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THE L-THEORY OF LAURENT EXTENSIONS AND GENUS 0 FUNCTION FIELDS

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Introduction

The stable classification of quadratic forms over a field is given by the Witt group. Topology, via surgery theory, has embedded the Witt groups in a general theory of forms over any ring with involution. In this paper we use geometrically inspired methods to make computations. General results on the L-theory of a Laurent polynomial extension are used to study the Witt groups of genus 0 function fields.

Given a ring A with involution $\tau : a \longrightarrow \bar{a}$ and a central unit $u \in A$ such that $\bar{u} = u$ let $A[t,t^{-1}]_u$ denote the Laurent extension ring $A[t,t^{-1}]$ with the involution $\bar{t} = ut^{-1}$, abbreviated to $A[t,t^{-1}]$ for $u = 1$. In Theorem 4.1 we establish the exact sequence

$$\begin{aligned} \dots \longrightarrow L^n_p(A) \xrightarrow{1-u} L^n_h(A) &\xrightarrow{i_!} L^n_h(A[t,t^{-1}]_u) \\ &\xrightarrow{B} L^{n-1}_p(A) \xrightarrow{1-u} L^{n-1}_h(A) \longrightarrow \dots \end{aligned}$$

expressing the free symmetric L-groups of $A[t,t^{-1}]_u$ in terms of the free and projective symmetric L-groups of A. For $u = 1$ the exact sequence of 4.1 becomes a splitting

$$L^n_h(A[t,t^{-1}]) \ = \ L^n_h(A) \oplus L^{n-1}_p(A) \ .$$

Although the exact sequence is stated and proved only for the symmetric L-groups L^* there is an entirely analogous treatment for the quadratic L-groups L_*.

In the case where $\frac{1}{2} \in A$ the symmetric and quadratic L-groups are equal, $L_* = L^*$. If A is a field $\mathbb{F}$ of characteristic different from 2 then $L_0(A) = L^0(A) = W(\mathbb{F})$, the Witt group of non-singular quadratic forms over $\mathbb{F}$ modulo hyperbolics. $W(\mathbb{F})$ is a Hermitian Witt group if the involution on $\mathbb{F}$ is non-trivial.

In §5 - §7 we apply the exact sequence of 4.1 to determine the Witt groups of genus 0 function fields. These are the fields of the form

$$\mathbb{F}\langle\lambda,\mu\rangle \ = \ \mathbb{F}(x)[y]/(y^2 - \mu x^2 + 4\lambda\mu)$$

for non-zero λ,μ in the ground field $\mathbb{F}$. Perhaps the most interesting of these fields is $\mathcal{F} = \mathbb{R}(x)[y]/(x^2 + y^2 + 1)$. Here, our methods recover the following result of Knebusch [9] as a special case:

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Theorem 5.3:

(i) There is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow L_0(\mathcal{F}) \longrightarrow \mathbb{Z}/2 \oplus \bigoplus_{\mathcal{I}} \mathbb{Z}/2 \longrightarrow 0$$

where the index set $\mathcal{I}$ is the points $p \in \mathcal{M}_1$, and $\mathcal{M}_1$ is the open Möbius band

$$\{\mathbb{C} - \{0\}/(z \sim -1/\bar{z})\} .$$

The first $\mathbb{Z}/2$ in the right hand sum carries the element $\langle 1 \rangle$.

(ii) The sequence above does not split. In particular the Stufe of $\mathcal{F}$ is 2.

The Stufe of a field $\mathcal{F}$ is the least integer n (necessarily a power of 2) such that -1 is a sum of n squares in $\mathcal{F}$. As a consequence the exponent of $W(\mathcal{F})$ is $2n$.

In §6 and §7 we give a more detailed analysis of these Witt groups, using a commutative square of injections of rings

$$\begin{array}{ccc} \mathbb{F}[x](y) & \longrightarrow & \mathbb{F}\langle \lambda, \mu \rangle \\ \big\downarrow & & \big\downarrow \\ \mathbb{K}[t, t^{-1}] & \longrightarrow & \mathbb{K}(t) \end{array}$$

for any elements λ, μ in $\mathbb{F}$ with λ non-zero and μ non-square, $\mathbb{K} = \mathbb{F}(\sqrt{\mu})$ a degree two Galois extension of $\mathbb{F}$, $\mathbb{F}[x](y) = \mathbb{F}[x, y]/(y^2 - \mu x^2 + 4\lambda\mu)$, and

$$x \ = \ t + \lambda t^{-1} \quad , \quad y \ = \ \sqrt{\mu}(t - \lambda t^{-1}) \quad , \quad t \ = \ (x + y/\sqrt{\mu})/2 \ .$$

The genus 0 function field $\mathbb{F}\langle \lambda, \mu \rangle$ is the fixed field of the involution on the rational function field $\mathbb{K}(t)$ defined by the extension of the Galois automorphism on $\mathbb{K}$ with

$$\bar{t} \ = \ (x - y/\sqrt{\mu})/2 \ = \ \lambda t^{-1} \ .$$

The basic result 7.14 defines a group V and proves the existence of an exact sequence

$$0 \longrightarrow L_0(\mathbb{K}[t, t^{-1}]_\lambda) \longrightarrow L_0(\mathbb{F}[x](y)) \longrightarrow V \longrightarrow L_3(\mathbb{K}[t, t^{-1}]_\lambda) \longrightarrow L_3(\mathbb{F}[x](y)) \longrightarrow 0 \ .$$

The Witt group $L_0(\mathbb{F}[x](y))$ is the kernel of the "second boundary map" from $L_0(\mathbb{F}\langle \lambda, \mu \rangle)$ to the Witt groups of its various completions, while the group $L_3(\mathbb{F}[x](y))$ is the cokernel. Hence this latter group serves as an explicit reciprocity law for these fields.

The exact sequence above is a special case of one of the sequences in the "Twisting Diagram (0.1)" appearing in Hambleton, Taylor and Williams [6]. The key new observation here is that $\mathbb{K}[t, t^{-1}]$ is an integral quadratic extension of $\mathbb{F}[x](y)$ (Lemma 6.4).

The reciprocity law arises as follows. There is the well known exact sequence of Jacobson

$$0 \longrightarrow L_0(\mathbb{K}(t)_\lambda) \longrightarrow L_0(\mathbb{F}\langle\lambda,\mu\rangle) \longrightarrow L_0(\mathbb{K}(t)).$$

(The referee points out that D.W. Lewis [11] has generalized this sequence.) Localization gives an exact sequence

$$0 \to L_0(\mathbb{K}[t,t^{-1}]_\lambda) \to L_0(\mathbb{K}(t)_\lambda) \to \bigoplus_{\mathcal{P}} L_0(\mathbb{K}[t,t^{-1}]_\lambda/\mathcal{P}) \xrightarrow{j} L_3(\mathbb{K}[t,t^{-1}]_\lambda) \to 0\ .$$

Thus the Hermitian Witt group $L_0(\mathbb{K}(t)_\lambda)$ is determined by the L-groups of the quotient fields $\mathbb{K}[t,t^{-1}]_\lambda/\mathcal{P}$ and the groups $L_*(\mathbb{K}[t,t^{-1}]_\lambda)$. Moreover, j above can be regarded as a reciprocity law for $L_0(\mathbb{K}(t)_\lambda)$.

It should be noted that our main exact sequence for a Laurent extension completely determines the groups $L_*(\mathbb{K}[t,t^{-1}]_\lambda)$ in terms of the Witt group of $\mathbb{K}$. The possibility of applying the techniques of L-theory to the study of the Witt groups of genus 0 function fields first came up in conversations with W. Jacob and A. Wadsworth. T. Y. Lam asked us about the special case $\mathcal{F}$ above. We thank the referee for drawing our attention to the recent work of Parimala [14], which contains (§§4-6) a thorough study of the Witt rings of conics by methods very different from ours. It seems to require some work to exhibit the precise relationship between her main results, in particular Theorem 6.2 of [14] and our Theorem 7.14.

In §8 we briefly consider the situation where the involution acts trivially on the field $\mathbb{F}$. Here, we obtain explicit reciprocity laws and by way of an example show:

Corollary 8.6: *If $\mathbb{R}[t,t^{-1}]$ is given the involution which is the identity on $\mathbb{R}$ and $\bar{t}=t^{-1}$ then the associated Hermitian Witt group is (naturally) isomorphic to the uncountable direct sum*

$$L_0(\mathbb{R}(t)) \;\cong\; \bigoplus_{S^1}\mathbb{Z}$$

where S^1 denotes the points of the unit circle.

We do not entirely understand the interplay between the L-groups and the real algebraic varieties in the statements of 5.3 and 8.6.

The quadratic L-groups $L_*(A)$ are the Wall [25] surgery obstruction groups. Shaneson [22] obtained a geometric splitting of the simple quadratic L-groups

$$L^s_n(A[t,t^{-1}]) \;=\; L^s_n(A)\oplus L^h_{n-1}(A)$$

for $A=\mathbb{Z}[\pi]$, $\bar{t}=t^{-1}$ using the realization theorem of [25] to express elements of $L^s_*(\mathbb{Z}[\pi][t,t^{-1}])=L^s_*(\mathbb{Z}[\pi\times\mathbb{Z}])$ as the surgery obstructions of normal maps with fundamental group $\pi\times\mathbb{Z}$ which are simple homotopy equivalences on the boundary, and applying the codimension 1 splitting theorem of Farrell and Hsiang [4]. The algebraic splitting theorem of Novikov [13] and Ranicki [15] for the quadratic L-groups $L^q_*(A[t,t^{-1}])$ ($\bar{t}=t^{-1},\ q=s,h$) was proved by an analysis of quadratic forms and formations over $A[t,t^{-1}]$, working by analogy with the analysis of invertible matrices over $A[t,t^{-1}]$ used by Bass, Heller and Swan [2] and Bass [1] to prove the algebraic splitting theorem for $K_1(A[t,t^{-1}])$. Certain discrepancies between the algebraic and geometric methods of proof were clarified in Ranicki [19]. The L-theory splitting theorem is proved here by algebra, using the expression in Ranicki [16],[17] of the L-groups as the cobordism groups of chain complexes with Poincaré duality. The codimension 1 transversality of the geometric proof is replaced by an L-theoretic version of the method of Mayer-Vietoris presentations used by Waldhausen [23],[24] in algebraic K-theory, which is an extension of the linearization trick of Higman [8]. The geometric approach of Wall [25,Thm.12.6] applies to the quadratic L-groups of the Laurent polynomial extension $A[t,t^{-1}]=\mathbb{Z}[\pi\times\mathbb{Z}]$ of $A=\mathbb{Z}[\pi]$, with $\bar{t}=ut^{-1}$ both for $u=+1$ and $u=-1$. See Ranicki [17, §7.6] for an account of the geometric background to quadratic L-theory splitting theorems for $A[t,t^{-1}]$ with $\bar{t}=\pm t^{-1}$, and also of the case $\bar{t}=t$. The splitting theorem proved here for the symmetric L-groups $L^*(A[t,t^{-1}])$ ($\bar{t}=t^{-1}$) was conjectured in Ranicki [16, §10].The symmetric L-groups $L^*(\mathbb{Z}[\pi])$ are not geometrically realizable, so that the geometric methods of proof do not apply. See Ranicki [21] for a further development of the L-theory of Laurent extensions.

We thank M. Knebusch for some very helpful comments. In particular he pointed out that the genus 0 function fields were first studied in depth by E. Witt [26].

Contents

§1. The K-theory of Laurent extensions

Let A be an associative ring with 1. We shall be working with left A-modules and positive A-module chain complexes C.

An A-module chain complex C is *finite* if it is a finite-dimensional complex of based f.g. (finitely generated) free A-modules. We assume that the reader is already familiar with the *torsion* $\tau(E)\in K_1(A)$ of a contractible finite complex E. The *(reduced) torsion* of a chain equivalence $f:C\longrightarrow D$ of finite complexes is defined as usual by

$$\tau(f)\ =\ \tau(C(f))\in \tilde{K}_1(A)\ =\ coker(K_1(\mathbb{Z})\longrightarrow K_1(A))$$

with $C(f)$ the algebraic mapping cone.

An A-module chain complex C is *finitely dominated* if there exists a domination $(D, f : C \longrightarrow D, g : D \longrightarrow C, h : gf \simeq 1 : C \longrightarrow C)$ by a finite complex D, or equivalently if it is chain equivalent to a finite-dimensional complex P of f.g. projective A-modules. The *projective class* of C is defined by

$$[C] \;=\; [P] \;=\; \sum_{r=0}^{\infty}(-)^r[P_r] \in K_0(A)$$

for any such P. An A-module chain complex C is *homotopy finite* if it is chain equivalent to a finite complex. The reduced projective class of a finitely dominated chain complex C is the *finiteness obstruction*

$$[C] \;=\; [P] \in \tilde{K}_0(A) \;=\; \mathit{coker}(K_0(\mathbb{Z}) \longrightarrow K_0(A)),$$

with $[C] = 0$ if and only if C is homotopy finite. See Ranicki [18] for the algebraic theory of finiteness obstruction of finitely dominated chain complexes.

Let $A[t, t^{-1}]$ be the Laurent polynomial extension ring, consisting of the polynomials $\sum_{j=-\infty}^{\infty} a_j t^j$ in a central invertible indeterminate t over A with the coefficients $a_j \in A$ such that $\{j \in \mathbb{Z} \mid a_j \neq 0\}$ is finite. The inclusion of rings

$$i : \; A \longrightarrow A[t, t^{-1}]$$

determines functors

$$i_! : (A - \text{modules}) \longrightarrow (A[t, t^{-1}] - \text{modules}); \quad F \longrightarrow i_! F \;=\; A[t, t^{-1}] \otimes_A F \;=\; F[t, t^{-1}],$$

$$i^! : \; (A[t, t^{-1}] - \text{modules}) \longrightarrow (A - \text{modules}); E \longrightarrow i^! E \;=\; E,$$

with A acting on $i^! E$ by the restriction of the $A[t, t^{-1}]$-action to $A \subset A[t, t^{-1}]$.

Given an A-module F let F^+, F^- denote the A-modules

$$F^+ \;=\; \sum_{j=0}^{\infty} t^j F, \quad F^- \;=\; \sum_{j=-\infty}^{-1} t^j F,$$

so that

$$i^! i_! F \;=\; F^+ \oplus F^-.$$

If F is a f.g. A-module and G is any A-module it is possible to express every $A[t, t^{-1}]$-module morphism $\alpha : F[t, t^{-1}] \longrightarrow G[t, t^{-1}]$ as a polynomial

$$\alpha \;=\; \sum_{j=-\infty}^{\infty} t^j \alpha_j \quad : \quad F[t, t^{-1}] \longrightarrow G[t, t^{-1}]$$

with the coefficients A-module morphisms $\alpha_j \in \mathit{Hom}_A(F,G)$ such that $\{j \in \mathbb{Z} \mid \alpha_j \neq 0\}$ is finite.

Given an isomorphism of based f.g. free $A[t,t^{-1}]$-modules

$$\alpha \;=\; \sum_{j=-N}^{M} t^j\alpha_j :\; F[t,t^{-1}]\longrightarrow G[t,t^{-1}]$$

$$(\alpha_j \in \mathit{Hom}_A(F,G),\quad M,N \geq 0)$$

there are defined f.g. projective A-modules

$$B^+_N(F,G) \;=\; G^+/\alpha(t^N F^+),\;\; B^-_M(F,G) \;=\; G^-/\alpha(t^{-M}F^-)$$

such that up to isomorphism

$$B^+_N(F,G) \;=\; \alpha(t^N F^-)\cap G^+,\;\; B^-_M(F,G) \;=\; \alpha(t^{-M}F^+)\cap G^-,$$

$$B^+_N(F,G)\oplus B^-_M(F,G) \;=\; \sum_{j=-M}^{N-1} t^j F,$$

and such that there are defined nilpotent endomorphisms

$$t :\; B^+_N(F,G)\longrightarrow B^+_N(F,G),\;\; t^{-1} :\; B^-_M(F,G)\longrightarrow B^-_M(F,G).$$

Bass, Heller and Swan [2] and Bass [1] showed that the torsion group $K_1(A[t,t^{-1}])$ of the Laurent polynomial extension $A[t,t^{-1}]$ fits into a split exact sequence

$$\begin{array}{l} 0\longrightarrow K_1(A)\xrightarrow{\;\;i_!\;\;} K_1(A[t,t^{-1}]) \\ \qquad\qquad \xrightarrow{(BB^+\;\,B^-)} K_0(A)\oplus \overline{Nil}_0(A)\oplus \overline{Nil}_0(A)\longrightarrow 0, \end{array}$$

with $\overline{Nil}_0(A)$ the reduced nilpotent class group. The injection $i_!$ is induced by the inclusion of rings $i : A\longrightarrow A[t,t^{-1}]$. The projections B, B^+, B^- are defined by

$$\begin{array}{rl} B :& K_1(A[t,t^{-1}])\longrightarrow K_0(A); \\ & \tau(\alpha)\longrightarrow [B^+_N(F,G)] - [\sum\limits_{j=0}^{N-1} t^j F] \;=\; [\sum\limits_{j=-M}^{-1} t^j F] - [B^-_M(F,G)], \\ B^+ :& K_1(A[t,t^{-1}])\longrightarrow \overline{Nil}_0(A); \tau(\alpha)\longrightarrow [B^+_N(F,G),t], \\ B^- :& K_1(A[t,t^{-1}])\longrightarrow \overline{Nil}_0(A); \tau(\alpha)\longrightarrow [B^-_M(F,G),t^{-1}]. \end{array}$$

Define the *doubly reduced* torsion group

$$\tilde{\tilde{K}}_1(A[t,t^{-1}]) \;=\; \mathit{coker}(K_1(\mathbb{Z}[t,t^{-1}])\longrightarrow K_1(A[t,t^{-1}])).$$

Proposition 1.1: *The doubly reduced torsion group fits into a split exact sequence*

$$0 \longrightarrow \tilde{K}_1(A) \xrightarrow{i_!} \tilde{\tilde{K}}_1(A[t,t^{-1}]) \xrightarrow{(BB^+\ B^-)} \tilde{K}_0(A) \oplus \overline{Nil}_0(A) \oplus \overline{Nil}_0(A) \longrightarrow 0.$$

Proof: This is just the quotient of the split exact sequence for $K_1(A[t,t^{-1}])$ by the split exact sequence for $K_1(\mathbb{Z}[t,t^{-1}])$, using $\overline{Nil}_0(\mathbb{Z}) = 0$. □

A *Mayer-Vietoris presentation* of an $A[t,t^{-1}]$-module chain complex E is a short exact sequence

$$\mathcal{E} :\ 0 \longrightarrow i_!C \xrightarrow{f-gt} i_!D \xrightarrow{h} E \longrightarrow 0,$$

with C, D A-module chain complexes and $f, g : C \longrightarrow D$ A-module chain maps. $\mathcal{E}$ is *finite* if E is a finite $A[t,t^{-1}]$-module chain complex and C, D are finite A-module chain complexes.

Let E be a finite n-dimensional $A[t,t^{-1}]$-module chain complex, and let F_r be the based f.g. free A-module generated by the $A[t,t^{-1}]$-module base of E_r $(0 \leq r \leq n)$, so that

$$E_r = i_!F_r = F_r[t,t^{-1}].$$

The *fundamental lattice of E* $\mathbb{N}(E)$ is the lattice of ordered $(2n+2)$-tuples

$$N = (N_0^+, N_1^+, \dots, N_n^+; N_0^-, N_1^-, \dots, N_n^-)$$

of elements $N_r^+, N_r^- \in \{0, 1, 2, \dots, \infty\}$ such that

$$d_E\Big(\sum_{j=-N_r^+}^{N_r^-} t^j F_r\Big) \subseteq \sum_{j=-N_{r-1}^+}^{N_{r-1}^-} t^j F_{r-1} \quad (1 \leq r \leq n),$$

with

$$N \leq N' \text{ if } N_r^+ \leq N_r'^+,\ N_r^- \leq N_r'^- \ (0 \leq r \leq n),$$

$$(N \cap N')_r^+ = \min(N_r^+, N_r'^+),\ \ (N \cap N')_r^- = \min(N_r^-, N_r'^-)$$

$$(N \cup N')_r^+ = \max(N_r^+, N_r'^+),\ \ (N \cup N')_r^- = \max(N_r^-, N_r'^-).$$

The lattice $\mathbb{N}(E)$ is non-empty, with maximal element

$$\infty = (\infty, \dots, \infty; \infty, \dots, \infty) \in \mathbb{N}(E).$$

Every element $N \in \mathbb{N}(E)$ determines a Mayer-Vietoris presentation of E

$$\mathcal{E}(N) :\ 0 \longrightarrow i_!C(N) \xrightarrow{f(N)-g(N)t} i_!D(N) \xrightarrow{h(N)} E \longrightarrow 0,$$

with $C(N), D(N) \subseteq i^! E$ the A-module subcomplexes defined by

$$D(N)_r \;=\; \sum_{j=-N_r^+}^{N_r^-} t^j F_r \subseteq i^! E_r \;=\; \sum_{j=-\infty}^{\infty} t^j F_r,$$

$$C(N)_r \;=\; D(N)_r \cap tD(N)_r \;=\; \sum_{j=-N_r^+ +1}^{N_r^-} t^j F_r,$$

and $f(N)$, $g(N)$, $h(N)$ the chain maps defined by

$$\begin{aligned} f(N) :& C(N) \longrightarrow D(N);\quad x \longrightarrow x, \\ g(N) :& C(N) \longrightarrow D(N);\quad x \longrightarrow t^{-1}x, \\ h(N) :& i_! D(N) \longrightarrow E;\quad a \otimes x \longrightarrow ax. \end{aligned}$$

An element $N \in \mathbb{N}(E)$ is *finite* if all of the N_r^+, N_r^- are finite, in which case $\mathcal{E}(N)$ is a finite Mayer-Vietoris presentation of E. Let $\mathbb{N}^f(E) \subset \mathbb{N}(E)$ be the sublattice of the finite elements. $\mathbb{N}^f(E)$ is non-empty, since the unique minimal element $N \in \mathbb{N}(E)$ is finite, with $N_n^+ \;=\; 0$, $N_n^- \;=\; 0$. Thus:

Proposition 1.2: (Waldhausen [23]) *Every finite $A[t, t^{-1}]$-module chain complex E admits a finite Mayer-Vietoris presentation.* □

Let $\mathbb{N}^+(E) \subset \mathbb{N}(E)$ be the sublattice of the elements

$$N^+ \;=\; (N_0^+, N_1^+, \ldots, N_n^+;\ \infty, \infty, \ldots, \infty) \in \mathbb{N}(E)$$

with N_r^+ $(0 \leq r \leq n)$ finite. Similarly, let $\mathbb{N}^-(E) \subset \mathbb{N}(E)$ be the subset of the elements

$$N^- \;=\; (\infty, \infty, \ldots, \infty; N_0^-, N_1^-, \ldots, N_n^-) \in \mathbb{N}(E)$$

with N_r^- $(0 \leq r \leq n)$ finite. There are evident forgetful maps

$$\begin{aligned} &\mathbb{N}^f(E) \longrightarrow \mathbb{N}^+(E); \\ &\quad N \;=\; (N_0^+, N_1^+, \ldots, N_n^+; N_0^-, N_1^-, \ldots, N_n^-) \longrightarrow \\ &\qquad\qquad N^+ \;=\; (N_0^+, N_1^+, \ldots, N_n^+; \infty, \infty, \ldots, \infty), \\ &\mathbb{N}^f(E) \longrightarrow \mathbb{N}^-(E); \\ &\quad N = (N_0^+, N_1^+, \ldots, N_n^+; N_0^-, N_1^-, \ldots, N_n^-) \longrightarrow \\ &\qquad\qquad N^- \;=\; (\infty, \infty, \ldots; N_0^-, N_1^- \ldots, N_n^-). \end{aligned}$$

Given $N^+ \in \mathbb{N}^+(E)$ and $N^- \in \mathbb{N}^-(E)$ there is defined a short exact sequence of A-module chain complexes

$$0 \longrightarrow t^{-N^+} F^+ \cap t^{N^-} F^- \longrightarrow t^{-N^+} F^+ \oplus t^{N^-} F^- \longrightarrow i^! E \longrightarrow 0\ .$$

If E is a contractible $A[t,t^{-1}]$-module chain complex then $i^!E$ is a contractible A-module chain complex, and the A-module chain complexes $t^{-N^+}F^+$, $t^{N^-}F^-$ are finitely dominated, with $t^{-N^+}F^+ \oplus t^{N^-}F^-$ chain equivalent to the finite chain complex $t^{-N^+}F^+ \cap t^{N^-}F^-$.

Proposition 1.3 (Ranicki [21]): *The projection $B : \tilde{\tilde{K}}_1(A[t,t^{-1}]) \longrightarrow \tilde{K}_0(A)$ is such that*

$$B(\tau(E)) \;=\; [t^{-N^+}F^+] \;=\; -[t^{N^-}F^-] \in \tilde{K}_0(A)$$

for any contractible finite $A[t,t^{-1}]$-module chain complex E and any $N^+ \in \mathbb{N}^+(E)$, $N^- \in \mathbb{N}^-(E)$. □

§2. **The L-theory of finite complexes**

We recall the L-theory of f.g. projective chain complexes with Poincaré duality. Readers familiar with Ranicki [16] may skip this section. In §3 we develop L-theory using chain complexes of infinitely generated based free A-modules.

Let A be an associative ring with 1, together with an involution

$$\tau \;:\; A \longrightarrow A \;;\; a \longrightarrow \bar{a}.$$

Given A-modules M, N define the $\mathbb{Z}$-module

$$M \otimes_A N \;=\; M \otimes_{\mathbb{Z}} N/\{ax \otimes y - x \otimes \bar{a}y \mid a \in A,\; x \in M,\; y \in N\}.$$

Given an A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on $C \otimes_A C$ by the transposition involution

$$T :\; C \otimes_A C \longrightarrow C \otimes_A C \;;\; x \otimes y \longrightarrow (-)^{pq} y \otimes x \quad (x \in C_p,\; y \in C_q),$$

and define the $\mathbb{Z}$-module chain complex

$$W^{\%}C \;=\; Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\; C \otimes_A C)$$

with W the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W :\; \cdots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0\;.$$

An *n-dimensional symmetric complex over A* (C,ϕ) is an n-dimensional A-module chain complex C together with a cycle $\phi \in (W^{\%}C)_n$. A *morphism* of n-dimensional symmetric complexes over A

$$(f,\chi) :\; (C,\phi) \longrightarrow (C',\phi')$$

is a chain map $f : C \longrightarrow C'$ together with a chain $\chi \in (W^{\%}C')_{n+1}$ such that

$$\phi' - f^{\%}(\phi) \ = \ d\chi \in (W^{\%}C')_n .$$

The morphism is a *homotopy equivalence* if $f : C \longrightarrow D$ is a chain equivalence.

Given a f.g. projective A-module M let M^* denote the dual A-module of A-module morphisms $f : M \longrightarrow A$, with A acting by

$$A \times M^* \longrightarrow M^* \ ; \ (a, f) \longrightarrow (x \longrightarrow f(x)\bar{a}) .$$

The natural A-module isomorphism

$$M \longrightarrow M^{**} \ ; \ x \longrightarrow (f \longrightarrow \overline{f(x)})$$

is used to identify

$$M^{**} \ = \ M .$$

For any f.g. projective A-modules M, N use the natural isomorphism

$$M \otimes_A N \longrightarrow Hom_A(M^*, N); \quad x \otimes y \longrightarrow (f \longrightarrow \overline{f(x)}.y)$$

to identify

$$M \otimes_A N \ = \ Hom_A(M^*, N) .$$

Thus for a f.g. projective A-module chain complex C a cycle $\phi \in W^{\%}C_n$ is defined by a collection of chains

$$\{\phi_s \in (C \otimes_A C)_{n+s} \ = \ \sum_r Hom_A(C^{n-r+s}, C_r) \mid s \geq 0\}$$

such that

$$d_C\phi_s + (-)^r \phi_s d_C^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T\phi_{s-1}) \ = \ 0 : \ C^{n-r+s-1} \longrightarrow C_r$$

$$(r, s \geq 0, \phi_{-1} \ = \ 0) .$$

A f.g. projective n-dimensional symmetric complex over A (C, ϕ) is *Poincaré* if the A-module chain map $\phi_0 : \ C^{n-*} \longrightarrow C$ is a chain equivalence, with C^{n-*} the n-dual f.g. projective A-module chain complex defined by

$$d_{C^{n-*}} \ = \ (-)^r (d_C)^* : \ (C^{n-*})_r \ = \ C^{n-r} \longrightarrow (C^{n-*})_{r-1} \ = \ C^{n-r+1}$$

$$(C^r \ = \ (C_r)^*) .$$

A f.g. projective *n-dimensional symmetric pair over A* $(f : \ C \longrightarrow D, \ (\delta\phi, \ \phi))$ is defined by a chain map $f : \ C \longrightarrow D$ from a f.g. projective $(n-1)$-dimensional A-module chain complex C to a f.g. projective n-dimensional A-module chain complex D, together with a cycle $(\delta\phi, \ \phi) \in C(f^{\%} : W^{\%}C \longrightarrow W^{\%}D)_n$. The symmetric pair is *Poincaré* if a certain A-module chain map $(\delta\phi,\ \phi)_0 :\ C(f)^{n-*}\longrightarrow D$ is a chain equivalence, in which case the *boundary* (C,ϕ) is an $(n-1)$-dimensional symmetric Poincaré complex over A. A *cobordism* of n-dimensional symmetric (Poincaré) complexes (C,ϕ), (C',ϕ') is an $(n+1)$-dimensional symmetric (Poincaré) pair

$$(\ (f\ f')\ :\ C\oplus C' \longrightarrow D\ ,\ (\delta\phi\,,\ \phi\oplus -\phi'))$$

with boundary $(C,\phi)\oplus(C',-\phi')$. Homotopy equivalent complexes are cobordant.

The projective (resp. free) symmetric L-group $L^n_p(A)$ (resp. $L^n_h(A)$) was defined in Ranicki [16] for $n\geq 0$ to be the cobordism group of f.g. projective (resp. finite) n-dimensional symmetric Poincaré complexes over A. See [16,§6] for the extension of the definition to the range $n<0$.

The *union* of adjoining cobordisms of n-dimensional symmetric complexes

$$\begin{aligned}\mathcal{C} &= (\ (f_C\ f_{C'})\ :\ C\oplus C' \longrightarrow D\ ,\ (\delta\phi\,,\ \phi\oplus -\phi')\ )\ ,\\ \mathcal{C}' &= (\ (f'_{C'}\ f'_{C''})\ :\ C'\oplus C'' \longrightarrow D'\ ,\ (\delta\phi'\,,\ \phi'\oplus -\phi'')\ )\end{aligned}$$

is the cobordism

$$\mathcal{C}'' \ =\ \mathcal{C}\cup_{(C',\ \phi')}\mathcal{C}' \ =\ (\ (f''_C\ f''_{C''})\ :\ C\oplus C'' \longrightarrow D''\ ,\ (\delta\phi''\,,\ \phi\oplus -\phi'')\ )$$

defined by

$$d_{D''} \ =\ \begin{pmatrix} d_D & (-)^{r-1}f_{C'} & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-)^{r-1}f'_{C'} & d_{D'} \end{pmatrix}$$

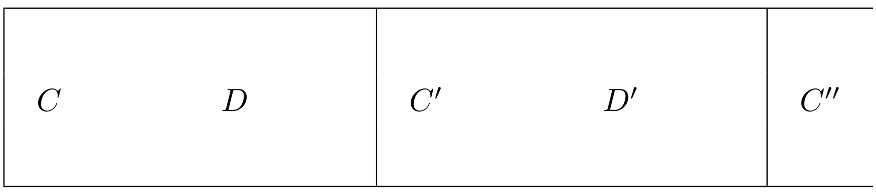
$$:\ D''_r \ =\ D_r\oplus C'_{r-1}\oplus D'_r \longrightarrow D''_{r-1} \ =\ D_{r-1}\oplus C'_{r-2}\oplus D'_{r-1},$$

$$f''_C \ =\ f_C\oplus 0\oplus 0 :\ C_r\longrightarrow D''_r \ =\ D_r\oplus C'_{r-1}\oplus D'_r,$$

$$f''_{C''} \ =\ 0\oplus 0\oplus f'_{C''} :\ C''_r\longrightarrow D''_r \ =\ D_r\oplus C'_{r-1}\oplus D'_r,$$

$$\delta\phi''_s \ =\ \begin{pmatrix} \delta\phi_s & 0 & 0 \\ (-)^{n-r}\phi'_s f^*_{C'} & (-)^{n-r+s+1}T\phi'_{s-1} & 0 \\ 0 & (-)^s f'_{C'}\phi'_s & \delta\phi'_s \end{pmatrix}$$

$$:\ D''^{n-r+s+1} \ =\ D^{n-r+s+1}\oplus C'^{n-r+s}\oplus D'^{n-r+s+1}\longrightarrow D''_r \ =\ D_r\oplus C'_{r-1}\oplus D'_r.$$



$$D'' \;=\; D\cup_{C'} D'$$

An n-dimensional symmetric (Poincaré) triad over A

$$X \;=\; \left(\begin{array}{ccc} \partial C & \longrightarrow & \partial_+ D \\ \big\downarrow & \Gamma & \big\downarrow \\ C & \longrightarrow & D \end{array}\;,\; \begin{array}{ccc} \partial\phi & \longrightarrow & \partial_+\delta\phi \\ \big\downarrow & \Phi & \big\downarrow \\ \phi & \longrightarrow & \delta\phi \end{array}\right)$$

is a relative (Poincaré) cobordism to 0 of the $(n-1)$-dimensional symmetric (Poincaré) pair over A

$$\partial X \;=\; (\partial C\longrightarrow\partial_+ D,\ (\partial_+\delta\phi,\partial\phi)),$$

as defined by a commutative square Γ of chain complexes and cycle $\Phi\in W^{\%}\Gamma_n$. See Ranicki [17, §1.3] for further details.

§3. The L-theory of locally finite complexes

The algebraic K- and L-theory of a ring A are defined using f.g. projective and f.g. free A-modules. The Laurent polynomial extension ring $A[t,t^{-1}]$ of a ring A is a countably infinitely generated free A-module. The splitting theorem of §4 expresses the L-groups for $A[t,t^{-1}]$ in terms of the L-groups of A by passing from chain complexes of f.g. free $A[t,t^{-1}]$-modules to locally finite chain complexes of countably infinitely generated free A-modules, and hence to chain complexes of f.g. projective A-modules. In each case only finite-dimensional chain complexes are considered. In §3 we develop the L-theory of the locally finite chain complexes of countably infinitely generated free A-modules.

A *morphism matrix* is a collection of A-module morphisms

$$\{f_{ij}\in Hom_A(M(j),N(i))|i\in I, j\in J\}$$

which we also write as

$$f \;=\; (f_{ij}):\ M \;=\; \{M(j)|j\in J\}\longrightarrow N \;=\; \{N(i)|i\in I\}\ .$$

Morphism matrices

$$f: M=\{M(j)\}\longrightarrow N=\{N(i)\}\ ,\ g: N=\{N(i)\}\longrightarrow P=\{P(k)\}$$

can be composed if and only if for each $x(j)\in M(j)$ and k the set $\{i\in I|g_{ki}f_{ij}(x(j))\neq 0\in P(k)\}$ is finite, in which case the composite morphism matrix $gf: M\longrightarrow N$ is defined by

$$(gf)_{kj} \;=\; \sum_{i\in I} g_{ki}f_{ij}\ :\ M(j)\longrightarrow N(k)\ .$$

The morphisms of direct sums of A-modules

$$M \;=\; \sum_{j\in J} M(j)\longrightarrow N \;=\; \sum_{i\in I} N(i)$$

are in one-one correspondence with the morphism matrices $f = \{f_{ij}\}$ such that for each $x(j) \in M(j)$ the set $\{i \in I \mid f_{ij}(x(j)) \neq 0 \in N(i)\}$ is finite, with the morphism corresponding to f given by

$$\begin{array}{rl} f \;=\; (f_{ij}) : M \;=\; \sum_{j\in J} M(j) \;\longrightarrow\; N \;=\; \sum_{i\in I} N(i) \;; \\ x \;=\; \sum_{j\in J} x(j) \;\longrightarrow\; f(x) \;=\; \sum_{i\in I}\sum_{j\in J} f_{ij}(x(j)) \end{array} .$$

If each $M(j)$ is a f.g. projective A-module this condition is equivalent to f being column finite, i.e. the set $\{i \in I | f_{ij} \neq 0\}$ being finite for each $j \in J$.

An A-module M is *countably free* (resp. *projective*) if it is expressed as a direct sum

$$M \;=\; \sum_{j\in J} M(j)$$

of f.g. free (resp. projective) A-modules $M(j)$, with J countable. An A-module morphism $f : M \longrightarrow N$ between countably free (resp. projective) A-modules is a column finite morphism matrix (f_{ij}). The morphism is *locally finite* if the morphism matrix is also row finite, i.e. the set $\{j \in J \mid f_{ij} \neq 0\}$ is finite for each $i \in I$.

Given countably projective A-modules M, N let $Hom_A^{lf}(M,N)$ be the subgroup of $Hom_A(M,N)$ consisting of the locally finite morphisms. If N is f.g. projective then

$$Hom_A^{lf}(M,N) \;=\; Hom_A(M,N).$$

Remark 3.1: A locally finite A-module morphism $f : M \longrightarrow N$ of countably projective A-modules can be an isomorphism without being a locally finite isomorphism, i.e. the inverse $f^{-1} : N \longrightarrow M$ need not be locally finite. For example, let $A = \mathbb{Z}$, $M = N = \sum_{i=0}^{\infty} \mathbb{Z}$ and consider the automorphism

$$f \;=\; \begin{pmatrix} 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots ,$$

such that the inverse

$$f^{-1} \;=\; \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$$

is not locally finite. □

An A-module chain complex C is *locally finite* if it is a finite-dimensional complex of countably projective A-modules with the differentials $d_C : C_r \longrightarrow C_{r-1}$ $(r \geq 1)$ locally finite. A chain map $f : C \longrightarrow D$ of locally finite complexes is *locally finite* if each $f : C_r \longrightarrow D_r$ $(r \geq 0)$ is locally finite. A *locally finite equivalence* is a chain equivalence in the locally finite category. A locally finite chain complex C is *locally finitely dominated* if there exists a finite domination (D, f, g, h) with f, g, h locally finite. A locally finite chain complex C is *locally homotopy finite* if it is locally finite equivalent to a finite complex D.

Locally finite duality is defined with respect to an involution $A \longrightarrow A; a \longrightarrow \bar{a}$ on the ground ring A, as in the f.g. projective case.

The *locally finite dual* of a countably free (resp. projective) A-module M is the countably free (resp. projective) A-module defined by

$$M^* = Hom_A^{lf}(M, A)$$

with A acting on M^* as in the f.g. projective case and $M^*(j) = M(j)^*$ $(j \in J)$. (If M is f.g. then M^* is the dual defined above). If M is countably free then choosing a base $(b_{j1}, b_{j2}, ..., b_{jk_j})$ for each $M(j)$ determines a locally finite isomorphism

$$M \longrightarrow M^*;\quad b_{jk} \longrightarrow (b_{j'k'} \longrightarrow \begin{cases} 1 & \text{if } (j,k) = (j',k') \\ 0 & \text{otherwise} \end{cases}).$$

For any countably projective M there is defined a natural locally finite isomorphism $M \longrightarrow M^{**}$ as in the f.g. projective case, which is used to identify $M^{**} = M$.

The *dual* of a morphism matrix $f : M \longrightarrow N$ is the morphism matrix $f^* : N^* \longrightarrow M^*$ with

$$(f^*)_{ji} = (f_{ij})^* : N^*(i) = N(i)^* \longrightarrow M^*(j) = M(j)^* .$$

The *locally finite dual* of a locally finite morphism $f : M \longrightarrow N$ is the locally finite morphism $f^* : N^* \longrightarrow M^*$ defined by

$$f^* : N^* \longrightarrow M^* ;\ g \longrightarrow (x \longrightarrow g(f(x))) ,$$

which has the dual morphism matrix. Locally finite duality defines an isomorphism

$$T : Hom_A^{lf}(M, N) \longrightarrow Hom_A^{lf}(N^*, M^*);\quad f \longrightarrow f^*,$$

with inverse

$$T : Hom_A^{lf}(N^*, M^*) \longrightarrow Hom_A^{lf}(M^{**}, N^{**}) = Hom_A^{lf}(M, N);\quad f \longrightarrow f^*.$$

The dual $f^* : N^* \longrightarrow M^*$ of an isomorphism of f.g. projective A-modules $f : M \longrightarrow N$ is also an isomorphism, with inverse

$$(f^*)^{-1} = (f^{-1})^* : M^* \longrightarrow N^*$$

the dual of the inverse $f^{-1} : N \longrightarrow M$.

Remark 3.2: Let $f : M \longrightarrow N$ be a locally finite morphism which is an A-module isomorphism. By 3.3 (below) the inverse $f^{-1} : N \longrightarrow M$ is locally finite if and only if the locally finite dual $f^* : N^* \longrightarrow M^*$ is an A-module isomorphism. In particular, if f^{-1} is not locally finite then f^* is not an isomorphism. For example, let $A = \mathbb{Z}$, $M = N = \sum_{i=0}^{\infty} \mathbb{Z}$ as in 3.1 and consider the $\mathbb{Z}$-module automorphism

$$f = \begin{pmatrix} 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$$

with locally finite dual

$$f^* = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \ .$$

In this case f^* is not an isomorphism, and the inverse of f

$$f^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$$

is not locally finite. In fact, f^* is a split monomorphism with cokernel $\mathbb{Z}$ which fits into a non locally finite direct sum system of $\mathbb{Z}$-modules

$$\sum_0^{\infty} \mathbb{Z} \underset{g}{\overset{f^*}{\rightleftarrows}} \sum_0^{\infty} \mathbb{Z} \ \underset{\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}}{\overset{(1\ 1\ ...)}{\rightleftarrows}} \ \mathbb{Z}$$

with

$$g = \begin{pmatrix} 0 & -1 & -1 & \cdots \\ 0 & 0 & -1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots .$$

This is the canonical example of the Eilenberg swindle in algebraic K-theory. □

The following result is a special case of the proper Whitehead theorem of Farrell, Taylor and Wagoner [5,4.2], proved here in a more chain homotopy theoretic way.

Proposition 3.3: A locally finite A-module chain map $f : C \longrightarrow D$ of n-dimensional locally finite A-module chain complexes is a locally finite equivalence if and only if both f and the locally finite n-dual $f^{n-*} : D^{n-*} \longrightarrow C^{n-*}$ are chain equivalences.

Proof : The algebraic mapping cone of f is a locally finite chain complex $E = C(f : C \longrightarrow D)$ with locally finite n-dual

$$E^{n-*} \ = \ S^{-1}C(f^{n-*} : D^{n-*} \longrightarrow C^{n-*})$$

with S^{-1} the desuspension by a dimension shift of 1, so that $(E^{n-*})_r = C(f^{n-*})_{r+1}$. (The *suspension* of an A-module chain complex C is the A-module chain complex SC with

$$d_{SC} \ = \ d_C \ : \ SC_r \ = \ C_{r-1} \longrightarrow SC_{r-1} \ = \ C_{r-2} \ .)$$

We have to prove that E is locally finite chain contractible if and only if both E and E^{n-*} are chain contractible. One way is easy: the locally finite n-dual of a locally finite chain contraction $\Gamma : 0 \simeq 1 : E \longrightarrow E$ is a (locally finite) chain contraction $\Gamma^{n-*} : 0 \simeq 1 : E^{n-*} \longrightarrow E^{n-*}$. Conversely, assume that there exist A-module chain contractions

$$\Gamma_1 \ : \ 0 \simeq 1 \ : \ E \longrightarrow E \ \ , \ \ \Gamma_2 \ : \ 0 \simeq 1 \ : \ E^{n-*} \longrightarrow E^{n-*} \ ,$$

so that

$$d\Gamma_1 + \Gamma_1 d = 1 : E_r \longrightarrow E_r \ \ , \ \ d^*\Gamma_2 + \Gamma_2 d^* = 1 \ : \ E^{n-r} \longrightarrow E^{n-r} \ .$$

The morphism matrices of Γ_1 and Γ_2 are column finite. The dual morphism matrices Γ_2^* are row finite, satisfying

$$d\Gamma_2^* + \Gamma_2^* d \ = \ 1 \ : \ E_r \longrightarrow E_r \ .$$

The morphism matrices defined by

$$\Delta \ = \ (\Gamma_1 - \Gamma_2^*)\Gamma_1 \ : \ E_r \longrightarrow E_{r+2}$$

are such that

$$\Gamma_2^* - \Gamma_1 \ = \ d\Delta - \Delta d \ : \ E_r \longrightarrow E_{r+1} \ .$$

Every morphism matrix Φ can be expressed as a sum $\Phi_1 + \Phi_2$ with Φ_1 column finite and Φ_2 row finite (e.g. by writing it as a sum of an upper triangular and a lower triangular morphism matrix). Express Δ in this way as $\Delta_1 + \Delta_2$. Since the differentials $d : E_r \longrightarrow E_{r-1}$ are locally finite the morphism matrices defined by

$$\Gamma \ = \ \Gamma_1 + d\Delta_1 - \Delta_1 d \ = \ \Gamma_2^* - d\Delta_2 + \Delta_2 d \ : \ E_r \longrightarrow E_{r+1}$$

are both row finite and column finite, and so define locally finite A-module morphisms. Thus Γ is a locally finite chain contraction of E. □

Example 3.4 Let $f : C \longrightarrow D$, $f' : C \longrightarrow D'$ be the locally finite A-module chain maps defined by

$$\begin{aligned}
& f \;=\; f' \;:\; C_0 \;=\; A \longrightarrow D_0 \;=\; D'_0 \;=\; \sum_0^\infty A \;;\; x \longrightarrow (x,0,0,\dots) \;, \\
& d \;:\; D_1 \;=\; \sum_0^\infty A \longrightarrow D_0 \;=\; \sum_0^\infty A \;; \\
& \qquad\qquad (x_0,x_1,x_2,\dots) \longrightarrow (x_1,x_2,x_3,\dots) \;, \\
& d' \;:\; D'_1 \;=\; \sum_0^\infty A \longrightarrow D'_0 \;=\; \sum_0^\infty A \;; \\
& \qquad\qquad (x_0,x_1,x_2,\dots) \longrightarrow (x_0,x_1-x_0,x_2-x_1,\dots) \;, \\
& C_r \;=\; 0 \text{ for } r \neq 0 \;,\; D_r \;=\; D'_r \;=\; 0 \text{ for } r \neq 0,1 \;.
\end{aligned}$$

Both f and f' are chain equivalences, inducing isomorphisms

$$f_* \;:\; H_*(C) \;=\; A \longrightarrow H_*(D) \;,\; f'_* \;:\; H_*(C) \;=\; A \longrightarrow H_*(D')$$

in homology. Also, f is a locally finite equivalence, with locally finite homotopy inverse $f^{-1} : D \longrightarrow C$ defined by

$$f^{-1} \;:\; D_0 \;=\; \sum_0^\infty A \longrightarrow C_0 \;=\; A \;;\; (x_0,x_1,x_2,\dots) \longrightarrow x_0 \;,$$

and D is locally homotopy finite. However, 3.3 shows that f' is not a locally finite equivalence, since the locally finite 1-dual $f'^{1-*} : D'^{1-*} \longrightarrow C^{1-*}$ induces A-module morphisms in homology

$$\begin{aligned}
f^* \;=\; 0 \;:\; H_0(D'^{1-*}) \;=\; A \longrightarrow H_0(C^{1-*}) \;=\; 0 \;, \\
f^* \;=\; 0 \;:\; H_1(D'^{1-*}) \;=\; 0 \longrightarrow H_1(C^{1-*}) \;=\; A
\end{aligned}$$

which are not isomorphisms, and D' is not locally finitely dominated. D' is essentially the example of 3.1,3.2 all over again. For $A = \mathbb{Z}$ it shows that the real line $\mathbb{R}$ is not proper homotopy equivalent to a point.

□

Remark 3.5 For any f.g. projective A-module $P = \mathrm{im}(p = p^2 : A^m \longrightarrow A^m)$ there is defined a locally finite resolution

$$0 \longrightarrow C_1 \xrightarrow{\;d\;} C_0 \xrightarrow{\;e\;} P \longrightarrow 0$$

with

$$d\ :\ C_1\ =\ \sum_0^\infty A^m \longrightarrow C_0\ =\ \sum_0^\infty A^m\ ;$$

$$(x_0,x_1,x_2,\dots) \longrightarrow (x_0,-p(x_0)+x_1,-p(x_1)+x_2,\dots)\ ,$$

$$e\ :\ C_0\ =\ \sum_0^\infty A^m \longrightarrow P\ ;$$

$$(x_0,x_1,x_2,\dots) \longrightarrow p(x_0+x_1+x_2+\dots)\ .$$

C is not locally finitely dominated, by the argument of 3.4 (the special case $P = A,\ m = 1\ ,p = 1$.) Every finitely dominated A-module chain complex is chain equivalent to a finite f.g. projective A-module chain complex, and hence to a locally finite n-dimensional complex (for some $n \geq 0$), although not necessarily to one which is locally finitely dominated. □

For any countably projective A-modules M, N define the abelian group

$$M \otimes_A^{lf} N\ =\ Hom_A^{lf}(M^*, N).$$

There is defined a natural inclusion

$$M \otimes_A N \longrightarrow M \otimes_A^{lf} N;\ x \otimes y \longrightarrow (f \longrightarrow \overline{f(x)}.y),$$

and every element of $M \otimes_A^{lf} N$ can be expressed as a locally finite infinite sum of elements $x \otimes y\ \in M \otimes_A N$. The duality isomorphism

$$T:\ Hom_A^{lf}(M^*, N) \longrightarrow Hom_A^{lf}(N^*, M)$$

can be identified with the transposition isomorphism

$$T:\ M \otimes_A^{lf} N \longrightarrow N \otimes_A^{lf} M\ ;\ x \otimes y \longrightarrow y \otimes x.$$

For any locally finite A-module chain complex C define the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$C \otimes_A^{lf} C\ =\ Hom_A^{lf}(C^*, C),$$

with $T \in \mathbb{Z}_2$ acting by the involution

$$T:\ C \otimes_A^{lf} C \longrightarrow C \otimes_A^{lf} C;\ x \otimes y \longrightarrow (-)^{pq} y \otimes x \quad (x \in C_p,\ y \in C_q),$$

which can also be written as

$$T:\ Hom_A^{lf}(C^*, C) \longrightarrow Hom_A^{lf}(C^*, C);\ f \longrightarrow (-)^{pq} f^*$$

$$(f \in Hom_A^{lf}(C^p, C_q)).$$

Let $W^{\%}_{lf}C$ denote the $\mathbb{Z}$-module chain complex

$$W^{\%}_{lf}C \ = \ Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes^{lf}_A C).$$

A *locally finite n-dimensional symmetric complex over A* (C, ϕ) is a locally finite A-module chain complex C together with a cycle $\phi \in (W^{\%}_{lf}C)_n$, as defined by a collection of locally finite A-module morphisms

$$\{\phi_s \in (C \otimes^{lf}_A C)_{n+s} \ = \ \sum_r Hom^{lf}_A(C^{n-r+s}, C_r) \mid s \geq 0\}$$

such that

$$\begin{aligned} d_C\phi_s + &(-)^r\phi_s d^*_C + (-)^{n+s-1}(\phi_{s-1} + (-)^s T\phi_{s-1}) \\ &= \ 0 : \ C^{n-r+s-1} \longrightarrow C_r \quad (r, s \geq 0, \ \phi_{-1} \ = \ 0) \ . \end{aligned}$$

A locally finite n-dimensional symmetric complex (C, ϕ) is *Poincaré* if the locally finite chain map $\phi_0 : C^{n-*} \longrightarrow C$ is a locally finite equivalence. Locally finite symmetric Poincaré pairs, cobordisms and triads are defined by analogy with the f.g. projective cases.

Proposition 3.6: *The cobordism group of locally finitely dominated (resp. locally homotopy finite) n-dimensional symmetric Poincaré complexes over A is the projective (resp. free) symmetric L-group $L^n_p(A)$ (resp. $L^n_h(A)$).*

Proof: An n-dimensional locally finitely dominated A-module chain complex C is locally finite equivalent to an n-dimensional f.g. projective A-module chain complex D, by the algebraic theory of finiteness obstruction of Ranicki [18]. The n-dual of a locally finite equivalence $f : C \longrightarrow D$ is a locally finite equivalence $f^{n-*} : D^{n-*} \longrightarrow C^{n-*}$, and there is induced a $\mathbb{Z}[\mathbb{Z}_2]$-module chain equivalence

$$f \otimes_A f : C \otimes^{lf}_A C \longrightarrow D \otimes^{lf}_A D \ = \ D \otimes_A D \ .$$

Thus there is a natural one-one correspondence between the locally finite equivalence classes of locally finitely dominated n-dimensional symmetric (Poincaré) complexes over A and finitely dominated n-dimensional symmetric (Poincaré) complexes over A. Similarly for pairs. □

Proposition 3.7: *There is a natural one-one correspondence between the locally finite equivalence classes of locally finite n-dimensional symmetric complexes over A and locally finite n-dimensional symmetric Poincaré pairs over A.*

Proof: This is just the locally finite version of the one-one correspondence of Ranicki [16,3.4] for the f.g. projective case, and proceeds as follows.

The *boundary* of a locally finite n-dimensional symmetric complex (C, ϕ) is the locally finite $(n-1)$-dimensional symmetric Poincaré complex

$$\partial(C, \phi) \ = \ (\partial C, \partial\phi)$$

with

$$\begin{aligned} d_{\partial C} \ &= \ \begin{pmatrix} d_C & (-)^r\phi_0 \\ 0 & (-)^r d_C^* \end{pmatrix} \ : \\ &\partial C_r \ = \ C_{r+1}\oplus C^{n-r} \longrightarrow \partial C_{r-1} \ = \ C_r \oplus C^{n-r+1}, \\ \partial\phi_0 \ &= \ \begin{pmatrix} (-)^{n-r-1}T\phi_1 & (-)^{r(n-r-1)} \\ 1 & 0 \end{pmatrix} \ : \\ &\partial C^{n-r-1} \ = \ C^{n-r}\oplus C_{r+1} \longrightarrow \partial C_r \ = \ C_{r+1}\oplus C^{n-r}, \\ \partial\phi_s \ &= \ \begin{pmatrix} (-)^{n-r+s-1}T\phi_{s+1} & 0 \\ 0 & 0 \end{pmatrix} \ : \\ &\partial C^{n-r+s-1} = C^{n-r+s}\oplus C_{r-s+1} \longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r} \quad (s\geq 1)\ . \end{aligned}$$

(We are assuming here that $d_{\partial C} : \partial C_0 \longrightarrow \partial C_{-1}$ is a locally finite split surjection.)

The locally finite n-dimensional symmetric Poincaré pair

$$\delta\partial(C,\phi) \ = \ (p_C \ = \ \textit{projection}: \ \partial C \longrightarrow C^{n-*}, \quad (0,\partial\phi))$$

is a null-cobordism of $\partial(C,\phi)$. The locally finite symmetric complex (C,ϕ) is Poincaré if and only if the boundary $\partial(C,\phi)$ is locally finite contractible.

Conversely, a locally finite n-dimensional symmetric Poincaré pair

$$X = (f : C \longrightarrow D, \ (\delta\phi,\phi))$$

determines the locally finite n-dimensional symmetric complex

$$X/\partial X \ = \ (D,\delta\phi)/C \ = \ (C(f),\delta\phi/\phi)$$

with

$$\begin{aligned} (\delta\phi/\phi)_s = &\begin{pmatrix} \delta\phi_s & 0 \\ (-)^{n-r}\phi_s f^* & (-)^{n-r+s+1}T\phi_{s-1} \end{pmatrix} \ : \\ &C(f)^{n-r+s+1} = D^{n-r+s+1}\oplus C^{n-r+s} \longrightarrow C(f)_r = D_r \oplus C_{r-1} \end{aligned}$$

which is locally finite equivalent to $\delta\partial(X/\partial X)$. □

We shall also need the relative version of Proposition 3.7:

Proposition 3.8: *There is a natural one-one correspondence between the locally finite equivalence classes of locally finite n-dimensional symmetric pairs over A and locally finite n-dimensional symmetric Poincaré triads over A.*

Proof: Define the *boundary* of a locally finite n-dimensional symmetric pair $(f : C \longrightarrow D$, $(\delta\phi,\phi))$ to be the locally finite $(n-1)$-dimensional symmetric Poincaré pair

$$\partial(f,(\delta\phi,\phi)) \ = \ (\partial f : \ \partial C \longrightarrow \partial_+ D, \ (\partial_+\delta\phi,\partial\phi))$$

with

$$d_{\partial_+ D} = \begin{pmatrix} d_D & (-)^r\delta\phi_0 & (-)^r f\phi_0 \\ 0 & (-)^r d_D^* & 0 \\ 0 & f^* & (-)^r d_C^* \end{pmatrix} :$$

$$\partial_+ D_r = D_{r+1} \oplus D^{n-r} \oplus C^{n-r-1} \longrightarrow \partial_+ D_{r-1} = D_r \oplus D^{n-r+1} \oplus C^{n-r},$$

$$\partial f = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} :$$

$$\partial C_r = C_{r+1} \oplus C^{n-r-1} \longrightarrow \partial_+ D_r = D_{r+1} \oplus D^{n-r} \oplus C^{n-r-1}.$$

The locally finite n-dimensional symmetric Poincaré triad

$$\delta\partial(f,(\phi,\partial\phi)) \; = \; \left(\begin{array}{ccc} \partial C & \xrightarrow{\partial f} & \partial_+ D \\ \big\downarrow p_C & & \big\downarrow p_D \\ C^{n-1-*} & \longrightarrow & C(f)^{n-*} \end{array} \; , \; \begin{array}{ccc} \partial\phi & \longrightarrow & \partial_+\delta\phi \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & 0 \end{array} \right)$$

is a null-cobordism of $\partial(f,(\delta\phi,\phi))$, with p_C, p_D the projections. As in the absolute case $(f,(\delta\phi,\phi))$ is a Poincaré pair if and only if the boundary $\partial(f,(\delta\phi,\phi))$ is a contractible pair.

Conversely, a locally finite n-dimensional symmetric Poincaré triad

$$X \; = \; \left(\begin{array}{ccc} \partial C & \longrightarrow & \partial_+ D \\ \big\downarrow & \Gamma & \big\downarrow \\ C & \longrightarrow & D \end{array} \; , \; \begin{array}{ccc} \partial\phi & \longrightarrow & \partial_+\delta\phi \\ \big\downarrow & \Phi & \big\downarrow \\ \phi & \longrightarrow & \delta\phi \end{array} \right)$$

determines an n-dimensional symmetric pair

$$X/\partial X \; = \; (C/\partial C \longrightarrow D/\partial_+ D, (\delta\phi/\partial_+\delta\phi, \phi/\partial\phi))$$

such that X is locally finite equivalent to the triad $\delta\partial_+(X/\partial X)$. □

Remark 3.9 It follows from the theory developed in §4 below that every n-dimensional f.g. projective symmetric Poincaré complex (C,ϕ) over A is locally finite equivalent to the boundary (C',ϕ') of a locally finite countably free $(n+1)$-dimensional symmetric Poincaré pair $(f' : C' \longrightarrow D', (\delta\phi',\phi'))$ over A, with f' a chain equivalence. This does not contradict 3.6 since in general f' is not a locally finite equivalence and D' is not locally finite dominated, so that it is not implied that (C,ϕ) is null-cobordant in the f.g. projective category. In particular, consider the 0-dimensional case and suppose given a nonsingular symmetric form $(P, \phi = \phi^* : P \longrightarrow P^*)$ over A with P f.g. projective (e.g. $P = A,\ \phi = 1$). Let $P = \mathrm{im}(p = p^2 : A^m \longrightarrow A^m)$ and let (A^m, θ) be the (singular) symmetric form defined by

$$\theta \; : \; A^m \longrightarrow (A^m)^* \; ; \; x \longrightarrow (y \longrightarrow \phi(px)(py)) \; .$$

Identifying $(A^m)^* = A^m$ by means of the A-module isomorphism

$$A^m \longrightarrow (A^m)^* \ ; \ (a_1, a_2, ..., a_m) \longrightarrow ((b_1, b_2, ..., b_m) \longrightarrow \sum_{k=1}^{m} b_k \bar{a}_k)$$

regard θ as an endomorphism of A^m. Use the dual of the locally finite resolution of 3.5

$$0 \longrightarrow \sum_0^\infty A^m \xrightarrow{\ d\ } \sum_0^\infty A^m \xrightarrow{\ e\ } P \longrightarrow 0$$

to define a locally finite countably free 1-dimensional symmetric complex (B, λ) over A by

$$\begin{aligned}
d_B \ = \ d^* \ : \ B_1 \ = \ \sum_0^\infty A^m \longrightarrow B_0 \ = \ \sum_0^\infty A^m \ ; \\
(x_0, x_1, x_2, \dots) \longrightarrow (x_0 - p^*(x_1), x_1 - p^*(x_2), x_2 - p^*(x_3), \dots) \ , \\
\lambda_0 \ : \ B^0 \ = \ \sum_0^\infty A^m \longrightarrow B_1 \ = \ \sum_0^\infty A^m \ ; \\
(x_0, x_1, x_2, \dots) \longrightarrow (\theta(x_0), \theta(x_1), \theta(x_2), \dots) \ , \\
\lambda_0 \ : \ B^1 \ = \ \sum_0^\infty A^m \longrightarrow B_0 \ = \ \sum_0^\infty A^m \ ; \\
(x_0, x_1, x_2, \dots) \longrightarrow (\theta(x_1), \theta(x_2), \theta(x_3), \dots) \ , \\
\lambda_1 \ : \ B^1 \ = \ \sum_0^\infty A^m \longrightarrow B_1 \ = \ \sum_0^\infty A^m \ ; \\
(x_0, x_1, x_2, \dots) \longrightarrow (-\theta(x_0), -\theta(x_1), -\theta(x_2), \dots) \ .
\end{aligned}$$

Use the construction of 3.7 to define the locally finite countably free 1-dimensional symmetric Poincaré pair

$$(f' : C' \longrightarrow D', (\delta\phi', \phi')) \ = \ \delta\partial(B, \lambda) \ ,$$

with

$$f' \ = \ \textit{projection} \ : \ C' \ = \ S^{-1}C(\lambda_0) \longrightarrow D' \ = \ B^{1-*} \ .$$

B is contractible but not locally finite so (cf. 3.1), with the locally finite 1-dual B^{1-*} a resolution of P, and C' is a locally finitely dominated countably free complex. f' is a chain equivalence, but not a locally finite equivalence (cf. 3.3). The composite

$$ef' \ : \ C' \xrightarrow{\ f'\ } B^{1-*} \xrightarrow{\ e\ } P$$

is chain homotopic to the locally finite equivalence $e' : C' \longrightarrow P$ defined by

$$\begin{aligned}
e' \ : \ C'_0 \ = \ B^1 \oplus B_1 \ = \ \sum_0^\infty A^m \oplus \sum_0^\infty A^m \longrightarrow P \ ; \\
((x_0, x_1, x_2, \dots), (y_0, y_1, y_2, \dots)) \longrightarrow p(x_0 - y_0) \ .
\end{aligned}$$

The composite $\phi e' : C' \longrightarrow P^*$ defines a locally finite equivalence of 0-dimensional symmetric Poincaré complexes

$$\phi e' \ : \ (C', \phi') \ = \ \partial(B, \lambda) \longrightarrow (P^*, \phi) \ .$$

This is an algebraic L-theoretic version of the Eilenberg swindle, corresponding to the expression of any compact closed n-manifold M as the boundary of the non-compact open $(n+1)$-manifold $M \times [0, \infty)$. □

§4. The L-theory of Laurent extensions

Given a central unit $u \in A$ such that $\bar{u} = u \in A$ let $A[t, t^{-1}]_u$ denote the Laurent polynomial extension ring $A[t, t^{-1}]$ with the involution $\bar{t} = ut^{-1}$. The inclusion of rings with involution $i : A \longrightarrow A[t, t^{-1}]_u$ determines a functor preserving duality involutions

$$\begin{aligned} i_! : \ (A\text{-}modules) &\longrightarrow (A[t, t^{-1}] - modules); \\ M &\longrightarrow i_! M \ = \ A[t, t^{-1}] \otimes_A M \ = \ M[t, t^{-1}], \end{aligned}$$

both for f.g. projective and countably based free modules. There are induced morphisms in the L-groups

$$\begin{aligned} i_! : \ L^n_q(A) &\longrightarrow L^n_q(A[t, t^{-1}]_u); \\ (C, \phi) &\longrightarrow A[t, t^{-1}] \otimes_A (C, \phi) \ \ (q = h, p). \end{aligned}$$

Define also morphisms

$$1 - u : \ L^n_p(A) \longrightarrow L^n_h(A); \ (C, \phi) \longrightarrow (C, \phi) \oplus (C, -u\phi) \oplus \partial(C, 0).$$

Theorem 4.1: *The free symmetric L-groups of $A[t, t^{-1}]_u$ fit into a long exact sequence*

$$\begin{aligned} \cdots \longrightarrow L^n_p(A) \xrightarrow{1-u} L^n_h(A) \xrightarrow{i_!} L^n_h(A[t, t^{-1}]_u) \\ \xrightarrow{B} L^{n-1}_p(A) \xrightarrow{1-u} L^{n-1}_h(A) \longrightarrow \cdots . \end{aligned}$$

□

The proof of 4.1 occupies all of §4.

The restriction of an $A[t, t^{-1}]$-module M is an A-module $i^! M$, as already defined in §1. If M is a based f.g. free $A[t, t^{-1}]$-module with base B then $i^! M$ is a countably based free A-module with base $t^j b$ $(j \in \mathbb{Z}, b \in B)$. The restriction of a morphism $f : M \longrightarrow N$ of based f.g. free $A[t, t^{-1}]$-modules is a locally finite morphism $i^! f : i^! M \longrightarrow i^! N$. The morphism of abelian groups defined by restriction

$$i^! : Hom_{A[t, t^{-1}]}(M, N) \longrightarrow Hom^{lf}_A(i^! M, i^! N)$$

is an injection. If F, G are the based f.g. free A-modules generated by the $A[t, t^{-1}]$-module bases of M, N and

$$f = \sum_{j=-\infty}^{\infty} f_j t^j : M = \sum_{j=-\infty}^{\infty} t^j F \longrightarrow N = \sum_{j=-\infty}^{\infty} t^j G$$

then

$$i^! f = \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & f_0 & f_{-1} & f_{-2} & \cdots \\ \cdots & f_1 & f_0 & f_{-1} & \cdots \\ \cdots & f_2 & f_1 & f_0 & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} :$$

$$i^! M = \cdots \oplus t^{-1} F \oplus F \oplus tF \oplus \cdots \longrightarrow i^! N = \cdots \oplus t^{-1} G \oplus G \oplus tG \oplus \cdots .$$

We identify

$$(i_! F)^* = i_!(F^*)$$

using the natural $A[t, t^{-1}]$-module isomorphism

$$i_!(F^*) \longrightarrow (i_! F)^* ;\ t^j f \longrightarrow (t^k x \longrightarrow f(x) u^j t^{k-j})$$

$$(j, k \in \mathbb{Z},\ f \in F^*,\ x \in F).$$

We identify

$$i^!(M^*) = (i^! M)^*$$

using the natural A-module isomorphism

$$\begin{aligned} & i^!(M^*) \longrightarrow (i^! M)^* ; \\ & \quad (f : M \longrightarrow A[t, t^{-1}]) \longrightarrow ([f]_0 : i^! M \longrightarrow A; x \longrightarrow [f(x)]_0), \end{aligned}$$

where $[a]_0 = a_0 \in A$ for $a = \sum_{j=-\infty}^{\infty} a_j t^j \in A[t, t^{-1}]$. The restriction of an $A[t, t^{-1}]$-module morphism

$$\alpha = \sum_{j=-\infty}^{\infty} \alpha_j t^j : M = \sum_{j=-\infty}^{\infty} t^j F \longrightarrow N^* = \sum_{j=-\infty}^{\infty} t^j G^* ;$$

$$x \longrightarrow (y \longrightarrow \sum_{j=-\infty}^{\infty} \alpha_j(x)(y) t^j) \quad (x \in F, y \in G)$$

is the locally finite A-module morphism

$$\begin{aligned} & i^! \alpha : i^! M \longrightarrow (i^! N)^* ; \\ & \quad t^j x \longrightarrow (t^k y \longrightarrow [\alpha(t^j x)(t^k y)]_0 = \alpha_{j-k}(x)(y) u^j) \end{aligned}$$

$$(x \in F,\ y \in G,\ j,k \in \mathbb{Z}).$$

The dual $A[t,t^{-1}]$-module morphism is given by

$$\alpha^* = \sum_{j=-\infty}^{\infty} \alpha_j^* u^j t^{-j} : N = \sum_{j=-\infty}^{\infty} t^j G \longrightarrow M^* = \sum_{k=-\infty}^{\infty} t^k F^*$$
$$y \longrightarrow (x \longrightarrow \overline{\alpha(x)(y)} = \sum_{j=-\infty}^{\infty} \overline{\alpha_j(x)(y)} u^j t^{-j}) \ \ (x \in F, y \in G).$$

For any finite $A[t,t^{-1}]$-module chain complex E there is defined a restriction map

$$i^! : E \otimes_{A[t,t^{-1}]_u} E = Hom_{A[t,t^{-1}]}(E^*, E) \longrightarrow i^! E \otimes_A^{lf} i^! E = Hom_A^{lf}(i^! E^*, i^! E).$$

The *restriction* of a finite n-dimensional symmetric complex (E,Θ) over $A[t,t^{-1}]_u$ is the locally finite n-dimensional symmetric complex over A

$$i^!(E,\Theta) = (i^! E, i^! \Theta).$$

Given a locally finite n-dimensional symmetric complex (C,ϕ) over A and a based free subcomplex $B \subseteq C$ define the locally finite n-dimensional symmetric complex over A

$$(C,\phi)/B = (C/B, p^{\%}(\phi)),$$

with $p : C \longrightarrow C/B$ the locally finite morphism defined by projection.

The *torsion* of a finite n-dimensional symmetric Poincaré complex (E,Θ) over A is defined by

$$\tau(E,\Theta) = \tau(\Theta_0 : E^{n-*} \longrightarrow E) \in \tilde{K}_1(A).$$

The K-theory map $B : \tilde{\tilde{K}}_1(A[t,t^{-1}]) \longrightarrow \tilde{K}_0(A)$ sends the torsion $\tau(E)$ of a contractible finite $A[t,t^{-1}]$-module chain complex E with $E_r = i_! F_r$ to

$$B(\tau(E)) = [t^{-N^+} F^+] \in \tilde{K}_0(A)$$

for any $N^+ \in \mathbb{N}^+(E)$ (1.3). The proof of 4.1 identifies the image under B of the torsion of a finite n-dimensional symmetric Poincaré complex (E,Θ) over $A[t,t^{-1}]_u$ with the finiteness obstruction of an explicitly constructed finitely dominated $(n-1)$-dimensional symmetric Poincaré complex $B(E,\Theta)$ over A,

$$B(\tau(E,\Theta)) = [B(E,\Theta)] \in \tilde{K}_0(A),$$

as follows:

Proposition 4.2:

i) *For any finite n-dimensional symmetric Poincaré complex (E,Θ) over $A[t,t^{-1}]_u$ with $E_r = i_! F_r$ and any $N^+ \in \mathbb{N}^+(E)$ there is defined a locally finite n-dimensional symmetric complex $i^!(E,\Theta)/t^{-N^+}F^+$ over A, such that the boundary $\partial(i^!(E,\Theta)/t^{-N^+}F^+)$ is a locally finitely dominated $(n-1)$-dimensional symmetric Poincaré complex over A with*

$$B(\tau(E,\Theta)) \;=\; [\partial(i^! E/t^{-N^+}F^+)] \in \tilde{K}_0(A).$$

ii) *The map $B : L^n_h(A[t,t^{-1}]_u) \longrightarrow L^{n-1}_p(A)$ is such that*

$$B(E,\Theta) \;=\; \partial(i^!(E,\Theta)/t^{-N^+}F^+) \in L^{n-1}_p(A).$$

□

Remark 4.3: A finite (resp. finitely dominated) n-dimensional symmetric Poincaré complex (C,ϕ) over A is *highly−connected* if $H_r(C) \;=\; 0$ for $2r < n-1$. See Ranicki [16, §2] for the details of the one-one correspondence between such highly-connected complexes and f.g. free (resp. f.g. projective) nonsingular $(-)^k$-symmetric forms ($n = 2k$) and formations ($n = 2k+1$) over A. If (E,Θ) is a highly-connected finite n-dimensional symmetric Poincaré complex over $A[t,t^{-1}]_u$ then the finitely dominated $(n-1)$-dimensional symmetric Poincaré complex $B(E,\Theta)$ constructed in 4.2 is also highly-connected. The corresponding passage from forms and formations over $A[t,t^{-1}]_u$ to formations and forms over A is the exact analogue for the symmetric case of the formulae of Ranicki [15] in the quadratic case. However, whereas the quadratic L-groups are 4-periodic $L_* \;=\; L_{*+4}$ and are represented by highly-connected complexes (corresponding to Witt groups of forms and formations) this is not in general the case for the symmetric L-groups L^* with $* \geq 2$.

□

4.1 and 4.2 are proved by first defining the relative L-groups for $1-u : L^*_p(A) \longrightarrow L^*_h(A)$ as the cobordism groups of "finitely balanced" symmetric Poincaré pairs over A, and then proving that these relative L-groups $L^*_{h,p}(1-u)$ are isomorphic to the absolute L-groups $L^*_h(A[t,t^{-1}]_u)$ of finite symmetric Poincaré complexes over $A[t,t^{-1}]_u$.

An n-dimensional symmetric pair over A

$$X \;=\; ((f\;\, g) :\; C \oplus C' \longrightarrow D,\; (\delta\phi, \phi \oplus -\phi'))$$

is *u-fundamental* if

$$(C',\phi') \;=\; (C, u\phi),$$

in which case it is the (algebraic) fundamental domain of a symmetric complex over $A[t,t^{-1}]_u$, defined as follows.

The *Laurent union* of a (locally finite) u-fundamental n-dimensional symmetric pair $X \;=\; ((f\;\, g) : C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -u\phi))$ over A is the (locally finite) n-dimensional symmetric complex over $A[t,t^{-1}]_u$

$$U(X) \;=\; (E,\Theta)$$

with

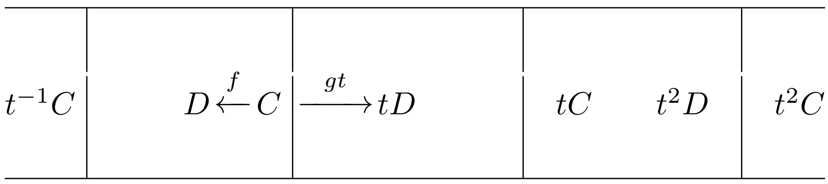
$$\begin{aligned} d_E \;=\; & \begin{pmatrix} d_D & (-)^{r-1}(f-gt) \\ 0 & d_C \end{pmatrix} : \\ E_r \;=\; & D_r[t,t^{-1}] \oplus C_{r-1}[t,t^{-1}] \longrightarrow E_{r-1} \;=\; D_{r-1}[t,t^{-1}] \oplus C_{r-2}[t,t^{-1}], \\ \Theta_s \;=\; & \begin{pmatrix} \delta\phi_s & (-)^s g\phi_s t \\ (-)^{n-r-1}\phi_s f^* & (-)^{n-r+s}T\phi_{s-1} \end{pmatrix} : \\ E^{n-r+s} \;=\; & D^{n-r+s}[t,t^{-1}] \oplus C^{n-r+s-1}[t,t^{-1}] \longrightarrow E_r \;=\; D_r[t,t^{-1}] \oplus C_{r-1}[t,t^{-1}]. \end{aligned}$$

The restriction of $U(X)$ is the locally finite n-dimensional symmetric complex over A defined by the countable union

$$i^! U(X) \;=\; \bigcup_{j=-\infty}^{\infty} t^j X$$

of the copies of X

$$t^j X \;=\; ((f\ g) : t^j C \oplus t^{j+1} C \longrightarrow t^{j+1} D, (u^j \delta\phi, u^j \phi \oplus -u^{j+1}\phi)) \ \ (j \in \mathbb{Z}).$$



The boundary $\partial U(X)$ is (locally finite) homotopy equivalent to the Laurent union $U(\partial X)$ of the boundary (locally finite) u-fundamental $(n-1)$-dimensional symmetric Poincaré cobordism $\partial X \;=\; ((\partial f\ \partial g) : \partial C \oplus \partial C \longrightarrow \partial_+ D, (\partial_+\delta\phi, \partial\phi \oplus -u\partial\phi))$

$$\partial U(X) \simeq U(\partial X)$$

(as defined in the proof of 3.8). Thus if X is a Poincaré pair over A then $U(X)$ is a Poincaré complex over $A[t,t^{-1}]_u$.

The Laurent union $U(X) \;=\; (E,\Theta)$ is the L-theory analogue of a Mayer-Vietoris presentation. Indeed, E is the algebraic mapping cone

$$E \;=\; C(f - gt : C[t,t^{-1}] \longrightarrow D[t,t^{-1}]).$$

Replacing D by the algebraic mapping cylinder D' of $(f\ g) : C \oplus C \longrightarrow D$ there is obtained an $A[t,t^{-1}]$-module chain complex E' chain equivalent to E, with a Mayer-Vietoris presentation

$$0 \longrightarrow i_! C \xrightarrow{f' - g't} i_! D' \longrightarrow E' \longrightarrow 0.$$

A u-fundamental n-dimensional (locally finite) symmetric pair over A

$$X = ((f\ g) : C \oplus C \longrightarrow D,\ (\delta\phi, \phi \oplus -u\phi))$$

is *(locally) finitely balanced* if it is (locally) finitely dominated and

$$[C] = [D] \in \tilde{K}_0(A),$$

in which case the Laurent union $U(X)$ is a homotopy (locally) finite symmetric complex over $A[t,t^{-1}]_u$. In particular, this is the case if C, D are finite.

Proposition 4.4: *The cobordism group of finitely balanced u-fundamental n-dimensional symmetric Poincaré pairs over A is the relative group $L^n_{h,p}(1-u)$ in the exact sequence*

$$\cdots \longrightarrow L^n_p(A) \xrightarrow{1-u} L^n_h(A) \xrightarrow{j} L^n_{h,p}(1-u) \xrightarrow{\partial} L^{n-1}_p(A) \longrightarrow \cdots,$$

with

$$\begin{aligned} j :& L^n_h(A) \longrightarrow L^n_{h,p}(1-u);\ (D,\delta\phi) \longrightarrow (0 \oplus 0 \longrightarrow D,\ (\delta\phi, 0 \oplus 0)), \\ \partial :& L^n_{h,p}(1-u) \longrightarrow L^{n-1}_p(A); \\ & \qquad ((f\ g) : C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -u\phi)) \longrightarrow (C,\phi). \end{aligned}$$

$L^n_{h,p}(1-u)$ is also the cobordism group of locally finitely balanced Poincaré pairs.

Proof: $L^n_{h,p}(1-u)$ is the cobordism group of homotopy finite n-dimensional symmetric Poincaré cobordisms over A of the type $(C \oplus C \oplus \partial C \longrightarrow D, (\delta\phi, \phi \oplus -u\phi \oplus \partial 0))$. Given such a cobordism define a finitely balanced u-fundamental n-dimensional symmetric Poincaré pair

$$\begin{aligned} &(C \oplus C \longrightarrow D', (\delta\phi', \phi \oplus -u\phi)) \\ &\quad = (C \oplus C \oplus \partial C \longrightarrow D, (\delta\phi, \phi \oplus -u\phi \oplus \partial 0)) \cup_{\partial(C,0)} (p_C : \partial C \longrightarrow C^{n+1-*}, (0, \partial 0)), \end{aligned}$$

with $p_C : \partial C \to C^{n+1-*}$ the projection. Conversely, given a finitely balanced u-fundamental n-dimensional symmetric Poincaré pair $(C \oplus C \to D, (\delta\phi, \phi \oplus -u\phi))$ define a homotopy finite n-dimensional symmetric Poincaré cobordism

$$\begin{aligned} &(C \oplus C \oplus \partial C \longrightarrow D'', (\delta\phi'', \phi \oplus -u\phi \oplus \partial 0)) \\ &\quad = (C \oplus C \longrightarrow D, (\delta\phi, \phi \oplus -u\phi)) \oplus (p_C : \partial C \longrightarrow C^{n+1-*}, (0, \partial 0)). \end{aligned}$$

Similarly for the locally finite case. □

We show that U defines an isomorphism

$$U :\ L^n_{h,p}(1-u) \longrightarrow L^n_h(A[t,t^{-1}]_u);\ X \longrightarrow U(X),$$

using the following L-theory analogue of the existence of finite Mayer-Vietoris presentations of finite $A[t,t^{-1}]$-module chain complexes (1.2):

Proposition 4.5: *Every finite n-dimensional symmetric Poincaré complex (E,Θ) over $A[t,t^{-1}]_u$ is homotopy equivalent to the Laurent union $U(X)$ of a locally finitely balanced u-fundamental n-dimensional symmetric Poincaré pair over A*

$$X \;=\; ((f\ g) : C\oplus C\longrightarrow D, (\delta\phi, \phi\oplus -u\phi))$$

with

$$B(\tau(E,\Theta)) \;=\; [C] \in \tilde{K}_0(A).$$

□

The isomorphism inverse to U is then defined by

$$U^{-1}:\ L^n_h(A[t,t^{-1}]_u)\longrightarrow L^n_{h,p}(1-u);\ (E,\Theta) \;=\; U(X)\longrightarrow X,$$

and B is given by

$$B:\ L^n_h(A[t,t^{-1}]_u)\longrightarrow L^{n-1}_p(A);\ (E,\Theta)\longrightarrow(C,\phi)$$

with $X = ((f\ g) : C\oplus C\longrightarrow D, (\delta\phi, \phi\oplus -u\phi))$ the locally finitely balanced Poincaré pair given by 4.5. In order to actually verify that U^{-1} and B are well-defined we need the relative version of 4.5 stated in 4.6 below.

The *algebraic mapping torus* of a morphism of (locally finite) n-dimensional symmetric complexes over A

$$(f,\chi):\ (C,u\phi)\longrightarrow(C,\phi)$$

is the (locally finite) $(n+1)$-dimensional symmetric complex over $A[t,t^{-1}]_u$

$$T(f,\chi) \;=\; U(Y)$$

defined by the Laurent union of the (locally finite) u-fundamental symmetric cobordism

$$Y \;=\; ((1\ f):\ C\oplus C\longrightarrow C,\ (\chi, \phi\oplus -u\phi)).$$

The underlying $A[t,t^{-1}]$-module chain complex is the algebraic mapping torus of $f: C\longrightarrow C$

$$T(f) \;=\; C(f-t : i_!C\longrightarrow i_!C).$$

The boundary $\partial T(f,\chi)$ is (locally finite) homotopy equivalent to the Laurent union $U(X)$ of the (locally finite) u-fundamental n-dimensional symmetric pair over A

$$X \;=\; \partial Y \;=\; ((\partial 1\ \partial f) : \partial C\oplus\partial C\longrightarrow\partial_+C, (\partial_+\chi, \partial\phi\oplus -u\partial\phi))$$

defined as in the proof of 3.8.

For any countably based free $A[t,t^{-1}]$-module M define a Mayer-Vietoris presentation

$$0\longrightarrow i_!i^!M \xrightarrow{\zeta(M)-t} i_!i^!M \xrightarrow{p(M)} M\longrightarrow 0$$

with

$$\zeta(M) : i^! M \longrightarrow i^! M;\ x \longrightarrow tx,$$

$$p(M) : i_! i^! M \longrightarrow M;\ a \otimes x \longrightarrow ax\quad (a \in A[t,t^{-1}], x \in M).$$

Define also a Mayer-Vietoris presentation of the dual countably based free $A[t,t^{-1}]$-module M^*

$$0 \longrightarrow i_! i^!(M^*) \xrightarrow{\zeta^*(M)-ut^{-1}} i_! i^!(M^*) \xrightarrow{p^*(M)} M^* \longrightarrow 0$$

with

$$\begin{aligned} \zeta^*(M) &= \zeta(M)^* : i^!(M^*) = (i^! M)^* \longrightarrow i^!(M^*); \\ & \qquad f \longrightarrow (x \longrightarrow f(tx)), \\ p^*(M) &= p(M^*) : i_! i^!(M^*) \longrightarrow M^*; \\ & \qquad a \otimes f \longrightarrow af\quad (a \in A[t,t^{-1}], f \in M^*). \end{aligned}$$

Proof of 4.5: Given a finite n-dimensional symmetric complex (E,Θ) over $A[t,t^{-1}]_u$ fix an element

$$N^+ = (N_0^+, N_1^+, \dots, N_n^+; \infty, \infty, \dots, \infty) \in \mathbb{N}^+(E).$$

Define a morphism of locally finite n-dimensional symmetric complexes over A

$$(\eta, 0) :\ (C, u\phi) \longrightarrow (C, \phi)$$

by

$$(C,\phi) = i^!(E,\Theta)/t^{-N^+}F^+,\ \eta :\ C \longrightarrow C;\ x \longrightarrow tx.$$

The algebraic mapping torus of $(\eta,0)$ is the locally finite $(n+1)$-dimensional symmetric complex over $A[t,t^{-1}]_u$

$$T(\eta,0) = U(Y)$$

defined by the Laurent union of the locally finite u-fundamental $(n+1)$-dimensional symmetric pair over A

$$Y = ((1\ \eta) : C \oplus C \longrightarrow C, (0, \phi \oplus -u\phi)).$$

We shall show that (E,Θ) is locally finite homotopy equivalent to the boundary $\partial T(\eta,0)$, and hence to the Laurent union $U(X)$ of the locally finitely balanced u-fundamental n-dimensional symmetric Poincaré cobordism $X = \partial Y$ over A, with $B(E,\Theta) = \partial(C,\phi)$.

The dual of the locally finite A-module morphism defined by projection

$$i^! E_r = i^! i_! F_r \longrightarrow C_r = t^{-N_r^+} F_r^-$$

is the locally finite A-module morphism defined by injection

$$C^r = t^{-N_r^+}(F^r)^- \longrightarrow i^! E^r = i^! i_! F^r.$$

The $A[t,t^{-1}]$-module morphisms $\eta - t : i_! C_r \longrightarrow i_! C_r$ are automorphisms (such that the inverses are not locally finite, cf. Remark 3.1). Also, there are defined Mayer-Vietoris presentations

$$0 \longrightarrow i_! C^r \xrightarrow{\eta^* - ut^{-1}} i_! C^r \xrightarrow{g} E^r \longrightarrow 0$$

with g the (non locally finite) $A[t,t^{-1}]$-module morphisms

$$g:\ i_!C^r \xrightarrow{\text{inclusion}} i_!i^!E^r \xrightarrow{p^*(E_r)} E^r \ =\ i_!F^r.$$

Define a (non locally finite) chain equivalence of locally finite $A[t,t^{-1}]$-module chain complexes

$$e:\ \partial T(\eta)\longrightarrow E$$

by

$$\begin{aligned} e\ =\ (0\ 0\ 0\ \Theta_0 g)\ :\partial T(\eta)_r\ &=\ i_!C_{r+1}\oplus i_!C_r\oplus i_!C^{n-r+1}\oplus i_!C^{n-r}\\ &\longrightarrow E_r\ =\ i_!F_r. \end{aligned}$$

The (non locally finite) $A[t,t^{-1}]$-module morphisms

$$h:\ i_!C_r \xrightarrow{\text{inclusion}} i_!i^!E_r \xrightarrow{p(E_r)} E_r\ =\ i_!F_r$$

are such that the $A[t,t^{-1}]$-module morphisms

$$e^{lf}\ =\ e+d_E(h\ 0\ 0\ 0)+(h\ 0\ 0\ 0)d_{\partial T(\eta)}:\ \partial T(\eta)_r\longrightarrow E_r$$

are locally finite. The locally finite $A[t,t^{-1}]$-module chain map

$$e^{lf}\ :\ \partial T(\eta)\longrightarrow E$$

is chain homotopic to e, defining a locally finite equivalence of homotopy finite n-dimensional symmetric Poincaré complexes over $A[t,t^{-1}]_u$

$$(e^{lf},0):\ \partial T(\eta,0)\longrightarrow(E,\Theta).$$

Define $\overline{N}^-\ =\ (\infty,\infty,\dots,\infty;\ \overline{N}_0^-,\overline{N}_1^-,\dots,\overline{N}_n^-)\in\mathbb{N}^-(E^{n-*})$ by $\overline{N}_r^-\ =\ N_{n-r}^+$ $(0\le r\le n)$. Let $N^-\ =\ (\infty,\infty,\dots,\infty;\ N_0^-,N_1^-,\dots,N_n^-)\in\mathbb{N}^-(E)$ be so large that

$$\Theta_0(t^{\overline{N}^-}(F^{n-*})^-)\subseteq t^{N^-}F^-,$$

so that $i^!\Theta_0: i^!E^{n-*}\longrightarrow i^!E$ restricts to a locally finite A-module chain map

$$\lambda\ =\ i^!\Theta_0|\ :\ t^{\overline{N}^-}(F^{n-*})^-\longrightarrow t^{N^-}F^-$$

and there is defined an exact sequence of locally finite A-module chain complexes

$$0\longrightarrow t^{N^-}F^-\cap t^{-N^+}F^+\longrightarrow t^{N^-}F^-\longrightarrow C\longrightarrow 0$$

with $t^{N^-}F^-\cap t^{-N^+}F^+$ finite. By 1.3 and 3.3 the algebraic mapping cone $C(\lambda)$ is locally finitely dominated with finiteness obstruction

$$[C(\lambda)]\ =\ -B(\tau(\Theta_0: E^{n-*}\longrightarrow E))\in\tilde{K}_0(A).$$

It now follows from the exact sequence

$$0 \longrightarrow t^{N^-}F^- \cap t^{-N^+}F^+ \longrightarrow C(\lambda) \longrightarrow C(\phi_0 : C^{n-*} \longrightarrow C) \longrightarrow 0$$

that

$$[\partial C] \ = \ -[C(\phi_0 : C^{n-*} \longrightarrow C)] \ = \ B(\tau(E,\Theta)) \in \tilde{K}_0(A).$$

□

An n-dimensional symmetric triad over A

$$X \ = \ \left(\begin{array}{ccc} \partial C \oplus \partial C' & \xrightarrow{(\partial f \ \partial g)} & \partial_+ D \\ \big\downarrow & & \big\downarrow \\ C \oplus C' & \xrightarrow{(f \ g)} & D \end{array} \ , \ \begin{array}{ccc} \partial\phi \oplus -\partial\phi' & \longrightarrow & \partial_+\delta\phi \\ \big\downarrow & & \big\downarrow \\ \phi \oplus -\phi' & \longrightarrow & \delta\phi \end{array}\right)$$

is *u-fundamental* if $\partial C \oplus \partial C' \to C \oplus C'$ is the sum of chain maps $\partial C \to C$, $\partial C' \to C'$ and

$$(\partial C' \longrightarrow C', (\phi', \partial\phi')) \ = \ (\partial C \longrightarrow C, u(\phi, \partial\phi)).$$

The *Laurent union* n-dimensional symmetric pair over $A[t,t^{-1}]_u$

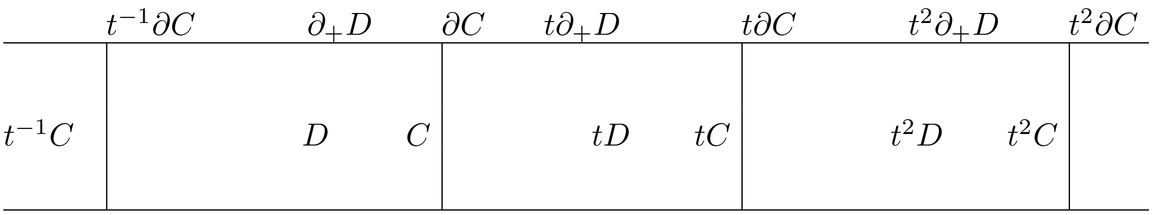
$$U(X) \ = \ (\partial E \longrightarrow E, (\Theta, \partial\Theta))$$

is defined by analogy with the absolute case $(\partial C \oplus \partial C \to \partial_+ D) \ = \ 0$, with

$$\begin{aligned} \partial E \ &= \ C(\partial f - \partial g t : \ \partial C[t,t^{-1}] \longrightarrow \partial_+ D[t,t^{-1}]), \\ E \ &= \ C(f - gt : \ C[t,t^{-1}] \longrightarrow D[t,t^{-1}]). \end{aligned}$$

As in the absolute case X is a fundamental domain of $U(X)$, and the restriction is obtained by glueing together all the copies $t^j X$ $(j \in \mathbb{Z})$ of X

$$i^! U(X) \ = \ \bigcup_{j=-\infty}^{\infty} t^j X.$$



A u-fundamental (locally finite) symmetric triad over A

$$X \ = \ \left(\begin{array}{ccc} \partial C \oplus \partial C & \longrightarrow & \partial_+ D \\ \big\downarrow & & \big\downarrow \\ C \oplus C & \longrightarrow & D \end{array} \ , \ \begin{array}{ccc} \partial\phi \oplus -u\partial\phi & \longrightarrow & \partial_+\delta\phi \\ \big\downarrow & & \big\downarrow \\ \phi \oplus -u\phi & \longrightarrow & \delta\phi \end{array}\right)$$

is *(locally) finitely balanced* if it is (locally) finitely dominated and

$$[C] \;=\; [D],\ [\partial C] \;=\; [\partial_+ D] \in \tilde{K}_0(A),$$

in which case the Laurent union $U(X)$ is a homotopy (locally) finite symmetric pair over $A[t,t^{-1}]_u$.

The *torsion* of a finite n-dimensional symmetric Poincaré pair $(f:\ \partial E \longrightarrow E, (\Theta, \partial\Theta))$ over A is defined by

$$\tau(f:\ \partial E \longrightarrow E,\ (\Theta,\partial\Theta)) \;=\; \tau((\Theta_0,\partial\Theta_0) : C(f)^{n-*} \longrightarrow E) \in \tilde{K}_1(A).$$

4.5 is the special case $(\partial E, \partial\Theta) \;=\; 0$, $\partial X \;=\; 0$ of the following relative version:

Proposition 4.6: *Every finite n-dimensional symmetric Poincaré pair over $A[t,t^{-1}]_u$ $(f:\ \partial E \longrightarrow E, (\Theta,\partial\Theta))$ is homotopy equivalent to the Laurent union $U(X)$ of a locally finitely balanced u-fundamental n-dimensional symmetric Poincaré triad X over A such that*

$$B(\tau(f:\ \partial E \longrightarrow E,\ (\Theta,\partial\Theta))) \;=\; [C] \in \tilde{K}_0(A).$$

□

Proof of 4.1: The Laurent union morphism

$$U:\ L^n_{h,p}(1-u) \longrightarrow L^n_h(A[t,t^{-1}]_u);\ X \longrightarrow U(X),$$

is onto by 4.5, and one-one by 4.6. □

§5. **An example**

Let $\mathcal{F} \;=\; \mathbb{R}(x)[y]/(x^2+y^2+1)$, the quotient field of $(\mathbb{R}[x,y]/(x^2+y^2+1))$, then we have

Lemma 5.1: $\mathcal{F}(i) \;=\; \mathbb{C}(t)$*, the rational function field, where i is, as usual, a primitive 4^{th}-root of 1.*

Proof: Define a map $\phi : \mathcal{F}(i) \to \mathbb{C}(t)$ by setting

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}(t - t^{-1})\\ \phi(y) &= \frac{i}{\sqrt{2}}(t + t^{-1}).\end{aligned}$$

Now, $\phi(\frac{1}{\sqrt{2}}(x - iy)) \;=\; t$, so ϕ is onto. Since both range and domain are fields ϕ is an isomorphism, and the lemma follows. □

Lemma 5.2: *Define an involution λ on $\mathbb{C}(t)$ by setting $\lambda(t) = -t^{-1}$, $\lambda(i) = -i$, $\lambda = id$ on $\mathbb{R} \subset \mathbb{C}$, then the fixed field of λ, $\mathbb{C}(t)^\lambda$ is $\mathcal{F}$.*

Proof: Directly we have that $\lambda(\phi(x)) = \phi(x)$, $\lambda(\phi(y)) = \phi(y)$. Hence, $\phi(\mathcal{F}) \subset \mathbb{C}(t)^\lambda$, and the result follows by a dimension count. □

The L-groups of a ring A with respect to the trivial involution are denoted simply by $L_*(A)$. When the involution is non-trivial, say τ, we write the corresponding L-groups as $L_*(A, \tau)$.

We now wish to study the Witt group $W(\mathcal{F}) = L_0(\mathcal{F})$ with respect to the trivial (identity) involution on $\mathcal{F}$. From now on we shall only be dealing with the L^h-groups, and so omit h from the terminology.

Theorem 5.3 (Knebusch [9,§11]):

(i) There is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow L_0(\mathcal{F}) \longrightarrow \mathbb{Z}/2 \oplus \bigoplus_{\mathcal{I}} \mathbb{Z}/2 \longrightarrow 0$$

where the index set $\mathcal{I}$ is the points $p \in \mathcal{M}_1$ and $\mathcal{M}_1$ is the open Möbius band

$$\{\mathbb{C} - \{0\}/(z \sim -1/\bar{z})\} \ .$$

The first $\mathbb{Z}/2$ carries the element $\langle 1 \rangle$.

(ii) The sequence above does not split. In particular the Stufe of $\mathcal{F}$ is 2.

Proof: There is an exact sequence due to N. Jacobson, (see [12,Appendix B])

5.4 $$0 \longrightarrow L_0(\mathbb{C}(t), \lambda) \longrightarrow L_0(\mathcal{F}) \longrightarrow L_0(\mathbb{C}(t)).$$

Again, quoting a result of Milnor ([12], Theorem 3.1, pg. 265) for any field $\mathcal{K}$ of characteristic different from 2 there is an exact sequence

5.5 $$0 \longrightarrow L_0(\mathcal{K}) \longrightarrow L_0(\mathcal{K}(t)) \longrightarrow \bigoplus_{\mathcal{P}} L_0(\mathcal{K}[t]/\mathcal{P}) \longrightarrow 0$$

where $\mathcal{P}$ runs over all the primes of $\mathcal{K}[t]$. In our case

$$L_0(\mathbb{C}(t)) = \left(\bigoplus_{z \in \mathbb{C}} L_0(\mathbb{C}) \right) \oplus L_0(\mathbb{C}) \quad \text{where } L_0(\mathbb{C}) = \mathbb{Z}/2.$$

Moreover, the involution λ on primes $(t - z)$ becomes $\lambda(t - z) = -\bar{z}t^{-1}(t + 1/\bar{z})$, and we see that λ is geometrically the map $z \leftrightarrow -1/\bar{z}$. In particular the extra $\mathbb{Z}/2$ can be identified with the $\mathbb{Z}/2$ over ∞ in S^2 and, clearly, this $\mathbb{Z}/2$ and the $\mathbb{Z}/2$ corresponding to the prime (t) are interchanged by λ as well. (There is room for possible confusion here. The inclusion $L_0(\mathbb{C}) \subset L_0(\mathcal{F})$ includes the $\mathbb{Z}/2$ as the sum of the two terms above.) □

Next, note that

Lemma 5.6: *(i)* $imL_0(\mathcal{F}) \hookrightarrow L_0(\mathbb{C}(t))$ *is contained in* $L_0(\mathbb{C}(t))^\lambda$.

(ii) If $\theta \in L_0(\mathbb{C}(t))$ *has the form* $\mu + \lambda(\mu)$ *then* $\theta \in imL_0(\mathbb{F})$.

Proof: (i) is clear. To see (ii), suppose $\mu = \sum\langle a_i\rangle$, so that

$$\mu + \lambda(\mu) = \sum\langle a_i\rangle\perp\langle\lambda(a_i)\rangle.$$

In our case there are only two invariants determining an element in $L_0(\mathbb{C}(t))$, the rank mod(2) and the discriminant. But $\langle a\lambda(a)\rangle\perp\langle 1\rangle$ and $\langle a\rangle\perp\langle\lambda(a)\rangle$ both have even rank and equal discriminant. Hence they are equivalent in the Witt group, and (ii) follows. □

This identifies the image of $L_0(\mathcal{F})$ in $L_0(\mathbb{C}(t))$. We now must analyze the kernel in the map 5.4. It is precisely at this point we require our results on Laurent extensions.

Consider first the localization exact sequence

$$L_*(\mathbb{C}[t,t^{-1}],\lambda)\to L_*(\mathbb{C}(t),\lambda)\to \bigoplus_{\mathcal{P}\ fixed} L_*(\mathbb{C}[t,t^{-1}]/\mathcal{P},\lambda)\to L_{*-1}(\mathbb{C}[t,t^{-1}],\lambda)\to\cdots.$$

(See e.g. [17,§3]). We have already seen that there are no fixed primes. Hence

$$L_*(\mathbb{C}[t,t^{-1}],\lambda) \cong L_*(\mathbb{C}(t),\lambda). \qquad (5.7)$$

Our main result 4.1 applies to give an exact sequence

$$\cdots\longrightarrow L_*(\mathbb{C},-)\xrightarrow{\langle 1\rangle\perp\langle 1\rangle} L_*(\mathbb{C},-)\longrightarrow L_*(\mathbb{C}[t,t^{-1}],\lambda)\longrightarrow L_{*-1}(\mathbb{C},-)\longrightarrow\cdots$$

where by $L_*(\quad,-)$ we mean the L-group with respect to complex conjugation. It is well known that

$$L_*(\mathbb{C},-) = \begin{cases}\mathbb{Z} & *\text{ even}\\ 0 & *\text{ odd}\end{cases}$$

and the generators are $\langle 1\rangle$ for $*=0$, $\langle i\rangle$ for $*=2$. Then $\circ(\langle 1\rangle\perp\langle 1\rangle)$ is just multiplication by 2 and we have

$$L_*(\mathbb{C}[t,t^{-1}],\lambda) = \begin{cases}\mathbb{Z}/2 & *\text{ even}\\ 0 & *\text{ odd.}\end{cases}$$

The existence of the exact sequence follows.

It remains to show that it does not split. To this end consider the inclusion

$$L_0(\mathbb{C}(t),\lambda)\longrightarrow L_0(\mathcal{F}),$$

which is obtained by taking real forms. A basis for $\mathbb{C}(t)$ over $\mathcal{F}$ is given by 1, i, so that the image of the Hermitian form $\langle 1\rangle$ is $\langle 1\rangle\perp\langle 1\rangle$. The proof is complete. □

Remark 5.8: We shall see in the next section that $\mathcal{F}$ is an example of a genus 0 function field. Indeed, the original motivation for presenting the exact sequence for Laurent extensions in the generality used here was to apply it to study these Witt groups. T.Y. Lam pointed out to one of us somewhat later that the field $\mathcal{F}$ above was a genus 0 function field, and asked us to determine its Stufe. □

Remark 5.9: The quotient field

5.10 $$\mathcal{F}_{n+1} \;=\; \mathbb{R}(x_1,\ldots,x_n)(y)[z]/(y^2+z^2+\sum x_j^2+1)$$

also satisfies $\mathcal{F}_{n+1}(i) \;=\; \mathbb{C}(x_1,\ldots,x_n,t)$ and $\mathcal{F}_{n+1}$ is the fixed field of

$$\lambda:\ \lambda(i) \;=\; -i,\ \ \lambda(x_j) \;=\; x_j,\ \ \lambda(t) \;=\; -(1+\sum x_j^2)t^{-1}.$$

Hence, it appears possible to apply the techniques above to obtain information about $L_*(\mathcal{F}_{n+1})$.

For example, when $n=1$ it is not hard to show that we have the diagram

5.11 $$\begin{array}{ccccc}
\bigoplus_{r\in\mathbb{R}}(\mathbb{Z}/2) & & & & \\
\downarrow & & & & \\
L_0(\mathbb{C}(t),\lambda) & \longrightarrow & L_0(\mathcal{F}_1) & \longrightarrow & L_0(\mathbb{C}(x)(t)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{\mathcal{P}\subset\mathcal{F}_1} L_0(\mathbb{C}(x)[t]/\mathcal{P},\lambda) & \longrightarrow & \bigoplus_{\mathcal{P}\subset\mathcal{F}_1} L_0(\mathcal{F}/\mathcal{P}) & \longrightarrow & \bigoplus_{\mathcal{Q}} L_0(\mathbb{C}(x)[t]/\mathcal{Q})
\end{array}$$

Thus the Stufe is ≤ 4 in this case. □

§6. **Genus 0 function fields**

We assume from now on that $\mathbb{F}$ is a field and $char(\mathbb{F})\neq 2$. For non-zero $\lambda,\ \mu\in\mathbb{F}$ let

$$\left\langle \begin{matrix}\lambda\ \mu\\ \mathbb{F}\end{matrix}\right\rangle \;=\; \mathbb{F}[i,j]/\{i^2=\lambda,\ j^2=\mu,\ ij=-ji\}$$

be a quaternion algebra over $\mathbb{F}$, and let

6.1 $$\mathbb{F}\langle\lambda,\mu\rangle \;=\; \mathbb{F}(x)[y]/(y^2-\mu(x^2-4\lambda))$$

be the associated transcendence degree 1 extension field of $\mathbb{F}$. $\mathbb{F}\langle\lambda,\mu\rangle$ is a degree 2 extension of the rational function field $\mathbb{F}(x)$. It is a standard result of algebraic number theory

(Hasse [7], p.480) that a genus 0 function field over a perfect field $\mathbb{F}$ is either rational over $\mathbb{F}$ (i.e. isomorphic to $\mathbb{F}(x)$) or else isomorphic to one of the type $\mathbb{F}\langle\lambda,\mu\rangle$. By an abuse of terminology, we shall call $\mathbb{F}\langle\lambda,\mu\rangle$ the genus 0 function fields. In the special case $\mathbb{F} = \mathbb{C}$ these are precisely the function fields of the genus 0 curves, i.e. the conics.

We assume in what follows that $y^2 - \mu x^2 + 4\mu\lambda$ is irreducible over $\mathbb{F}$. Note that y is integral over the polynomial subring $\mathbb{F}[x] \subset \mathbb{F}(x)$ so that $\mathbb{F}[x](y) = \mathbb{F}[x,y]/(y^2 - \mu(x^2 - 4\lambda))$ injects into $\mathbb{F}\langle\lambda,\mu\rangle$.

Lemma 6.2: $\mathbb{F}[x](y) \subset \mathbb{F}\langle\lambda,\mu\rangle$ *is the integral closure of* $\mathbb{F}[x]$ *in* $\mathbb{F}\langle\lambda,\mu\rangle$. *Consequently it is a Dedekind domain.*

(The proof is the usual one: first check traces, and then norms.)

The quotient fields $\mathbb{F}[x](y)/\mathcal{P}$ as $\mathcal{P}$ runs over the prime ideals of $\mathbb{F}[x](y)$ have a special relationship with the quaternion algebra originally used to define $\mathbb{F}\langle\lambda,\mu\rangle$. Specifically, we have

Lemma 6.3: *Let* $\mathcal{P}$ *be a prime of* $\mathbb{F}[x](y)$, *then*

$$(\mathbb{F}[x](y)/\mathcal{P}) \otimes_{\mathbb{F}} \left\langle \begin{matrix} \lambda\ \mu \\ \mathbb{F} \end{matrix} \right\rangle \;=\; M_2(\mathbb{F}[x](y)/\mathcal{P}).$$

Proof: $(xj + 2ij)^2 \;=\; \mu x^2 - 4\lambda\mu \;=\; y^2$ in $\mathbb{F}[x](y)/\mathcal{P}$, so

$$(xj + 2ij - y)(xj + 2ij + y) \;=\; 0$$

in $\mathbb{F}[x](y)/\mathcal{P} \otimes_{\mathbb{F}} \left\langle \begin{smallmatrix} \lambda\ \mu \\ \mathbb{F} \end{smallmatrix} \right\rangle$ and the result follows. □

It is not hard to see that the converse is also true, namely, if $\mathbb{K}$ is a finite extension of $\mathbb{F}$ and $\mathbb{K} \otimes_{\mathbb{F}} \left\langle \begin{smallmatrix} \lambda\ \mu \\ \mathbb{F} \end{smallmatrix} \right\rangle = M_2(\mathbb{K})$, then $\mathbb{K}$ is the quotient of $\mathbb{F}[x](y)$ by a prime ideal.

Clearly, every genus 0 function field is a degree two extension of a pure transcendental extension $\mathbb{F}(x)$. But it is also true that either $\mathbb{F}\langle\lambda,\mu\rangle$ is itself pure transcendental, or a degree two extension of it is so. Indeed, if we set $\mathbb{K} = \mathbb{F}(\sqrt{\mu})$, then we have

Lemma 6.4: $\mathbb{K}[x](y) \;=\; \mathbb{K}[t,\ t^{-1}]$ *where* $2\sqrt{\mu}t \;=\; y + \sqrt{\mu}\cdot x$.

Proof: Set $2\sqrt{\mu}\cdot z \;=\; (y - \sqrt{\mu}\cdot x)$, then

$$4\mu\cdot t\cdot z \;=\; y^2 - \mu x^2 \;=\; -4\lambda\mu$$

and $t^{-1} \in \mathbb{K}[x](y)$. On the other hand, x and y are obtained in terms of t, z, over $\mathbb{K}$, so the lemma follows. □

From this it follows that the field $\mathbb{K}(t)$ is a degree 2 extension of $\mathbb{F}(x)(y)$ provided that μ is a non-square in $\mathbb{F}$. Indeed, if an automorphism

6.5 $$\tau^\lambda : \mathbb{K}(t) \longrightarrow \mathbb{K}(t)$$

is defined by the identity on $\mathbb{F}$ and

$$\tau^\lambda(t) \;=\; \lambda t^{-1} \quad , \quad \tau^\lambda(\sqrt{\mu}) \;=\; -\sqrt{\mu},$$

then the assignations $x \longrightarrow t+\lambda t^{-1}$, $y \longrightarrow \sqrt{\mu}(t-\lambda t^{-1})$ define an isomorphism of $\mathbb{F}\langle\lambda,\mu\rangle$ to the fixed field $\mathbb{K}(t)^{\tau^\lambda}$ of τ^λ. In the terminology of §4 the ring $\mathbb{K}[t,t^{-1}]$ with the involution τ^λ is denoted by $\mathbb{K}[t,t^{-1}]_\lambda$, and

$$L_*(\mathbb{K}[t,t^{-1}],\tau^\lambda) \;=\; L_*(\mathbb{K}[t,t^{-1}]_\lambda)\ .$$

As in §5 we only consider the L^h-groups, and so omit the superscript h. Since the characteristic of the ground field $\mathbb{F}$ is $\neq 2$ there is no difference between the quadratic L-groups L_* and the symmetric L-groups L^*.

We summarize the discussion so far with the diagram of fields and rings of integers

6.6 $$\begin{array}{ccccc}
 & & \mathbb{F}[x] & \longrightarrow & \mathbb{F}(x) \\
 & & \big\downarrow & & \big\downarrow \\
\mathbb{F} & \longrightarrow & \mathbb{F}[x](y) & \longrightarrow & \mathbb{F}(x)(y) \\
\big\downarrow \tau & & \big\downarrow \tau^\lambda & & \big\downarrow \tau^\lambda \\
\mathbb{K} & \longrightarrow & \mathbb{K}[t,t^{-1}] & \longrightarrow & \mathbb{K}(t)
\end{array}$$

where the labels on the vertical arrows denote the Galois automorphisms.

Finally, we need

Lemma 6.7: *If λ is not a norm from $\mathbb{K}$ the ring $\mathbb{F}[x](y)$ is a principal ideal domain.*

Proof: Let $\mathcal{P}$ be a prime ideal of $\mathbb{F}[x](y)$. Then $\mathbb{K}[t,t^{-1}]\mathcal{P} \subset \mathbb{K}[t,t^{-1}]$ either remains prime or splits as a product $\mathcal{Q}\tau(\mathcal{Q})$ since the extension is unramified. If it splits as a product, since $\mathcal{Q}$ is principal, we can write $\mathcal{Q} \;=\; (p(t))$, and thus

$$\mathbb{K}[t,t^{-1}]\mathcal{P} \;=\; (p(t)\tau(p(t))),$$

so $p(t)\tau(p(t)) \in \mathbb{F}[x](y)$ generates $\mathcal{P}$. If it remains prime we have $\mathbb{K}[t,t^{-1}]\mathcal{P} \;=\; (h(t))$, and $\tau(h(t)) \;=\; uh(t)$ for some unit $u \in \mathbb{K}[t,t^{-1}]$. The units in $\mathbb{K}[t,t^{-1}]$ are all of the form kt^i with $k \neq 0$ in $\mathbb{K}$. Thus, applying τ again we obtain

$$h(t) \;=\; k\tau(k)\lambda^i h(t),$$

so $k\tau(k)\lambda^i = 1$, and i must be even since λ is not a norm. Let $h_1(t) = t^{i/2}h(t)$, then $\tau(h_1(t)) = k\lambda^{i/2}h_1(t)$, and, since the norm of $k\lambda^{i/2}$ is 1, we must have $k\lambda^{i/2} = \frac{\tau(\theta)}{\theta}$ for some $\theta \in \mathbb{K}$. If we set $h_2(t) = \tau(\theta)h_1(t)$ then $h_2(t)$ is invariant under τ and must generate $\mathcal{P}$. The proof is complete. □

Corollary 6.8: *If $\lambda \in \mathbb{F}$ is not a norm from $\mathbb{K}$ then the map*

$$L_0(\mathbb{F}[x](y)) \longrightarrow L_0(\mathbb{F}\langle\lambda,\mu\rangle)$$

is an injection.

Proof: If $\theta \in L_0(\mathbb{F}[x](y))$ is in the kernel, then θ is represented by a form $(\mathbb{F}[x](y)^r,\ A)$ which becomes hyperbolic over $\mathbb{F}\langle\lambda,\mu\rangle$. In particular there is a projective kernel in $(\mathbb{F}[x](y)^r,\ A)$. But since $\frac{1}{2} \in \mathbb{F}$, and projectives are free, it follows that the form is already hyperbolic over $\mathbb{F}[x](y)$. □

Remark: This is a special case of Corollary 3.3 pg. 93, of [12]. □

§7. **The Witt groups of Genus 0 function fields**

Our study of the Witt groups $W(\mathbb{F}\langle\lambda,\mu\rangle) = L_0(\mathbb{F}\langle\lambda,\mu\rangle)$ for the fields $\mathbb{F}\langle\lambda,\mu\rangle$ discussed in §6 is based on the following diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & L_0(\mathbb{K}[t,t^{-1}],\tau^\lambda) & \longrightarrow & L_0(\mathbb{F}[x](y)) & \longrightarrow & L_0(\mathbb{K}[t,t^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L_0(\mathbb{K}(t),\tau^\lambda) & \longrightarrow & L_0(\mathbb{F}\langle\lambda,\mu\rangle) & \longrightarrow & L_0(\mathbb{K}(t)) \\
7.1 & & \downarrow & & \downarrow & & \downarrow\uparrow \\
0 & \longrightarrow & \bigoplus_{\mathcal{P}\in J} L_0(\frac{\mathbb{K}[t,t^{-1}]}{\mathcal{P}},\tau^\lambda) & \longrightarrow & \bigoplus_{\mathcal{P}\in V} L_0(\frac{\mathbb{F}[x](y)}{\mathcal{P}}) & \longrightarrow & \bigoplus_{\mathcal{Q}\in W} L_0(\frac{\mathbb{K}[t,t^{-1}]}{\mathcal{Q}}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & L_3(\mathbb{K}[t,t^{-1}],\tau^\lambda) & \longrightarrow & L_3(\mathbb{F}[x](y)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where

1. J is the set of primes $\mathcal{P} \subset \mathbb{F}[x](y)$ which do not split in $\mathbb{K}[t,t^{-1}]$,

2. V is the set of primes $\mathcal{P} \subset \mathbb{F}[x](y)$,

3. W is the set of primes $\mathcal{Q} \subset \mathbb{K}[t, t^{-1}]$,

4. the columns are the localization exact sequences,

5. the two middle rows are exact, since the upper one is just the Jacobson exact sequence, and the lower one is a sum of Jacobson sequences.

Remark 7.2: The top row in 7.1 is also exact and $L_0(\mathbb{K}[t, t^{-1}], \tau^\lambda)$ embeds in $L_0(\mathbb{F}[x](y))$ since all the groups include into the corresponding L-groups for the quotient fields.

The fact that the bottom row is exact is more difficult. This was originally shown in Hambleton, Taylor and Williams [6] and Ranicki [20]. The reader familiar with L-theory can skip much of what follows in §7, but the proof given here is essentially self contained.

From our main results on the L-groups of Laurent polynomial extensions we have, following the notation established in 6.6,

Lemma 7.3: $L_1(\mathbb{K}[t, t^{-1}], \tau^\lambda) \;=\; L_3(\mathbb{K}[t, t^{-1}], \tau^\lambda) \;=$

$$\mathit{Ker}\,((\langle 1\rangle \perp - \langle\lambda\rangle)\cdot :\; L_2(\mathbb{K}, \tau) \longrightarrow L_2(\mathbb{K}, \tau))\,.$$

Likewise

$$L_0(\mathbb{K}[t, t^{-1}], \tau^\lambda) \;=\; L_0(\mathbb{K}, \tau)/\mathit{im}(\langle 1\rangle \perp - \langle\lambda\rangle).$$

Example 7.4: Suppose $\lambda = -1$, $\mathbb{F} = \mathbf{Q}$, $\mathbb{K} = \mathbf{Q}(i)$, then $L_0(\mathbb{K}, \tau) = \mathbb{Z} \oplus \bigoplus_{p \equiv 3(4)} \mathbb{Z}/2$, and the ring structure gives that the action of $\langle -1\rangle$ is multiplication by -1 on the $\mathbb{Z}$ and the identity on the $\mathbb{Z}/2$'s. Consequently, $\mathit{Ker}(\langle 1\rangle \perp - \langle -1\rangle)$ is $\bigoplus_{p\equiv 3(r)} \mathbb{Z}/2$, *coker* $= \mathbb{Z}/2 \oplus \left(\bigoplus_{p\equiv 3(4)} \mathbb{Z}/2\right)$.

Lemma 7.5: $L_0(\mathbb{K}[t, t^{-1}]) \;=\; L_0(\mathbb{K}) \oplus L_0(\mathbb{K})$ ($\bar{t} = t$). *Moreover* τ^λ *(thought of now as a Galois automorphism) acts on* $L_0(\mathbb{K}[t, t^{-1}])$ *in terms of this representation by the formula*

$$\tau^\lambda(a,\; b) \;=\; (\tau(a),\; \langle\lambda\rangle \cdot \tau(b)).$$

Proof: This is direct from Milnor's theorem (5.5) and the localization exact sequence. (See Ranicki [17,§5] for another approach). The generators are from $L_0(\mathbb{K}) \hookrightarrow L_0(\mathbb{K}[t, t^{-1}])$ induced by inclusion, for the first summand, and elements of the form $\langle\theta t\rangle$ for the second, where $\theta \in \mathbb{K}$. In particular $\langle\theta\rangle \mapsto \langle\theta t\rangle$ gives the inclusion for the second summand. Finally, the effect of τ^λ on $\langle\theta t\rangle$ is $\langle\tau(\theta)\lambda t^{-1}\rangle \sim \langle\tau(\theta)\lambda t\rangle$ and the lemma follows. □

Corollary 7.6: $L_0(\mathbb{K}[t, t^{-1}])^{\tau^\lambda} \;=\; L_0(\mathbb{K})^\tau \oplus L_0(\mathbb{K})^{\lambda\tau}$.

Lemma 7.7: *Assume* λ *is not a norm from* $\mathbb{K}$. *Let* $\theta \in L_0(\mathbb{K}[t, t^{-1}]/\mathcal{P})$ *be the image of* $\mu \in L_0(\mathbb{F}[x](y)/\mathcal{P})$, *then there is an element* $\alpha \in L_0(\mathbb{F}\langle\lambda, \mu\rangle)$ *with*

$$p(\alpha) \;=\; \mu + z \quad \textit{where } z \in\; \mathit{im}(L_0(\mathbb{K}[t, t^{-1}], \tau^\lambda)).$$

Proof: We have seen that $\mathcal{P} = (p)$ is principal, and suppose that $\theta = \langle f \rangle$ with $f \in \mathbb{F}[x](y)/(p)$ is a generator in this image. Then we can represent a lifting of $\langle f \rangle$ as $\langle q(x,y)p \rangle$ (at least up to lower degree) since $q(x,y)$ can always be choosen to represent f with its degree less than that of p. This reduces us to the consideration of the situation at primes of strictly smaller degree than p.

To complete the induction we need to consider also the case where $\mathcal{P}$ splits in $\mathbb{K}[t,t^{-1}]$. For such a prime let $\mathcal{Q}$, $\tau(\mathcal{Q})$ represent the primes over $\mathcal{P}$. Then

$$\mathbb{K}[t,t^{-1}]/\mathcal{Q} \;\cong\; \mathbb{K}[t,t^{-1}]/\tau(\mathcal{Q}) \;\cong\; \mathbb{F}[x](y)/\mathcal{P},$$

and

$$L_0(\mathbb{F}[x](y)/\mathcal{P}) \longrightarrow L_0(\mathbb{K}[t,t^{-1}]/\mathcal{Q}) \oplus L_0(\mathbb{K}[t,t^{-1}]/\tau(\mathcal{Q}))$$

is the diagonal map. But, if $\langle\theta\rangle \in L_0(\mathbb{K}[t,t^{-1}]/\mathcal{Q})$ is represented by $\sum_i \langle\bar\theta_i\rangle \in L_0(\mathbb{K}(t))$, then $\langle\theta\rangle \in L_0(\mathbb{K}[t,t^{-1}]/\tau(\mathcal{Q}))$ is represented by $\sum_i \langle\tau\bar\theta_i\rangle$. On the other hand, consider the term $\langle\bar\theta\rangle\perp\langle\tau\bar\theta\rangle \in L_0(\mathbb{K}(t))$. By an obvious change of variables we have

7.8 $$\langle\bar\theta\rangle\perp\langle\tau\bar\theta\rangle \;=\; \langle\bar\theta\rangle + \langle\tau\bar\theta\rangle\perp\langle\bar\theta\tau\bar\theta(\bar\theta+\tau\bar\theta)\rangle$$

and hence comes from $L_0(\mathbb{F}\langle\lambda,\mu\rangle)$. The lemma follows. □

Corollary 7.9: *Let $f \in \mathbb{F}$, then the class $\langle f\rangle\perp\langle\lambda f\rangle \in L_0(\mathbb{K})^{\lambda\tau}$ is in the image from $L_0(\mathbb{F}\langle\lambda,\mu\rangle)$.*

Proof: From 7.8, we have that

$$\langle ft\rangle\perp\langle\lambda ft\rangle \sim \langle fx\rangle\perp\langle\lambda fx\rangle$$

in $L_0(\mathbb{K}(t))$ and this is in the asserted group. □

More generally we may use the Scharlau transfer to obtain precise information about $L_0(\mathbb{F}[x](y))$. By Theorem 3.3 of Lam [10], pg. 201, there is an exact sequence

7.10 $$L_0(\mathbb{F}\langle\lambda,\mu\rangle) \xrightarrow{\iota_*} L_0(\mathbb{K}(t)) \xrightarrow{tr_*} L_0(\mathbb{F}\langle\lambda,\mu\rangle)$$

where tr_* is the Scharlau transfer associated with the $\mathbb{F}\langle\lambda,\mu\rangle$-linear homomorphism

$$\phi:\ \mathbb{K}(t)\longrightarrow\mathbb{F}\langle\lambda,\mu\rangle,\quad \phi(a+b\sqrt{\mu}) \;=\; b.$$

Note that ϕ restricts to $\phi| \;:\; \mathbb{K}\longrightarrow\mathbb{F}$, so we have

Lemma 7.11: *tr_* restricts to give a homomorphism $L_0(\mathbb{K}[t,t^{-1}]) \xrightarrow{tr_*} L_0(\mathbb{F}[x](y))$, the following diagram commutes,*

$$\begin{array}{ccc} L_0(\mathbb{K}) & \xrightarrow{tr_*} & L_0(\mathbb{F}) \\ \downarrow \iota & & \downarrow \iota \\ L_0(\mathbb{K}[t,t^{-1}]) & \xrightarrow{tr_*} & L_0(\mathbb{F}[x](y)) \end{array}$$

and $tr_*(\langle k\rangle) \; = \; tr_*(f+g\sqrt{\mu}) \; = \; \langle g\rangle(\langle 1\rangle\perp\langle -N(k)\rangle)$, *while*

$$tr_*(\langle kt\rangle) \; = \; \langle xg\sqrt{\mu}+yf\rangle(\langle 1\rangle\perp\langle -\lambda N(k)\rangle)$$

after including $L_0(\mathbb{F}[x](y))$ *in* $L_0(\mathbb{F}\langle\lambda,\mu\rangle)$.

Proof: For the form $\langle k\rangle$ we have

$$\phi(A+B\sqrt{\mu})(f+g\sqrt{\mu})(C+D\sqrt{\mu}) \; = \; (AD+BC)f+(AC+BD\mu)g$$

which associates to the matrix

$$\Delta_1 \; = \; \begin{pmatrix} g & f \\ f & \mu g \end{pmatrix}.$$

Since $Det\Delta_1 \; = \; -N(k)$, the first formula follows. To obtain the second formula, note from 6.4 that $t = \frac{1}{2}(x + \frac{\sqrt{\mu}}{\mu}y)$, so $tr_*(\langle kt\rangle)$ is given by the matrix of

$$\frac{1}{2}\left[(AC+\mu BD)(\frac{fy}{\mu}+xg)+(AD+BC)(xf+yg)\right]$$

which is

$$\Delta_2 \; = \; \frac{1}{2}\begin{pmatrix} \frac{fy}{\mu}+xg & xf+yg \\ xf+yg & fy+\mu xg \end{pmatrix}.$$

Also

$$Det\Delta_2 \; = \; \frac{1}{4}\left(\frac{f^2-g^2\mu}{\mu}\right)(y^2-\mu x^2) \; = \; -\lambda N(k)$$

is a unit in $\mathbb{F}[x](y)$, and 7.11 follows. □

Remark 7.12: The image of $L_0(\mathbb{F})$ in $L_0(\mathbb{F}[x](y))$ in 7.11 is exactly the quotient

$$L_0(\mathbb{F})/\left\{L_0(\mathbb{F})\cdot(\langle\mu\rangle\perp\langle 1\rangle)(\langle 1\rangle\perp-\langle\lambda\rangle)\right\}.$$

This is a direct consequence of the commutativity of the diagram obtained by amalgamating the exact sequence for a Laurent extension with the Jacobson sequence

7.13

$$\begin{array}{ccccc}
L_0(\mathbb{K},\tau) & \xrightarrow{\langle 1\rangle\perp\langle\mu\rangle} & L_0(\mathbb{F}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\
L_0(\mathbb{K},\tau) & \xrightarrow{\langle 1\rangle\perp\langle\mu\rangle} & L_0(\mathbb{F}) & \longrightarrow & L_0(\mathbb{K}) \\
\downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\
L_0(\mathbb{K}[t,t^{-1}],\tau^\lambda) & \longrightarrow & L_0(\mathbb{F}[x](y)) & \longrightarrow & L_0(\mathbb{K}[t,t^{-1}])
\end{array}$$

where the map γ is multiplication by $\langle 1\rangle\perp -\langle\lambda\rangle$.

Let $V\subset L_0(\mathbb{K}[t,t^{-1}])$ be the kernel of tr_*. Putting all this together we have

Theorem 7.14: *There is an exact sequence*

$$0{\longrightarrow}L_0(\mathbb{K}[t,t^{-1}],\ \tau^\lambda){\longrightarrow}L_0(\mathbb{F}[x](y)){\longrightarrow}V{\longrightarrow}L_3(\mathbb{K}[t,t^{-1}],\ \tau^\lambda){\longrightarrow}L_3(\mathbb{F}[x](y)){\longrightarrow}0.$$

□

(This is basically the snake lemma, which is applicable because of 7.7.)

7.14 provides, in general, an effective means of studying the Witt group of a genus 0 function field. The group $L_3(\mathbb{F}[x](y))$ plays the role of a reciprocity law, measuring the extent to which the "second boundary map" is surjective.

§8. The case $\mathbb{F}$ acted on trivially by τ

Consider the case of the ring $A = \mathbb{F}[t,t^{-1}]$ where $\mathbb{F}$ is a field of characteristic different from 2, with the involution $\tau : a{\longrightarrow}\bar{a}$ given by $\bar{a} = a$ for $a\in\mathbb{F}$, $\bar{t} = t^{-1}$. The L-groups of A are given by

$$L_*(A) = \begin{cases} L_0(\mathbb{F}) & *\equiv 0,1(4) \\ 0 & *\equiv 2,3(4)\end{cases}$$

by our main theorem (4.1) and $L_*(\mathbb{F})=0$ for $*\equiv 1,2,3(4)$. We wish to use this result to study the (Hermitian) Witt group of the function field $\mathbb{F}(t)$ under the associated involution $\tau:\mathbb{F}(t){\longrightarrow}\mathbb{F}(t)$.

Set $x=t+t^{-1}$ so $\mathbb{F}(t)^\tau=\mathbb{F}(x)$, and $(t-t^{-1})^2=x^2-4$. It follows that we can write

$$\mathbb{F}(t) = \mathbb{F}(x)(\sqrt{x^2-4}) = \mathbb{F}(x)(\sqrt{(x-2)(x+2)}).$$

$\mathbb{F}[t,t^{-1}]$ is Dedekind, so it is the integral closure of $\mathbb{F}[x]$ in $\mathbb{F}(t)$, and the extension ramifies at exactly the two primes $(x-1)$, $(x+1)$.

We apply the L-theory localization sequence of Carlsson and Milgram [3] and Ranicki [17,§3] to obtain the two exact sequences

8.1 $$0{\longrightarrow}L_0(\mathbb{F}){\longrightarrow}L_0(\mathbb{F}(t),\tau)\xrightarrow{\partial_0}\bigoplus_{\mathcal{P}\in\mathcal{I}}L_0^{tor}(\mathbb{F}[t,t^{-1}]/\mathcal{P},\tau){\longrightarrow}0,$$

and

8.2 $$0{\longrightarrow}L_2(\mathbb{F}(t),\tau)\xrightarrow{\partial_2}\bigoplus_{\mathcal{P}\in\mathcal{I}}L_2^{tor}(\mathbb{F}[t,t^{-1}]/\mathcal{P},\tau){\longrightarrow}L_0(\mathbb{F}){\longrightarrow}0.$$

Here $\mathcal{I}$ is the set of primes in $\mathbb{F}[x]$ which either ramify or remain prime in $\mathbb{F}[t,t^{-1}]$. Away from the ramified primes we have that $L_0^{tor}(\mathbb{F}[t,t^{-1}]/\mathcal{P},\tau) = L_2^{tor}(\mathbb{F}[t,t^{-1}]/\mathcal{P},\tau)$ is the ordinary L-theory of the quotient under the induced involution. Here, the isomorphism from L_0^{tor} to L_2^{tor} is given by $\langle\theta\rangle \mapsto \langle\{t-t^{-1}\}\theta\rangle$ on generators.

Likewise, for the Hermitian Witt groups of the function fields we have:

Lemma 8.3: *Let $\theta \in \mathbb{F}(x)$ then*

(1) The forms $\langle\theta\rangle$ generate $L_0(\mathbb{F}(t),\tau)$,

(2) $L_0(\mathbb{F}(t),\tau) \cong L_2(\mathbb{F}(t),\tau)$, the isomorphism being given explicitly by

$$\langle\theta\rangle \mapsto \langle(t-t^{-1})\theta\rangle,$$

(3) $L_*^{tor}(\mathbb{F}[t,t^{-1}]/(t-1),\tau) = \begin{cases} L_0(\mathbb{F}) & * \equiv 2(4) \\ 0 & * \equiv 0,1,3(4), \end{cases}$ *and similarly for*

$$L_*^{tor}(\mathbb{F}[t,t^{-1}]/(t+1),\tau),$$

(4) $\partial_2(\langle(t-t^{-1})\theta\rangle) \;=\; \langle\{t-t^{-1}\}\rangle\partial_0(\langle\theta\rangle) + \langle\theta(1)\rangle_{(t-1)} + \langle\theta(-1)\rangle_{(t+1)}.$

Proof: 8.3(1) and 8.3(2) are well known. 8.3(3) is true because, while $(t\pm1)$ are both prime, and both invariant – as ideals – under τ, they do not have invariant generators. In fact $\tau(t\pm1) = \mp t^{-1}(t\pm1)$. In tracing the definition of the torsion form, the coefficient $\mp t^{-1}$ evaluated in the quotient determines the type of the corresponding L-group. In these cases $\mp t^{-1} = -1$ at both primes, so parities reverse, and 8.3(3) follows.

To prove 8.3(4) it suffices to evaluate $\partial_2(\langle t-t^{-1}\rangle)$. At the prime $t-1$ we write this as $-(t+1)/(t-1)$, after dividing by $(t-1)(t^{-1}-1)$. Similarly, at $t+1$ it can be written $(t-1)/(t+1)$, and the result follows directly. □

Corollary 8.4: *The Hermitian Witt group*

$$W(\mathbb{F}(t),\tau) \cong \bigoplus_{\mathcal{P}\in\mathcal{I}} W(\mathbb{F}[t,t^{-1}]/\mathcal{P},\tau) \bigoplus W(F).$$

Here, $\mathcal{I}$ is the set of non-ramified, non-split primes in $\mathbb{F}[x]$. Moreover, the isomorphism above is natural.

(The point is that the map in 8.1 $L_0(\mathbb{F})\longrightarrow L_0(\mathbb{F}(t),\tau)$ is split by the composite

$$\partial_2 \cdot \langle t-t^{-1}\rangle : L_0(\mathbb{F}(t),\tau)\longrightarrow L_0(\mathbb{F})_{(t-1)} \oplus L_0(\mathbb{F})_{(t+1)}$$

using 8.3(4). The subscripts are labels, not localizations. Also, $W \;=\; L_0$.)

Example 8.5: Let $\mathbb{F} = \mathbb{R}$, the field of real numbers. The primes of $\mathbb{R}[x]$ are $(x - r)$ or $(x^2 - ax + b)$ with $a^2 - 4b < 0$. All primes of the second type have $\mathbb{R}[x]/\mathcal{P} = \mathbb{C}$, the complex numbers, so they split in $\mathbb{R}[t, t^{-1}]$. But primes of the first type split only if $r^2 < 4$. Hence the non-split primes are the $(x - r)$ with $|r| \leq 2$, where the two endpoints $(x \pm 2)$ are both ramified. It follows that using the reciprocity law above we have

Corollary 8.6: *The Hermitian Witt group of* $\mathbb{R}(t)$ *is naturally isomorphic to* $\bigoplus_{S^1} \mathbb{Z}$.

Proof: From 8.3, 8.4 we obtain $L_0(\mathbb{R}(t), \tau) = \bigoplus_{[-2,2]} \mathbb{Z}$, with an identification of the two $\mathbb{Z}$'s at the endpoints. But that is the result. □

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