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SYSTEMS OF INTEGRAL EQUATIONS ON A HALF LINE WITH KERNELS DEPENDING ON THE DIFFERENCE OF ARGUMENTS

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CONTENTS

Introduction

1. The present paper is devoted to an exposition of the foundations of th theory of systems of integral equations of the following type:

$$\chi_p(t) - \sum_{q=1}^{n}\int_0^\infty k_{pq}(t-s)\,\chi_q(s)\,ds = f_p(t) \quad (0 \leqslant t < \infty;\ p = 1, 2, \ldots, n), \qquad ((
$$

where $k_{pq}(t)$ and $f_p(t)$ $(p, q = 1, 2, \cdots, n)$ are given functions, and $\chi_p(t)$ $(p = 1, 2, \cdots, n)$ are unknown. Concerning the functions $k_{pq}(t)$ $(p, q = 1, 2, \cdot\cdot$ it is supposed throughout in the following that they are of class $L_1(-\infty, \infty)$. cerning the functions $f_p(t)$ and $\chi_p(t)$ $(p = 1, 2, \cdots, n)$, the first mentioned a given, and the second are sought, in one of the Banach spaces E^+ (see §2), cluding in particular the Banach space $L_1(0, \infty)$ and its adjoint space—the s of bounded measurable functions $M(0, \infty)$.

Using vector-matrix symbolism, the system of equations (0.1) may be written more briefly in the form

$$\chi(t)-\int_0^\infty k(t-s)\chi(s)\,ds=f(t)\qquad (0\leqslant t<\infty). \tag{A}$$

With the sole exception of §13, during the course of the whole paper it will be supposed that the following condition holds:

$$\det(I-\mathscr{K}(\lambda))\neq 0\qquad(-\infty<\lambda<\infty), \tag{0.2}$$

where

$$\mathscr{K}(\lambda)=\int_{-\infty}^{\infty}k(t)\,e^{i\lambda t}\,dt. \tag{0.3}$$

Under ordinary circumstances, by using Fourier integral theory, the theory of the integral equation

$$\chi(t)-\int_{-\infty}^{\infty}k(t-s)\chi(s)\,ds=f(t)\qquad(-\infty<t<\infty), \tag{0.4}$$

which differs from equation (A) in that both limits of integration are infinite, so that the equation is considered on the whole, rather than half, of the real axis, does not present any difficulty.

In fact, if, for example, one supposes that $f_p(t)\in L_1(-\infty,\infty)$ $(p=1,\cdots,n)$ and that the functions $\chi_p(t)$ are sought in the same class $L_1(-\infty,\infty)$, then the Fourier transform of both sides of equation (0.4) may be taken. Taking it, one obtains

$$(I-\mathscr{K}(\lambda))X(\lambda)=\mathscr{F}(\lambda), \tag{0.5}$$

where

$$X(\lambda)=\int_{-\infty}^{\infty}\chi(t)\,e^{i\lambda t}\,dt,\qquad \mathscr{F}(\lambda)=\int_{-\infty}^{\infty}f(t)\,e^{i\lambda t}\,dt\qquad(-\infty<\lambda<\infty).$$

From (0.5) it follows that

$$X(\lambda)=(I-\mathscr{K}(\lambda))^{-1}\mathscr{F}(\lambda). \tag{0.6}$$

On the other hand, if condition (0.2) holds, from the known theorem of Wiener (see §3, Theorem W) one may assert the existence of a matrix $l(t)=\|l_{pq}(t)\|_1^n$ with elements $l_{pq}(t)\in L_1(-\infty,\infty)$ such that

$$(I-\mathscr{K}(\lambda))^{-1}=I+\int_{-\infty}^{\infty}l(t)\,e^{i\lambda t}\,dt.$$

This relation permits rewriting (0.6) thus

$$\chi(t)=f(t)+\int_{-\infty}^{\infty}l(t-s)f(s)\,ds. \tag{0.7}$$

Consequently, if (0.2) holds, then for arbitrary right-hand side $f(t)$ with coordinates $f_p(t)\in L_1(-\infty,\infty)$, equation (0.4) possesses one and only one solution $\chi(t)$ with

coordinates $\chi_p(t)\in L_1(-\infty,\infty)$, which is given explicitly by Formula (0.7).

As soon as the matrix-function $f(t)$ has been obtained, the proposition ju
formulated can be easily extended to take care of the solution of equation (0.
other function spaces, in particular the space $M(-\infty,\infty)$ or to its subspace o
bounded continuous functions, and others.

In spite of the similarity of appearance of equations (A) and (0.4), the th
of equation (A) is much more difficult—in particular, condition (0.2) does not
antee the solvability of equation (A) in this or the other space.

2. The last mentioned circumstance occurs already in the case of the sc
equation (A).

As was shown in [1], equation (A) possesses one and only one solution χ
for any given $f\in E^+$ if and only if

$$1-\mathscr{K}(\lambda)\neq 0\qquad(-\infty<\lambda<\infty)\qquad ($$

and the number

$$\varkappa=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d_\lambda\arg(1-\mathscr{K}(\lambda))$$

is zero.

It was shown further in [1] that when (0.8) holds, and $\kappa>0$, then the no
homogeneous equation (A) may always be solved, but not uniquely, because i
this case the homogeneous equation

$$\varphi(t)-\int_0^\infty k(t-s)\varphi(s)\,ds=0\qquad(0\leqslant t<\infty)$$

possesses, in all spaces $E^+(L_1(0,\infty), M(0,\infty)$, etc.), the same κ–dimensio
manifold of solutions, and all these solutions are absolutely continuous and
proach zero as $t\to\infty$.

If the condition (0.8) holds and the number $\kappa<0$, then for each $f(t)\in E$
the equation (A) will possess a solution if and only if

$$\int_0^\infty f(t)\psi(t)\,dt=0,$$

where $\psi(t)$ is an arbitrary function of the κ–dimensional set of all solutions
the transposed homogeneous equation

$$\psi(t)-\int_0^\infty k(s-t)\psi(s)\,ds=0.$$

In proving these results, an essential rôle is played by the representatio
the function $(1-\mathscr{K}(\lambda))^{-1}$, when condition (0.8) holds, in the following form

$$(1-\mathscr{K}(\lambda))^{-1}=\mathfrak{G}_+(\lambda)\,\mathfrak{G}_-(\lambda)\qquad(-\infty<\lambda<\infty),\qquad ($$

where the factors $\mathfrak{G}_\pm(\lambda)$ are of the form

$$\mathfrak{G}_+(\lambda)=1+\int_0^\infty\gamma(t)\,e^{i\lambda t}\,dt,\qquad \gamma(t)\in L_1(0,\infty)\qquad(\operatorname{Im}\lambda\geqslant 0),$$

$$\mathfrak{G}_-(\lambda) = 1 + \int_0^\infty \gamma_\tau(t) e^{-i\lambda t}\, dt, \qquad \gamma_\tau(t) \in L_1(0, \infty) \qquad (\operatorname{Im}\lambda \geq 0),$$

and at least one of these functions is not identically zero in its half space of definition.

It may be shown that in all cases where (A) is solvable, one of its solutions is given by

$$\chi(t) = f(t) + \int_0^\infty \gamma(t, s) f(s)\, ds \qquad (0 \leq t < \infty), \tag{0.10}$$

where

$$\gamma(t, s) = \gamma(t-s) + \gamma_\tau(s-t) + \int_0^\infty \gamma(t-r)\gamma_\tau(s-r)\, dr,$$

and where $\gamma(t) = \gamma_\tau(t) = 0$ for $t < 0$.

3. In the case of the matrix equation, $n > 1$, the rôle of the representation (0.9) is played by the representation

$$(I - \mathscr{K}(\lambda))^{-1} = \mathfrak{N}_+(\lambda) \begin{pmatrix} \zeta^{\varkappa_1} & 0 & \dots & 0 \\ 0 & \zeta^{\varkappa_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \zeta^{\varkappa_n} \end{pmatrix} \mathfrak{N}_-(\lambda); \qquad \zeta = \frac{\lambda - i}{\lambda + i} \qquad (-\infty < \lambda < \infty) \tag{0.11}$$

which will be referred in this paper as the "left standard factorization" of the matrix $\mathfrak{M}(\lambda) = (I - \mathcal{K}(\lambda))^{-1}$, where $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ are certain whole numbers, and $\mathfrak{N}_\pm(\lambda)$ are certain matrix functions, holomorphic in the interior of, and continuous even on the boundary of, corresponding half planes $\Pi_\pm$,[1] and having non-zero determinants.

In this representation the numbers $\kappa_1, \kappa_2, \cdots, \kappa_n$ are uniquely determined.

In the scalar case ($n = 1$) the possibility of the factorization (0.9) follows rather easily from the known theorems of Wiener and Wiener-Lévy (see [1]) about functions with absolutely convergent Fourier integrals.

The authors do not concern themselves with the proof of the existence of the left standard factorization by means of harmonic analysis, under the general hypotheses (0.2), (0.3). Here this will be avoided; use will only be made of several facts from the theory of integral equations which are related to the factorization (0.11), which may be proved without appeal to the general theory of linear operators in Banach spaces, in particular of the property of normal extension of equation (A), and specifically of the fact that the homogeneous equation (B) possesses, in all spaces $E^+_{(n\times 1)}$, the same set of solutions, constituting a linear manifold of finite dimensionality α.

1) The notation $\Pi_\pm$ refers to the closed half planes $\operatorname{Im}\lambda \geq 0$ and $\operatorname{Im}\lambda \leq 0$, together with the point.

Among other things, it is shown that the number α coincides with the sun positive indices κ_j, while the sum of the absolute values of all negative κ_j i dimension of the linear manifold of all solutions $\psi(t)$ of the transpose equati

$$\psi(t) - \int_0^\infty k'(t-s)\psi(s)\, ds = 0 \qquad (0 \leq t < \infty)$$

($k'(t)$ being the transpose matrix of $k(t)$).

The factorization (0.11) permits the extension to the general case of $n >$ of formula (0.10), and also leads to other results.

It is emphasized that the fundamental theorem concerning the factorizatio (0.11), which is used (see §7) to establish the fundamental propositions concerning the integral equations (A) and (B), plays the same essential rôle in la considerations.

4. The carrying out of the factorization (0.11) is based on the solution o homogeneous Hilbert problem for the matrix function $\mathfrak{M}(\lambda) = (I - \mathcal{K}(\lambda))^{-1}$ (see §6). After the fundamental papers of Hilbert [2], [3] and Plemelj [4], and thei important extensions in the works of N. I. Mushelišvili and N. P. Vekua (see [and [6]), the existential part of the solution of this problem appeared to have reached a definitive aspect.

However, these considerations turned out to be insufficient for our preser purposes, because in all of them it is always supposed that the elements of th fundamental matrix are functions which satisfy a Hölder condition. In view of it was necessary for us to establish anew the fundamental facts concerning th homogeneous Hilbert problem, employing new analytic and abstract functional means.

Among these, many constructions in §6 are analogous to the correspondin constructions used by the authors quoted, and this fact simplified our task a great deal.

The "partial indices" κ_j ($j = 1, 2, \cdots, n$) play a fundamental rôle in the theory of equation (A); in the homogeneous Hilbert problem they were first intı duced by N. I. Mushelišvili and N. P. Vekua [7].

The theory of the integral equation (A) (see §9) reveals a new meaning oı partial indices.

To each positive index[1] κ_j there corresponds a set $\phi_1^{(j)}(t), \phi_2^{(j)}(t), \cdots$ $\cdots, \phi_{\kappa_j}^{(j)}(t)$ of solutions of equation (B) in $L_1(0, \infty)$, all absolutely continuo and approaching zero as $t \to \infty$, and such that

1) For reasons concerning certain definitions (see §7) in this paper they are call "left indices" (or "left exponents").

$$\frac{d\varphi_k^{(j)}(t)}{dt} = \varphi_{k+1}^{(j)}(t), \quad \varphi_k(0) = 0 \quad (k = 1, 2, \ldots, \varkappa_j - 1) \text{ and } \varphi_{\varkappa_j}(0) \neq 0.$$

These sets, called "D-chains", form a basis for the solutions of equation (B) in all spaces E^+.

Analogously, to the negative partial indices κ_j there correspond D-chains of length $|\kappa_j|$ of solutions of the transpose equation (B′).

In the paper there occur new results concerning the partial indices which are due to the young mathematicians Yu. V. Šmul′yan [8], [9], and G. N. Čebutarev [10]. These results can be immediately extended to our Hilbert problem. Further developments are to be found in sections 8 and 11.

According to this theorem, the indices κ_j $(j = 1, 2, \cdots, n)$ are not altered by an arbitrary "sufficiently small" perturbation of the kernel matrix $k(t)$ (or what is the same, of the matrix $\mathfrak{M}(\lambda)$) only in case they have the following values

$$\varkappa_1 = \varkappa_2 = \ldots = \varkappa_r = q + 1, \qquad \varkappa_{r+1} = \ldots = \varkappa_n = q,$$

where the integers q and r are defined by the relation

$$\varkappa = qn + r \qquad (0 \leqslant r < n),$$

where

$$\varkappa = \frac{1}{2\pi} \int_{-\infty}^{\infty} d_\lambda \arg \det \mathfrak{M}(\lambda).$$

The method of proof of this theorem may be applied in the general case of the Hilbert problem for one or for several closed contours, in the usual formulation.

It is remarked that in recent times the problem of the factorization of matrix functions $\mathfrak{M}(\lambda)$ $(-\infty < \lambda < \infty)$ of the type considered by us has played an important role in inverse problems of the quantum theory of dispersion (see [12] and [13]).

It should also be noticed that the results of the first thirteen sections may be readily extended so as to apply also to the discrete analogue of equation (A) (compare the infinite system of linear equations (14.1)) and to the related problem of factorization for the homogeneous Hilbert problem on the unit circle, involving matrix functions whose elements are expandable in uniformly convergent Fourier series (for these matters see also [14] and [1]).

5. In this paper we put aside the problem of determining all the cases in which equation (A) may be explicitly solved. However, one such case is considered, namely that in which $k(t)$ is a triangular matrix function. In this case it is shown in section 11 how the left standard factorization (0.11) may be actually carried out, which permits the determination of all solutions of the homogeneous equation (B), and also of the simple and the generalized resolvents of equation (A). The explicit solution of equation (A) is also achieved when $\mathcal{K}(\lambda)$ is a rational function matrix, because then the standard factorization of $(I - \mathcal{K}(\lambda))^{-1}$ may be ac-

complished by means of simple algebraic operations [6].

However, this does not cover all the cases of equation (A) occurring in cations which admit explicit solutions. To verify this it is only necessary to the papers of V. V. Sobolev [15] and S. Chandrasekhar [16] who give a n of cases in which the integro-differential equations of radiation energy tran may be effectively solved. On the other hand, these equations are equivale certain systems of integral equations of type (0.1), when the so called disp indicatrix is a polynomial in the cosine of the dispersion angle, which is a in the papers of the authors mentioned.

6. The central rôle in the theory of the integral equations (A) and (B) played by the "factorization idea". This idea first made its appearance in paper of Wiener–Hopf [17] on the solution of the homogeneous scalar equati when $k(t)$ is an even function satisfying

$$k(t) = O(e^{-c|t|}) \quad \text{or} \quad e^{c|t|}k(t) \in L_2(-\infty, \infty), (c > 0).$$

Further developments are to be found in the papers of Reissner [18], V Fok [19] and I. M. Rapoport [20]. This last mentioned author first called at to the relation between the theory of the scalar equation (A) and the homog and non-homogeneous Hilbert problems for a straight line. The discovery o a relation, together with the use of the known results of F. D. Gahov [21] f non-homogeneous Hilbert problem, enabled I. M. Rapoport to remove the rest of type (0.12) and to obtain general propositions about equation (A) without suming the function $k(t)$ to be even.

However, the complete results (some of which have been published alr for the scalar equation (A), considered in various spaces E^+ under the sin hypothesis that the function $k(t) \in L_1(-\infty, \infty)$ and satisfies (0.8), were o by one the authors in the paper [1], which is of particular importance for th sent paper.

It is remarked that certain of the basic propositions of the theory expo here were presented by the authors to the Third All-Soviet Mathematical Co [22].

§1. Auxiliary considerations from the general theory of linear operato the following we shall have occasion to employ some relatively new tools c tional analysis, a full discussion of which may be found in [11]. Here the r will be presented without proofs.

All operators mentioned in the present section will be supposed to (wi further mention being made of the fact) operate on a Banach space $\mathfrak{B}$ and linear continuous operators. By $\mathfrak{B}^\dagger$ we shall designate the adjoint space c i.e., the space of all linear continuous functionals on $\mathfrak{B}$.

If A is an operator then $A^\dagger$ will denote the adjoint operator of A, i.e., if $\phi \in \mathfrak{B}^\dagger$ and $f = A^\dagger \phi$, then $\phi(Ax) = f(x)$ for every x in $\mathfrak{B}$.

The subspace $\mathfrak{Z}(A)$ consisting of all solutions of the equation $Ax = 0$ is called *the null space of the operator* A. The subspace of all zeros of the operator $A^\dagger$ is denoted by $\mathfrak{Z}^\dagger(A)$.

The dimensions of the subspaces $\mathfrak{Z}(A)$, $\mathfrak{Z}^\dagger(A)$ are denoted respectively by $\alpha(A)$ and $\beta(A)$.

If the numbers $\alpha(A)$ and $\beta(A)$ are finite, then the difference

$$\varkappa(A) = \alpha(A) - \beta(A)$$

is called the *index* of the operator A.

We shall say that the operator A possesses an inverse if $\alpha(A) = 0$, and $A\mathfrak{B} = \mathfrak{B}$. For only in this case does there exist an operator A such that

$$AA^{-1} = A^{-1}A = I,$$

where I is the identity transformation of the space $\mathfrak{B}$.

As usual, the operator A is said to possess a normal extension, provided that for each $y \in \mathfrak{B}$ the equation $Ax = y$ has a solution if and only if $f(y) = 0$ for all $f \in \mathfrak{Z}^\dagger(A)$.

The last condition is equivalent to the fact that the range of values of the operator A is closed. The normal extensibility of one of the operators A, $A^\dagger$ carries along with it the normal extensibility of the other.

In order to check the normal extensibility of an operator $A^\dagger$ it is not necessary to have recourse to the second adjoint space $(\mathfrak{B}^\dagger)^\dagger$ and to the second adjoint operator $(A^\dagger)^\dagger$. For it is true that (see [23] or [24]) that the operator $A^\dagger$ is normally extensible, i.e., that the set of its values $A^\dagger\mathfrak{B}^\dagger$ is closed in $\mathfrak{B}^\dagger$ if and only if this set coincides with the set of all $f \in \mathfrak{B}^\dagger$ which are orthogonal to $\mathfrak{Z}(A) : f(\mathfrak{Z}(A)) = 0$; in other words, the equation $A^\dagger g = f (f \in \mathfrak{B}^\dagger)$ has a solution if and only if $f(x) = 0$ for every $x \in \mathfrak{Z}(A)$.

The operator A is called a Φ*–operator* provided that it is normally extensible and that both numbers $\alpha(A)$ and $\beta(A)$ are finite.

If one of the operators A, $A^\dagger$ is a Φ–operator then the other operator is also a Φ–operator.

The following important propositions hold for Φ–operators: (see section 2 of [11]):

A) *To each Φ–operator A there corresponds a positive number ρ such that every operator B With $\|B\| < \rho$, has the property that the operator $A + B$ is also a Φ–operator and further*

$$\varkappa(A+B) = \varkappa(A) \text{ and } \alpha(A+B) \leqslant \alpha(A).$$

B) *If A is a Φ–operator and T is a completely continuous operator then the*

operator $A + T$ is also a Φ–operator and

$$\varkappa(A+T) = \varkappa(A).$$

C) *The product AB of two Φ–operators is also a Φ–operator, and*

$$\varkappa(AB) = \varkappa(A) + \varkappa(B).$$

Propositions A) and B) above can be extended to the case when only one the numbers $\alpha(A)$ and $\beta(A)$ is finite.

§2. **General theorem about equation (A).** We shall consider several spac of complex valued functions defined on the positive half axis.

We shall employ the following notation. As usual, $L^{(p)}(0, \infty)$ $(p \geq 1)$ wi designate the space of all measurable functions $f(t)$ $(0 \leq t < \infty)$ possessing a finite integral for $|f(t)|^p$ over the half axis, the norm in this space being giv

$$\|f\|_{L^{(p)}(0,\ \infty)} = \left(\int_0^\infty |f(t)|^p\, dt\right)^{\frac{1}{p}}.$$

The letter M^+ will designate the adjoint space to the space $L^+ = L^{(1)}(0, \infty$ is well known, M^+ consists of all bounded functions $f(t)$ $(0 \leq t < \infty)$, the nor being defined by

$$\|f\|_{M^+} = \sup_{0 \leqslant t < \infty} \text{ess}\, |f(t)|.$$

The spaces M_c^+ and M_u^+ are subspaces of the space M^+. The first one sists of continuous functions, and the second of uniformly continuous functio

Further, C^+ denotes the subspace of all continuous functions $f(t) \in M_c^+$ $(0 \leq t < \infty)$ for which the following limit exists:

$$f(\infty) = \lim_{t \to \infty} f(t),$$

and C_0^+ designates the subspace of all $f(t) \in C^+$ for which $f(\infty) = 0$.

It is remarked that in the spaces C^+ and C_0^+ the norm may be equivalen defined as follows:

$$\|f\|_{C^+} = \max_{0 \leqslant t \leqslant \infty} |f(t)|.$$

In the sequel, the letter E^+ will be used to stand for one of the spaces

$$L^{(p)}(0,\ \infty)\,(p \geqslant 1), \qquad C_0^+ \subset C^+ \subset M_u^+ \subset M_c^+ \subset M^+ \qquad ($$

and the letter D^+ for one of the spaces $L^{(p)}(0, \infty)$ $(p \geq 1)$, M^+.

To each space D^+ corresponds its adjoint space D_r^+, which coincides $L^{(q)}(0, \infty)$ $(q^{-1} + p^{-1} = 1)$, if $D^+ = L^{(p)}(0, \infty)$ $(p > 1)$; and with M^+ if $D^+ =$ and with L^+; if $D^+ = M^+$.

Analogously, one may introduce the spaces of functions $L^{(p)}(-\infty, \infty)$ (p and $M(-\infty, \infty)$, defined on the whole real axis.

The letter D will be used to designate one of these spaces.

Let B be a linear set. Then the set of all n–dimensional column vector

$f=(f_1, f_2, \cdots, f_n)$ with coordinates in B $(f_j \in B)$ will be designated by $B_{(n\times 1)}$. If B is a Banach space then $B_{(n\times 1)}$ will become a Banach space also, once a norm is defined in it by the relation

$$\|f\| = \sum_{j=1}^{n} \|f_j\|_B.$$

The set of all square matrices of the nth order with elements in B will be denoted by $B_{(n\times n)}$.

The *intersection* of two Banach spaces B_1 and B_2 is the set of all elements common to B_1 and B_2, the norm being defined by

$$\|f\| = \|f\|_{B_1} + \|f\|_{B_2} \qquad (f \in B_1 \cap B_2).$$

In view of its definition, the intersection of two Banach spaces is again a Banach space.

To each matrix function $k(t) \in L_{(n\times n)}$ $(=L^{(1)}_{(n\times n)})$ we associate three operators [1]:

$$\widetilde{K}f = \int_{-\infty}^{\infty} k(t-s) f(s)\, ds \qquad (-\infty \leqslant t < \infty;\ f \in D_{(n\times 1)}), \tag{2.2}$$

$$Kf = \int_{0}^{\infty} k(t-s) f(s)\, ds \qquad (0 \leqslant t < \infty;\ f \in E^{+}_{(n\times 1)}), \tag{2.3}$$

$$\hat{K}f = \int_{0}^{\infty} k(t+s) f(s)\, ds \qquad (0 \leqslant t < \infty;\ f \in E^{+}_{(n\times 1)}). \tag{2.4}$$

It is easily shown (see [1] and [11]) that

$$\widetilde{K} D_{(n\times 1)} \subset D_{(n\times 1)}, \qquad K E^{+}_{(n\times 1)} \subset E^{+}_{(n\times 1)} \ \text{ and } \ \hat{K} E^{+}_{(n\times 1)} \subset E^{+}_{(n\times 1)}.$$

In particular, in their corresponding spaces the operators $\widetilde{K}$, K, and $\hat{K}$ are linear and bounded, because their norms satisfy the inequalities

$$\|\widetilde{K}\|_D \leqslant n \max_{i,\, j=1,\, 2,\, \ldots,\, n} \|k_{ij}(t)\|_L; \qquad \|K\|_{E^+} \leqslant n \max_{i,\, j=1,\, 2,\, \ldots,\, n} \|k_{ij}(t)\|_{L^+} \tag{2.5}$$

and

1) The convergence of the improper integrals (2.2), (2.3), (2.4) in the spaces $D_{(n\times 1)}$, $E^{+}_{(n\times 1)}$, with the exception of $L^{(p)}_{(n\times 1)}(0,\infty)$, $L^{(p)}_{(n\times 1)}(-\infty,\infty)$ $(p>1)$, is understood in the usual sense.

In the exceptions mentioned, the convergence of the integrals is meant in a special sense, for example

$$\int_{-\infty}^{\infty} k(t-s) f(s)\, ds = \underset{N,\, M\to\infty}{\mathrm{l.\ i.\ m.}} \int_{-N}^{M} k(t-s) f(s)\, ds,$$

where l.i.m. is the limit in the sense of the metric of $L^{(p)}_{(n\times 1)}(-\infty,\infty)$. For more details concerning this, see §10 of [11].

$$\|\hat{K}\|_{E^+} \leqslant n \cdot \max_{i,\, j=1,\, 2,\, \ldots,\, n} \|k_{ij}(t)\|_{L^+},$$

where $k_{ij}(t)$ $(i, j = 1, 2, \cdots, n)$ are the elements of the matrix $k(t)$. It may shown without difficulty that the operator $\hat{K}$ is *completely continuous* on ea space $E_{(n\times 1)}$.

Similar properties are possessed by the operators (2.3) and (2.4), consi as acting on the intersection of two spaces (2.1).

The following theorem holds:

Theorem 2.1. *Suppose that the matrix function* $k(t) \in L_{(n\times n)}$ *satisfie*

$$\det(I - \mathscr{K}(\lambda)) \neq 0, \qquad (-\infty < \lambda < \infty),$$

where $\mathfrak{K}(\lambda)$ *is the Fourier transform of* $k(t)$. *Then each of the homogeneou equations*

$$\varphi(t) - \int_{0}^{\infty} k(t-s)\varphi(s)\, ds = 0 \qquad (0 \leqslant t < \infty),$$

$$\psi(t) - \int_{0}^{\infty} k'(s-t)\psi(s)\, ds = 0 \qquad (0 \leqslant t < \infty)$$

possesses, on each space $D^{+}_{(n\times 1)}$, *not more than a finite number of linearl independent solutions.*

The nonhomogeneous equation

$$g(t) - \int_{0}^{\infty} k(t-s) g(s)\, ds = f(s) \qquad (0 \leqslant t < \infty),$$

where $f(t) \in D^{+}_{(n\times 1)}$, has at least one solution $g(t)$ *in the space* $D^{+}_{(n\times 1)}$ *only if the following condition holds:* [1]

$$\int_{0}^{\infty} f(s)\psi'(s)\, ds = 0,$$

whenever $\psi(s)$ *is an arbitrary solution of equation* (B') *in the space* $D^{+}_{(n}$

Proof. To equation (A) we associate the equation

$$g(t) - \int_{-\infty}^{\infty} k(t-s) g(s)\, ds = f(t) \qquad (g,\ f \in D_{(n\times 1)};\ -\infty < t < \infty),$$

which may be written briefly as follows: $g - \widetilde{K}g = f$.

Suppose at first that the vector function $f(t) \in L_{(n\times 1)}$, and that the tion (2.7) has a solution $g(t)$ in the space $L_{(n\times 1)}$.

Taking the Fourier transform of both members of (2.7), we obtain

$$(I - \mathscr{K}(\lambda))\, \mathscr{G}(\lambda) = \mathscr{F}(\lambda) \qquad (-\infty < \lambda < \infty),$$

where $\mathfrak{G}(\lambda)$, $\mathfrak{F}(\lambda)$ and $\mathfrak{K}(\lambda)$ are, respectively, the Fourier transforms of th

1) The vector $\psi(t)$ we suppose to be represented as a column vector; by ψ' denote the same vector, but written as a matrix having a single row.

vector functions $g(t)$, $f(t)$, and the matrix function $k(t)$.

According to Wiener's theorem, about which more will be said in §3, there exists a matrix function $q(t) \in L_{(n\times n)}$ such that

$$(I - \mathscr{K}(\lambda))^{-1} = I + Q(\lambda) = I + \int_{-\infty}^{\infty} q(t) e^{i\lambda t}\, dt. \tag{2.8}$$

From (2.8) it follows that

$$\mathscr{G}(\lambda) = (I - \mathscr{K}(\lambda))^{-1} \mathscr{F}(\lambda) = \mathscr{F}(\lambda) + Q(\lambda)\mathscr{F}(\lambda)$$

or, what is the same

$$g(t) = f(t) + \int_{-\infty}^{\infty} q(t-s) f(s)\, ds. \tag{2.9}$$

It is easily seen that, conversely, for each $f \in L_{(n\times 1)}$ the vector function $g(t)$ constructed by means of (2.9) will also belong to $L_{(n\times 1)}$ and will be a solution of equation (2.7).

Moreover, it may be immediately verified, that, for an arbitrary right-hand term $f \in D_{(n\times 1)}$, formula (2.9) defines a vector function $g(t)$ belonging to the space $D_{(n\times 1)}$, which is the unique solution of equation (2.7).

Thus, in each space $D_{(n\times 1)}$ the operator $\widetilde{I} - \widetilde{K}$, where $\widetilde{I}$ is the identity operator in the space $D_{(n\times 1)}$, has a bounded inverse operator.

Consider, on the other hand, the space D^+_{II}, consisting of pairs of vector functions $\mathfrak{f}(t) = \{f_1(t), f_2(t)\}$ belonging to the space $D^+_{(n\times 1)}$ $(f_j \in D^+_{(n\times 1)}; j = 1, 2)$, with the norm defined by

$$\|\mathfrak{f}\| = \|f_1\|_{D^+} + \|f_2\|_{D^+}.$$

The space D^+_{II} is equivalent to the space $D_{(n\times 1)}$ in view of the following isomorphism: to each $f \in D_{(n\times 1)}$ corresponds $\mathfrak{f} \in D^+_{II}$ which is defined by

$$f_1(t) = f(t), \qquad f_2(t) = f(-t) \qquad (0 \leqslant t < \infty).$$

In view of this equivalence, the operator $\widetilde{I} - \widetilde{K}$ may be thought of as an operator acting on D^+_{II}, in such a way that if

$$\mathfrak{f} = \widetilde{K}\mathfrak{g},$$

then

$$f_1(t) = \int_0^{\infty} k(t-s) g_1(s)\, ds + \int_0^{\infty} k(t+s) g_2(s)\, ds,$$

$$(0 \leqslant t < s)$$

$$f_2(t) = \int_0^{\infty} k(-t-s) g_1(s)\, ds + \int_0^{\infty} k(-t+s) g_2(s)\, ds,$$

or, in abbreviated notation

$$f_1 = K g_1 + \hat{K} g_2,$$

$$f_2 = \hat{K}_1 g_1 + K_1 g_2,$$

or, in still another way,

$$\widetilde{K} = \begin{pmatrix} K & \hat{K} \\ \hat{K}_1 & K_1 \end{pmatrix}.$$

Together with the operator $\widetilde{K}$ we shall consider also the operators $\widetilde{B}$ an which are defined by the equations

$$\widetilde{B} = \begin{pmatrix} K & 0 \\ 0 & K_1 \end{pmatrix}, \qquad \widetilde{T} = \begin{pmatrix} 0 & \hat{K} \\ \hat{K}_1 & 0 \end{pmatrix}.$$

The operator K_1 has for kernel the transpose of the kernel of the operat The operator $\hat{K}_1$ is of the same type as the operator $\hat{K}$, and consequently th operator $\widetilde{T}$ is completely continuous.

The operator $\widetilde{I} - \widetilde{B}$ is expressible as the sum of the inverse operator o and the completely continuous operator $\widetilde{T}$.

Thus, $\widetilde{I} - \widetilde{B}$ is a Φ-operator, since $\kappa(\widetilde{I} - \widetilde{B}) = 0$, and hence the opera $\widetilde{I} - \widetilde{B}$ is the direct sum of the operators $I - K$ and $I - K_1$, where I is the ic transformation of the space $D^+_{(n\times 1)}$, each of which act on the space $D^+_{(n\times 1)}$ both of these operators $I - K$ and $I - K_1$ must be Φ-operators, and

$$\varkappa(\widetilde{I} - \widetilde{B}) = \varkappa(I - K) + \varkappa(I - K_1) = 0.$$

From what was said concerning the definition of Φ-operators follows tl truth of the theorem for all spaces $D_{(n\times 1)}$, with the exception of the space

In the excluded case $D^+_{(n\times 1)} = M^+_{(n\times 1)}$, the theorem is also valid becau the operator $I - K$ may be considered as the operator adjoint to the operator where

$$K^{\dagger}\varphi = \int_0^{\infty} k'(s-t)\varphi(s)\, ds,$$

and it is already known that the operator $I - K$ is a Φ-operator on the spac $L^+_{(n\times 1)}$.

In concluding this paragraph we shall make two remarks.

1. The theorem proved remains valid if in its formulation the operator is replaced by the operator $A - K$ [1], where A is a constant nonsingular m and the condition (2.6) is replaced by the condition

$$\det(A - \mathscr{K}(\lambda)) \neq 0 \qquad (-\infty < \lambda < \infty).$$

The truth of this remark is obvious; because multiplying the operator A by the nonsingular constant matrix A^{-1} one is led to an operator of the typ sidered above.

2. From the proof of the theorem it follows that its hypotheses imply t operator $I - K$ is a Φ-operator (see the definition in §1) in any space $E^+_{(n}$ and also in any intersection of any two spaces $E^+_{(n\times 1)}$.

1) $A - K$ is an abbreviation for the operator $AI - K$.

§3. **Auxiliary considerations from the theory of Fourier integrals.** This section is devoted to certain results from the theory of Fourier integrals which will be needed in the sequel.

The letter L will designate the linear normed ring of all complex valued, measurable, absolutely integrable functions $f(t)$ $(-\infty < t < \infty)$, the norm being defined by

$$\|f\|_L = \int_{-\infty}^{\infty} |f(t)|\,dt$$

and the product $f = f_1 * f_2$ of two elements f_1 and f_2 in L is defined as their convolution

$$f(t) = \int_{-\infty}^{\infty} f_1(s) f_2(t-s)\,ds = \int_{-\infty}^{\infty} f_1(t-s) f_2(s)\,ds.$$

As is well known

$$\|f_1 * f_2\|_L \leq \|f_1\|_L \cdot \|f_2\|_L.$$

We shall denote by $L^+(L^-)$ the subring of all functions of L which vanish for $t < 0$ $(t > 0)$.

Whenever a lower case letter is used to designate an element of L then the corresponding capital letter will used to denote its Fourier transform.

For example, if $f \in L$, then

$$\mathcal{F}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,dt \qquad (-\infty < \lambda < \infty).$$

It will be recalled that $\mathfrak{F}(\lambda) \to 0$ as $\lambda \to \pm\infty$, for any $f \in L$.

Defining $\mathfrak{F}(\pm\infty) = 0$, we may consider the Fourier transform $\mathfrak{F}(\lambda)$ as a continuous function defined on the closed axis $\{-\infty, \infty\}$, obtained by identifying the ends of the real axis.

It is well known that the Fourier transform of the convolution $f * g$ of two functions $f, g \in L$ is the product of the Fourier transforms of the factors, i.e., if $h = f * g$, then

$$\mathcal{H}(\lambda) = \mathcal{F}(\lambda)\mathcal{G}(\lambda).$$

Thus, the set of all functions $\mathfrak{F}(\lambda)$ $(f \in L)$ forms a certain ring $\mathfrak{R}^0$ of continuous functions defined on the closed real axis $\{-\infty, \infty\}$.

From the equalities

$$(\lambda - \alpha)^{-n} = -\frac{i^n}{n!}\int_0^{\infty} t^n e^{-i\alpha t} e^{i\lambda t}\,dt \qquad (\operatorname{Im}\alpha < 0),$$

$$(\lambda - \alpha)^{-n} = \frac{i^n}{n!}\int_{-\infty}^{0} t^n e^{-i\alpha t} e^{i\lambda t}\,dt \qquad (\operatorname{Im}\alpha > 0)$$

it follows that all functions of the type $(\lambda - \alpha)^{-n}$ $(\operatorname{Im}\alpha \neq 0;\ n = 1, 2, \cdots)$ belong to the ring $\mathfrak{R}^0$.

Consequently, a rational function $R(\lambda)$ belongs to the ring $\mathfrak{R}^0$ if and on
$R(\infty) = 0$ and all its poles are non-real.

As is well known, all functions $f(t)$ which vanish for $t < 0$ and are of th
$f(t) = e^{-t}p(t)$, with $p(t)$ an arbitrary polynomial, form a dense subset of the
space $L^+ \subset L$. Observe that the transforms of these functions are certain pol
mials in $(\lambda - i)^{-1}$. It is easy to see that the functions obtained by replacing
$-t$ are dense in L^-, and that their Fourier transforms are polynomials in $(\lambda +$

Since the space L is the direct sum of the subspaces L^+ and L^-, from
preceding it follows that every function $\psi(t) \in L$ may be approximated (in th
of the metric of L) arbitrarily closely by functions of L whose Fourier trans
are rational functions.

Let us denote by $\mathfrak{R}$ the ring obtained by extending $\mathfrak{R}^0$ by one dimensio
adding to it all the constants. In other words, $\mathfrak{R}$ is the ring of functions $\mathfrak{F}($
the form

$$\mathfrak{F}(\lambda) = c + \int_{-\infty}^{\infty} f(t) e^{i\lambda t}\,dt \qquad (-\infty \leq \lambda \leq \infty,\ \ f \in L).$$

Functions of the ring $\mathfrak{R}$ will be denoted by gothic letters.

In the following, an essential rôle will be played by the following result
to N. Wiener (see [25]).

Theorem W. *If the function* $\mathfrak{F}(\lambda) \in \mathfrak{R}$ *does not vanish at any point of th*
closed line $\{-\infty, \infty\}$ then the function $\mathfrak{F}^{-1}(\lambda)$ *also belongs to the ring* $\mathfrak{R}$.

Let us denote by $\Pi_+(\Pi_-)$ the closed upper (lower) half plane $\operatorname{Im}\lambda \geq 0$ (
together with the point $\lambda = \infty$.

By $\mathfrak{R}^+(\subset \mathfrak{R})$ we shall designate the subring of functions $\mathfrak{F}(\lambda)$ of the f

$$\mathfrak{F}(\lambda) = c + \int_0^{\infty} f(t) e^{i\lambda t}\,dt \qquad (f \in L^+,\ c \text{ — a number}$$

and by $\mathfrak{R}^{0+}$ the set of functions $\mathfrak{F} \in \mathfrak{R}^{0+}$ for which $c = 0$.

An analogous interpretation is to be given to the symbols $\mathfrak{R}^-$, $\mathfrak{R}^{0-}$.

The right-hand side of (3.1) makes sense for an arbitrary complex $\lambda \in \Pi$
In view of this, every function $\mathfrak{F}(\lambda) \in \mathfrak{R}^+$ may be thought of as being defin
means of a function which is holomorphic inside Π_+ and which is continuou
to the boundary. An analogous statement may be made for functions of $\mathfrak{R}^-$,
rôle of the half plane Π_+ being taken over by the half plane Π_-.

For the rings $\mathfrak{R}^{\pm}$ one has the following theorem, which is analogous to
Wiener's theorem (see [26], pp. 60–63) [1].

1) The simplest and most elegant proof of the Theorems W and W_+ is that by th
method of normed rings of I. M. Gelfand [27].

Theorem W_+. *If the function $\mathfrak{F}(\lambda) \in \mathfrak{R}^+$ does not vanish at any point of the half plane Π_+ then the function $\mathfrak{F}^{-1}(\lambda)$ also belongs to the ring $\mathfrak{R}^+$.*

An analogous formulation gives Theorem W_-.

An immediate verification readily substantiates the truth of the following proposition (see [1], section 1).

Lemma 3.1. *If the function*

$$\mathfrak{F}(\lambda) = \mathfrak{F}(\infty) + \int_0^\infty f(t)\, e^{i\lambda t}\, dt \qquad (f \in L^+),$$

belongs to $\mathfrak{R}^+$, and has the value zero at a ($\operatorname{Im} a > 0$), then the function $\mathfrak{G}(\lambda) = (\lambda - a)^{-1}\, \mathfrak{F}(\lambda)$ belongs to the ring $\mathfrak{R}^{0+}$, that is

$$\mathfrak{G}(\lambda) = \int_0^\infty g(t)\, e^{i\lambda t}\, dt \qquad (g \in L^+).$$

The functions $f(t)$ and $g(t)$ are related by the equations

$$i\frac{dg}{dt} - ag(t) = f(t), \qquad ig(0) = \mathfrak{F}(\infty).$$

Theorem W_+ may be used to prove the following proposition (see [1], section 1).

Lemma 3.2. *Suppose that $\mathfrak{F}, \mathfrak{G} \in \mathfrak{R}$ and satisfy the following conditions: 1) $\mathfrak{G}(\lambda) \neq 0$ $(-\infty \leq \lambda < \infty)$ and 2) every zero of the function $\mathfrak{G}(\lambda)$ interior to the half plane Π_+ is also a zero (of not smaller multiplicity) of the function $\mathfrak{F}(\lambda)$. Then the function $\mathfrak{F}(\lambda) / \mathfrak{G}(\lambda) \in \mathfrak{R}^+$.*

If, in particular, $\mathfrak{F}(\lambda) \in \mathfrak{R}^{0+}$, then also $\mathfrak{F}(\lambda) / \mathfrak{G}(\lambda) \in \mathfrak{R}^{0+}$.

Let us introduce now a simple concept which will play an important rôle in the following.

Suppose that $\Phi(\lambda)$ is a function which is continuous on the whole closed axis $\{-\infty, \infty\}$ and never vanishes there. Then the *index of the function* $\Phi(\lambda)$ is defined to be the integer

$$\operatorname{ind} \Phi = \frac{1}{2\pi} [\arg \Phi(\lambda)]_{-\infty}^{\infty} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d_\lambda \arg \Phi(\lambda).$$

If the function $\Phi(\lambda)$ admits of an extension which is holomorphic in the interior of and is continuous on the boundary of $\Pi_+(\Pi_-)$; then, by Rouché's theorem, ind Φ gives the number of zeros (minus the number of zeros) of Φ inside $\Pi_+(\Pi_-)$, each zero being counted as many times as its multiplicity.

If $R(\lambda)$ is a rational function from $\mathfrak{R}$, with $\mathfrak{R}(\infty) \neq 0$, then ind R equals the difference between the number of zeros and the number of poles of the function R inside Π_+; or, what is the same, the difference between the number of poles and the number of zeros of the function R inside Π_-.

§4. Calculation of the index of equation (A). In this section we shall prove the following assertion:

Theorem 41. *Suppose that the matrix function $k(t) \in L_{(n\times n)}$ satisfies condition*

$$\det(A - \mathscr{K}(\lambda)) \neq 0 \qquad (-\infty < \lambda < \infty),$$

where A is a constant, nonsingular matrix. Then the index of the operator U in the space $E^+_{(n\times 1)}$, as well as in the intersection of any two $E^+_{(n\times 1)}$ spa[ces] may be computed by means of the formula

$$\varkappa(U) = -\operatorname{ind} \det(A - \mathscr{K}(\lambda)).$$

The proof of the theorem will be preceded by an easily provable remark.

Suppose that K_1 and K_2 are operators, defined on the space $E^+_{(n\times 1)}$ b[y] equations

$$K_j\varphi = \int_0^\infty k_j(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty;\ j = 1, 2),$$

where $k_j(t) \in L_{(n\times n)}$ $(j = 1, 2)$. Then the product of the operators, $K_1 K_2$, [dif]fers from the operator

$$K\varphi = \int_0^\infty k(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty),$$

where

$$k(t) = \int_{-\infty}^{\infty} k_1(t-s)\, k_2(s)\, ds,$$

by a completely continuous operator.

Proof of the theorem. The proof will be divided into three parts.

1) Suppose at first that the matrices A and $k(t)$ are triangular. Consi[der] the matrices $\kappa_\mu(t)$ and A_μ, depending on a complex parameter μ, which ar[e de]fined by

$$k_\mu(t) = \mu k(t) + (1-\mu)\,\| k_{pq}(t)\,\delta_{pq} \|_1^n; \qquad A_\mu = \mu A + (1-\mu)\,\| a_{pq}\delta_{pq} \|_1^n,$$

where $k_{pq}(t)$ and $a_{pq}(t)$ $(p, q = 1, 2, \cdots, n)$ are the elements of the matric[es] and A, respectively.

Consider the operator

$$U_\mu\varphi = A_\mu\varphi(t) - \int_0^\infty k_\mu(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty)$$

in the space $E^+_{(n\times 1)}$.

Since for an arbitrary complex μ:

$$\det(A_\mu - \mathscr{K}_\mu(\lambda)) = \det(A - \mathscr{K}(\lambda)) \neq 0 \qquad (-\infty < \lambda < \infty),$$

it follows from Theorem 2.1 that for an arbitrary complex number μ the oper[ator] U_μ is a ϕ-operator. The function $\kappa(\mu) = \kappa(U_\mu)$, which is defined on the wh[ole] complex plane, is continuous and takes on only integral values; consequent[ly] must be a constant. In particular,

$$\varkappa(U) = \varkappa(1) = \varkappa(0); \quad \text{that is} \quad \varkappa(U) = \varkappa(U_0).$$

The matrices $k_0(t)$ and A_0, which are obtained from the matrices $k_\mu(t)$ and A_μ by putting $\mu = 0$, are diagonal. Thus the operator U_0 is decomposed into a direct sum of n scalar operators $a_{jj}I - K_{jj}$ $(j = 1, 2, \cdots, n)$, each of which acts on the space E^+ and has the form

$$(a_{jj}I - K_{jj})\varphi = a_{jj}\varphi(t) - \int_0^\infty k_{jj}(t-s)\varphi(s)\,ds \quad (0 \leqslant t < \infty;\ j = 1, 2, \ldots, n).$$

Consequently the index of the operator U_0 is the sum of the indices of the operators $a_{jj}I - K_{jj}$ $(j = 1, 2, \cdots, n)$. Taking into account the formula for the index of a scalar operator (see [1]), we obtain

$$\varkappa(U_0) = -\sum_{j=1}^{n} \operatorname{ind}(a_{jj} - \mathscr{K}_{jj}(\lambda)) = -\operatorname{ind}\prod_{j=1}^{n}(a_{jj} - \mathscr{K}_{jj}(\lambda)) = -\operatorname{ind}\det(A - \mathscr{K}(\lambda)).$$

Hence

$$\varkappa(U) = -\operatorname{ind}\det(A - \mathscr{K}(\lambda)).$$

II) Consider the second case, when the matrix function $k(t)$ is such that its Fourier transform $\mathscr{K}(\lambda)$ is a rational matrix function.

From the theory of polynomial λ–matrices [26], it follows that the rational matrix function $\mathfrak{M}(\lambda) = A - \mathscr{K}(\lambda)$ may be represented as a product of rational matrix functions $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{D}\,(\in \mathfrak{R}_{(n\times n)})$:

$$\mathfrak{M}(\lambda) = \mathfrak{M}_1(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{M}_2(\lambda).$$

Where $\mathfrak{D}(\lambda)$ is a diagonal matrix, and $\mathfrak{M}_j(\lambda)$ $(j = 1, 2)$ are the products of certain constant matrixes and certain triangular matrix functions in $\mathfrak{R}_{(n\times n)}$, whose determinants are unity.

Thus, the matrices $\mathfrak{M}_j(\lambda)$ have the form

$$\mathfrak{M}_j(\lambda) = A_j - \int_{-\infty}^{\infty} h_j(t)\,e^{i\lambda t}\,dt \qquad (h_j \in L_{(n\times n)};\quad j = 1, 2)$$

and

$$\mathfrak{M}_j(\lambda) = \prod_{k=1}^{m_j}\left(B_{jk} - \int_{-\infty}^{\infty} l_{jk}(t)\,e^{i\lambda t}\,dt\right) \qquad (l_{jk} \in L_{(n\times n)};\quad j = 1, 2),$$

where A_j $(j = 1, 2)$ and B_{jk} $(j = 1, 2;\ k = 1, 2, \cdots, m_j)$ are constant nonsingular matrices, $l_{jk}(t)$ $(j = 1, 2;\ k = 1, 2, \cdots, m_j)$ are triangular matrices, some of which are identically zero; but for which

$$\det\left(B_{jk} - \int_0^\infty l_{jk}(t)\,e^{i\lambda t}\,dt\right) \equiv 1 \qquad (j = 1, 2;\quad k = 1, 2, \ldots, m_j). \tag{4.3}$$

From what has been said above it follows that the operators U_{jk} $(j = 1, 2;\ k = 1, 2, \cdots, m_j)$ defined by

$$U_{jk}\varphi = B_{jk}\varphi(t) - \int_0^\infty l_{jk}(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty),$$

are Φ–operators, since condition (4.3) implies that

$$\varkappa(U_{jk}) = 0 \qquad (j = 1, 2;\quad k = 1, 2, \ldots, m_j).$$

The operators

$$Y_j\varphi = A_j\varphi(t) - \int_0^\infty h_j(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty;\quad j = 1, 2)$$

differ from the corresponding products of operators

$$\prod_{k=1}^{m_j} U_{kj} \qquad (j = 1, 2)$$

by completely continuous terms; consequently they are Φ–operators, it bein[g] called that according to assertion C) of section 1 concerning the index of th[e] duct of operators one has that

$$\varkappa(Y_j) = 0 \qquad (j = 1, 2).$$

The operator

$$Y\varphi = B\varphi(t) - \int_0^\infty p(t-s)\,\varphi(s)\,ds \qquad (0 \leqslant t < \infty),$$

where

$$\mathfrak{D}(\lambda) = B - \int_{-\infty}^{\infty} p(t)\,e^{i\lambda t}\,dt \qquad (p(t) \in L_{(n\times n)}),$$

is a Φ–operator, because

$$\det\mathfrak{D}(\lambda) = \det(A - \mathscr{K}(\lambda)) \neq 0 \qquad (-\infty < \lambda < \infty).$$

In particular, from what was shown in I) one has that $\kappa(\gamma) = -\operatorname{ind}\det\mathfrak{D}(\lambda)$.

The operators $Y_1 Y Y_2$ and U differ from each other by a completely co[ntin]uous operator, hence

$$\varkappa(U) = \varkappa(Y_1 Y Y_2) = \varkappa(Y) = -\operatorname{ind}\det\mathfrak{D}(\lambda) = -\operatorname{ind}\det(A - \mathscr{K}(\lambda)).$$

III) Let us now consider the general case. According to assertion A) o[f sec]tion 1, there exists a positive number δ such that any linear bounded opera[tor] satisfying the inequality

$$\|U - V\| < \delta,$$

will also satisfy $\kappa(U) = \kappa(V)$.

Select the matrix function $l(t) = \|l_{pq}(t)\|_1^n\ (\in L_{(n\times n)})$ such that its transform $\mathfrak{L}(\lambda)$ is a rational matrix function and that

$$\|k_{pq}(t) - l_{pq}(t)\|_L < \varepsilon \qquad (p, q = 1, 2, \ldots, n).$$

Let the number $\epsilon > 0$ be chosen sufficiently small, so that: a) the oper[ator] V, defined by the equality $V\phi = A\phi - \int_0^\infty l(t-s)\,ds$, $(0 \leq t < \infty)$, satisfie[s] inequality (4.4); B) the determinants $(A - \mathfrak{L}(\lambda))$ does not vanish on the real axis; and C)

$$\text{ind det}\,(A - \mathscr{K}(\lambda)) = \text{ind det}\,(A - \mathscr{L}(\lambda)). \qquad (4.5)$$

In view of (4.4), $\kappa(U) = \kappa(V)$, and from what was proved in II)

$$\varkappa(V) = -\text{ind det}\,(A - \mathscr{L}(\lambda)).$$

Employing (4.5), we conclude that

$$\varkappa(U) = -\text{ind det}\,(A - \mathscr{K}(\lambda)).$$

The theorem is proved.

§5. Coincidence of the solution of the homogeneous equation (B) in different classes of functions. The purpose of the present brief section is the proof of the following theorem:

Theorem 5.1. *If the matrix function $k(t)$ belonging to the class $L_{(n\times n)}$ satisfies condition* (0.2) *then the homogeneous equation*

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds = 0 \qquad (0 \leqslant t < \infty)$$

has one and the same set of solutions in all spaces $E^+_{(n\times 1)}$.

Proof. Denote by $\mathfrak{E}$ one of the spaces $L^{(p)}_{(n\times 1)}(0,\infty)$ $(p \geq 1)$, $C^+_{0\,(n\times 1)}$ and by $\mathfrak{E}_{12}$ the Banach space which is the intersection of two arbitrary spaces $\mathfrak{E}_1$ and $\mathfrak{E}_2$ of $\mathfrak{E}$. Let us recall that in $\mathfrak{E}_{12}$ the norm is introduced by means of the equation

$$\|\varphi\| = \|\varphi\|_1 + \|\varphi\|_2 \qquad (\varphi \in \mathfrak{E}_1 \cap \mathfrak{E}_2).$$

Denote by U_j $(j = 1, 2)$ the operator defined on the space $\mathfrak{E}_j$ $(j = 1, 2)$ by the equality

$$U_j\varphi = \varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds \qquad (0 \leqslant t < \infty), \qquad (5.1)$$

and by $\hat{U}$ the part of the operator U_1 which acts in the space $\mathfrak{E}_{12}$. According to Theorem 2.1 and remark 2° to this theorem, the operators U_1, U_2, and $\hat{U}$ are Φ–operators.

Since $\mathfrak{E}_{12} \subset \mathfrak{E}_j$ (j = 1, 2), one has that

$$\alpha(\hat{U}) \leqslant \alpha(U_j) \qquad (j = 1, 2). \qquad (5.2)$$

The set $\mathfrak{E}_{12}$ is dense in each of the spaces $\mathfrak{E}_j$ $(j = 1, 2)$ in the sense of the norm of each of these spaces. Consequently, $\mathfrak{E}^+_j \subset \mathfrak{E}^+_{12}$ $(j = 1, 2)$. From this last relation it follows that

$$\beta(\hat{U}) \geqslant \beta(U_j) \qquad (j = 1, 2). \qquad (5.3)$$

Comparing the inequalities (5.2) and (5.3) with the fact that the indices of the operators are the same: $\kappa(U_j) = \kappa(\hat{U})$ $(j = 1, 2)$, we obtain that

$$\alpha(U_j) = \alpha(\hat{U}) \text{ and } \beta(U_j) = \beta(\hat{U}) \qquad (j = 1, 2).$$

Finally, the relations $\mathfrak{Z}(\hat{U}) \subseteq \mathfrak{Z}(U_j)$ $(j = 1, 2)$ lead us to the equation

$$\mathfrak{Z}(U_1) = \mathfrak{Z}(U_2). \qquad (\text{:}$$

It remains for us to prove that equation (5.4) continues to hold when $\mathfrak{E}$ denote one of the spaces $M^+_{(n\times 1)}$, $M^+_{c\,(n\times 1)}$, $M^+_{u\,(n\times 1)}$, and $C^+_{(n\times 1)}$. In order to do th all that is needed is to prove that equation (B) has one and the same set of so tions in the spaces $L^+_{(n\times 1)}$ and $M^+_{(n\times 1)}$. Let us suppose that U_1 denotes the operator defined by (5.1) on the space $L^+_{(n\times 1)}$, and that U_2 denotes the opera defined by the same equation on the space $M^+_{(n\times 1)}$. All solutions of equation in $L^+_{(n\times 1)}$ are bounded continuous functions, because they belong to the spac $C^+_{0\,(n\times 1)}$. Thus

$$\alpha(U_1) \leqslant \alpha(U_2). \qquad (\text{:}$$

Considering the transposed equation (B′) it is easy to establish that

$$\beta(U_1) = \alpha(U_1^\dagger) \geqslant \alpha_L(U_1^\dagger) = \beta(U_2),$$

where the subscript L attached to α means that the operator $U_1^\dagger$, defined on space $M^\dagger_{(n\times 1)}$, is now considered on the space $L^\dagger_{(n\times 1)}$.

Consequently

$$\beta(U_1) \geqslant \beta(U_2). \qquad (\text{:}$$

Comparing inequalities (5.5) and (5.6) with the equation $\kappa(U_1) = \kappa(U_2)$, we ol

$$\alpha(U_1) = \alpha(U_2), \qquad \beta(U_1) = \beta(U_2).$$

Taking into account, finally, the relation $\mathfrak{Z}(U_1) \subseteq \mathfrak{Z}(U_2)$, it follows that

$$\mathfrak{Z}(U_1) = \mathfrak{Z}(U_2).$$

The theorem is proved.

§6. The homogeneous Hilbert problem in the ring of functions $\mathfrak{R}_{(n\times 1)}$.

1. As is well known, the homogeneous Hilbert problem on the closed real ax. $\{-\infty, \infty\}$ for a continuous nonsingular function matrix $\mathfrak{M}(\lambda)$ consists in the d mination of all pairs of vector functions $(\Phi_+(\lambda), \Phi_-(\lambda))$ holomorphic inside a continuous up to the boundary of, the corresponding half planes Π_+ and Π_-, are related to each other by the equation

$$\mathfrak{M}(\lambda)\Phi_+(\lambda) = \Phi_-(\lambda). \qquad (\text{(}$$

In the pair (Φ_+, Φ_-), the function Φ_1 is uniquely determined by the func Φ_+. Hence, when speaking from now on about the solution of the Hilbert probl we shall always have in mind the determination of the function $\Phi_+(\lambda)$.

We shall give below the solution of the homogeneous Hilbert problem (6.1 for a nonsingular matrix function $\mathfrak{M}(\lambda)$ belonging to $\mathfrak{R}_{(n\times n)}$. It should be no that in this case the vector functions $\Phi_\pm(\lambda)$ satisfying (6.1) always belong to corresponding classes $\mathfrak{R}_{(n\times 1)}$.

Without loss of generality we may put $\mathfrak{M}(\infty) = I$, i.e., suppose that the fu tion matrix $\mathfrak{M}(\lambda)$ has the form

$$\mathfrak{M}(\lambda) = I - \mathscr{K}(\lambda),$$

where $\mathcal{K}(\lambda) \in \mathfrak{R}_{(n\times n)}$.

First of all we shall prove a lemma which plays a fundamental rôle in what follows.

Lemma 6.1. *If the vector function* $\Phi_+(\lambda) \in \mathfrak{R}^+_{(n\times 1)}$ *is a solution of the problem*

$$(I - \mathcal{K}(\lambda))\,\Phi_+(\lambda) = \Phi_-(\lambda), \tag{6.2}$$

has the value zero at λ_0 ($\operatorname{Im}\lambda_0 \geq 0$), *then the vector function* $(\lambda-\lambda_0)^{-1}\,\Phi_+(\lambda)$ *belongs to the class* $\mathfrak{R}^{0+}_{(n\times 1)}$ *and is also a solution of the Hilbert problem* (6.2).

Proof. In fact, if $\operatorname{Im}\lambda_0 > 0$, Lemma 3.1 implies that $(\lambda-\lambda_0)^{-1}\,\Phi_+(\lambda) \in \mathfrak{R}^{0+}_{(n\times 1)}$.

The vector function $(\lambda-\lambda_0)^{-1}\,\Phi(\lambda) \in \mathfrak{R}^{0-}_{(n\times 1)}$, since $(\lambda-\lambda_0)^{-1} \in \mathfrak{R}^{0+}$. Thus, upon multiplying equation (6.2) by $(\lambda-\lambda_0)^{-1}$ we see that the vector function $(\lambda-\lambda_0)^{-1}\Phi_+(\lambda)$ is a solution of the problem (6.2).

Consider now the second case: $\operatorname{Im}\lambda_0 = 0$. Let us denote by $f(t)$ the solution of the differential equation

$$f'(t) + i\lambda_0 f(t) = \varphi(t), \qquad f(0) = c, \tag{6.3}$$

where

$$\Phi_+(\lambda) = c + \int_0^\infty \varphi(t)\, e^{i\lambda t}dt \qquad (\varphi \in L^+_{(n\times 1)}).$$

Obviously

$$f(t) = ce^{-i\lambda_0 t} + \int_0^t e^{-i\lambda_0(t-s)}\varphi(s)\,ds.$$

The vector function $f(t)$ is bounded and continuous, and

$$\lim_{t\to\infty} e^{i\lambda_0 t}f(t) = c + \int_0^\infty e^{i\lambda_0 s}\varphi(s)\,ds = \Phi_+(\lambda_0) = 0.$$

Therefore, $f(t) \in C^+_{0\,(n\times 1)}$.

Equation (6.2) is equivalent to the following:

$$\int_0^\infty e^{i\lambda t}\left(\varphi(t) - \int_0^\infty k(t-s)\,\varphi(s)\,ds\right)dt = \int_0^\infty k(t)\, ce^{i\lambda t}dt,$$

from which it follows that

$$\varphi(t) - \int_0^\infty k(t-s)\,\varphi(s)\,ds = k(t)\,c \qquad (0 \leqslant t < \infty).$$

Inserting, in the last equation, the expression for $\phi(t)$ given by (6.3), we obtain

$$f'(t) + i\lambda_0 f(t) - \int_0^\infty k(t-s)\,(f'(s) + i\lambda_0 f(s))\,ds = k(t)\,c \qquad (0 \leqslant t < \infty)$$

that is

$$\frac{d}{dt}(e^{i\lambda_0 t}f(t)) - \int_0^\infty e^{i\lambda_0(t-s)}k(t-s)\,\frac{d}{ds}(e^{i\lambda_0 s}f(s))\,ds = e^{i\lambda_0 t}k(t)\,c \qquad (0 \leqslant t < \text{c}$$

Let us integrate the last equation between the limits t and ∞. The doul integral obtained in this process is absolutely convergent. Interchanging the of integration in this double integral we obtain

$$-e^{i\lambda_0 t}f(t) - \int_0^\infty\int_{t-s}^\infty e^{i\lambda_0 u}k(u)\,du\,\frac{d}{ds}(e^{i\lambda_0 s}f(s))\,ds = \int_t^\infty e^{i\lambda_0 u}k(u)\,c\,du.$$

Integrating the second member of the left-side by parts, we find

$$e^{i\lambda_0 t}f(t) - \int_0^\infty e^{i\lambda_0 t}k(t-s)\,f(s)\,ds = 0 \qquad (0 \leqslant t < \infty)$$

or, what is the same,

$$f(t) - \int_0^\infty k(t-s)\,f(s)\,ds = 0 \qquad (0 \leqslant t < \infty).$$

In view of the fact that equation (B) has one and the same set of solutio all spaces $E^+_{(n\times 1)}$, we conclude that $f \in L^+_{(n\times 1)}$.

From equation (6.3) it is obvious that

$$\mathcal{F}_+(\lambda) = \int_0^\infty f(t)\, e^{i\lambda t}dt = (\lambda-\lambda_0)^{-1}\Phi_+(\lambda) \in \mathfrak{R}^{0+}_{(n\times 1)}.$$

Finally, applying the Fourier transformation to equation (6.4) we deduce the vector function $\mathfrak{F}_+(\lambda) = (\lambda-\lambda_0)^{-1}\,\Phi_+(\lambda)$ is a solution of the problem (6. The lemma is proved.

2. In section 11 it will be shown that the matrix function $\mathfrak{M}(\lambda)$ may be plied by a suitable integral power of the quotient $\frac{\lambda-i}{\lambda+i}$ in such a way that the equation

$$\phi(t) - \int_0^\infty k'(s-t)\,\phi(s)\,ds = 0 \qquad (0 \leqslant t < \infty)$$

has only the zero solution in $L^+_{(n\times 1)}$.

Thus, for our purposes, without loss of generality we may assume that e tion (B′) has only the zero solution.

Under this assumption, the following assertions are valid.

1) *The problem* (6.2) *has* n *solutions in* $\mathfrak{R}^+_{(n\times n)}$, *whose values at infin are linearly independent.*

Under the stated restrictions concerning the matrix function $\mathfrak{M}(\lambda)$, the e

$$g(t) - \int_0^\infty k(t-s)\,g(s)\,ds = f(t) \qquad (0 \leqslant t < \infty)$$

has, for arbitrary right-hand term $f(t) \in L^+_{(n\times 1)}$, at least one solution $g(t) \in$

The matrix equation

$$\gamma(t)-\int_0^\infty k(t-s)\gamma(s)\,ds=k(t)\qquad(0\leqslant t<\infty)\tag{6.5}$$

reduces to n equations of type (A); consequently, it has at least one solution $\gamma(t)$ with elements belonging to the space $L^+_{(n\times 1)}$. Let us define $\gamma(t)=0$ $(0\leq t<\infty)$ and denote by $b(t)$ the function defined by the equation

$$b(t)=-k(t)-\int_0^\infty k(t-s)\gamma(s)\,ds\quad(-\infty<t<0);\qquad b(t)=0\quad(0\leqslant t<\infty).$$

Then equation (6.5) may be written in the form

$$\gamma(t)-\int_{-\infty}^\infty k(t-s)\gamma(s)\,ds=k(t)+b(t)\qquad(-\infty<t<\infty).$$

Applying the Fourier transformation to both sides, we obtain

$$\mathfrak{M}(\lambda)\Gamma(\lambda)=\mathscr{K}(\lambda)+\mathscr{B}(\lambda),$$

where the function matrices $\Gamma(\lambda)$ and $\mathscr{B}(\lambda)$ belong respectively to the rings $\Re^{0+}_{(n\times n)}$ and $\Re^{0-}_{(n\times n)}$. Adding $\mathfrak{M}(\lambda)$ to both sides of the last equation, we obtain

$$\mathfrak{M}(\lambda)\cdot(I+\Gamma(\lambda))=I+\mathscr{B}(\lambda).$$

From this it follows that the columns of the function matrix $I+\Gamma(\lambda)$ are solutions of the problem (6.2), and their values at infinity are linearly independent, because $I+\Gamma(\infty)=I$.

2) *The multiplicity of any zero λ_0 $(\operatorname{Im}\lambda_0\geq 0)$ of a solution $\Phi_+(\lambda)$ $(\in\Re^+_{(n\times 1)})$ of the problem* (6.2) *does not exceed the number α of linearly independent solutions of the equation:*

$$\varphi(t)-\int_0^\infty k(t-s)\varphi(s)\,ds=0\qquad(0\leqslant t<\infty;\quad\varphi\in L^+_{(n\times 1)}).$$

Let m be the multiplicity of a zero λ_0 $(\operatorname{Im}\lambda_0\geq 0)$ of a solution $\Phi_+(\lambda)$ $(\in\Re^+_{(n\times 1)})$ of the problem (6.2).

Then the vector function

$$\Phi_j(\lambda)=(\lambda-\lambda_0)^{-j}\Phi_+(\lambda)\qquad(j=1,2,\ldots,m),$$

according to Lemma 6.1, belongs to the class $\Re^{0+}_{(n\times 1)}$ and is a solution of the problem (6.2), that is

$$\Phi_j(\lambda)=\int_0^\infty e^{i\lambda t}\varphi_j(t)\,dt\quad(\varphi_j\in L^+_{(n\times 1)}),\quad\mathfrak{M}(\lambda)\Phi_j\in\overset{0}{\Re}{}^-_{(n\times 1)}\quad(j=1,2,\ldots,m).$$

From this it follows that

$$\int_0^\infty e^{i\lambda t}\left(\varphi_j(t)-\int_0^\infty k(t-s)\varphi_j(s)\,ds\right)dt=0\qquad(j=1,2,\ldots,m)$$

or, what is the same

$$\varphi_j(t)-\int_0^\infty k(t-s)\varphi_j(s)\,ds=0\qquad(0\leqslant t<\infty;\quad i=1,2,\ldots,m).$$

Since the functions $\Phi_j(\lambda)$ are linearly dependent, so are the functions ϕ_j and thus $m\leq\alpha$.

3. Let us denote by $\mathfrak{F}_+(\lambda)$ $(\in\Re^+_{(n\times n)})$ an arbitrary matrix function whos column vectors are solutions of the problem (6.2). It is obvious that

$$\mathfrak{M}(\lambda)\mathfrak{F}_+(\lambda)=\mathfrak{F}_-(\lambda),\tag{$$

where $\mathfrak{F}_-(\lambda)\in\Re^-_{(n\times n)}$.

A matrix function such as $\mathfrak{F}_+(\lambda)$ will be called a *solution* of the problem (
In the sequel attention will be restricted only to those solutions $\mathfrak{F}_+(\lambda)$ of the trix problem (6.6) for which $\det\mathfrak{F}_+(\infty)\neq 0$.

Let us denote by k_j $(j=1,2,\cdots,n)$ the multiplicity of the zero at $\lambda=i$ jth column of the solution $\mathfrak{F}_+(\lambda)$ of the matrix problem (6.6); the n dimension vector $\mathbf{k}=(k_1,k_2,\cdots,k_n)$ will be called the index of the solution $\mathfrak{F}_+(\lambda)$. Th indices of the solutions of the matrix problem (6.6) may be ordered lexicograph cally, i.e., we shall say that the index $\mathbf{k}=(k_1,k_2,\cdots,k_n)$ is greater than th index $\mathbf{l}=(l_1,l_2,\cdots,l_n)$ if $k_1>l_1$, of if there exists a whole number $p\,(<n)$, that $k_j=l_j$ $(j=1,2,\cdots,p)$ and $k_{p+1}>l_{p+1}$.

The indices of all the solutions in $\Re^+_{(n\times n)}$ of the matrix problem (6.6) are bounded, since according to property 2) the coordinates of the indices of a sol tion of the matrix problem (6.6) never exceed α. Consequently, among all the i dices of all solutions of (6.6) which are regular at infinity and belong to $\Re^+_{(n\times}$ there must be a greatest. Let us denote it by $\kappa=(\kappa_1,\kappa_2,\cdots,\kappa_n)$. It is obviou that $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_n$.

All solutions of the matrix problem (6.6) with index κ will be referred to *standard* solutions.

Theorem 6.1. *Suppose that the matrix function $\mathfrak{F}_+(\lambda)$ $(\in\Re^+_{(n\times n)})$ is a st ard solution of the matrix problem* (6.6). *Then for every λ in the upper half pla with the exception of $\lambda=i$, the determinant of $\mathfrak{F}_+(\lambda)$ does not vanish:*

$$\det\mathfrak{F}_+(\lambda)\neq 0\qquad(\lambda\in\Pi_+,\ \lambda\neq i).$$

The multiplicity of the zero of $\det\mathfrak{F}_+(\lambda)$ at $\lambda=i$ is equal to

$$\sum_{j=1}^n\varkappa_j=-\operatorname{ind}\det\mathfrak{M}(\lambda),$$

and, consequently, the determinant of the matrix function $\mathfrak{F}_-(\lambda)=\mathfrak{M}(\lambda)\mathfrak{F}_+(\lambda)$ never vanishes on the lower half plane:

$$\det\mathfrak{F}_-(\lambda)\neq 0\qquad(\lambda\in\Pi_-).$$

Proof. Let us denote by $\Phi_j(\lambda)$ $(j=1,2,\cdots,n)$ the column vectors of a sta ard solution $\mathfrak{F}_+(\lambda)$.

Suppose that at a certain point $\lambda_0 (\mathrm{Im}\,\lambda_0 \geq 0)$ the determinant of $\mathfrak{F}_+(\lambda)$ has the value zero.

Then there exist numbers $c_1, c_2, \cdots, c_p (p \leq n;\ c_p \neq 0)$ such that the vector function

$$\Phi(\lambda) = \sum_{j=1}^{p} c_j \Phi_j(\lambda),$$

is a solution of the problem (6.2) which vanishes at λ_0. According to Lemma 6.1, the vector function

$$\hat{\Phi}(\lambda) = \frac{\lambda - i}{\lambda - \lambda_0} \Phi(\lambda)$$

also belongs to $\mathfrak{R}^+_{(n\times 1)}$ and is a solution of (6.2). In particular, at $\lambda = i$ the vector function $\hat{\Phi}(\lambda)$ has order [1] greater than κ_p.

Substituting in the matrix $\mathfrak{F}_+(\lambda)$ the column vectors of the vector function $\Phi_p(\lambda)$ by those of $\hat{\Phi}(\lambda)$ one obtains a solution of the matrix problem (6.6) which is regular at infinity and whose index is greater than κ, which is impossible.

The first assertion of the theorem is proved.

Let us now prove the second assertion. If at $\lambda = i$ the determinant of $\mathfrak{F}_+(\lambda)$ had order greater than $\kappa_1 + \kappa_2 + \cdots + \kappa_n$ then there would exist numbers $c_1, c_2, \cdots, c_p (p \leq n,\ c_p \neq 0)$ such that the vector function

$$\Psi(\lambda) = \sum_{j=1}^{p} c_j (\lambda - i)^{-\kappa_j} \Phi_j(\lambda) \qquad (6.7)$$

would vanish at the point $\lambda = i$. Multiplying the equation (6.7) termwise by $(\lambda - i)^{\kappa_p}$, we find that the vector function

$$\hat{\Psi}(\lambda) = (\lambda - i)^{\kappa_p} \Psi(\lambda)$$

belonging to the class $\mathfrak{R}^+_{(n\times 1)}$, would be a solution of the problem (6.2) having, at $\lambda = i$, an order greater than κ_p. Replacing the column vectors of the matrix $\Phi_p(\lambda)$ by those of the matrix function $\hat{\Phi}(\lambda)$ one obtains a matrix function, regular at infinity, which is a solution of the matrix problem (6.6) and has index greater than κ, which is impossible.

Since, in view of what has already been proved

$$\det \mathfrak{F}_-(\lambda) = \det \mathfrak{M}(\lambda) \cdot \det \mathfrak{F}_+(\lambda) \neq 0 \qquad (-\infty \leqslant \lambda \leqslant \infty)$$

and that

$$\mathrm{ind}\det \mathfrak{F}_-(\lambda) = \mathrm{ind}\det \mathfrak{M}(\lambda) + \mathrm{ind}\det \mathfrak{F}_+(\lambda),$$

to be complete the proof of the theorem it only remains to prove that $\det \mathfrak{F}_-(\lambda)$ is never zero for $\mathrm{Im}\,\lambda < 0$.

Suppose the contrary, that is, that for some $\lambda_0 (\mathrm{Im}\,\lambda_0 < 0)$ one has $\det \mathfrak{F}_-(\lambda_0) = 0$.

1) The order of $\Phi(\lambda)$ at $\lambda = \lambda_0$ is zero if $\Phi(\lambda_0) \neq 0$, and is equal to the multiplicity of the zero of the vector function $\Phi(\lambda)$ at $\lambda = \lambda_0$, when $\Phi(\lambda_0) = 0$.

Then there exist numbers $c_1, c_2, \cdots, c_p (p \leq n;\ c_p \neq 0)$ such that the vecto function

$$\Omega(\lambda) = \sum_{j=1}^{p} c_j \Psi_j(\lambda),$$

constructed from the column vectors of the matrix $\mathfrak{F}_-(\lambda)$, vanishes at λ_0. Sinc the vector function $(\lambda - i)(\lambda - \lambda_0)^{-1}\Omega(\lambda)$ belongs to the class $\mathfrak{R}^-_{(n\times 1)}$, the ve tor function

$$X(\lambda) = \frac{\lambda - i}{\lambda - \lambda_0} \sum_{j=1}^{p} c_j \Phi_j(\lambda),$$

also belongs to $\mathfrak{R}^-_{(n\times 1)}$, and is a solution of the problem (6.2). In particular, X has order greater than κ_p at $\lambda = i$. Replacing, in the matrix function $\mathfrak{F}_+(\lambda)$, th vector columns of $\Phi_p(\lambda)$ by the vector columns of $X(\lambda)$, we arrive at a contradiction, similarly to what was done previously. The theorem is proved.

Theorem 6.2. *Every vector function* $\Phi_+(\lambda)$ *of the form*

$$\Phi_+(\lambda) = \sum_{j=1}^{n} (a_{j0} + a_{j1}(\lambda - i)^{-1} + \cdots + a_{j\kappa_j}(\lambda - i)^{-\kappa_j}) \Phi_j(\lambda), \qquad (6.$$

where the a_{jk} *are arbitrary complex numbers, and the* $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, n)$ *a column vectors of a standard solution* $\mathfrak{F}_+(\lambda)$ *of the matrix function* (6.6), *is a solution of the problem* (6.2).

Conversely, every solution $\Phi_+(\lambda)$ *of the problem* (6.2) *has the form* (6.8), *consequently, belongs to* $\mathfrak{R}^+_{(n\times 1)}$.

Proof. The truth of the first assertion of the theorem is obvious. Let us ceed to the proof of the second one. Suppose that $\Phi_+(\lambda)$ is a solution of the lem (6.2). In view of (6.6) the equality

$$\mathfrak{M}(\lambda)\Phi_+(\lambda) = \Phi_-(\lambda)$$

is equivalent to the following:

$$\mathfrak{F}_+^{-1}(\lambda)\Phi_+(\lambda) = \mathfrak{F}_-^{-1}(\lambda)\Phi_-(\lambda).$$

From the preceding relations it follows that the vector function $\mathfrak{F}_+^{-1}(\lambda)\Phi$ may be continued analytically to the whole lower half plane Π_-. At all points the half plane Π_+, with the exception of $\lambda = i$, the matrix function $\mathfrak{F}_+^{-1}(\lambda)$ is holomorphic, and at $\lambda = i$ its jth row vector has a pole of order $\kappa_j (j = 1, 2, \cdots$

Consequently, the vector function $\mathfrak{F}_+^{-1}(\lambda)\Phi_+(\lambda)$ is holomorphic througho the half plane Π_+, with the exception of $\lambda = i$, and at $\lambda = i$ its jth coordinat has a pole of order not greater than $\kappa_j (j = 1, 2, \cdots, n)$. Hence

$$\mathfrak{F}_+^{-1}(\lambda)\Phi_+(\lambda) = \mathcal{P}\left(\frac{1}{\lambda - i}\right),$$

where $\mathcal{P}(z)$ is a polynomial vector function, whose coordinates are polynomia of degree $\leq \kappa_j (j = 1, 2, \cdots, n)$.

Thus the solution $\Phi_+(\lambda)$ may be represented in the form

$$\Phi_+(\lambda) = \mathfrak{F}_+(\lambda)\mathcal{P}\left(\frac{1}{\lambda - i}\right),$$

which is just another way of writing (6.8).

The theorem is proved.

Theorem 6.3. *Any solution $\Phi_+(\lambda)$ of the problem (6.2), which has order m at $\lambda = i$, where $\kappa_p \geq m > \kappa_{p+1}$, $(1 \leq p \leq n;\ \kappa_{n+1} = 0)$ must be of the form*

$$\Phi_+(\lambda) = \sum_{j=1}^{p} (a_{j0} + a_{j1}(\lambda - i)^{-1} + \ldots + a_{j\kappa_j - m}(\lambda - i)^{-\kappa_j + m})\,\Phi_j(\lambda), \qquad (6.9)$$

where $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, m)$—are the column vectors of an arbitrary standard solution of the matrix problem (6.6).

Proof. Suppose that $\Phi_+(\lambda)$ is a solution of the problem (6.2), having order m at $\lambda = i$, where $\kappa_p \geq m \geq \kappa_{p+1}$. Consider the vector functions $\hat{\Phi}_j(\lambda)$, defined by the equalities

$$\hat{\Phi}_j(\lambda) = (\lambda - i)^{-\kappa_j}\,\Phi_j(\lambda) \qquad (j = 1, 2, \ldots, n).$$

It is obvious that the vector functions $\hat{\Phi}_j(\lambda)$ $(j = 1, 2, \cdots, n)$ belong to class $\mathfrak{R}^+_{(n \times 1)}$, because the determinant of which they are column vectors never vanishes on the upper half plane.

According to Theorem 6.2 the vector function $\Phi_+(\lambda)$ may be represented as follows

$$\Phi_+(\lambda) = \sum_{j=1}^{n} (a_{j0}(\lambda - i)^{\kappa_j} + a_{j1}(\lambda - i)^{\kappa_j - 1} + \ldots + a_{j\kappa_j})\,\hat{\Phi}_j(\lambda). \qquad (6.10)$$

Putting $\lambda = i$ in this equation, one obtains

$$\sum_{j=1}^{n} a_{j\kappa_j}\hat{\Phi}_j(i) = 0,$$

from which one has that

$$a_{j\kappa_j} = 0 \qquad (j = 1, 2, \ldots, n).$$

Dividing both sides of equation (6.10) by $(\lambda - i)$ and then setting $\lambda = i$, one obtains

$$a_{j\kappa_j - 1} = 0 \qquad (j = 1, 2, \ldots, n).$$

Repeating this operation m times, one obtains

$$\Phi_+(\lambda) = \sum_{j=1}^{n} (a_{j0}(\lambda - i)^{\kappa_j - m} + a_{j1}(\lambda - i)^{\kappa_j - m - 1} + \ldots + a_{j\kappa_j - m})\,\hat{\Phi}_j(\lambda),$$

which yields (6.9). The theorem is proved.

Remark. According to Theorem 6.1, any standard solution $\mathfrak{F}_+(\lambda)$ of the matrix problem (6.6) has the property that the multiplicity of the zero at $\lambda = i$ of the determinant $\det \mathfrak{F}_+(\lambda)$ is equal to minus $\operatorname{ind} \det \mathfrak{M}(\lambda)$.

It is to be observed that from any solution $\mathfrak{F}_+(\lambda)$ of the matrix problem (6.6) which is regular at infinity, and whose determinant has a zero, of multiplicity $-\operatorname{ind} \det \mathfrak{M}(\lambda)$, at $\lambda = i$, one may, by a simple sequence of operations, obtain a standard solution of (6.6).

Indeed, without loss of generality, it may be supposed that the columns $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, n)$ of the matrix function $\mathfrak{F}_+(\lambda)$ are so ordered that their respective orders k_j $(j = 1, 2, \cdots, n)$ at $\lambda = i$ are non-increasing.

If

$$\sum_{j=1}^{n} k_j = -\operatorname{ind} \det \mathfrak{M}(\lambda), \qquad (6.$$

then it is easy to show that $\mathfrak{F}_+(\lambda)$ is a standard solution of the problem (6.6). If, on the other hand, $\Sigma k_j < -\operatorname{ind} \det \mathfrak{M}(\lambda)$, then there exist numbers $c_1, c_2, \cdot \cdots, c_p$ $(p \leq n;\ c_p \neq 0)$ such that the vector function

$$\Phi(\lambda) = \sum_{j=1}^{p} c_j(\lambda - i)^{-k_j}\,\Phi_j(\lambda) \qquad (6.$$

vanishes at $\lambda = i$. Multiplying equation (6.12) by $(\lambda - i)^{k_p}$ it follows that the tor function

$$\hat{\Phi}(\lambda) = (\lambda - i)^{k_p}\,\Phi(\lambda) \qquad (\in \mathfrak{R}^+_{(n \times 1)}),$$

is a solution of the problem (6.2), having order $\hat{k}_p > k_p$ at $\lambda = i$. Replacing, ir matrix function $\mathfrak{F}_p(\lambda)$, the columns of $\Phi_p(\lambda)$ by the columns of $\hat{\Phi}(\lambda)$, one obt a solution $\mathfrak{F}_+(\lambda)$, regular at infinity, of the problem (6.6), for which $\Sigma\hat{k}_j$ is gre than before. If it equals $-\operatorname{ind} \det \mathfrak{M}(\lambda)$ then it is the desired standard solutio

If it is still less than the number in question; then, repeating the mention operation a finite number $(\leq -\operatorname{ind} \det \mathfrak{M}(\lambda) - \Sigma k_j)$ of times, one finally arrive a standard solution of the problem (6.6).

§7. Fundamental standard factorization theorem. 1. Definition: *A left s ard factorization* of a nonsingular continuous matrix function $\mathfrak{M}(\lambda)$ $(-\infty < \lambda < \infty$ is a representation of this matrix in the form

$$\mathfrak{M}(\lambda) = \mathfrak{N}_+(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{N}_-(\lambda), \qquad (7$$

where $\mathfrak{D}(\lambda)$ is a diagonal matrix function

$$\mathfrak{D}(\lambda) = \left\| \left(\frac{\lambda - i}{\lambda + i}\right)^{\kappa_j} \delta_{jk} \right\|_1^n,$$

$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ are certain integers, and the matrix functions $\mathfrak{N}_\pm(\lambda)$ admit analytic continuations, holomorphic in the interior of and continuous up to the boundary of, the corresponding half planes $\Pi_\pm$, such that the determinants of these continuations is not zero:

$$\det \mathfrak{N}_+(\lambda) \neq 0\ (\lambda \in \Pi_+), \quad \det \mathfrak{N}_-(\lambda) \neq 0 \qquad (\lambda \in \Pi_-).$$

If the factors $\mathfrak{N}_\pm(\lambda)$ are interchanged in (7.1), then the resultant factoriz of the matrix function $\mathfrak{M}(\lambda)$ will be called a right standard factorization.

The factors $\mathfrak{D}(\lambda)$ in the left and right standard factorization will be refer to as *diagonal factors*.

It is readily seen that each left (right) factorization of a matrix function $\mathfrak{M}$ generates a right (left factorization of the matrix functions $\mathfrak{M}(-\lambda)$, $\mathfrak{M}^1(\lambda)$, $\mathfrak{M}^{-1}$

1) Indeed, all the considerations involved in the proof of Theorem 6.11 may be us in dealing with a matrix function $\mathfrak{F}_+(\lambda)$ which possesses property (6.11). In view of t the equality $\mathfrak{M}^{-1}(\lambda) = \mathfrak{F}_+(\lambda)\,\mathfrak{F}_-^{-1}(\lambda)$ immediately yields a left standard factorization (the next paragraph) of the matrix $\mathfrak{M}^{-1}(\lambda)$ with left indices k_j $(j = 1, 2, \cdots, n)$.

From Theorem 7.1 concerning the independence of the indices relative to the fact zation it follows that $k_j = \kappa_j$ $(j = 1, 2, \cdots, n)$.

obtained respectively by replacing λ by $-\lambda$ in (7.1), taking the transpose of both sides of (7.1), and taking the inverse of both sides of (7.1).

One has the following

Theorem 7.1. *Suppose that the matrix function $\mathfrak{M}(\lambda)$ $(\in \mathfrak{R}_{(n\times n)})$ has a left (right) factorization. Then all the left (right) standard factorizations of the matrix function $\mathfrak{M}(\lambda)$ possess exactly the same diagonal factor.*

Proof. Let us suppose that the following are two left standard factorizations of the matrix function $\mathfrak{M}(\lambda)$:

$$\mathfrak{M}(\lambda)=\mathfrak{N}_+(\lambda)\mathfrak{D}(\lambda)\mathfrak{N}_-(\lambda) \tag{7.2}$$

and

$$\mathfrak{M}(\lambda)=\widetilde{\mathfrak{N}}_+(\lambda)\widetilde{\mathfrak{D}}(\lambda)\widetilde{\mathfrak{N}}_-(\lambda), \tag{7.3}$$

where

$$\mathfrak{D}(\lambda)=\left\| \left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_p}\delta_{pq}\right\|_1^n \quad\text{and}\quad \widetilde{\mathfrak{D}}(\lambda)=\left\| \left(\frac{\lambda-i}{\lambda+i}\right)^{\widetilde{\varkappa}_p}\delta_{pq}\right\|_1^n.$$

From equations (7.2) and (7.3) it follows that

$$\mathfrak{Q}_+(\lambda)\mathfrak{D}(\lambda)=\widetilde{\mathfrak{D}}(\lambda)\mathfrak{Q}_-(\lambda), \tag{7.4}$$

where

$$\mathfrak{Q}_+(\lambda)=\widetilde{\mathfrak{N}}_+^{-1}(\lambda)\mathfrak{N}_+(\lambda), \qquad \mathfrak{Q}_-(\lambda)=\mathfrak{N}_-(\lambda)\widetilde{\mathfrak{N}}_-^{-1}(\lambda). \tag{7.5}$$

Equation (7.4) is equivalent to

$$q_{pq}^+(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_q}=\left(\frac{\lambda-i}{\lambda+i}\right)^{\widetilde{\varkappa}_p}q_{qp}^-(\lambda) \qquad (p,\ q=1,\ 2,\ \ldots,\ n),$$

where the functions $q_{pq}^{\pm}(\lambda)$ are elements of the matrices $\Omega_{\pm}(\lambda)$.

In all cases, when $\kappa_q>\widetilde{\kappa}_q$, one has the equality

$$q_{pq}^+(\lambda)=q_{pq}^-(\lambda)=0 \qquad (-\infty<\lambda<\infty).$$

Indeed, the equations

$$q_{pq}^+(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_q-\widetilde{\varkappa}_p}=q_{pq}^-(\lambda) \qquad (p,\ q=1,\ 2,\ \ldots,\ n)$$

imply that the functions

$$q_{pq}^+(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_q-\widetilde{\varkappa}_p}, \tag{7.6}$$

are holomorphic in the upper half plane and bounded at infinity, and may be continued analytically to the whole lower half plane. Hence the functions (7.6) must reduce to constants. In view of the fact that they vanish for $\lambda=i$, we obtain $q_{pq}^+(\lambda)\equiv q_{pq}^-(\lambda)\equiv 0$.

Suppose that κ_r is the largest number among the numbers $\kappa_1, \kappa_2, \cdots, \kappa_n$ which is not equal to the corresponding one of the numbers $\widetilde{\kappa}_1, \widetilde{\kappa}_2, \cdots, \widetilde{\kappa}_n$ that is

$$\varkappa_j=\widetilde{\varkappa}_j \quad (j=1,\ 2,\ \ldots,\ r-1) \text{ and } \varkappa_r\neq\widetilde{\varkappa}_r.$$

Without loss of generality it may be supposed that $\kappa_r>\widetilde{\kappa}_r$. Then it follows easily that the inequality $\kappa_q>\widetilde{\kappa}_q$ holds for all subscripts p and q satisfying

$$p=r,\ r+1,\ r+2,\ \ldots,\ n; \qquad q=1,\ 2,\ \ldots,\ r$$

and that

$$q_{pq}^+(\lambda)\equiv 0 \qquad (p=r,\ r+1,\ \ldots,\ n;\quad q=1,\ 2,\ \ldots,\ r).$$

From the preceding equations we obtain that every minor of order r, constr out of the first r columns of the matrix $q_+(\lambda)$ is identically zero. Thus, by Lapl determinant expansion,

$$\det\mathfrak{Q}_+(\lambda)\equiv 0,$$

which is impossible. Consequently

$$\varkappa_j=\widetilde{\varkappa}_j \quad (j=1,\ 2,\ \ldots,\ n) \text{ and } \mathfrak{D}(\lambda)=\widetilde{\mathfrak{D}}(\lambda).$$

The theorem proved implies that if a nonsingular matrix function $\mathfrak{M}(\lambda)$ po sesses a left (right) standard factorization, then the exponents $\kappa_j (j=1,2,\cdots$ are uniquely determined by the matrix $\mathfrak{M}(\lambda)$. The exponents $\kappa_1, \kappa_2, \cdots, \kappa_n$ w be called, respectively, the left (right) factorization indices (or exponents) of matrix function $\mathfrak{M}(\lambda)$.

In section 11 it will be shown, by a consideration of the standard factoriz of triangular matrices, that in general the left and right indices of a matrix do coincide.

On the other hand, from equation (7.1) it follows that always

$$\sum_{j=1}^{n}\varkappa_j=\operatorname{ind}\det\mathfrak{M}(\lambda).$$

2. In the course of the proof of the inequalities $\kappa_j=\widetilde{\kappa}_j$ $(j=1,2,\cdots,n)$ was shown in passing that the equations

$$q_{pq}^+(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_q-\varkappa_p}=q_{pq}^-(\lambda) \qquad ($$

imply that

$$q_{pq}^+(\lambda)\equiv q_{pq}^-(\lambda)\equiv 0.$$

It is to be observed that if $\kappa_p=\kappa_q$, then (7.7) implies that the functions $q_{pq}^{\pm}(\lambda)$ reduce to constants.

If, finally, $\kappa_p<\kappa_q$, then from (7.7) it follows that the functions appearin the left-hand side of (7.7) are holomorphic in the whole upper half plane, with exception of the point $\lambda=i$, at which they have a pole of order $\leq\kappa_p-\kappa_q$, an may be continued analytically to the whole lower half plane. Thus, they must polynomials in $(\lambda-i)^{-1}$ of degree $\leq\kappa_p-\kappa_q$. Hence

$$q_{pq}^+(\lambda)=q_{pq}\left(\frac{1}{\lambda+i}\right),$$

where $q_{pq}(z)$ is a polynomial of degree $\leq\kappa_p-\kappa_q$.

Taking equation (7.5) into account, this leads to the:

Theorem 7.2. *If the nonsingular continuous matrix function* $\mathfrak{M}(\lambda)$ $(-\infty<\lambda<\infty)$ *possesses a left standard factorization* (7.1), *then the most general factors* $\widetilde{\mathfrak{N}}_{\pm}(\lambda)$ *in such a factorization are given by the formulas*

$$\widetilde{\mathfrak{N}}_{+}(\lambda)=\mathfrak{N}_{+}(\lambda)\Omega(\lambda),\quad \widetilde{\mathfrak{N}}_{-}(\lambda)=\Omega^{-1}(\lambda)\mathfrak{N}_{-}(\lambda), \tag{7.8}$$

where $\Omega(\lambda)$ *is an arbitrary nonsingular matrix function, whose elements* $q_{jk}(\lambda)$ $(j, k=1, 2, \cdots, n)$ *satisfy the following conditions:*

1) $q_{jk}(\lambda)\equiv 0$ *if* $\kappa_k>\kappa_j$,
2) $q_{jk}(\lambda)$ *is constant if* $\kappa_k=\kappa_j$
3) $q_{jk}(\lambda)$ *is a polynomial in* $(\lambda+i)^{-1}$ *of degree* $\leq\kappa_j-\kappa_k$, *if* $\kappa_k<\kappa_j$.

Thus, the matrix $\Omega(\lambda)$ has the form

$$\Omega(\lambda)=\begin{Vmatrix} Q_1 & 0\ldots 0\\ * & Q_2\ldots 0\\ \cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ * & *\ldots Q_n\end{Vmatrix},$$

where Q_j $(j=1, 2, \cdots, n)$ is a nonsingular constant square matrix, the asterisks denote the places occupied by matrices with elements which are polynomials in $(\lambda+i)^{-1}$, of the appropriate degrees; and, finally, the determinant of $\Omega(\lambda)$ is the number

$$\det\Omega(\lambda)=\det Q_1\det Q_2\ldots\det Q_n.$$

Let us notice that the considerations involved in proving the theorem have yielded the fact that if (7.2) and (7.3) are left standard factorizations of $\mathfrak{M}(\lambda)$, then the factors $\mathfrak{N}_{\pm}(\lambda)$ are connected by (7.8). It is easily verified that, no matter what the nonsingular matrix $\Omega(\lambda)$, satisfying the specified conditions, one obtains from any left standard factorization (7.2) a new left standard factorization by means of (7.8).

An analogous theorem holds for the right standard factorization.

3. The following important theorem holds

Theorem 7.3. *Every nonsingular matrix function* $\mathfrak{M}(\lambda)\in\mathfrak{R}_{(n\times n)}$ *possesses a left (right) standard factorization and for each such factorization the factors* $\mathfrak{N}_{\pm}(\lambda)\in\mathfrak{R}^{\pm}_{(n\times n)}$.

Proof. Without loss of generality we may suppose that $\mathfrak{M}(\infty)=I$. According to the Theorem W of N. Wiener, the matrix function $\mathfrak{M}^{-1}(\lambda)$ may be represented in the form

$$\mathfrak{M}^{-1}(\lambda)=I-\mathscr{K}(\lambda),$$

where $\mathscr{K}(\lambda)\in\mathfrak{R}^{0}_{(n\times n)}$.

Consider first the case when the equation

$$\psi(t)-\int_0^\infty k'(t-s)\psi(s)\,ds=0\qquad(0\leqslant t<\infty) \tag{B}$$

has only the zero solution in the space $E^{+}_{(n\times 1)}$.

Let $\mathfrak{B}_{+}(\lambda)$ $(\in\mathfrak{R}^{+}_{(n\times n)})$ denote an arbitrary standard solution of the matrix Hilbert problem

$$\mathfrak{M}(\lambda)\mathfrak{B}_{+}(\lambda)=\mathfrak{B}_{-}(\lambda), \tag{7.}$$

by $\Phi_j(\lambda)$ $(j=1, 2, \cdots, n)$ the column vectors, and by $(\kappa_1, \kappa_2, \cdots, \kappa_n)$ the exp[onents] of such a solution.

The matrix function $\mathfrak{B}_{+}(\lambda)$ may be represented in the form

$$\mathfrak{B}_{+}(\lambda)=\mathfrak{N}_{+}(\lambda)\mathfrak{D}(\lambda),$$

where the matrix function $\mathfrak{N}_{+}(\lambda)$ belongs to the ring $\mathfrak{R}^{+}_{(n\times n)}$ and $\det\mathfrak{N}_{+}(\lambda)\neq 0$ $(\lambda\in\Pi_{+})$, and the matrix function $\mathfrak{D}(\lambda)$ is of the form

$$\mathfrak{D}(\lambda)=\left\Vert\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_p}\delta_{pq}\right\Vert_1^n.$$

By what was proved before, $\det\mathfrak{B}_{-}(\lambda)\neq 0$ $(\lambda\in\Pi_{-})$, consequently, accord[ing] to the Theorem W of Wiener which was recalled before, the matrix function $\mathfrak{N}_{-}($ $=\mathfrak{B}_{-}^{-1}(\lambda)\in\mathfrak{R}^{-}_{(n\times n)}$.

Using equation (7.9) we obtain easily a left standard factorization of the [ma]trix function $\mathfrak{M}(\lambda)=(I-\mathscr{K}(\lambda))^{-1}$:

$$\mathfrak{M}(\lambda)=\mathfrak{N}_{+}(\lambda)\mathfrak{D}(\lambda)\mathfrak{N}_{-}(\lambda)\qquad(-\infty\leqslant\lambda\leqslant\infty).$$

Let us now consider the general case. Let β denote the number of linear independent solutions of equation (B′).

Consider the matrix function $k_1(t)$, with elements in the space L, whose Fourier transform satisfies the equation

$$I-\mathscr{K}_1(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{\beta}(I-\mathscr{K}(\lambda)).$$

The equation

$$\chi(t)-\int_0^\infty k_1'(s-t)\chi(s)\,ds=0\qquad(0\leqslant t<\infty) \tag{7.}$$

unlike equation (B′), does not have nonzero solutions in the space $L^{+}_{(n\times 1)}$. [...] suppose that equation (7.10) did have a solution $\chi(\not\equiv 0)$, and apply the Fouri[er] transformation to both sides of equation (7.10); we obtain that

$$(I-\mathscr{K}_1'(-\lambda))X(\lambda)=\Omega_{-}(\lambda)\qquad(-\infty\leqslant\lambda\leqslant\infty;\ \Omega_{-}\in\mathfrak{R}^{-}_{(n\times 1)})$$

or, what is the same

$$(I-\mathscr{K}'(-\lambda))\left(\frac{\lambda-i}{\lambda+i}\right)^{\beta}X(\lambda)=\Omega_{-}(\lambda).$$

The vector functions

$$\Psi_j(\lambda)=\frac{(\lambda-i)^{\beta-j}}{(\lambda+i)^{\beta}}X(\lambda)\qquad(j=0, 1, 2, \ldots, \beta)$$

are linearly independent, belong to the ring $\mathfrak{R}^{0+}_{(n\times 1)}$, and satisfy

$$(I-\mathscr{K}'(-\lambda))\Psi_j(\lambda)=(\lambda-i)^{-j}\Omega_{-}(\lambda)\qquad(j=0, 1, 2, \ldots, \beta),$$

where $(\lambda-i)^{-j}\Omega_{-}(\lambda)\in\mathfrak{R}^{0-}_{(n\times 1)}$. The last relation is equivalent to

$$\int_0^\infty e^{i\lambda t}\left(\psi_j(t)-\int_0^\infty k'(s-t)\psi_j(s)\,ds\right)dt=0\qquad(j=0, 1, 2, \ldots, \beta).$$

Thus, equation (B′) has $\beta+1$ linearly independent solutions ψ_j $(j=0, 1, 2, \cdots, \beta)$, which is impossible.

According to what was shown above, the matrix function $(I-\mathcal{K}_1(\lambda))^{-1}$ possesses a left standard factorization:

$$(I-\mathcal{K}_1(\lambda))^{-1}=\mathfrak{N}_+(\lambda)\widetilde{\mathfrak{D}}(\lambda)\mathfrak{N}_-(\lambda).$$

Multiplying both sides of the last equation by $(\lambda-i)^{\beta}(\lambda+i)^{\beta}$, one obtains a left standard factorization of the matrix function $\mathfrak{M}(\lambda)$:

$$\mathfrak{M}(\lambda)=\mathfrak{N}_+(\lambda)\mathfrak{D}(\lambda)\mathfrak{N}_-(\lambda),$$

where

$$\mathfrak{D}(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{\beta}\widetilde{\mathfrak{D}}(\lambda).$$

To complete the proof of the theorem it remains only to remark that, in view of Theorem (7.2), the factors $\widetilde{\mathfrak{N}}_\pm(\lambda)$ of an arbitrary left standard factorization of the matrix function $\mathfrak{M}(\lambda)$ are related to the $\mathfrak{N}_\pm(\lambda)$ just constructed by the equations

$$\widetilde{\mathfrak{N}}_+(\lambda)=\mathfrak{N}_+(\lambda)\mathfrak{D}_+(\lambda), \qquad \widetilde{\mathfrak{N}}_-(\lambda)=\mathfrak{D}_-(\lambda)\mathfrak{N}_-(\lambda),$$

where $\mathfrak{D}_\pm(\lambda)$ is a rational matrix function belonging also to $\mathfrak{R}^{\pm}_{(n\times n)}$.

4. If all the left exponents of the nonsingular matrix function $\mathfrak{M}(\lambda)$ are zero, then the left factorization of $\mathfrak{M}(\lambda)$ has the form

$$\mathfrak{M}(\lambda)=\mathfrak{N}_+(\lambda)\mathfrak{N}_-(\lambda). \tag{7.11}$$

Obviously, in equation (7.11) the factor $\mathfrak{N}_+(\lambda)$ may be normalized by the requirement that

$$\mathfrak{N}_+(\infty)=I. \tag{7.12}$$

A left standard factorization (7.11), where the factor $\mathfrak{N}_+(\lambda)$ is normalized by (7.12), will be called a left canonical factorization of the matrix function $\mathfrak{M}(\lambda)$.

Similarly, a right standard factorization

$$\mathfrak{M}(\lambda)=\widetilde{\mathfrak{N}}_-(\lambda)\widetilde{\mathfrak{N}}_+(\lambda),$$

in which the factor $\widetilde{\mathfrak{N}}_-(\infty)$ is normalized by the requirement that

$$\mathfrak{N}_-(\infty)=I,$$

will be called a right canonical factorization.

The factors $\mathfrak{N}_\pm(\lambda)$ of a left (right) canonical factorization of a matrix function $\mathfrak{M}(\lambda)\in\mathfrak{R}_{(n\times n)}$ are uniquely determined.

Moreover, if the continuous nonsingular matrix function $\mathfrak{M}(\lambda)$ possesses a left canonical factorization (7.11), then for every one of its representations in the form

$$\mathfrak{M}(\lambda)=\mathfrak{G}_+(\lambda)\mathfrak{G}_-(\lambda) \qquad (-\infty<\lambda<\infty;\ \mathfrak{G}_+(\infty)=I),$$

where the matrix functions $\mathfrak{G}_\pm(\lambda)$ may be continued analytically so that they are holomorphic in the interior and continuous up to and including the boundary of the corresponding half planes $\Pi_\pm$, and such that at least one of these analytic continuations never vanishes, one must have

$$\mathfrak{G}_\pm(\lambda)=\mathfrak{N}_\pm(\lambda). \tag{7.13}$$

In fact, suppose for definiteness that $\det\mathfrak{G}_+(\lambda)\neq 0$ $(\lambda\in\Pi_+)$. Then the equation

$$\mathfrak{G}_+^{-1}(\lambda)\mathfrak{N}_+(\lambda)=\mathfrak{G}_-(\lambda)\mathfrak{N}_-^{-1}(\lambda) \qquad (-\infty<\lambda<\infty)$$

implies that the matrix function $\mathfrak{G}_+^{-1}(\lambda)\mathfrak{N}_+(\lambda)$, which is holomorphic in the upp half plane and bounded at infinity, possesses an analytic continuation to the wl lower half plane. Hence $\mathfrak{G}_+^{-1}(\lambda)\mathfrak{N}_+(\lambda)$ must be a constant matrix. Further, sinc $\mathfrak{G}_+^{-1}(\infty)\mathfrak{N}_+(\infty)=I$, one obtains that

$$\mathfrak{G}_+^{-1}(\lambda)\mathfrak{N}_+(\lambda)=\mathfrak{G}_-(\lambda)\mathfrak{N}^{-1}(\lambda)=I$$

which is (7.13).

§8. Factorization of hermitian and pseudopositive matrix functions. 1. B $\mathfrak{M}^*(\lambda)$ we shall denote the hermitian conjugate of the matrix function $\mathfrak{M}(\lambda)$, that $\mathfrak{M}^*(\lambda)=\overline{\mathfrak{M}'(\lambda)}$, where the bar above $\mathfrak{M}'(\lambda)$ stands for the operation of taking the complex conjugates of the elements of $\mathfrak{M}'(\lambda)$.

A matrix function $\mathfrak{H}(\lambda)$ is called Hermitian provided that $\mathfrak{H}(\lambda)=\mathfrak{H}^*(\lambda)$. As customary, the real part of the matrix function $\mathfrak{M}(\lambda)$ denotes the matrix

$$\mathfrak{M}_R(\lambda)=\frac{\mathfrak{M}(\lambda)+\mathfrak{M}^*(\lambda)}{2},$$

and the imaginary part of the matrix $\mathfrak{M}(\lambda)$ denotes the matrix

$$\mathfrak{M}_J(\lambda)=\frac{\mathfrak{M}(\lambda)-\mathfrak{M}^*(\lambda)}{2i}.$$

The matrix functions $\mathfrak{M}_R(\lambda)$, $\mathfrak{M}_J(\lambda)$ are, obviously, Hermitian.

A matrix function $\mathfrak{H}(\lambda)$ is called definite provided that for all values λ of closed real line $\{-\infty, \infty\}$ the quadratic form

$$\xi^*\mathfrak{H}(\lambda)\xi,$$

where ξ is an arbitrary n-dimensional column vector, and ξ^* is the correspon row vector with complex conjugate elements, has only real values $(\neq 0)$ havin constant sign.

Obviously, every definite matrix function is hermitian.

In order that the hermitian matrix function $\mathfrak{H}(\lambda)$ be definite it is necessar and sufficient that $\mathfrak{H}(\lambda)$ be nonsingular and that $\mathfrak{H}(\lambda)$ be definite in at least c point.

One has the following

Theorem 8.1. *If the real or the imaginary part of the nonsingular matrix fi tion $\mathfrak{M}(\lambda)$ $(\in\mathfrak{R}_{(n\times n)})$ is definite, then all left (right) exponents of the matrix are zero.*

Proof. For definiteness it will be supposed that the matrix function $\mathfrak{M}_R($ is pseudopositive, that is

$$\xi^*\mathfrak{M}_R(\lambda)\xi>0 \quad \text{for} \quad \xi\neq 0 \qquad (-\infty<\lambda<\infty). \qquad (\xi$$

The matrix function $\mathfrak{M}(\lambda)$ possesses a left standard factorization, and th

$$\mathfrak{M}(\lambda)\mathfrak{F}_-(\lambda)=\mathfrak{N}_+(\lambda)\mathfrak{D}(\lambda) \qquad (\mathfrak{F}_-(\lambda)=\mathfrak{N}_-^{-1}(\lambda))$$

or

$$\mathfrak{M}(\lambda)\,\mathfrak{f}_j(\lambda) = \mathfrak{n}_j(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_j} \qquad (j = 1, 2, \ldots, n), \tag{8.2}$$

where $\mathfrak{f}_j(\lambda)\,(\in \Re^-_{(n\times 1)})$ are the column vectors of the matrix $\mathfrak{F}_-(\lambda)$, and $\mathfrak{n}_j(\lambda)$ $(\in \Re^+_{(n\times 1)})$ are the column vectors of the matrix $\mathfrak{N}_+(\lambda)$. Multiplying equation (8.2) by the vector function $\mathfrak{f}^*_j(\lambda)\,(\in \Re^+_{(n\times 1)})$ one obtains that

$$\mathfrak{f}^*_j(\lambda)\,\mathfrak{M}(\lambda)\,\mathfrak{f}_j(\lambda) = \mathfrak{f}^*_j(\lambda)\,\mathfrak{n}_j(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_j} \qquad (j = 1, 2, \ldots, n). \tag{8.3}$$

All functional values of the functions

$$\mathfrak{f}^*_j(\lambda)\,\mathfrak{M}(\lambda)\,\mathfrak{f}_j(\lambda) = \mathfrak{f}^*_j(\lambda)\,\mathfrak{M}_R(\lambda)\,\mathfrak{f}_j(\lambda) + i\mathfrak{f}^*_j(\lambda)\,\mathfrak{M}_J(\lambda)\,\mathfrak{f}_j(\lambda)$$

are different from zero and lie in the right half plane, consequently

$$\operatorname{ind} \mathfrak{f}^*_j(\lambda)\,\mathfrak{M}(\lambda)\,\mathfrak{f}_j(\lambda) = 0.$$

From equation (8.3) it follows that

$$\operatorname{ind} \mathfrak{f}^*_j(\lambda)\,\mathfrak{n}_j(\lambda) = -\varkappa_j.$$

The left-hand side of the last equation is negative, because $\mathfrak{f}^*_j(\lambda)\,\mathfrak{n}_j(\lambda) \in \Re^+$. Therefore

$$\varkappa_j \leqslant 0 \qquad (j = 1, 2, \ldots, n). \tag{8.4}$$

Let $\zeta_j(\lambda)$ $(j = 1, 2, \cdots, n)$ denote the eigenvalues of the matrix $\mathfrak{M}(\lambda)$. As is well known from (8.1) it follows that all the numbers $\zeta_j(\lambda)$ lie strictly in the right half plane, and hence

$$\operatorname{ind} \zeta_j(\lambda) = 0 \qquad (j = 1, 2, \ldots, n).$$

Since

$$\det \mathfrak{M}(\lambda) = \prod_{j=1}^{n} \zeta_j(\lambda),$$

we conclude that

$$\operatorname{ind} \det \mathfrak{M}(\lambda) = 0. \tag{8.5}$$

However, on the other hand

$$\operatorname{ind} \det \mathfrak{M}(\lambda) = \sum_{j=1}^{n} \varkappa_j,$$

thus (8.4) and (8.5) imply that

$$\varkappa_j = 0 \qquad (j = 1, 2, \ldots, n).$$

The theorem is proved.

Theorem 8.2. *In order that the nonsingular matrix function* $\mathfrak{H}(\lambda) \in \Re_{(n\times n)}$ *possess a representation of the form*

$$\mathfrak{H}(\lambda) = \mathfrak{F}_+(\lambda)\,\mathfrak{F}^*_+(\lambda), \tag{8.6}$$

in which the matrix function $\mathfrak{F}_+(\lambda) \in \Re^+_{(n\times n)}$ *and* $\det \mathfrak{F}_+(\lambda) \neq 0\ (\lambda \in \Pi_+)$, *it is necessary and sufficient that* $\mathfrak{H}(\lambda)$ *be positive definite.*

Proof. The necessity is obvious. Let us proceed to the proof of the sufficiency. Suppose that the matrix function $\mathfrak{H}(\lambda)$ is positive definite, then according

to Theorem 8.1 all of its left exponents must be zero. Therefore $\mathfrak{H}(\lambda)$ possess a left canonical factorization

$$\mathfrak{H}(\lambda) = \mathfrak{N}_+(\lambda)\,\mathfrak{N}_-(\lambda) \qquad (\mathfrak{N}_+(\infty) = I, \quad \mathfrak{N}_-(\infty) = \mathfrak{H}(\infty)).$$

Consider the matrix functions $\mathfrak{F}_\pm(\lambda)$, defined by the equations

$$\mathfrak{F}_+(\lambda) = \mathfrak{N}_+(\lambda)\,A, \qquad \mathfrak{F}_-(\lambda) = A^{-1}\,\mathfrak{N}_-(\lambda),$$

where A is any matrix such that $A^2 = \mathfrak{H}(\infty)$.

Then, obviously

$$\mathfrak{H}(\lambda) = \mathfrak{F}_+(\lambda)\,\mathfrak{F}_-(\lambda), \qquad \mathfrak{F}_+(\infty) = \mathfrak{F}_-(\infty) = A. \tag{8}$$

Taking the conjugate of both sides of (8.7) we obtain that

$$\mathfrak{H}(\lambda) = \mathfrak{F}^*_-(\lambda)\,\mathfrak{F}^*_+(\lambda).$$

Comparing the last equation with (8.6) it follows that

$$\mathfrak{F}^*_+(\lambda)\,\mathfrak{F}^{-1}_-(\lambda) = \mathfrak{F}_+(\lambda)\,(\mathfrak{F}^*_-(\lambda))^{-1}.$$

Since the left-hand side of the last equation belongs to the ring $\Re^-_{(n\times n)}$, and t right-hand side belongs to the ring $\Re^+_{(n\times n)}$, it follows that each of them must e one and the same constant matrix. In view of the fact that $\mathfrak{F}^*_+(\infty)\,\mathfrak{F}^{-1}_-\infty = I$, w obtain the equation

$$\mathfrak{F}_-(\lambda) = \mathfrak{F}^*_+(\lambda) \quad \text{and} \quad \mathfrak{F}_+(\lambda) = \mathfrak{F}^*_-(\lambda),$$

which proves the theorem.

Remark. It is readily seen that in the representation (8.6) the factor $\mathfrak{F}_+(\lambda$ $\Re^+_{(n\times n)}$ is uniquely determined, up to multiplication on the right by a constant unitary matrix.

2. Let $\mathfrak{H}(\lambda) \in \Re_{(n\times n)}$ be a nonsingular hermitian matrix. It possesses a standard factorization

$$\mathfrak{H}(\lambda) = \mathfrak{N}_+(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{N}_-(\lambda).$$

Applying the operation of conjugation to both sides of the last equation, o obtains another factorization of the matrix function

$$\mathfrak{H}(\lambda) = \mathfrak{N}^*_-(\lambda)\,\mathfrak{D}^*(\lambda)\,\mathfrak{N}^*_+(\lambda).$$

Letting κ_j $(j = 1, 2, \cdots, n)$ be the left exponents of the matrix function $\mathfrak{H}$ and recalling that $\mathfrak{D}(\lambda) = \mathfrak{D}^*(\lambda)$, it follows that

$$\varkappa_j = -\varkappa_{n-j+1} \qquad (j = 1, 2, \ldots, n).$$

This implies that the number n_+ of positive left exponents of $\mathfrak{H}(\lambda)$ coinci with the number n_- of all negative left exponents of the same matrix function. analogous assertion is valid for the right exponents of the matrix function $\mathfrak{H}(\lambda$

Before formulating a theorem which gives bounds on the numbers $n_\pm$, ano remark is in order. Since the matrix function $\mathfrak{H}(\lambda)$ is nonsingular and depends tinuously on λ, the number of positive and negative square terms in the hermi form $\xi^*\,\mathfrak{H}(\lambda)\,\xi$ (ξ a column vector) does not depend upon λ. Let us denote the numbers, respectively, by p and q.

Theorem 8.3. *The number of left (right) exponents of one and the same si* *of the hermitian nonsingular matrix function* $\mathfrak{H}(\lambda)$ *does not exceed* $\min(p, q)$, *and hence the number of zero exponents of* $\mathfrak{H}(\lambda)$ *is not less than* $|p-q|$.

Proof. The matrix function $\mathfrak{H}^{-1}(\lambda)$ admits the left standard factorization

$$\mathfrak{H}^{-1}(\lambda)=\mathfrak{R}_+(\lambda)\mathfrak{D}(\lambda)\mathfrak{R}_-(\lambda),$$

in which the numbers n_+, n_- are the same as for the matrix $\mathfrak{H}(\lambda)$. Let us denote by $\Phi_j(\lambda)(\in \Re^+_{(n\times 1)})$ $(j=1, 2, \cdots, n_+)$ the first n column vectors of the matrix function $\mathfrak{R}_+(\lambda)\,\mathfrak{D}(\lambda)$.

Recall that the vector functions $\Phi_j(\lambda)$ $(j=1, 2, \cdots, n)$ are solutions of the problem

$$\mathfrak{H}(\lambda)\Phi_+(\lambda)=\Phi_-(\lambda),$$

having linear independent values of each real λ, and vanishing at $\lambda=i$.

Let $\mathcal{E}_n$ be an n dimensional vector space.

Define in $\mathcal{E}_n$ a scalar product by means of the formula

$$(x, y)=y^*\mathfrak{H}(\lambda_0)x \qquad (x, y\in\mathcal{E}_n),$$

where λ_0 is a fixed real number. With respect to the scalar product just introduced, the space $\mathcal{E}_n$ has an indefinite metric.

Let us show that

$$(\Phi_j(\lambda_0),\quad \Phi_k(\lambda_0))=0 \qquad (j, k=1, 2, \ldots, n_+). \tag{8.8}$$

Indeed, the function $\chi(\lambda)=\Phi_j^*(\lambda)\,\mathfrak{H}(\lambda)\,\Phi_k(\lambda)$ belongs to both the rings $\Re^+$ and $\Re^-$ since it may be represented in the form

$$\chi(\lambda)=\Phi_j^*(\lambda)\,(\mathfrak{H}(\lambda)\,\Phi_k(\lambda))=(\mathfrak{H}(\lambda)\,\Phi_j(\lambda))^*\,\Phi_k(\lambda),$$

where Φ_j^*, $\mathfrak{H}\Phi_k\in\Re^-_{(n\times 1)}$ and Φ_k, $\mathfrak{H}\Phi_j\in\Re^+_{(n\times 1)}$. Consequently, $\chi(\lambda)$ must be a constant. Recalling that $\Phi_j(i)=0$ $(j=1, 2, \cdots, n_+)$ we obtain that $\chi(\lambda)\equiv 0$. This implies equation (8.8).

Thus, the linear envelope of the vectors $\Phi_j(\lambda_0)$ $(j=1, 2, \cdots, n_+)$ contains an isotropic subspace of $\mathcal{E}_n$. Since the dimension of such an isotropic subspace must be $\le \min(p, q)$, one has that $n_+\le\min(p, q)$.

The proof of the following theorem is left to the reader.

Theorem 8.4. *In order that the matrix function* $\mathfrak{H}(\lambda)$, *satisfying the conditions of the previous theorem, admit a representation of the form*

$$\mathfrak{H}(\lambda)=\mathfrak{F}_+(\lambda)A\mathfrak{F}_+^*(\lambda),$$

where $\mathfrak{F}_+(\lambda)\in\Re^+_{(n\times n)}$, $\det\mathfrak{F}_+(\lambda)\neq 0$, $(\lambda\in\Pi_+)$, *and* A *is an arbitrary nonsingular constant hermitian matrix, whose associated quadratic form has p positive squares and q negative squares, it is necessary and sufficient that all the left exponents of* $\mathfrak{H}(\lambda)$ *be zero.*

Theorems 8.2, 8.3 and 8.4 and the method of proof of these theorems, in the case of hermitian matrix functions, whose elements are functions defined on the unit circle and satisfying a Hölder condition, were first published by Yu. L. Smul'yan [9] (in this connection see also section 14).

§9. **Homogeneous integral equation (B).** 1. The following concepts are ne for the exact formulation of our fundamental result concerning the integral equa

$$\varphi(t)-\int_0^\infty k(t-s)\varphi(s)\,ds=0 \qquad (0\leqslant t<\infty). \quad ($$

A sequence of vector functions $\phi_0(t), \phi_1(t), \cdots, \phi_{\nu-1}(t)$ from the space $L^+_{(n\times 1)}$ will be called a D-chain of length ν provided that

1) The vector functions $\phi_j(t)$ $(j=0, 1, 2, \cdots, \nu)$ are absolutely continuo in any finite interval;

2) $\phi_{j+1}(t)=\frac{d}{dt}\phi_j(t)$, $\phi_j(0)=0$ $(j=0, 1, 2, \cdots, \nu-2)$ and

3) $\phi_{\nu-1}(0)\neq 0$.

Condition 2) means that

$$\varphi_j(t)=\int_0^t\varphi_{j+1}(s)\,ds \qquad (j=0, 1, 2, \ldots, \nu-2).$$

If we set

$$\Phi_j(\lambda)=\int_0^\infty e^{i\lambda t}\varphi_j(t)\,dt \qquad (j=0, 1, 2, \ldots, \nu-1),$$

then it follows that

$$\Phi_j(\lambda)=(-i\lambda)^j\Phi_0(\lambda) \qquad (j=0, 1, 2, \ldots, \nu-1),$$

from which, in particular, it follows that the vector functions $\phi_j(t)$ $(j=0, 1, 2, \cdots, \nu$ are linearly independent.

In order to study the integral equation (B), let us introduce the function b $(b(-t)\in L^+_{(n\times 1)})$, defined by the equations

$$b(t)=-\int_0^\infty k(t-s)\varphi(s)\,ds \quad (-\infty<t<\infty) \text{ and } b(t)=0 \qquad (0\leqslant t<\infty$$

and define $\phi(t)=0$ $(-\infty<t<0)$. Obviously, now equation (B) may be rewritt

$$\varphi(t)-\int_{-\infty}^\infty k(t-s)\varphi(s)\,ds=b(t) \qquad (0\leqslant t<\infty).$$

Applying the Fourier transform to both sides of this equation, we obtain

$$(I-\mathscr{K}(\lambda))\,\Phi(\lambda)=\mathscr{B}(\lambda), \quad ($$

where the vector functions $\phi(\lambda)$ and $\mathcal{B}(\lambda)$ belong respectively to the rings $\Re^{0+}_{(n}$ and $\Re^{0-}_{(n\times 1)}$. These considerations are reversible; consequently, the set of th Fourier transforms of all solutions of equation (B) coincides with the set of a solutions $\phi(\lambda)$ $(\in\Re^+_{(n\times 1)})$ of the problem (9.1), satisfying the additional cond

$$\phi(\infty)=0.$$

Consider first the case when the homogeneous transposed equation

$$\psi(t)-\int_0^\infty k'(s-t)\psi(s)\,ds=0 \qquad (0\leqslant t<\infty)$$

has only the zero solution. Then, from Theorem 6.2, the general solution of problem (9.1) can be obtained from the formula

$$\Phi(\lambda)=\sum_{j=1}^{n}\left(a_0^{(j)}+a_1^{(j)}(\lambda-i)^{-1}+\dots+a_{\varkappa_j}^{(j)}(\lambda-i)^{\varkappa_j}\right)\Phi_j(\lambda),$$

where $a_k^{(j)}$ $(k=0,1,\cdots,\kappa_j;\ j=1,2,\cdots,n)$ are arbitrary complex numbers, $\Phi_j(\lambda)$ $(j=1,2,\cdots,n)$ are column vectors of an arbitrary standard solution of the matrix problem (6.6), and $\kappa_1,\kappa_2,\cdots,\kappa_n$ $(\kappa_j\geq 0)$ are the left exponents of the matrix function $(I-\mathcal{K}(\lambda))^{-1}$. Since the vectors $\Phi_j(\infty)$ $(j=1,2,\cdots,n)$ are linearly independent and

$$\Phi(\infty)=\sum_{j=1}^{n}a_0^{(j)}\Phi_j(\infty),$$

the condition $\Phi(\infty)=0$ will hold if and only if $a_0^{(j)}=0$ $(j=1,2,\cdots,n)$. Thus the Fourier transform of the general solution of equation (B) is of the form

$$\Phi(\lambda)=\sum_{j=1}^{n}\left(a_1^{(j)}(\lambda-i)^{-1}+\dots+a_{\varkappa_j}^{(j)}(\lambda-i)^{-\varkappa_j}\right)\Phi_j(\lambda),$$

where $a_k^{(j)}$ are arbitrary complex numbers.

The vector functions

$$\mathcal{G}_{jk}(\lambda)=i^k(\lambda-i)^{-k}\Phi_j(\lambda)\qquad(j=1,2,\dots,n;\quad k=1,2,\dots,\varkappa_j)$$

are linearly independent; and, therefore, the vector functions $g_{jk}(t)$ $(\in L^+_{(n\times 1)})$, defined by

$$\mathcal{G}_{jk}(\lambda)=\int_0^\infty g_{jk}(t)e^{i\lambda t}\,dt\qquad(j=1,2,\dots,n;\quad k=1,2,\dots,\varkappa_j),$$

form a basis for all solutions of equation (B). The vector functions $g_{jk}(t)$ $(k=1,2,\cdots,\kappa_j)$ are related to the functions $g_j(t)$ defined by the equations

$$\Phi_j(\lambda)=a_j+\int_0^\infty g_j(t)e^{i\lambda t}\,dt\qquad(j=1,2,\dots,n),$$

by the formulas

$$\left(\frac{d}{dt}-1\right)g_{j1}(t)=g_j(t),\quad g_{j1}(0)=a_j,$$

$$\left(\frac{d}{dt}-1\right)g_{jk}(t)=g_{jk-1}(t),\quad g_{jk}(0)=0\qquad(k=2,3,\dots,\varkappa_j).$$

In these equations the subscript j takes on integral values from 1 to n.

Besides the vector functions $g_{jk}(t)$ $(j=1,2,\cdots,n;\ k=1,2,\cdots,\kappa_j)$, another basis set of solutions of equation (B) is the set of functions

$$\varphi_{j0}(t)=g_{j\varkappa_j},\quad \varphi_{j1}(t)=\varphi_{j0}'(t)=g_{j\varkappa_j}(t)+g_{j\varkappa_j-1}(t),\ \dots,$$

$$\varphi_{j\varkappa_j-1}(t)=\varphi_{j0}^{(\varkappa_j-1)}(t)=g_{j\varkappa_j}(t)+\binom{\varkappa_j-1}{1}g_{j\varkappa_j-1}(t)+\dots+g_{j1}(t)$$

$$(j=1,2,\dots,n).$$

Since, in view of Theorem 5.1, the vector functions

$$\phi_{jk}(t)\ (j=1,2,\cdots,n;\ k=0,1,\cdots,\kappa_j-1)$$

all belong to the space $E^+_{(n\times 1)}$, one has that

$$\varphi_{jk-1}(t)=\int_0^t\varphi_{jk}(s)\,ds=-\int_t^\infty\varphi_{jk}(s)\,ds\quad(k=1,2,\dots,\varkappa_j-1;\quad j=1,2,\dots,$$

and

$$\varphi_{j\varkappa_j-1}(t)=a_j+\int_0^t\varphi_{j\varkappa_j}(s)\,ds=-\int_t^\infty\varphi_{j\varkappa_j}(s)\,ds\qquad(j=1,2,\dots,n),$$

where

$$\varphi_{j\varkappa_j}(t)=g_{j\varkappa_j}(t)+\binom{\varkappa_j}{1}g_{j\varkappa_j-1}(t)+\dots+g_j(t).$$

Thus, each vector function $\Phi_j(\lambda)$ $(j=1,2,\cdots,n)$ for which $\kappa_j>0$, genera a D-chain $\phi_{j0},\phi_{j1},\cdots,\phi_{j\kappa_j-1}$ of length κ_j of solutions of equation (B).

Let us now consider the general case, when equation (B) has $\beta(\neq 0)$ linea independent solutions. Obviously, the left exponent κ_n of the matrix function $I-\mathcal{K}(\lambda)$ is negative.

Let us adjoin to problem (9.1) the problem

$$(I-\mathcal{K}(\lambda))\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_n}\widetilde{\Phi}(\lambda)=\mathcal{F}(\lambda).\qquad(9.$$

Every solution $\Phi(\lambda)\in\Re^+_{(n\times 1)})$ of the problem (9.1) generates a solution

$$\widetilde{\Phi}(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{-\varkappa_n}\Phi(\lambda)\qquad(9.$$

of the problem (9.2). The converse is not always true. A solution $\widetilde{\Phi}(\lambda)(\in\Re^+_{(n\times}$ of problem (9.2) generates, by means of formula (9.3), a solution $\Phi(\lambda)\in\Re^+_{(n\times 1}$ of problem (9.1) if and only if $\widetilde{\Phi}(\lambda)$ has a zero of multiplicity not less than $-\kappa$ at $\lambda=i$.

The left exponents of the matrix function $\left(\frac{\lambda-i}{\lambda+i}\right)^{-\kappa_n}(I-\mathcal{K}(\lambda))^{-1}$ are precis the non-negative numbers $\kappa_j-\kappa_n$ $(j=1,2,\cdots,n)$. Let us assume that among th numbers κ_j $(j=1,2,\cdots,n)$ there is at least one that is positive; denote by κ_p the smallest of such positive numbers. Then the general solution $\Phi(\lambda)$ of the p lem (9.1), which vanishes at infinity and has a zero of multiplicity not less tha $-\kappa_n$ at $\lambda=i$, is given by

$$\widetilde{\Phi}(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{-\varkappa_n}\sum_{j=1}^{p}\left(a_1^{(j)}(\lambda-i)^{-1}+a_2^{(j)}(\lambda-i)^{-2}+\dots+a_{\varkappa_j}^{(j)}(\lambda-i)^{-\varkappa_j}\right)\Phi_j($$

Accordingly, the Fourier transform of the general solution of equation (B) h then the form

$$\Phi(\lambda)=\sum_{j=1}^{p}\left(a_1^{(j)}(\lambda-i)^{-1}+a_2^{(j)}(\lambda-i)^{-2}+\dots+a_{\varkappa_j}^{(j)}(\lambda-i)^{-\varkappa_j}\right)\Phi_j(\lambda),$$

where $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, n)$ are vector columns of a standard solution of the problem

$$(I - \mathscr{K}(\lambda))\left(\frac{\lambda - i}{\lambda + i}\right)^{\varkappa n} \mathfrak{F}_+(\lambda) = \mathfrak{F}_-(\lambda)$$

or the vector columns of a matrix function $\mathfrak{G}_+(\lambda)$, related to the factors in the left standard factorization of the matrix function $(I - \mathcal{K}(\lambda))^{-1}$,

$$(I - \mathscr{K}(\lambda))^{-1} = \mathfrak{R}_+(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{R}_-(\lambda),$$

by means of the formula

$$\mathfrak{G}_+(\lambda) = \mathfrak{R}_+(\lambda)\left\| \left(\frac{\lambda - i}{\lambda + i}\right)^{\max(\varkappa_j,\, 0)} \right\|_1^n .$$

Proceeding in this fashion, as was done in the case $\beta = 0$, we can construct, starting from the functions $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, p)$ exactly p D-chains of solutions of equation (B), whose lengths are respectively κ_j $(j = 1, 2, \cdots, p)$.

We have thus

Theorem 9.1. *If among the left exponents of the matrix function* $(I - \mathcal{K}(\lambda))^{-1}$ $(\in \mathfrak{R}_{(n\times n)})$ *there is at least one which is positive, and* κ_p *is the smallest of these, then the equation*

$$\varphi(t) - \int_0^\infty k(t - s)\,\varphi(s)\,ds = 0 \qquad (0 \leqslant t < \infty)$$

has exactly

$$\alpha = \sum_{j=1}^{p} \varkappa_j$$

linearly independent solutions; which, suitably chosen, generate p D-chains of solutions of length κ_j $(j = 1, 2, \cdots, p)$.

An analogous assertion holds for the solutions of the equation

$$\psi(t) - \int_0^\infty k'(s - t)\,\psi(s)\,ds = 0 \qquad (0 \leqslant t < \infty),$$

the rôle of the numbers $\kappa_1, \kappa_2, \cdots, \kappa_p$ now being played by the numbers $-\kappa_n$, $-\kappa_{n-1}, \cdots, -\kappa_q$, where κ_q is the "first" of the negative left exponents of the matrix function $I - \mathcal{K}(\lambda)$.

2. The above construction of D-chains of solutions of equation (B) was obtained by means of a standard solution of the matrix problem (6.6), but such a construction can also be carried out, in the following way, independently of the standard solutions.

Suppose that $\mathfrak{Z}_\alpha$ is the α-dimensional linear set of Fourier transforms $\Phi(\lambda)$ of all solutions of the homogeneous equation (B).

Let us denote by k_1 the largest multiplicity of the zero of any vector function $\Phi(\lambda) \in \mathfrak{Z}_\alpha$ at $\lambda = i$. The subset of all vector functions $\Phi(\lambda) \in \mathfrak{Z}_\alpha$ having at $\lambda = i$ a zero of multiplicity not less than $p\,(\leq k_1)$, will be denoted by $\mathfrak{L}_{k_1 - p}$.

Then one has

$$\mathscr{L}_0 \subset \mathscr{L}_1 \subset \cdots \subset \mathscr{L}_{k_1 - 1} \subset \mathscr{L}_{k_1} = \mathfrak{Z}_\alpha,$$

because $\Phi(\lambda) \in \mathfrak{L}_j$, implies that $(\lambda - i)^{-1}\Phi \in \mathfrak{L}_{j+1}$.

Let us denote by $\Phi_{0j}(\lambda)$ $(j = 1, 2, \cdots, p_0)$ an arbitrary basis for the linea set $\mathfrak{L}_0$. By $\Phi_{1j}(\lambda)$ $(j = 1, 2, \cdots, p_1;\ p_1 \geq 2p_0)$ let us denote a basis for the li set $\mathfrak{L}_0$ such that the first p_0 of its vector functions coincide respectively wit the vector functions $\Phi_{0j}(\lambda)$ $(j = 1, 2, \cdots, p_0)$, the next p_0 vector functions Φ $(j = p_0 + 1, \cdots, 2p_0)$ satisfy the relations

$$\Phi_{1,\, j+p_0}(\lambda) = (\lambda - i)^{-1}\Phi_{0j}(\lambda) \qquad (j = 1, 2, \ldots, p_0),$$

and the remaining ones (if there are any) are chosen arbitrarily.

In the linear set $\mathfrak{L}_m$ $(m = 2, \cdots, k_1)$ let us choose a basis

$$\Phi_{mj}(\lambda) \ (j = 1, 2, \cdots, p_m;\ p_m \geq 2p_{m-1}),$$

whose first $2p_m$ vector functions satisfy the relations

a) $\Phi_{mj}(\lambda) = \Phi_{m-1,\, j}(\lambda) \quad (j = 1, 2, \ldots, p_{m-1}),$

б) $\Phi_{mj}(\lambda) = (\lambda - i)^{-1}\Phi_{m-1,\, j}(\lambda) \quad (j = p_{m-1} + 1, \ldots, 2p_{m-1}).$

If $p_m \geq 2p_{m-1}$, then the remaining vector functions $\Phi_{mj}(\lambda)$ $(j = 2p_{m-1} + 1, \cdots, p_{\prime}$ are arbitrary.

Obviously, each vector function $\Phi_{mj}(\lambda)$ which was left arbitrary in the pı ceding process generates a D-chain of solutions of equation (B).

It should be observed that a basis for the subspace $\mathfrak{Z}_\alpha$ may be split up i D-chains in various ways. The way chosen above is characterized by the pro that it yields a minimum number of D-chains.

Theorem 9.2. *In order that every D-chain of solutions of equation* (B) *c sist of not more than one vector function it is necessary and sufficient that al solutions* $\phi(t)$ *of equation* (B) *differ from zero at* $t = 0$.

Proof. In fact, if there exists at least one D-chain of solutions of equa (B) consisting of at least two functions, then, by definition, the first of these functions must vanish at $t = 0$.

Conversely, suppose that $\phi_0(t)$ $(\in E^+_{(n\times 1)})$ is a solution of equation (B) such that $\phi_0(0) = 0$. Since all solutions of equation (B) are absolutely contir ous, the vector function $\phi_0(t)$ may be represented in the form

$$\varphi_0(t) = \int_0^t \varphi_1(s)\,ds,$$

where $\phi_1(t)$ is a vector function belonging to the space $L^+_{(n\times n)}$.

Let us prove that the vector function $\phi_1(t)$ must be a solution of equati (B). Hence, in view of this there is a D-chain of solutions of equation (B) c taining at least the two solutions $\phi_0(t)$ and $\phi_1(t)$.

Changing variables in (B) yields

$$\varphi_0(t)-\int_{-\infty}^{t} k(u)\varphi_0(t-u)\,du=0 \qquad (0\leqslant t<\infty).$$

Differentiating the last equation, it follows that

$$\varphi_0'(t)-\int_{-\infty}^{t} k(u)\varphi_0'(t-u)\,du=0 \qquad (0\leqslant t<\infty),$$

which, in turn, means that

$$\varphi_0'(t)-\int_{0}^{\infty} k(t-s)\varphi_0'(s)\,ds=0 \qquad (0\leqslant t<\infty).$$

The theorem is proved.

Remark. From Theorem 9.1, in the scalar case, one has the following assertion, which was mentioned in the introduction.

If the function $k(t)\in L$ and the function $1-\mathcal{K}(\lambda)$ does not vanish on the closed axis, then:

1) If $\kappa=-\operatorname{ind}(1-\mathcal{K}(\lambda))\geq 0$, equation (B) has exactly κ linearly independent solutions in E^+, which may be chosen so as to form a single D-chain; equation (B′) then has only the zero solution and; finally, the nonhomogeneous equation (A) then has a solution for an arbitrarily given right-hand side.

2) $\kappa<0$, equation (B) has only the zero solution in E^+, and equation (B′) has exactly $|\kappa|$ linearly independent solutions, which form a single D-chain, if suitably chosen.

These results were first obtained in [1].

§10. **Stability theorems for systems of exponents.** 1. In the ring $\Re$ there is a natural definition of a norm. If

$$\mathfrak{F}(\lambda)=\mathfrak{F}(\infty)+\int_{-\infty}^{\infty} e^{i\lambda t}f(t)\,dt \quad (f\in L),$$

then one sets

$$\|\mathfrak{F}\|=\|\mathfrak{F}(\infty)\|+\|f(t)\|_L.$$

Clearly, $\|\mathfrak{F}\|\geq|\mathfrak{F}(\lambda)|$ for all $\lambda\in(-\infty,\infty)$. For any $\mathfrak{M}(\lambda)\in\Re_{(n\times n)}$ let us define

$$\|\mathfrak{M}\|=n\max_{j,k=1,2,\ldots,n}\|\mathfrak{m}_{jk}\|,$$

and then $\Re_{(n\times n)}$ will be a Banach space.

Further, $\Re_{(n\times n)}$ may be considered as a normed ring, because, as is readily seen, for any $\mathfrak{M}_1, \mathfrak{M}_2\in\Re_{(n\times n)}$:

$$\|\mathfrak{M}_1\cdot\mathfrak{M}_2\|\leqslant\|\mathfrak{M}_1\|\cdot\|\mathfrak{M}_2\|. \qquad (10.1)$$

It should be noticed that, if $g(\lambda)\in\Re$, $\mathfrak{M}\in\Re_{(n\times n)}$ then

Theorem 10.1. *Suppose that the nonsingular matrix function $\mathfrak{M}(\lambda)\in\Re_{(n}$ Then there exists a number $\delta(>0)$ such that any matrix function $\mathfrak{N}(\lambda)\in\Re_{(n}$ in the δ-neighborhood of $\mathfrak{M}:\|\mathfrak{M}-\mathfrak{N}\|<\delta$, is also nonsingular and for each i p its left (right) exponents satisfy*

$$\sum_{\varkappa_j(\mathfrak{M})>p}(\varkappa_j(\mathfrak{M})-p)\geqslant\sum_{\varkappa_j(\mathfrak{N})>p}(\varkappa_j(\mathfrak{N})-p). \qquad (1($$

Proof. Define

$$\mathfrak{M}_1(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{-p}\mathfrak{M}(\lambda),$$

where p is any integer satisfying the inequality

$$\varkappa_n(\mathfrak{M})\leqslant p\leqslant\varkappa_1(\mathfrak{M}). \qquad (1($$

Obviously

$$\varkappa_j(\mathfrak{M}_1)=\varkappa_j(\mathfrak{M})-p \quad (j=1,\ 2,\ \ldots,\ n).$$

Let us introduce the operator U, acting on $L^+_{(n\times 1)}$, defined by

$$U\varphi=A\varphi(t)-\int_0^{\infty}k(t-s)\varphi(s)\,ds \quad (0\leqslant t<\infty),$$

where

$$\mathfrak{M}_1^{-1}(\lambda)=A-\int_{-\infty}^{\infty}k(t)e^{i\lambda t}\,dt \quad (-\infty<\lambda<\infty).$$

In view of Theorem 9.1

$$\alpha(U)=\sum_{\varkappa_j(\mathfrak{M})>p}(\varkappa_j(\mathfrak{M})-p).$$

From proposition A) in section 1, there exists a number $\rho_p(>0)$, such th each linear operator $V(VL^+_{(n\times 1)}\subseteq L^+_{(n\times 1)})$ which satisfies

$$\|U-V\|<\rho_p, \qquad (1($$

also satisfies

$$\alpha(V)\leqslant\alpha(U). \qquad (1($$

Let $\delta(>0)$ denote a number which is smaller than all the quantities

$$\rho_p\left\|\left(\frac{\lambda-i}{\lambda+i}\right)^{-p}\right\|^{-1}(\|\mathfrak{M}_1^{-1}\|^2+\rho_p\|\mathfrak{M}_1^{-1}\|)^{-1} \quad (\varkappa_n(\mathfrak{M}_1)\leqslant p\leqslant\varkappa_1(\mathfrak{M}_1))$$

and so small that any matrix function in the δ-neighborhood of the matrix $\mathfrak{M}_1$ is nonsingular.

Let us show that δ satisfies the requirements made in the theorem.

Suppose that $\mathfrak{N}(\lambda)\in\Re_{(n\times n)}$ and satisfies $\|\mathfrak{M}-\mathfrak{N}\|<\delta$.

Then for any p in the interval (10.3) the matrix function $\mathfrak{N}_1(\lambda)=(\frac{\lambda-i}{\lambda+i})^{-p}$ is nonsingular and satisfies the condition

$$\|\mathfrak{M}_1-\mathfrak{N}_1\|<\eta=(\|\mathfrak{M}_1^{-1}\|^2+\rho_p\|\mathfrak{M}_1^{-1}\|)^{-1}\rho_p.$$

Consider the operator V, acting on the space $L^+_{(n\times 1)}$, defined by the equation

$$V\varphi=A_1\varphi(t)+\int^{\infty}k_1(t-s)\varphi(s)\,ds\quad(0\leqslant t<\infty),$$

where

$$\mathfrak{N}_1^{-1}(\lambda)=A_1+\int_{-\infty}^{\infty}e^{i\lambda t}k_1(t)\,dt\quad(k_1(t)\in L_{(n\times n)}).$$

According to the inequality (2.5): $\|U-V\|\leq\|\mathfrak{M}_1^{-1}(\lambda)-\mathfrak{N}_1^{-1}(\lambda)\|$, and since

$$\|\mathfrak{M}_1^{-1}(\lambda)-\mathfrak{N}_1^{-1}(\lambda)\|<\frac{\eta\|\mathfrak{M}_1^{-1}\|^2}{1-\eta\|\mathfrak{M}_1^{-1}\|}=\rho_p,$$

it follows that the operator V satisfies the inequality (10.4). Thus the operator V also satisfies the inequality (10.5). In view of Theorem 9.1, this relation is equivalent to (10.2). Hence inequality (10.2) holds for all integers p in the interval (10.3). In particular, for $p=\kappa_1(\mathfrak{M})$ and $p=\kappa_n(\mathfrak{M})$ this inequality furnishes

$$\varkappa_1(\mathfrak{M})\geqslant\varkappa_1(\mathfrak{N})\quad\text{and}\quad\varkappa_n(\mathfrak{M})\leqslant\varkappa_n(\mathfrak{N}).\tag{10.6}$$

Since the δ-neighborhood of the matrix function $\mathfrak{M}$ consists of nonsingular matrix functions, then for each matrix function $\mathfrak{N}$ in this neighborhood one has $\kappa(\mathfrak{M})=\kappa(\mathfrak{N})$, i.e.,

$$\sum_{j=1}^{n}\varkappa_j(\mathfrak{M})=\sum_{j=1}^{n}\varkappa_j(\mathfrak{N}).$$

Together with (10.6) this implies that, for any p which lies outside the interval $(\kappa_n(\mathfrak{M}),\kappa_1(\mathfrak{M}))$, the equality sign holds in the desired relations. The theorem is proved.

2. Let us denote by $\mathfrak{S}_n$ the set of all ordered sequences $\{\kappa_j\}_1^n$ of integers $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_n$. Let $\{\kappa_j\}_1^n$ and $\{\kappa_j'\}_1^n$ be two sequences in $\mathfrak{S}_n$; we shall say that the second sequence is obtained from the first by means of an *elementary operation* provided that for certain integers p and q $(1\leq p<q\leq n)$ it is true that:

$$\varkappa_p'=\varkappa_p-1,\ \varkappa_q'=\varkappa_q+1,\ \varkappa_j'=\varkappa_j\quad\text{for}\quad j\neq p,\ q.$$

Further, we shall write

$$\{\varkappa_j\}_1^n\succ\{\varkappa_j'\}_1^n.\tag{10.7}$$

if either the sequence $\{\kappa_j'\}_1^n$ coincides with the sequence $\{\kappa_j\}_1^n$, or is obtained from it by means of elementary operations. It should be noticed that (10.7) implies that

$$\sum_{j=1}^{n}\varkappa_j=\sum_{j=1}^{n}\varkappa_j'.$$

Relation (10.7) serves to introduce a partial ordering of $\mathfrak{S}_n$.

If $\{\kappa_j\}_1^n\in\mathfrak{S}_n$, then its mean is defined to be the sequence $\{\hat{\kappa}_j\}_1^n$, defined by the equations

$$\hat{\varkappa}_1=\hat{\varkappa}_2=\ldots=\hat{\varkappa}_r=q+1,\quad\varkappa_{r+1}=\varkappa_{r+2}=\ldots=\varkappa_n=q,$$

where the integers q and r are in turn defined by the equation

$$\sum_{j=1}^{n}\varkappa_j=nq+r\quad(0\leqslant r<n).$$

It is easily seen that always

$$\{\varkappa_j\}_1^n\succ\{\hat{\varkappa}_j\}_1^n.$$

Further, the sequence $\{\hat{\kappa}_j\}_1^n$ is minimal among all sequences $\{\kappa_j'\}$ satisfying (1

It will be left to the reader to verify the following assertion:

Let $\{\kappa_j\}_1^n$ and $\{\kappa_j'\}_1^n$ be two sequences from $\mathfrak{S}_n$, satisfying condition (10

A necessary and sufficient that for any integer p one have

$$\sum_{\varkappa_j>p}\varkappa_j\geqslant\sum_{\varkappa_j'>p}\varkappa_j',$$

is that $\{\kappa_j\}_1^n\succ\{\kappa_j'\}_1^n$.

This assertion serves to justify the proposition that Theorem 10.1 is equi

lent to the following:

Theorem 10.1′. *Let $\mathfrak{M}(\lambda)\in\mathfrak{R}_{(n\times n)}$ be a nonsingular matrix function. Th there exists a number $\delta(>0)$ such that every matrix function $\mathfrak{N}(\lambda)\in\mathfrak{R}_{(n\times n)}$ the δ-neighborhood: $\|\mathfrak{M}-\mathfrak{N}\|<\delta$ is also nonsingular and*

$$\{\varkappa_j(\mathfrak{M})\}_1^n\succ\{\varkappa_j(\mathfrak{N})\}_1^n\ (\succ\{\hat{\varkappa}_j(\mathfrak{M})\}).$$

3. We shall say that a system of left (right) exponents for a nonsingular m trix function $\mathfrak{M}(\lambda)\in\mathfrak{R}_{(n\times n)}$ is *stable*, provided that there exists $\delta>0$ such any matrix function, which belongs to the δ-neighborhood of the matrix

$$\mathfrak{M}:\ \|\mathfrak{M}-\mathfrak{N}\|<\delta,$$

also has the same system of left (right) exponents as $\mathfrak{M}$.

It will be shown below that the stability of a system of exponents $\{\kappa_j(\mathfrak{M})$ of a matrix function $\mathfrak{M}(\lambda)\in(\mathfrak{R}_{(n\times n)})$ is completely determined by its arithme structure, to wit: the system of exponents $(\kappa_1(\mathfrak{M}),\kappa_2(\mathfrak{M}),\cdots,\kappa_n(\mathfrak{M}))$ is stabl and only if $\kappa_1(\mathfrak{M})-\kappa_n(\mathfrak{M})\leq 1$, i.e., when it coincides with its mean.

In other words, one has the

Theorem 10.2. *Suppose that the nonsingular matrix function $\mathfrak{M}(\lambda)\in\mathfrak{R}_{(n}$ and that $\kappa=\operatorname{ind}\det\mathfrak{M}(\lambda)$. Then the system of left (right) exponents $(\kappa_1,\kappa_2,\cdots,$ of the matrix function $\mathfrak{M}(\lambda)$ is stable if and only if*

$$\varkappa_1=\varkappa_2=\ldots=\varkappa_r=q+1,\quad\varkappa_{r+1}=\varkappa_{r+2}=\ldots=\varkappa_n=q,$$

where the integers q and r are determined by the equation

$$\varkappa=qn+r,\quad 0\leqslant r<n.$$

Proof. The sufficiency of the condition is immediate from Theorem 10.2.

Let us proceed to the proof of the necessity.

It is obvious that it suffices to prove that if

$$\varkappa_1(\mathfrak{M}) - \varkappa_n(\mathfrak{M}) \geqslant 2,$$

then a sufficiently small variation of the matrix function will produce a change in at least one left index $\kappa_j(\mathfrak{M})$. Without loss of generality it may be supposed that all exponents $\kappa_j(\mathfrak{M})$ $(j = 1, 2, \cdots, n)$ are non-negative.

Let the matrix function $\mathfrak{F}_+(\lambda)$ be a standard solution of the matrix problem

$$\mathfrak{M}(\lambda)\mathfrak{F}_+(\lambda) = \mathfrak{F}_-(\lambda).$$

It is readily seen that the column vectors $\Phi_j(\lambda)$ $(j = 1, 2, \cdots, n)$ of the matrix function $\mathfrak{F}_+(\lambda)$ have zeros of order $\kappa_j = \kappa_j(\mathfrak{M})$, respectively, at $\lambda = i$. Let us introduce a new matrix function $\mathfrak{F}_+(\lambda; \epsilon)$, which is obtained from $\mathfrak{F}_+(\lambda)$ by replacing its first column vector by the column vector

$$\Phi_1(\lambda;\ \varepsilon) = \Phi_1(\lambda) - \varepsilon\left(\frac{\lambda - i}{\lambda + i}\right)^{\varkappa_1 - 1} c_0,$$

where $\epsilon(\neq 0)$ is a complex number, and c_0 is a vector, the value of the vector function $\left(\frac{\lambda - i}{\lambda + i}\right)^{-k_n} \Phi_n(\lambda)$ at $\lambda = i$.

Obviously, $\mathfrak{F}_+(\lambda; \epsilon)$ belongs to the ring $\mathfrak{R}^+_{(n\times n)}$. If the absolute value of ϵ is sufficiently small, then

$$\det \mathfrak{F}_+(\lambda;\ \varepsilon) \neq 0 \ \ (-\infty \leqslant \lambda \leqslant \infty) \text{ and } \operatorname{ind}\det \mathfrak{F}_+(\lambda;\ \varepsilon) = \operatorname{ind}\det \mathfrak{M}(\lambda) = \varkappa.$$

Let us define the matrix function $\mathfrak{M}_\epsilon(\lambda)$ by setting

$$\mathfrak{M}_\varepsilon(\lambda) = \mathfrak{F}_+(\lambda;\ \varepsilon)\mathfrak{F}_-^{-1}(\lambda).$$

The matrix function $\mathfrak{M}_\epsilon(\lambda)$ is nonsingular and belongs to the ring $\mathfrak{R}_{(n\times n)}$; the norm of the difference $\mathfrak{M}(\lambda) - \mathfrak{M}_\epsilon(\lambda)$ may be made arbitrarily small by choosing ϵ suitably, because

$$\| \mathfrak{M} - \mathfrak{M}_\varepsilon \| \leqslant |\varepsilon| \, \| \mathfrak{F}_-^{-1} \| \cdot \left\| \left(\frac{\lambda - i}{\lambda + i}\right)^{\varkappa_1 - 1} \right\| \cdot \| c_0 \| \cdot n,$$

where $\| c_0 \|$ is the maximum absolute value of the coordinates of the vector c_0.

It is also obvious that $\operatorname{ind}\det \mathfrak{M}_\epsilon(\lambda) = \kappa$.

The matrix function $\mathfrak{F}_+(\lambda; \epsilon)$ is a solution of the matrix problem

$$\mathfrak{M}_\varepsilon^{-1}(\lambda)\mathfrak{F}_+(\lambda;\ \varepsilon) = \mathfrak{F}_-(\lambda). \tag{10.8}$$

Since the sum of the multiplicities of the zeros of the columns of $\mathfrak{F}_+(\lambda; \epsilon)$ at $\lambda = i$ equals $\kappa - 1\ (\neq \kappa)$, the solution $\mathfrak{F}_+(\lambda; \epsilon)$ of the problem (10.8) is not standard. Let us transform the matrix $\mathfrak{F}_+(\lambda; \epsilon)$ so as to obtain a standard solution of (10.8). In order to do this, multiply the first column vector of the matrix $\mathfrak{F}_+(\lambda; \epsilon)$ by the function $\epsilon^{-1}\left(\frac{\lambda - i}{\lambda + i}\right)^{-\kappa_1 - 1 + \kappa_n}$ and add it to the last column of this matrix function. The same transformation is to be applied to the matrix function $\mathfrak{F}_-(\lambda)$. Let the resultant matrices be denoted, respectively, by $\widetilde{\mathfrak{F}}_\pm(\lambda; \epsilon)$. Clearly, $\widetilde{\mathfrak{F}}_\pm(\lambda; \epsilon) \in \mathfrak{R}^\pm_{(n\times n)}$. Further, the last column vector of $\widetilde{\mathfrak{F}}_+(\lambda; \epsilon)$ has a zero of multiplicity $\kappa_n + 1$ at $\lambda = i$.

The matrix function $\widetilde{\mathfrak{F}}_+(\lambda; \epsilon)$ has the property that the sum of the multiplicities of the zeros of its columns at $\lambda = i$ equals κ; consequently, it is a standard solution of the problem (10.8).

Thus, by means of a sufficiently small variation of $\mathfrak{M}(\lambda)$ one obtains a ma function $\mathfrak{M}_\epsilon(\lambda)$, whose left exponents differ from the left exponents of the matr $\mathfrak{M}(\lambda)$, in fact:

$$\varkappa_1(\mathfrak{M}_\varepsilon) = \varkappa_1(\mathfrak{M}) - 1,\ \ \varkappa_n(\mathfrak{M}_\varepsilon) = \varkappa_n(\mathfrak{M}) + 1 \text{ and } \varkappa_j(\mathfrak{M}_\varepsilon) = \varkappa_j(\mathfrak{M})\ \ (j = 2, \ldots, n$$

The theorem is proved.

Remark 1. Let us generalize 10.1′ and 10.2 as follows:

a) *Given any sequence of numbers* $\{\kappa_j'\}_1^n$ *satisfying the "inequality"*

$$\{\kappa_j(\mathfrak{M})\}_1^n \succ \{\kappa_j'\}_1^n,$$

there is, in an arbitrary δ*–neighborhood of a nonsingular matrix function* $\mathfrak{M}(\lambda)$ $(\in \mathfrak{R}_{(n\times n)})$ *a nonsingular matrix function* $\mathfrak{N}(\lambda)$ $(\in \mathfrak{R}_{(n\times n)})$ *such that* $\kappa_j(\mathfrak{N}) =$ $(j = 1, 2, \cdots, n)$.

b) *Given a matrix function* $\mathfrak{M}(\lambda)$ $(\in \mathfrak{R}_{(n\times n)})$, *in any arbitrarily small nei borhood of it there always exists a matrix function with a stable system of left and right exponents.*

Indeed, in the course of the proof of Theorem 10.2 it was shown that, by v slightly a matrix function with unstable exponents, one could increase one of left exponents by unity and decrease one of the other exponents also by unity. repeating this procedure a finite number of times one finally will arrive at a m which differs slightly from $\mathfrak{M}(\lambda)$ and which possesses the system of exponent $(\kappa_1', \kappa_2', \cdots, \kappa_n')$ given beforehand. Assertion a) is therefore proved. In partic in an arbitrary neighborhood of $\mathfrak{M}$ there is a matrix function $\mathfrak{N}$ with a stable s of left exponents. If the matrix function $\mathfrak{N}$ has an unstable system of right ex nents, then by altering it slightly one may make sure that its right exponents l come stable, while the left exponents remain as they were before.

Assertion b) therefore has also been proved.

Remark 2. A slight variant of the proof of Theorem 10.1, and of Theorem as well, furnishes these theorems in a stronger formulation, (see [13]), corres ing to the following wider definition of the δ–neighborhood of an element $\mathfrak{M}($ the ring $\mathfrak{R}_{(n\times n)}$ as the set of all $\mathfrak{N}(\lambda) \in \mathfrak{R}_{(n\times n)}$ such that

$$\max_{\lambda,\, j,\, k} |\mathfrak{m}_{jk}(\lambda) - \mathfrak{n}_{jk}(\lambda)| < \frac{\delta}{n},$$

where the maximum is taken over all $\lambda \in \{-\infty, \infty\}$ and $j, k = 1, 2, \cdots, n$.

It should be noticed that this new definition of a δ–neighborhood of a ma $\mathfrak{M} \in \mathfrak{R}_{(n\times n)}$ corresponds to the following definition of a norm

$$\| \mathfrak{M} \| = n \max_{\lambda,\, j,\, k} |\mathfrak{m}_{jk}(\lambda)|,$$

under which $\mathfrak{R}_{(n\times n)}$ is no longer a complete linear normed vector space.

§11. Exponents of triangular matrices and their factorization. 1. A mat is called a left (right) triangular matrix provided that all its elements above (

the principal diagonal are zero.

If $\mathfrak{M}(\lambda) \in \mathfrak{R}_{(n\times n)}$ is a left nonsingular triangular matrix, then the transposed matrix $\mathfrak{M}'(\lambda)$ is right triangular, and the left exponents of one of the matrices are right exponents for the other matrix. If we denote by $\mathfrak{M}_{,}(\lambda)$ the matrix obtained from $\mathfrak{M}(\lambda)$ by reflection across the second diagonal, then we can again assert that the left exponents of one of the matrices $\mathfrak{M}(\lambda)$ and $\mathfrak{M}_{,}(\lambda)$ are right exponents for the other, and that $\mathfrak{M}_{,}(\lambda)$, like $\mathfrak{M}(\lambda)$, is a left triangular matrix.

In view of this, we may always restrict our attention to the left triangular matrices $\mathfrak{M}(\lambda)$ and to their left exponents, which will be denoted by $\kappa_j(\mathfrak{M})$ $(j = 1, 2, \cdots, n)$.

We shall denote by $k_j(\mathfrak{M})$ $(j = 1, 2, \cdots, n)$ the indices of the diagonal elements of the nonsingular triangular matrix $\mathfrak{M}(\lambda)$:

$$k_j = k_j(\mathfrak{M}) = \operatorname{ind} \mathfrak{m}_{jj}(\lambda) \qquad (j = 1, 2, \ldots, n).$$

Obviously

$$\varkappa_1(\mathfrak{M}) + \ldots + \varkappa_n(\mathfrak{M}) = k_1(\mathfrak{M}) + \ldots + k_n(\mathfrak{M}) = \operatorname{ind}\det\mathfrak{M}(\lambda).$$

One has the

Theorem 11.1. *All left (right) exponents κ_j $(j = 1, 2, \cdots, n)$ of a triangular nonsingular matrix function $\mathfrak{M}(\lambda)$ $(\in \mathfrak{R}_{(n\times n)})$ lie between the largest, $k_{\max}$, and the smallest, $k_{\min}$, of the indices k_j $(j = 1, 2, \cdots, n)$ of the diagonal elements of $\mathfrak{M}(\lambda)$:*

$$k_{\min} \leqslant \varkappa_j \leqslant k_{\max} \qquad (j = 1, 2, \ldots, n).$$

If $\mathfrak{M}(\lambda)$ is a left triangular matrix function and the numbers κ_j $(j = 1, 2, \cdots, n)$ do not decrease (do not increase), then, when suitably ordered, they coincide with the left (right) exponents of the matrix function $\mathfrak{M}(\lambda)$.

The corresponding theorem for Hilbert's problem for matrix functions of the second order $(n = 2)$, relative to the class of functions satisfying a Hölder condition, was first discovered by G. N. Čebotarev [10].

Proof. For definiteness let us suppose that $\mathfrak{M}(\lambda)$ is a left triangular matrix function, and that the theorem has to be proved for the left exponents $\kappa_j(\mathfrak{M})$ $(j = 1, 2, \cdots, m)$. Consider the triangular matrix function

$$\mathfrak{N}(\lambda) = \left(\frac{\lambda - i}{\lambda + i}\right)^{-k_{\max}} \mathfrak{M}(\lambda) \quad (\in \mathfrak{R}_{(n\times n)}), \tag{11.1}$$

which is such that

$$\varkappa_j(\mathfrak{N}) = \varkappa_j(\mathfrak{M}) - k_{\max} \quad \text{and} \quad k_j(\mathfrak{N}) = k_j(\mathfrak{M}) - k_{\max} \qquad (j = 1, 2, \ldots, n).$$

The matrix $\mathfrak{N}(\lambda)$ corresponds to the integral equation

$$A\varphi(t) - \int_0^\infty l(t-s)\,\varphi(s)\,ds = 0 \qquad (0 \leqslant t < \infty), \tag{11.2}$$

where the triangular matrices $A = \| a_{pq} \|_1^n$ and $l(t) = \| l_{pq}(t) \|_1^n \in L_{(n\times n)}$ are

determined by

$$\mathfrak{N}(\lambda) = A - \int_0^\infty e^{i\lambda t} l(t)\,dt \qquad (\lambda \in \Pi_+).$$

Equation (11.2) may be rewritten as a system of scalar equations of the fo[llowing] lowing form:

$$\left.\begin{aligned}
&a_{11}\varphi_1(t) - \int_0^\infty l_{11}(t-s)\,\varphi_1(s)\,ds = 0,\\
&a_{22}\varphi_2(t) - \int_0^\infty l_{22}(t-s)\,\varphi_2(s)\,ds = -a_{21}\varphi_1(t) + \int_0^\infty l_{21}(t-s)\,\varphi_1(s)\,ds,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{nn}\varphi_n(t) - \int_0^\infty l_{nn}(t-s)\varphi_n(s)ds = -\sum_{j=1}^{n-1} a_{nj}\varphi_j(t) + \sum_{j=1}^{n-1}\int_0^\infty l_{nj}(t-s)\,\varphi_j(s)\,ds,
\end{aligned}\right\} \quad (1$$

where, since the matrix A is nonsingular, all $a_{jj} \neq 0$ $(j = 1, 2, \cdots, n)$.

Since

$$\operatorname{ind}(a_{11} - \mathscr{L}_{11}(\lambda)) = \operatorname{ind} \mathfrak{n}_{11}^{-1}(\lambda) = -k_1(\mathfrak{N}) \geqslant 0,$$

the first equation of the system (11.3) (see section 9) has only the solution $\phi \equiv 0$. Replacing ϕ_1 by zero in the second equation and using the fact that

$$\operatorname{ind}(a_{22} - \mathscr{L}_{22}(\lambda)) = \operatorname{ind} \mathfrak{n}_{22}^{-1}(\lambda) = -k_2(\mathfrak{N}) \geqslant 0,$$

one obtains that also $\phi_2 \equiv 0$. Continuing in this manner, one arrives at the co[n]clusion that the system (11.3) has the unique solution $\phi_1 \equiv \phi_2 \equiv \phi_3 \equiv \cdots \equiv \phi_n$ According to a theorem this means that all the left exponents of the matrix fun[c]tion $\mathfrak{N}(\lambda)$ are non-negative, i.e.,

$$\varkappa_j(\mathfrak{M}) \leqslant k_{\max} \qquad (j = 1, 2, \ldots, n).$$

In order to prove the inequalities

$$\varkappa_j(\mathfrak{M}) \geqslant k_{\min} \qquad (j = 1, 2, \ldots, n) \tag{1:}$$

let us define the matrix function $\mathfrak{N}(\lambda)$, not as in (11.1), but as follows:

$$\mathfrak{N}(\lambda) = \left(\frac{\lambda - i}{\lambda + i}\right)^{-k_{\min}} \mathfrak{M}(\lambda) \quad (\in \mathfrak{R}_{(n\times n)}), \tag{1}$$

and consider the corresponding system of equations (11.2) associated with thi[s] matrix. Since now

$$\operatorname{ind}(a_{jj} - \mathscr{L}_{jj}(\lambda)) = \operatorname{ind} \mathfrak{n}_{jj}^{-1}(\lambda) - k_j(\mathfrak{N}) = -(k_j - k_{\min}) \leqslant 0 \quad (j = 1, 2, \ldots,$$

then each of the homogeneous equations

$$a_{jj}\varphi(t) - \int_0^\infty l_{jj}(t-s)\,\varphi(s)\,ds = 0 \qquad (0 \leqslant t < \infty)$$

has exactly $k_j - k_{\min}$ linearly independent solution in L^+, and for each $f \in$

each of the nonhomogeneous equations

$$a_{jj}\varphi(t)-\int_0^\infty l_{jj}(t-s)\varphi(s)\,ds=f(t)\qquad(0\leqslant t<\infty)$$

always has a solution in L^+. From this it follows at once that the exact number $\alpha(\mathfrak{N})$ of linearly independent solutions of the system (11.2) is equal to

$$\sum_{j=1}^{n}k_j(\mathfrak{N})=\operatorname{ind}\det\mathfrak{N}(\lambda).$$

In view of Theorem 9.1, all exponents of $\mathfrak{N}(\lambda)$ are non-negative, from which (11.4) follows. The first assertion of the theorem is proved.

Let us prove the second assertion of the theorem by induction.

For $n = 1$ the assertion is trivial. Let us suppose that the assertion has been proved for all triangular matrix functions of order $m \leq n-1$, and let us prove that it continues to hold for a given matrix $\mathfrak{M}(\lambda)$ of order n. Again let us construct the matrix

$$\mathfrak{N}(\lambda)=\left(\frac{\lambda-i}{\lambda+i}\right)^{-k_1}\mathfrak{M}(\lambda)\quad(\in\mathfrak{R}_{(n\times n)})$$

and consider the corresponding system of integral equations (11.2). Since

$$\operatorname{ind}(a_{11}-\mathscr{L}_{11}(\lambda))=\operatorname{ind}\mathfrak{n}_{11}^{-1}(\lambda)=k_1-k_1=0,$$

the first equation of (11.2) implies that $\phi_1\equiv 0$.

Replacing ϕ_1 by zero in the remaining equations of (11.2), one sees that the resulting system of equations corresponds to a triangular matrix $\mathfrak{N}_1(\lambda)$ which can be obtained from the matrix $\mathfrak{N}(\lambda)$ by striking out its first column and its first row. By the induction hypothesis, the exponents of the matrix $\mathfrak{N}_1(\lambda)$ are the numbers

$$\varkappa_1'=k_n-k_1,\ \ \varkappa_2'=k_{n-1}-k_1,\ \ \ldots,\ \ \varkappa_{n-1}'=k_2-k_1.$$

Consequently, the system of equations (11.2), by Theorem 9.1, has a basis of solutions, which may be arranged in $n-1$ chains of lengths $\kappa_1', \kappa_2', \cdots, \kappa_{n-1}'$. Since, by the first assertion of the theorem, all exponents of $\mathfrak{N}(\lambda)$ are non-negative, the exponents of $\mathfrak{N}(\lambda)$ must be the numbers $\kappa_1', \kappa_2', \cdots, \kappa_{n-1}', 0$, and consequently the exponents of $\mathfrak{M}(\lambda)$ must be:

$$\varkappa_1=k_1+\varkappa_1'=k_n,\ \ \ldots,\ \ \varkappa_{n-1}=\varkappa_{n-1}'+k_1=k_2,\ \ \varkappa_n=k_1.$$

The theorem is proved.

From Theorem 11.1 and equation (11.1) it immediately follows that:

Conclusion: *If $\mathfrak{M}(\lambda)$ is a nonsingular triangular matrix function and the indices k_j $(j = 1, 2, \cdots, n)$ of its diagonal elements differ from each other by not more than unity, then, arranged in non-increasing order, these numbers are a system of left, and of right, exponents $\kappa_j(\mathfrak{M})$ $(j = 1, 2, \cdots, n)$ of the matrix $\mathfrak{M}(\lambda)$, which is thus seen to be stable.*

2. Let us show that the left (right) standard factorization of a triangular matrix function $\mathfrak{M}(\lambda)$ may be effectively carried out, as soon as the standard factori-

zation of its diagonal elements has been accomplished.

Without loss of generality it may be supposed that $\mathfrak{M}(\infty) = I$ and that

$$k_j(\mathfrak{M})\geqslant 0\qquad(j=1,\ 2,\ \ldots,\ n).\qquad(1$$

In view of Theorem 7.3, applied in the special case $n = 1$, all the diagona functions $\mathfrak{m}_{jj}(\lambda)$ $(j = 1, 2, \cdots, n)$ of the matrix function $\mathfrak{M}(\lambda)$ admit the standa factorization

$$\mathfrak{m}_{jj}(\lambda)=\mathfrak{g}_j^+(\lambda)\,\mathfrak{g}_j^-(\lambda)\qquad(j=1,\ 2,\ \ldots,\ n),\qquad(1$$

where the factors $\mathfrak{g}_j^{\pm}(\lambda)\in\mathfrak{R}^{\pm}$ have the following properties: $\mathfrak{g}_j^-(\lambda)\neq 0$ $(\lambda \in \Gamma$ and $\mathfrak{g}_j^+(\lambda)$ does not vanish on the halfplane Π_+, with the exception of $\lambda = i$, it has a zero of multiplicity k_j $(j = 1, 2, \cdots, n)$.

As soon as the factorization (11.7) has been carried out, one can easily co struct a nonsingular triangular solution $\mathfrak{F}_+(\lambda)$ of the matrix Hilbert problem:

$$\mathfrak{M}^{-1}(\lambda)\,\mathfrak{F}_+(\lambda)=\mathfrak{F}_-(\lambda).\qquad(11$$

In fact, the first column of the solution $\mathfrak{F}_+(\lambda) = \|\,\mathfrak{f}_{pq}^+(\lambda)\,\|_1^n$ is determine by the following system of equations

$$\left.\begin{aligned}&\mathfrak{a}_{11}(\lambda)\,\mathfrak{f}_1^+(\lambda)=\mathfrak{f}_1^-(\lambda),\\&\mathfrak{a}_{22}(\lambda)\,\mathfrak{f}_2^+(\lambda)=-\mathfrak{a}_{21}(\lambda)\,\mathfrak{f}_1^+(\lambda)+\mathfrak{f}_2^-(\lambda),\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&\mathfrak{a}_{nn}(\lambda)\,\mathfrak{f}_n^+(\lambda)=-\mathfrak{a}_{n1}(\lambda)\,\mathfrak{f}_1^+(\lambda)-\ldots-\mathfrak{a}_{nn-1}(\lambda)\,\mathfrak{f}_{n-1}^+(\lambda)+\mathfrak{f}_n^-(\lambda).\end{aligned}\right\}\qquad(11$$

Here, to simplify the writing, the second subscript has been omitted from functions $\mathfrak{f}_{p1}(\lambda)$ $(p = 1, 2, \cdots, n)$.

Since $\mathfrak{a}_{11}(\lambda) = \mathfrak{m}_{11}^{-1}(\lambda)$, the functions

$$\mathfrak{f}_1^+(\lambda)=\mathfrak{g}_1^+(\lambda)\ \text{ and }\ \mathfrak{f}_1^-(\lambda)=\mathfrak{g}_1^-(\lambda),\qquad(11.$$

obviously satisfy the first equation of (11.9). In order to determine the functio $\mathfrak{f}_2^+(\lambda)$ let us substitute in the second equation of (11.9) the value of $\mathfrak{f}_1^+(\lambda)$ already obtained, and replace $\mathfrak{a}_{22}(\lambda)$ by $\mathfrak{m}_{22}^{-1}(\lambda) = \mathfrak{g}_2^+(\lambda)\,\mathfrak{g}_2^-(\lambda)$; then

$$\mathfrak{g}_2^+(\lambda)\,\mathfrak{f}_2^+(\lambda)=\mathfrak{g}_2^-(\lambda)\,\mathfrak{f}_2^-(\lambda)-\mathfrak{g}_2^-(\lambda)\,\mathfrak{a}_{21}(\lambda)\,\mathfrak{g}_2^+(\lambda).$$

Since $\mathfrak{g}_2^-(\lambda)\,\mathfrak{f}_\lambda^-(\lambda)\in\mathfrak{R}^-$, we conclude that

$$\mathfrak{g}_2^+(\lambda)\,\mathfrak{f}_2^+(\lambda)=-P_+(\mathfrak{g}_2^-\mathfrak{a}_{21}\mathfrak{g}_2^+),$$

where P_+ is a projection operator, mapping the ring $\mathfrak{R}$ on the ring $\mathfrak{R}^+$, i.e.,

$$P_+\Big(c+\int_{-\infty}^{\infty}e^{i\lambda t}h(t)\,dt\Big)=c+\int_0^{\infty}e^{i\lambda t}h(t)\,dt\qquad(h\in L).$$

Thus

$$\mathfrak{f}_2^+(\lambda)=-\mathfrak{g}_2^+(\lambda)\,P_+(\mathfrak{g}_2^-\mathfrak{a}_{21}\mathfrak{g}_1^+).$$

Proceeding in this fashion, one obtains all the functions $\mathfrak{f}_{j1}^+(\lambda)$ $(j = 1, 2, \cdots,$ The first function of the second column of $\mathfrak{F}_+(\lambda)$ is zero. In order to determin the remaining functions appearing in the second column of the matrix $\mathfrak{F}_+(\lambda)$ w can construct a system analogous to (11.9). It is clear that the matrix function this new system can be obtained from $\mathfrak{M}^{-1}(\lambda)$ by striking out its first column and its first row.

By the process outlined above one can obtain all the elements of the matrix function $\mathfrak{F}_+(\lambda)$ which is a solution of the problem (11.8).

Since the functions $g_j^+(\lambda)$ $(j = 1, 2, \cdots, n)$ are the diagonal elements of $\mathfrak{F}_+(\lambda)$, the determinant of the matrix function $\mathfrak{F}_+(\lambda)$ has a zero of multiplicity $k_1 + k_2 + \cdots + k_n$ at $\lambda = i$. In view of the fact that

$$k_1 + k_2 + \ldots + k_n = \operatorname{ind} \det \mathfrak{M}(\lambda),$$

we arrive at the conclusion that, after a finite number of operations, which were explained in detail in section 6, one can transform $\mathfrak{F}_+(\lambda)$ into a standard solution of the problem (11.8). This completes the left standard factorization of the matrix function $\mathfrak{F}_+(\lambda)$.

3. Let us examine the case $n = 2$ further. In this case the matrix function $\mathfrak{F}_+(\lambda)$ has the form

$$\mathfrak{F}_+(\lambda) = \begin{Vmatrix} g_1^+(\lambda) & 0 \\ \Omega(\lambda)\, g_2^+(\lambda) & g_2^+(\lambda) \end{Vmatrix}, \qquad (11.11)$$

where

$$\Omega(\lambda) = -P_+(g_2^- g_1^+ a_{12}).$$

The function $\Omega(\lambda)$ may have a zero at $\lambda = i$.

In all cases, the function $\Omega^{-1}(\lambda)$ may be uniquely represented in the form

$$\Omega^{-1}(\lambda) = Q_0\left(\frac{1}{\lambda - i}\right) + R_1(\lambda),$$

where $Q_0(\mu)$ is a polynomial of degree $q_0 \geq 0$, and $R_1(\lambda)$ is a function which is regular at $\lambda = i$, and which vanishes there: $R_1(i) = 0$.

If $R_1(\lambda) \not\equiv 0$, then the function $R_1^{-1}(\lambda)$ may be uniquely represented in the form

$$R_1^{-1}(\lambda) = Q_1\left(\frac{1}{\lambda - i}\right) + R_2(\lambda),$$

where $R_2(\lambda)$ is a polynomial of degree $q_1 \geq 1$, and $R_2(\lambda)$ is a function which is regular at $\lambda = i$, and which vanishes there: $R_2(i) = 0$.

If $R_2(\lambda) \not\equiv 0$, the process may be continued to obtain a representation of $R_2^{-1}(\lambda)$ as a sum: $Q_2(\frac{1}{\lambda - i}) + R_3(\lambda)$, etc.

Thus, one may always formally expand the function Ω in a finite or infinite continued fraction

$$\Omega(\lambda) = \cfrac{1}{Q_0(\mu) + \cfrac{1}{Q_1(\mu) + \ldots}} \qquad \left(\mu = \frac{1}{\lambda - i}\right), \qquad (11.12)$$

where $Q_j(\mu)$ is a polynomial of degree q_j $(j = 0, 1, \cdots)$, and $q_0 \geq 0$, $q_j \geq 1$ $(j = 1, 2, \cdots)$. Let us put

$$s_0 = q_0, \quad s_1 = q_0 + q_1, \quad s_2 = q_0 + q_1 + q_2, \ \ldots .$$

If the continued fraction (11.12) is finite (Ω is a rational function) and Q_2 is the last denominator of this fraction, then we will set $q_{l+1} = \infty$. Then the fol lowing rule holds:

If $s_0 \geq k_1 - k_2$, then $\kappa_1(\mathfrak{M}) \geq \max(k_1, k_2)$, $\kappa_2(\mathfrak{M}) = \min(k_1, k_2)$; if s_0, $s_0 + s_1$, $s_1 + s_2, \cdots, s_{l-1} + s_l < k_1 - k_2$, and $s_l + s_{l+1} \geq k_1 - k_2$, then the left ex nents $\kappa_1(\mathfrak{M})$ and $\kappa_2(\mathfrak{M})$ coincide respectively with the largest and the smallest the numbers $k_1 - s_l$ and $k_2 - s_l$.

The first such rule was given by G. N. Čebotarev [10], in considering the Hilbert problem for one and for several smooth contours. Repeating the conside tions of this author word by word, one arrives at the rule stated here.

Besides, for the integer l, for the l-th convergent of the continued fractic (11.12) and for R_l and R_{l+1}, it is easy to obtain an explicit expression in terr of the factor $\mathfrak{N}_+(\lambda)$ of the standard factorization of the matrix $\mathfrak{M}(\lambda)$ (see [6]).

Observe that, if $s_0 < k_1 - k_2$, inasmuch as $s_0 \geq 0$, the numbers k_1 and k must be right exponents of the matrix $\mathfrak{M}(\lambda)$. According to G. N. Čebotarev's rul they must coincide with the left exponents of $\mathfrak{M}(\lambda)$ only in two cases, namely: 1) $l = 0$, $s_0 = 0$, or if $s_l = k_1 - k_2$.

On the other hand, for arbitrary integers $k_1 > k_2$ and $d\,(0 \leq d \leq \frac{1}{2}(k_1 - k_2))$ one may always construct a left triangular matrix $\mathfrak{M}(\lambda)$ with diagonal indices k_1, k_2 and left exponents $k_1 - d$, $k_2 - d$.

Indeed, supposing, for simplicity, that $k_2 \geq 0$, for example, that $\mathfrak{M}(\lambda)$ is o the form (11.11), where $g_j^+ \in \mathfrak{R}^+$ $(j = 1, 2)$ have in Π_+ a unique zero at $\lambda = i$ multiplicity k_j $(j = 1, 2)$, and

$$\Omega(\lambda) = \cfrac{1}{Q_0(\mu) + \cfrac{1}{Q_1(\mu)}} \qquad \left(\mu = \frac{1}{\lambda - i}\right),$$

where $Q_0(\mu)$ is a polynomial of degree d, and $Q_1(\mu)$ is a polynomial of degre $q_1 > k_1 - k_2 - 2d$, so chosen that $\Omega(\lambda)$ has no real poles. Then, by G. N. Čeb tarev's rule, $l = 0$, $s_0 = d$, and, finally, the exponents of $\mathfrak{M}(\lambda)$ are equal to κ_1 and $\kappa_2 - d$.

§12. Simple and generalized resolvent of equation (A). 1. In this section shall consider the question of the determination of the solution of the equation

$$\varphi(t) - \int_0^\infty k(t-s)\,\varphi(s)\,ds = f(t) \qquad (0 \leqslant t < \infty) \qquad ($$

in case a factorization of the matrix function $(I - \check{\mathcal{K}}(\lambda))^{-1}$ is known. It will be supposed throughout that the matrix function $I - \check{\mathcal{K}}(\lambda)$ is nonsingular.

At first, we shall consider the case when the equations (B), (B′) possess only the zero solution. In this case, in view of the fundamental Theorem 2.1, equation (A) has, for each right-hand side $f(t) \in E^+_{(n \times 1)}$, a unique solution

$\phi(t) \in E^{+}_{(n\times 1)}$.

Let $f(t)$ be a vector function from $L^{+}_{(n\times 1)}$, and $\phi(t)\ (\in L^{+}_{(n\times 1)})$ be the corresponding solution of equation (A). By setting

$$b(t) = -\int_0^\infty k(t-s)\varphi(s)\,ds \qquad (-\infty < t < 0),$$

$$\varphi(t) = f(t) = b(-t) = 0 \qquad (-\infty < t < \infty),$$

equation (A) may be rewritten

$$\varphi(t) - \int_{-\infty}^{\infty} k(t-s)\varphi(s)\,ds = f(t) + b(t) \qquad (0 \leqslant t < \infty).$$

Let us apply the Fourier transformation to both sides of this equation, to obtain that

$$(I - \mathscr{K}(\lambda))\,\Phi(\lambda) = \mathscr{F}(\lambda) + \mathscr{B}(\lambda), \tag{12.1}$$

where $\Phi(\lambda)$, $\mathfrak{F}(\lambda) \in \Re^{0+}_{(n\times 1)}$, and $\mathfrak{B}(\lambda) \in \Re^{0-}_{(n\times 1)}$.

In view of the conditions imposed on the matrix function $I - \mathscr{K}(\lambda)$, the matrix function $(I - \mathscr{K}(\lambda))^{-1}$ admits the left canonical factorization

$$(I - \mathscr{K}(\lambda))^{-1} = \mathfrak{G}_{+}(\lambda)\,\mathfrak{G}_{-}(\lambda). \tag{12.2}$$

From (12.1) and (12.2) it follows that

$$\mathfrak{G}_{+}^{-1}\Phi = \mathfrak{G}_{-}\mathscr{F} + \mathfrak{G}_{-}\mathscr{B}$$

and since Φ, $\mathfrak{F} \in \Re^{0+}_{(n\times 1)}$ and $\mathfrak{B} \in \Re^{0}_{(n\times 1)}$, this implies that

$$\mathfrak{G}_{+}^{-1}\Phi = P_{+}(\mathfrak{G}_{-}\mathscr{F}),$$

where P_{+} ,as in section 11) is a projection operator mapping $\Re_{(n\times 1)}$ into $\Re^{+}_{(n\times 1)}$, i.e.,

$$P_{+}\Big(c + \int_{-\infty}^{\infty} e^{i\lambda t}h(t)\,dt\Big) = c + \int_0^\infty e^{i\lambda t}h(t)\,dt \qquad (h \in L_{(n\times 1)}).$$

Thus, given $f(t) \in L^{+}_{(n\times 1)}$, the Fourier transform of the solution of equation (A) is given by the formula

$$\Phi = \mathfrak{G}_{+}P_{+}(\mathfrak{G}_{-}\mathscr{F}). \tag{12.3}$$

Let us show that the operator $(I-K)^{-1}$ may also be represented by means of a certain matrix resolvent kernel. In order to obtain an analytical expression for the resolvent, one has to apply the "inverse" of the Fourier transformation to both sides of equation (12.3). Putting

$$\mathfrak{G}_{+}(\lambda) = 1 + \int_0^\infty \gamma(t)\,e^{i\lambda t}\,dt \qquad (\gamma(t) \in L^{+}_{(n\times n)}) \tag{12.4}$$

and

$$\mathfrak{G}_{-}(\lambda) = 1 + \int_0^\infty \gamma_{\tau}(t)\,e^{-i\lambda t}\,dt \qquad (\gamma_{\tau}(t) \in L^{+}_{(n\times n)}), \tag{12.5}$$

one obtains that

$$P_{+}(\mathfrak{G}_{-}\mathscr{F}) = \int_0^\infty g(t)\,e^{i\lambda t}\,dt,$$

where

$$g(t) = f(t) + \int_0^\infty \gamma_{\tau}(t-r)\,f(r)\,dr;$$

it being understood that $\gamma_{\tau}(t) = \gamma(t) = 0\ (-\infty < t < 0)$.

Once this has been done, it follows easily from (12.3) that

$$\varphi(t) = g(t) + \int_0^\infty \gamma(t-r)\,g(r)\,dr.$$

Substituting here the expression for $g(t)$ we obtain

$$\varphi(t) = f(t) + \int_0^\infty \gamma(t, s)\,f(s)\,ds,$$

where the kernel $\gamma(t, s)$ $(0 \le t,\ s < \infty)$ is given by

$$\gamma(t, s) = \gamma(t-s) + \gamma_{\tau}(s-t) + \int_0^\infty \gamma(t-r)\,\gamma_{\tau}(s-r)\,dr \qquad (0 \leqslant t,\ s < \infty).$$

or, what is the same

$$\gamma(t, s) = \gamma(t-s) + \gamma_{\tau}(s-t) + \int_0^{\min(t,\, s)} \gamma(t-r)\,\gamma_{\tau}(s-r)\,dr \qquad (0 \leqslant t,\ s < \infty); \tag{12.6}$$

the kernel $\gamma(s, t)$, being the resolvent of equation (A), satisfies the two functional equations:

$$\left.\begin{aligned} \gamma(t, s) &= k(t-s) + \int_0^\infty k(t-r)\,\gamma(r, s)\,dr, \\ \gamma(t, s) &= k(t-s) + \int_0^\infty \gamma(t, r)\,k(r-s)\,dr, \end{aligned}\right\} \qquad (0 \leqslant s,\ t < \infty); \tag{12.7}$$

where each equation is meant in the sense that it holds for arbitrary $t \geq 0$ $(s \geq 0)$ for almost all $s \geq 0$ $(t \geq 0)$. Analogously to what happens in the scalar case (see [1]) it may be estiblished that a kernel $\gamma(t, s)$ of type (12.6) generates a bound operator Γ in any space $E^{+}_{(n\times 1)}$. The relations (12.7) are the analytic equivalen of the fact that, in $L^{+}_{(n\times 1)}$

$$(I-K)(I+\Gamma) = I, \qquad (I+\Gamma)(I-K) = I, \tag{12.8}$$

and, as soon as they are established, from them follows the truth of (12.8) for an arbitrary space $E^{+}_{(n\times 1)}$.

It should be noticed that, although the kernel $\gamma(t, s)$ has an entirely different structure than the kernel $k(t-s)$, both of them are entirely determined by their values on the axes $t=0$ and $s=0$; in fact

$$\gamma(t, s)=\gamma(t-s, 0)-\gamma(0, s-t)+\int_0^\infty \gamma(t-r, 0)\gamma(0, s-r)dr \quad (0\leqslant t, s<\infty). \tag{12.9}$$

For the understanding of this last equation it must be remembered that $\gamma(t) = \gamma(t, 0)$ and $\gamma(s) = \gamma(0, s)$ $(0 \leq s, t < \infty)$ (see (12.6)). In writing this equation it was supposed that $\gamma(t, 0) = \gamma(0, t) = 0$ for $t<0$.

We have thus established the following proposition:

Theorem 12.1. *If to the matrix function $k(t) \in L_{(n\times n)}$ there corresponds a nonsingular matrix function $I - \mathcal{K}(\lambda)$ $(-\infty<\lambda<+\infty)$ and the equations* (B), (B′) *have only the zero solution in the space $E^+_{(n\times 1)}$, then, given $f \in E^+_{(n\times 1)}$, the solution $\phi \in E^+_{(n\times 1)}$ is given by the formula*

$$\varphi(t)=f(t)+\int_0^\infty \gamma(t, s)f(s)ds,$$

where $\gamma(t, s)$ $(0\leq t, s<\infty)$ is a matrix kernel such that the matrix functions $\gamma(t, 0)$ and $\gamma(0, t)$ belong to $L^+_{(n\times n)}$ and that $\gamma(t, s)$ is expressible in terms of $\gamma(t, 0)$ and $\gamma(0, t)$ by means of equation (12.9).

Remark. Setting $s=0$ and $t=0$, respectively, in the equations (12.7), one obtains that the matrix functions $\gamma(t)$ and $\gamma_\tau(t)$ are the unique solutions of class $L^+_{(n\times n)}$ of the equations

$$\left.\begin{aligned}\gamma(t)-\int_0^\infty k(t-s)\gamma(s)ds&=k(t) \qquad (0\leqslant t<\infty),\\ \gamma_\tau(t)-\int_0^\infty k'(s-t)\gamma_\tau(s)ds&=k(-t) \qquad (0\leqslant t<\infty).\end{aligned}\right\} \tag{12.10}$$

Therefore, if the matrix function $k(t)$ generates an operator

$$K\varphi=\int_0^\infty k(t-s)\varphi(s)ds$$

(in the space $L^+_{(n\times 1)}$) with norm less than unity, then $\gamma(t)$ (and similarly $\gamma_\tau(t)$) may be obtained as a Neumann series

$$\gamma(t)=\sum_{n=0}^{\infty} k_n(t), \tag{12.11}$$

where

$$k_0(t)=k(t),\ k_n(t)=\int_0^\infty k(t-s)k_{n-1}(s)ds \qquad (n=1, 2, \ldots)$$

and the convergence in (12.11) is understood in the sense of the metric of the

space $L^+_{(n\times 1)}$.

The condition $\|K\|<1$ is always fulfilled, provided that

$$\|k_{ij}(t)\|_L<\frac{1}{n} \quad (i, j=1, 2, \ldots, n), \text{ where } k(t)=\|k_{pq}(t)\|_1^n.$$

It should be observed that, as soon as equations (12.10) have been solved, one way or another, then equations (12.4) and (12.5) serve to determine the fact in the left canonical factorization of the matrix function $(I-\mathcal{K}(\lambda))^{-1}$.

2. Consider the solution of equation (A) in the special case when

$$f(t)=ae^{i\zeta t},$$

where a is an n-dimensional constant vector and $\operatorname{Im}\zeta>0$.

Denote by $\chi_\zeta(t)$ the solution of equation (A) for this particular right-hand side. As before, we shall suppose that the hypotheses of the theorem proved ar fulfilled.

Since $\exp(i\zeta t) \in L$, one may use the formula

$$X_\zeta(\lambda)=\mathfrak{G}_+(\lambda)P_+(\mathfrak{G}_-(\lambda)\mathscr{F}_-(\lambda)). \tag{12.1}$$

In the case under consideration

$$\mathscr{F}(\lambda)=a\int_0^\infty e^{i\lambda t}e^{i\zeta t}dt=ai(\lambda+\zeta)^{-1}$$

and, therefore

$$\mathfrak{G}_-(\lambda)\mathscr{F}(\lambda)=i\frac{\mathfrak{G}_-(\lambda)-\mathfrak{G}_-(-\zeta)}{\lambda+\zeta}a+i\frac{\mathfrak{G}_-(-\zeta)}{\lambda+\zeta}a.$$

Since the matrix function $\mathfrak{G}_-(\lambda)-\mathfrak{G}_-(-\zeta)$ belongs to the ring $\Re^-_{(n\times n)}$ an vanishes at $\lambda=-\zeta$, the first term on the right-hand side of the last equation is vector function in $\Re^-_{(n\times 1)}$. On the other hand, $(\lambda+\zeta)^{-1}a \in \Re^{0+}_{(n\times 1)}$. Thus, in present case

$$P_+(\mathfrak{G}_-\mathscr{F})=i\frac{\mathfrak{G}_-(-\zeta)}{\lambda+\zeta}a$$

and (12.12) yields

$$X_\zeta(\lambda)=\int_0^\infty e^{i\lambda t}\chi_\zeta(t)dt=i\frac{\mathfrak{G}_+(\lambda)\mathfrak{G}_-(-\zeta)}{\lambda+\zeta}a \qquad (\lambda\in\Pi_+). \tag{12.12}$$

As in the scalar case (see [1]), it may be proved that, when $\operatorname{Im}\lambda>0$, equa tion (12.12′) holds also for real ζ.

It should be noticed that all the considerations of this section are almost identical to the corresponding considerations for the scalar case (see [1]). All the assertions of [1], section 8 remain valid for the matrix situation considered here, but will be omitted for lack of space.

3. Let us now consider the general case, when equations (B), (B′) may

possess nontrivial solutions. Although in this case equation (A) does not have a resolvent in the usual sense, still it may be shown that if equation (A) does possess a solution, then one of its solutions is given by a formula similar to (12.3).

In fact, suppose that equation (A) does possess a solution for a certain $f \in L^+_{(n\times 1)}$. Then the Fourier transform $\Phi(\lambda)$ of a solution of equation (A) satisfies the equation

$$(I-\mathcal{K}(\lambda))\Phi(\lambda)=\mathcal{F}(\lambda)+\mathcal{B}(\lambda). \qquad (12.13)$$

Let us now use the left standard factorization of the matrix $(I-\mathcal{K}(\lambda))^{-1}$:

$$(I-\mathcal{K}(\lambda))^{-1}=\mathfrak{N}_+(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{N}_-(\lambda).$$

Let us denote by $\mathfrak{G}_\pm(\lambda)$ the matrix functions which are defined by

$$\mathfrak{E}_+(\lambda)=\mathfrak{N}_+(\lambda)\left\|\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_j^+}\delta_{jk}\right\|_1^n, \qquad \mathfrak{E}_-(\lambda)=\left\|\left(\frac{\lambda-i}{\lambda+i}\right)^{\varkappa_j^-}\delta_{jk}\right\|_1^n\mathfrak{N}_-(\lambda),$$

where $\kappa_j (j=1,2,\dots,n)$ are the left exponents of the matrix function $(I-\mathcal{K}(\lambda))^{-1}$, and $\kappa_j^+=\max(\kappa_j,0)$ and $\kappa_j^-=\min(\kappa_j,0)$.

It is obvious that $\mathfrak{G}_\pm(\lambda)$ belong respectively to the rings $\mathfrak{R}^\pm_{(n\times n)}$, and that

$$(I-\mathcal{K}(\lambda))^{-1}=\mathfrak{E}_+(\lambda)\,\mathfrak{E}_-(\lambda). \qquad (12.14)$$

Let us denote by κ_q the smallest positive left exponent of the matrix function $(I-\mathcal{K}(\lambda))^{-1}$. In view of what was shown in section 6, the Fourier transform of the general solution of the homogeneous equation (B) may be represented in the form

$$\mathfrak{E}_+(\lambda)\,\mathcal{P}\left(\frac{1}{\lambda-i}\right),$$

where $\mathcal{P}(\frac{1}{\lambda-i})$ is a polynomial vector function, whose components $p_j(\frac{1}{\lambda-i})$ are

$$p_j\left(\frac{1}{\lambda-i}\right)=\sum_{k=1}^{\varkappa_j}a_k^{(j)}(\lambda-i)^{-k} \quad (j=1,2,\dots,p);$$

$$p_j\left(\frac{1}{\lambda-i}\right)\equiv 0 \quad (j=p+1,\ p+2,\ \dots,\ n). \qquad (12.15)$$

If the vector function $\Phi_1(\lambda)$ is the Fourier transform of a solution of equation (A), then the Fourier transform $\hat{\Phi}(\lambda)$ of the general solution of equation (A) is given by the equation

$$\hat{\Phi}(\lambda)=\Phi_1(\lambda)-\mathfrak{E}_+(\lambda)\,\mathcal{P}\left(\frac{1}{\lambda-i}\right), \qquad (12.16)$$

where $\mathcal{P}(\frac{1}{\lambda-i})$ is a polynomial vector function, possessing the properties (12.15).

Let us select the solution $\phi(t)\in L^+_{(n\times 1)}$ of equation (A) is such a way that the vector function $\mathfrak{G}_+^{-1}(\lambda)\,\Phi(\lambda)\in\mathfrak{R}^{0+}_{(n\times n)}$, where

$$\Phi(\lambda)=\int_0^\infty e^{i\lambda t}\varphi(t)\,dt.$$

This selection may actually be carried out by putting, in equation (12.16), the polynomial $\mathcal{P}(\frac{1}{\lambda-i})$ equal to the principal part of the matrix function $\mathfrak{G}_+^{-1}(\lambda)\,\Phi_1(\lambda)$ in the neighborhood of $\lambda=i$. Obviously, the coordinates of the vector function $\mathcal{P}(\frac{1}{\lambda-i})$ have the desired form (12.15).

A comparison of equations (12.13) and (12.14) shows that

$$\mathfrak{E}_+^{-1}(\lambda)\,\Phi(\lambda)=\mathfrak{E}_-(\lambda)\,\mathcal{F}(\lambda)+\mathfrak{E}_-(\lambda)\,\mathcal{B}_-(\lambda),$$

and since $\mathfrak{G}_+^{-1}(\lambda)\,\Phi\in\mathfrak{R}^{0+}_{(n\times 1)}$, $\mathfrak{G}_-\mathcal{B}_-\in\mathfrak{R}^{0}_{(n\times 1)}$ one concludes that

$$\mathfrak{E}_+^{-1}(\lambda)\,\Phi(\lambda)=P_+(\mathfrak{E}_-\mathcal{F}).$$

Consequently, in all cases, whenever equation (A) with $f\in L^+_{(n\times 1)}$ is solvab the Fourier transform of one of its solutions is given by

$$\Phi(\lambda)=\mathfrak{E}_+P_+(\mathfrak{E}_-\mathcal{F})$$

or by

$$\Phi(\lambda)=\mathfrak{G}_+P_+(\mathfrak{G}_-\mathcal{F}), \qquad (12.1$$

where

$$\mathfrak{G}_+(\lambda)=\mathfrak{E}_+(\lambda)\,\mathfrak{E}_+^{-1}(\infty) \quad\text{and}\quad \mathfrak{G}_-(\lambda)=\mathfrak{E}_-^{-1}(\infty)\,\mathfrak{E}_-(\lambda).$$

Repeating the considerations employed in obtaining equation (12.6) for the resolvent of equation (A) in the case when the homogeneous equations (B) and (I have only the zero solution, one verifies that equation (12.17) may be shown to lead to

$$\varphi(t)=f(t)+\int_0^\infty\gamma(t,s)f(s)\,ds. \qquad (12.1$$

The matrix kernel $\gamma(t,s)$ $(0\le t,\ s<\infty)$ is defined by

$$\gamma(t,s)=\gamma(t-s)+\gamma_\tau(t-s)+\int_0^\infty\gamma(t-r)\,\gamma_\tau(s-r)\,dr \quad (0\leqslant t,\ s<\infty), \qquad (12.1$$

where

$$\mathfrak{G}_+(\lambda)=1+\int_0^\infty\gamma(t)\,e^{i\lambda t}\,dt, \qquad \mathfrak{G}_-(\lambda)=1+\int_0^\infty\gamma_\tau(t)\,e^{i\lambda t}\,dt, \qquad (12.2$$

and

$$\gamma(t)=\gamma_\tau(t)=0 \quad (-\infty<t<0). \qquad (12.2$$

Let us now show that equation (12.18) furnishes a solution of equation (A) in any space $E^+_{(n\times 1)}$ whenever $f\in E^+_{(n\times 1)}$ and

$$\int_0^\infty f(s)\,\psi_j'(s)\,ds=0 \quad (j=1,\ 2,\ \dots,\ \beta), \qquad (12.2$$

where $\psi_j(s)$ $(j=1,2,\cdots,\beta)$ is a complete system of linearly independent solutions of equation (B′).

Let us denote, for brevity, by Γ the operator generated by the kernel $\gamma(t, s$ recalling that this operator Γ is a bounded operator acting on $E^+_{(n\times 1)}$. Let us denote by $\omega_j(t)$ $(j=1,2,\cdots,\beta)$ a system of vector functions in the intersection $E_\cap$ of these spaces, such that

$$(\psi_k,\ \omega_j)=\int_0^\infty\psi_k'(s)\,\omega_j(s)\,ds=\delta_{jk} \quad (j,\ k=1,\ 2,\ \dots,\ \beta).$$

It is readily seen that, for any $f \in L^+_{(n\times 1)}$, the difference

$$f_1 = f - \sum_{j=1}^{\beta} (f, \psi_j)\omega_j$$

belongs to the space $L^+_{(n\times 1)}$ and that $(f_1, \psi_j) = 0$ $(j = 1, 2, \cdots, \beta)$. Recalling that the equation

$$\varphi - K\varphi = f_1$$

is solvable, one obtains one of its solutions by means of (12.18)

$$\varphi = f_1 + \Gamma f_1.$$

Hence

$$(I - K)(I + \Gamma) f_1 - f_1 = 0.$$

Substituting the value of the vector function f_1 in the last equation one obtains

$$(I - K)(I + \Gamma) f - f = \sum_{j=1}^{\beta} (f, \psi_j)\chi_j, \qquad (12.23)$$

where the vector functions $\chi_j(t)$ $(j = 1, 2, \cdots, \beta)$ are determined by

$$\chi_j = (I - K)(I + \Gamma)\omega_j - \omega_j \qquad (j = 1, 2, \ldots, \beta)$$

and belong to all spaces $E^+_{(n\times 1)}$.

Making use of the arbitrariness of the function $f \in L^+_{(n\times 1)}$ in (12.23) one obtains that

$$\gamma(t, s) = k(t - s) + \int_0^\infty k(t - r)\gamma(r, s)\,dr + l(t, s),$$

where

$$l(t, s) = \left\| \sum_{j=1}^{\beta} \psi_{jp}(s)\chi_{jq}(t) \right\|_1^n,$$

and ψ_{jp} and χ_{jq} $(p, q = 1, 2, \cdots, n)$ are the coordinates, respectively, of the vector ψ_j and χ_j $(j = 1, 2, \cdots, \beta)$.

Suppose now that $f(t)$ is an arbitrary function in $E^+_{(n\times 1)}$, satisfying equation (12.22).

Then, upon multiplying both sides of (12.23) on the right by the vector function $f(t)$ and integrating the resulting equation between the limits zero and infinity, one obtains that

$$(I - K)(I + \Gamma) f = f.$$

Thus the vector function

$$\varphi(t) = f(t) + \int_0^\infty \gamma(t, s) f(s)\,ds \quad (\in E^+_{(1\times n)})$$

is indeed a solution of equation (A).

We have proved the following theorem:

Theorem 12.2. *If the matrix function $I - \mathcal{K}(\lambda)$ $(k(t) \in L)$ is nonsingular, th for every vector function $f(t) \in E^+_{(n\times 1)}$, satisfying the solvability conditions (12.22), one of the solutions of equation (A) is given by the formula*

$$\varphi(t) = f(t) + \int_0^\infty \gamma(t, s) f(s)\,ds \quad (\in E^+_{(n\times 1)}),$$

where the "generalized resolvent" $\gamma(t, s)$ is defined by equations (12.19), (12.2 and (12.21).

§13. Spectrum of the operator K in the space $E^+_{(n\times 1)}$. Let us recall that every matrix function $k(t) \in L_{(n\times n)}$ generates, in the space $E^+_{(n\times 1)}$, a linear, bounded operator, defined by the equation

$$K\varphi = \int_0^\infty k(t - s)\varphi(s)\,ds \qquad (0 \leq t < \infty).$$

From Theorem 2.1 it follows that, for an arbitrary complex number ζ, satisf the condition

$$\det\left(\zeta I - \int_{-\infty}^{\infty} k(t)e^{i\lambda t}\,dt\right) \neq 0 \qquad (-\infty < \lambda < \infty), \qquad (13.$$

the operator $\zeta I - K$ is a Φ-operator. Consequently, any number ζ, satisfying (13.1), is a Φ-number [1)] of the operator K. From among the above Φ-points ζ the points in the spectrum of the operator K are those and only those points for which at least one of the numbers

$$\alpha(\zeta) = \alpha(\zeta I - K), \qquad \beta(\zeta) = \beta(\zeta I - K)$$

is not zero.

In the scalar case (see [1]) it was shown that all points not satisfying condition (13.1) are points of the spectrum of the operator K; further, these points are not Φ-points, nor $\Phi_\pm$-points, of the operator K.

In the present matrix situation the following, similar, proposition is valid.

Theorem 13.1. *The spectrum of the operator K, generated by a matrix $k(t)$ $\in L_{(n\times n)}$, and acting on the space $E^+_{(n\times 1)}$, consists of all points ζ satisfying (13.1) for which at least one of the numbers $\alpha(\zeta)$, $\beta(\zeta)$ is not zero (these poin are ϕ-points of the operator K) and all points ζ for which $\det(\zeta I - \mathcal{K}(\lambda)) = 0$ These last are neither Φ-points nor $\Phi_\pm$-points of the operator K.*

Proof. Taking into account the remarks made before the statement of the theorem, it only remains to prove the second part of the theorem.

Let us suppose that the point ζ_0, at which the determinant $\det(\zeta_0 I - \mathcal{K}($ vanishes for some real number λ_0, is a Φ-point of the operator K. Then there

1). A point of the complex plane, ζ, is called a Φ-point of the operator A acting on a Banach space $\mathfrak{B}$, provided that the operator $\zeta I - A$ is a Φ-operator. If the operat $\zeta I - A$ possesses a normal extension, and least one of the numbers $\alpha(\zeta)$, $\beta(\zeta)$ is finit then the point ζ is called a $\Phi_\pm$-point.

exists a $\delta > 0$, such that for every operator K_1 satisfying the inequality

$$\| K_1 - K \| < \delta, \tag{13.2}$$

the point ζ_0 is also a Φ–point.

Let us select the matrix function $r(t) = \| r_{jk}(t) \|_1^n \in L_{(n\times n)}$ so that its Fourier transform $\mathcal{R}(\lambda)$ is a rational matrix function satisfying, firstly

$$\| K - R \| < \frac{\delta}{2},$$

where R is the operator defined by the equality

$$R\varphi = \int_0^\infty r(t-s)\varphi(s)\,ds \qquad (0 \leqslant t < \infty),$$

and, secondly, in the circle $| \zeta - \zeta_0 | < \frac{\delta}{2}$ there is at least one root ζ_1 of the equation $\det(R(\lambda_0) - \zeta I) = 0$.

Then the operator $K_1 = R + (\zeta_0 - \zeta_1) I$ will satisfy (13.2). Consequently, the point ζ_0 will be a Φ–point of this operator, or, what is the same, the operator $\zeta_1 I - R$ will be a Φ–operator.

The rational matrix function $\zeta_1 I - \mathcal{R}(\lambda)$ may be represented in the form

$$\zeta_1 I - \mathcal{R}(\lambda) = \mathfrak{M}_1(\lambda)\,\mathfrak{D}(\lambda)\,\mathfrak{M}_2(\lambda), \tag{13.3}$$

where $\mathfrak{D}(\lambda)$ ($\in \mathfrak{R}_{(n\times n)}$) is a diagonal rational matrix function, and the matrices $\mathfrak{M}_j(\lambda)$ ($\in \mathfrak{R}_{(n\times n)}$) among other things are such that

$$\det \mathfrak{M}_j(\lambda) = 1 \qquad (j = 1,\ 2;\ -\infty \leqslant \lambda \leqslant \infty).$$

The matrices $\mathfrak{M}_j(\lambda)$ ($j = 1, 2$) correspond to the Φ–operators

$$U_j\varphi = A_j\varphi(t) - \int_0^\infty m_j(t-s)\varphi(s)\,ds \qquad (0 \leqslant t < \infty;\ j = 1,\ 2),$$

where

$$\mathfrak{M}_j^{-1}(\lambda) = A_j - \int_0^\infty m_j(t)\,e^{i\lambda t}\,dt \qquad (j = 1,\ 2;\ m_j \in L_{(n\times n)}).$$

The operator

$$U\varphi = B\varphi - \int_0^\infty l(t-s)\varphi(s)\,ds \qquad (0 \leqslant t < \infty),$$

where

$$\mathfrak{D}(\lambda) = B - \int_0^\infty l(t)\,e^{i\lambda t}\,dt \qquad (l \in L_{(n\times n)}),$$

is also a Φ–operator; because, in view of (13.3), it differs from the product of the Φ–operators $U_1(\zeta_1 I - R)U_2$ by a completely continuous term.

Since the matrices $B = \| b_{jj}\delta_{jk} \|_1^n$ and $l(t) = \| l_{jj}(t) \|_1^n$ are diagonal matrices, the Φ–operator U reduces to the direct sum of n scalar Φ–operators V_j, each of which acts on the space E according to the formulas:

$$V_j\varphi = b_{jj}\varphi(t) - \int_0^\infty l_{jj}(t-s)\varphi(s)\,ds \qquad (0 \leqslant t < \infty;\ j = 1,\ 2,\ \ldots,\ n).$$

From the equality of the determinants of the matrix functions $\mathfrak{D}(\lambda)$ and $\zeta_1 - \mathcal{R}(\lambda)$ it follows that at least one of the functions

$$b_{jj} - \int_{-\infty}^{\infty} l_{jj}(t)\,e^{i\lambda t}\,dt \qquad (j = 1,\ 2,\ \ldots,\ n)$$

must be zero at $\lambda = \lambda_0$. This contradicts the fact that all operators V_j ($j = 1, 2, \cdots,$ are Φ–operators (see [1]).

Similarly, it may be proved that the point ζ_0 can not be a $\Phi_\pm$–point of the operator K. The theorem is proved.

§14. Factorization of a matrix function on a circle, and discrete analogue equation (A). As the analogue of the normed ring $\mathfrak{R}$ of functions on the closed axis $\{-\infty, \infty\}$ one has to consider, on the circle $|\zeta| = 1$, the ring $(\mathfrak{R})$ of all absolutely convergent series

$$\mathfrak{a}(\zeta) = \sum_{k=-\infty}^{\infty} a_k\zeta^k \qquad (|\zeta| = 1)$$

with the ordinary multiplication, and with the norm

$$\| \mathfrak{a} \| = \sum_{k=-\infty}^{\infty} |a_k|.$$

In a similar manner as the ring $\mathfrak{R}$ was split into subrings $\mathfrak{R}^+$ and $\mathfrak{R}^-$ one may split the ring $(\mathfrak{R})$ into subrings $(\mathfrak{R}^+)$, $(\mathfrak{R}^-)$, where $(\mathfrak{R}^+)$ and $(\mathfrak{R}^-)$ consis of those functions $\mathfrak{a}(\zeta) \in (\mathfrak{R})$ for which, $a_k = 0$ for $k < 0$ and, respectively, $a_k = 0$ for $k > 0$.

Thus, each function $\mathfrak{a}(\zeta) \in (\mathfrak{R}^+)$ admits a unique continuous extension $\mathfrak{a}($ to the circle $C_+ : |z| \leq 1$, holomorphic inside this circle, to wit

$$\mathfrak{a}(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (|z| \leqslant 1),$$

and a similar assertion holds relative to the extension of the function $\mathfrak{a}(\zeta)$ to t "circle" $C_- : |z| \geq 1$.

Wiener's Theorems W and $W_\pm$ concerning the rings $\mathfrak{R}$ and $\mathfrak{R}^\pm$ carry over co pletely to the rings $(\mathfrak{R})$ and $(\mathfrak{R}^\pm)$. As a matter of fact, Wiener first proved his Theorem W for the ring $(\mathfrak{R})$.

It should be noticed that all theorems concerning the factorization of matrix functions $\mathfrak{M}(\lambda) \in \mathfrak{R}_{(n\times n)}$ carry over to matrices $\mathfrak{A}(\zeta) \in (\mathfrak{R}_{(n\times n)})$, the proofs becoming somewhat simpler and transparent.

By analogy to the definition of section 7, a left standard factorization of a continuous nonsingular matrix function $\mathfrak{A}(\zeta)$ ($|\zeta| = 1$) will mean a representatic of this matrix in the form

$$\mathfrak{A}(\zeta) = \mathfrak{B}_+(\zeta)\begin{pmatrix} \zeta^{\varkappa_1} & 0 \ldots 0 \\ 0 & \zeta^{\varkappa_2} \ldots 0 \\ \cdot & \cdot \;\cdot\;\cdot\;\cdot \\ 0 & 0 \ldots \zeta^{\varkappa_n} \end{pmatrix}\mathfrak{B}_-(\zeta),$$

where $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ are certain integers, and the matrix functions $\mathfrak{B}^{\pm}(\zeta)$ admit analytic continuations $\mathfrak{B}^{\pm}(z)$ which are holomorphic inside of and continuous up to and including the boundary of, respectively, the circles $C_{\pm}$, and also are such that their determinants are not zero there:

$$\det \mathfrak{B}^+(z) \neq 0 \quad (|z| \leqslant 1), \qquad \det \mathfrak{B}^-(z) \neq 0 \quad (|z| \geqslant 1).$$

Similarly to what was proved in Theorem 7.1, one can show here that the numbers $\kappa_1, \kappa_2, \cdots, \kappa_n$ are uniquely determined by the matrix $\mathfrak{A}(\zeta)$; these integers are called the left exponents of the matrix function $\mathfrak{A}(\zeta)$.

Analogously, one defines a right standard factorization and the right exponents of a matrix function $\mathfrak{A}(\zeta)$.

The fundamental Theorem 7.3 has the following counterpart.

Theorem 14.1. *Every nonsingular matrix function* $\mathfrak{A}(\zeta) \in (\Re_{(n\times n)})$ *admits a left (right) standard factorization and, in all possible factorizations, the factors* $\mathfrak{B}_{\pm}(\lambda) \in (\Re^{\pm}_{(n\times n)})$.

In analogy to what was proved in section 8, it may be asserted that if the matrix function $\mathfrak{A}(\zeta) + \mathfrak{A}^*(\zeta)$ is positive definite then all left (right) exponents of $\mathfrak{A}(\zeta)$ must be zero.

For hermitian nonsingular matrix functions $\mathfrak{A}(\zeta) \in (\Re_{(n\times n)})$ one has Yu. L. Šmul'ian's theorem, to the effect that the number of positive, as well as the number of negative, left (right) exponents of a nonsingular hermitian matrix function $\mathfrak{A}(\zeta)$ does not exceed $\min(p, q)$, where p and q, respectively, are the number of positive and negative squares of the form $\xi^* \mathfrak{A}(\zeta)\, \xi$. Strictly speaking, Yu. L. Šmul'ian proved his theorem under the tacit assumption that $\mathfrak{A}(\zeta)$ admitted a standard factorization.

It is reasonable to expect that all the results of the remaining paragraphs carry over to matrix functions $\mathfrak{A}(\zeta) \in (\Re_{(n\times n)})$. In particular, Theorems 10.1' and 10.2 concerning the stability of systems of exponents remain valid. [1)]

Just as the theorems concerning the factorization of the matrix functions $\mathfrak{M}(\lambda) \in \Re_{(n\times n)}$ lead to a series of important results and at the same time serve as a basis for certain results, proved independently elsewhere, of the theory of the integral equation (A), the theorems concerning the factorization of the matrix functions $\mathfrak{A}(\zeta)$ $(|\zeta| = 1)$ allow one to prove, and in certain cases serve as a basis for, important propositions of the theory of infinite systems of equations

$$\sum_{k=0}^{\infty} A_{j-k}x_k = c_j \qquad (j = 0, 1, 2, \ldots), \qquad (14.$$

where A_j $(j = 0, \pm 1, \pm 2, \cdots)$ are matrices of a fixed order n, c_j $(j = 1, 2, \cdots)$ are n dimensional column vectors, and x_k are unknown n dimensional column vectors.

Supposing that the elements of the matrices A_j $(j = 1, 2, \cdots)$ are absolutel convergent series, we may associate to the system (14.1) the matrix function

$$\mathfrak{A}(\zeta) = \sum_{j=-\infty}^{\infty} A_j \zeta^j.$$

As in the corresponding continuous case, the study of the system (14.1) ma be based essentially on considering it simultaneously in a whole family of spac $(l_p)_{(n\times 1)} (p \geq 1)$, $(m)_{(n\times 1)}$, $(c)_{(n\times 1)}$, $(c_0)_{(n\times 1)}$. By the notation $(E_{(n\times 1)})$ we shall understand any of these spaces. Let us explain the notation further.

As usual, by (l_p) $(p \geq 1)$, we shall understand the Banach space of all sequences $\xi = \{\xi_j\}_0^\infty$ of complex numbers such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converge the norm being defined by $\|\xi\|_p = (\sum_{j=0}^{\infty} |\xi_j|^p)^{\frac{1}{p}}$. By (m) we shall denote the Banach space of all sequences of complex numbers $\xi = \{\xi_j\}_0^\infty$ which are bound in absolute value, with the norm $\|\xi\|_m = \sup_{0\leq j<\infty} |\xi_j|$; and, finally, by (c) and (c_0) the subspaces of (m) consisting of all convergent, and convergent to zero, respectively, subsequences. If (E) is any of these spaces, the notation $(E)_{(n\times}$ will stand for the linear space of all sequences of n elements, where each eleme for $p = 1, 2, \cdots, n$, belong to (E).

Almost all the propositions which were established for the integral equatio (A), (B) and (B') have valid analogues in the theory of the system (14.1) and its associated homogeneous systems

$$\sum_{k=0}^{\infty} A_{j-k}\, u_k = 0 \qquad (j = 0, 1, 2, \ldots) \qquad (14.$$

and

$$\sum_{k=0}^{\infty} A'_{k-j}\, v_k = 0 \qquad (j = 0, 1, 2, \ldots), \qquad (14.$$

where A_j' are the transpose matrices of the A_j $(j = 0, 1, 2, \cdots)$.

Thus, for example, the condition (0.2) has as counterpart

$$\det \mathfrak{A}(\zeta) \neq 0 \qquad (|\zeta| = 1)$$

and the following propositions hold if it is satisfied:

1. In all spaces $(E_{(n\times 1)})$ the system (14.2) (the system (14.3)) has the same solutions. The solutions of the system (14.2) (the system (14.3)) form a

1). Further, in the note [28] the authors have shown that the method of proof of these theorems applies also to the exponents in the matrix Hilbert problem for one or several contours in the usual formulation involving functions satisfying a Hölder condition.

finite dimensional set, whose dimension coincides with the sum of the positive (absolute value of the negative) left exponents of the matrix $\mathfrak{A}^{-1}(\zeta)$.

2. If the sequence $\{c_j\}_0^\infty \in (E_{(n\times 1)})$, then the system (14.1) has a solution $\{x_j\}_0^\infty \in (E_{(n\times 1)})$ if and only if one has that [1]

$$\sum_{j=0}^{\infty}(c_j,\ u_j)=0,$$

where $\{u_j\}_0^\infty \in (E_{(n\times 1)})$ is any solution of the system (14.3).

Hence, the system (14.1) is solvable in $(E_{(n\times 1)})$, for any given $\{c_j\}_0^\infty \in \in (E_{(n\times 1)})$, whenever all right exponents of the matrix $\mathfrak{A}(\zeta)$ are nonpositive, and has a unique solution if all these exponents are zero.

Suppose that all the left exponents of $\mathfrak{A}^{-1}(\zeta)$ are zero and that

$$\mathfrak{A}^{-1}(\zeta)=\Gamma_+(\zeta)\Gamma_-(\zeta) \qquad (\Gamma_\pm(\zeta)\in(\mathfrak{R}^\pm)_{(n\times n)})$$

is a left standard factorization of the matrix $\mathfrak{A}^{-1}(\zeta)$. If

$$\Gamma_+(\zeta)=\sum_{j=0}^{\infty}\gamma_j^{(1)}\zeta^j, \qquad \Gamma_-(\zeta)=\sum_{j=0}^{\infty}\gamma_j^{(2)}\zeta^{-j},$$

where γ_j^1 and γ_j^2 $(j=1,2,\cdots)$ are certain n-dimensional matrices, then the solution of the system (14.1) is always given by the formula

$$x_j=\sum_{k=0}^{\infty}\gamma_{jk}c_k \qquad (j=0,\ 1,\ \ldots), \tag{14.4}$$

where the matrices γ_{jk} are determined by the system

$$\gamma_{jk}=\sum_{r=0}^{\min(j,k)}\gamma_{j-r}^{(1)}\gamma_{k-r}^{(2)} \qquad (j,\ k=0,\ 1,\ 2,\ \ldots).$$

The "generalized resolvent matrix" $\|\gamma_{jk}\|_0^\infty$ furnishes, by means of (14.4), a solution of the system (14.1), provided one exists; this matrix may be constructed no matter what the left exponents of the matrix $\mathfrak{A}^{-1}(\zeta)$ happen to be. To see this it is only necessary to proceed similarly to what was done in number 3 of section 12.

If $u=\{u_j\}_0^\infty$ is a solution of system (14.2) such that

$$u_0=u_1=\ldots=u_{\varkappa-1}=0,$$

then, as is easily seen, the sequence $\{u_{j+r}\}_{j=0}^\infty$ $(r=1,2,\cdots,\kappa)$ will also be a solution of the system (14.2). If $u_\kappa\neq 0$, we will say that the solution u generates a d-chain of length κ of solutions of the system (14.2).

Proposition 1 above may be strengthened by adding that if $\kappa_1,\kappa_2,\cdots,\kappa_r$ are all the positive left exponents of the matrix $\mathfrak{A}^{-1}(\zeta)$, then the system (14.2) has exactly r solutions, generating, respectively, d-chains of solutions of (14.2) of lengths $\kappa_1,\kappa_2,\cdots,\kappa_r$, and that the solutions appearing in these d-chains form a basis for all the solutions of the system (14.2) (analogue of Theorem 9.1).

1). If x, y are n-dimensional vectors, then (x, y) denotes the sum of all the products of the corresponding coordinates of the vectors x and y.

It is left to the reader to formulate discrete analogues of other proposition relative to the integral equations (A) and (B).

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