) + \ldots + a_r(x_r - 1),\ (\lambda_G^2)\ "\quad (p.45);$$

this would establish the isomorphism between G/G' and $I + I(G)/I(G)^2$ [San 3].

Part (ii), that $H' = G \cap 1 + I(G)I(H)$, is proved for free groups in [Fox; Gru 1] where it is attributed to [Schu. 1]. There is a characterisation of H', based on what are now known as Fox derivatives, in [Schu. 1] and in [Bla; Lyn] to whom Fox also refers the result. But the ideal $I(G)I(H)$ does not enter into these articles. (Schumann's treatment of this point is more clearly and concisely expressed in [Schu.2]. My thanks to R. Griess for providing me with [Bla], an undergraduate thesis which translates [Schu. 1] into the language of Fox derivatives.)

This identity is the key step in establishing the isomorphism between H/H' and $\mathbb{Z}GI(H)/I(G)I(H)$. Although Higman does not state this isomorphism, he effectively uses it in the case $H = G'$ in his proof of Theorem 14; in the case $H = G$, this is the previous isomorphism. For G free, the isomorphism appears in [Gru 1] and, for the general case, in [Oba 1; Whitc.]. Both groups may be interpreted as right $\mathbb{Z}G$-modules; that the isomorphism respects this structure is discussed in [Gru 1, 2; Oba 2;

San 1, 2; Gru-Ros]. A consequence of this observation is that $H/[H,G]$ is isomorphic to $\mathbb{Z}GI(H)/(I(G)I(H) + I(H)I(G))$, a fact related to part (iii) of Higman's Corollary.

In view of the subsequent history of the isomorphism problem in the metabelian case, it is instructive to quote the proof of Theorem 14 for the case of rational integral coefficients:

"Theorem 14. Let G be a group whose second derived group consists of the identity alone, and the c-th group of whose upper central series is the whole group G. Then a group of normalised units of finite order in R(G,C) is isomorphic to a subgroup of G.

By Theorem 11, a group of normalised units of finite order in R(G,C) is finite; it therefore suffices to prove theorem 14 in case the quotient field of C is a finite extension of the rational field.

Let E be a unit of finite order in R(G,C). The image of E in the natural homomorphism of R(G,C) on R(G/G',C) is a trivial unit in the latter group ring, since G/G' is Abelian. If furthermore E is normalised, then this image is an element of G/G'. Let g_1 be any element of this residue class modulo G'. We have then

$$E = g_1 + \sum_{i=1}^{r} \phi_i(h_i - 1) \ ;$$

where $h_1,\dots,h_r$, of orders $n_1,\dots,n_r$, are a basis of the Abelian group G'. If the sum of the coefficients of the elements in ϕ_i is ρ_i then $\phi_i \equiv \rho_i \mod \lambda_G$, and therefore

$$E \equiv g_1 + \sum_{i=1}^{r} \rho_i(h_i - 1) \mod \lambda_G\lambda_{G'}. \tag{5}$$

Let us for the sake of clarity first suppose that we are dealing with the ring of rational integers as coefficient ring C, and treat the general case afterwards. Then from equation (5) we have,

$$\begin{aligned} E &\equiv g_1 + \sum_{i=1}^{r} \rho_i (h_i - 1) \mod \lambda_G\lambda_{G'} \\ &\equiv g_1 + \sum_{i=1}^{r} (1 + h_i + \dots + h_i^{\rho_i - 1})(h_i - 1) \mod \lambda_G\lambda_{G'}, \\ &\equiv g_1 + \sum_i g_1 h_1^{\rho_1} \dots h_{i-1}^{\rho_{i-1}} (h_i^{\rho_i} - 1) \mod \lambda_G\lambda_{G'}, \\ &\equiv g_1 h_1^{\rho_1} \dots h_r^{\rho_r} \mod \lambda_G\lambda_{G'}. \end{aligned}$$

That is to say, every normalised unit E of finite order in R(G,C) satisfies a congruence

$$E \equiv g \mod \lambda_G\lambda_{G'}. \tag{6}$$

This congruence determines g uniquely. For by the corollary to Lemmas 1 and 2, $g \equiv g' \mod \lambda_G\lambda_{G'}$ implies that gg'^{-1} is in the derived group of G', - that is to say, is the identity. Moreover, if we have also $E_1 \equiv g_1 \mod \lambda_G\lambda_{G'}$, then

$$EE_1 \equiv gg_1 \mod \lambda_G\lambda_{G'},$$

so that the correspondence $E \to g$ provides a homomorphism of any group of normalised units of finite order on a subgroup of G. To show that it is an isomorphism, we must show that if

$$E \equiv 1 \mod \lambda_G\lambda_{G'}, \tag{7}$$

and E is a unit of finite order, then $E = 1$. By hypothesis G has the upper central series $Z_0 = 1, Z_1, \ldots, Z_c = G$. We proceed by induction on c; as if $c = 1$, G' consists of the identity alone, $\lambda_{G'}$ is the ideal (0), and the assertion is therefore trivial. Assume then that the assertion is true of the factor group G/Z_1. Let $\bar{E}$ be the unit of $R(G/Z_1, C)$ corresponding to E. Then (7) implies

$$\bar{E} \equiv 1 \mod \lambda_{G/Z_1}\lambda_{(G/Z_1)'}$$

and so by hypothesis $\bar{E} = 1$, or $E \equiv 1 \mod \lambda_{Z_1}$. It follows that the sum of the coefficients of elements of Z_1 in E is 1, so that not all of these coefficients are zero. Since Z is the centrum of G, it follows from Theorem 10 that E is trivial, say $E = z$. But we have already shown that (6) determines g uniquely, and therefore by (7), $z = 1$. Thus Theorem 14 is true if C is the ring of rational integers." (pp.46-48).

In the case of a more general coefficient ring C, a basis $w_0 = 1, w_1, \ldots, w_s$ of C over $\mathbb{Z}$ is chosen and the fact that CG is the direct sum of the $\mathbb{Z}Gw_i$ used to allow the proof to proceed along similar lines (pp.49-50).

Jackson, a student of Higman, set out to remove the nilpotency restriction in Theorem 14 in [Jac 1, 2] (his paper and thesis are much the same, the paper having an additional section on nilpotent groups). Jackson follows the lines of the above proof but his concluding steps (reduction argument, use of an integral representation in a special case) are much more involved and are not generally understood.

Recent authors have written on various classes of metabelian groups, often using ideas like those of Jackson, to show that, in the group of normalised units, the subgroup G has a torsion-free normal complement. For a group ring with this property, it is immediate that Higman's strong form of the isomorphism conjecture is correct. The following cases have yielded a positive result: the symmetric group of degree 3 [Den], the alternating group of degree 4 [All-Hob], dihedral groups of odd degree [Miy], groups having a normal abelian subgroup of index 2 [Pas-Smi], groups with a commutator factor group of exponent 2, 3, 4 or 6 [Sek; Cli-Seh-Wei; Rog] or of odd exponent [Cli-Seh-Wei] (this last was reported on at this Tagung).

Higman concludes the section with Corollaries asserting the correctness of the isomorphism conjecture for finite metabelian nilpotent groups. The case of finite metabelian groups was proved in [Whitc.]. All the elements of Whitcomb's proof appear in the above proof of Theorem 14. Whitcomb is able to derive his conclusion using these arguments alone by means of symmetry as he deals only with group bases and not with arbitrary groups of normalised units of finite order (the symmetry argument is spelled out in [Seh 1]). The special case of a nilpotent group of class 2 was resolved in [Pas 1] and in [Sak] for a more general coefficient ring.

6. <u>An Example</u> (pp.52-62)

"In this section we consider the detailed application of the theorems of the last three sections to a particular group ring. We take one of the simplest groups not of prime power order, namely the group G generated by two elements a, b, subject to the relations:-

$$a^p = b^{p-1} = 1; \quad b^{-1}ab = a^r ;$$

where p is an odd prime, and r a primitive number modulo p" (p.52).

The section illustrates certain techniques of Higman in studying representations and provides an early example of an explicit integral representation.

He begins by describing the absolutely irreducible representations of G : $p-1$ linear representations and one of degree $p-1$, which he obtains as a summand of the linear representation corresponding to the doubly transitive permutation representation of G on the cosets of the subgroup generated by b. These provide a decomposition of KG into the direct sum of two algebras : A_1, isomorphic to the group ring KG^* where G^* is the factor group of G with respect to the (normal) subgroup generated by a; and A_2, "the total matrix algebra of index $p-1$" (p.53). K and its ring of integers R are not explicitly defined but K can be taken to be any field of algebraic numbers containing $(p-1)$-th roots of unity. A_1 is obtained by uniting "the components corresponding to representations of degree 1, into a single component" (p.53). This is a technique he had introduced in section 3 and used there in the example of non-isomorphic groups having isomorphic rational group algebras. It allows the decomposition to be effected over smaller fields such as the rational field.

Most of the section is taken up with locating the integral group ring RG in this decomposition. The "first component" (p.54) of an element of RG (that is, its component in A_1) is in RG^*; any member of RG^* is a first component of such an element. Thus, interest centres on the second components. Here the problem is finding a basis for the representation, an integral one, in which the representing matrices are in "the most convenient form" (p.54).

Before giving an explicit basis, he analyses the representation theoretically:

"an element in $R(G,k)$ with first component zero, and second component a matrix whose elements are integers divisible by p is in $R(G,C)$, provided only that our representation has been chosen so that the matrices representing elements of G have integral elements. That is to say, what we have to consider is the form the representation takes modulo p. Now over the Galois field of order p, the group algebra of G ceases to be semi-simple. We have in fact $(a-1)^p \equiv a^p-1 = 0 \mod p$; and since, if g is any element in G, $g(a-1) = (a^s-1)g = (a-1).(a^{s-1}+...+a+1)g$, the ideal $(a-1)$

is nilpotent. As a matter of fact it forms the radical of the group algebra, and therefore the quotient algebra with respect to the radical is isomorphic to the group algebra of a cyclic group of order p-1. It is, in particular, commutative, and any representation of G in the Galois field is therefore equivalent to a representation by matrices whose elements above the main diagonal are zero[1].

(1) Cf. B.L. van der Waerden, Moderne Algebra, vol. II, ch. XVII, § 121." (pp.54-55). The radical of such a group algebra was not known in general at the time; it was determined in the case of a normal Sylow p-subgroup in [Bru; Lom].

His first approximation to a convenient basis is that obtained from the permutation representation; its elements are denoted $x_1, x_2, \ldots, x_{p-1}$ and correspond to the cosets of a^i, $i = 1, 2, \ldots, p-1$. By realising each pE_{ij} as representing an element of RG, he concludes that all elements of KG having zero first component and second component of the form pA, A an integral matrix, are in RG.

Next he changes basis

$$" \; y_\alpha = \sum_{\beta=1}^{p-1} \binom{p-\beta}{a} x_\beta \,, \quad \alpha = 1, \ldots, p-1 \; " \qquad (p.56).$$

He notes that, by properties of the binomial coefficients, this is effected by "an integral unimodular matrix" (p.57). Using this basis, he finds that, modulo p, the second components of a and b are given respectively by the matrices

$$" \begin{Vmatrix} 1 & 0 & & & 0 \\ 1 & 1 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ 0 & & & 1 & 1 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} r^* & 0 & & & 0 \\ * & r^{*2} & & & \\ & & \ddots & & \\ & & & \ddots & 0 \\ * & & & * & r^{*\,p-1} \end{Vmatrix} " \qquad (p.58).$$

(here r^* is the multiplicative inverse of r modulo p).

Further analysis then leads to the conclusion:

"<u>An element of R(G,k) is in R(G,C) if and only if its component in A_1 is $\phi(b^*)$ in $R(G^*,C)$, and its component in A_2 is a matrix with integer elements, those above the main diagonal being congruent to zero, those in it to $\phi(r^*)$, $\phi(r^{*2})$, ..., $\phi(r^{*\,p-1})$ respectively modulo p</u>" (pp.59-60).

Here b^* is the image of b in the factor group G^* and $\phi(b^*)$ is the expression for an element of A_1 as a function ϕ of b^*, the function ϕ then defining the diagonal entries of the component in A_2 as described. It is easy to see (and Higman observes on p.59) that the projection of ℤG with respect to the primitive central idempotent corresponding to A_2 is the ring of matrices

$$\begin{pmatrix} \mathbb{Z} & p\mathbb{Z} & & & p\mathbb{Z} \\ & \mathbb{Z} & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot \; p\mathbb{Z} \\ \mathbb{Z} & & & & \mathbb{Z} \end{pmatrix} ,$$

an important hereditary order.

The group G is metacyclic; the integral representation of the triangular form given here is a special case of that in [Gal-Rei-Ull]. For $p = 3$, Higman has already provided such a representation in section 5, a and b being sent respectively to

$$\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} .$$

Up to signs and choice of generators, these same matrices are used in [Al-S] and by Roggenkamp in lectures on [Rog]; on the other hand, the representation in [Hug-Pea] is not triangular.

From the description of RG in the decomposed KG, conclusions are drawn about the units of RG:

"We have then obviously:-

<u>The group U of units in $R(G,C)$ has a self-conjugate subgroup U_o consisting of all units whose component in A_1 is 1. U_o is isomorphic to the group of all matrices with determinant unity and elements in C, those above the main diagonal being congruent to zero and those in it to unity modulo p. The factor group U/U_o is isomorphic to the unit group of $R(G^*,C)$</u>." (p.60).

Next he describes normalised units and explicitly verifies that a group of normalised units of finite order is isomorphic to a subgroup of G. The section ends with a calculation showing that the same is true for the integral group ring of any subgroup of G which contains a and in particular for "the dihedral group of order $2p$" (p.62). In [Hug-Pea] the units are described in the case $p = 3$ with rational integer coefficients. In [Col], there is some discussion of the isomorphism problem for a wider class of metacyclic groups than that treated here.

7. Group Rings of Groups without Elements of Finite Order (pp.63-78)

The question of which torsion free groups have trivial Whitehead group is one which draws the active interest of many mathematicians not primarily algebraists. Having trivial Whitehead group implies having only trivial units in the integral group ring. In his paper, Higman had identified a large class of torsion-free groups with the latter property, those which he called "indicable throughout". This term means that every non-trivial subgroup maps homomorphically onto the infinite cyclic group.

The term is not used in the thesis, its place being taken by Condition II which is the same except for the refinement that only finitely generated subgroups need be surveyed (as this is sufficient for the application). In the paper, he showed the class to be closed under extensions and free products and to include such important groups as free and free abelian groups. He also recognised that the nature of the ring of coefficients was unimportant as long as it had no zero divisors and that, in this case, RG had no zero divisors.

The class of groups satisfying Condition II behaves the same way. It is contained in a larger class introduced in the thesis, that described by Condition I (p.65), a class again with the same formal properties and the same consequences for the group ring. These formal properties for the two conditions (closure with respect to extensions, direct products and free products) are the assertions of the final numbered theorems of the thesis, Theorems 17, 18 and 19 respectively. Theorems 15 and 16 state that RG has no zero divisors and no non-trivial units if G satisfies Conditions I and II respectively.

Condition I, given in an equivalent form below, is a complicated one involving what he calls the isolated product set of subsets of the group. He finds it "very unwieldy in application" (p.78) and could not determine whether it held for groups which did not satisfy Condition II (an example of such a group is obtained from $G = \langle x, y : x^3 = y^2 \rangle$ as (p.77) the free product of two copies of G with the subgroup generated by yx^{-1} and $(yx^{-1})^6x^{-3}$ amalgamated; this is a finitely generated torsion-free perfect group, and this suffices).

The relevance of Condition I to the group ring is to be found in the

"Lemma. If in any two non-vacuous finite subsets X, Y of G, we can find a pair x, y of elements having an isolated product, then R(G,K) has no zero divisors. If furthermore whenever X or Y has more than one element, we can choose two such pairs, then R(G,K) has only trivial units." (p.64).

Here "isolated product" means that the element xy can be expressed as a product of an element from X and one from Y in no other way. At present the term "unique product" is used, and a group satisfying the first hypothesis is called a unique product (or u.p.) group [Rud-Sch]. The second hypothesis defines what is now called a two unique products (or t.u.p.) group; it appears in [Kem] and is applied to group rings in [Sch-Wei]. Higman states (p.65) that the second hypothesis is equivalent to Condition I. He also writes "Whether or not the hypothesis of the second half of the lemma follows from the hypothesis of the first half, I do not know" (p.65). It has recently been established in [Str] that the two are equivalent.

Much of the section is taken up in providing additional illustrations of groups satisfying Condition II. He gives three types. The first (p.72) is a group generated by elements $a_1,\ldots,a_n$ subject to relations, for $s = r + 1,\ldots,n$ (r fixed), of the form: $a_s^{\lambda_s}$ is a non-trivial word in $a_1,\ldots,a_{s-1}$. He cites as "the simplest case of the above type (the group $\langle x, y : x^m = y^n \rangle$ which) if m is prime to n (is) the group of the torus knot" (p.74).

Secondly he takes "another class of knots, - those formed by the process of doubling a simple circuit - their groups are generated by generators a, b, subject to the relation

$$a^{2n+1} = b\, a^n\, b^{-2}\, a^n\, b \text{" (p.74).}$$

"As a third example, consider the group G generated by $a_1, a_2, \ldots$ ad inf., subject to

$$a_{i-1} = a_i\, a_{i+1}\, a_i^{-1}\, a_{i-1}^{-1}\,, \quad i = 2, 3, \ldots \text{ ad inf." (p.76).}$$

He shows that G is perfect, locally free and imbeddable in a finitely presented group which also satisfies Condition II.

Towards the end of the chapter comes a conjecture which has proven to be an important focus in the development of the theory of group rings:

"It is obvious that neither condition I nor condition II can hold in a group having elements of finite order. It is a plausible hypothesis, however, that if a group has no elements of finite order then its group ring is without zero divisors and has only trivial units." (p.77).

Acknowledgement

My thanks to Bridget Iveson for her excellent typing.

Bibliography

Citations in the text give the first three (or more) letters of the name(s) of the author(s). Multiple articles are numbered according to the ordering in this bibliography. References are supplied here to reviews from Mathematical Reviews and occasionally from Zentralblatt für Mathematik und ihre Grenzgebiete and Jahrbuch über die Fortschritte der Mathematik.

Albert, A. A., Structure of Algebras, Amer. Math. Soc., New York, 1939. Zbl.23.199.

Allen, P. J.; Hobby, C., A characterization of the units in $Z[A_4]$, J. Algebra, to appear.

Al-Sohebani, A.A., On the units of certain group rings, Ph.D. thesis, Univ. of Birmingham, 1978.

Berman, S. D., On certain properties of integral group rings. (Russian), Dokl. Akad. Nauk SSSR (N. S.)91(1953), 7-9. MR 15-99.

Berman, S. D., On the equation $x^m = 1$ in an integral group ring. (Russian), Ukrain. Mat. Ž. 7(1955), 253-261. MR 17-1048.

Berman, S. D., On certain properties of group rings over the field of rational numbers. (Russian), Užgorod. Gos. Univ. Naučn. Zap. Him. Fiz. Mat. 12(1955), 88-110. MR 20 #3920.

Berman, S. D., Representations of finite groups. (Russian), Algebra 1964, p.83-122, Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1966 = Itogi Nauki - Seriya Matematika. Alg. Top. Geom. 3. MR 25 #2973.

Berman, S. D.; Rossa, A. R., Integral group rings of finite and periodic groups. (Russian), Algebra and Math. Logic : Studies in Algebra. (Russian), p.44-53, Izdat. Kiev. Univ., Kiev, 1966. MR 35 #265.

Blanchfield, R. C., Application of free differential calculus to the theory of groups, Senior thesis, Princeton Univ., 1949.

Bourbaki, N., Algèbra Linéaire, Hermann, Paris, 1947. MR 9-406.

Bovdi, A. A., Group Rings. (Textbook). (Russian), University publishers, Užgorod, 1974. Zbl.339.16004.

Brauer, R., Representations of finite groups, Lectures on Modern Mathematics, Vol. I, pp.133-175, Wiley, New York, 1963. MR 31 #2314. Also in: Richard Brauer : Collected Papers, Vol. I-III, The MIT Press, Cambridge MA, 1980.

Brummund, H., Über Gruppenringe mit einem Koeffizientenkörper der Characteristik p, Dissertation, Münster 1939, Emsdetten, Heinrich und J. Lechte, Münster, 1939. Zbl.22.119. F.d.M.65_{II},1136.

Cliff, G. H; Sehgal, S. K.; Weiss, A. R., Units of integral group rings of metabelian groups, to appear.

Cohn, J. A.; Livingstone, D. On groups of the order p^3, Canad. J. Math. 15(1963), 622-624. MR 27 #3700.

Cohn, J. A.; Livingstone, D., On the structure of group algebras. I., Canad. J. Math. 17(1965), 583-593. MR 31 #3514.

Cohn, P. M., Generalization of a theorem of Magnus, Proc. London Math. Soc. (3)2(1952), 297-310. MR 14-532.

Coleman, D. B., Finite groups with isomorphic group algebras, Trans. Amer. Math. Soc. 105(1962), 1-8. MR 26 #191.

Curtis, C. W.; Reiner, I., Representation theory of finite groups and associative algebras, Interscience - Wiley, New York, 1962. MR 26 #2519.

Dennis, R. K., The structure of the unit group of group rings, Ring theory, II, pp.103-130, Dekker, New York, 1977. MR 56 #3047.

Eichler, M. Über die Einheiten der Divisionsalgebren, Math. Ann. 114(1937), 635-654. Zbl.17.244.

Farkas, D. R., Group rings: an annotated questionnaire, Comm. Algebra 8 (1980), 585-602.

Fox, R. H., Free differential calculus. I. Derivation in the free group ring, Ann. of Math. (2)57(1953), 547-560. MR 14-843.

Franz, W., Überdeckungen topologischer Komplexe mit hyperkomplexen Systemen, J. Reine Angew. Math. 173(1935), 174-184. Zbl.12.127.

Franz, W., Über die Torsion einer Überdeckung, J. Reine Angew. Math. 173(1935), 245-254. Zbl.12.127.

Franz, W., (review of Whitehead, J. H. C., Simplicial spaces, nuclei and m-groups), Zbl. Math. 22(1940), 407-408.

Freudenthal, H., Der Einfluss der Fundamentalgruppe auf die Bettischen Gruppen, Ann. of Math. (2)47(1946), 274-316. MR 8-166.

Galovich, S; Reiner, I.; Ullom, S., Class groups for integral representations of metacyclic groups, Mathematika 19(1972), 105-111. MT 48 #4087.

Gruenberg, K. W., Resolutions by relations, J. London Math. Soc. 35(1960), 481-494. MR 23 #A 3165.

Gruenberg, K. W., Some cohomological topics in group theory, Queen Mary College, London, 1967. MR 41 #3608.

Gruenberg, K. W.; Roseblade, J. E., The augmentation terminals of certain locally finite groups, Canad. J. Math. 24(1972), 221-238. MR 45 #3538.

Higman, G., The units of group-rings, Proc. London Math. Soc. (2)46(1940), 231-248. MR 2-5. Zbl.25.243. F.d.M. 66_{I}, 104.

Higman, G. Units in group rings, D. Phil. thesis, Oxford Univ., 1940.

Higman, G., (Review of Passman, Isomorphic groups and group rings), Math. Reviews 33(1967), 244.

Hopf, H., Über die Bettischen Gruppen, die zu einer beliebigen Gruppe gehoren, Comment. Math. Helv. 17(1945), 39-79. MR 6-279.

Hughes, I.; Pearson, K. R., The group of units of the integral group ring ZS_3, Canad. Math. Bull. 15(1972), 529-534. MR 48 #4089.

Jackson, D. A., On a problem in the theory of integral group rings, D. Phil. thesis, Oxford Univ., 1967/1968.

Jackson, D. A., The groups of units of the integral group rings of finite metabelian and finite nilpotent groups, Quart. J. Math. Oxford Ser. (2)20(1969), 319-331. MR 40 #2766.

Kemperman, J. H. B., On complexes in a semigroup, Nederl. Akad. Wetensch. Proc. Ser.A. 59 = Indag. Math. 18(1956), 247-254. MR 18-14. MR 19-13.

Lombardo-Radice, L., Intorno alle algebre legate ai gruppi di ordine finito. Nota 2ª, Rend. Sem. Mat. Roma (4)3(1939), 239-256. MR 1-258. Zbl.23.103.

Lyndon, R. C., Cohomology theory of groups with a single defining relation, Ann. of Math. (2) 52 (1950), 650-665. MR 13-819.

Miyata, T., On the units of the integral group ring of a dihedral group, J. Math. Soc. Japan, to appear.

Newman, M.; Taussky, O., On a generalization of the normal basis in abelian algebraic number fields, Comm. Pure Appl. Math. 9(1956), 85-91. MR 17-829.

Obayashi, T., Solvable groups with isomorphic group algebras, J. Math. Soc. Japan 18(1966), 394-397. MR 33 #5750.

Obayashi, T., Integral group rings of finite groups, Osaka J. Math. 7(1970), 253-266. MR 43 #2122.

Passman, D. S., Isomorphic groups and group rings, Pacific J. Math. 15(1965), 561-583. MR 33 #1381.

Passman, D. S., The Algebraic Structure of Group Rings, Wiley - Interscience, New York, 1977. Zbl.368.16003.

Passman, D. S.; Smith, P. F., Units in integral group rings, to appear. Abstract in: Abstracts Amer. Math. Soc. 1(1980), 280.

Perlis, S.; Walker, G., Abelian group algebras of finite order, Trans. Amer. Math. Soc. 68(1950), 420-426. MR 11-638.

Polcino Milies, F. C., Anéis de Grupos, Instituto de Matemâtica e Estatistica, Univ. de São Paulo, São Paulo, 1976. Zbl.407.16008.

Reidemeister, K., Überdeckungen von Komplexen, J. Reine Angew. Math. 173(1935), 164-173. Zbl.12.126.

Rham, G. de, Sur les complexes avec automorphismes, Comment. Math. Helv. 12(1940), 191-211. Zbl.22.408.

Roggenkamp, K. W., to appear.

Rossa, A. R., On integral group rings of finite groups. (Russian), Tezicy Dokl. Soobšč. Užgorod. Gos. Univ. Ser. Fiz.-Mat. Nauk 1964, 4-5.

Rudin, W.; Schneider, H., Idempotents in group rings, Duke Math. J. 31(1964), 585-602. MR 29 #5119.

Saksonov, A. I., Group rings of finite groups. I, Publ. Math. Debrecen 18(1971), 187-209. MR 46 #5425.

Sandling, R., The modular group rings of p-groups, Ph.D. thesis, Univ. of Chicago, 1969.

Sandling, R., Note on the integral group ring problem, Math. Z. 124(1972), 255-258. MR 45 #3594.

Sandling, R., Group rings of circle and unit groups, Math. Z. 140(1974), 195-202. MR 52 #3217.

Schilling, O., Einheitentheorie in rationalen hyperkomplexen Systemen, J. Reine Angew. Math. 175(1936), 246-251. Zbl.15.2.

Schneider, H.; Weissglass, J., Group rings, semigroup rings and their radicals, J. Algebra 5(1967), 1-15. MR 35 #4317.

Schumann, H.-G., Über Moduln und Gruppenbilder, Math. Ann. 114(1937), 385-413. Zbl.16.294.

Schumann, H.-G., Zum Beweis des Hauptidealsatzes, Abh. Math. Sem. Hansische Univ. 12(1937), 42-47. Zbl.16.103.

Sehgal, S. K., On the isomorphism of integral group rings. II, Canad. J. Math. 21(1969), 1182-1188. MR 42 #392.

Sehgal, S. K., Topics in Group Rings, Marcel Dekker, New York, 1978. Zbl.411.16004.

Sekiguchi, K., Units in integral group rings, Proceedings of the 12th Symposium on Ring Theory (Hokkaido Univ., Sapporo, 1979), pp. 39-50, Okayama Univ., Okayama, 1980.

Speiser, A., Die Theorie der Gruppen von endlicher Ordnung. (Dritte Auflage?), Springer, Berlin, (1937?). Zbl.17.153.

Strojnowski, A., A note on u.p. groups, Comm. Algebra 8(1980), 231-234.

Takahashi, S., Some properties of the group ring over rational integers of a finite group, Notices Amer. Math. Soc 12(1965), 463.

Taussky, O., Matrices of rational integers, Bull. Amer. Math. Soc. 66(1960), 327-345. MR 22 #10994.

Taussky-Todd, O., Olga Taussky-Todd, Number Theory and Algebra, pp. xxxiv - xlvi, Academic Press, New York, 1977. MR 58 #10227.

Waerden, B. L. van der, Moderne Algebra II, Springer, Berlin, 1931. Zbl.2.8.

Whitcomb, A., The group ring problem, Ph.D. thesis, Univ. of Chicago, 1968.

Whitehead, J. H. C., Simplicial spaces, nuclei and m-groups, Proc. London. Math. Soc. (2)45(1939), 243-327. Zbl.22.407. Also in: The mathematical works of J. H. C. Whitehead, Vol. II. Complexes and Manifolds, pp. 99-183, Pergamon Press, Oxford, 1962. MR 30 #4667b.

Zalesskiĭ, A. E.; Mihalev, A. V., Group rings. (Russian), Current problems in mathematics, Vol. 2 (Russian), pp. 5-118, Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1973. MR 54 #2723.