Alexander Polynomials of Non-locally-flat Knots

GREG FRIEDMAN

ABSTRACT. We generalize the classical study of Alexander polynomials of smooth or PL locally-flat knots to PL knots that are not necessarily locally-flat. We introduce three families of generalized Alexander polynomials and study their properties. For knots with point singularities, we obtain a classification of these polynomials that is complete except for one special low-dimensional case. This classification extends existing classifications for PL locally-flat knots. For knots with higher-dimensional singularities, we further extend the necessary conditions on the invariants. We also construct several varieties of singular knots to demonstrate realizability of certain families of polynomials as generalized Alexander polynomials. These constructions, of independent interest, generalize known knot constructions such as frame spinning and twist spinning.

CONTENTS

1. INTRODUCTION

Background. One of the central motivations for studying knots and their invariants, including Alexander polynomials, is the central role that knots play in the understanding of the geometry of subvarieties of real codimension two. Thus, for a subpseudomanifold, X^{n-2}, of a manifold W^n, e.g. for a complex divisor of a complex manifold, the local geometry of X in W is classically described in terms of link pairs. Therefore, there is a large classical topological and algebraic geometric literature (e.g. [27], [22], [23], [24]) which studies the topology of the non-singular knots that arise as the link pairs of isolated singular points. However, in general, the singularities that arise naturally in (both high- and low-dimensional) topological and algebraic geometric situations (see, e.g., [5], [6], [12]) cannot be assumed to be isolated, and the corresponding link pairs of points of the singularities will consist of knotted sphere pairs which are themselves singular embeddings.

In the non-singular case, more specifically, the focus of knot theory historically has been the study of smooth or locally-flat codimension two knots, that is embeddings of S^{n-2} in S^n which are differentiable or piecewise-linear such that the neighborhood pair of any image point is PL-homeomorphic to an unknotted ball pair $D^{n-2} \subset D^n$. Furthermore, much effort has gone into the study of invariants of knots, algebraic objects which can be assigned to knots and which are identical for equivalent knots. Prominent among these invariants are the Alexander polynomials which are elements, up to similarity class, of the ring of integral Laurent polynomials

$$\Lambda := \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}].$$

They can be defined in may ways, one of which is as follows: By Alexander duality, the knot complement $C := S^n - S^{n-2}$ is a homology circle and hence possesses an infinite cyclic cover, $\tilde{C}$. The homology of $\tilde{C}$ with rational coefficients, $H_i(\tilde{C}; \mathbb{Q})$, has the structure of a module over $\Gamma := \mathbb{Q}[t, t^{-1}]$, where the action of t is given by the covering translation. These modules can be shown to be Γ-torsion modules which, since Γ is a principal ideal domain, possess square presentation matrices. The determinants of these matrices are elements of Γ which can be "normalized" by "clearing denominators" to elements of Λ whose coefficients are relatively prime, collectively. These normalized determinants are the Alexander polynomials. An equivalent approach would be to begin with the homology modules of C taken with a local coefficient system $\mathit{\Gamma}$, which is given by stalk Γ and action determined by factoring the fundamental group to the group of covering translations.

In [19], Levine completely characterized the Alexander polynomials of PL-locally-flat knots. (Some of these results were known somewhat earlier for low dimensions; see [7], [32], [17], [15].) If we represent the polynomial corresponding to the homology group in dimension i by p_i, he showed that the following conditions are necessary and sufficient for the collection $\{p_i\}$, $0 < i < n-1$, to be the Alexander polynomials of such a knot $S^{n-2} \subset S^n$ (in the other dimensions the polynomials are trivial):

1. $p_i(t) \sim p_{n-i-1}(t^{-1})$ ("$\sim$" denotes similarity in Λ),
2. $p_i(1) = \pm 1$,
3. if $n = 2q + 1$, q even, then $p(-1)$ is an odd square.

In this paper, we study the generalization of these invariants and their properties to various classes of knots which are not necessarily locally-flat, that is *non-locally-flat* (we also sometimes refer to knots which are definitely not locally-flat as *singular*). For knots with only point singularities, we establish necessary and sufficient conditions generalizing those of Levine and which form a complete characterization in all dimensions save $n = 5$ (and even for $n = 5$, we come close to a complete characterization; see below). For knots with more general singularities, we further generalize the necessary conditions, and we study several methods for realizing given sets of polynomials, including a construction of independent interest in the study of smooth knots which generalizes twist spinning [40], superspinning [4], and frame spinning [30].

We now outline our results section by section.

Section 2: Polynomial Algebra is a preliminary section in which we develop some fundamental results of what we call *polynomial algebra* by analogy with homological algebra. In particular, to each torsion Γ-module there is associated an element of Γ, up to similarity class. This is the determinant of the presentation matrix of the module or, equivalently, the product of its torsion coefficients (recall that $\Gamma = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ is a principal ideal domain). We develop some relationships among the polynomials associated to the modules of an exact sequence.

In *Section 3: Sphere Knots with Point Singularities and Locally-flat Disk Knots*, we first show that the study of the complement of a knot with a point singularity is homologically equivalent to the study of the complement of a locally-flat proper disk knot whose boundary sphere knot is the link pair of the singular point. In fact, by a technique of Milnor and Fox [26], the same is true of a sphere knot with any finite number of point singularities, and the boundary sphere knot will be the knot sum of the link pair knots of all the singular points. In this context, we define a family of three polynomials: λ_i, corresponding to the homology module of the cover of the disk knot complement, C; ν_i, corresponding to the boundary sphere knot complement, X; and μ_i, corresponding to the relative homology of the cover of the pair (C, X). Furthermore, there is a natural factorization of these polynomials: $\nu_i \sim a_i b_i$, $\lambda_i \sim b_i c_i$, and $\mu_i \sim c_i a_{i-1}$. With this notation we prove the following theorem:

Theorem 1.1 (Theorem 3.28)**.** *For $n \neq 5$ and $0 < i < n-1$, $0 < j < n-2$, the following conditions are necessary and sufficient for λ_i, μ_i, and ν_j to be the polynomials associated to the Γ-modules $H_i(\tilde{C}; \mathbb{Q})$, $H_i(\tilde{C}, \tilde{X}; \mathbb{Q})$, and $H_j(\tilde{X}; \mathbb{Q})$ of a locally-flat proper disk knot $D^{n-2} \subset D^n$: There exist polynomials $a_i(t)$, $b_i(t)$, and $c_i(t)$, primitive in Λ, such that*

1. (a) *$\nu_i \sim a_i b_i$.*

(b) $\lambda_i \sim b_i c_i$.
(c) $\mu_i \sim c_i a_{i-1}$.

2. (a) $c_i(t) \sim c_{n-i-1}(t^{-1})$.
(b) $a_i(t) \sim b_{n-i-2}(t^{-1})$.
3. $a_i(1) = \pm 1, b_i(1) = \pm 1, c_i(1) = \pm 1$, $a_0(t) = 1$.
4. *If $n = 2q + 1$ and q is even, then there exist an integer ρ and an integer $\omega \geq 0$ such that $((1-t)^\omega \rho)/\pm c_q(t)$ is the discriminant of a skew Hermitian form $A \times A \to Q(\Lambda)/\Lambda$ on a finitely-generated Λ-module, A, on which multiplication by $t-1$ is an isomorphism (or equivalently, $c_q(t) = \det[M(t)]$, where $M(t) = (-1)^{q+1}(R^{-1})'\tau R t - \tau'$ for integer matrices τ and R such that R has non-zero determinant and $(R^{-1})'\tau R$ is an integer matrix (here $'$ indicates transpose; see Section **3.6** for more details).*

For a locally-flat proper disk knot $D^3 \subset D^5$ (the case $n = 5$), these conditions are all necessary. Furthermore, we can construct knots which satisfies both these conditions and the added, perhaps unnecessary, condition that $|c_2(-1)|$ be an odd square.

In Section ***3.3***, we prove the necessity of the duality and normalization conditions (2) and (3) by a generalization of Levine's technique in [19] by

(i) constructing an appropriately generalized Seifert surface,
(ii) using the surface to construct the cover by a cut-and-paste procedure,
(iii) deducing from the homology modules of the pieces of the construction the form of the presentation matrices of the desired modules, and
(iv) exploiting the properties of an integer linking pairing between the homology of the Seifert surface and that of its complement to show that these matrices have the requisite properties to induce those claimed for the polynomials.

In Section ***3.5***, we prove the sufficiency of these conditions, modulo condition (4), by employing various explicit constructions using surgery and relative surgery. In particular, we show complete sufficiency under the added (unnecessary) condition that if $n = 2q + 1$ and q is even, then $|c_q(-1)|$ is an odd square.

Section ***3.6*** contains a study of the additional issues which are involved in characterizing the "middle dimension polynomial", $c_q(t)$, for $n = 2q + 1$ and q even. The necessity of condition (4) is a consequence of the existence of a skew Hermitian form on the module $\ker(\partial_* : H_q(\tilde{C}, \tilde{X}; \mathbb{Q}) \to H_{q-1}(\tilde{X}; \mathbb{Q}))$ which we deduce from the Blanchfield pairing [1]. The realization of a given polynomial $c_q(t)$, for $n \neq 5$, is deduced as a consequence of the following more general theorem:

Proposition 1.2 (Proposition 3.21)**.** *Let A be a finitely generated $\mathbb{Z}$-torsion free Λ-module on which multiplication by $t-1$ is an automorphism and on which there is a non-degenerate conjugate linear $(-1)^{q+1}$-Hermitian pairing $\langle\, , \rangle : A \times A \to Q(\Lambda)/\Lambda$. Then there exists a disk knot $D^{n-2} \subset D^n$, $n = 2q + 1$, $q > 2$, such that:*

1. $H_q(\tilde{C}) \cong A$,
2. $H_i(\tilde{C}) = 0$, $0 < i < n-1$, $i \neq q$,
3. $H_i(\tilde{X}) = 0$, $0 < i < n-2$, $i \neq q-1$,

4. *$H_{q-1}(\tilde{X}) = 0$ is a $\mathbb{Z}$-torsion module,*
5. *$H_i(\tilde{C}, \tilde{X}) = 0$, $0 < i < n-1$, $i \neq q$,*
6. *the form on $H_q(\tilde{C})$ is given by $\langle\, ,\rangle$. (Note that $H_q(\tilde{X}) = 0$ implies that $H_q(\tilde{C}) \cong A \cong \ker(\partial_*)$ in the long exact sequence).*

The impediment to a complete characterization in dimension $n = 5$ is a consequence of special difficulties associated with low-dimensional surgery and is related to an open problem of Levine's in the study of pairings on low-dimensional locally-flat sphere knots [21].

In Section 4: Knots with More General Singularities, we consider the case of a sphere knot $K = S^{n-2} \subset S^n$ with singular set Σ, which need no longer consist solely of isolated points. It remains useful to study not the actual knot complement, $S^n - K$, but the homotopy equivalent complement of the locally-flat restriction of the knot to $S^n - N(\Sigma)$, where $N(\Sigma)$ is an open regular neighborhood of Σ. Then, we again obtain a boundary "knot" which is the complement in $\partial\bar{N}(\Sigma)$ of its intersection with the knot K. Accordingly, we can again define three sets of polynomials (corresponding to the boundary, absolute, and relative homology modules of the covers) which again have natural factorizations $\nu_i \sim a_i b_i$, $\lambda_i \sim b_i c_i$, $\mu_i \sim c_i a_{i-1}$. In this setting, by further generalizing the above techniques and by employing a number of homological algebra computations, we show in Section ***4.2*** that the necessary conditions of Theorem 1.1 generalize as follows:

Theorem 1.3 (Theorem 4.16)**.** *Let $\nu_j(t)$, $\lambda_i(t)$, and $\mu_i(t)$, $0 < j < n-2$ and $0 < i < n-1$, denote the Alexander polynomials corresponding to $H_j(\tilde{X})$, $H_i(\tilde{C})$, and $H_i(\tilde{C}, \tilde{X})$, respectively, of a knotted $S^{n-2} \subset S^n$. We can assume these polynomials to be primitive in Λ. Then, there exist polynomials $a_i(t)$, $b_i(t)$, and $c_i(t)$, primitive in Λ, such that*

1. *$\nu_j(t) \sim a_j(t) b_j(t)$,*
2. *$\lambda_i(t) \sim b_i(t) c_i(t)$,*
3. *$\mu_i(t) \sim c_i(t) a_{i-1}(t)$,*
4. *$a_i(t) \sim b_{n-2-i}(t^{-1})(t-1)^{\tilde{\beta}_i}$,*
5. *$c_i(t) \sim c_{n-1-i}(t^{-1})$,*
6. *$b_i(1) = \pm 1$,*
7. *$c_i(1) = \pm 1$,*
8. *if $n = 2q+1$, then $c_q(t)$ is the determinant of a matrix of the form $(R^{-1})'\tau R t - (-1)^{q+1}\tau'$ where τ and R are matrices such that R has non-zero determinant.*

Furthermore, if $\mathfrak{H}_q = H_q(\tilde{C}; \mathbb{Q}) / \ker(H_q(\tilde{C}; \mathbb{Q}) \to H_q(\tilde{C}, \tilde{X}; \mathbb{Q}))$ and $n = 2q+1$, there is a $(-1)^{q+1}$-Hermitian pairing $\langle\, ,\rangle : \mathfrak{H}_q \times \mathfrak{H}_q \to Q(\Gamma)/\Gamma$ which has a matrix representative

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' t R^{-1}}$$

with respect to an appropriate basis.

In this setting of general singularities, realization of polynomials is more difficult because the allowable set of polynomials will depend subtly on the properties of the singular set, its link pairs, and its embedding. However, in Section ***4.3***, we employ several constructions available for creating locally-flat knots including the *frame spinning* of Roseman [30] and our own generalization to *frame twist-spinning*. Together, these include as special cases the superspinning of Cappell [4] and the twist spinning of Zeeman [40]. By adapting these techniques and generalizing them to knots with singularities, it is possible to construct singular knots and to obtain some realization results here as well. In particular, for any manifold M which can be embedded with framing in S^{n-2}, we construct classes knots $S^{n-2} \subset S^n$ whose singular sets are M.

Furthermore, we calculate the polynomials of the knots so constructed based upon the polynomials of the knots being spun and the homology properties of the manifolds, M, they are being spun about. Let λ_i^σ, μ_i^σ, and ν_i^σ denote the polynomials of a frame spun knot; λ_i^τ, μ_i^τ, and ν_i^τ the polynomials of a frame twist-spun knot; and λ_i, μ_i, and ν_i the polynomials of the knot K being spun. Let Σ be the singular set of the knot K. Denote the Betti numbers of Σ by $\mathfrak{b}_i$, let B_i be the i^{th} Betti number of M^k, and let $\tilde{\beta}_i$ be the reduced Betti number of $M \times \Sigma$. Suppose that

$$H_j(M^k; \Gamma|_{M^k}) \cong \Gamma^{\mathfrak{B}_j} \oplus \bigoplus_\ell \frac{\Gamma}{(\zeta_{j\ell})}$$

(see Section ***4.3.2*** for the definition of this local coefficient system on M) and that the torsion coefficients of the boundary knot of K which are relatively prime to $t-1$ are denoted by $\nu_{i\ell}$, so that $\nu_i = (t-1)^{\mathfrak{b}_i} \prod_\ell \nu_{i\ell}$. Similarly, let

$$\lambda_i = \prod_\ell \lambda_{i\ell} \quad \text{and} \quad \mu_i = (t-1)^{\tilde{\mathfrak{b}}_i} \prod_\ell \mu_{i\ell}\,.$$

Then, we show in Sections ***4.3.1*** and ***4.3.2*** that:

$$\lambda_i^\sigma(t) = \prod_{\ell=1}^{m-2} [\lambda_\ell(t)]^{B_{i-\ell}}$$

$$\mu_i^\sigma(t) = (t-1)^{\tilde{B}_{i-1}} \prod_{\ell=0}^{m-2} [\mu_\ell(t)]^{B_{i-\ell}}$$

$$\nu_i^\sigma(t) = \prod_{\ell=0}^{m-3} [\nu_\ell(t)]^{B_{i-\ell}}$$

$$\lambda_j^\tau(t) = \prod_{\substack{r+s=j\\ s>0}} \left((\prod_\ell \lambda_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri}, \lambda_{sl}) \right) \cdot \prod_{\substack{r+s=j-1\\ s>0}} \left(\prod_{i,\ell} d(\zeta_{ri}, \lambda_{sl}) \right)$$

$$\mu_j^\tau(t) = (t-1)^{\bar\beta_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} \left(\left(\prod_\ell \mu_{m-s-1,\ell}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\bar\zeta_{ri}, \mu_{m-s-1,\ell}\right)\right)$$
$$\cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \left(\prod_{i,\ell} d(\bar\zeta_{ri}, \mu_{m-s-1,\ell})\right)$$
$$\nu_j^\tau(t) = (t-1)^{\beta_j} \prod_{r+s=j} \left(\prod_\ell \nu_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri}, \nu_{sl})\right) \cdot \prod_{r+s=j-1} \left(\prod_{i,\ell} d(\zeta_{ri}, \nu_{sl})\right).$$

From these formulas, we then deduce the following realization theorems:

Proposition 1.4 (Proposition 4.23)**.** *Let M^k be a manifold which embeds in S^{n-2} with trivial normal bundle with framing ϕ and such that $n-k>3$. Let Σ be a single point. Let B_i denote the ith Betti number of M, and let $\mathfrak{b}_i$ and $\bar\beta_i$ denote the i^{th} reduced Betti numbers of Σ and $M\times\Sigma$, respectively. Suppose that we are given any set of polynomials, $a_i(t)$, $b_i(t)$, $c_j(t)$ and $c'_\ell(t)$, $0<i<n-k-2$, $0<j<n-k-1$, and $0<\ell<n-1$, which satisfy:*

1. $a_i(t) \sim b_{n-k-2-i}(t^{-1})$,
2. $c_i(t) \sim c_{n-k-1-i}(t^{-1})$,
3. $c'_i(t) \sim c'_{n-1-i}(t^{-1})$,
4. $b_i(1) = \pm 1$,
5. $c_i(1) = \pm 1$,
6. $c'_i(1) = \pm 1$,
7. *if $n-k=2p+1$, p even, $p\neq 2$, then $c_p(t)$ is the determinant of a matrix of the form $(R^{-1})'\tau R t - (-1)^{q+1}\tau'$ where τ and R are integer matrices such that R has non-zero determinant and $(R^{-1})'\tau R$ is an integer matrix; if $n-k=2p+1$, p even, $p=2$, then $|c_p(-1)|$ is an odd square,*
8. *if $n=2q+1$, q even, then $|c'_q(-1)|$ is an odd square.*

Then there exists a knotted $S^{n-2}\subset S^n$ with singular set M and Alexander subpolynomials $a_i^\sigma(t)$, $b_i^\sigma(t)$, and $c_i^\sigma(t)$ satisfying

$$a_i^\sigma(t) \sim (t-1)^{\bar\beta_i} \prod_{\ell=1}^{m-2} [a_\ell(t)]^{B_{i-\ell}},$$
$$b_i^\sigma(t) \sim \prod_{\ell=1}^{m-2} [b_\ell(t)]^{B_{i-\ell}},$$
$$c_i^\sigma(t) \sim c'_i(t) \prod_{\ell=1}^{m-2} [c_\ell(t)]^{B_{i-\ell}}.$$

Theorem 1.5 (Theorem 4.27)**.** *Let M^k, $n-k>3$, be a manifold which embeds in S^{n-2} with trivial normal bundle with framing ϕ. Given a map*

$\tau : M \to S^1$, let $\mathfrak{B}_i$ be the rank of the free part and $\zeta_{i\ell}$ be the torsion invariants of the Γ-modules $H_i(M;\Gamma|_M)$. (These modules are independent of the knot being spun in the construction.) If $\gamma \in \Gamma$, then let $\bar{\gamma} \in \Gamma$ be such that $\bar{\gamma}(t) = \gamma(t^{-1})$. If K is a knot $S^{m-2} \subset S^m$ with Alexander invariants $\lambda_{i\ell}$, $\mu_{i\ell}$, and $\nu_{i\ell}$ and with singular set Σ with reduced Betti numbers $\mathfrak{b}_i$, then there exists a frame twist-spun knot $\sigma_M^{\phi,\tau}(K)$ with singular set $M \times \Sigma$ (whose reduced Betti numbers we denote $\tilde{\beta}_i$) and with Alexander polynomials given for $j > 0$ by:

$$\lambda_j^\tau(K) \sim \prod_{\substack{r+s=j\\ s>0}} \left((\prod_{\ell} \lambda_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri}, \lambda_{sl}) \right) \cdot \prod_{\substack{r+s=j-1\\ s>0}} \left(\prod_{i,\ell} d(\zeta_{ri}, \lambda_{sl}) \right),$$

$$\mu_j^\tau(K) \sim (t-1)^{\tilde{\beta}_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} \left((\prod_{\ell} \mu_{m-s-1,\ell}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\bar{\zeta}_{ri}, \mu_{m-s-1,\ell}) \right)$$

$$\cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \left(\prod_{i,\ell} d(\bar{\zeta}_{ri}, \mu_{m-s-1,\ell}) \right),$$

$$\nu_j^\tau(t) \sim (t-1)^{\tilde{\beta}_j} \prod_{r+s=j} \left(\prod_{\ell} \nu_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri}, \nu_{sl}) \right) \cdot \prod_{r+s=j-1} \left(\prod_{i,\ell} d(\zeta_{ri}, \nu_{sl}) \right).$$

In particular, by frame twist-spinning knots with a single point as their singular set, we obtain knots with M as their singular sets.

Remark 1.6. In fact, we can create a knot with a single point as its singular set and with (nearly) any given set of allowable invariants by the results of Section 3. Putting this together with the above theorem, we know exactly what kinds of polynomials can be realized as those of frame twist-spun knots with singular set M, modulo our ability to compute the homology $H_j(M;\Gamma|_M)$ and our difficulty with the polynomial $c_2(t)$ of a disk knot $D^3 \subset D^5$.

Finally, we form singular knots by the suspension of locally-flat or singular knots and compute their polynomials (λ_i^Σ, μ_i^Σ, and ν_i^Σ) from those of the original the knots (λ_i, μ_i, and ν_i). This is done in Section ***4.3.3***, where we obtain the following result:

Proposition 1.7 (Proposition 4.30)**.** *With the notation as above,*

1. $\lambda_i^\Sigma \sim \lambda_i \sim b_i c_i$
2. $\mu_i^\Sigma \sim \mu_{i-1} \sim c_{i-1} a_{i-2}$
3. $\nu_i^\Sigma \sim \lambda_i \mu_i \sim a_{i-1} b_i c_i^2$.

This work originally appeared as part of the author's dissertation [9]. In further papers, we study the intersection homology analogues of Alexander polynomials for non-locally-flat knots (see [9], [8]). I thank my advisor, Sylvain Cappell, for all of his generous and invaluable guidance.

2. Polynomial Algebra

Let $\Gamma = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ be the ring of Laurent polynomials with rational coefficients. In other words, the elements of Γ are polynomials $\sum_{i\in\mathbb{Z}} a_i t^i$, such that each $a_i \in \mathbb{Q}$ and $a_i = 0$ for all but a finite number of i. Γ is a principal ideal domain [19, §1.6]. Unless otherwise specified, we will generally not distinguish between elements of Γ and their similarity classes up to unit. In this introductory section, we study some basic facts, which will be used often, concerning torsion Γ-modules and their associated polynomials (the determinants of their square presentation matrices). In analogy with homological algebra for modules, we refer to this theory of the behavior of the associated polynomials as *polynomial algebra*.

Let

$$\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}],$$

the ring of Laurent polynomials with integer coefficients. Then $\Gamma = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We call a polynomial in Λ *primitive* if its set of non-zero coefficients have no common divisor except for ± 1. Any element of Γ has an associate in Γ which is a primitive polynomial in Λ: Any element $at^i \in \Gamma$ is a unit and, in particular then, any $a \in \mathbb{Q}$. So given an element of Γ, we can first clear denominators and then divide out any common divisors without affecting similarity (associate) class in Γ. We will often choose to represent an element of Γ (technically, its associate class) by such a primitive element of Λ.

Proposition 2.1. *Suppose we have an exact sequence of finitely generated torsion Γ-modules*

$$0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} M_n \xrightarrow{d_n} 0, \tag{2.1}$$

and suppose that Δ_i is the determinant of a square presentation matrix of M_i (which we will refer to as the polynomial associated to the module). Then, taking $\Delta_{n+1} = 1$ if n is odd, the alternating product

$$\prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}}{\Delta_{2i}} \in \mathbb{Q}(t)$$

is equal to a unit of Γ, and, in particular, with a consistent choice of normalization within associate classes for the elementary divisors of the M_i (in the language of [13]), this product is equal to 1.

Proof. It is well known (see, for example, [13, p. 225]) that a finitely generate torsion module over a principal ideal domain can be decomposed as the direct sum of cyclic torsion summands of orders $p_j^{k_j}$, the p_j not necessarily distinct primes in the ground ring and the k_j positive integers, also not necessarily distinct. Furthermore, we know that this decomposition is unique in the sense that the p_j are determined up to associate class, but the cyclic summands $\Gamma/(p_j^{k_j})$, being independent of the choice of p_j within the associate class, are uniquely determined. Hence, in particular, each M_i has a square presentation matrix of the form

$$\begin{pmatrix} p_{i_1}^{k_{i_1}} & 0 & \cdots & 0 \\ 0 & p_{i_2}^{k_{i_2}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & p_{i_{m_i}}^{k_{i_{m_i}}} \end{pmatrix},$$

and $\Delta_i = \prod_{j=1}^{m_i} p_{i_j}^{k_{i_j}}$. Since we have a finite number of modules, each finitely generated, we have a finite number of primes of Γ occurring in the elementary divisors and in these matrices. We are free to choose these primes so that if two are in the same associate class, then they are, in fact, the same element of Γ, and we assign an order so that we may speak of the collection of distinct primes $\{p_j\}_{j=1}^m$ which occur. Let $M_i(p_j)$ be the summand of M_i which is the direct sum of cyclic modules of order a power of p_j. This may be a trivial summand. Then each M_i decomposes as

$$M_i \cong M_i(p_1) \oplus M_i(p_2) \oplus \cdots \oplus M_i(p_m),$$

and, if we set $\Delta_i(p_j)$ to be the determinant of the presentation matrix of $M_i(p_j)$, then $\Delta_i = \prod_j \Delta_i(p_j)$ and $\Delta_i(p_j) = p_j^k$ where k is the sum of the powers of p_j which occur in the elementary divisors of M_i.

Lemma 2.2. *Let r and s be powers of distinct (non-associate) prime elements of Γ. Then the only Γ-module morphism $f : \Gamma/(r) \to \Gamma/(s)$ is the 0 map.*

Proof. Suppose f is such a map. Then, letting elements of Γ stand for their classes in $\Gamma/(s)$ and $\Gamma/(r)$ where appropriate, we have

$$0 = f(0) = f(r1) = rf(1) = ra$$

for some $a \in \Gamma/(s)$. But $ra = 0$ implies that $ra = sb$ in Γ for some $b \in \Gamma$. Since $s|sb$ but no prime divisor of s divides r, we must have $s|a$ so that $a = sc$ for some $c \in \Gamma$. But then $f(1) = a = sc = 0$, for some c. This implies that f is the 0 map because f is completely determined by the image of the generator. ∎

Corollary 2.3. *With the notation above, the only Γ-module morphisms $f : M_i(p_k) \to M_j(p_\ell)$, for $k \neq \ell$, are the 0 maps.*

Proof. This follows immediately from the lemma since the map on each summand must be 0. ∎

Corollary 2.4. *For any p_j, the sequence*

$$0 \xrightarrow{e_0} M_1(p_j) \xrightarrow{e_1} M_2(p_j) \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} M_n(p_j) \xrightarrow{e_n} 0 \tag{2.2}$$

is exact, where the maps e_i are the restrictions of the the maps d_i to the direct summands $M_i(p_j)$.

Proof. First, we note that these maps are well-defined: Any element $a \in M_i$ can be represented as $\sum_\ell a_\ell$ where $a_\ell \in M_i(p_\ell)$, and so if $a_j \in M_i(p_j)$, we can identify it via the inclusion of the summand with $0+\cdots+0+a_j+0+\cdots+0 \in M_i$, which we will also call a_j. Then $d_i(a_j)$ is represented by a sum $\sum_\ell b_\ell$, $b_\ell \in M_{i+1}(p_\ell)$. If r_ℓ is the projection of M_i to $M_i(p_\ell)$, then $r_\ell d_i(a_j) = b_\ell$. But this gives a Γ-module morphism $M_i(p_j) \to M_{i+1}(p_\ell)$ and hence $b_\ell = 0$ if $\ell \neq j$ by the preceding corollary. In other words, the image of the summand $M_i(p_j)$ under d_i lies in $M_{i+1}(p_j)$. Thus the maps of this sequence are well-defined.

Next since $d_{i+1}d_i = 0$, we also have $e_{i+1}e_i = 0$.

It remains to show that $\mathrm{Ker}(e_i) \subset \mathrm{Im}(e_{i-1})$. Suppose that $e_i(a_j) = 0$, $a_j \in M_i(p_j)$. Then we have also $d_i(a_j) = 0$, since we have already observed that d_i takes a_j to 0 in all of the other summands $M_{i-1}(p_\ell)$, $\ell \neq j$. But since (2.1) is exact, there is an element c in M_{i-1} such that $d_{i-1}(c) = a_j$, and we have $c = \sum_\ell c_\ell$, $c_\ell \in M_{i-1}(p_\ell)$ and $d_{i-1}(c) = \sum_\ell d_{i-1}(c_\ell) = a_j$. Since we know $d_{i-1}(c_\ell) \in M_i(p_\ell)$, we must have $d_{i-1}(c_j) = a_j$ and $d_{i-1}(c_\ell) = 0$, $\ell \neq j$. But then $a_j = e_{i-1}(c_j) \in \mathrm{Im}(e_{i-1})$. ◻

Note that this lemma together with its corollary allows us to write the exact sequence (2.1) as the direct sum of exact sequences of the form (2.2).

We will prove Proposition 2.1 in the special case that the exact sequence in its statement has the form of that in equation (2.2). The proposition will then follow for the general case by the formula

$$\prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}}{\Delta_{2i}} = \prod_{i=1}^{\lceil n/2 \rceil} \frac{\prod_j \Delta_{2i-1}(p_j)}{\prod_j \Delta_{2i}(p_j)} = \prod_j \prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}(p_j)}{\Delta_{2i}(p_j)}.$$

So it remains to prove that the exact sequence (2.2) implies that

$$\prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}(p_j)}{\Delta_{2i}(p_j)} \quad \text{is a unit of } \Gamma.$$

In particular, with our choice of consistent p_j's within the associated classes, this product will be 1. For this, recall that we have already observed that $\Delta_i(p_j) =$

$p_j^{k_i(p_j)}$, where $k_i(p_j)$ is the sum of the powers of p_j which occur in the elementary divisors of M_i. Therefore

$$\prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}(p_j)}{\Delta_{2i}(p_j)} = p_j^{k(p_j)},$$

where $k(p_j) = \sum_{i=1}^{\lceil n/2 \rceil} (-1)^{i+1} k_i(p_j)$. We claim that $k(p_j) = 0$, which will complete the proof.

Of course, each Γ-module $M_i(p_j)$ has the underlying structure of a rational vector space if we forget about the t action, and similarly the exact sequence (2.2) can be regarded as an exact sequence of vector spaces over $\mathbb{Q}$. Suppose $p_j = \sum_{\ell=a}^{b} c_\ell t^\ell$, where a and b are finite integers, $c_a \neq 0$ and $c_b \neq 0$ (we can always find such a representation of an element of Γ). Define $\|p_j\| = b - a$. Then the dimension of $\Gamma/(p_j)$ as a rational vector space is $\|p_j\|$, and, more generally, the dimension of $\Gamma/(p_j^k)$ as a vector space is $\|p_j^k\| = k\|p_j\|$, for any non-negative integer k. Therefore, the dimension of $M_i(p_j)$ as a rational vector space must be $k_i(p_j)\|p_j\|$. But since (2.2) is an exact sequence of vector spaces,

$$\begin{aligned} 0 &= \sum_i (-1)^{i+1} \dim(M_i(p_j)) = \sum_i (-1)^{i+1} k_i(p_j)\|p_j\| \\ &= \|p_j\| \sum_i (-1)^{i+1} k_i(p_j) = \|p_j\| k(p_j). \end{aligned}$$

Since $\|p_j\| \neq 0$ (else p_j would be a unit of Γ and $\Gamma/(p_j^k)$ trivial), we must have $k(p_j) = 0$ as claimed. This completes the proof.

Note that had we not fixed the p_j within their associate classes, the product

$$\prod_{i=1}^{\lceil n/2 \rceil} \frac{\Delta_{2i-1}}{\Delta_{2i}}$$

would not necessarily be 1, but it would still follow from minor adjustments to the arguments above that it would be a unit of Γ. □

Corollary 2.5. *With the notation and assumptions as above, each $\Delta_i = \delta_i \delta_{i+1}$ where $\delta_{i+1} | \Delta_{i+1}$ and $\delta_i | \Delta_{i-1}$. Furthermore, if we represent the Δ_i by the elements in their similarity classes in Γ which are primitive in Λ, the δ_i will also be primitive in Λ.*

Proof. Let $Z_i \subset M_i$ denote the kernel of d_i. Then we have the short exact sequences

$$0 \longrightarrow Z_i \longrightarrow M_i \longrightarrow Z_{i+1} \longrightarrow 0$$

Let δ_i be the determinant of a square presentation matrix of Z_i. Then, applying the above proposition for various choices of i, we obtain $\Delta_i = \delta_i \delta_{i+1}$ up to associate classes, as well as $\Delta_{i-1} = \delta_{i-1}\delta_i$ and $\Delta_{i+1} = \delta_{i+1}\delta_{i+2}$. This proves the first part of the corollary. For the second, recall that we can always find an element in the associate class of δ_i in Γ which is primitive in Λ, and this choice will be unique up to associate class in Λ. Similarly for δ_{i+1}. But the product of two primitive elements of Λ is again primitive in Λ (the argument of [11, §3.10] for $\mathbb{Z}[t]$ extends easily), so that, with this choice, $\delta_i\delta_{i+1}$ is a primitive element of Λ which is equal to Δ_i up to associativity in Λ. $\square$

This corollary will be used often in what follows.

For convenience, we introduce the following notation. Suppose $\Delta_i \in \Gamma$. We will refer to an *exact sequence of polynomials*, denoted by

$$\longrightarrow \Delta_{i-1} \longrightarrow \Delta_i \longrightarrow \Delta_{i+1} \longrightarrow ,$$

to mean a sequence of polynomials such that each $\Delta_i \sim \delta_i\delta_{i+1}$, $\delta_i \in \Gamma$. As we have seen, such a sequence arises in the case of an exact sequence of torsion Γ-modules, M_i, and, in that case, the factorization of the polynomials is determined by the maps of the modules as in Corollary 2.5. In particluar, each δ_i is the polynomial of the module $\ker(M_i \to M_{i+1})$.

Observe that knowledge of two thirds of the terms of an exact sequence of polynomials (for example, all Δ_{3i} and Δ_{3i+1}, $i \in \mathbb{Z}$) and the common factors of those terms (the δ_{3i+1}), allows us to deduce the missing third of the sequence ($\Delta_{3i+2} = \delta_{3i+2}\delta_{3i+3} = (\Delta_{3i+1}/\delta_{3i+1}) \cdot (\Delta_{3i+3}/\delta_{3i+4})$).

Note also that for any bounded exact sequence of polynomials (or even a half-bounded sequence), the collections $\{\Delta_i\}$ and $\{\delta_i\}$ carry the same information. That is, suppose that one (or both) end(s) of the polynomial sequence is an infinite number of 1's (by analogy to extending any bounded or half-bounded exact module sequence to an infinite number of 0 modules). Clearly, the Δ_i can be reconstructed from the δ_i by $\Delta_i \sim \delta_i\delta_{i+1}$. On the other hand, if Δ_0 is the first nontrivial term in the polynomial sequence, then $\delta_0 \sim 1$, $\delta_1 \sim \Delta_0$, and $\delta_i \sim \Delta_{i-1}/\delta_{i-1}$ for all $i > 1$. Similar considerations hold for a sequence which is bounded on the other end. Therefore, we will often study properties of the polynomials Δ_i in an exact sequence by studying the δ_i instead. We will refer to the δ_i as the *subpolynomials* of the sequence and to the process of determining the subpolynomials from the polynomials as "dividing in from the outside of the sequence".

3. Sphere Knots with Point Singularities and Locally-flat Disk Knots

3.1. Introduction. Our goal in this section is to study the Alexander polynomials of a knot with isolated singularities. More specifically, let $\alpha : S^{n-2} \hookrightarrow S^n$, $n > 3$ be a PL-embedding such that for $x \in \alpha(S^{n-2})$, the link pair of x in $(S^n, \alpha(S^{n-2}))$ is PL-homeomorphic to the standard unknotted sphere pair except at finitely many x, where it may be a knotted sphere pair. Henceforth, we will dispense with α unless necessary and refer simply to the knot pair $(S^n, K = \alpha(S^{n-2}))$ or the n-knot K. Just as in the classical locally flat case, Alexander duality tells us that the homology of the knot complement $S^n - K$ is that of a circle, and this allows us to study the infinite cyclic cover of the knot complement and its homology regarded as a module over $\Gamma = \mathbb{Q}[H_1(S^n - K)] = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$. We can then study the Alexander invariants of these modules.

We begin by seeing that the study of the homological properties of the complements of sphere knots with isolated singularities reduces to the study of the complements of locally-flat disk knots. This study of disk knots starts by emulating J. Levine's study of Alexander invariants for the locally flat sphere knots [19]. In Section ***3.3***, we introduce two sets of polynomial invariants, λ_q^i and μ_q^i, corresponding to certain absolute and relative homology modules and show that they satisfy certain duality and normalization conditions. From these, we arrive at the corresponding definitions and properties for the Alexander polynomials λ_i and μ_i (see Section ***3.4***).

In Section ***3.5***, we turn to the realization of locally-flat disk knots with given polynomial invariants which satisfy the properties obtained in Section ***3.3***. We show that any allowable set of λ_i can be realized, first for a knotted D^2 in D^4 (Section ***3.5.1***) and then for arbitrary $D^{n-2} \subset D^n$, $n > 4$ (Section ***3.5.2***). In Section ***3.5.3***, we show that we can nearly completely characterize all three sets of Alexander polynomials which can occur for a locally flat disk knot (the polynomials λ_i and μ_i, which we have already mentioned, plus the Alexander polynomials of the boundary locally-flat sphere knot). The barrier to a complete classification, at that point, is a certain polynomial factor shared by λ_q and μ_q for knotted $D^{2q-1} \subset D^{2q+1}$, q an even integer.

In Section ***3.6***, we take up the study of this middle-dimensional polynomial factor. We show that it is related to a certain Hermitian self-pairing induced by the Blanchfield pairing on the middle-dimension homology modules. We establish the realizability of such pairings in disk knots and then study the relationship between the Alexander polynomial factors and the presentation matrices of the modules and their pairings. This allows us to state necessary and sufficient conditions for this polynomial factor for $n \neq 5$.

Finally, in Section ***3.7***, we gather together the results of Section 3. Theorem 3.28 states a complete set of necessary and sufficient conditions for Alexander polynomials for locally-flat disk knots $D^{n-2} \subset D^n$, $n \neq 5$. For $n = 5$, the classification is nearly complete, but we obtain only a partial characterization of the middle dimensional polynomial factor.

3.2. The knot complement. For technical simplicity, we will often study not the knot complement but rather a version of the the homotopy equivalent "knot exterior". For locally flat knots this is the exterior of an open tubular (PL-regular) neighborhood of the knot. Similarly, we can consider the exterior of a regular neighborhood of our singular knot.

First, assume that the knot, K, has only one singular point, x. Then the neighborhood Star(x) of x in S^n is a knotted ball pair (D^n, D^{n-2}) which is (PL-homeomorphic to) the cone on Link(x), which is a knotted sphere pair $\partial(D^n, D^{n-2}) = (S^{n-1}, S^{n-3}) = (S^{n-1}, k)$, where we let k denote the locally flat (n-1)-knot of the sphere pair. Since the cone point no longer remains when we consider only the knot complement, we can retract what remains of the complement in Star(x) out to the boundary and see that our knot complement is homotopy equivalent to the complement of a locally-flat knotted disk pair (D^n, D^{n-2}) where this D^n is the complement of the open disk neighborhood of x in S^n. This knotted disk pair in fact provides a null-knot cobordism of the slice knot k, and the study of the knot complement reduces (up to homotopy equivalence) to the study of the cobordism complement $D^n - D^{n-2}$, which we shall denote by C. If desired, we can also retract this complement to the complement of an open tubular (regular) neighborhood of the locally-flatly embedded knotted disk in analogy with the usual notion of knot exteriors. See Figures 1a–1g.

If K has multiple isolated singularities, x_i, the situation is slightly more complicated but similar. Fox and Milnor's [26] analysis of the case of a two-sphere with isolated singularities embedded in four-space carries over to higher dimensions. In particular, we can choose a PL-arc, p, embedded in S^{n-2} which traverses each singular point (where here we confuse S^{n-2} with K). Then a regular neighborhood, N, of p is again a disk pair (D^n, D^{n-2}) whose boundary is a knotted sphere pair $(S^{n-1}, S^{n-3}) = (S^{n-1}, k)$, where the knot k is the knot sum $\sum_i k(x_i)$ of the knots of each sphere pair, Link(x_i). As in the case of a single singular point, the knot complement is homotopy equivalent to the complement of the disk pair which is obtained from the sphere pair by removing the open regular neighborhood of p. This can be seen as follows: First retract the star neighborhoods of the x_i in the knot complement radially away from the cone points, x_i, as in the last paragraph. The portion of $S^n - K$ remaining in the interior of N then consists of a disjoint set of standard ball pairs (D^n, D^{n-2}) whose boundaries lie in ∂N except for two opposing sides (thinking of the balls as cubes) which lie in the link pairs of x_i and x_{i+1} and can be identified as neighborhoods there of points of $k(x_i)$ and $k(x_{i+1})$, respectively. But once we have gone over to the knot complement and hence removed the (n-2)-balls, their complements easily retract out to $\partial N - (\partial N \cap K)$. Once again, our study is reduced to the complement of a knotted disk pair which forms the null-cobordism of a slice knot. Henceforth, we refer to the knotted disk L in D^n, $\partial L = k \in \partial D^n$. See Figures 2a–2e.

By this discussion, our study of the homological properties of $S^n - K$ reduces to a study of the homological properties of $D^n - L$.

3.3. Necessary conditions on the Alexander invariants.

3.3.1. *Alexander invariants.* We now undertake a study of the Alexander polynomials of the complements of locally-flat knotted disks following the pattern of Levine's [19] study of the Alexander polynomials of locally flat sphere knots. In

Figure 1a

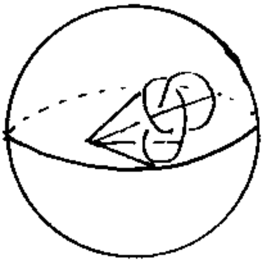

Figure 1d

Figure 1b

Figure 1e

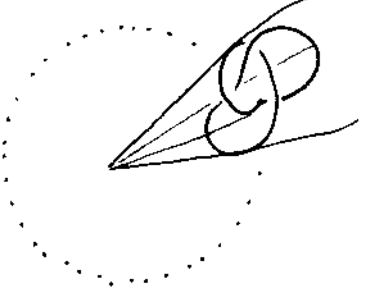

Figure 1c

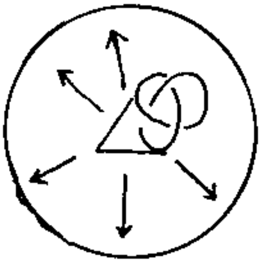

Figure 1f

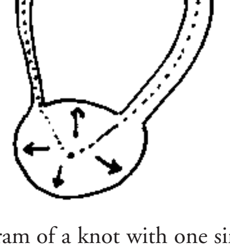

Figure 1g

1a. Schematic diagram of a knot with one singular point. 1b. A close-up of the neighborhood of the singular point. 1c–d. Slightly less schematic diagrams of the neighborhood of the singular point. 1e–f. Schematic and slightly less schematic diagrams of pushing the knot complement away from the singular point. 1g. Once we have removed a neighborhood of the knot, we are left with the knot exterior.

Figure 2a

Figure 2b

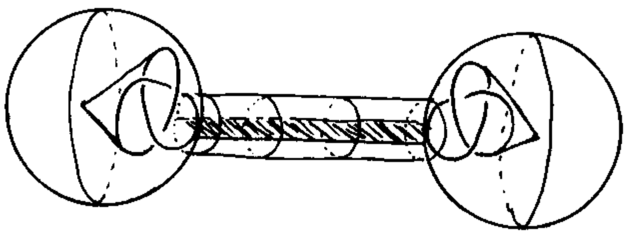

Figure 2c

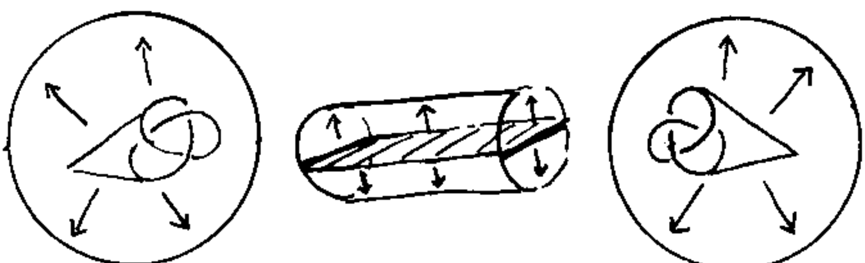

Figure 2d

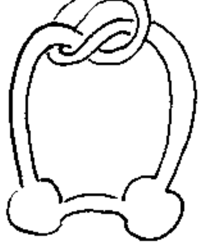

Figure 2e

2a. Schematic diagram of a knot with two singular points. 2b. A close-up of the neighborhood of a path p connecting the singular points. 2c. A slightly less schematic diagram of the neighborhoods of p in the knot and in the ambient space. 2d. Pushing the knot complement away from the knot in a neighborhood of p. 2e. The knot exterior.

particular, let C be the disk-knot complement $D^n - L$ and let $\tilde{C}$ be the infinite cyclic cover associated with the kernel of the abelianization $\pi_1(C) \to \mathbb{Z}$. Let t denote a generator for the covering translation and Λ the group ring $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$. The homology groups of $\tilde{C}$ are finitely generated Λ-modules since C has a finite polyhedron as a deformation retract, and the rational homology groups $H_*(\tilde{C}; \mathbb{Q}) \cong H_*(\tilde{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$ are, therefore, finitely generated modules over the principal ideal domain $\Gamma = \mathbb{Q}[t, t^{-1}] \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, letting $M_R(\lambda)$ denote the R-module of rank 1 with generator of order λ, $H_q(\tilde{C}; \mathbb{Q}) \cong \bigoplus_{i=1}^{k} M_\Gamma(\lambda_q^i)$ (note that "i" is here an index, not a power). Furthermore, we can choose the λ_q^i so that: 1) The λ_q^i are primitive in Λ but are unique up to associate class in Γ, and 2) $\lambda_q^{i+1} | \lambda_q^i$. For $0 < q < n-1$, the λ_q^i are called the *Alexander invariants* of the knot complement. We will also consider the relative homology modules $H_*(\tilde{C}, \tilde{X}; \mathbb{Q})$, where X is the complement of k in $S^{n-1} = \partial D^n$ and $\tilde{X}$ is its infinite cyclic covering. It will be clear from our construction that $\tilde{X}$ and the cover of X in $\tilde{C}$ are equivalent. Then $H_*(\tilde{C}, \tilde{X}; \mathbb{Q})$ has the same properties listed above for $H_*(\tilde{C}; \mathbb{Q})$ and its own Alexander invariants $\{\mu_q^i\}$, $0 < q < n-1$.

We will prove the following theorem concerning necessary conditions on these polynomials:

Theorem 3.1. *Let $p = n - 1 - q$. With $\{\lambda_q^i\}$ and $\{\mu_q^i\}$ as above for a knotted disk pair (D^n, D^{n-2}), $n \geq 3$, the following properties hold:*
1. *$\lambda_q^{i+1} | \lambda_q^i$ and $\mu_q^{i+1} | \mu_q^i$ in Λ;*
2. *$\lambda_q^i(1) = \pm 1$, $\mu_q^i(1) = \pm 1$;*
3. *$\lambda_q^i(t) \sim \mu_p^i(t^{-1})$ in Λ, where $\sim$ denotes associativity of elements in Λ, i.e. $a \sim b$ implies $a = \pm t^k b$ for some k.*

The proof of the theorem is given over the following sections.

3.3.2. *Construction of the covering.* We begin by finding Γ-module presentations for $H_*(\tilde{C}; \mathbb{Q})$ and $H_*(\tilde{C}, \tilde{X}; \mathbb{Q})$ by generalizing the usual technique of studying the Mayer-Vietoris sequence for the infinite cyclic cover obtained from cutting and pasting along a Seifert surface.

Proposition 3.2. *Given a knotted disk $L \in D^n$, there exists an $(n-1)$-dimensional connected bicollared submanifold $V \in D^n$ such that $\partial V = L \cup F$, where F is a Seifert surface for k in ∂D^n.*

Proof. Letting T be a regular tubular neighborhood of L in D^n, there is a map $f : T - L \to S^1$ given by projection on the fibers. (A trivialization of the disk bundle is provided by the restricting the trivialization of the disk bundle constituting the tubular neighborhood of the locally-flat sphere knot obtained by gluing our disk knot and its mirror image along the boundary knots). As in the construction of the usual Seifert surfaces, this map can be extended to the rest of ∂D^n so that the inverse image there of a regular value, x, of S^1 is a Seifert surface for k after throwing away extraneous components. In fact, the map can be easily modified so as to avoid extraneous components (the extraneous components will be bicollared close manifolds of ∂D^n such that the fiber in the collar of each point maps to an arc of S^1 containing x, and so on each fiber we can reverse the map to run around S^1 the other way, avoiding x). Now we wish to extend f to the rest of $D^n - ((T - L) \cup \partial D^n)$.

The obstructions to this extension lie in

$$H^{i+1}(C, T \cup \partial D^n; \pi_i(S^1)) \cong H^{i+1}(C, \partial C; \pi_i(S^1))$$

(see [2, p. 54]). We know that $\pi_i(S^1) = 0$ for $i \neq 1$ and $\mathbb{Z}$ for $i = 1$ so we need only calculate:

$$H^2(C, \partial C; \mathbb{Z}) \cong H_{n-2}(C) \cong H^1(L, \partial L = k) \cong H^1(D^{n-2}, \partial D^{n-2}) = 0,$$

since $n > 3$. The first two isomorphisms are due to Lefschetz duality and Alexander duality for a ball [28, p. 426]. Therefore, the obstruction is 0, so the extension exists, and we can take V as the inverse image of a regular value in S^1 after, again, throwing away extraneous components. If x is no longer a regular value, we can instead choose a new regular value, y, in an ϵ-neighborhood of x such that $f^{-1}(y) \cap \partial D^n$ is isotopic (in ∂D^n) to $f^{-1}(x)$ and hence gives "the same" Seifert surface for k. It is clear that V has the desired boundary. $\square$

We can now construct $\tilde{C}$ in the usual way by cutting along V to create a manifold Y whose boundary is $(D^n - V) \cap S^{n-1}$ together with two copies of V, V_+ and V_-, identified along L and then by pasting together a countably infinite number of disjoint copies (Y_i, V_+^i, V_-^i), $-\infty < i < \infty$, of $(Y - L, V_+ - L, V_- - L)$ by identifying $V_+^i - L$ with $V_-^{i+1} - L$ for all i. Then $\tilde{X}$ is the submanifold resulting from the restriction of this construction to $\partial D^n \cap (Y_i, V_+^i, V_-^i)$. $\tilde{X}$ is thus the usual infinite cyclic cover constructed for a classical knot complement as claimed. Note that, just as in that case, $H_q(D^n - V) \cong H_q(Y)$ due to homotopy equivalence. We also denote $Y \cap \partial D^n$ by Z and have $H_q(\partial D^n - F) \cong H_q(Z)$.

The usual considerations (see, e.g., [19]) now allow us to set up the Mayer-Vietoris sequences for $\tilde{C}$ and $(\tilde{C}, \tilde{X})$:

$$\longrightarrow H_q(V; \mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{d_1} H_q(Y; \mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{e_1} H_q(\tilde{C}; \mathbb{Q}) \longrightarrow \tag{3.1}$$

and

$$\longrightarrow H_q(V, F; \mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{d_2} H_q(Y, Z; \mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{e_2} H_q(\tilde{C}, \tilde{X}; \mathbb{Q}) \longrightarrow \tag{3.2}$$

We will see that d_i, $i = 1, 2$, is a monomorphism for $0 \leq q < n - 1$. Hence e_i is an epimorphism, $0 < q < n - 1$, and the d_i provide presentation matrices for the homology modules of the covers as Γ-modules. That the d_i are square matrices in this range follows from the following proposition.

Proposition 3.3.

1. $H_q(V;\mathbb{Q}) \cong H_q(Y;\mathbb{Q})$, $0 \le q < n-1$,
2. $H_q(V,F;\mathbb{Q}) \cong H_q(Y,Z;\mathbb{Q})$, $0 \le q < n-1$.

Proof.

1. In the proof, we assume rational coefficients while omitting mention for the sake of notational convenience. $H_q(Y) \cong H_q(D^n - V) \cong H^p(V,F)$, $p+q = n-1$, by Alexander duality for the ball, while $H_q(V) = H^p(V,\partial V)$ by Lefschetz duality. So we must show that $H^p(V,F) \cong H^p(V,\partial V)$. Recall that $\partial V = F \cup_k L \cong F \cup_k D^{n-2}$. From the reduced Mayer-Vietoris sequence, we get immediately that $H^k(F) \cong H^k(\partial V)$ for $k < n-3$, and the top of the sequence is

$$\begin{tikzcd} {} & H^{n-2}(F) \arrow[l, "0"'] \arrow[d, "="'] & H^{n-2}(\partial V) \arrow[l] \arrow[d, "\cong"'] & H^{n-3}(S^{n-3}) \arrow[l] \arrow[d, "\cong"'] & H^{n-3}(F) \arrow[l] \arrow[d, "="'] & H^{n-3}(\partial V) \arrow[l] \arrow[d, "="'] & {} \arrow[l, "0"'] \\ {} & 0 \arrow[l] & \mathbb{Q} \arrow[l] & \mathbb{Q} \arrow[l] & H^{n-3}(F) \arrow[l] & H^{n-3}(\partial V) \arrow[l] & {} \arrow[l, "0"'] \end{tikzcd}$$

because F is an $(n-2)$-manifold with boundary and ∂V is a closed $(n-2)$-manifold. Since the map $\mathbb{Q} \to \mathbb{Q}$ must be an isomorphism, so must be the map $H^{n-3}(\partial V) \to H^{n-3}(F)$. Therefore, $H^k(F) \cong H^k(\partial V)$ for $k < n-2$. Now turning to the long exact sequences of the pairs, the inclusion $(V,F) \hookrightarrow (V,\partial V)$ and naturality give a commutative diagram

$$\begin{tikzcd} H^k(F) \arrow[d] & H^k(V) \arrow[l] \arrow[d] & H^k(V,F) \arrow[l] \arrow[d] & H^{k-1}(F) \arrow[l] \arrow[d] & H^{k-1}(V) \arrow[l] \arrow[d] \\ H^k(\partial V) & H^k(V) \arrow[l] & H^k(V,\partial V) \arrow[l] & H^{k-1}(\partial V) \arrow[l] & H^{k-1}(V). \arrow[l] \end{tikzcd}$$

By the five-lemma, $H^k(V,F) \cong H^k(V,\partial V)$ for $k < n-2$. For $k = n-2$, we can use the facts that $H^{n-2}(F) = 0$ and $H^{n-1}(V) = 0$, since F is an $(n-2)$-manifold with boundary and V is an $(n-1)$-manifold with boundary, and that $H^{n-2}(\partial V) \to H^{n-1}(V,\partial V)$ is an isomorphism $\mathbb{Q} \to \mathbb{Q}$ for similar reasons. These allow us to extract the following commutative diagram near the top of the sequence:

$$\begin{tikzcd}
0 & H^{n-2}(V) \arrow[l] \arrow[d] & H^{n-2}(V,F) \arrow[l] \arrow[d] & H^{n-3}(F) \arrow[l] \arrow[d] & H^{n-3}(V) \arrow[l] \arrow[d] \\
0 & H^{n-2}(V) \arrow[l] & H^{n-2}(V,\partial V) \arrow[l] & H^{n-3}(\partial V) \arrow[l] & H^{n-3}(V). \arrow[l]
\end{tikzcd}$$

Again using the five-lemma, we conclude that $H^k(V,F) \cong H^k(V,\partial V)$ for $k < n-1$.

2. For $q = 0$, the statement is obvious as all of the spaces are connected. Otherwise, from part (1), $H_q(V) \cong H_q(Y)$, $0 \leq q < n-1$, and from the perfect linking pairings L' and L'' (see Section 3.3.3, below), these are dually paired to $H_{n-p-1}(Y,Z)$ and $H_{n-p-1}(V,F)$, respectively, for $0 < q < n-1$. From the induced perfect rational pairings we get the statement of part (2). □

Returning to the maps d_i; $i = 1,2$; it follows from the construction of the covering and the action of the covering translation, t, that the maps can be written as

$$\begin{aligned} d_i(\alpha \otimes 1) &= i_{-*}(\alpha) \otimes t - i_{+*}(\alpha) \otimes 1 \\ &= t(i_{-*}(\alpha) \otimes 1) - i_{+*}(\alpha) \otimes 1, \end{aligned}$$

where $i_\pm$ correspond to the identification maps of (V,F) to $(V_\pm, F_\pm)$ and $\alpha \in H_q(V;\mathbb{Q})$ or $H_q(V,F;\mathbb{Q})$ according to whether $i = 1$ or 2.

3.3.3. *Linking Numbers.* We now turn to the linking pairings on these homology groups. Let $p = n-1-q$. There are perfect (modulo torsion) pairings

$$L' : H_p(V,F) \otimes H_q(Y) \to \mathbb{Z} \tag{3.3}$$

$$L'' : H_p(Y,Z) \otimes H_q(V) \to \mathbb{Z}. \tag{3.4}$$

These are the usual geometric linking pairings which are induced, via some isomorphisms, by the classical Lefschetz dual intersection pairing $H_i(S^n,Z) \otimes H_{n-i}(S^n - Z) \to \mathbb{Z}$, for $Z \subset S^n$. In particular, given the disk D^n and a closed subpolyhedron $B \subset D^n$ which meets $\partial D^n = S^{n-1}$ regularly, let $A = B \cup \partial D^n =$

$B \cup S^{n-1}$. We also think of $D^n = D^n_+$ as the top hemisphere of $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$. Then a linking pairing

$$L' : H_i(B, B \cap S^{n-1}) \times H_{n-i-1}(D^n - B) \to \mathbb{Z}, \tag{3.5}$$

$0 < i < n-1$, can be defined by applying the following isomorphisms and then applying the intersection pairing:

$$\begin{aligned}
H_i(B, B \cap S^{n-1}) &\cong H_i(B \cup S^{n-1}, S^{n-1}) && \text{by excision}\\
&= H_i(A, S^{n-1}) && \text{by definition of } A\\
&\cong H_i(A) && \text{by the long exact sequence of } (A, S^{n-1})\\
&\cong H_{i+1}(D^n, A) && \text{by the long exact sequence of } (D^n, A)\\
&\cong H_{i+1}(S^n, A \cup D^n_-) && \text{by excision,}
\end{aligned}$$

and

$$\begin{aligned}
H_{n-i-1}(D^n - B) &\cong H_{n-i-1}(D^n - A) && \text{by the homotopy equivalence of}\\
& && D^n - A \text{ and } D^n - B\\
&= H_{n-i-1}(S^n - A \cup D^n_-) && \text{since } D^n - A = S^n - A \cup D^n_-.
\end{aligned}$$

The linking pairing

$$L'' : H_i(D^n - B, (D^n - B) \cap S^{n-1}) \times H_{n-1-i}(B) \to \mathbb{Z}$$

can be obtained by considering an open regular neighborhood, N, of B in D^n. Then N deformation retracts to B and $(D^n - B, (D^n - B) \cap S^{n-1})$ deformation retracts to $(D^n - N, (D^n - N) \cap S^{n-1})$. So, if $U = D^n - N$, then $H_*(B) \cong H_*(D^n - U)$ and $H_*(D^n - B, (D^n - B) \cap S^{n-1}) \cong H_*(U, U \cap S^{n-1})$. Then we can apply L' with U in place of B in equation (3.5).

See [9, Appendix] for more details on the construction of these linking pairings.

By taking tensor products, these pairings extend to perfect pairings from the rational homology groups to $\mathbb{Q}$. Let $\{\alpha_i^p\}$, $\{\beta_i^q\}$, $\{\gamma_i^p\}$, and $\{\delta_i^q\}$ represent dual bases for $H_p(V, F)$, $H_q(Y)$, $H_p(Y, Z)$, and $H_q(V)$, all modulo torsion, so that

$$L'(\alpha_i^p \otimes \beta_j^q) = L''(\gamma_i^p \otimes \delta_j^q) = \delta_{ij}. \tag{3.6}$$

These collections also form bases then for the rational homology groups that result by tensoring with $\mathbb{Q}$, and the relations (3.6) hold under the induced perfect rational pairing.

Given $r \in H_p(V, F; \mathbb{Q})$ and $s \in H_q(V; \mathbb{Q})$, we also have the relation

$$L'(r \otimes i_{-*}(s)) = L''(i_{+*}(r) \otimes s). \tag{3.7}$$

This can be seen as follows: we can choose the inclusion maps $i_\pm : (V, F) \hookrightarrow (Y, Z)$ as isotopies which push V out along its collar in one direction or the other. Then any chain representing s gets pushed into Y under i_- and the the linking form is the intersection of this chain with a chain, R, representing the isomorphic image of r in $H_{p+1}(D^n, V \cup S^{n-1}; \mathbb{Q})$ (see [9, Appendix]). The latter chain can be taken as some chain in D^n whose boundary, lying in $V \cup S^{n-1}$, is a chain representing r. Now, under the isotopy which takes V to $i_+(V)$ and $i_-(V)$ to V, the chain representing s gets pushed back into V and R gets pushed into a chain in D^n whose boundary, lying in $Y \cup S^{n-1}$, is i_+ of the chain representing r. In particular, this latter chain represents $i_{+*}(r) \in H_p(Y, Z)$. Thus this isotopy induces maps which take $i_{-*}(s)$ to s and r to $i_{+*}(r)$, but since the geometric relationship between the chains is unaffected by the isotopy, the intersection number is unaffected. The formula then follows immediately using the definitions of L' and L'' as geometric linking pairings (again, see [9, Appendix] for more details). Similarly, we get

$$L'(r \otimes i_{+*}(s)) = L''(i_{-*}(r) \otimes s). \tag{3.8}$$

The final property of linking numbers which we will need is that given r and s as above

$$L'(r \otimes i_{-*}(s)) - L'(r \otimes i_{+*}(s)) = r \cap s \tag{3.9}$$

$$L''(i_{-*}(r) \otimes s) - L''(i_{+*}(r) \otimes s) = r \cap s, \tag{3.10}$$

where $r \cap s$ is the intersection pairing on V. The proof is analogous to that in the usual case [19, p. 542].

3.3.4. *The proof of Theorem 3.1.*

We can now complete the proof of the theorem: With the bases $\{\alpha_i^p\}$, $\{\beta_i^q\}$, $\{\gamma_i^p\}$, and $\{\delta_i^q\}$ as above, $\{\alpha_i \otimes 1\}$, etc., give bases of $H_p(V, F; \mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma$, etc. Let

$$\begin{aligned} i_{+*}(\delta_j^q) &= \sum_i \lambda_{ij}^q \beta_i^q\,, \\ i_{-*}(\delta_j^q) &= \sum_i \sigma_{ij}^q \beta_i^q\,, \\ i_{+*}(\alpha_j^q) &= \sum_i \mu_{ij}^q \gamma_i^q\,, \\ i_{-*}(\alpha_j^q) &= \sum_i \tau_{ij}^q \gamma_i^q\,. \end{aligned}$$

Note that the λ, σ, μ, and τ will all be integers (by the chain map interpretation of $i_\pm$ and the fact that the α and δ were initially chosen as generators of the torsion free parts of the appropriate integral homology groups). Then

$$\begin{aligned} d_1(\delta_j^q \otimes 1) &= \sum_i (t\sigma_{ij}^q - \lambda_{ij}^q)(\beta_i^q \otimes 1)\,, \\ d_2(\alpha_j^q \otimes 1) &= \sum_i (t\tau_{ij}^q - \mu_{ij}^q)(\gamma_i^q \otimes 1)\,, \end{aligned}$$

and we obtain presentation matrices

$$P_1^q(t) = (t\sigma_{ij}^q - \lambda_{ij}^q),$$
$$P_2^q(t) = (t\tau_{ij}^q - \mu_{ij}^q)$$

for $H_q(\tilde{C};\mathbb{Q})$ and $H_q(\tilde{C},\tilde{X};\mathbb{Q})$. As we have already seen, these matrices will be square for $0 < q < n-1$ by Proposition 3.3.

Applying the perfect linking pairing gives

$$L'(\alpha_k^p \otimes i_{+*}(\delta_j^q)) = \sum_i \lambda_{ij}^q L'(\alpha_k^p \otimes \beta_i^q) = \lambda_{kj}^q,$$
$$L'(\alpha_k^p \otimes i_{-*}(\delta_j^q)) = \sum_i \sigma_{ij}^q L'(\alpha_k^p \otimes \beta_i^q) = \sigma_{kj}^q,$$
$$L''(i_{+*}(\alpha_j^q) \otimes \delta_k^p) = \sum_i \mu_{ij}^q L''(\gamma_i^q \otimes \delta_k^p) = \mu_{kj}^q,$$
$$L''(i_{-*}(\alpha_j^q) \otimes \delta_k^p) = \sum_i \tau_{ij}^q L''(\gamma_i^q \otimes \delta_k^p) = \tau_{kj}^q,$$

and so by (3.7) and (3.8), we have $\sigma_{jk}^q = \mu_{kj}^p$ and $\lambda_{jk}^q = \tau_{kj}^p$. This implies that $P_1^q(t) = -tP_2^p(t^{-1})'$, where $'$ indicates transpose. Further,

$$P_1^q(1) = (\sigma_{ij}^q - \lambda_{ij}^q) = (-\alpha_i^p \cap \delta_j^q),$$
$$P_2^q(1) = (\tau_{ij}^q - \mu_{ij}^q) = (-\alpha_i^q \cap \delta_j^p),$$

by (3.9) and (3.10). Since this is the (modulo torsion) intersection pairing $\cap : H_p(V) \otimes H_q(V,F) \cong H_p(V) \otimes H_q(V,\partial V) \to \mathbb{Z}$, which is non-singular modulo torsion, these matrices are non-singular which implies that the maps d_i of the Mayer-Vietoris sequence (3.1) are injective as claimed above. In fact, as the matrix of a perfect intersection pairing over $\mathbb{Z}$ of the free summands of the relevant integral homology modules, the matrix is unimodular with determinant ± 1.

With only minor modifications, the conclusion of the theorem is now obtained just as in [19, §2.8] by looking at the i^{th} order minors of $P_1^q(t)$ and $P_2^p(t)$ and applying the properties of modules over principal ideal domains. In particular, if Δ_i^q and $\bar{\Delta}_i^p$ are the greatest common divisors of the i^{th} order minors of $P_1^q(t)$ and $P_2^p(t)$, respectively, then they are elements of Λ, and $\lambda_q^i \sim \Delta_i^q/\Delta_{i+1}^q$ and $\mu_p^i \sim \bar{\Delta}_i^p/\bar{\Delta}_{i+1}^p$ in Λ. Furthermore, by the properties of $P_1^q(t)$ and $P_2^p(t)$ proven above, $\Delta_i^q(1) = \pm 1$, $\bar{\Delta}_i^p(1) = \pm 1$, and $\Delta_i^q(t) \sim \bar{\Delta}_i^p(t^{-1})$ in Λ. These imply that $\lambda_q^i(1) = \pm 1$, $\mu_p^i(1) = \pm 1$, and $\lambda_q^i(t) \sim \mu_p^i(t^{-1})$ in Λ. □

3.4. Some corollaries: Definition of Alexander polynomials.

Corollary 3.4. *With the notations as above, if the boundary slice knot k is j-simple, meaning that $S^{n-1} - k$ has the homotopy of a circle for dimensions less than or equal to j, then for $0 < i < j+1$ and $n-j-3 < i < n-1$,*

1. $\lambda_q^i(1) = \pm 1$,
2. $\lambda_q^i(t) \sim \lambda_{n-q-1}^i(t^{-1})$.

Proof. By [18, p.14], the simplicity condition implies that we can modify the Seifert surface, F, to be j-connected without changing F near its boundary. We can then use this Seifert surface to redefine the extension of the map f of Proposition 3.2 on S^{n-1} so that it yields this Seifert surface on the sphere. We then extend to the interior of $D^n - L$ as in Proposition 3.2 to get a new V with $\partial V = F \cup L$.

Since $\pi_i(F) = 0$ for $0 < i < j+1$, we get $H_i(F) = 0$ in the same range, and so $H_i(V;\mathbb{Q}) \cong H_i(V,F;\mathbb{Q})$ by the long exact sequence of the pair. This isomorphism also holds for $n-j-3 < i < n-1$. In fact, by Lefschetz duality, $\tilde{H}_i(F;\mathbb{Q}) \cong H_{n-i-2}(F,\partial F;\mathbb{Q}) \cong H_{n-i-2}(F,S^{n-3};\mathbb{Q})$, since F is $(n-2)$-dimensional and we are using field coefficients. Thus $H_i(F,S^{n-3};\mathbb{Q}) = 0$ for $n-j-3 < i < n-2$ and $H_i(F) = 0$, $n-j-3 < i < n-3$ by the long exact sequence of the pair $(F,\partial F)$. $H_{n-2}(F) = 0$ since F is an $(n-2)$-manifold with boundary, and, since the top of the sequence is (suppressing the coefficients)

$$\begin{tikzcd}
H_{n-2}(F) \arrow[r] \arrow[d, "\cong"'] & H_{n-2}(F,S^{n-3}) \arrow[r] \arrow[d, "\cong"'] & H_{n-3}(S^{n-3}) \arrow[r] \arrow[d, "\cong"'] & H_{n-3}(F) \arrow[r] \arrow[d, "\cong"'] & H_{n-3}(F,S^{n-3}) \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q} \arrow[r] & ? \arrow[r] & 0
\end{tikzcd},$$

we must have that $H_{n-3}(F) = 0$ also. Again by the long exact sequence of the pair, we get $H_i(V;\mathbb{Q}) \cong H_i(V,F;\mathbb{Q})$, $n-j-3 < i < n-1$.

So, in this range, the perfect linking pairing L' can be defined

$$L' : H_p(V;\mathbb{Q}) \otimes H_q(Y;\mathbb{Q}) \to \mathbb{Q}$$

using the isomorphisms $H_i(V) \cong H_i(V,F)$. From here, the proof follows as above and as in [19] using the dual bases of $H_p(V;\mathbb{Q})$ and $H_q(Y;\mathbb{Q})$ in the relevant dimensions. ◻

Corollary 3.5. *If we define the* Alexander polynomials $\lambda_q(t)$ *and* $\mu_q(t)$ *as the primitive polynomials in* Λ *determined up to similarity class by the determinants of the square presentation matrices of* $H_i(\tilde{C})$ *and* $H_i(\tilde{C},\tilde{X})$ *as* Γ*-modules, then*

1. $\lambda_q(1) = \pm 1$ *and* $\mu_q(1) = \pm 1$
2. $\lambda_q(t) \sim \mu_{n-p-1}(t^{-1})$.

Proof. From our earlier definitions, $\lambda^q = \prod \lambda_j^q$ and $\mu^q = \prod \mu_j^q$. The corollary now follows immediately. ◻

Corollary 3.6. *Let $\nu_i(t)$, $\lambda_i(t)$, and $\mu_i(t)$, $0 < i < n-2$, be the Alexander polynomials corresponding to $H_i(\tilde{X})$, $H_i(\tilde{C})$, and $H_i(\tilde{C}, \tilde{X})$, respectively. Then*

$$\prod_{i>0} \frac{\mu_{2i-1}(t)\nu_{2i-1}(t)\lambda_{2i}(t)}{\mu_{2i}(t)\nu_{2i}(t)\lambda_{2i-1}(t)} = 1, \tag{3.11}$$

where, for this formula only, we define the polynomials to be 1 for $i > n-2$.

Proof. The Alexander polynomials are given by the determinants of the presentation matrices of the terms of the exact sequence of Γ-modules

$$\longrightarrow \tilde{H}_i(\tilde{X};\mathbb{Q}) \longrightarrow \tilde{H}_i(\tilde{C};\mathbb{Q}) \longrightarrow \tilde{H}_i(\tilde{C},\tilde{X};\mathbb{Q}) \longrightarrow .$$

We know that each term is finitely generated as a Γ-module, so the corollary follows immediately from Proposition 2.1, the triviality of $\tilde{H}_0(\tilde{X};\mathbb{Q})$ (since $\tilde{X}$ is connected), and the triviality of $H_{n-2}(\tilde{X};\mathbb{Q})$ (by classical knot theory). ◻

Corollary 3.7. *With the notation above, $\lambda_{n-2}(t)$ divides $\lambda_1(t^{-1})$.*

Proof. From the proof of the last corollary and Corollary 2.5, $\lambda_{n-2}(t)$ divides $\mu_{n-2}(t)$, but $\mu_{n-2}(t) \sim \lambda_1(t^{-1})$. ◻

We can also use these methods to obtain this well-known fact:

Corollary 3.8. *A classical slice 1-knot ($S^1 \subset S^3$) has Alexander polynomial of the form $\nu_1(t) \sim p(t)p(t^{-1})$.*

Proof. We take $n = 4$ for our disk knot pair, so that the boundary slice knot will be a knotted S^1 in S^3. Then the only non-trivial Alexander polynomials are $\nu_1(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$. From Corollary 3.6,

$$\nu_1(t) \sim \frac{\mu_2(t)\lambda_1(t)}{\mu_1(t)\lambda_2(t)} \sim \frac{\lambda_1(t^{-1})\lambda_1(t)}{\lambda_2(t^{-1})\lambda_2(t)}.$$

From here we can proceed more or less as in [26]: Let $d(t)$ be the greatest common divisor of $\lambda_1(t)$ and $\lambda_2(t)$ so that $\lambda_1(t) = d(t)a(t)$, $\lambda_2(t) = d(t)b(t)$, and $a(t)$ and $b(t)$ are relatively prime. Then

$$\nu_1(t) \sim \frac{d(t)d(t^{-1})a(t)a(t^{-1})}{d(t)d(t^{-1})b(t)b(t^{-1})} = \frac{a(t)a(t^{-1})}{b(t)b(t^{-1})}.$$

Similarly, now let $c(t)$ be the greatest common divisor of $a(t^{-1})$ and $b(t)$ so that $a(t^{-1}) = p(t^{-1})c(t)$ and $b(t) = q(t)c(t)$. Then

$$\nu_1(t) = \frac{p(t)p(t^{-1})}{q(t)q(t^{-1})},$$

and the numerator and denominator are now relatively prime. But $\nu_1(t)$ is actually a polynomial so $q(t) \sim 1$ and $\nu_1(t) \sim p(t)p(t^{-1})$. ◻

3.5. Realization of given polynomials. In this section, we obtain results on the realization of knots with prescribed Alexander polynomials. The construction of a knot $D^2 \subset D^4$ with a given polynomial is done by hand to get a feel for the geometric concepts involved. This lays the foundation for realization theorems in higher dimensions.

Throughout this section, we continue to use λ_q, μ_q, and ν_q as defined in Corollary 3.6.

3.5.1. *Realizing* $\boldsymbol{\lambda_i}$ *for* $\mathbf{D^2} \subset \mathbf{D^4}$.

Theorem 3.9. *Given any polynomial $p(t) \in \Lambda$ such that $p(1) = \pm 1$, there exists a knotted D^2 in D^4 with $\lambda_1(t) \sim p(t)$ and $\lambda_2(t) \sim 1$.*

Proof. For definiteness, let us normalize $p(t)$ so that

$$p(t) = \sum_{i=0}^{m} a_i t^i, \quad p(1) = 1,$$

and $p(0) \neq 0$. We will construct a knotted disk with $H_2(\tilde{C}) = 0$ and $H_1(\tilde{C}) \cong \Lambda/p(t)$.

We begin by embedding a 2-disk, L, in $S^1 \times D^3$, so that, in a neighborhood of a boundary point which is homeomorphic to the half-space $\mathbb{R}^{4+}$, L is embedded as a standard disk. In other words, $\partial D^2 = S^1$ is an unknotted circle within a neighborhood of a point of $\partial(S^1 \times D^3) = S^1 \times S^2$, $\text{int}(L)$ lies in $\text{int}(S^1 \times D^3)$, and $(L, \partial L)$ is null-homotopic in $(S^1 \times D^3, \partial(S^1 \times D^3))$. We also let ∂L bound a disk F in $S^1 \times S^2$ so that $F \cup L$ bounds a manifold V homeomorphic to a standard D^3 such that $\text{int}(V)$ lies in $\text{int}(S^1 \times D^3)$. Let $C_0 = S^1 \times D^3 - L$, and let $\tilde{C}_0$ be the infinite cyclic covering associated with the kernel of the homomorphism $\pi_1(\tilde{C}_0) \to \mathbb{Z}$ defined by intersection number with V. Forming the infinite cyclic cover by cutting along V, it is clear that $H_2(\tilde{C}_0) = 0$ and that $H_1(\tilde{C}_0) \cong \Lambda$, where we can take as generator, α, the lift of a circle representing a generator of $\pi_1(S^1 \times D^3)$ and which does not intersect F.

We will prove the following lemma below:

Lemma 3.10. *There exists an embedding $f : S^1 \hookrightarrow S^1 \times S^2 - \partial L$ which lifts to an embedding $g : S^1 \hookrightarrow \tilde{C}_0$ which represents the element $\lambda(t)\alpha \in H_1(\tilde{C}_0)$. Furthermore, f can be chosen isotopic to the standard embedding which takes S^1 to $S^1 \times x_0$ for some $x_0 \in S^2$.*

Now let

$$S = f(S^1) \in S^1 \times S^2 = \partial(S^1 \times D^3).$$

We will attach a 2-handle along S. In particular, there is a neighborhood $S \times D^2$ of S in $\partial(S^1 \times D^3)$ which we identify with half of the boundary

$$\partial(I^2 \times I^2) = (S^1 \times D^2) \cup (D^2 \times S^1).$$

If H denotes the handle, then $(S^1 \times D^3) \cup_{S\times D^2} H \cong D^4$, and we claim that L, which is now knotted in D^4, is the desired knotted disk.

Still assuming the lemma, it remains to show only that we get the desired homology of the cover. Notice that S lifts to an infinite number of disjoint embeddings, S_i, in $\tilde{C}_0$ which correspond to $t^i\lambda(t)\alpha$, $-\infty < i < \infty$. If we attach an infinite number of handles sewn along $S_i \times D^2$, we will obtain an infinite cyclic covering of $D^4 - L$, which we denote by $\tilde{C}$. That $\tilde{C}$ has the desired homology follows from the reduced Mayer-Vietoris sequence:

$$\longrightarrow \bigoplus_{i=-\infty}^{\infty} H_2(S^1) \longrightarrow \Big(\bigoplus_{i=-\infty}^{\infty} H_2(D^4)\Big) \oplus H_2(\tilde{C}_0) \longrightarrow H_2(\tilde{C})$$
$$\longrightarrow \bigoplus_{i=-\infty}^{\infty} H_1(S^1) \xrightarrow{d} \Big(\bigoplus_{i=-\infty}^{\infty} H_1(D^4)\Big) \oplus H_1(\tilde{C}_0) \longrightarrow H_1(\tilde{C})$$
$$\longrightarrow \bigoplus_{i=-\infty}^{\infty} \tilde{H}_0(S^1) \longrightarrow \Big(\bigoplus_{i=-\infty}^{\infty} \tilde{H}_0(D^4)\Big) \oplus \tilde{H}_0(\tilde{C}_0) \longrightarrow \tilde{H}_0(\tilde{C}) \longrightarrow .$$

The first two terms and the last are zero, as are $H_i(D^4)$, $i = 1, 2$, and $\tilde{H}_0(\tilde{C}_0)$. The map $\bigoplus_{i=-\infty}^{\infty} \tilde{H}_0(S^1) \to \bigoplus_{i=-\infty}^{\infty} \tilde{H}_0(D^4)$ is an isomorphism $\Lambda \to \Lambda$, and $\bigoplus_{i=-\infty}^{\infty} H_1(S^1)$ and $H_1(\tilde{C}_0)$ are both isomorphic to Λ. Since we know that the generators of $H_1(S_i)$ map onto the the generators $t^i\lambda(t)\alpha$, the map d must be an injection. Hence, we can conclude from this information that $H_2(\tilde{C}) = 0$ and $H_1(\tilde{C}) \cong \Lambda/(\lambda(t)\Lambda)$, which is the desired result. ◻

Proof of Lemma 3.10. We wish to embed a circle S into $S^1 \times S^2$ so that it will be isotopic to a standard circle and so that that some lifting will represent $\lambda(t)\alpha$, where $\lambda(t) = \sum_{i=0}^{m} a_i t^i$ and α is some generator of $H_1(\tilde{C}_0)$. It is possible, and simpler for visualization purposes, to embed the circle into the standard solid torus $S^1 \times D^2 \subset S^1 \times S^2$ obtained by removing a neighborhood of some $S^1 \times x$, $x \in S^2$. We are also free to take L in the theorem so that ∂L is a circle concentric to a standard meridian inside this solid torus. Then F can be taken as the disk which fills in this circle. Note that α can be taken as a lift of a longitude, ℓ, which does not intersect F.

We will construct S primarily by running around the boundary $S^1 \times S^1$ of the solid torus with ever-increasing meridional angle. To be precise, we begin by choosing an orientation for the longitude, $\ell = S^1 \times 0$, which does intersect F, so that its lifts will be arcs running from $\tilde{x}$ to $t\tilde{x}$, where $\tilde{x}$ is the lift of a point of the longitude. Now, choose a point x_0 which lies in $S^1 \times S^1$ on the meridian concentric to ∂L and F. We begin by running an arc around the torus $|a_0|$ times, choosing the direction to agree with the that of ℓ if $a_0 > 0$ or to disagree if $a_0 < 0$, while the meridional angle increases slightly to avoid self intersection. Then run the arc into the interior through F in the direction of ℓ and then back out to the boundary of the torus. It is clear that this can be done in such a way that the radial retraction of the arc to the torus will continue to be an embedding with increasing meridional angle. Now, follow the same procedure for each of the a_i, doing nothing but the final step of crossing F if $a_i = 0$. Clearly we can choose the rate of increase of the meridional angle so that we never complete a full cycle meridionally. Lastly, after wrapping around the torus the a_mth time, we run the arc back through F m times against the direction of ℓ (i.e. so that it links with ∂F m more times but the total linking number will be 0), still with increasing meridional angle, and then connect it back to the starting point along a meridian.

To see that S is isotopic to the standard longitude S^1, first observe that our construction allows us to isotop S out to the torus $S^1 \times S^1$. In the torus, the *homotopy type* of S is $(1,1)$ since $\lambda(1) = 1$ and by the method of construction. Now by [29, p. 25], S is ambient isotopic in the torus to the standard representation of the $(1,1)$ homotopy class, and this ambient isotopy can be extended to a neighborhood $S^1 \times S^1 \times [-1,1]$ of the torus in $S^1 \times S^2$ (indeed, just perform the isotopy, itself, on $S^1 \times S^1 \times [-1,0]$ and then its reverse on $S^1 \times S^1 \times [0,1]$). But the standard representation of $(1,1)$ is clearly ambient isotopic to the standard $S^1 \times 0 \subset S^1 \times D^2$ displacing it radially in each meridional disk.

It also apparent from the construction that S will lift to the proper element of $H_1(\tilde{C}_0)$. In fact, by considering the usual cut and paste construction of the infinite cyclic covering, $\tilde{C}_0$ looks like an infinite number of $S^1 \times D^4$'s glued together, and, as remarked above, the generator of $H_1(\tilde{C}_0)$ corresponds to the generator of one of these "solid tori" and projects down to a generator of the homology of $H_1(S^1 \times D^4)$ which does not intersect V (or hence, F). So, by this construction, if we lift the starting point of S to its covering point in $\tilde{C}_0$ corresponding to the t^0 copy of $S^1 \times D^4$, then S lifts to an arc which runs around this $S^1 \times D^4$, parallel to α, a_0 times in the correct direction, then crosses into the t^1 copy of $S^1 \times D^4$ and runs around it a_1 times parallel to $t\alpha$ and so on. After finishing its circuits in the t^m copy of $S^1 \times D^4$, it returns straight back to its starting point. Evidently, this lift represents the homology class $\lambda(t)\alpha$ as desired. □

Corollary 3.11. *The conditions $\lambda_2(1) = \pm 1$, $\lambda_1(1) = \pm 1$, and $\lambda_2(t) | \lambda_1(t^{-1})$ completely characterize all of the Alexander polynomials, λ_i, of a disk knot $D^2 \subset D^4$ and hence of a singularly knotted 2-sphere in S^4.*

Proof. We know that these conditions are necessary. To show that they are sufficient, let $\lambda_1(t) = \lambda_2(t^{-1})r(t)$. Then we must also have $r(1) = \pm 1$. It follows from the preceding theorem that we can find a knotted $D^2 \subset D^4$ whose Alexander polynomials are $r(t)$ and 1 in dimensions 1 and 2, respectively. Taking the cone on the boundary sphere pair gives a singular knot with the same Alexander polynomial. We can also find a locally-flat knot whose first and second Alexander polynomials are $\lambda_2(t^{-1})$ and $\lambda_2(t)$, respectively [19, §4]. Then the knot sum of these two knots has the desired Alexander polynomials since Alexander polynomials multiply under knot sum. ◻

Note that for a knot $D^2 \subset D^4$ we have now completely classified *all* of the Alexander polynomials, since $\mu_1(t) \sim \lambda_2(t^{-1})$, $\mu_2 \sim \lambda_1(t^{-1})$, and ν_1 is completely determined by Corollary 3.6.

3.5.2. *Realizing* $\boldsymbol{\lambda_i}$ *for* $\mathbf{D^{n-2} \subset D^n}$, $\mathbf{n \geq 5}$. We next turn to realizing the Alexander polynomials, $\lambda_i(t)$, for n-disk knots, $n \geq 5$. Our arguments are split into two propositions. The first provides the realizability directly for the lower dimensional polynomials. The second provides the realizability in higher dimensions by constructing knots with the appropriate dual polynomials $\mu_i(t^{-1})$ in the lower dimensions. The first result has been shown already by Sumners in his thesis using similar methods (see [38], [37]). The second result on the higher dimensional λ_i was also shown there but by different methods. Consequences of our specific construction will be used again in the proof of Theorem 3.18 below.

Proposition 3.12. *Given a polynomial $p(t)$ such that $p(1) = \pm 1$ and integers q and n such that $1 \leq q \leq (n-2)/2$ and $n \geq 5$, there exists a knotted $D^{n-2} \subset D^n$ such that $\lambda_q(t) \sim p(t)$ and $\lambda_i(t) \sim 1$ for $0 < i < n-1$, $i \neq q$, where $\lambda_i(t)$ is the Alexander polynomial corresponding to $H_i(\tilde{C})$, $\tilde{C}$ the knotted disk complement.*

Proof. We can normalize $p(t)$ so that $p(1) = 1$. It suffices to construct a disk knot such that $H_q(\tilde{C}) \cong \Lambda/(p(t))$ and $H_i(\tilde{C}) = 0$ for $0 < i < n-1$, $i \neq q$. The proof is a variation of that of Levine [19] for locally-flat knots $S^{n-2} \subset S^n$.

We begin by embedding an $(n-2)$-disk, L, in $S^q \times D^{n-q}$, so that, in a neighborhood of a boundary point which is homeomorphic to the half-space $\mathbb{R}^{n+}$, L is embedded as a standard disk. In other words, $\partial D^{n-2} = S^{n-3}$ is an unknotted sphere within a neighborhood of a point in $\partial(S^q \times D^{n-q}) = S^q \times S^{n-q-1}$, $\text{int}(L)$ lies in $\text{int}(S^q \times D^{n-q})$, and $(L, \partial L)$ is null-homotopic in $(S^q \times D^{n-q}, \partial(S^q \times D^{n-q}))$. We also let ∂L bound a disk F in $S^q \times S^{n-q-1}$ so that $F \cup L$ bounds a manifold V, homeomorphic to a standard D^{n-1}, such that $\text{int}(V)$ lies in $\text{int}(S^q \times D^{n-q})$. Let $C_0 = S^q \times D^{n-q} - L$, and let $\tilde{C}_0$ be the infinite cyclic covering associated with the kernel of the homomorphism $\pi_1(\tilde{C}_0) \to \mathbb{Z}$ defined by intersection number with V. Similarly, we have the covering $\tilde{X}_0$ of $X_0 = S^q \times S^{n-q-1} - \partial L$. Forming the infinite cyclic covers by cutting and pasting along V, it is clear that

$$\tilde{H}_i(\tilde{C}_0) \cong \begin{cases} \Lambda, & i = q, \\ 0, & i \neq q, \end{cases}$$

$$\tilde{H}_i(\tilde{X}_0) \cong \begin{cases} \Lambda, & i = q,\ n{-}q{-}1, \\ 0, & i \neq q,\ n{-}q{-}1. \end{cases}$$

In fact $H_q(\tilde{C}_0) \cong H_q(\tilde{X}_0)$, and we can take the lift of a sphere representing a generator of $\pi_q(S^q \times S^{n-q-1}) \cong \mathbb{Z}$ which does not intersect F as a Λ-module generator, α, of both modules. ($\tilde{X}_0$ is the connected sum along the boundaries of an infinite number of copies of $S^q \times S^{n-q-1}$ − {the open neighborhood of a point}, and $\tilde{C}_0$ is the boundary connected sum of an infinite number of copies of $S^q \times D^{n-q}$.)

The Hurewicz map

$$h_q : \pi_q(\tilde{X}_0) \to H_q(\tilde{X}_0) \cong H_q(\tilde{C}_0)$$

is an epimorphism. This follows immediately if $q = 1$ by the abelianization map. For $q > 1$, we note that $\pi_1(\tilde{X}_0) = 0$ using the Van Kampen theorem, and then the Hurewicz theorem applies. Since $\pi_q(\tilde{X}_0)$ is isomorphic to a subgroup of $\pi_q(X_0)$, we can represent any element of $H_q(\tilde{C}_0)$ by the lift of an embedded sphere in X_0 to $\tilde{X}_0$. The embedding is possible since $2q < n - 1$. In particular, we choose an embedded sphere, S, in X_0 whose lift represents $p(t)\alpha$ in $H_q(\tilde{C}_0)$. We will attach a handle to $S^q \times D^{n-q}$ along S to create a new manifold which will turn out to be D^n. The image of L under the modification is the desired knotted disk.

In particular, the dimensions are sufficient for us to embed $S^q \times D^{n-q-1}$ as a tubular neighborhood of S in $\partial(S^q \times D^{n-q})$, and we identify this neighborhood with the first term of the boundary

$$\partial(D^n) = \partial(D^{q+1} \times D^{n-q-1}) = (S^q \times D^{n-q-1}) \cup_{S^q \times S^{n-q-2}} (D^{q+1} \times S^{n-q-2})$$

to form the new manifold $\Delta \cong (S^q \times D^{n-q}) \cup_{S \times D^{n-q-1}} D^n$. On the boundary, this gives the surgery which transforms $S^q \times S^{n-q-1}$ into

$$\partial\Delta = (S^q \times S^{n-q-1} - S \times D^{n-q-1}) \cup_{S \times S^{n-q-2}} (D^{q+1} \times S^{n-q-2}).$$

We first show that Δ is in fact isomorphic to D^n.

Since $p(1) = 1$, S represents the generator of $\pi_q(S^q \times D^{n-q}) \cong \mathbb{Z}$ and hence the generator of $H_q(S^q \times D^{n-q}) \cong \mathbb{Z}$. The reduced Mayer-Vietoris sequence immediately gives us that $\tilde{H}_i(\Delta) = 0$ for all i. On the boundary, following Levine [19, p. 547], we can choose S isotopic, in $S^q \times S^{n-q-1}$, to the standard embedded $S^q \times x_0$, $x_0 \in S^{n-q-1}$, provided $q < (n-2)/2$, and the modified boundary is then diffeomorphic to S^{n-1}. In fact, we can extend the isotopy on the boundary radially into $S^q \times D^{n-q}$. Then the standard $(q+1)$-handle attachment to $S^q \times D^{n-q}$ along a tubular neighborhood of $S^q \times x_0$ yields the n-disk.

For $q = (n-2)/2$, we have $n \geq 6$, and we will show that Δ is a disk through an application of the h-cobordism theorem [31]. First, it must be that $\dim(S^q \times S^{n-q-1}) = n-1$ is odd, and since $n \geq 6$, $S^q \times S^{n-q-1}$ is simply-connected. It then follows from simply-connected surgery theory (see [3, IV.2.13]) that $\tilde{H}_i(\partial\Delta) = 0$ for $i \leq (n-2)/2$ and then from Poincaré duality that

$$\tilde{H}_i(\partial\Delta) \cong \begin{cases} Z, & i = n-1\,, \\ 0, & i \neq n\,. \end{cases}$$

Furthermore, $\partial\Delta$ is simply connected by the Van Kampen theorem since $D^{q+1} \times S^{n-q-2}$ and $S^q \times S^{n-q-1} - S \times D^{n-q-1}$ are both simply-connected, the latter because $S^q \times S^{n-q-1}$ is simply connected and $S \times D^{n-q-1}$ is homotopy equivalent to a subset of codimension > 2. We now wish to show that $\partial\Delta$ is homotopy equivalent to $\Delta - B_\epsilon(x)$, where $B_\epsilon(x)$ is a small open ball neighborhood of a point, x, in $\operatorname{int}(\Delta)$. Δ is also simply connected by an easy application of the Van-Kampen theorem, and therefore so is $\Delta - B_\epsilon(x)$, which is homotopy equivalent to $\Delta - \{\text{a point}\}$. Since $\tilde{H}_i(\Delta - B_\epsilon(x)) = 0$ except in dimension $n-1$ (by an easy long exact sequence argument for the pair $(\Delta, \Delta - B_\epsilon(x))$), the inclusion $\imath : \partial\Delta \to \Delta - B_\epsilon(x)$ induces an isomorphism of H_i, $i \neq n-1$. It also induces the isomorphism in dimension $n-1$ from the long exact sequence of the pair, since

$$H_n(\Delta - B_\epsilon(x), \partial\Delta) \cong H^0(\Delta - B_\epsilon(x), \partial\bar{B}_\epsilon(x)) = 0,$$

using Lefschetz duality, and

$$\begin{aligned} H_{n-1}(\Delta - B_\epsilon(x), \partial\Delta) &\cong H^1(\Delta - B_\epsilon(x), \partial B_\epsilon(x)) \\ &\cong \operatorname{Hom}(H_1(\Delta - B_\epsilon(x), \partial B_\epsilon(x)), \mathbb{Z}) = 0, \end{aligned}$$

using Lefschetz duality, the universal coefficient theorem, the reduced long exact sequence of the pair, and the simple-connectivity of each term of the pair. Therefore, $\imath : \partial\Delta \to \Delta - B_\epsilon(x)$ is a homotopy equivalence by the Whitehead theorem, since it is a homology equivalence of simply connected spaces. That $\imath : \partial\bar{B}_\epsilon(x) \cong S^{n-1} \to \Delta - B_\epsilon(x)$ is a homotopy equivalence follows similarly, and the h-cobordism theorem applies to tell us that $\Delta - B_\epsilon(x) \cong S^{n-1} \times I$. Filling $B_\epsilon(x)$ back in gives us that $\Delta \cong D^n$ as claimed.

Finally, letting C denote $\Delta - L$, we need to show that $\tilde{C}$ has the desired homology modules. But we can form $\tilde{C}$ by attaching an infinite number of handles to $\tilde{C}_0$, attached along the infinite number of lifts of S which represent the homology elements $t^i p(t)\alpha$ obtained from $p(t)\alpha$ by the actions of the covering transformations. Then it is immediate from the Mayer-Vietoris sequence that

$$\tilde{H}_i(\tilde{C}) \cong \begin{cases} \Lambda/(p(t)), & i = q, \\ 0, & i \neq q, \end{cases}$$

which completes the proof of the proposition. □

Proposition 3.13. *Given a polynomial $p(t)$ such that $p(1) = \pm 1$ and integers q and n such that $(n-2)/2 < q < n-2$ and $n \geq 5$, there exists a knotted $D^{n-2} \subset D^n$ such that $\lambda_q(t) \sim p(t)$ and $\lambda_i(t) \sim 1$ for $0 < i < n-1$, $i \neq q$, where $\lambda_i(t)$ is the Alexander polynomial corresponding to $H_i(\tilde{C})$, $\tilde{C}$ the knotted disk complement.*

Proof. It suffices to construct a disk knot such that $H_q(\tilde{C}) \cong \Lambda/(p(t))$ and $H_i(\tilde{C}) = 0$ for $0 < i < n-1$, $i \neq q$. In fact, letting X denote $C \cap \partial D^n$ and $p = n - q - 1$ (so that $1 < p < n/2$), we construct a disk knot that

$$H_i(\tilde{C}, \tilde{X}) \cong \begin{cases} \Lambda/(p(t^{-1})), & i = p\,, \\ 0, & 0 < i < n-1,\ i \neq p, \end{cases}$$

which will suffice since the Alexander polynomials corresponding to $H_i(\tilde{C})$ and $H_{n-i-1}(\tilde{C}, \tilde{X})$ are related by $\lambda_i(t) \sim \mu_{n-i-1}(t^{-1})$ according to Theorem 3.1. We normalize $p(t)$ so that $p(1) = 1$.

We begin by embedding an $(n-2)$-disk, L, in $D^p \times S^{n-p}$, so that, in a neighborhood of a boundary point which is homeomorphic to the half-space $\mathbb{R}^{n+}$, L is embedded as a standard disk. In other words, $\partial D^{n-2} = S^{n-3}$ is an unknotted sphere within a neighborhood of a point in $\partial(D^p \times S^{n-p}) = S^{p-1} \times S^{n-p}$, $\text{int}(L)$ lies in $\text{int}(D^p \times S^{n-p})$, and $(L, \partial L)$ is null-homotopic in $(D^p \times S^{n-p}, \partial(D^p \times S^{n-p}))$. We also let ∂L bound a disk F in $S^{p-1} \times S^{n-p}$ so that $F \cup L$ bounds a manifold, V, homeomorphic to a standard D^{n-1}, such that $\text{int}(V)$ lies in $\text{int}(D^p \times S^{n-p})$. Let $C_0 = D^p \times S^{n-p} - L$, and let $\tilde{C}_0$ be the infinite cyclic covering associated with the kernel of the homomorphism $\pi_1(\tilde{C}_0) \to \mathbb{Z}$ defined by intersection number with V. Similarly, we have the covering $\tilde{X}_0$ of $X_0 = S^{p-1} \times S^{n-p} - \partial L$. Forming the infinite cyclic covers by cutting and pasting along V, it is clear that

$$\tilde{H}_i(\tilde{C}_0) \cong \begin{cases} \Lambda, & i = n-p\,, \\ 0, & i \neq n-p\,; \end{cases}$$

$$\tilde{H}_i(\tilde{X}_0) \cong \begin{cases} \Lambda, & i = p-1,\ n-p\,, \\ 0, & i \neq p-1,\ n-p\,. \end{cases}$$

In fact, $H_{n-p}(\tilde{C}_0) \cong H_{n-p}(\tilde{X}_0)$, and we can take as the generator, α, of both modules the lift of a sphere representing a generator of $H_{n-p}(S^{p-1} \times S^{n-p}) \cong \mathbb{Z}$ and which does not intersect F. ($\tilde{X}_0$ is the connected sum along the boundaries of a countably infinite number of copies of $D^p \times S^{n-p} - \{$the open neighborhood of a point$\}$, and $\tilde{C}_0$ is isomorphic to the boundary connected sum of an infinite number of copies of $D^p \times S^{n-p}$.) By the long exact sequence of the pair,

$$H_i(\tilde{C}_0, \tilde{X}_0) \cong \begin{cases} \Lambda, & i = p\,, \\ 0, & i \neq p\,. \end{cases}$$

Now for a lemma:

Lemma 3.14. *The Hurewicz map $h_p : \pi_p(\tilde{C}_0, \tilde{X}_0) \to H_p(\tilde{C}_0, \tilde{X}_0)$ is an epimorphism.*

Proof. We note first that $\tilde{C}_0$ is simply connected because it is the universal abelian cover of C_0 whose fundamental group is $\pi_1(C_0) = \mathbb{Z}$. This last statement is true because we can decompose C_0 into $N - L$, where N is the contractible neighborhood of the boundary in which we have embedded L, and $D^p \times S^{n-p} - N$. The latter is homotopy equivalent to $D^p \times S^{n-p}$, and $\pi_1(D^p \times S^{n-p}) = 0$ due to the range of p. $N - L$ is homotopy equivalent to the complement of the trivial sphere pair (S^n, S^{n-2}), and so $\pi_1(N - L) = \mathbb{Z}$. Since $n \geq 5$, an easy application of the Van-Kampen theorem proves the claim.

Then, using our knowledge of the homology of $\tilde{C}_0$ and the Hurewicz theorem [2, §VII.10], $\pi_i(\tilde{C}_0) = 0$, $i < n - p$. This implies by the long exact homotopy sequence that $\pi_i(\tilde{C}_0, \tilde{X}_0) \cong \pi_{i-1}(\tilde{X}_0)$, $1 < i < n - p$. Furthermore, $H_i(\tilde{C}_0, \tilde{X}_0) \cong H_{i-1}(\tilde{X}_0)$, $1 < i < n - p$, for the same homological reasons. Now, as in the proof of Proposition 3.12, $h_{p-1} : \pi_{p-1}(\tilde{X}_0) \to H_{p-1}(\tilde{X}_0)$ is an epimorphism, and since $p < n - p$, we have the following commutative diagram as a piece of the "homotopy-homology ladder":

$$\begin{tikzcd}
\pi_p(\tilde{C}_0, \tilde{X}_0) \arrow[r, "\cong"] \arrow[d, "h_p"'] & \pi_{p-1}(\tilde{X}_0) \arrow[d, "h_{p-1}"', "\text{onto}"] \\
H_p(\tilde{C}_0, \tilde{X}_0) \arrow[r, "\cong"] & H_{p-1}(\tilde{X}_0).
\end{tikzcd}$$

The truth of the lemma is now apparent. □

Since $\pi_p(\tilde{C}_0, \tilde{X}_0) \cong \pi_p(C_0, X_0)$, there is therefore a map $(D^p, S^{p-1}) \longrightarrow (C_0, X_0)$ whose lift represents the element $p(t^{-1})\beta$, where β is some generator of $H_p(\tilde{C}_0, \tilde{X}_0)$ as a Λ-module. Let (D, S) represent the image of the disk-sphere pair. Since $p < n/2$, we can choose (D, S) to be an embedded disk in C_0 whose boundary is an embedded sphere in X_0 and such that $\text{int}(D) \subset \text{int}(C_0)$. Note that, chasing the exact sequences around, this boundary must lift to the element $p(t^{-1})\alpha \in H_{p-1}(\tilde{X}_0)$, for some generator, α, of $H_{p-1}(\tilde{X}_0)$ as a Λ-module. Let R be an open tubular (regular) neighborhood of (D, S). We will show that $(D^p \times S^{n-p} - R, L)$, denoted by (Δ, L), is our desired knotted disk pair.

We begin by showing that $D^p \times S^{n-p} - R$ is the n-disk. As in the proof of Proposition 3.12, the fact that $p(1) = 1$ implies that S represents a generator of $\pi_{p-1}(S^{p-1} \times S^{n-p}) \cong \mathbb{Z}$, and hence, using the long exact homotopy sequence, (D, S) represents a generator of

$$\pi_p(D^p \times S^{n-p}, S^{p-1} \times S^{n-p}) \cong H_p(D^p \times S^{n-p}, S^{p-1} \times S^{n-p}) \cong \mathbb{Z}.$$

Hence (D, S) is homotopic in $(D^p \times S^{n-p}, S^{p-1} \times S^{n-p})$ to $D^p \times x_0$ for some $x_0 \in S^{n-p}$. If $2(p + 1) \leq n$, then this can be taken as an ambient isotopy, and then clearly $D^p \times S^{n-p} - R \cong D^p \times D^{n-p} \cong D^n$.

If $p = n/2 - 1/2$, then $2(p - 1) + 2 = n - 1$, so there is still an ambient isotopy of the boundary, $S^{p-1} \times S^{n-p}$, which takes S to $S^{p-1} \times x_0$ for some $x_0 \in S^{n-p}$, and this isotopy can be extended to an ambient isotopy of all of $D^p \times S^{n-p}$, [31]. When we form $\Delta = D^p \times S^{n-p} - R$, the new boundary will therefore be

$$(S^{p-1} \times S^{n-p} - S^{p-1} \times x_0 \times D^{n-p}) \cup_{S^{p-1} \times x_0 \times S^{n-p-1}} (D \times S^{n-p-1}) \cong S^{n-1},$$

since this is the standard torus decomposition of S^{n-1}. We will next show that Δ is contractible. Then, since the manifold Δ will be a homotopy n-disk bounded by an $n-1$ sphere, $\Delta \cup \bar{c}(\partial\Delta)$, (where $\bar{c}(\partial\Delta)$ indicates the closed cone on the boundary), will be a homotopy n-sphere. But $n \geq 5$, so $\Delta \cup \bar{c}(\partial\Delta)$ is in fact a true sphere by the Poincaré conjecture and Δ will be a true n-disk. It remains to show that Δ is contractible. Δ is simply-connected because, for p as given, $\pi_1(D^p \times S^{n-p}) = 0$ and $\dim(D) = p < n-2$. Together, these imply by a general position argument that $\pi_1(D^p \times S^{n-p} - D) \cong \pi_1(D^p \times S^{n-p} - R) = 0$, as well. To compute the homology of Δ, we observe that $H_i(\Delta) = 0$, $i \geq n$, since Δ is an n-manifold with boundary; $H_n(\Delta, \partial\Delta) \cong H_{n-1}(\partial\Delta) \cong \mathbb{Z}$, since $\partial\Delta \cong S^{n-1}$ and these are generated by the orientation classes; and $H_i(\Delta) \cong H_i(\Delta, \partial\Delta)$, $0 < i < n$, by the long exact sequence of the pair. By excision and homotopy equivalence,

$$\begin{aligned} H_i(\Delta, \partial\Delta) &\cong H_i(D^p \times S^{n-p}, (S^{p-1} \times S^{n-p}) \cup \bar{R}) \\ &\cong H_i(D^p \times S^{n-p}, (S^{p-1} \times S^{n-p}) \cup D). \end{aligned}$$

By the Mayer-Vietoris sequence, and using the fact that S is a generator of $\pi_{p-1}(S^{p-1} \times S^{n-p})$ and hence of $H_{p-1}(S^{p-1} \times S^{n-p})$,

$$\tilde{H}_i((S^{p-1} \times S^{n-p}) \cup D) \cong \begin{cases} \mathbb{Z}, & i = n-p,\ n-1, \\ 0, & i \neq n-p,\ n-1. \end{cases}$$

But the generators of $H_i((S^{p-1} \times S^{n-p}) \cup D)$ in dimensions $n-p$ and n, respectively, are the generators of $H_i(S^{p-1} \times S^{n-p})$ in the same dimensions (note that D has no simplices of dimension $> p$). The former is also a generator of $H_{n-p}(D^p \times S^{n-p})$, and the latter is the boundary of the orientation class of $H_n(D^p \times S^{n-p}, S^{p-1} \times S^{n-p})$. Therefore, using these isomorphisms, the long exact sequence yields that $H_{n-p}(D^p \times S^{n-p}, (S^{p-1} \times S^{n-p}) \cup D) = 0$, $0 < i < n$, which, by our calculations, shows that $\tilde{H}_i(\Delta) = 0$, $i > 0$. Therefore, by the Whitehead Theorem, Δ is contractible, and we have finished proving that $\Delta \cong D^n$.

For the last step in the proof of the proposition, we begin by fixing some notation. Let C denote $\Delta - L$. We can lift $(D, S) \subset (C_0, X_0)$ to an infinite number of copies (D_i, S_i), $-\infty < i < \infty$, corresponding to the translates of a lift of (D, S) under the covering translations, and similarly we lift the neighborhood R to an infinite number of R_i. Let $\tilde{D}$, $\tilde{S}$, and $\tilde{R}$ denote the disjoint unions $\amalg_i D_i$, $\amalg_i S_i$, and $\amalg_i R_i$, respectively. Then $\tilde{C}_0 - \tilde{R}$ covers $C_0 - R \cong C$. Furthermore, let X denote the manifold $(X_0 - S \times D^{n-p}) \cup_{S \times S^{n-p-1}} (D \times S^{n-p-1})$ which results as the new complement of ∂L in $\partial\Delta$. Then the cover $\tilde{X}$ corresponds to $(\tilde{X}_0 - \tilde{R}) \cup_{\tilde{S}\times S^{n-p-1}} (\tilde{D}\times S^{n-p-1})$. We show that the homology of $(\tilde{C},\tilde{X})$ is as desired.

By excision, $H_i(\tilde{C},\tilde{X}) \cong H_i(\tilde{C}_0, \tilde{X}_0 \cup \tilde{R})$. Let us denote $\tilde{X}_0 \cup \tilde{R}$ by $\tilde{Y}$. Since

$$\tilde{H}_i(\tilde{X}_0) \cong \begin{cases} \Lambda, & i = p-1,\ n-p\,, \\ 0, & i \neq p-1,\ n-p\,, \end{cases}$$

and since the lifts S_i represent $t^i\lambda(t^{-1})\alpha \in H_p(\tilde{X}_0)$, an easy Mayer-Vietoris sequence argument gives

$$\tilde{H}_i(\tilde{Y}) \cong \begin{cases} \dfrac{\Lambda}{p(t^{-1})}, & i = p-1\,, \\ \Lambda, & i = n-p\,, \\ 0, & i \neq p-1,\ n-p\,. \end{cases}$$

Then using

$$\tilde{H}_i(\tilde{C}_0) \cong \begin{cases} \Lambda, & i = n-p\,, \\ 0, & i \neq n-p\,, \end{cases}$$

the long exact sequence of the pair $(\tilde{C}_0,\tilde{Y})$ gives $H_p(\tilde{C}_0,\tilde{Y}) = \Lambda/p(t^{-1})$. The only other part of the sequence that bears checking is where we have

$$\begin{tikzcd} H_{n-p+1}(\tilde{C}_0) \arrow[r] \arrow[d] & H_{n-p+1}(\tilde{C}_0,\tilde{Y}) \arrow[r] \arrow[d] & H_{n-p}(\tilde{Y}) \arrow[r, "i"] \arrow[d] & H_{n-p}(\tilde{C}_0) \arrow[r] \arrow[d] & H_{n-p}(\tilde{C}_0,\tilde{Y}) \arrow[r] \arrow[d] & H_{n-p-1}(\tilde{Y}) \arrow[d] \\ 0 \arrow[r] & ? \arrow[r] & \Lambda \arrow[r] & \Lambda \arrow[r] & ? \arrow[r] & 0. \end{tikzcd}$$

But the isomorphism $H_{n-p}(\tilde{Y}) \cong \Lambda$ comes from the isomorphism $H_{n-p}(\tilde{X}_0) \xrightarrow{\cong} H_{n-p}(\tilde{Y})$ induced by inclusion in the Mayer-Vietoris sequence we used, and we already know that $H_{n-p}(\tilde{X}_0) \xrightarrow{\cong} H_{n-p}(\tilde{C}_0)$ is induced by inclusion (see above). Therefore, the map i is an isomorphism and

$$H_i(\tilde{C},\tilde{X}) \cong \begin{cases} \dfrac{\Lambda}{p(t^{-1})}, & i = p\,, \\ 0, & 0 < i < n-1,\ i \neq p\,, \end{cases}$$

as claimed. $\square$

Putting the results of these propositions together yields the following classification of polynomials which can be realized as the Alexander polynomials, λ_i, of a disk knot.

Theorem 3.15. *Given polynomials $p_i(t) \in \Lambda$, $0 < i < n-1$, $n \geq 4$, such that $p_i(1) = \pm 1$ for each i and $p_{n-2}(t) | p_1(t^{-1})$, there exists a knotted embedding $S^{n-2} \hookrightarrow S^n$ with at most isolated point singularities such that the Alexander polynomials, $\lambda_i(t)$, of the knot are the given polynomials.*

Proof. The case $n = 4$ has already been show. Suppose $n \geq 5$. The necessity of the conditions on the $\lambda_i(t)$ has been shown above in Section ***3.3***. For the sufficiency, let $p_1(t) = p_{n-1}(t^{-1})r(t)$. By [19], there is a locally-flat knot $S^{n-2} \subset S^n$ whose first and $(n-2)$nd Alexander polynomials are $p_{n-1}(t^{-1})$ and $p_{n-1}(t)$, respectively, and whose other Alexander polynomials are all 1. By Propositions 3.12 and 3.13, above, we can form $n-3$ separate knotted disk pairs such that the first pair has first Alexander polynomial $r(t)$, the i^{th} pair has i^{th} Alexander polynomial $\lambda_i(t)$, $1 < i < n-2$, and all the rest of the Alexander polynomials are 1. Then, taking the cone on the boundary of each knotted disk pair gives a knotted sphere pair with point singularity, $S^{n-2} \subset S^n$, and with the same Alexander polynomials. Taking the knot sum of all of these knots (with the connections being made in the neighborhoods of locally-flat points of the embeddings) gives the desired knot because Alexander polynomials multiply under knot sum. $\square$

As a corollary and sample application, we can re-prove the following known result concerning the Alexander polynomials of locally-flat slice sphere knots.

Corollary 3.16. *For any $n \geq 3$ and collection of polynomials $p_i(t) \in \Lambda$, $0 < i \leq \lfloor (n-1)/2 \rfloor$, such that $p_i(1) = \pm 1$ and, if n is odd, $p_{(n-1)/2}(t) \sim r(t)r(t^{-1})$ for some $r(t)$, there is a locally flat slice knot $S^{n-2} \subset S^n$ whose* i*th Alexander polynomials, $\lambda_i(t)$, $0 < i \leq \lfloor (n-1)/2 \rfloor$, are the $p_i(t)$. These conditions on $p_{(n-1)/2}(t)$ are also necessary. (Note that this also determines the Alexander polynomials for $\lfloor (n-1)/2 \rfloor < i < n-1$, as well, since $\lambda_i(t) \sim \lambda_{n-i-1}(t^{-1})$ for locally flat knots.)*

Proof. The necessity that $p_i(1) = \pm 1$ is proven in [19].

We construct $\lfloor (n-1)/2 \rfloor$ distinct locally flat slice knots such that the i^{th} Alexander polynomial of the i^{th} knot is $p_i(t)$ and the rest of the Alexander polynomials (for $i \leq \lfloor (n-1)/2 \rfloor$) are 1. Then our desired knot is the knot sum of these, since Alexander polynomials multiply under knot sum and the knot sum of slice knots is slice.

Consider the long exact sequence of the pair $(\tilde{C}, \tilde{X})$ for the complement of a knotted disk pair $D^{n-1} \subset D^{n+1}$. By Theorem 3.15, there is such a knotted disk pair whose Alexander polynomial corresponding to $H_i(\tilde{C})$ is $p_i(t)$ and such that $H_p(\tilde{C}) = 0$ for all other p, $0 < p < n$. This implies by Corollary 3.5 that $H_{n-i}(\tilde{C}, \tilde{X})$ has Alexander polynomial $p_i(t^{-1})$ and all other $H_p(\tilde{C}, \tilde{X}) = 0$. For $i < (n-1)/2$, we obtain immediately from the long exact sequence of the pair that the boundary knot with complement X has the desired homology. In fact, since $n-i > i+1$ in this case, the exact sequence implies that $H_i(\tilde{X}) \cong H_i(\tilde{C})$ and $H_p(\tilde{X}) = 0$, $0 < p \leq \lfloor (n-1)/2 \rfloor$, $p \neq i$. So the boundary knot is the desired slice knot with $\lambda_i(t) \sim p_i(t)$, $\lambda_{n-i-1}(t) \sim p(t^{-1})$, and no other non-trivial Alexander polynomials.

For $i = (n-1)/2$, the necessity that $\lambda_{(n-1)/2}(t) \sim r(t)r(t^{-1})$ follows just as in Corollary 3.8 for the case of the classical slice knots, where $n = 3$, by using the product formula (3.11) which relates the Alexander polynomials corresponding to the homology modules of $\tilde{X}$, $\tilde{C}$, and $(\tilde{C}, \tilde{X})$. Note that for slice knots, this condition along with $p_{(n-1)/2}(1) = \pm 1$ implies Levine's necessary condition (d) for Alexander polynomials [19].

For the construction, we consider the knotted disk pair whose $(n-1)/2$-th Alexander polynomial corresponding to $H_{(n-1)/2}(\tilde{C})$ is $r(t)$ and whose other Alexander polynomials, corresponding to the other $H_p(\tilde{C})$, are all trivial. Such a disk pair exists by the theorem and the fact that we must have $r(1) = \pm 1$ in order to have $p_{(n-1)/2}(1) = \pm 1$. Then the Alexander polynomials corresponding to the $H_p(\tilde{C}, \tilde{X})$ are $r(t^{-1})$ for $p = (n+1)/2$ and trivial otherwise. It then follows from the long exact sequence of the pair $(\tilde{C}, \tilde{X})$ that $H_p(\tilde{X}) = 0$ for $0 < p < n-1$, $p \neq (n-1)/2$. Lastly, from the product formula of Corollary 3.6, which relates the three sets of Alexander polynomials, it must be that the Alexander polynomial $\lambda_{(n-1)/2}(t) \sim r(t)r(t^{-1})$.

For the cases $i < (n-1)/2$, we can also note the existence of the given slice knots by observing that our procedure for creating disk pairs with given Alexander polynomials in this range restricts on the boundary to Levine's procedure [19] for creating knotted sphere pairs with the same prescribed Alexander polynomials. $\square$

3.5.3. *Realization of all Alexander polynomials.* So far, we have stated all of our realizability conditions for disk knots in terms of the Alexander polynomials λ_i which correspond to the Γ modules $H_i(\tilde{C}; \mathbb{Q})$. We now turn to a characterization which simultaneously involves all of the Alexander polynomials we have discussed: λ_i, μ_i, and ν_i, which correspond, respectively, to $H_i(\tilde{C}; \mathbb{Q})$, $H_i(\tilde{C}, \tilde{X}; \mathbb{Q})$, and $H_i(\tilde{X}; \mathbb{Q})$. It will prove more natural, however, to consider the corresponding *subpolynomials* (see Section 2). In fact, the long exact reduced homology sequence of the pair $(\tilde{C}, \tilde{X})$ yields an exact polynomial sequence

$$1 \to \lambda_{n-2} \to \mu_{n-2} \to \nu_{n-3} \to \cdots \to \nu_1 \to \lambda_1 \to \mu_1 \to 1$$

with all polynomials in primitive form. By the discussion in Section 2, this gives rise to a sequence of primitive polynomials of the form

$$1 \to c_{n-2} \to c_{n-2}a_{n-3} \to a_{n-3}b_{n-3} \to \cdots \to a_1 b_1 \to b_1 c_1 \to c_1 \to 1.$$

As noted there, knowledge of the a_i, b_i, and c_i is equivalent to knowledge of the λ_i, μ_i, and ν_i. While we have been referring to λ_i, μ_i, and ν_i as the Alexander polynomials of the disk, we will refer to a_i, b_i, and c_i as the *Alexander subpolynomials*.

With this notation, we can observe the following lemma:

Lemma 3.17. *For a locally-flat disk knot $D^{n-2} \subset D^n$, $c_i(t) \sim c_{n-i-1}(t^{-1})$ for $0 < i < n-1$, $a_j(t) \sim b_{n-j-2}(t^{-1})$ for $0 < j < n-2$, and each of these polynomials evaluates to ± 1 at $t = 1$.*

Proof. The last statement follows from the fact that each of the a_i, b_i, and c_i is a primitive polynomial in Λ which divides another primitive polynomial which evaluates to ± 1 at $t = 1$. The other results follow by induction from the outside of the sequence to the inside using Theorem 3.1 and Corollary 3.5 (which states that $\lambda_i(t) \sim \mu_{n-i-1}(t^{-1})$) together with Levine's [19] necessary conditions for the Alexander polynomials of the locally-flat boundary sphere knot (which states that $\nu_i(t) \sim \nu_{(n-1)-i-1}(t^{-1})$). First, we note that $c_{n-2}(t) \sim \lambda_{n-2}(t) \sim \mu_1(t^{-1}) \sim c_1(t^{-1})$. Next, $\mu_{n-2}(t) \sim \lambda_1(t^{-1})$, but $\mu_{n-2}(t) \sim c_{n-2}(t)a_{n-3}(t)$ and $\lambda_1(t^{-1}) \sim b_1(t^{-1})c_1(t^{-1})$. It follows that $a_{n-3}(t) \sim b_1(t^{-1})$. The lemma is established by continuing this procedure to the middle of the exact polynomial sequence. $\square$

In these terms, we can now completely classify the Alexander polynomials of disk knots with the exception of some extra middle-dimensional information which we will study in the next section. For now, we impose one unnecessary condition for the purpose of collecting the results that derive from our work in this section.

Theorem 3.18. *Let $\nu_i(t)$, $\lambda_i(t)$, and $\mu_i(t)$ denote the Alexander polynomials of a knotted $D^{n-2} \subset D^n$ corresponding to $H_i(\tilde{X})$, $H_i(\tilde{C})$, and $H_i(\tilde{C}, \tilde{X})$, respectively, and suppose $n \geq 4$. Recall that we can assume these to be primitive in Λ. Let $q_i(t)$, $0 < i \leq \lfloor (n-2)/2 \rfloor$; $r_i(t)$, $0 < i \leq \lfloor (n-2)/2 \rfloor$; and $p_i(t)$, $0 < i \leq \lfloor (n-1)/2 \rfloor$, be polynomials in Λ satisfying the following properties:*

1. (a) $q_i(1) = \pm 1$ *;*
   (b) $r_i(1) = \pm 1$ *;*
   (c) $p_i(1) = \pm 1$ *; and hence, in particular, they must each also be primitive in Λ.*

2. *There exist polynomials $a_i(t)$, $b_i(t)$, and $c_i(t)$, primitive in Λ, such that*
   (a) $q_i(t) \sim a_i(t)b_i(t)$, $0 < i < \lfloor (n-2)/2 \rfloor$ *;*
   (b) $q_{(n-2)/2}(t) \sim b_{(n-2)/2}(t)b_{(n-2)/2}(t^{-1})$, *if n is even ;*
   (c) $r_i(t) \sim b_i(t)c_i(t)$, $0 < i \leq \lfloor (n-2)/2 \rfloor$ *;*
   (d) $p_i(t) \sim c_i(t)a_{i-1}(t)$, $0 < i \leq \lfloor (n-1)/2 \rfloor$ *(taking $a_0(t) = 1$) ;*
   (e) $c_{(n-1)/2}(t) \sim c_{(n-1)/2}(t^{-1})$, *if n is odd ;*
   (f) $c_{(n-1)/2}(-1) = \pm$an odd square *if $n = 2k+1$, k even, and $c_{(n-1)/2}(t)$ is in normal form (defined below).*

Then, there exists a knotted $D^{n-2} \subset D^n$ such that $\nu_i(t) \sim q_i(t)$, $\lambda_i(t) \sim r_i(t)$, and $\mu_i(t) \sim p_i(t)$ in the relevant ranges. Note that this determines all of the Alexander polynomials using $\nu_i(t) \sim \nu_{n-i-2}(t^{-1})$ and $\lambda_i(t) \sim \mu_{n-i-1}(t^{-1})$. Furthermore, these conditions are necessary except for condition (2f).

Proof. Most of the necessity has already been shown either above or in [19]. Conditions (2e) and (2b) are necessary by Lemma 3.17 since $n - ((n-1)/2) - 1 =$

$(n-1)/2$ is an integer for n odd, and, if n is even, $(n-2)/2$ is an integer, $n-((n-2)/2)-2=(n-2)/2$, and $\nu_{(n-2)/2} \sim a_{(n-2)/2}(t)b_{(n-2)/2}(t)$. Condition (2f) is not necessary and will be weakened in the following section. Following Levine [19], the normal form for $c_{(n-1)/2}(t)$ is the member, $c(t)$, of its similarity class in Λ which satisfies $c(t)=c(t^{-1})$ and $c(1)=1$. Due to the other conditions on $c_{(n-1)/2}(t)$, it is always possible to find such a similar polynomial with these properties (see [19, §1.5]).

To show that there is such a knotted disk, we will find a series of knots whose only nontrivial Alexander polynomials are the $a_i(t)$, $b_i(t)$, or $c_i(t)$ in the correct dimesnions and then take a knot sum. In this case, by a knot sum of two knots we mean the following: Suppose we have two knotted disk pairs $D_1^{n-2} \subset D_1^n$ and $D_2^{n-2} \subset D_2^n$. We can first take the connected sum of the D_i^n to form a new disk, D^n, in which the two knotted $(n-2)$-disks are embedded disjointly. We can then connect the two knotted disks by an unknotted tube, $T \cong D^1 \times D^{n-2}$, which connects a neighborhood of a point of ∂D_1^{n-2} in D_1^{n-2} with a neighborhood of a point of ∂D_2^{n-2} in D_2^{n-2} with the reverse orientation. The knot sum is the closure of

$$(D_1^{n-2} \cup D_2^{n-2} \cup T) - (\mathrm{int}(T) \cup (T \cap (D_1^{n-2} \cup D_2^{n-2} \cup \partial D^n)).$$

On the boundary this is the usual knot sum of the knots given by ∂D_1^{n-2} and ∂D_2^{n-2} in ∂D^n. From the usual Mayer-Vietoris considerations, the Alexander polynomials multiply under this knot sum. (Alternatively, we could define knot sum by coning on the boundary of the knotted disk pairs to create possibly non-locally flat sphere pairs, taking the usual knot sum with the connections in neighborhoods of locally flat points, and following our original procedure for turning such a singular knot back into a knotted disk pair.)

For the remainder of the proof, the term "lower dimensional Alexander polynomial" will refer to the Alexander polynomials in the lower dimensions listed above. We make this definition in order to avoid repetition of conditions that arise from the duality in the upper dimensions.

The knotted disk pairs with $\nu_j(t) \sim \lambda_j(t) \sim b_j(t)$ for a single j, $0 < j < \lfloor (n-2)/2 \rfloor$, and all other lower dimensional Alexander polynomials trivial is constructed in the proof of Corollary 3.16, as is the case where $\nu_{(n-2)/2}(t) \sim b_{(n-2)/2}(t)b_{(n-2)/2}(t^{-1})$, $\lambda_{(n-2)/2}(t) \sim b_{(n-2)/2}(t)$ and all other lower dimensional Alexander polynomials are trivial.

To construct the knotted disk pair with $\lambda_j(t) \sim \mu_j(t) \sim c_j(t)$ for a single j, $0 < j \le \lfloor (n-1)/2 \rfloor$, and all other lower dimensional Alexander polynomials trivial, we have stipulated the sufficient conditions [19] to construct a locally flat knot $S^{n-2} \subset S^n$ with $c_j(t)$ as its only non-trivial lower dimensional Alexander polynomial in the usual sense. Then, we can take the trivial slicing of this knot by excising a ball neighborhood of a point of the knot. Then $\lambda_j(t) \sim c_j(t)$; all of the $\nu_i(t) \sim 1$; $\mu_j(t) \sim c_j(t)$, by the long exact sequence; and all other lower dimensional Alexander polynomials are trivial.

To construct the knotted disk pair with $\mu_{j+1}(t) \sim \nu_j(t) \sim a_j(t)$ for a single j, $0 < j < \lfloor (n-2)/2 \rfloor$, and all other lower dimensional Alexander polynomials trivial, we recall our procedure from Proposition 3.13. As in that proof, we can construct a knotted disk pair with $\mu_{j+1}(t) \sim a_j(t)$, $0 < j < \lfloor (n-2)/2 \rfloor$, and all other $\mu_i(t)$ and all lower dimension $\lambda_i(t)$ trivial. But the construction restricted to the boundary was a surgery which we can check to be equivalent to Levine's method [19] for creating a knotted $S^{n-1} \subset S^{n-3}$ whose only lower dimensional Alexander polynomial is $\nu_j(t) \sim a_j(t)$. Therefore, this knotted disk pair is the desired one.

By taking the knots sums of these constructions as indicated above, we obtain our desired knot. $\square$

3.6. The middle dimension polynomial. We now turn to the case of realizing Alexander polynomials in the middle dimension of a $(2q+1)$-disk knot, q even. In particular, for $q > 2$, we give a characterization of the polynomials $c(t) \in \Lambda$ such that there exists a locally-flat knotted disk pair $D^{2q-1} \subset D^{2q+1}$ such that $c(t)$ is the Alexander polynomial factor shared by $H_q(\tilde{C})$ and $H_q(\tilde{C}, \tilde{X})$. Equivalently, $c(t)$ is the Alexander polynomial associated to the modules $\ker(\partial_*)$ and $\operatorname{cok}(i_*)$ in the long exact sequence of the pair

$$\longrightarrow H_k(\tilde{X}) \xrightarrow{i_*} H_k(\tilde{C}) \longrightarrow H_k(\tilde{C}, \tilde{X}) \xrightarrow{\partial_*} H_{k-1}(\tilde{X}) \longrightarrow .$$

We will show, in particular, that for any realizable $c(t)$ there exists such a knot with all other Alexander polynomials (and Alexander subpolynomials) equal to 1 so that $c(t)$ will be the only non-trivial Alexander polynomial of $H_*(\tilde{C})$ and $H_*(\tilde{C}, \tilde{X})$. We can then use the usual procedure of taking connected sums of disk knots to combine this with other Alexander polynomials.

We will in fact show something more. We will realize entire Λ-modules and intersection pairings. First, we need a few definitions. Following Levine [21], we say that a Λ module, A, is of *type K* if it is finitely generated and multiplication by $t-1$ induces an automorphism of A. It is a standard fact, see e.g. [21], that the Alexander modules of locally-flat sphere knots must be of type K. The standard proof following Milnor [25] extends easily to disk knots. We provide it here to add the few words relevant for the cases we will consider.

Lemma 3.19. *Let $D^{n-2} \subset D^n$ be a locally flat disk knot. Then the Λ-modules $H_i(\tilde{X})$, $H_i(\tilde{C})$, and $H_i(\tilde{C}, \tilde{X})$, $i > 0$, are all of type K.*

Proof. That the modules are finitely generate follows from the usual argument stemming from the fact that there is a one-to-one correspondence between generators of the chain complexes of the knot exteriors (which are finite complexes) and the generators of the chain complexes of the infinite cyclic covers as Λ-modules. Specifically, we choose one lift of each simplex.

Now, let W stand for C, X, or the pair (C, X). Then we have an exact sequence

$$0 \longrightarrow C_i(\tilde{W}) \xrightarrow{t-1} C_i(\tilde{W}) \longrightarrow C_i(W) \longrightarrow 0,$$

which generates the long exact homology sequence

$$\longrightarrow H_i(\tilde{W}) \xrightarrow{t-1} H_i(\tilde{W}) \longrightarrow H_i(W) \xrightarrow{\partial_*} . \tag{3.12}$$

But by Alexander duality (Alexander duality for a ball), X and C are homology circles, and it is easy to see that (C, X) is a homology ball. Therefore, it is immediate for $i \geq 2$ that $t - 1$ is an automorphism of the homology groups of the covers of C and X, and in all dimensions $i > 0$ for $W = (C, X)$.

For the remaining cases, we note that the long exact sequence must terminate as

$$\xrightarrow{0} H_1(W) \xrightarrow{\cong} H_0(\tilde{W}) \xrightarrow{0} H_0(\tilde{W}) \xrightarrow{\cong} H_0(W) \longrightarrow 0.$$

The rightmost isomorphism is induced by the projection of a point that generates $H_0(\tilde{W})$ to a point that generates $H_0(W)$. To see the other isomorphism, observe that in the diagram chase that defines the boundary map of the long exact sequence of homology, the generator of $H_1(W)$, a meridian of the boundary sphere knot (disk knot), gets lifted to a 1-chain in the cover whose boundary is $(t - 1)x$ for some point x in $\tilde{W}$ representing an element of $C_0(\tilde{W})$. This is the image of x under the map $t - 1$. Thus the image of a generator of $H_1(W)$ goes to a generator of $H_0(\tilde{W})$ under the boundary map of the long exact sequence. The sequence now shows that $t - 1$ is also a homology automorphism of the cover for $i = 1$. ◻

3.6.1. *The Blanchfield pairing.* We will also need the *Blanchfield pairing* on the infinite cyclic cover. We summarize its construction and properties following [21]. More details can be found in the references cited there. (Note: for notational convenience we introduce the symbol $(\,,)$ to represent the general Blanchfield pairing and reserve $\langle\,,\rangle$ for the induced middle dimensional self-pairing.)

For M a compact m-dimensional PL-manifold with boundary which admits a regular cover with group of covering transformations π, one first defines an intersection pairing on the chain groups of the covers, $C_q(\tilde{M}, \partial\tilde{M}) \times C_{m-q}(\tilde{M}^1) \to \mathbb{Z}[\pi]$, $\alpha \times \beta \to \alpha \cdot \beta$, where the chain groups are thought of as (left) $\mathbb{Z}[\pi]$ modules and M^1 represents the dual complex to the triangulation of M. This pairing is bilinear over $\mathbb{Z}$ and satisfies

1. $(g\alpha) \cdot \beta = g(\alpha \cdot \beta)$, for $g \in \pi$,
2. $\alpha \cdot \beta = (-1)^{q(m-q)}\overline{\beta \cdot \alpha}$, where the bar denotes the antiautomorphism of $\mathbb{Z}[\pi]$ induced by $\bar{g} = g^{-1}$ for $g \in \pi$,
3. $(\partial\alpha) \cdot \beta = (-1)^q \alpha \cdot (\partial\beta)$.

This induces a pairing on the appropriate homology groups.

Now assume $\pi = \mathbb{Z}$ and that $\alpha \in H_q(\tilde{M}, \partial\tilde{M})$ and $\beta \in H_{m-q-1}(\tilde{X})$ are Λ-torsion elements represented by chains $z \in C_q(\tilde{M}, \partial\tilde{M})$ and $w \in C_{m-q-1}(\tilde{M}^1)$. Then $z = (1/\lambda)\partial c$ for some $\lambda \in \Lambda = \mathbb{Z}[\mathbb{Z}]$ and $c \in C_{q+1}(\tilde{M}, \partial\tilde{M})$. Define $(\alpha, \beta) = (1/\lambda)c \cdot w \bmod \Lambda$. This induces a well-defined pairing $H_q(\tilde{M}, \partial\tilde{M}) \times$

$H_{m-q-1}(\tilde{M}) \to Q(\Lambda)/\Lambda$, where $Q(\Lambda)$ is the field of fractions of Λ. The pairing $(\,,)$ is conjugate linear, meaning that it is additive in each variable and $(\lambda\alpha,\beta) = \lambda(\alpha,\beta) = (\alpha,\bar{\lambda}\beta)$. The conjugation on elements of Λ is induced by $\bar{t} = 1/t$. Furthermore, if $m = 2q+1$, $(\,,)$ induces a self-pairing $\langle\,,\rangle : H_q(\tilde{M}) \times H_q(\tilde{M}) \to Q(\Lambda)/\Lambda$ by $\langle\alpha,\beta\rangle = (j_*\alpha,\beta)$, where $j_* : H_q(\tilde{M}) \to H_q(\tilde{M},\partial\tilde{M})$ is the map of the long exact sequence. This pairing is $(-1)^{q+1}$-Hermitian, meaning that $\langle\alpha,\beta\rangle = (-1)^{q+1}\overline{\langle\beta,\alpha\rangle}$.

We observe that, in the case of a disk knot, the arguments of [21, §5] carry over to show that $(\,,)$ is a non-singular pairing on the Z-torsion free parts of $H_q(\tilde{C})$ and $H_{n+1-q}(\tilde{C},\partial\tilde{C})$. If $n = 2q+1$, the induced pairing $\langle\,,\rangle$ on $H_q(\tilde{M})$ further induces a nondegenerate (though possibly singular) conjugate linear $(-1)^{q+1}$-Hermitian pairing on the $\mathbb{Z}$-torsion free part of $\operatorname{coim}(j_*)$ (though we will keep the same notation $\langle\,,\rangle$): To see that this is well-defined, we observe that if $\alpha+\beta,\gamma \in H_q(\tilde{M})$, $\beta \in \ker(j_*)$, then

$$\langle\alpha+\beta,\gamma\rangle = (j_*(\alpha+\beta),\gamma) = (j_*\alpha,\gamma) = \langle\alpha,\gamma\rangle\,.$$

By the Hermitian property, similar considerations hold for the second argument so that $\langle\,,\rangle$ only depends on $H_q(\tilde{M})/\ker(j_*)$. For the non-degeneracy, note that the non-singularity of $(\,,)$ implies that for every non-zero, non-$\mathbb{Z}$ torsion element $j_*\alpha \in H_q(\tilde{M},\partial\tilde{M})$, there is a non-$\mathbb{Z}$ torsion $\gamma \in H_q(\tilde{M})$ such that $\langle\alpha,\gamma\rangle = (j_*\alpha,\gamma) \neq 0$. But from the well-definedness argument above, if $\gamma \in \ker(j_*)$, then $\langle\alpha,\gamma\rangle = 0$. So γ has non-zero image when projected into $\operatorname{coim}(j_*)$. This establishes the non-degeneracy since such a γ exists for all such $j_*\alpha$.

In the arguments above, we can replace $H_i(\tilde{C},\partial\tilde{C})$ with $H_i(\tilde{C},\tilde{X})$, $i < n-2$, as follows: First observe that

$$\partial\tilde{C} = \tilde{X} \cup_{S^{n-3}\times\mathbb{R}} D^{n-2}\times\mathbb{R}\,,$$

so that $H_i(\partial\tilde{C},\tilde{X}) = H_i(D^{n-2},S^{n-3})$, by excision and homotopy equivalence. Therefore, by the long exact sequence of the pair, the map induced by inclusion, $j_* : H_i(\tilde{X}) \to H_i(\partial\tilde{C})$, is an isomorphism for $i < n-3$ and onto for $i = n-3$. Using long exact sequences and the five-lemma, this implies that $H_i(\tilde{C},\tilde{X}) \cong H_i(\tilde{C},\partial\tilde{C})$, $i < n-2$.

Summarizing part of this discussion gives the following result:

Proposition 3.20. *Let $D^{n-2} \subset D^n$ be a disk knot, $n = 2q+1$, $k > 0$. Let $f(A)$ denotes the Λ-module A modulo its $\mathbb{Z}$-torsion. Then $H_q(\tilde{C})$ and $H_q(\tilde{C},\tilde{X})$ are Λ-modules of type K, and the non-singular pairing $(\,,) : f(H_q(\tilde{C},\tilde{X})) \times f(H_q(\tilde{C})) \to Q(\Lambda)/\Lambda$ induces a nondegenerate conjugate linear $(-1)^{q+1}$-Hermitian pairing $f(\operatorname{coim}(j_*)) \times f(\operatorname{coim}(j_*)) \to Q(\Lambda)/\Lambda$.*

3.6.2. *Realization of middle dimensional pairings.* We will establish a converse to Proposition 3.20:

Proposition 3.21. *Let A be a $\mathbb{Z}$-torsion free Λ module of type K with a non-degenerate conjugate linear $(-1)^{q+1}$-Hermitian pairing $\langle\,,\rangle : A \times A \to Q(\Lambda)/\Lambda$. Then there exists a disk knot $D^{n-2} \subset D^n$, $n = 2q + 1$, $q > 2$, such that:*

1. $H_q(\tilde{C}) = A$,
2. $H_i(\tilde{C}) = 0$, $0 < i < n - 1$, $i \neq q$,
3. $H_i(\tilde{X}) = 0$, $0 < i < n - 2$, $i \neq q - 1$,
4. $H_{q-1}(\tilde{X}) = 0$ *is a* $\mathbb{Z}$*-torsion module,*
5. $H_i(\tilde{C}, \tilde{X}) = 0$, $0 < i < n - 1$, $i \neq q$,
6. *the pairing on* $H_q(\tilde{C})$ *is given by* $\langle\,,\rangle$. *(Note that* $H_q(\tilde{X}) = 0$ *implies that* $H_q(\tilde{C}) \cong A \cong \mathrm{coim}(j_*)$ *in the long exact sequence).*

Proof. By [21, Proposition 12.5], given such an A and $\langle\,,\rangle$, there exists a smooth compact $(2q + 1)$-dimensional manifold, C, such that $\pi_1(C) = \mathbb{Z}$, $H_q(\tilde{C}) = A$, $H_i(\tilde{C}) = 0$ for $i \neq 0, q$, and the given pairing $\langle\,,\rangle$ corresponds to the pairing on $H_q(\tilde{C})$. The proof consists of first being able to write the defining matrices for the presentation of A and the pairing with respect to the basis of presentation in certain forms, which follows from [21, Proposition 12.3] and the remarks before [21, Proposition 12.5] because A is of type K; and then ([21, Lemma 12.2]) constructing C using the matrix information to attach appropriate q-handles to

$$C_0 = (\#_{i=1}^m S^q \times D^{q+1})\#(S_1 \times D^{2q}),$$

where the presentation matrix has size $m\times m$ and $\#_{i=1}^m S^q \times D^{q+1}$ denotes the connected sum of m copies of $S^q \times D^{q+1}$. This C will be our disk knot complement.

We observe that C is a homology circle: $H_1(C) = \pi_1(C) = \mathbb{Z}$ as above, and the triviality in the remaining dimensions, $i > 1$, follows from Milnor's exact sequence (3.12) and A being of type K. As Levine notes in Proposition 12.6 of [21], we also have $\mathbb{Z} = \pi_1(C) \cong \pi_1(C - K) \cong \pi_1(\partial C)$, where K is the $(q + 1)$-dimensional subcomplex formed from the cores of the handles added onto C_0: $C - K$ deformation retracts to ∂C, and the claim follows from general position since $q > 2$. Thus, we can add a 2-handle onto C along a generator of $\pi_1(\partial C)$ to obtain a manifold which is contractible (using the Hurewicz and Whitehead theorems) with simply-connected boundary, hence a disk by [33]. If $D^2 \times D^n$ is the attached handle, then our disk knot is $0 \times D^n$, the "cocore" of the handle. Clearly then C is the knot's exterior with modules and pairings as claimed.

It remains to show that properties (3), (4), and (5) hold. Again from the proof of [21, Proposition 12.6], $H_i(\partial\tilde{C}) = 0$ for $i \neq 0, q-1, q, 2q-1$, and $H_i(\tilde{C}, \partial\tilde{C}) = 0$, $i \neq q - 1, q, 2q - 1$. The argument uses the Hurewicz theorem, a version of Poincaré duality for coverings ([25] and [21]), and a universal coefficient short exact sequence for torsion Λ-modules. As noted above, $j_* : H_i(\tilde{X}) \to H_i(\partial\tilde{C})$ is an isomorphism for $i < n - 3$, so $H_i(\tilde{X}) = 0$, $i < q - 1$, and therefore

$$H_i(\tilde{X}) = 0\,, \quad q < i < n - 2,$$

by the duality of sphere knot modules [21]. Similarly,

$$H_i(\tilde{C}, \tilde{X}) = 0\,, \quad i < q \text{ or } q < i < n-1,$$

using the long exact sequence of the pair $(\tilde{C}, \tilde{X})$.

At this point we have all of the Alexander modules 0 except for $H_q(\tilde{C})$, $H_q(\tilde{C}, \tilde{X})$, $H_{q-1}(\tilde{X})$, and $H_q(\tilde{X})$. But $H_q(\tilde{X})$ must be 0 because the non-degeneracy of the pairing on C implies that the map

$$j_* : H_q(\tilde{C}) \to H_q(\tilde{C}, \tilde{X})$$

of the long exact sequence must be injective. It now follows from Levine's duality properties for the Alexander modules of locally-flat sphere knots (see [21]) that $H_{q-1}(\tilde{X})$ is a $\mathbb{Z}$-torsion module. ◻

3.6.3. *Matrix representations of the middle dimension module and its pairing; Characterization of the middle dimensional polynomial in these terms.* It is also useful to study these middle-dimensional Alexander modules using presentation matrices. We first examine the form that these matrices take. From the proof of Corollary 2.5, we know that $c_q(t)$ is the determinant of the presentation matrix of the kernel of the map

$$\partial_* : H_q(\tilde{C}, \tilde{X}; \mathbb{Q}) \to H_{q-1}(\tilde{X}; \mathbb{Q})$$

in the long exact sequence of the pair. Let us denote this kernel module by H. Equivalently, it is the determinant of the presentation matrix of the isomorphic coimage of the map

$$p_* : H_q(\tilde{C}; \mathbb{Q}) \to H_q(\tilde{C}, \tilde{X}; \mathbb{Q}).$$

We will refer to this module as $\bar{H}$.

To obtain a presentation matrix for H (or $\bar{H}$), recall the Mayer-Vietoris sequences (3.1) and (3.2) used to obtain the presentation matrices for the Alexander polynomials. The long exact sequences of the *rational* homology of the pairs (V, F) and (Y, Z) must split at each term as exact sequences of vector spaces; in other words, each is isomorphic to an exact sequence of vector spaces of the form

$$\longrightarrow A \oplus B \to B \oplus C \to C \oplus D \longrightarrow \;.$$

This splitting and exactness is preserved under the tensor product with the free module Γ over $\mathbb{Q}$. Hence we obtain the following diagram which commutes owing to the obvious commutativity at the chain level induced by the maps in the Mayer-Vietoris sequence and by naturality of the homology functor. The 0 terms arise by truncation, using our knowledge that the Mayer-Vietoris sequences break into short exact sequences.

(3.13)

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & H_q(F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r] \arrow[d] & H_q(Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r] \arrow[d] & H_q(\bar{X};\mathbb{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_q(V;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "d_1"] \arrow[d, "r"'] & H_q(Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "e_1"] \arrow[d, "s"'] & H_q(\bar{C};\mathbb{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_q(V,F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "d_2"] \arrow[d, "\partial_*"'] & H_q(Y,Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "e_2"] \arrow[d, "\partial_*"'] & H_q(\bar{C},\bar{X};\mathbb{Q}) \arrow[r] \arrow[d, "\partial_*"'] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Let E and G denote, respectively, the kernels of the boundary maps ∂_* in $H_q(V,F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma$ and $H_q(Y,Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma$. Let J, K, and L be the respective cokernels of the boundary maps of which E, G, and H are the kernels. Then, by the snake lemma, we obtain an exact sequence

$$0 \longrightarrow E \xrightarrow{\ d\ } G \longrightarrow H \longrightarrow J \longrightarrow K \longrightarrow L \longrightarrow 0.$$

But note that by the splitting of the two leftmost (non-zero) vertical sequences in the diagram (3.13), J and K are direct summands of $H_{q-1}(F;\mathbb{Q}) \otimes \Gamma$ and $H_{q-1}(Z;\mathbb{Q}) \otimes \Gamma$, respectively. Hence the injectivity of the map $H_{q-1}(F;\mathbb{Q}) \otimes \Gamma \to H_{q-1}(Z;\mathbb{Q}) \otimes \Gamma$ in the Mayer-Vietoris sequence implies that the induced map $J \to K$ must also be injective. Therefore, we get an exact sequence

$$0 \longrightarrow E \xrightarrow{\ d\ } G \longrightarrow H \longrightarrow 0.$$

This sequence gives a presentation for H. In fact, E and G are certainly free Γ-modules (each being a rational vector space tensored with Γ over $\mathbb{Q}$), and the matrix representing d gives a presentation matrix for H. Note that the matrix for d is a submatrix (which we can arrange to be the upper left submatrix) of the matrix representing d_2. The generators of E and G are the elements $\{e_i \otimes 1\}$ and $\{g_i \otimes 1\}$, where $\{e_i\}$ and $\{g_i\}$ are the generators of the direct summands of $H_k(V,F;\mathbb{Q})$ and $H_q(Y,Z;\mathbb{Q})$ which are the images of $H_q(V;\mathbb{Q})$ and $H_q(Y;\mathbb{Q})$ under the projection maps of the exact sequences of the pairs. Furthermore, d must be represented by a square matrix: If it had more columns than rows, then there would be more generators than relations in H which is impossible since we know that H is a Γ-torsion module; and if it had more rows than columns, then since the elements in the summand E map only into the summand G and d_2 is square, d_2 would be forced to have determinant 0, which is also impossible as we saw in the proof of Theorem 3.1. Hence the matrix of d gives a square presentation of H, which we can take to be the upper left $m \times m$ submatrix of d_2, by changing bases if necessary. Similar considerations give the isomorphic presentation of the coimages

$$0 \longrightarrow \bar{E} \xrightarrow{\bar{d}} \bar{G} \longrightarrow \bar{H} \longrightarrow 0.$$

From the termwise splitting of the leftmost column of (3.13) before tensoring with Γ, there exist vectors space summands $\tilde{E}$ and $\tilde{\bar{E}}$ in $H_q(V, F; \mathbb{Q})$ and $H_q(V; \mathbb{Q})$, respectively, such that $E = \tilde{E} \otimes \Gamma$ and $\bar{E} = \tilde{\bar{E}} \otimes \Gamma$. Furthermore, r can be written as $\tilde{r} \otimes \mathrm{id}$, where $\tilde{r} : H_q(V; \mathbb{Q}) \to H_q(V, F; \mathbb{Q})$ is the map of the long exact sequence induced by inclusion (and induces the isomorphism of the summands $\tilde{E} \cong \tilde{\bar{E}}$). We can make similar conclusions about G in the second column of (3.13) and carry over all of the bar and tilde notations. Identifying quotient vector spaces with summands, for convenience, we obtain the diagram:

$$\begin{tikzcd}
\tilde{\bar{E}} \subset H_q(V; \mathbb{Q}) \arrow[r] \arrow[d, "\tilde{r}"'] & H_q(Y; \mathbb{Q}) \supset \tilde{\bar{G}} \arrow[d, "\tilde{s}"'] \\
\tilde{E} \subset H_q(V, F; \mathbb{Q}) \arrow[r] & H_q(Y, Z; \mathbb{Q}) \supset \tilde{G}.
\end{tikzcd}$$

We will now choose suitable bases for $\tilde{E}$, $\tilde{\bar{E}}$, $\tilde{G}$, and $\tilde{\bar{G}}$. Consider now the *integral* homology groups and long exact sequence maps $\tilde{r}_{\mathbb{Z}} : H_q(V) \to H_q(V, F)$ and $\tilde{s}_{\mathbb{Z}} : H_q(Y) \to H_q(Y, Z)$. As abelian groups, each of these is the direct sum of its free part and its torsion part, and we can choose bases so that maps between the free summands are represented by diagonal matrices ordered so that all of the zero diagonal entries are moved to the bottom right [28, §11]. Clearly then when we tensor with $\mathbb{Q}$, we get the maps in the above diagram with the vector space summands $\tilde{E}$, $\tilde{\bar{E}}$, $\tilde{G}$, and $\tilde{\bar{G}}$ being represented by the $\mathbb{Q}$ spans of the first m basis elements of the groups, i.e. we can now choose bases $\{\alpha_i\}$, $\{\beta_i'\}$, $\{\gamma_i\}$, $\{\delta_i'\}$, of the free parts of $H_q(V, F)$, $H_q(Y)$, $H_q(Y, Z)$, $H_q(V)$, such that, upon tensoring with $\mathbb{Q}$, the first m elements of each basis will span $\tilde{E}$, $\tilde{\bar{G}}$, $\tilde{G}$, and $\tilde{\bar{E}}$, respectively, and the maps $\tilde{r}_{\mathbb{Z}} \otimes \mathbb{Q}$ and $\tilde{s}_{\mathbb{Z}} \otimes \mathbb{Q}$ induce the appropriate vector space isomorphisms. Furthermore, $\{\alpha_i \otimes 1\}_{i=1}^m$, $\{\beta_i' \otimes 1\}_{i=1}^m$, $\{\gamma_i \otimes 1\}_{i=1}^m$, $\{\delta_i' \otimes 1\}_{i=1}^m$ now span E, $\bar{G}$, G, and $\bar{E}$.

We claim also that with these choices $\tilde{E}$ and $\tilde{\bar{G}}$ are dual with respect to the linking pairing L' (see Section ***3.3***) and $\tilde{\bar{E}}$ and $\tilde{G}$ are dual with respect to L'', which will allow us to perform changes of bases of $\tilde{\bar{G}}$ (to $\{\beta_i\}$) and $\tilde{\bar{E}}$ (to $\{\delta_i\}$)

such that

$$L'(\alpha_i \otimes \beta_j) = L''(\gamma_i \otimes \delta_j) = \delta_{ij},$$

$1 \le i \le m$. The changes of bases can be taken to be integrally unimodular (see below).

We proceed by first proving that the duals to the $\{\gamma_i\}_{j=1}^m$ under L'' span $\bar{\bar{E}} \subset H_q(V;\mathbb{Q})$. To see this, we first observe that, up to sign, $L''([v], \bar{s}([y])) = L'(\bar{r}([v]), [y])$ for $[v] \in H_q(V;\mathbb{Q})$ and $[y] \in H_q(Y;\mathbb{Q})$. This follows by considering the definition of the linking pairings. If v and y are chains representing $[v]$ and $[y]$, then they also represent $\bar{r}[v]$ and $\bar{s}[y]$ (as relative chains modulo the chain complexes $C_q(F)$ and $C_q(V)$). Then $L''([v], \bar{s}([y]))$ is the intersection number of y with a chain in D^n whose boundary is v, while $L'(\bar{r}([v]), [y])$ is the intersection number of v with a chain in D^n whose boundary is y. By the properties of intersection numbers, these agree up to sign. Now suppose that v is an element of $H_q(V;\mathbb{Q})$ which lies in the summand $\ker(\bar{r})$ and that $\{\bar{s}^{-1}\gamma_i\}_{i=1}^m$ are elements of $H_q(Y)$ which map onto the γ_i. Then $0 = L'(\bar{r}(v), \bar{s}^{-1}\gamma_i) = L''(v, \gamma_i)$. Therefore, $\ker(\bar{r})$ is orthogonal to $\bar{G}$ under L''. Thus, the dual subspace to $\bar{G}$, spanned by $\{\delta_i\}_{i=1}^m$, must lie outside $\ker(\bar{r})$ and project onto an m-dimensional subspace of $\operatorname{coim}(\bar{r}) = \bar{\bar{E}}$. But $\dim(\bar{\bar{E}}) = \dim(\bar{E}) = m$ by isomorphism and $\dim(\bar{E}) = \dim(\bar{G})$ because the map d was a square presentation. This proves that $\bar{\bar{E}}$ and $\bar{G}$ are dual.

It also follows from the discussion of the last paragraph that we must have $\delta_i \in \ker(\bar{r})$ for $i > m$: Suppose not. Without loss of generality, suppose $\delta_{m+1} \notin \ker(\bar{r})$. Then, in the rational vector space $\bar{\bar{E}} \otimes \mathbb{Q}$, there will be (at least) $m+1$ linearly independent vectors, $\{\delta_i\}_{i=1}^m$, which do not lie in the kernel. But since the kernel has dimension $n - m$ (rationally), the span of $\{\delta_i\}_{i=1}^m$ must intersect the kernel. Therefore there is a vector $v \in \ker(\bar{r}) \otimes \mathbb{Q}$ such that

$$v = \sum_{i=1}^{m+1} n_i \delta_i, \quad n_i \in \mathbb{Q}.$$

Furthermore, there must be some n_j, $j \le m$, such that $n_j \ne 0$ (else $v = n_{m+1}\delta_{m+1} \notin \ker(\bar{r}) \otimes \mathbb{Q}$). Then $L''(v, \alpha_j) = n_j \ne 0$, contrary to the results of the last paragraph. Therefore, $\delta_i \in \ker(\bar{r})$ for $i > m$. Now, since each δ_i' is an integral linear combination of the $\{\delta_i\}$ (since each is a basis for $H_q(V)$), the same must be true under the projection to $\bar{\bar{E}}$, i.e. the projection of each δ_i' is an integral linear combination of the projections of the $\{\delta_i\}$. But since $\delta_i \in \ker(\bar{r})$ for $i > m$, each projected δ_i' is a linear combination of the projections of $\{\delta_i\}_{i=1}^m$. Since the projected $\{\delta_i'\}_{i=1}^m$ form a basis for $\bar{\bar{E}}$, it follows that the projections $\{\delta_i\}_{i=1}^m$ also form a basis for $\bar{\bar{E}}$. In particular, we see that $\bar{\bar{E}}$ is integrally dual to $\bar{G}$ (and hence also rationally when tensored with $\mathbb{Q}$). In what follows, we shall also refer to the projections of the $\{\delta_i\}_{i=1}^m$ into $\bar{\bar{E}}$ as $\{\delta_i\}_{i=1}^m$.

Similar considerations apply for the other case to show that $\bar{\bar{G}}$ with basis $\{\beta_i\}_{i=1}^m$ is dual to $\bar{E}$.

Next, we can apply our previous notations, procedures, and results (see Section 3.3) to these modules to obtain the formulae:

$$\begin{aligned}
i_{+*}(\delta_j) &= \sum_i \lambda_{ij}\beta_i\,,\\
i_{-*}(\delta_j) &= \sum_i \sigma_{ij}\beta_i\,,\\
i_{+*}(\alpha_j) &= \sum_i \mu_{ij}\gamma_i\,,\\
i_{-*}(\alpha_j) &= \sum_i \tau_{ij}\gamma_i\,,\\
L'(\alpha_k \otimes i_{+*}(\delta_j)) &= \sum_i \lambda_{ij}L'(\alpha_k \otimes \beta_i) = \lambda_{kj}\,,\\
L'(\alpha_k \otimes i_{-*}(\delta_j)) &= \sum_i \sigma_{ij}L'(\alpha_k \otimes \beta_i) = \sigma_{kj}\,,\\
L''(i_{+*}(\alpha_j) \otimes \delta_k) &= \sum_i \mu_{ij}L''(\gamma_i \otimes \delta_k) = \mu_{kj}\,,\\
L''(i_{-*}(\alpha_j) \otimes \delta_k) &= \sum_i \tau_{ij}L''(\gamma_i \otimes \delta_k) = \tau_{kj},
\end{aligned}$$

where all of the indices run only to m and everything is of dimension q. We get presentation matrices

$$\begin{aligned}
P_1(t) &= (t\sigma_{ij} - \lambda_{ij})\,,\\
P_2(t) &= (t\tau_{ij} - \mu_{ij})
\end{aligned}$$

for $\bar{H}$ and H, and we know that $\sigma_{jk} = \mu_{kj}$ and $\lambda_{jk} = \tau_{kj}$.

We are further furnished with one more relation between the matrices μ and τ. Let $R = (R_{ij})$ be the matrix representation of $\bar{r}|\bar{\bar{E}}$. Let v_i be a chain representing $\delta_i \in \bar{\bar{E}}$, $1 \le i \le m$, and observe that the same chain (modulo chains in F) represents $\bar{r}(\delta_i) \in \bar{E}$. Thus, using chains interchangeably with their appropriate homology classes,

$$\begin{aligned}
L''(i_{+*}(\bar{r}\delta_j) \otimes \delta_i) &= L''(i_+(v_j) \otimes v_i)\\
&= (-1)^{q+1}L''(i_-(v_i) \otimes v_j) = (-1)^{q+1}L''(i_{-*}(\bar{r}\delta_i) \otimes \delta_j),
\end{aligned}$$

where the middle equality comes from the usual geometry of the isotopies obtained by "pushing along the bicollar" (see Section 3.3.3), and the sign change is the usual sign change in the commutativity formula for a linking pairing induced by an intersection pairing (see [9, Appendix]). But

$$\begin{aligned} L''(i_{+*}(\bar{r}\delta_j) \otimes \delta_i) &= L''(i_{+*}\Big(\sum_{k=1}^{m} R_{kj}\alpha_k\Big) \otimes \delta_i) \\ &= \sum_{k=1}^{m} R_{kj} L''(i_{+*}(\alpha_k) \otimes \delta_i) \\ &= \sum_{k=1}^{m} R_{kj}\mu_{ik}. \end{aligned}$$

Similarly, we get that

$$L''(i_{-}(\bar{r}\delta_i) \otimes \delta_j) = \sum_{k=1}^{m} R_{ki}\tau_{jk}.$$

This yields the matrix equations

$$\mu \cdot R = (-1)^{q+1}(\tau \cdot R)' = (-1)^{q+1} R' \cdot \tau',$$

and we can conclude the following:

Proposition 3.22. *The Γ-module H has a presentation matrix of the form $\tau t - (-1)^{q+1} R'\tau' R^{-1}$, where R is the matrix of the map $\bar{\bar{E}} \to \bar{E}$ induced by $\bar{r} : H_q(V) \to H_q(V, F)$. $\bar{H}$ has presentation matrix $(-1)^{q+1}(R^{-1})'\tau R t - \tau'$.*

Remark 3.23. Both of these presentation matrices have the same determinant, up to sign, as expected.

In this situation, we can say the following about the matrix of the pairing $\langle\, ,\rangle : \bar{H} \times \bar{H} \to Q(\Lambda)/\Gamma$:

Proposition 3.24. *In the above situation, taking $\{B_i\}_{i=1}^m$ as the generators of $\bar{H}$, where B_i is the image of the $\{\beta_i \otimes 1\} \in \bar{\bar{E}} \otimes \Gamma = \bar{E}$ in $\bar{H}$, a matrix representative of the pairing $\langle\, ,\rangle : \bar{H} \times \bar{H} \to Q(\Lambda)/\Lambda$ is given by*

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1} t\tau' R^{-1}}.$$

Proof. The proof follows closely that of [21, Proposition 14.3]. We choose particular lifts of V and Y which adjoin (i.e. any path from $t^{-1}Y$ to Y must cross V, identifying t as the covering translation) and identify $\delta_i \in V$ with $\delta_i \otimes 1$, which we will call $\tilde{\delta}_i$ for convenience. Set $p(t)\tilde{\delta}_i = \delta_i \otimes p(t)$ for $p(t) \in \Gamma$. We treat the other bases similarly.

Since $i_{+*}(\tilde{\delta}_i) = \sum_i \lambda_{ji}\tilde{\beta}_j$ and $i_{-*}(\tilde{\delta}_i) = \sum_j \sigma_{ji}\tilde{\beta}_j$ are induced by homotopies, there are chains c_i and c_i' such that

$$\partial c_i = \tilde{\delta}_i - \sum_j \lambda_{ji}\tilde{\beta}_j\,,$$
$$\partial t c_i' = \tilde{\delta}_i - \sum_j \sigma_{ji} t\tilde{\beta}_j\,.$$

Thus

$$\begin{aligned}\partial(tc_i' - c_i) &= -\sum_i t\sigma_{ji}\tilde{\beta}_j + \sum_i \lambda_{ji}\tilde{\beta}_j \\ &= \sum_i (\lambda_{ji} - t\sigma_{ji})\tilde{\beta}_j.\end{aligned}$$

As usual, let λ and σ denote the matrices (λ_{ji}) and (σ_{ji}). Let $\Delta(t) = \det(\lambda - t\sigma)$ and $M(t) = \Delta(t)(\lambda' - t\sigma')^{-1}$, i.e. the matrix of cofactors of $(\lambda' - t\sigma')$. Thus

$$\delta_{jk}\Delta(t) = \sum_i M_{ki}(t)(\lambda_{ji} - t\sigma_{ji}), \tag{3.14}$$

so that

$$\begin{aligned}\Delta(t)\tilde{\beta}_k &= \sum_j \delta_{jk}\Delta(t)\tilde{\beta}_j \\ &= \sum_{i,j} M_{ki}(t)(\lambda_{ji} - t\sigma_{ji})\tilde{\beta}_j \\ &= \sum_i M_{ki}(t)\partial(tc_i' - c_i) \\ &= \partial\Big(\sum_i M_{ki}(t)(tc_i' - c_i)\Big).\end{aligned}$$

Now, as outlined above, to compute $\langle B_k, B_\ell\rangle$, we choose representative chains for the B_i (denoting both the chains and classes by the same symbol for simplicity) and find a chain c such that $\partial c = p(t)B_k$ for some $p(t) \in \Lambda$. Then

$$\langle B_k, B_\ell\rangle = \frac{1}{p(t)} c \cdot B_\ell \qquad \mod \Lambda\,.$$

Based upon the computations above, we can take $p(t) = \Delta(t)$ and $c(t) = \sum_i M_{ki}(t)(tc_i' - c_i)$ from which

$$\langle B_k, B_\ell\rangle = \frac{\sum_i M_{ki}(t)(t(c_i' \cdot B_\ell) - (c_i \cdot B_\ell))}{\Delta(t)}.$$

Since the c_i, c_i', and B_i all lie in the same lift of Y, the intersection numbers in this formula are the ordinary intersection numbers in $Y \subset S^n$ and are thus the same as the usual linking numbers of the chains ∂c_i and $\partial c_i'$ with B_ℓ. Since a chain representing $\tilde{\delta}_i$ represents $\sum_j R_{ji}\tilde{\alpha}_j$ in $\tilde{E}$, we get

$$\begin{aligned}
c_i \cdot B_\ell &= L''\Big(\sum_j R_{ji}\tilde{\alpha}_j, \tilde{\beta}_\ell\Big) - \sum_j \lambda_{ji}\ell(\tilde{\beta}_j, \tilde{\beta}_\ell) \\
&= R_{\ell i} - \sum_j \lambda_{ji}\ell(\tilde{\beta}_j, \tilde{\beta}_\ell) \\
c_i' \cdot B_\ell &= L''\Big(\sum_j R_{ji}\tilde{\alpha}_j, \tilde{\beta}_\ell\Big) - \sum_j \sigma_{ji}\ell(\tilde{\beta}_j, \tilde{\beta}_\ell) \\
&= R_{\ell i} - \sum_j \sigma_{ji}\ell(\tilde{\beta}_j, \tilde{\beta}_\ell),
\end{aligned}$$

where $\ell(\tilde{\beta}_j, \tilde{\beta}_\ell)$ is the linking number in S^n of chains representing β_j and β_ℓ. Thus,

$$\begin{aligned}
\langle B_k, B_\ell\rangle &= \sum_i \frac{M_{ki}(t)}{\Delta(t)}\Big[(t-1)R_{\ell i} + \sum_j(\lambda_{ji} - t\sigma_{ji})\ell(\tilde{\beta}_j, \tilde{\beta}_\ell)\Big] \\
&= \sum_i \frac{M_{ki}(t)(t-1)R_{\ell i}}{\Delta(t)} + \sum_{ij} \frac{M_{ki}(t)(\lambda_{ji} - t\sigma_{ji})\ell(\tilde{\beta}_j, \tilde{\beta}_\ell)}{\Delta(t)} \\
&= \sum_i \frac{M_{ki}(t)(t-1)R_{\ell i}}{\Delta(t)} + \sum_j \delta_{jk}\ell(\tilde{\beta}_j, \tilde{\beta}_\ell),
\end{aligned}$$

where we have used equation (3.14) to simplify in the last step. Since $\ell(\tilde{\beta}_j, \tilde{\beta}_\ell)$ is an integer,

$$\langle B_k, B_\ell\rangle = \sum_i \frac{M_{ki}(t)(t-1)R_{\ell i}}{\Delta(t)} \operatorname{mod} \Lambda.$$

Thus the matrix of the pairing is given by

$$\begin{aligned}
\frac{t-1}{\Delta(t)}M(t)R' &= (t-1)(\lambda' - t\sigma')^{-1}R' \\
&= (t-1)(\tau - (-1)^{q+1}tR'\tau'R^{-1})^{-1}R' \\
&= \frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}t\tau'R^{-1}}
\end{aligned}$$

□

Conversely, suppose we are given integer matrices τ and R such that R has non-zero determinant, $(R^{-1})'\tau R$ is an integer matrix, and $\det[M(1)] = \pm 1$, where M is the matrix $M(t) = (-1)^{q+1}(R^{-1})'\tau R t - \tau'$. Let A be the Λ-module whose presentation matrix is $M(t)$, i.e. $A = \Lambda/M\Lambda$. Then

$$N(t) = \frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' t R^{-1}} = \frac{1-t}{(R^{-1})'M(t)'}$$

determines a nondegenerate $(-1)^{q+1}$-Hermitian form $\langle\, ,\rangle : A \times A \to Q(\Lambda)/\Lambda$ by $\langle a_1, a_2\rangle = a_1' N(t)\bar{a}_2$. (For a more general discussion of the construction of which this is a minor modification, see [39, §1].) A simple calculation shows that $N(t)$ is $(-1)^{q+1}$-Hermitian. The pairing is well-defined because if $a_1 = 0$ in A, then $a_1 \in M(t)\Lambda$ so that it can be represented as $M(t)a_0$. Then $\langle a_1, a_2\rangle = a_1' N(t) a_2 = (M(t)a_0)'N(t)\bar{a}_2 = (1-t)a_0' M(t)'(M(t)')^{-1}R'\bar{a}_2 = (1-t)a_0'R'\bar{a}_2 \in \Lambda$. For the non-degeneracy, the work of Blanchfield [1, pp. 350-1] implies that $N_0(t) = [(R^{-1})'M(t)']^{-1} = [M(t)']^{-1}R'$ is a non-singular Γ-module pairing $B \times B \to Q(\Gamma)/\Gamma$, where $B = \Gamma/[N_0(t)']^{-1}\Gamma$, provided this is a Γ-torsion module. But since R is rationally unimodular, $\Gamma/[N_0(t)']^{-1}\Gamma = \Gamma/M(t)R^{-1}\Gamma = \Gamma/M(t)\Gamma = A \otimes \mathbb{Q}$. Hence, B is Γ-torsion because A is Λ-torsion. Thus $N_0(t)$ can have no rows or columns composed completely of elements of Γ, hence of Λ. This together with the fact that $(t-1)$ is an isomorphism on A (which is clearly of type K) shows that the pairing $N(t)$ is non-degenerate.

Given any module and pairing as defined in the last paragraph, it is realizable as the middle-dimensional module and pairing of a disk knot $D^{2q-1} \subset D^{2q+1}$, $q > 2$, by Proposition 3.21. Thus, we have proven:

Theorem 3.25. *A polynomial $c(t) \in \Lambda$ can be realized as the Alexander subpolynomial factor shared by $H_q(\tilde{C})$ and $H_q(\tilde{C}, \tilde{X})$ for the locally-flat knotted disk pair $D^{2q-1} \subset D^{2q+1}$, $q > 2$, if and only if $c(t) = \det[M(t)]$, where $M(t) = (-1)^{q+1}(R^{-1})'\tau R t - \tau'$ for integer matrices τ and R, such that R has non-zero determinant, $(R^{-1})'\tau R$ is an integer matrix, and $\det[M(1)] = \pm 1$.*

Remark 3.26. If the boundary knot is trivial, then we will have $R = I$, and we expect our formulae to look like those in [21] for the middle-dimensional duality of a sphere knot. That these formulae do not agree identically is due to two differences in conventions: The first is that we have chosen to use Levine's original convention of [19] for which map to label i_- and which to label i_+ (these choices are reversed in [21]). The second is that while we have employed presentation matrices acting on the left, so that the matrix A corresponds to the module $\Lambda^k/A\Lambda^k$, in [21] Levine allows his presentation matrices to act on the right so that A corresponds to $\Lambda^k/\Lambda^k A$. Thus our presentation matrices are transposed compared to those in [21].

3.6.4. *Characterization of the middle dimension polynomial in terms of pairings.* An alternative way of formulating Theorem 3.25 is the following:

Theorem 3.27. *A primitive polynomial $c(t) \in \Lambda$ can be realized as the Alexander polynomial factor shared by $H_q(\tilde{C})$ and $H_q(\tilde{C}, \tilde{X})$ for the locally-flat knotted disk* *pair* $D^{2q-1} \subset D^{2q+1}$, $q > 2$, *if and only if* $c(1) = \pm 1$ *and there exist an integer* ρ *and a non-negative integer* ω *such that* $((t-1)^{\omega}\rho)/\pm c(t)$ *is the discriminant of a* $(-1)^{q+1}$*-Hermitian form on a* Λ*-module of type* K.

Proof. If $c(t)$ is the Alexander subpolynomial in primitive form, we know that $c(1) = \pm 1$, $c(t^{-1}) \sim c(t) = \det[(-1)^{q+1}(R^{-1})'\tau Rt - \tau']$, and

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' tR^{-1}}$$

is the matrix of a form of the given type on a Λ-module of type K. Letting $|\tau|$ stand for the number of rows (or columns) of the square matrix τ, the discriminant of the form is

$$\det\left[\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' tR^{-1}}\right] = \frac{(t-1)^{|\tau|}\det(R)}{\pm c(t)}. \tag{3.15}$$

Setting $\rho = \det(R)$ and $\omega = |\tau|$ proves the claim in this direction.

Conversely, suppose that we are given a primitive polynomial $p(t) \in \Lambda$ such that $p(1) = \pm 1$ and there exist ρ and ω such that $((t-1)^{\omega}\rho)/p(t)$ is the discriminant, D, of a $(-1)^{q+1}$-Hermitian form on a Λ-module of type K. Then by Propositions 3.21 and 3.24, the module and pairing can be realized as an appropriate middle-dimensional knot pairing such that the module has a presentation matrix of the form $(-1)^{q+1}(R^{-1})'\tau Rt - \tau'$ and the pairing has a matrix of the form

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' tR^{-1}}.$$

The associated Alexander polynomial is then $c(t) = \det((-1)^{q+1}(R^{-1})'\tau Rt - \tau')$, while the discriminant is

$$D = \det\left[\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' tR^{-1}}\right].$$

Thus we have

$$\begin{aligned} c(t) &\sim \frac{(t-1)^{|\tau|}\det(R)}{D} && \text{as in the last paragraph} \\ &\sim \frac{p(t)(t-1)^{|\tau|}\det(R)}{(t-1)^{\omega}\rho} && \text{by assumption} \\ &\sim p(t)(t-1)^{|\tau|-\omega}\left(\frac{\det(R)}{\rho}\right). \end{aligned}$$

But since we know that both $c(1)$ and $p(1)$ are equal to ± 1, we must have $\omega = |\tau|$ and $\rho = \det(R)$, so that $c(t) \sim p(t)$ and $p(t)$ is an Alexander polynomial of the desired type. ◻

For the case where q is odd, we already know from Sections 3.3 and 3.5 that these polynomials must be completely classified as those such that $c(1) = \pm 1$ and $c(t) \sim c(t^{-1})$. I do not know of such a similarly straightforward classification for the case where q is even, although we will show that the previously imposed condition that $|c(-1)|$ be a square is not necessary. In fact, we will show that any quadratic polynomial, $c(t) \in \Lambda$, satisfying

1. $c(1) = \pm 1$,
2. $c(t) \sim c(t^{-1})$

can be realized. It is easy to show that any such polynomial has the form $at^2 + (\pm 1 - 2a)t + a$. Now, we can just take

$$R = \begin{pmatrix} \pm 1 + 4a & 0 \\ 0 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} c(t) &= \det[(R^{-1})'\tau R t - (-1)^{q+1}\tau'] = \det \begin{pmatrix} at + a & 1 \\ (\pm 1 + 4a)t + 1 & t + 1 \end{pmatrix} \\ &= at^2 + (\pm 1 - 2a)t + a. \end{aligned}$$

Note that $c(-1) = 4a \pm 1$, so that, by choosing a suitably, we can realize any odd number as $c(-1)$. Observe that $c(-1)$ must be odd for any $c(t)$ satisfying conditions (1) and (2) above (see [19]).

For $q > 2$, we can now replace condition 2f of Theorem 3.18 with the necessity statement of Theorem 3.25 or Theorem 3.27. The constructibility follows by taking an appropriate connected sum with the knots constructed in Proposition 3.21.

For $n = 2q + 1$, $q = 2$, the methods employed above break down. The difficulties in this case are clearly related to the difficulties of classifying the Z-torsion part of the dimension-one Alexander module of a locally-flat knot $S^2 \subset S^4$ (see [21]).

3.7. Conclusion. We summarize our results on the Alexander polynomials of locally-flat disk knots, or equivalently, sphere knots with point singularities.

Theorem 3.28. *For $n \neq 5$ and $0 < i < n - 1$, $0 < j < n - 2$, the following conditions are necessary and sufficient for λ_i, μ_i, and ν_j, to be the polynomials associated to the Γ-modules $H_i(\tilde{C}; \mathbb{Q})$, $H_i(\tilde{C}, \tilde{X}; \mathbb{Q})$, and $H_j(\tilde{X}; \mathbb{Q})$ of a locally flat disk knot $D^{n-2} \subset D^n$ or a knot $S^{n-2} \subset S^n$ with point singularities (see Section 3.3 for the definitions of C and X) : There exist polynomials $a_i(t)$, $b_i(t)$, and $c_i(t)$, primitive in Λ, such that*

1. (a) $\nu_i \sim a_i b_i$;
   (b) $\lambda_i \sim b_i c_i$;
   (c) $\mu_i \sim c_i a_{i-1}$;

2. (a) $c_i(t) \sim c_{n-i-1}(t^{-1})$;
   (b) $a_i(t) \sim b_{n-i-2}(t^{-1})$;
3. $a_i(1) = \pm 1$, $b_i(1) = \pm 1$, $c_i(1) = \pm 1$, $a_0(t) = 1$.
4. *If $n = 2q + 1$ and q is even, then there exist an integer ρ and a non-negative integer ω such that $((t-1)^{\omega}\rho)/\pm c_q(t)$ is the discriminant of a $(-1)^{q+1}$-Hermitian form on a Λ-module of type K (or equivalently, $c_q(t) = \det[M(t)]$, where $M(t) = (-1)^{q+1}(R^{-1})'\tau R t - \tau'$ for integer matrices τ and R such that R has non-zero determinant and $(R^{-1})'\tau R$ is an integer matrix). See Section* ***3.6*** *for definitions and more details.*

For a locally-flat disk knot $D^3 \subset D^5$ or a knot $S^3 \subset S^5$ with point singularities, these conditions are all necessary. Furthermore, we can construct any knot which satisfies both these conditions and the added, perhaps unnecessary, condition that $|c_2(-1)|$ be an odd square.

Proof. This is simply a conglomeration of the results of this section. Note that the duality statements of (2) follow from the duality results of Section ***3.3*** and some simple polynomial algebra (see Lemma 3.17 in Section ***3.5.3***). □

Remark 3.29. For a locally-flat $D^1 \subset D^3$, the boundary modules are all trivial in dimensions greater than 0. In fact the only nontrivial Alexander modules will be $H_1(\tilde{C};\mathbb{Q}) \cong H_1(\tilde{C},\tilde{X};\mathbb{Q})$, and the only non-trivial polynomial $c_1 \sim \lambda_1 \sim \mu_1$ is completely classified by $c_1(t) \sim c_1(t^{-1})$ and $c_1(1) = \pm 1$. Noting that the complement of a locally-flat 1-disk knot is the same as that of the S^1 knot obtained by coning on the boundary (such a cone remains locally-flat at all points), this follows by Levine's conditions [19]. These conditions are equivalent to the conditions stated above, taking $n = 3$, although we have not proved here that any such knot can be constructed. (The necessity could follow from our proof for higher dimensional knots as the assumption $n > 3$ was imposed only to focus our attention on knots which could have point singularities.)

4. Knots with More General Singularities

4.1. Introduction. We now study the Alexander polynomials of non-locally-flat knots with singularities more general than the point singularities of the last section. To be specific, let $\alpha : S^{n-2} \hookrightarrow S^n$, $n \geq 3$, be a PL-embedding which is locally-flat except on a singular set $\Sigma_{n-k} \subset S^{n-2} \subset S^n$. Note that if we view S^n as a PL-stratified space with singular locus S^{n-2}, then Σ will be a subpolyhedron of dimension less than $n-3$ (see Section 5 of [9] or [8] for a more detailed discussion of knots as stratified spaces).

By analogy with Section 3, we can study the homology modules of the infinite cyclic covers of the complement of the knot in the exterior of a regular neighborhood of the singularity and of the complement of the intersection of the knot with the boundary of this regular neighborhood (see Section ***4.2.1***). We can also study the relative homology of the pair. These will all be torsion Γ-modules, and thus we again obtain three sets of polynomials to study: ν_i, λ_i, and μ_i.

In Section **4.2.2**, we show that λ_i and μ_i satisfy a modified version of the duality and normalization results for disk knots (Theorem 3.1). In fact, the results are the same except for the appearance of a power of $t-1$ as a factor in each μ_i. In Section **4.2.3**, we show that the ν_i also satisfy self-duality and normalization conditions which generalize Levine's conditions for the boundary locally-flat sphere knots of Section 3. Again there are extra $t-1$ factors to account for.

In Section **4.2.4**, we discuss the factorization of the Alexander polynomials into subpolynomials and rephrase our results in that context. In Section **4.2.5**, we show that the Alexander polynomials are all trivial for sufficiently large dimension index i.

4.2. Necessary conditions on the Alexander invariants.

4.2.1. *Geometric preliminaries.* Letting $K = \alpha(S^{n-2})$, $S^n - K$ is a homology circle by Alexander duality, and again, just as in Section 3, we can study the rational homology of its infinite cyclic cover viewed as a module over $\Gamma = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$. The Alexander polynomials are the determinants of the presentation matrices of these modules. We begin with some geometric preliminaries and notations.

As usual, we use the homotopy equivalent knot complement or knot exterior as it suits our needs (the knot exterior being the complement in S^n of an open regular neighborhood of K). As already stipulated, K is locally flat away from Σ so that each point in $K - \Sigma$ has a distinguished neighborhood homeomorphic to $D^{n-2} \times c(S^1)$ (where $c(X)$ denotes the open cone on X). Let D denote the manifold $S^n - N(\Sigma)$, where $N(X)$ is the open regular neighborhood of X, and let $\partial D = S = \partial\overline{N(\Sigma)}$. Note that the boundary of the knot exterior is the union of two pieces: a circle bundle in D over $S^{n-2} - N(\Sigma)$ and the exterior in S of an open neighborhood of $S \cap K$ in S. These pieces are joined along their intersection, a circle bundle in S over $S \cap K$ which is the boundary of the closed regular neighborhood of $S \cap K$ in S.

We now construct a version of the Seifert surface in this context.

Lemma 4.1. *There is a retract $R : S^n - K \to S^1$, where S^1 is a given PL-meridian of the knot K.*

Proof. Let $i : S^1 \to S^n - K$ be the inclusion of the meridian S^1. Since $S^n - K$ is a homology circle and the meridians generate its first homology group, i_* is a homology isomorphism in all dimensions. Hence, $\tilde{H}_*(S^n - K, S^1) \cong \tilde{H}^*(S^n - K, S^1) = 0$. But by Eilenberg-MacLane theory, since S^1 is a $K(\mathbb{Z}, 1)$ and since $(S^n - K, S^1)$ as a simplicial pair can also be considered a CW pair, this implies that the identity map $S^1 \to S^1$ can be extended to a map $R : S^n - K \to S^1$ (see [34, 8.1.12]). $\square$

Proposition 4.2. *With D and S as above, there is a bicollared $(n-1)$-manifold $V \subset D$, such that $\partial V = (K \cap D) \cup F$, where F is a bicollared $(n-2)$-manifold in S with $\partial F = K \cap S$.*

Proof. Consider the regular neighborhood $\overline{N(K \cap D)}$ in D which is a 2-disk bundle over $K \cap D$. Let γ be a fiber of the boundary circle bundle $\partial\overline{N(K \cap D)}$ in D. Then γ, with the proper choice of orientation, generates $H_1(S^n - N(K)) = \mathbb{Z}$. In fact, γ certainly has linking number ± 1 with K, corresponding to the intersection point of K with the obvious 2-disk in S^n which bounds γ and makes up a fiber of the regular neighborhood of K. Since the linking pairing $H_1(S^n - K) \otimes H_{n-2}(K) \cong \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ is perfect (see [9, Appendix]) any element of $H_1(S^n - K)$ which maps to 1 under the pairing must be a generator.

Now, by the lemma, there is a retract $R : S^n - K$ to γ which, by restricting to $\partial\overline{N(K \cap D)}$ in D, provides a homotopy trivialization of this circle bundle and hence of the disk bundle $N(K \cap D)$ in D by extending in the obvious way to the interior of the bundle. This homotopy trivialization is homotopic to an actual trivialization, i.e. a projection $\partial\overline{N(K \cap D)} \cong (K \cap D) \times S^1 \to S^1$, and by the homotopy extension principle, we can obtain a map r, homotopic to R, such that $r|\partial\overline{N(K \cap D)}$ is the projection to S^1. Consider now r restricted to $\partial(S^n - N(K))$. We wish to obtain our Seifert surface, V, by taking the transverse inverse image of a generic point under a PL-approximation of r, but first we must take care to avoid getting excess boundary components.

We can first take a PL-approximation to $r|\partial(S^n - N(K))$ which remains the projection on $(K \cap D) \times S^1$. Now, we take the transverse inverse image in $\partial S^n - N(K)$ of a sufficiently generic point, say y, of S^1, which gives us a bi-collared $(n-2)$-submanifold. One component of this submanifold consists of the union of $(K \cap D) \times y \subset (K \cap D) \times S^1$ and a manifold $F \subset S$, with the union taken along their common boundary $(K \cap S) \times y \subset (K \cap S) \times S^1$. This can be seen by considering F to be a component of the transverse inverse image of the restriction to $S - S \cap N(K)$ of the PL-approximation to $r|\partial(S^n - N(K))$. Unfortunately, there may be excess closed components of the inverse image in S, but these can be removed by replacing the approximation to r with the map to S^1 determined by the connected bicollared submanifold consisting of the main component discussed above (in particular, the map which takes the submanifold to the point $y \in S^1$, the hemispheres of the bicollar to the two halves of the circle, and the rest of $\partial(S^n - N(K))$ to the point antipodal to y). Since $S^n - N(K) \sim_{h.e.} S^n - K$ is a homology circle, $H^2(S^n - N(K), \partial(S^n - N(K))) \cong H_{n-2}(S^n - N(K)) = 0$, (recall $n \geq 4$), and therefore there is no obstruction to extending this new map to a map $r : S^n - N(K) \to S^1$. Now we take the transverse inverse image of a PL-approximation to r at y or another sufficiently close point and discard excess components to obtain a bicollared submanifold in $S^n - N(K)$ which will have the desired properties once we extend it trivially to the interior of the disk bundle $\overline{N(K \cap D)}$ in D. ◻

We now establish some notation. We have already denoted $V \cap S$ by F. Let $Y = D - V$, $Z = Y \cap S$, $W = V \cup \overline{N(\Sigma)}$, and $\Omega = Y \cup \overline{N(\Sigma)}$. Note that both W and

Ω contain Σ. We observe that $D-(K\cap D)$ is homotopy equivalent to S^n-K, and so we can consider the homology of either to study the Alexander polynomials.

We begin our study of the Alexander invariants with the following observations and definitions: Let C be the knot complement $D-(K\cap D)\sim_{h.e.} S^n-K$, and let $\tilde{C}$ be the infinite cyclic cover associated with the kernel of the abelianization $\pi_1(C)\to\mathbb{Z}$. Letting t denote a generator of the covering translation, the homology groups of $\tilde{C}$ are finitely generated Λ-modules ($\Lambda=\mathbb{Z}[\mathbb{Z}]=\mathbb{Z}[t,t^{-1}]$) since C has a finite polyhedron as a deformation retract, and the rational homology groups $H_*(\tilde{C};\mathbb{Q})\cong H_*(\tilde{C})\otimes_{\mathbb{Z}}\mathbb{Q}$ are finitely generated modules over the principal ideal domain $\Gamma=\mathbb{Q}[t,t^{-1}]\cong\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$. Therefore, letting $M_R(\lambda)$ denote the R-module of rank 1 with generator of order λ, $H_q(\tilde{C};\mathbb{Q})\cong\bigoplus_{i=1}^k M_\Gamma(\lambda_{q_i})$, where we can choose the λ_{q_i} so that: 1) The λ_{q_i} are primitive in Λ but are unique up to associate class in Γ and 2) $\lambda_{q_{i+1}}|\lambda_{q_i}$. For $0<q<n-1$, these are called the *Alexander invariants* of the knot complements. The polynomial $\lambda_q=\prod_{i=1}^k\lambda_{q_i}$, which is also primitive in Λ, is the Alexander polynomial of the knot complement. We will also consider the relative homology modules $H_*(\tilde{C},\tilde{X};\mathbb{Q})$, where X is the complement in S of $K\cap S$ (the "link complement", as S is the link of Σ) and $\tilde{X}$ is its infinite cyclic covering. It will be clear from our construction that $\tilde{X}$ and the cover of X in $\tilde{C}$ are equivalent. Then $H_q(\tilde{C},\tilde{X};\mathbb{Q})$ has the same properties listed above for $H_q(\tilde{C};\mathbb{Q})$ and its own Alexander invariants $\{\mu_{q_i}\}$, $0<q<n-1$, and relative Alexander polynomial $\mu_q=\prod_i\mu_{q_i}$.

4.2.2. *Duality and normalization theorem.* We will prove the following theorem analogous to that already established in Section 3 for the case of a point singularity:

Theorem 4.3. *Let $p+q=n-1$ with knot and notation as above. The following properties hold:*

1. $\lambda_p(1)=\pm1$,
2. $\mu_q(t)\sim\lambda_p(t^{-1})(t-1)^{\tilde{B}_{q-1}}$, *where $\sim$ denotes associativity of elements in Λ (i.e., $a\sim b$ if and only if $a=\pm t^kb$ for some k) and $\tilde{B}_i$ is the i^{th} reduced Betti number of Σ (i.e., the Betti number of the reduced homology).*

The proof will occupy the next several pages. We begin the proof by finding Γ-module presentations for $H_*(\tilde{C};\mathbb{Q})$ and $H_*(\tilde{C},\tilde{X};\mathbb{Q})$ by studying the Mayer-Vietoris sequences for the infinite cyclic cover obtained by cutting and pasting along the Seifert surface $V\subset D$.

We construct $\tilde{C}$ as in Section 3 by first cutting D open along V to create a manifold, Y', which is homotopy equivalent to Y and whose boundary is Z plus two copies of V, V_+ and V_-, identified along $K\cap D$, and by then pasting together a countably infinite number of disjoint copies (Y^i,V_+^i,V_-^i), $-\infty<i<\infty$, of $(Y'-K,V_+-K,V_--K)$ by identifying V_+^i-K with $V_-^{i+1}-K$ for all i. Then $\tilde{X}$ is the sub-manifold resulting from looking at the restriction of this construction to

$S \cap (Y^i, V^i_+, V^i_-)$. $\tilde{X}$ is thus an infinite cyclic cover of X as claimed. We note once again that $H_i(D - V) \cong H_i(Y)$ and that $H_i(S - F) \cong H_i(Z)$.

The usual considerations (see [19] and Section **3.3.2**) now allow us to set up the Mayer-Vietoris sequences for $\tilde{C}$ and $(\tilde{C}, \tilde{X})$:

$$\to H_q(V;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{d^1_q} H_q(Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{e^1_q} H_q(\tilde{C};\mathbb{Q}) \to, \tag{4.1}$$

$$\to H_q(V,F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{d^2_q} H_q(Y,Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \xrightarrow{e^2_q} H_q(\tilde{C},\tilde{X};\mathbb{Q}) \to . \tag{4.2}$$

We will see later that d^i_q, $i = 1,2$, is a monomorphism for $0 \le q < n-1$. Hence e^i_q is an epimorphism, $0 < q < n-1$, and the d^i_q provide presentation matrices for the homology modules of the covers. In fact, the surjectivity of the e^i_1 and the equivalent injectivity of the d^i_0 follows from standard connectedness considerations or by replacing homology with reduced homology, so we need only show injectivity of d^i_q, $0 < q < n-1$. That the d^1_q are square matrices in this range follows from:

Proposition 4.4. *$H_i(Y;\mathbb{Q}) \cong H_i(V;\mathbb{Q})$, $0 < i < n-1$.*

Proof. In the proof (and often from here out) we suppress the rational coefficients for simplicity of notation.

On the one hand, for $0 < i < n-1$:

$$\begin{aligned} H_i(Y) &\cong H_i(D - V) && \text{by definition of } Y, \\ &\cong H_i(D - (V \cup S)) && \text{by homotopy equivalence}, \\ &\cong H_i(S^n - W) && \text{from the definition of } W, \\ &\cong H_{n-i-1}(W) && \text{by Alexander duality}. \end{aligned}$$

On the other hand, for $0 < i < n-1$:

$$\begin{aligned} &H_i(V) \\ &\cong H_{n-i-1}(V, \partial V) && \text{Poincaré Duality and the universal coefficient theorem}, \\ &\cong H_{n-i-1}(W, \partial V \cup \overline{N(\Sigma)}) && \text{by excision}, \\ &\cong H_{n-i-1}(W, K \cup \overline{N(\Sigma)}) && \text{by the definitions of the spaces}, \\ &\cong H_{n-i-1}(W, S^{n-2}) && \text{by homotopy equivalence } (\overline{N(\Sigma)} \text{ collapses to } \Sigma), \\ &\cong H_{n-i-1}(W) && \text{for } i \ne 1 \text{ by the reduced long exact sequence of the pair}. \end{aligned}$$

For $i = 1$, we examine the top of the long exact sequence of the pair (W, S^{n-2}):

$$\begin{aligned} &\longrightarrow H_{n-1}(W) \longrightarrow H_{n-1}(W, S^{n-2}) \longrightarrow H_{n-2}(S^{n-2}) \\ &\qquad \longrightarrow H_{n-2}(W) \longrightarrow H_{n-2}(W, S^{n-2}) \longrightarrow H_{n-3}(S^{n-2}) \longrightarrow . \end{aligned}$$

Of course $H_{n-3}(S^{n-2}) = 0$ and $H_{n-2}(S^{n-2}) \cong \mathbb{Q}$. We claim that $H_{n-1}(W) \cong 0$ and $H_{n-1}(W, S^{n-2}) \cong \mathbb{Q}$. This will suffice because any injection $\mathbb{Q}$ to $\mathbb{Q}$ must be an isomorphism.

$H_{n-1}(W, S^{n-2}) \cong \mathbb{Q}$ because, as above, $H_{n-1}(W, S^{n-2}) \cong H_{n-1}(V, \partial V)$, which is isomorphic to $\mathbb{Q}$ since V is a connected $(n-1)$-manifold with boundary. To see that $H_{n-1}(W) \cong 0$, consider the Mayer-Vietoris sequence of $W \cong V \cup_F \overline{N(\Sigma)}$. We know that

1. $H_{n-1}(V) = 0$, since V is an $(n-1)$-manifold with boundary;
2. $H_{n-1}(\overline{N(\Sigma)}) = 0$, since $\overline{N(\Sigma)}$ is homotopy equivalent to a complex (Σ) of dimension less than $n-1$; and
3. $H_{n-2}(F) = 0$, since F is an $(n-2)$-manifold with boundary. Therefore, $H_{n-1}(W) = 0$ by the Mayer-Vietoris sequence.

Therefore, $H_i(V) \cong H_{n-i-1}(W) \cong H_i(Y)$ for all i, $0 < i < n-1$. $\square$

We will show later that it is also true that $H_i(Y, Z; \mathbb{Q}) \cong H_i(V, F; \mathbb{Q})$, $0 < i < n-1$, and so the maps d_q^2 also give square presentation matrices.

It follows from the construction of the covering and the action of the covering translation, t, that the maps can be written as

$$\begin{aligned} d_q^i(\alpha \otimes 1) &= i_{-*}(\alpha) \otimes t - i_{+*}(\alpha) \otimes 1 \\ &= t(i_{-*}(\alpha) \otimes 1) - i_{+*}(\alpha) \otimes 1, \end{aligned}$$

where $\alpha \in H_q(V; \mathbb{Q})$ or $H_q(V, F; \mathbb{Q})$ according to whether $i = 1$ or 2, and $i_\pm$ correspond to the identification maps of (V, F) to $(V_\pm, F_\pm)$ obtained by pushing chains out along the bicollar.

To identify these maps (and their matrices) more specifically, we will turn from the context of D, which was useful for the the geometric construction, back to the context of S^n, which will be more useful for the following algebraic constructions. In particular, we will make use of the facts that

1. $H_i(Y) \cong H_i(S^n - W)$ since the two spaces are homotopy equivalent,
2. $H_i(V, F) \cong H_i(W, \overline{N(\Sigma)}) \cong H_i(W, \Sigma)$ by excision and homotopy equivalence,
3. $H_i(V) \cong H_i(S^n - \Omega)$ by homotopy equivalence, and
4. $H_i(Y, Z) \cong H_i(\Omega, \overline{N(\Sigma)}) \cong H_i(\Omega, \Sigma)$ by excision and homotopy equivalence.

We also need to define maps $j_\pm : W \to \Omega$ which extend the maps $i_\pm$ which push V out along its collar isotopically. Let $N'(\Sigma)$ be another regular neighborhood of Σ such that $\overline{N'(\Sigma)}$ lies in the interior of $N(\Sigma)$. Then the closure of $N(\Sigma) - N'(\Sigma)$ is a collar of $\partial\overline{N(\Sigma)}$ using the "generalized annulus property" (see [35, Proposition 1.5]). Define $j_\pm$ to be $i_\pm$ on V and the identity on $N'(\Sigma)$. Extend it to $N(\Sigma) - N'(\Sigma)$ as the homotopy induced on $\partial\overline{N(\Sigma)}$ by $i_\pm$. It is easily seen that with the canonical identifications of homology groups above, $i_{\pm*} : H_i(V) \to H_i(Y)$ corresponds to $j_{\pm*} : H_i(S^n - \Omega) \to H_i(S^n - W)$ and $i_{\pm*} : H_i(V, F) \to H_i(Y, Z)$ corresponds to $j_{\pm*} : H_i(W, \Sigma) \to H_i(\Omega, \Sigma)$. This follows by making the correct identifications at the chain level. Of course we also get maps $j_{\pm *} : H_i(W) \to H_i(\Omega)$. Therefore, to study the matrices d^1 and d^2 we can use

$$\begin{aligned} d^i(\alpha \otimes 1) &= j_{-*}(\alpha) \otimes t - j_{+*}(\alpha) \otimes 1 \\ &= t(j_{-*}(\alpha) \otimes 1) - j_{+*}(\alpha) \otimes 1, \end{aligned}$$

where $\alpha \in H_q(S^n - \Omega; \mathbb{Q})$ or $H_q(W, \Sigma; \mathbb{Q})$ according to whether $i = 1$ or 2.

We will make use of the rational perfect linking pairings (suppressing the "$\mathbb{Q}$" in the homology notation)

$$\begin{aligned} L' &: H_p(W) \otimes H_q(S^n - W) \to \mathbb{Q} \\ L'' &: H_p(\Omega) \otimes H_q(S^n - \Omega) \to \mathbb{Q}, \end{aligned}$$

$p + q = n - 1$ and $0 < p < n - 1$, which derive from the perfect intersection pairings

$$\begin{aligned} \bigcap &: H_{p+1}(S^n, W) \otimes H_q(S^n - W) \to \mathbb{Q} \\ \bigcap &: H_{p+1}(S^n, \Omega) \otimes H_q(S^n - \Omega) \to \mathbb{Q} \end{aligned}$$

and the isomorphisms $H_{p+1}(S^n, W) \cong H_p(W)$ and $H_{p+1}(S^n, \Omega) \cong H_p(\Omega)$, $0 < p < n - 1$, obtained from the long exact sequences of the pairs. Note, to be technically precise, there is the issue that Ω is an open set and not a closed subcomplex of S^n, but we can get around this by replacing $Y = D - V$ with the homotopy equivalent $D - N(V)$ (the neighborhood taken in D) and then forming Ω from this Y and $\overline{N(\Sigma)}$ as above. This new Ω will be homotopy equivalent to the old one but have the benefit of being a closed subcomplex. We will usually avoid the distinction since the two versions are equivalent for homological purposes. Recall also that these pairings are induced (after tensoring with the rationals) from perfect pairings on the integral homology groups modulo their torsion subgroups.

Given $r \in H_p(W; \mathbb{Q})$ and $s \in H_q(S^n - \Omega; \mathbb{Q})$, we have

$$L'(r \otimes j_{-*}(s)) = L''(j_{+*}(r) \otimes s). \tag{4.3}$$

This can be seen as follows: Any chain representing s (which lies in $W - \overline{N(\Sigma)}$ by the definition of Ω) gets pushed into $S^n - W$ under j_- and the the linking form is the intersection of this chain with a chain, R, representing the isomorphic image of r in $H_{p+1}(S^n, W; \mathbb{Q})$ (see [9, Appendix]). The latter chain can be taken as some chain R in S^n whose boundary is a chain representing r. Now, under the isotopy of S^n which takes W to $j_+(W)$ and $j_-(W)$ to W, the chain representing s gets pushed back into W and R gets pushed into a chain in S^n whose boundary is j_+ of the chain representing r (which represents $j_{+*}(r) \in H_p(\Omega)$). Thus this isotopy induces maps which take $j_{-*}(s)$ to s and r to $j_{+*}(r)$, but since the geometric relationship between the chains is unaffected by isotopy, the intersection number is unaffected. The formula then follows immediately using the definitions of L' and L'' in the appendix. Similarly, we get

$$L'(r \otimes j_{+*}(s)) = L''(j_{-*}(r) \otimes s). \tag{4.4}$$

We will need one other property of linking numbers. Given r and s as above

$$L'(r \otimes j_{-*}(s)) - L'(r \otimes j_{+*}(s)) = r \cap s\,, \tag{4.5}$$

$$L''(j_{-*}(r) \otimes s) - L''(j_{+*}(r) \otimes s) = r \cap s\,, \tag{4.6}$$

where $r \cap s$ is the intersection pairing of r and s on W. The geometric proof is analogous to that in the usual case [19, p. 542]. Recall that $r \in H_p(W;\mathbb{Q}) \cong H_p(V,\partial V;\mathbb{Q})$ (see the proof of Proposition 4.4) and $s \in H_q(S^n - \Omega;\mathbb{Q}) \cong H_q(V;\mathbb{Q})$. We claim that this intersection pairing is equivalent to the perfect intersection pairing $\cap : H_p(V,\partial V;\mathbb{Q}) \otimes H_q(V;\mathbb{Q}) \to \mathbb{Q}$ (using $V \sim_{h.e.} V - \partial V$). In either case the intersection pairing is given by the sum of signed point intersections (assuming general position) of chains in the manifold $W - N(\Sigma)$. If $\bar{s}$ is a chain representing s, then, since it lies in $S^n - \Omega \subset V$, it also represents the corresponding class in $H_q(V;\mathbb{Q})$. Meanwhile, by tracing back what happens at the chain level in the equations of the second half of the proof of Proposition 4.4, any chain $\bar{r}$ representing r also represents its image under the isomorphism $H_p(W;\mathbb{Q}) \cong H_p(V,\partial V;\mathbb{Q})$. But none of this affects the geometric intersection, and the choice of chain is irrelevant since the intersection pairing is well-defined up to homology. Therefore, the pairings correspond under the isomorphisms.

We now show the following:

Proposition 4.5. *$H_m(W,\Sigma;\mathbb{Q}) \cong H_m(\Omega,\Sigma;\mathbb{Q})$, $0 < m < n-1$, and hence $H_m(V,F;\mathbb{Q}) \cong H_m(Y,Z;\mathbb{Q})$ and d_m^2 is a square matrix in the same range.*

Proof. Again we suppress the "$\mathbb{Q}$" in the proof for notational convenience. We begin with the claim that

$$H_m(W,\Sigma) \cong H_m(W) \oplus \tilde{H}_{m-1}(\Sigma)$$

and

$$H_m(\Omega,\Sigma) \cong H_m(\Omega) \oplus \tilde{H}_{m-1}(\Sigma)$$

for $0 < m < n-1$. This will follow from the fact that, for $0 < m < n-1$, the inclusion map i_* of each of the long exact reduced homology sequences of the pairs (W,Σ) and (Ω,Σ) is the 0 map. For $i_* : H_m(\Sigma) \to H_m(W)$, $0 < m < n-2$, this follows because the inclusion map can be factored $\Sigma \hookrightarrow K = S^{n-2} \hookrightarrow W$ since $\Sigma \subset K \subset W$. Then i_* factors through $H_m(S^{n-2})$ which is 0 in the appropriate range. For $m = n-2$, the equation still holds from the long exact sequence since Σ has dimension $n-k$, $k \geq 4$, so that $H_{n-2}(\Sigma) \cong 0$. The idea is the same for $i_* : H_m(\Sigma) \to H_m(\Omega)$ except that a little more care must be taken to identify the S^{n-2} that the inclusion factors through. This can be done by employing one of the maps $j_\pm$ to K. Since the $j_\pm$ are ends of isotopies they take the $(n-2)$-sphere K to another $(n-2)$-sphere. But by the properties of the map, $j_\pm K \subset \Omega$ and $\Sigma \subset j_\pm K$. Thus, we can conclude the homology arguments just as in the previous case.

Now, from the proof of Proposition 4.4, we have $H_m(W) \cong H_{n-m-1}(V)$, $0 < m < n-1$; we know that $H_{n-m-1}(V) \cong H_{n-m-1}(S^n - \Omega)$ by homotopy equivalence; and there is a perfect linking pairing between $H_{n-m-1}(S^n - \Omega)$ and $H_m(\Omega)$ which gives an isomorphism since each is a finitely generated vector space. Putting these together with the above identities establishes the proposition. ◻

We next study the maps d_1^q and d_2^q using $j_{\pm *}$. Recall that

$$\begin{aligned} d_q^i(\alpha \otimes 1) &= j_{-*}(\alpha) \otimes t - j_{+*}(\alpha) \otimes 1 \\ &= t(j_{-*}(\alpha) \otimes 1) - j_{+*}(\alpha) \otimes 1, \end{aligned} \tag{4.7}$$

where $\alpha \in H_q(S^n - \Omega; \mathbb{Q})$ or $H_q(W, \Sigma; \mathbb{Q})$ according to whether $i = 1$ or 2.

Let $\{\alpha_i^p\}$, $\{\beta_i^q\}$, $\{\gamma_i^p\}$, and $\{\delta_i^q\}$ represent dual bases for $H_p(W;\mathbb{Z})$, $H_q(S^n - W;\mathbb{Z})$, $H_p(\Omega;\mathbb{Z})$, and $H_q(S^n - \Omega;\mathbb{Z})$, all modulo torsion, so that

$$L'(\alpha_i^p \otimes \beta_j^q) = L''(\gamma_i^p \otimes \delta_j^q) = \delta_{ij}, \tag{4.8}$$

where δ_{ij} is here the delta function (i.e. 1 if $i = j$ and 0 otherwise). These collections then also form bases for the rational homology groups that result by tensoring with $\mathbb{Q}$, and the relations (4.8) hold under the induced perfect rational pairing.

Let $\{\xi_i^p\}$ be a basis for $\tilde{H}_p(\Sigma)$. Then, letting $\{\bar{\alpha}_i^p\}$ and $\{\bar{\xi}_i^{p-1}\}$ represent the bases $\{\alpha_i^p\}$ and $\{\xi_i^{p-1}\}$ under their isomorphic images as direct summands in $H_p(W, \Sigma)$ (see the proof of Proposition 4.5), $\{\bar{\alpha}_i^p\}$ and $\{\bar{\xi}_i^{p-1}\}$ taken together form a basis for $H_p(W, \Sigma)$. Similarly, we define $\{\hat{\gamma}_i^p\}$ together with $\{\hat{\xi}_i^{p-1}\}$ forming a basis for $H_p(\Omega, \Sigma)$.

Let

$$\begin{aligned} j_{+*}(\delta_j^q) &= \sum_i \lambda_{ij}^q \beta_i^q, \\ j_{-*}(\delta_j^q) &= \sum_i \sigma_{ij}^q \beta_i^q, \\ j_{+*}(\bar{\alpha}_j^q) &= \sum_i \mu_{ij}^q \hat{\gamma}_i^q + \sum_i e_{ij}^q \hat{\xi}_i^{q-1}, \end{aligned}$$

$$j_{-*}(\bar{\alpha}_j^q) = \sum_i \tau_{ij}^q \hat{\gamma}_i^q + \sum_i f_{ij}^q \hat{\xi}_i^{q-1},$$
$$j_{+*}(\bar{\xi}_j^{q-1}) = \sum_i \phi_{ij}^q \hat{\gamma}_i^q + \sum_i g_{ij}^q \hat{\xi}_i^{q-1},$$
$$j_{-*}(\bar{\xi}_j^{q-1}) = \sum_i \psi_{ij}^q \hat{\gamma}_i^q + \sum_i h_{ij}^q \hat{\xi}_i^{q-1},$$

where the first two equations are maps $H_q(S^n - \Omega) \to H_q(S^n - W)$ and the rest are maps $H_q(W, \Sigma) \to H_q(\Omega, \Sigma)$. Note that the λ, σ, μ, and τ will all be integers (by the chain map interpretation of $j_\pm$ and the fact that the α and δ were initially chosen as generators of the torsion free parts of the appropriate integral homology groups).

Lemma 4.6. *In the equations above, all of the e_{ij} and f_{ij} are* 0 *and each $g_{ij} = h_{ij} = \delta_{ij}$ (i.e.,* 1 *if $i = j$ and* 0 *otherwise).*

Proof. The proof comes from studying the action of $j_\pm$ on chain representatives of the $\bar{\alpha}$ and the $\bar{\xi}$.

First, since the $\bar{\alpha}$ come from $H_q(W)$ under the standard map $H_q(W) \to H_q(W, \Sigma)$ induced by projection of chain complexes, each $\bar{\alpha}$ can be represented by a chain a mod $C_*(\Sigma)$ (where $C_*(\Sigma)$ is the chain complex of Σ) and such that a is a cycle *in* W. But since $j_\pm$ is induced by an isotopy which fixes Σ, the image of each $\bar{\alpha}$ should also have such a representation, i.e. $j_\pm a$ is a cycle in Ω. But by a similar argument from the geometric underpinnings of the boundary map $\partial_* : H_q(\Omega, \Sigma) \to \bar{H}_{q-1}(\Sigma)$, each $\hat{\xi}$ is represented by a chain x mod $C_*(\Sigma)$ with ∂x a non-zero cycle representing a basis element of $H_{q-1}(\Sigma)$. Since none of these can occur in the image of a under $j_\pm$ (since $\partial j_\pm a = j_\pm \partial a = 0$), each of the e and f must be zero.

For the last pair of maps, we observe similarly that each $\bar{\xi}$ is represented by a chain y mod $C_*(\Sigma)$ with ∂y a non-zero cycle representing a basis element of $H_{q-1}(\Sigma)$. We have $\partial j_\pm y = j_\pm \partial y = \partial y$ since $j_\pm$ fixes Σ. Since $\bar{\xi}$ and $\hat{\xi}$ are both induced by the same basis for $\bar{H}_{q-1}(\Sigma)$ and since the $\hat{\xi}$ component of any element of $H_*(\Omega, \Sigma)$ is determined by its image under the boundary map ∂_*, it is clear that each $\bar{\xi}$ maps to an element whose component in the summand $\bar{H}_{*-1}(\Sigma)$ of $H_*(\Omega, \Sigma)$ is the corresponding $\hat{\xi}$. $\square$

Thus we have

$$j_{+*}(\delta_j^q) = \sum_i \lambda_{ij}^q \beta_i^q,$$
$$j_{-*}(\delta_j^q) = \sum_i \sigma_{ij}^q \beta_i^q,$$
$$j_{+*}(\bar{\alpha}_j^q) = \sum_i \mu_{ij}^q \hat{\gamma}_i^q,$$

$$j_{-*}(\bar{\alpha}_j^q) = \sum_i \tau_{ij}^q \hat{\gamma}_i^q,$$
$$j_{+*}(\bar{\xi}_j^{q-1}) = \sum_i \phi_{ij}^q \hat{\gamma}_i^q + \hat{\xi}_j^{q-1},$$
$$j_{-*}(\bar{\xi}_j^{q-1}) = \sum_i \psi_{ij}^q \hat{\gamma}_i^q + \hat{\xi}_j^{q-1}.$$

Using (4.7) we have therefore that the matrices d_q^1 can be written as

$$d_q^1(t) = (t\sigma_{ij}^q - \lambda_{ij}^q),$$

while d_q^2 has the form

$$\begin{pmatrix} P^q & 0 \\ Q^q & R^q \end{pmatrix},$$

where P^q is the matrix given by

$$P^q = (t\tau_{ij}^q - \mu_{ij}^q)$$

and R^q is the block sum of $\tilde{B}_{q-1}$ copies of $t-1$ along the diagonal, where $\tilde{B}_{q-1}$ is the reduced $(q-1)$-Betti number of Σ.

Since the Alexander polynomials in which we are interested are the determinants of these presentation matrices, the ϕ_{ij} and ψ_{ij} of Q are at present irrelevant (since by elementary linear algebra, $\det(d_q^2) = \det(P^q)\det(R^q) = \det(P^q)(t-1)^{\tilde{B}_{q-1}}$). To determine the relationships amongst the λ, μ, σ, and τ, we identify the chains representing the $\bar{\alpha}$ with those representing α as above, and similarly for the $\hat{\gamma}$ and the γ. Then we can apply the linking pairings to get:

$$L'(\alpha_k^p \otimes j_{+*}(\delta_j^q)) = \sum_i \lambda_{ij}^q L'(\alpha_k^p \otimes \beta_i^q) = \lambda_{kj}^q,$$
$$L'(\alpha_k^p \otimes j_{-*}(\delta_j^q)) = \sum_i \sigma_{ij}^q L'(\alpha_k^p \otimes \beta_i^q) = \sigma_{kj}^q,$$
$$L''(j_{+*}(\alpha_j^q) \otimes \delta_k^p) = \sum_i \mu_{ij}^q L''(\gamma_j^q \otimes \delta_k^p) = \mu_{kj}^q,$$
$$L''(j_{-*}(\alpha_j^q) \otimes \delta_k^p) = \sum_i \tau_{ij}^q L''(\gamma_j^q \otimes \delta_k^p) = \tau_{kj}^q.$$

We can now use (4.3) and (4.4) to obtain $\sigma_{jk}^q = \mu_{kj}^p$ and $\lambda_{jk}^q = \tau_{kj}^p$. This implies that $P^q(t) = -td_1^p(t^{-1})'$, where $'$ indicates transpose.

It remains only to prove that the d_q^i, $0 < q < n-1$, are non-singular and $\det(d_q^1(1)) = \pm 1$. As noted above, the first will show that the d_q^i are presentation matrices. The theorem will then follow by taking determinants.

Lemma 4.7. *The* d_q^i, $0 < q < n-1$, *are non-singular, and, in particular,* $\det(d_q^1(1)) = \pm 1$.

Certainly if d_q^1 is nonsingular, then d_p^2 is nonsingular, since, up to sign, the determinant of d_p^2 will be $(t-1)^{\bar{B}_{p-1}}$ times some power of t times the determinant of $d_q^1(t^{-1})$, and the last will be nonsingular if $d_q^1(t)$ is. Therefore, it remains to show that the d_q^1, $0 < q < n-1$, are non-singular. We will show that $d_q^1(1)$ has determinant ± 1 which will establish the claim.

$$\begin{aligned} d_q^1(1) &= (\sigma_{ij}^q - \lambda_{ij}^q) \\ &= (L'(\alpha_i^p \otimes j_{-*}(\delta_j^q)) - L'(\alpha_i^p \otimes j_{+*}(\delta_j^q)), \end{aligned}$$

but by the properties of the linking pairings above, this is the matrix of the perfect intersection pairing between $H_p(W;\mathbb{Q})$ and $H_q(S^n - \Omega;\mathbb{Q})$, which is equivalent to the perfect intersection pairing between $H_p(V,\partial V;\mathbb{Q})$ and $H_q(V;\mathbb{Q})$. In fact, since we have chosen generators corresponding to the generators of the integral homology groups modulo torsion and since

$$\begin{aligned} H_p(W;\mathbb{Z}) &\cong H_p(W,\partial W;\mathbb{Z}), \\ H_q(S^n - \Omega;\mathbb{Z}) &\cong H_q(V;\mathbb{Z}) \end{aligned}$$

(the latter by homotopy equivalence and the former as in the proof of Proposition 4.4), this is the matrix of the *integral* perfect intersection pairing between the torsion free parts $H_p(V,\partial V;\mathbb{Z})$ and $H_q(V;\mathbb{Z})$. Therefore, this matrix is unimodular over $\mathbb{Z}$ and has determinant ± 1. □

4.2.3. *Polynomials of the boundary "knot".* We now wish to study the properties of the polynomials associated to the homology of the infinite cyclic cover of

$$X = \partial\overline{N(\Sigma)} - K\,,$$

or, in other words, the complement of $K \cap S$ in S. Note that $(S, K \cap S)$ is a locally-flat manifold pair. If Σ were a point singularity, this would be the boundary sphere knot of a slicing locally-flat disk knot (see Section 3). Note, however, that for the case of multiple point singularities, we here diverge slightly from our previous treatment. Instead of linking the point singularities with an arc and considering the regular neighborhood of that arc, we instead consider the regular neighborhood of the collection of points. This will consist of a collection of balls, and, in this case, $(S, K \cap S)$ will be a collection of locally-flat sphere knots in disjoint spheres.

In order to study these “boundary knots” or “link knots”, we begin by examining the kernels of the boundary maps of the vertical exact sequences in the following commutative diagram in which $0 < k < n-1$:

$$
\begin{tikzcd}
 & {} \arrow[d, "a"] & {} \arrow[d, "b"] & {} \arrow[d, "c"] & \\
{} \arrow[r, "0"] & H_k(V;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "d^1"] \arrow[d, "r"] & H_k(Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "e^1"] \arrow[d, "s"] & H_k(\tilde{C};\mathbb{Q}) \arrow[r, "0"] \arrow[d, "u"] & {} \\
{} \arrow[r, "0"] & H_k(V,F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "d^2"] \arrow[d, "\partial_*"] & H_k(Y,Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "e^2"] \arrow[d, "\partial'_*"] & H_k(\tilde{C},\tilde{X};\mathbb{Q}) \arrow[r, "0"] \arrow[d, "\partial''_*"] & {} \\
{} \arrow[r, "f^3"] & H_k(F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "d^3"] \arrow[d, "a"] & H_k(Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \arrow[r, "e^3"] \arrow[d, "b"] & H_k(\tilde{X};\mathbb{Q}) \arrow[r, "f^3"] \arrow[d, "c"] & {} \\
 & {} & {} & {} & .
\end{tikzcd}
\tag{4.9}
$$

The top two rows are the exact rows of the Mayer-Vietoris sequences constructed in Section *4.2.2*. The bottom row is the Mayer-Vietoris sequence of the restriction to S of the construction which gives us the top row. The columns are the usual long exact sequences of the pairs in which the left two have been tensored with Γ over $\mathbb{Q}$. This preserves exactness since the initial sequence consists of free modules, in fact vector spaces. Commutativity of the diagrams is obvious at the chain level. The 0 maps are a consequence of the non-singularity of d^1 and d^2 (see Section *4.2.2*).

Using the exactness, this diagram induces the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{cok} a \arrow[r] \arrow[d, "\cong"'] & \operatorname{cok} b \arrow[r] \arrow[d, "\cong"'] & \operatorname{cok} c \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \ker \partial_* \arrow[r] & \ker \partial'_* \arrow[r] & \ker \partial''_* \arrow[r] & 0.
\end{tikzcd}
\tag{4.10}
$$

That the left vertical map is an isomorphism follows readily from the isomorphisms of $\ker\partial_*$ to $\operatorname{im} r$ from exactness, the canonical isomorphism of $\operatorname{coim} r$ to $\operatorname{im} r$ induced by r, and the isomorphism $\operatorname{coim} r$ to $\operatorname{cok} a$ induced by $\ker r = \operatorname{im} a$ from the exactness. The other vertical isomorphisms follow similarly, and so the sequences are isomorphic. Furthermore, the long exact sequences of the *rational* homology of the pairs (V, F) and (Y, Z), as exact sequences of vector spaces, must split at each term; in other words, each is isomorphic to an exact sequence of vector spaces of the form $\to A\oplus B \to B\oplus C \to C\oplus D \to$. This splitting and exactness is preserved under the tensor product with Γ over $\mathbb{Q}$ so that each of the left two kernels and cokernels in diagram (4.9) is isomorphic to a direct rational vector space summand of the appropriate homology module tensored with Γ. Thus each of the left four terms is a free Γ-module, and once we show that the rows are exact and the rightmost maps are surjective, the leftmost non-trivial maps will give us a presentation matrix for $\ker\partial''_* \cong \operatorname{cok} a$. For notational convenience, we relabel to get the sequence

$$E \longrightarrow G \longrightarrow H \tag{4.11}$$

but leave ourselves free to think of these modules as kernels, images, cokernels, or coimages as the proper contexts allow.

Lemma 4.8. *The following sequence is exact:*

$$E \xrightarrow{\ d\ } G \xrightarrow{\ e\ } H \longrightarrow 0. \tag{4.12}$$

Proof. Thinking of E, G, and H as the appropriate kernels, $\operatorname{im}(d) \subset \ker(e)$ because d and e are induced by d^2 and e^2 and $e^2d^2 = 0$ by the exactness of the rows of (4.9).

We next show that $\ker(e) \subset \operatorname{im}(d)$. Again we think of E, G, and H as the appropriate kernels. We will examine the following piece of (4.9):

$$\begin{array}{ccccc} H_k(V,F;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma & \xrightarrow{d^2} & H_k(Y,Z;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma & \xrightarrow{e^2} & H_k(\tilde{C},\tilde{X};\mathbb{Q}) \\ \partial_*\big\downarrow & & \partial'_*\big\downarrow & & \partial''_*\big\downarrow \\ H_{k-1}(F;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma & \xrightarrow{d^3} & H_{k-1}(Z;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma & \xrightarrow{e^3} & H_{k-1}(\tilde{X};\mathbb{Q}). \end{array} \tag{4.13}$$

Using the definitions of E, G, and H and the splittings of the left two vertical columns of (4.9), we can write this isomorphically as

$$\begin{array}{ccccc} \begin{smallmatrix} E \\ \oplus \\ A \end{smallmatrix} & \xrightarrow{d^2} & \begin{smallmatrix} G \\ \oplus \\ B \end{smallmatrix} & \xrightarrow{e^2} & C \\ \partial_*\big\downarrow & & \partial'_*\big\downarrow & & \partial''_*\big\downarrow \\ \begin{smallmatrix} A \\ \oplus \\ P \end{smallmatrix} & \xrightarrow{d^3} & \begin{smallmatrix} B \\ \oplus \\ Q \end{smallmatrix} & \xrightarrow{e^3} & R, \end{array} \tag{4.14}$$

where the label changes are the obvious ones, and, of course, $H \subset C$. Note that A and B are the kernels of the maps a and b at the bottom of (4.9). Thus since they are all kernels of the appropriate vertical maps, $d^2(E) \subset G$ and $d^3(A) \subset B$. Therefore d^2 and d^3 each have the block forms $\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$ in the appropriate bases. By the commutativity, the lower right submatrix of d^2 is the upper left submatrix of d^3, at least up to equivalence under change of bases; the upper left submatrix of d^2 is the matrix d of (4.12) in the statement of the lemma. Observe that d must be represented by a square matrix: if it had more rows than columns, d^2 would have determinant 0 which is impossible since d^2 is nonsingular; if it had more columns than rows, then, using the fact that d is also the lower right submatrix of a similar block decomposition of d^1 (by simply moving the whole argument up one level of the grid (4.9)), d^1 would have determinant 0 which is also impossible. But d^2 is square and so its lower right submatrix, say δ (so that $d^2 = \begin{pmatrix} d & 0 \\ Y & \delta \end{pmatrix}$), is also square. In addition, d and δ must each be nonsingular since d^2 is.

Now, let $x \in G$ be also in $\ker(e)$. Since $e : G \to H$ is a restriction of e^2, $e^2(x) = 0$. Therefore, by the exactness of the Mayer-Vietoris sequence, $x \in \operatorname{im}(d^2)$ and so $x = d^2(\epsilon + \alpha)$, where $\epsilon \in E$ and $\alpha \in A$. We need to show that $\alpha = 0$ so that $x \in \operatorname{im}(d) = \operatorname{im}(d^2|_E)$. Since $x \in \ker(\partial'_*)$ and $\epsilon \in \ker(\partial_*)$,

$$0 = \partial'_* x = \partial'_* d^2(\epsilon + \alpha) = d^3\partial_*(\epsilon + \alpha) = d^3\partial_*\alpha.$$

But $\partial_*\alpha \in \ker(a) = A$ so that $d^3\partial_*\alpha = \delta\partial_*\alpha$. Since δ is nonsingular, hence injective, $\partial_*\alpha$ must be 0, but this is only possible if $\alpha = 0$ since $\alpha \in A$ and A is mapped injectively under ∂_*. This completes the proof that $\ker(e) \subset \operatorname{im}(d)$.

Lastly, we show that e is surjective, this time treating E, G, and H as the appropriate cokernels. We can make use of the following fact of homological algebra [14, p.3]: In any exact category, given the commutative diagram with exact rows

$$\begin{tikzcd} W \arrow[r] \arrow[d, "w"'] & X \arrow[r] \arrow[d, "x"'] & Y \arrow[r] \arrow[d, "y"'] & Z \arrow[d, "z"'] \\ W' \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd} \tag{4.15}$$

and such that $z : Z \to Z'$ is injective, the induced sequence

$$\operatorname{cok} w \longrightarrow \operatorname{cok} x \longrightarrow \operatorname{cok} y$$

is exact.

From (4.9), we have a commutative diagram with exact rows

$$\begin{tikzcd}
H_{k+1}(Z;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma \arrow[r, "e^3"] \arrow[d, "b"'] & H_{k+1}(\tilde{X};\mathbb{Q}) \arrow[r, "f^3"] \arrow[d, "c"'] & H_k(F;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma \arrow[d] \\
H_k(Y;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma \arrow[r, "e^2"] & H_k(\tilde{C};\mathbb{Q}) \arrow[r, "0"] & H_{k-1}(V;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma.
\end{tikzcd} \tag{4.16}$$

We can truncate this to get a diagram

$$\begin{tikzcd}
H_{k+1}(Z;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma \arrow[r, "e^3"] \arrow[d, "b"'] & H_{k+1}(\tilde{X};\mathbb{Q}) \arrow[r, "f^3"] \arrow[d, "c"'] & \operatorname{im} f^3 \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
H_k(Y;\mathbb{Q})\otimes_{\mathbb{Q}}\Gamma \arrow[r] & H_k(\tilde{C};\mathbb{Q}) \arrow[r, "0"] & 0 \arrow[r] & 0,
\end{tikzcd} \tag{4.17}$$

which is still commutative (an easy verification) with exact rows. The map in the last column is an isomorphism, so the fact quoted above gives an exact sequence

$$\operatorname{cok} b \xrightarrow{e} \operatorname{cok} c \longrightarrow \operatorname{cok} f.$$

In this case, $\operatorname{cok}(f)$ is clearly 0 and so e is surjective. $\square$

Thus d gives a presentation matrix for H which we will now study. From here on, the $\mathbb{Q}$ in the homology notation will once again be implied but not written. We will also use E_k, G_k, H_k, and d_k when we mean to think of the groups as the appropriate kernels in the appropriate dimensions and $\mathfrak{E}_k$, $\mathfrak{G}_k$, $\mathfrak{H}_k$, and $\mathfrak{d}_k$ when we think of them as the appropriate cokernels. For $\mathfrak{E}_k$ and $\mathfrak{G}_k$, we also sometimes make the identification of the cokernels with appropriate direct summands of $H_k(V)\otimes\Gamma$ and $H_k(Y)\otimes\Gamma$ which maps onto the cokernels under the projection to them (since E_k and G_k are submodules, they can be automatically identified as summands). Note that d_k is the restriction of d^2 to E_k, while $\mathfrak{d}_k$ can be thought of as d^1 acting on the summand $\mathfrak{E}_k$ followed by the projection to the summand $\mathfrak{G}_k$.

From the splitting of the leftmost column of (4.9) before tensoring with Γ, there exist vectors space summands $\tilde{E}_k$ and $\tilde{\mathfrak{E}}_k$ in $H_k(V,F)$ and $H_k(V)$, respectively, such that $E_k = \tilde{E}_k\otimes\Gamma$ and $\mathfrak{E}_k = \tilde{\mathfrak{E}}_k\otimes\Gamma$. Furthermore, r can be written as $\tilde{r}\otimes\mathrm{id}$, where $\tilde{r}: H_k(V)\to H_k(V,F)$ is the map of the long exact sequence induced by inclusion (and induces an isomorphism of the summands $\tilde{E}_k\cong\tilde{\mathfrak{E}}_k$). We can make similar conclusions about G in the second column of (4.9) and carry over all of the tilde notations.

For what follows, it is once again simpler to make the identifications of Section ***4.2.2***:: $H_k(V;\mathbb{Q})\cong H_k(S^n-\Omega;\mathbb{Q})$, $H_k(Y;\mathbb{Q})\cong H_k(S^n-W;\mathbb{Q})$, $H_k(V,F;\mathbb{Q})\cong H_k(W,\Sigma;\mathbb{Q})$, and $H_k(Y,Z;\mathbb{Q})\cong H_k(\Omega,\Sigma;\mathbb{Q})$, but for convenience we maintain all of the other labels, both of submodules and maps, making the suitable identifications. We continue to use $\{\alpha_i^k\}$, $\{\beta_i^k\}$, $\{\gamma_i^k\}$, and $\{\delta_i^k\}$ as bases for $H_k(W)$, $H_k(S^n - W)$, $H_k(\Omega)$, and $H_k(S^n - \Omega)$, respectively, appropriately dually paired, and $\{\xi_i^k\}$ as a basis for $\tilde{H}_k(\Sigma)$. Recall that $H_k(W,\Sigma) \cong H_k(W) \oplus \tilde{H}_{k-1}(\Sigma)$ and $H_k(\Omega,\Sigma) \cong H_k(\Omega) \oplus \tilde{H}_{k-1}(\Sigma)$, $0 < k < n-1$, from the proof of Proposition 4.5. We observe that $\bar{E}_k \subset H_k(W) \subset H_k(W,\Sigma)$ and $\bar{G}_k \subset H_k(\Omega) \subset H_k(\Omega,\Sigma)$. In fact, since we have the diagram

$$\text{(4.18)} \qquad \begin{tikzcd} H_k(S^n - \Omega) \arrow[r, "\cong"] \arrow[d] & H_k(V) \arrow[d, "\bar{r}"] \\ H_k(W,\Sigma) \arrow[r, "\cong"] & H_k(V,F), \end{tikzcd}$$

where the top isomorphism is induced by homotopy equivalence and r is induced by the chain projection, and since $\bar{E}_k \cong \operatorname{im}(r)$, then any element $\varepsilon \in \bar{E}_k$ can be represented by the image of a cycle (mod $C(F)$) in V which is thus a cycle in W. Therefore, the image of ε under $\partial_* : H_k(W,\Sigma) \to H_{k-1}(\Sigma)$ is 0, which implies that ε lies in the the $H_k(W)$ summand. The argument for $\bar{G}_k \subset H_k(\Omega) \subset H_*(\Omega,\Sigma)$ is the same. Thus $\bar{E}_k$ and $\bar{G}_k$ are contained in the summands spanned by the $\{\alpha_i^k\}$ and $\{\gamma_i^k\}$, respectively. We can now prove the following lemma:

Lemma 4.9. *$\bar{E}_p \subset H_p(W,\Sigma)$ and $\bar{\mathfrak{G}}_q \subset H_q(S^n, W)$ are perfectly dually paired under L', $p+q = n-1$; $\bar{\mathfrak{E}}_p \subset H_p(S^n - \Omega)$ and $\bar{G}_q \subset H_q(\Omega,\Sigma)$ are perfectly dually paired under L'', $p+q = n-1$.*

Proof. We begin with the latter:

By the preceding discussion and without loss of generality, let us assume that the $\{\gamma_i^q\}$ are chosen so that the first m form a basis $\{\bar{g}_i^q\}_{i=1}^m$ for $\bar{G}$. We claim that the sub-basis $\{\delta_i^p\}_{i=1}^m$ in $H_p(S^n - \Omega)$, which is dual to the $\{\bar{g}_i^q\}_{i=1}^m = \{\gamma_i^q\}_{i=1}^m$ under L'', can be taken as the basis for $\bar{\mathfrak{E}}_p$ under the projection from $H_p(S^n - \Omega)$.

To see this, we first observe that, up to sign,

$$L''([v], \bar{s}([y])) = L'(\bar{r}([v]), [y])$$

for $[v] \in H_p(S^n - \Omega)$ and $[y] \in H_q(S^n - V; \mathbb{Q})$. This follows by considering the definition of the linking pairings. If v and y are chains representing $[v]$ and $[y]$, then they also represent $\bar{r}[v]$ and $\bar{s}[y]$ (as relative chains). Then $L''([v], \bar{s}([y]))$ is the intersection number of y with a chain in S^n whose boundary is v, while $L'(\bar{r}([v]), [y])$ is the intersection number of v with a chain in S^n whose boundary is y. By the properties of intersection numbers, these agree.

Now suppose that v is an element of $H_p(S^n - \Omega)$ which lies in $\ker(\bar{r})$ and that $\{\bar{\mathfrak{g}}_i\}_{i=1}^m$ are basis elements of $\bar{\mathfrak{G}}_q$ which map onto the $\bar{g}_i$. Then $0 = L'(\bar{r}(v), \bar{\mathfrak{g}}_i) = L''(v, \bar{g}_i)$. Therefore, the intersection of $\ker(\bar{r})$ and the dual space to $\bar{G}_q$ is 0.

Thus, the dual subspace to $\tilde{G}_q$, spanned by $\{\delta_i^p\}_{i=1}^m$, can be chosen as a sub-basis for $\tilde{\mathfrak{E}}_p$ under projection. In other words, $\{\delta_i^p + \ker(\tilde{r})\}_{i=1}^m$ is a basis for a linear subspace of $\tilde{\mathfrak{E}}_p$ and $L''(\delta_i^p + \ker(\tilde{r}), \gamma_j) = \delta_{ij}$. We will be done once we show that $\dim(\tilde{\mathfrak{E}}_p) = m$. Since the above gives $\dim(\tilde{\mathfrak{E}}_p) \geq m$, we need only show $\dim(\tilde{\mathfrak{E}}_p) \leq m$

Observe that the same arguments, suitably but easily modified, apply to a basis $\{\tilde{\alpha}_i^p\}_{i=1}^{\mu}$ for $\tilde{E}_p$ (where we have taken a subbasis of the $\{\alpha_i^p\}$ which span $H_p(W)$) to show that the duals $\{\beta_i^q\}_{i=1}^{\mu}$ span a subspace of $\tilde{\mathfrak{G}}_q$ under the projection, and $\mu \leq \dim \tilde{\mathfrak{G}}_q$. But then we have

$$\dim \tilde{\mathfrak{E}}_p = \dim \tilde{E}_p = \mu \leq \dim \tilde{\mathfrak{G}}_q = \dim \tilde{G}_q = m,$$

which is what we needed to show.

This establishes the duality of $\tilde{\mathfrak{E}}_p$ and $\tilde{G}_q$. The other statement follows similarly. $\square$

Using this lemma and once again the fact that

$$L''([v], \tilde{s}([y])) = L'(\tilde{r}([v]), [y])$$

for $[v] \in H_p(S^n - \Omega)$ and $[y] \in H_q(S^n - V; \mathbb{Q})$, we can choose bases $\{\tilde{\alpha}_i^k\}$, $\{\tilde{\beta}_i^k\}$, $\{\tilde{\gamma}_i^k\}$, and $\{\tilde{\delta}_i^k\}$ of $\tilde{E}_k$, $\tilde{\mathfrak{G}}_k$, $\tilde{G}_k$, and $\tilde{\mathfrak{E}}_k$ such that:

1. $L'(\tilde{\alpha}_i^p \otimes \tilde{\beta}_j^q) = L''(\tilde{\gamma}_i^p \otimes \tilde{\delta}_j^q) = \delta_{ij}$, the Kronecker delta function, $p+q = n-1$;
2. $\tilde{r}(\tilde{\delta}_i^k) = \tilde{\alpha}_i^k$ and $\tilde{s}(\tilde{\beta}_i^k) = \tilde{\gamma}_i^k$.

In fact, we can, for example, start with a basis $\{\gamma_i^q\}$ for $\tilde{G}_q$, dualize it to a basis for $\tilde{\mathfrak{E}}_p$, push these to a basis for $\tilde{E}_p$ under r, and then dualize again to get a basis for $\tilde{\mathfrak{G}}_q$. That s applied to these last basis elements returns us to our initial basis is easy to check using the duality and that $L''([v], \tilde{s}([y])) = L'(\tilde{r}([v]), [y])$.

With this choice of bases, the diagram

$$\begin{array}{ccc}
\tilde{\mathfrak{E}}_k \otimes \Gamma = \mathfrak{E}_k & \xrightarrow{\mathfrak{d}_k} & \tilde{\mathfrak{G}}_k \otimes \Gamma = \mathfrak{G}_k \\
{\scriptstyle \tilde{r}\otimes \mathrm{id} = r}\Big\downarrow{\scriptstyle\cong} & & {\scriptstyle \tilde{s}\otimes \mathrm{id} = s}\Big\downarrow{\scriptstyle\cong} \\
\tilde{E}_k \otimes \Gamma = E_k & \xrightarrow{d_k} & \tilde{G}_k \otimes \Gamma = G_k
\end{array}$$

makes it clear that as matrices $d_k = \mathfrak{d}_k$.

We can now establish duality for the polynomials of the modules H_k.

Proposition 4.10. $d_p(t) = -td_q(t^{-1})'$, $p + q = n - 1$, *where $'$ indicates transpose.*

Proof. By the immediately preceding comment, it suffices to show that $d_p(t) = -t\mathfrak{d}_q(t^{-1})'$, where the bases of the modules have been chosen as in the preceding discussion.

The proof is essentially the same as that of Theorem 4.3:

$$\begin{aligned} d_q(\tilde{\alpha} \otimes 1) &= \tilde{j}_{-*}(\tilde{\alpha}) \otimes t - \tilde{j}_{+*}(\tilde{\alpha}) \otimes 1 \\ &= t(\tilde{j}_{-*}(\tilde{\alpha}) \otimes 1) - \tilde{j}_{+*}(\tilde{\alpha}) \otimes 1, \end{aligned}$$

where $\tilde{\alpha} \in \tilde{E}_q$ and $\tilde{j}_\pm$ indicates the restriction of $j_\pm$ to $\tilde{E}_q$; and, similarly,

$$\begin{aligned} \mathfrak{d}_q(\tilde{\delta}) \otimes 1) &= \tilde{\mathfrak{j}}_{-*}(\tilde{\delta} \otimes t - \tilde{\mathfrak{j}}_{+*}(\tilde{\delta}) \otimes 1 \\ &= t(\tilde{\mathfrak{j}}_{-*}(\tilde{\delta}) \otimes 1) - \tilde{\mathfrak{j}}_{+*}(\tilde{\delta}) \otimes 1, \end{aligned}$$

where $\tilde{\delta} \in \tilde{\mathfrak{E}}_q$ and $\tilde{\mathfrak{j}}_\pm$ indicates $j_\pm$ restricted to the summand $\tilde{\mathfrak{E}}_q$ followed by projection to $\tilde{\mathfrak{G}}_q$ of $H_q(S^n - W)$.

Let

$$\begin{aligned} \tilde{j}_{+*}(\tilde{\alpha}^q_j) &= \sum_i \mu^q_{ij} \tilde{\gamma}^q_i, \\ \tilde{j}_{-*}(\tilde{\alpha}^q_j) &= \sum_i \tau^q_{ij} \tilde{\gamma}^q_i, \\ \tilde{\mathfrak{j}}_{+*}(\tilde{\delta}^q_j) &= \sum_i \lambda^q_{ij} \tilde{\beta}^q_i, \\ \tilde{\mathfrak{j}}_{-*}(\tilde{\delta}^q_j) &= \sum_i \sigma^q_{ij} \tilde{\beta}^q_i. \end{aligned}$$

Then d_q and $\mathfrak{d}_q$ have the forms

$$\begin{aligned} d_q(t) &= (t\tau^q_{ij} - \mu^q_{ij}), \\ \mathfrak{d}_q(t) &= (t\sigma^q_{ij} - \lambda^q_{ij}). \end{aligned}$$

To determine the relationships amongst the λ, μ, σ, and τ, we use the linking pairings to get:

$$\begin{aligned} L'(\tilde{\alpha}^p_k \otimes \tilde{\mathfrak{j}}_{+*}(\tilde{\delta}^q_j)) &= \sum_i \lambda^q_{ij} L'(\tilde{\alpha}^p_k \otimes \tilde{\beta}^q_i) = \lambda^q_{kj}, \\ L'(\tilde{\alpha}^p_k \otimes \tilde{\mathfrak{j}}_{-*}(\delta^q_j)) &= \sum_i \sigma^q_{ij} L'(\tilde{\alpha}^p_k \otimes \tilde{\beta}^q_i) = \sigma^q_{kj}, \\ L''(\tilde{j}_{+*}(\tilde{\alpha}^q_j) \otimes \tilde{\delta}^p_k) &= \sum_i \mu^q_{ij} L''(\gamma^q_j \otimes \tilde{\delta}^p_k) = \mu^q_{kj}, \\ L''(\tilde{j}_{-*}(\tilde{\alpha}^q_j) \otimes \tilde{\delta}^p_k) &= \sum_i \tau^q_{ij} L''(\tilde{\gamma}^q_j \otimes \tilde{\delta}^p_k) = \tau^q_{kj}. \end{aligned}$$

Once we establish that our previous equations

$$L'(\alpha \otimes j_{-*}(\delta)) = L''(j_{+*}(\alpha) \otimes \delta)\,, \tag{4.19}$$

$$L'(\alpha \otimes j_{+*}(\delta)) = L''(j_{-*}(\alpha) \otimes \delta)\,, \tag{4.20}$$

for $\alpha \in H_p(W;\mathbb{Q})$ and $\delta \in H_q(S^n - \Omega;\mathbb{Q})$, are still applicable for the restricted pairings with $\alpha \in \bar{E}_p$ and $\delta \in \bar{\mathfrak{E}}_q$, we can employ them to obtain $\sigma^q_{jk} = \mu^p_{kj}$ and $\lambda^q_{jk} = \tau^p_{kj}$. This will imply that $d_p(t) = -t\,\mathfrak{d}_q(t^{-1})'$, and the proposition will be proved.

We begin once again with the observation that $j_{\pm*}$ takes elements of $\bar{E}_k$ to elements of $\bar{G}_k$ so that for $\alpha \in \bar{E}_p$ and $\delta \in \bar{\mathfrak{E}}_q$,

$$L''(j_{\pm*}(\alpha) \otimes \delta) = L''(\bar{j}_{\pm*}(\alpha) \otimes \delta)$$

simply as a matter of making the obvious restrictions. On the other hand $j_{\pm*}(\delta)$ might have components in both $\bar{\mathfrak{G}}_q$ and its complementary summand. Since $\bar{\mathfrak{j}}_{\pm*}$ is $j_{\pm*}$ followed by projection to $\bar{\mathfrak{G}}_q$, we can write $j_{\pm*}(\delta) = \bar{\mathfrak{j}}_{\pm*}(\delta) + x$, where x lies in the $\ker(s)$. But E_p and $\ker(s)$ are orthogonal as in the proof of Lemma 4.9, so we have, for α and δ as above,

$$L'(\alpha \otimes j_{-*}(\delta)) = L'(\alpha \otimes \bar{\mathfrak{j}}_{\pm*}(\delta) + x) = L'(\alpha \otimes \bar{\mathfrak{j}}_{\pm*}(\delta)) + L'(\alpha \otimes x) = L'(\alpha \otimes \bar{\mathfrak{j}}_{\pm*}(\delta)).$$

Putting these together with (4.19) and (4.20) gives the desired

$$L'(\alpha \otimes \bar{\mathfrak{j}}_{-*}(\delta)) = L''(\bar{j}_{+*}(\alpha) \otimes \delta)\,,$$
$$L'(\alpha \otimes \bar{\mathfrak{j}}_{+*}(\delta)) = L''(\bar{j}_{-*}(\alpha) \otimes \delta)$$

for $\alpha \in \bar{E}_p$ and $\delta \in \bar{\mathfrak{E}}_q$. □

Corollary 4.11. $\det(d_p(t)) \sim \det(d_q(t^{-1}))$, $p + q = n - 1$, *where* $\sim$ *indicated the similarity relationship for polynomials in* Γ.

Proof. This follows immediately from Proposition 4.10 by taking determinants. □

Theorem 4.12. *Recall that* $S = \partial\overline{N(\Sigma)}$, $X = S - (K \cap S)$, *and* $\tilde{X}$ *is the infinite cyclic covering of* X. *Let* $\nu_i(t)$, $0 < i < n-2$, *be the Alexander polynomials of* $K \cap S$ *in* S. *In other words,* $\nu_i(t)$ *is the determinant of the presentation matrix of the* Γ*-module* $H_i(\tilde{X};\mathbb{Q})$. *Then* $\nu_i(t) = r_i(t)(t-1)^{\tilde{B}_i}$, *where* $\tilde{B}_i$ *is the* i^{th} *reduced Betti number of* Σ; $r_P(t) \sim r_Q(t^{-1})$, $P + Q = n - 2$; *and, if* $\nu_i(t)$ *is taken primitive in* Λ, *then* $r_i(1) = \pm 1$.

Proof. We will make us of the long exact sequence

$$\longrightarrow H_i(\tilde{X}) \xrightarrow{u_i} H_i(\tilde{C}) \xrightarrow{v_i} H_i(\tilde{C},\tilde{X}) \xrightarrow{\partial_{i*}}, \tag{4.21}$$

in which we continue to suppress the $\mathbb{Q}$'s which indicate *rational* homology. Observe that the Γ-module structure is preserved trivially at the chain level, by interpreting t as the covering transformation, so that this is an exact sequence of $\Gamma-$*modules*. Since the $H_i(\tilde{C})$ and $H_i(\tilde{C},\tilde{X})$ are $\Gamma-$torsion modules for $i \le n-2$ by Theorem 4.3, the $H_i(\tilde{X})$ must also be $\Gamma-$torsion modules for $i < n-2$. Thus, recalling that any module over a principal ideal domain can be given a square presentation matrix, $\nu_i(t)$ will be well defined as the determinant of that of $H_i(\tilde{X})$. (Equivalently, we can think of $\nu_i(t)$ as $\prod \nu_{i_j}(t)$ where $H_i(\tilde{X}) = \bigoplus_j \Gamma/(\nu_{i_j}(t))$.)

Recall that, by Corollary 2.5, we know that whenever we have an exact sequence of torsion Γ-modules, say

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4,$$

then the determinant of the presentation matrix of M_2 is the product of the determinants of the presentation matrices of $\ker(f_2)$ and $\ker(f_3) = \operatorname{im}(f_2)$.

Let c_k be the determinant of the matrix d_k above. With $\lambda_k(t)$, $\mu_k(t)$, and $\nu_k(t)$ all as above, we have then that $c_k|\mu_k$ and $c_k|\lambda_k$, each because d_k is the presentation matrix of H_k, the kernel of ∂_{k*}. Further, we must then have that $\mu_k(t)/c_k(t)$ is the determinant of the kernel of u_{k-1} and $\lambda_k(t)/c_k(t)$ is the determinant of the kernel of v_k. Thus

$$\nu_k(t) \sim \frac{\lambda_k(t)}{c_k(t)} \frac{\mu_{k+1}(t)}{c_{k+1}(t)}.$$

Recall that $\mu_q(t) \sim \lambda_p(t^{-1})(t-1)^{\tilde{B}_{q-1}}$, $p+q = n-1$, and we have just proven in Corollary 4.11 that $c_q(t) \sim c_p(t^{-1})$. Thus

$$\frac{\mu_q(t)}{c_q(t)} \sim \frac{\lambda_p(t^{-1})(t-1)^{\tilde{B}_{q-1}}}{c_p(t^{-1})}.$$

Further, since $(t-1)^{\tilde{B}_{k-1}}|\mu_k(t)$ but $(t-1) \nmid \lambda_k(t)$ (because $\lambda(1) = \pm 1$), it follows from the above formula for the decomposition of the determinants that $(t-1)^{\tilde{B}_{k-1}}|\nu_{k-1}(t)$. Therefore, if we take $p+q = n-1$, $P = p$, $Q = q-1$, and

$r_k(t) = \nu_k(t)/(t-1)^{\tilde{B}_k}$, then

$$\begin{aligned} r_P(t^{-1}) &\sim \frac{\nu_P(t^{-1})}{(t^{-1}-1)^{\tilde{B}_P}} \sim \frac{1}{(t^{-1}-1)^{\tilde{B}_p}} \cdot \frac{\lambda_p(t^{-1})}{c_p(t^{-1})} \cdot \frac{\mu_{p+1}(t^{-1})}{c_{p+1}(t^{-1})} \\ &\sim \frac{1}{(t^{-1}-1)^{\tilde{B}_p}} \cdot \frac{\mu_q(t)}{c_q(t)(t-1)^{\tilde{B}_{q-1}}} \cdot \frac{\lambda_{q-1}(t)(t^{-1}-1)^{\tilde{B}_p}}{c_{q-1}(t)} \\ &\sim \frac{1}{(t-1)^{\tilde{B}_{q-1}}} \cdot \frac{\mu_q(t)}{c_q(t)} \cdot \frac{\lambda_{q-1}(t)}{c_{q-1}(t)} \\ &\sim \frac{\nu_Q(t)}{(t-1)^{\tilde{B}_Q}} \sim r_Q(t). \end{aligned}$$

Lastly, we know, again from Corollary 4.11, that we can take each c_k, λ_k/c_k, and μ_k/c_k to be primitive in Λ, which will make λ_k, μ_k, and ν_k primitive in Λ. Since, in that case, $\lambda_k(1) = \pm 1$, we must have each of its factors $c_k(1)$ and $c_k(1)/c_k(1)$ equal to ± 1. Also, since

$$\frac{\mu_k(t)}{(t-1)^{\tilde{B}_{k-1}}} = \lambda_{n-k-1}(t) \quad \text{and} \quad c_k(t)$$

are equal to ± 1 at 1, so must be

$$\frac{\mu_k(t)}{(t-1)^{\tilde{B}_{k-1}}(t)\Delta_k(t)}.$$

But then

$$r_k(t) = \frac{\nu_k(t)}{(t-1)^{\tilde{B}_k}} = \frac{1}{(t-1)^{\tilde{B}_k}} \frac{\lambda_k(t)}{\Delta_k(t)} \frac{\mu_{k+1}(t)}{\Delta_{k+1}(t)}$$

must also be primitive in Λ and evaluate to ± 1 at 1. □

Remark 4.13. Note that when $n = 2q+2$, our duality results and the proof of the theorem imply that $r_q(t)$ is similar to a polynomial of the form $p(t)p(t^{-1})$, $p \in \Lambda$.

Remark 4.14. In the case where the singularity Σ is a point, the results of this section reduce to well-known facts about locally-flat sphere slice knots (see [19], [26], [20]).

4.2.4. *The subpolynomials.* The same algebraic considerations, which we applied in Sections ***3.5.3*** and ***3.7*** to split the three sets of Alexander polynomials of a disk knot into three sets of subpolynomials and to show that these subpolynomials satisfy their own duality relationships, readily generalize to the case of a knot with more general singularities. Note that all of the $(t-1)$ factors are shared between the relative and boundary polynomials.

Furthermore, if $c_k(t)$ is the polynomial factor shared by $H_k(\tilde{C})$ and $H_k(\tilde{C}, \tilde{X})$ (i.e. the polynomial of $\ker\left[\partial_*'' : H_k(\tilde{C}, \tilde{X}; \mathbb{Q}) \to H_{k-1}(\tilde{X}; \mathbb{Q})\right]$), then, for a knot $S^{2q-1} \subset S^{2q+1}$, we can generalize the necessary conditions we obtained for the middle dimension polynomial, $c_q(t)$, of a disk knot in Section ***3.6***. In fact, if we replace integral homology and integral pairings with rational homology and rational pairings, the computations of the presentation and pairing matrices goes through unchanged. It is only necessary to note that, in this context, the pairings L' and L'' again induce perfect pairings between certain kernels and coimages (or cokernels) of diagram (4.9), but this is shown in Section ***4.2.3***. Therefore, we have the following proposition:

Proposition 4.15. *$H_q = \ker(\partial_*'' : H_q(\tilde{C}, \tilde{X}; \mathbb{Q}) \to H_{q-1}(\tilde{X}; \mathbb{Q}))$ has a presentation matrix of the form $\tau t - (-1)^{q+1} R'\tau' R^{-1}$, where R is the matrix of the map $\tilde{\mathfrak{E}} \to \bar{E}$ induced by $\tilde{r} : H_q(V) \to H_q(V, F)$. $\mathfrak{H}_q = H_q(\tilde{C}; \mathbb{Q}) / \ker(H_q(\tilde{C}; \mathbb{Q}) \to H_q(\tilde{C}, \tilde{X}; \mathbb{Q}))$ has presentation matrix $(-1)^{q+1}(R^{-1})'\tau R t - \tau'$. Furthermore, there is $(-1)^{q+1}$-Hermitian pairing $\langle\,,\rangle : \mathfrak{H}_q \times \mathfrak{H}_q \to Q(\Gamma)/\Gamma$ with matrix representative*

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' t R^{-1}}$$

with respect to the appropriate basis.

All of these necessary conditions on the polynomials can now be summarized in the following theorem. The duality conditions on the Alexander subpolynomials follows from that on the Alexander polynomials as in the proof of Lemma 3.17. The only change, in fact, is the need to keep special track of the $(t-1)$ factors, but, as already noted, we know that these must all divide the a_i.

Theorem 4.16. *Let $\nu_j(t)$, $\lambda_i(t)$, and $\mu_i(t)$, $0 < j < n-2$ and $0 < i < n-1$, denote the Alexander polynomials corresponding to $H_j(\tilde{X})$, $H_i(\tilde{C})$, and $H_i(\tilde{C}, \tilde{X})$, respectively, of a knotted $S^{n-2} \subset S^n$. We can assume these polynomials to be primitive in Λ. Then there exist polynomials $a_i(t)$, $b_i(t)$, and $c_i(t)$, primitive in Λ, such that*

1. $\nu_j(t) \sim a_j(t) b_j(t)$,
2. $\lambda_i(t) \sim b_i(t) c_i(t)$,
3. $\mu_i(t) \sim c_i(t) a_{i-1}(t)$,
4. $a_i(t) \sim b_{n-2-i}(t^{-1})(t-1)^{\bar{B}_i}$,
5. $c_i(t) \sim c_{n-1-i}(t^{-1})$,
6. $b_i(1) = \pm 1$,
7. $c_i(1) = \pm 1$,
8. *if $n = 2q+1$, then $c_q(t)$ is the determinant of a matrix of the form $(R^{-1})'\tau R t - (-1)^{q+1}\tau'$, where τ and R are matrices such that R has non-zero determinant.*

Furthermore, if $n = 2q+1$, there is a $(-1)^{q+1}$-Hermitian pairing $\langle\,,\rangle : \mathfrak{H}_q \times \mathfrak{H}_q \to Q(\Gamma)/\Gamma$ with matrix representative

$$\frac{t-1}{(R^{-1})'\tau - (-1)^{q+1}\tau' t R^{-1}}$$

with respect to an appropriate basis, where $\mathfrak{H}_q = \operatorname{cok}(c)$ *in diagram* (4.9) *above.*

Remark 4.17. Note that $H_0(\tilde{C};\mathbb{Q}) \cong \mathbb{Q} \cong \Gamma/(t-1)$, since $\tilde{C}$ is connected and the action of t on $H_0(\tilde{C};\mathbb{Q})$ is trivial. Similarly, $H_0(\tilde{X};\mathbb{Q}) \cong \bigoplus \mathbb{Q} \cong \bigoplus \Gamma/(t-1)$, where the number of summands is equal to the number of components of Σ. And of course, $H_0(\tilde{C},\tilde{X}) = 0$. Therefore, by the long exact polynomial sequence of the knot, it is consistent in the above theorem to take $a_0(t) \sim (t-1)^{\bar{B}_0}$.

4.2.5. *High dimensions.* For completeness, we observe the following concerning the triviality of the knot homology modules in the dimensions above those which we have treated in detail. We maintain the notation above.

Proposition 4.18.

1. $H_i(\tilde{X};\mathbb{Q}) \cong 0$ *for* $i \geq n-2$.
2. $H_i(\tilde{C};\mathbb{Q}) \cong 0$ *for* $i \geq n-1$.
3. $H_i(\tilde{C},\tilde{X};\mathbb{Q}) \cong 0$ *for* $i \geq n-1$.

Proof. The assertion for $H_i(\tilde{C},\tilde{X};\mathbb{Q})$ will follow from the other two and the long exact sequence of the pair.

The proposition is trivial for $i \geq n-1$ in the case of $\tilde{X}$ and for $i \geq n$ in the case of $\tilde{C}$ because $\tilde{X}$ and $\tilde{C}$ are noncompact manifolds of respective dimensions $n-1$ and n.

To show that $H_{n-1}(\tilde{C};\mathbb{Q}) \cong 0$, we can employ Assertion 9 of Milnor [25], which states that for M a compact triangulated $n-$manifold, $\tilde{M}$ the infinite cyclic cover, there is a perfect orthogonal pairing to $\mathbb{Q}$ of $H_{i-1}(\tilde{M};\mathbb{Q}) \cong H^{i-1}(\tilde{M};\mathbb{Q})$ and $H_{n-i}(\tilde{M},\partial\tilde{M};\mathbb{Q}) \cong H^{n-i}(\tilde{M},\partial\tilde{M};\mathbb{Q})$, provided that $H_*(\tilde{M};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$. We can take $i = 1$ and $M = C$ (replacing C by the homotopy equivalent knot exterior to get compactness). Then $H_0(\tilde{C},\partial\tilde{C};\mathbb{Q}) \cong 0$, which implies $H_{n-1}(\tilde{C};\mathbb{Q}) \cong 0$, provided $H_*(\tilde{M};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$. But Assertion 5 of the same paper states that this holds if M is a homology circle over $\mathbb{Q}$, which we know to be true by Alexander duality.

The same argument holds to show that $H_{n-2}(\tilde{X};\mathbb{Q}) \cong 0$ provided $H_*(\tilde{X};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$. We know that $H_i(\tilde{X};\mathbb{Q})$ is 0 for $i \geq n-1$ and that it is a torsion Γ-module for $0 < i < n-2$, which implies that it is finite dimensional over $\mathbb{Q}$ in this dimension range. $H_0(\tilde{X};\mathbb{Q})$ is also finite dimensional, being equal in dimension to the finite number of components of Σ. Therefore, it only remains to show that $H_{n-2}(\tilde{X};\mathbb{Q})$ is finite dimensional over $\mathbb{Q}$. For this, we will show directly that $H_{n-1}(\tilde{C},\tilde{X};\mathbb{Q}) = 0$. Then, because $H_{n-2}(\tilde{C};\mathbb{Q})$ is finite dimensional (in fact a torsion Γ-module), the result for $H_{n-2}(\tilde{X};\mathbb{Q})$ will follow from the long exact sequence of the pair.

To prove that $H_{n-1}(\tilde{C},\tilde{X};\mathbb{Q}) = 0$, we once again employ the Mayer-Vietoris sequence of the covering:

$$\longrightarrow H_{n-1}(V,F;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \longrightarrow H_{n-1}(Y,Z;\mathbb{Q}) \otimes_{\mathbb{Q}} \Gamma \longrightarrow H_{n-1}(\tilde{C};\tilde{X};\mathbb{Q}) \xrightarrow{0} .$$

The last map can be taken as the zero map because we know that

$$H_{n-2}(V,F;\mathbb{Q}) \longrightarrow H_{n-2}(Y,Z;\mathbb{Q})$$

is injective from the proof of Theorem 4.3. So the proof will be complete if we show that $H_{n-1}(Y,Z;\mathbb{Q}) = 0$. But we saw in Section ***4.2*** that $H_{n-1}(Y,Z;\mathbb{Q}) \cong H_{n-1}(\Omega,\Sigma;\mathbb{Q})$, and, since Σ is a complex of dimension at most $n-4$, this is isomorphic to $H_{n-1}(\Omega;\mathbb{Q})$. Hence it suffices to show that this group is 0.

Consider the long exact sequence (with rational coefficients suppressed in the notation)

$$H_n(\Omega) \longrightarrow H_n(S^n) \longrightarrow H_n(S^n, \Omega) \longrightarrow H_{n-1}(\Omega) \longrightarrow H_{n-1}(S^n).$$

$H_{n-1}(S^n) = 0$ and $H_n(S^n) \cong \mathbb{Q}$, trivially, and $H_n(\Omega) = 0$ by Alexander duality. Furthermore, by Lefschetz duality, $H_n(S^n, \Omega) \cong H^0(S^n - \Omega) \cong \mathbb{Q}$ since $S^n - \Omega \sim_{h.e.} V$ is connected. Therefore, the long exact sequence reduces to

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow H_{n-1}(\Omega; \mathbb{Q}) \longrightarrow 0.$$

Since any injective map $\mathbb{Q} \longrightarrow \mathbb{Q}$ must be an isomorphism, the result follows. $\square$

4.3. Constructions. For the case of knots with general singularities, realization of polynomials is more difficult than it was for the case of point singularities because the allowable set of polynomials may depend subtly on the properties of the singular set, its link pairs, and its embedding. However, in the following sections, we will employ several constructions to create knots with singularities and to compute their Alexander polynomials. These will provide at least partial realization results.

In Section ***4.3.1***, we will use the *frame spinning* of Roseman [30] (generalized to spin non-locally-flat knots) to construct knots with certain kinds of manifold singularities. In Section ***4.3.2***, we further generalize this construction to create *frame twist-spinning*. Together, these procedures include as special cases the superspinning of Cappell [4] and the twist spinning of Zeeman [40]. In Section ***4.3.3***, we construct knots by suspension.

4.3.1. *Frame spinning.* To construct some examples of knots with a given singular stratum, we will employ the technique of *frame spinning*, which was introduced by Roseman in [30] and studied further by Suciu [36] and Klein and Suciu [16]. It generalizes the earlier techniques of spinning and the superspinning of Cappell [4]. We begin by describing this procedure.

Let K be a knot $S^{m-2} \subset S^m$, and let M^k be a k-dimensional framed submanifold of S^{m+k-2} with framing ϕ. Suppose that S^{m+k-2} is embedded in S^{m+k} by the standard (unknotted) embedding. Roughly speaking, the frame spun knot $\sigma_M^{\phi}(K)$ is formed by removing a standard disk pair (D^m, D^{m-2}) at each point of M and replacing it with the disk knot obtained by removing a neighborhood of a nonsingular point of the knot K.

More specifically, let (D^m_-, D^{m-2}_-) be an unknotted open disk pair which is the open neighborhood pair of a point which does not lie in the singularity of the embedding of the knot $K \subset S^m$. Let $(D^m_+, D^{m-2}_+) = (S^m, K) - (D^m_-, D^{m-2}_-)$. This is a disk knot, possibly non-locally-flat, with the unknotted locally-flat sphere pair as boundary. Let $M^k \times D^{m-2}$ be the normal bundle of $M^k \in S^{m+k-2}$ determined by the trivialization ϕ. Finally, writing S^{m+k} as $S^{m+k} = S^{m+k-2} \times D^2 \cup_{S^{m+k-2} \times S^1}$

$D^{m+k-1} \times S^1$, we let $S^{m+k-2} \times 0$ in the first factor represent the unknot in which M is embedded.

Now define $\sigma_M^\phi(K)$ to be the $(m+k-2)$-sphere

$$(S^{m+k-2} - M^k \times \text{int } D^{m-2}) \cup_{M^k \times S^{m-3}} M^k \times D_+^{m-2}$$

embedded in the $(m+k)$-sphere

$$(S^{m+k} - M^k \times \text{int } (D^{m-2} \times D^2)) \cup_{M^k \times S^{m-1}} M^k \times D_+^m.$$

This construction corresponds to removing, for each point of M, the trivial disk pair (D^m, D^{m-2}), which is the fiber of the normal bundle of M, and replacing it with the knotted disk pair (D_+^m, D_+^{m-2}). In the above references, K is always assumed to be a locally-flat knot, but there is nothing to prevent us from applying this construction to a non-locally-flat knot so long as we are careful to embed (D_-^m, D_-^{m-2}) away from the singularity. Observe that, in the case where M^k is the sphere S^k with the standard unknotted embedding and bundle framing, $\sigma_M^\phi(K)$ is the superspin of K (see [4]).

Let $n = m + k$. We obtain the following:

Proposition 4.19. *Let M^k be a manifold which can be embedded in S^{n-2} with trivial normal bundle. Then there exists a knot $S^{n-2} \subset S^n$ with M as its only singular stratum.*

Proof. Let K be any knot $S^{m-2} \subset S^m$ whose singular set constists of a single point. Let ϕ be a trivialization for the normal bundle of the embedding of $M \in S^{n-2}$. Then $\sigma_M^\phi(K)$ provides an example. □

To study the Alexander polynomials that occur from such constructions, we first need a geometric formula for the exterior of a frame-spun knot. This is provided, as follows, by Suciu in [36], although we adopt our own notations. Throughout this section, let $X(\cdot)$ denote the exterior of a knot and $\tilde{X}(\cdot)$ the corresponding infinite cyclic covering. $X(K)$, the exterior of the knot K, is homeomorphic to the exterior of the induced disk knot (D_+^m, D_+^{m-2}). Its intersection with the exterior of the induced boundary sphere pair is $D^{m-2} \times S^1$ because the induced boundary sphere pair of the disk knot is unknotted. Let $M^k \times \text{int}(D^{m-1})$ represent the intersection of the tubular neighborhood of M^k in S^n with $D^{n-1} \times 0 \in D^{n-1} \times S^1$, the exterior of the trivial knot $S^{n-2} \subset S^n$. It can be seen that

$$X(\sigma_M^\phi(K)) = (D^{n-1} - M^k \times \text{int}(D^{m-1})) \times S^1 \cup_{M^k \times D^{m-2} \times S^1} M^k \times X(K).$$

In the following lemma, we use Cov[$\cdot$] to denote the infinite cyclic covering where the tilde notation would be unwieldly.

Lemma 4.20.

$$
\begin{aligned}
&\tilde{X}(\sigma_M^{\phi}(K))\\
&\quad = \mathrm{Cov}\Big[(D^{n-1} - M^k \times \mathrm{int}\,(D^{m-1})) \times S^1\Big] \cup_{\mathrm{Cov}[M^k \times D^{m-2} \times S^1]} \mathrm{Cov}\Big[M^k \times X(K)\Big]\\
&\quad \sim_{h.e.} \Big(D^{n-1} - M^k \times \mathrm{int}\,(D^{m-1})\Big) \times \mathbb{R} \cup_{M^k \times D^{m-2} \times \mathbb{R}} M^k \times \tilde{X}(K)\,.
\end{aligned}
$$

Proof. As observed in [16], if V is a Seifert surface for the knot K and we define

$$\sigma_M^{\phi}(V) = (D^{n-1} - M^k \times \mathrm{int} D^{m-1}) \cup_{M^k \times D^{m-2}} M^k \times V,$$

then $\sigma_M^{\phi}(V)$ is a Seifert surface for $\sigma_M^{\phi}(K)$ (Note that if K is not locally-flat then we mean the knot exteriors and Seifert surfaces in the sense of Section ***4.2***). Using this Seifert surface we can form the infinite cyclic cover $\tilde{X}(\sigma_M^{\phi}(K))$ by the usual "cut and past" construction. From this, the first equation follows by considering what the construction does on each piece. The second equation follows from the observation that the covering space can be obtained by "unwrapping" S^1 to $\mathbb{R}$. ◻

Remark 4.21. The ability to create the Seifert surface in this manner relies heavily on the following fact: While the particular framing of M may serve to "spin" the knots K tangentially to S^{n-2}, the knots are never "twisted". No rotation takes place along the meridians circling S^{n-2} in S^n. Thus, contrary to a remark of Roseman [30], frame spinning can not yield instances of Zeeman's *twist spinning* [40]. In cases involving twisting, it is not always possible to get the Seifert surfaces to "line up" so that they may be connected by a disk (although this can happen in special cases, particularly with fibered knots where the Seifert surfaces can be forced to align by "rotating them around the fibration"). However, see the following section (Section ***4.3.2***), in which we introduce a method to obtain such twisting.

We can now use a Mayer-Vietoris sequence to study the Alexander modules of $\sigma_M^{\phi}(K)$. In particular, we have the rational exact sequence (in which we suppress the $\mathbb{Q}$'s from the notation)

$$\to \tilde{H}_i(M^k \times D^{m-2} \times \mathbb{R}) \to \tilde{H}_i(D^{n-1} \times \mathbb{R}) \oplus \tilde{H}_i(M^k \times \tilde{X}(K)) \to \tilde{H}_i(\tilde{X}(\sigma_M^{\phi}(K))) \to,$$

in which we have used the homotopy equivalence of $D^{n-1} - M^k \times \mathrm{int}\,(D^{m-1})$ and D^{n-1} to replace $\tilde{H}_i((D^{n-1} - M^k \times \mathrm{int}\,(D^{m-1})) \times \mathbb{R}; \mathbb{Q})$ with $\tilde{H}_i(D^{n-1} \times \mathbb{R}; \mathbb{Q})$. This sequence simplifies in the obvious manner to

$$\longrightarrow \tilde{H}_i(M^k; \mathbb{Q}) \xrightarrow{i_*} \tilde{H}_i(M^k \times \tilde{X}(K); \mathbb{Q}) \longrightarrow \tilde{H}_i(\tilde{X}(\sigma_M^{\phi}(K)); \mathbb{Q}) \longrightarrow\,. \tag{4.22}$$

From this, we will prove:

Proposition 4.22. *Let B_i be the i^{th} Betti number of M^k. Let $\lambda_j(t)$ be the j^{th} Alexander polynomial of K and $\lambda_i^\sigma(t)$ the i^{th} Alexander polynomial of $\sigma_M^\phi(K)$. Then, for $0 < i < n-1$,*

$$\lambda_i^\sigma(t) = \prod_{\ell=1}^{m-2} [\lambda_\ell(t)]^{B_{i-\ell}}.$$

Proof. We first study $H_i(M^k \times \tilde{X}(K); \mathbb{Q})$, which, by the Künneth theorem, is

$$H_i(M^k \times \tilde{X}(K); \mathbb{Q}) \cong \bigoplus_{j+\ell=i} H_j(M^k; \mathbb{Q}) \otimes_{\mathbb{Q}} H_\ell(\tilde{X}(K); \mathbb{Q}).$$

Now let B_i be the i^{th} Betti number of M^k, and, for a rational vector space A, let A^i denote the direct sum of i copies of A. Since $A \otimes_{\mathbb{Q}} \mathbb{Q}^i = A^i$,

$$H_i(M^k \times \tilde{X}(K); \mathbb{Q}) \cong \bigoplus_{j+\ell=i} [H_\ell(\tilde{X}(K); \mathbb{Q})]^{B_j}. \tag{4.23}$$

To establish the Γ-module structure, it is evident from the geometry that if

$$\alpha \otimes_{\mathbb{Q}} \beta \in H_j(M^k; \mathbb{Q}) \otimes_{\mathbb{Q}} H_\ell(\tilde{X}(K)),$$

then $t(\alpha \otimes_{\mathbb{Q}} \beta) = \alpha \otimes_{\mathbb{Q}} (t\beta)$. Therefore, equation (4.23) is a Γ-module isomorphism.

For the knot $K : S^{m-2} \hookrightarrow S^m$, recall that

1. $H_0(\tilde{X}(K); \mathbb{Q}) = \mathbb{Q}$,
2. $H_i(\tilde{X}(K); \mathbb{Q}) = 0$, $i \geq m-1$,

by Proposition 4.18. Taking this into account,

$$H_i(M^k \times \tilde{X}(K); \mathbb{Q}) \cong \bigoplus_{j+\ell=i, \ell<m-1} [H_\ell(\tilde{X}(K); \mathbb{Q})]^{B_j}.$$

We next study the map $i_* : H_i(M^k; \mathbb{Q}) \longrightarrow H_i(M^k \times \tilde{X}(K); \mathbb{Q})$ in the Mayer-Vietoris sequence (4.22). From the geometric constructions above and consideration of the chain maps used to define the Mayer-Vietoris sequence and Künneth theorem, i_* is the map which takes an element $\alpha \in H_i(M^k; \mathbb{Q})$ to $\alpha \otimes \{*\}$ in the submodule $H_i(M^k; \mathbb{Q}) \otimes H_0(\tilde{X}(K); \mathbb{Q})$ of $H_i(M^k \times \tilde{X}(K); \mathbb{Q})$, where $\{*\}$ is a point representing the generator of $H_0(\tilde{X}(K); \mathbb{Q})$. It follows that i_* is injective, and thus the Mayer-Vietoris sequence gives

$$H_i(\tilde{X}(\sigma_M^\phi(K)); \mathbb{Q}) \cong \bigoplus_{j+\ell=i, 0<\ell<m-1} [H_\ell(\tilde{X}(K); \mathbb{Q})]^{B_j}.$$

So, if $\lambda_j(t)$ is the j^{th} Alexander polynomial of K, then the i^{th} Alexander polynomial of $\sigma_M^{\phi}(K)$ is

$$\lambda_i^{\sigma}(t) = \prod_{\ell=1}^{m-2} [\lambda_\ell(t)]^{B_{i-\ell}}$$

since the polynomial associated to a direct sum of torsion Γ-modules is the product of the polynomials associated to the summands. $\square$

Now assume that the knot K has singular set, Σ. Then $\sigma_M^{\phi}(K)$ will have singular set $\Sigma \times M$ stratified by $(\Sigma \times M)_i = \Sigma_{i-k} \times M$. We can use our previous duality results (Theorem 4.3) to calculate the relative Alexander polynomials of the pair given by the spun knot complement and the link pair complement of $\Sigma \times M$. In particular, let B_i continue to denote the i^{th} Betti number of M, let $\tilde{\mathfrak{b}}_i$ denote the i^{th} reduced Betti number of Σ, and let $\tilde{\beta}_i$ denote the i^{th} reduced Betti number of $M \times \Sigma$. Then, for $i > 0$,

$$\begin{aligned}
\mu_i^{\sigma}(t) &\sim (t-1)^{\tilde{\beta}_{i-1}} \lambda_{n-1-i}^{\sigma}(t^{-1}) \\
&= (t-1)^{\tilde{\beta}_{i-1}} \prod_{\ell=1}^{m-2} [\lambda_\ell(t^{-1})]^{B_{n-1-i-\ell}} \\
&\sim (t-1)^{\tilde{\beta}_{i-1}} \prod_{\ell=1}^{m-2} \left[\frac{\mu_{m-1-\ell}(t)}{(t-1)^{\tilde{\mathfrak{b}}_{m-2-\ell}}} \right]^{B_{n-1-i-\ell}}.
\end{aligned}$$

Rather than explore the relations among these Betti numbers directly, we can simplify this formula by alternatively studying the relative Alexander polynomial directly using a Mayer-Vietoris sequence, as we did for the $\lambda_i^{\sigma}(t)$. The one significant difference is that the relative homology module for K in dimension 0 is $H_0(\tilde{X}(K), \mathrm{Cov}(X \cap \overline{N(\Sigma)})) = 0$ so that instead of all of the maps i_* of the Mayer-Vietoris sequence being injective, they are all 0 instead. Since $\tilde{H}_i(M^k; \mathbb{Q}) \cong (\Gamma/(t-1))^{\tilde{B}_{i-1}}$, because the Γ action of t on the cover $M^k \times \mathbb{R}$ induces the identity map on the homology, we can conclude by polynomial algebra that

$$\mu_i^{\sigma}(t) = (t-1)^{\tilde{B}_{i-1}} \prod_{\ell=0}^{m-2} [\mu_\ell(t)]^{B_{i-\ell}}$$

for $i > 0$. Note that $\mu_0 = 1$.

Letting $\nu_i(t)$ denote the i^{th} Alexander polynomial of the link pair of Σ for the knot K, the i^{th} Alexander polynomial of the link pair of $\Sigma \times M$ for the knot $\sigma_M^{\phi}(K)$ is easily derived from the Künneth theorem to be

$$\nu_i^{\sigma}(t) = \prod_{\ell=0}^{m-3} [\nu_\ell(t)]^{B_{i-\ell}}.$$

Note that $\nu_0(t) = (t-1)^{\tilde{\mathfrak{b}}_0+1}$.

As we know, the Alexander polynomials of $\sigma_M^{\phi}(K)$ factor into Alexander subpolynomials $a_i^{\sigma}(t)$, $b_i^{\sigma}(t)$, and $c_i^{\sigma}(t)$. It is an exercise with the long exact sequences to show that this factorization is preserved under the spinning, modulo some minor extra complication in the $t-1$ factors. In other words,

$$b_i^{\sigma}(t) = \prod_{\ell=1}^{m-2} [b_\ell(t)]^{B_{i-\ell}} \tag{4.24}$$

$$c_i^{\sigma}(t) = \prod_{\ell=1}^{m-2} [c_\ell(t)]^{B_{i-\ell}} \tag{4.25}$$

$$a_i^{\sigma}(t) = (t-1)^{\tilde{\beta}_i} \prod_{\ell=1}^{m-2} \left[\frac{a_\ell(t)}{(t-1)^{\tilde{\mathfrak{b}}_\ell}} \right]^{B_{i-\ell}}. \tag{4.26}$$

For example, to perform the calculation for the c_i, let $\tilde{L}(K)$ represent the the infinite cyclic covering of the intersection of the knot exterior $X(K)$ with the closed neighborhood of the singularity $\bar{N}(\Sigma)$ (i.e. the "link exterior", the usual subset for our relative homology modules) and similarly for $\sigma_M^{\phi}(K)$. The above calculations show that the kernel module of the map of the long exact sequence

$$H_i(\tilde{X}(\sigma_M^{\phi}(K));\mathbb{Q}) \longrightarrow H_i(\tilde{X}(\sigma_M^{\phi}(K)), \tilde{L}(\sigma_M^{\phi}(K));\mathbb{Q})$$

is isomorphic to the kernel of the map

$$\bigoplus_{j+\ell=i,\ 0<\ell<m-1} \left[H_\ell(\tilde{X}(K);\mathbb{Q})\right]^{B_j} \xrightarrow{f} \bigoplus_{j+\ell=i,\ \ell<m-1} \left[H_\ell(\tilde{X}(K), \tilde{L}(K);\mathbb{Q})\right]^{B_j}$$

because we know that $H_i(\tilde{X}(\sigma_M^{\phi}(K));\mathbb{Q})$ maps trivially to the other summand of

$$H_i(\tilde{X}(\sigma_M^{\phi}(K)), \tilde{L}(\sigma_M^{\phi}(K));\mathbb{Q}),$$

which consists of a sum of $\mathbb{Q}$'s with trivial Γ-action, i.e. $\Gamma/(t-1)$'s. (The triviality of this part of the map, f, is a result of the splitting of maps of torsion modules into their p-primary summands (see the proof of Proposition 2.1).) But from the Künneth theorem, this map is induced by the usual map $p_* : H_\ell(\tilde{X}(K);\mathbb{Q}) \to H_\ell(\tilde{X}(K), \tilde{L}(K);\mathbb{Q})$ in the long exact homology sequence tensored with the identity map on the homology of M. Since we are working with rational homology, this tensor product is an exact functor and so the kernel of the map as a rational vector space is $\bigoplus H_j(M) \otimes \ker p_*$. But the Γ-module structure is also evidently preserved, acting trivially on the $H_j(M)$ factors and with the action on $\ker p_*$ induced by that on $H_\ell(\tilde{X}(K);\mathbb{Q})$. Passing from the modules to the polynomials gives the above equation for $c_i^\sigma(t)$. The other equations are handled similarly modulo their $(t-1)$ factors, these terms being accounted for separately by consideration of what the $t-1$ factors must be according to the duality formulas of Theorem 4.3.

Perhaps a simpler way to look at what happens to the Alexander polynomials of a frame spun knot is the following interpretation which follows readily (with a little checking) from the calculations above. Take, for example, the polynomial $\lambda_i(t)$ of the knot K for some i. This polynomial will be a factor of $\lambda_j^\sigma(t)$ a number of times equal to the the Betti number B_{j-i} of M. So if, for example, we take $M = S^k$, each $\lambda_i(t)$ will appear exactly twice as a factor of the $\lambda_j^\sigma(t)$, once in its "native" dimension i and once k dimensions higher. Similar consideration apply for all of the other polynomials and subpolynomials modulo the $t-1$ terms which can be computed at the end by tallying the reduced Betti numbers.

By taking Σ to be a point we can therefore construct a knot with singularity M and certain specified polynomials as follows:

Proposition 4.23. *Let M^k be a manifold which embeds in S^{n-2} with trivial normal bundle with framing ϕ and such that $n-k>3$. Let Σ be a single point. Let B_i denote the i^{th} Betti number of M, and let $\tilde{\mathfrak{b}}_i$ and $\tilde{\beta}_i$ denote the i^{th} reduced Betti numbers of Σ and $M\times\Sigma$, respectively. Suppose that we are given any set of polynomials, $a_i(t)$, $b_i(t)$, $c_j(t)$ and $c'_\ell(t)$, $0<i<n-k-2$, $0<j<n-k-1$, and $0<\ell<n-1$, which satisfy:*

1. $a_i(t) \sim b_{n-k-2-i}(t^{-1})$,
2. $c_i(t) \sim c_{n-k-1-i}(t^{-1})$,
3. $c'_i(t) \sim c'_{n-1-i}(t^{-1})$,
4. $b_i(1) = \pm 1$,
5. $c_i(1) = \pm 1$,
6. $c'_i(1) = \pm 1$,
7. *if $n-k=2p+1$, p even, $p\neq 2$, then $c_p(t)$ is the determinant of a matrix of the form $(R^{-1})'\tau R t - (-1)^{q+1}\tau'$, where τ and R are integer matrices such that R has non-zero determinant and $(R^{-1})'\tau R$ is an integer matrix; if $n-k=2p+1$, p even, $p=2$, then $|c_p(-1)|$ is an odd square,*
8. *if $n=2q+1$, q even, then $|c'_q(-1)|$ is an odd square.*

Then there exists a knotted $S^{n-2}\subset S^n$ with singular set M and Alexander subpolynomials $a_i^\sigma(t)$, $b_i^\sigma(t)$, and $c_i^\sigma(t)$ satisfying

$$a_i^\sigma(t) \sim (t-1)^{\tilde{\beta}_i} \prod_{\ell=1}^{m-2} \left[a_\ell(t)\right]^{B_{i-\ell}},$$

$$b_i^\sigma(t) \sim \prod_{\ell=1}^{m-2}[b_\ell(t)]^{B_{i-\ell}},$$

$$c_i^\sigma(t) \sim c_i'(t)\prod_{\ell=1}^{m-2}[c_\ell(t)]^{B_{i-\ell}}.$$

(*The first equation comes from equation* (4.26) *by taking into account that* Σ *is being taken as a point. Note that this also implies that* $\tilde{\beta}_j$ *is just the reduced* j^{th} *Betti number of* M.)

Fist, we need one lemma regarding disk knots which was not available before our discussion of frame spinning. It will be proven below.

Lemma 4.24. *For any* $n \geq 4$*, there exists a locally-flat disk knot,* $D^{n-2} \subset D^n$*, with non-trivial boundary knot and with all Alexander polynomials equal to* 1*. Equivalently, there exists a sphere knot,* $S^{n-2} \subset S^n$*, with a point singularity and with all Alexander polynomials equal to* 1*.*

Proof of the Proposition. By the results of Section 3 and Lemma 4.24, there exists a sphere knot, K, with a single point singularity and with the desired polynomials $a(t)$, $b(t)$, and $c(t)$ (if the construction of Section 3 yields a locally-flat knot, we can take the knot sum with the knot of the lemma). Then, by the calculations above, $\sigma_M^\phi(K)$ has the desired $a_i^\sigma(t)$, $b_i^\sigma(t)$, and $c_i^\sigma(t)$ except for the $c_i'(t)$ factors. To get the latter, we can take the knot sum with a locally-flat knot $S^{n-2} \subset S^n$ which has the $c_i'(t)$ as its Alexander polynomials. Such a knot exists by [19]. $\square$

Proof of Lemma 4.24. It is well known that there exist nontrivial locally-flat knots $S^1 \subset S^3$ whose Alexander polynomials are trivial but whose knot groups are nontrivial (see, for example, [29]). For $n > 4$, we can now frame spin one of these knots about the sphere S^{n-4} with the trivial framing and embedding in S^{n-3} to obtain a locally-flat knot $K : S^{n-3} \hookrightarrow S^{n-1}$. For $n = 4$, we can simply choose K to be the knot with which we started. In each case the knot is still nontrivial because superspinning preserves knot groups by Cappell [4]. Next, we convert K to a disk knot, $L : D^{n-3} \hookrightarrow D^{n-1}$, by removing a trivial disk pair neighborhood of a point on the knot, just as in the first step of the frame spinning construction. Lastly, we take as our desired disk knot the product of the disk knot, L, with an interval I, $L \times I : D^{n-3} \times I \hookrightarrow D^{n-1} \times I$. Since its exterior is homotopy equivalent to the exterior of L, all of its Alexander polynomials $\lambda_i(t)$ are trivial. By duality, the μ_i are also trivial. The ν_i are then trivial by polynomial algebra from the long exact sequence of the knot pair. The boundary knot is the knot sum $K\#(-K)$, where $-K$ denotes the reflection of K. Therefore, by the Van Kampen theorem, the group of the boundary knot is nontrivial and so $L \times I$ is, in fact, knotted.

We have thus produced a disk knot, $L \times I$, with the desired properties. To obtain the desired sphere knot with point singularity, we simply take the cone on the boundary of the disk knot. □

4.3.2. *Frame twist-spinning.* We now slightly generalize the frame-spinning construction to include "twisting". In the special case where we frame twist-spin about a circle, S^1, embedded with standard framing in S^{n-2}, we will obtain the twist-spun knots of Zeeman [40].

Before beginning the construction, we recall that one alternative way to compute Alexander modules and hence Alexander polynomials is the following: Rather than considering the infinite cyclic cover of the knot complement and its homology with rational coefficients, we can instead consider the homology of the knot complement with a certain local coefficient system with Γ as the stalk. If α is an element of the fundamental group of the knot complement and $\ell(\alpha)$ denotes the linking number of α with the knot, then the action of the fundamental group on the stalk module is given by $\alpha(\gamma) = t^{\ell(\alpha)}\gamma$, and this completely determines the coefficient system which we shall call $\mathit{\Gamma}$. It is not hard to see that the (simplicial or singular) chain complex of Γ-modules determined by this coefficient system on the knot complement is equivalent to the chain complex of the infinite cyclic cover with rational coefficients. Thus, if X stands for the knot complement, the homology modules $H_*(X; \mathit{\Gamma})$ and $H_*(\tilde{X}; \mathbb{Q})$ are isomorphic. (See, for example, [10] for a related discussion of the relationship between homology with local coefficients and homology of covering spaces).

The procedure for forming a frame twist-spun knot from a lower dimensional knot is similar to the procedure for frame spinning except that we add a "longitudinal twist" to the gluing. To set up the proper language, we adapt some notation from Section 6 of Zeeman's paper, [40], in which he introduces twist spinning. Following Zeeman, if we consider the unit sphere S^{m-1} in the Euclidean space $\mathbb{R}^m = \mathbb{R}^{m-2} \times \mathbb{R}^2$, then we can define the latitude for a point $y \in S^{m-1}$ as its projection onto $\mathbb{R}^{m-2}$ and its longitude as the angular polar coordinate of the projection of y onto the $\mathbb{R}^2$ term. Hence the latitude is always well-defined, while the longitude is either undefined or a unique point of S^1 dependent on whether or not y lies in the sphere S^{m-3} that is the intersection of S^{m-1} with $\mathbb{R}^{m-2} \times 0$. Notice that in the case where the longitude in undefined, the point on the sphere is uniquely determined by its latitude (just as on a standard globe). As in Zeeman's paper, to simplify the notation in abstract cases, we will simply refer to the latitude-longitude coordinates, (z, θ), in either case.

Now, just as for frame spinning, we choose a knot $K \subset S^m$ and form the pair $(D^m_+, D^{m-2}_+) = (S^m, K) - (D^m_-, D^{m-2}_-)$ by removing a trivial (unknotted) disk pair. We can then identify the trivial boundary sphere pair (S^{m-1}, S^{m-3}) with the unit sphere of the preceding paragraph and its intersection with $\mathbb{R}^{m-2} \times 0$. Thus, each boundary point in (S^{m-1}, S^{m-3}) can be described by its latitude and longitude coordinates $(z, \theta) \in D^{m-2} \times S^1$. Then $M^k \times (D^m_+, D^{m-2}_+)$ gives a bundle of knots, and the points in $\partial[M^k \times (D^m_+, D^{m-2}_+)]$ have coordinates (x, z, θ), where $x \in M$ and (z, θ) are the latitude-longitude coordinates of $\partial(D^m, D^{m-2})$.

Similarly, given an embedding of $M^k \subset S^{m+k-2}$ with framing ϕ, where S^{m+k-2} is the $(m+k-2)$-sphere embedded in S^{m+k} with the standard normal bundle, we form

$$(S^{m+k}, S^{m+k-2}) - M^k \times \mathrm{int}(D^{m-2} \times D^2, D^{m-2})$$

as in the frame spinning construction (Section *4.3.1*). Again the boundary can be identified as $M^k \times (S^{m-1}, S^{m-3})$, and the framing ϕ, together with the trivial framing of S^{m+k-2} in S^{m+k}, allows us to assign to this boundary the same (x, z, θ)-coordinates.

Given a map $\tau : M^k \to S^1$, we can form the frame twist-spun knot $\sigma_M^{\phi,\tau}(K)$ as

$$[(S^{m+k}, S^{m+k-2}) - M^k \times \mathrm{int}(D^{m-2} \times D^2, D^{m-2})] \cup_f [M^k \times (D^m_+, D^{m-2}_+)],$$

where f is the attaching homeomorphism of the boundaries

$$f : \partial[M^k \times (D^m_+, D^{m-2}_+)] \to \partial[(S^{m+k}, S^{m+k-2}) - M^k \times \mathrm{int}(D^{m-2} \times D^2, D^{m-2})]$$

which, identifying each with $M^k \times (S^{m-1}, S^{m-3})$ as above, takes $(x, z, \theta) \to (x, z, \theta + \tau(x))$, where we define the addition in the last coordinate as the usual addition on S^1. The map f is clearly well-defined on $M^k \times (S^{m-1} - S^{m-3})$ and also on $M^k \times S^{m-3}$, if we ignore the undefined longitude coordinate. To see that this is a well-defined continuous map overall, simply observe that on each sphere $* \times (S^{m-1}, S^{m-3})$, the map is just the rotation by angle $\tau(x)$ of the longitude coordinate induced by the rotation in the second factor of $\mathbb{R}^{m-2} \times \mathbb{R}^2$. Considered along with the continuity of τ, f is obviously a homeomorphism.

Roughly speaking, we are once again removing a bundle of trivial knots over M and replacing it with a bundle of non-trivial knots. The new element is the longitudinal twist determined by τ. The framing ϕ employed in non-twist frame spinning dictates how the trivial bundle of knots over M is attached "latitudinally", while the addition of "twist" allows us to alter the attachment "longitudinally". As an example, if M is taken as the standard circle $S^1 \subset S^{m+k-2}$ with ϕ the trivial framing, then $\sigma^{\phi}_{S^1}(K)$ gives us the spun knot of Artin, but if $\tau : S^1 \to S^1$ is a map of degree k, then $\sigma^{\phi,\tau}_{S^1}(K)$ is the k-twist spun knot of Zeeman [40]. Note also that if τ is the trivial map, which will always be the case if M is simply-connected, then the frame twist-spin $\sigma^{\phi,\tau}_{S^1}(K)$ is simply the standard frame-spin $\sigma^{\phi}_{S^1}(K)$.

We next wish to compute the Alexander polynomials of the frame twist-spun knots. First, we recall the following basic facts of algebra, some of which we have used before: Since Γ is a principle ideal domain, any Γ-module can be written as

$\Gamma^{\mathfrak{B}} \oplus (\bigoplus_i \Gamma/(p_i))$ for some $\mathfrak{B} \geq 0$ and $p_i \in \Gamma$, $p_i \neq 0$. It is also sometimes assumed that the p_i satisfy $p_i|p_{i+1}$ or some other similar formula simply to provide a normalization, but we will not impose that condition here. If $b = 0$, then $\prod_i p_i$ is what we have been calling the polynomial associated to the module. We will sometimes refer to the p_i as the invariants or torsion invariants of the module.

We also recall that for $p, q \neq 0$, $\Gamma/(p) \otimes_\Gamma \Gamma/(q) \cong \Gamma/(p) *_\Gamma \Gamma/(q) \cong \Gamma/(d(p,q))$, where $\otimes_\Gamma$ and $*_\Gamma$ represent the tensor and torsion products over the ring Γ, respectively, and $d(p,q)$ is the greatest common divisor of p and q in Γ. In addition, for any Γ-module A, $\Gamma \otimes_\Gamma A \cong A$ and $\Gamma *_\Gamma A \cong 0$. Since we will always assume Γ as our ground ring in the following, we will often simply use $\otimes$ and $*$ to mean the respective products over Γ. Observe that the distributivity of $\otimes$ and $*$ over $\oplus$ allow us to calculate the tensor and torsion products of any two Γ-modules $A \cong \Gamma^{\mathfrak{B}_A} \oplus (\bigoplus_i \Gamma/(p_i))$ and $B \cong \Gamma^{\mathfrak{B}_B} \oplus (\bigoplus_i \Gamma/(q_i))$. If we let A^i stand for the direct sum of i copies of the module A and $T(A)$ stand for the torsion summand of the module A (i.e., $T(A) \cong \bigoplus_i \Gamma/(p_i)$), then we obtain the following formulas:

$$A \otimes B \cong \Gamma^{\mathfrak{B}_A + \mathfrak{B}_B} \oplus T(A)^{\mathfrak{B}_B} \oplus T(B)^{\mathfrak{B}_A} \oplus \Big(\bigoplus_{i,j} \frac{\Gamma}{d(p_i, q_j)}\Big)$$

$$A * B \cong \Big(\bigoplus_{i,j} \frac{\Gamma}{d(p_i, q_j)}\Big).$$

We will also be using the fact that an exact sequence of Γ-torsion modules can be split into the direct sum of exact sequences of the p-primary summands of the modules (see the proof of Proposition 2.1).

With these formulas in hand, we can compute the Alexander modules of frame twist-spun knots. Suppose that K is the knot $S^{m-2} \subset S^m$ which is to be spun and that its Alexander modules are $H_j(S^m - S^{m-2}; \mathit{\Gamma}) \cong \bigoplus_\ell \Gamma/p_{j\ell}$ (recall that these will always be Γ-torsion modules). We wish to compute the homology modules $H_j(S^n - S^{n-2}; \mathit{\Gamma})$, where $n = m + k$, $S^{n-2} \subset S^n$ is the frame twist-spun knot $\sigma_M^{\phi,\tau}(K)$, and $\mathit{\Gamma}$ is the local coefficient system as discussed above. Using the above description of the spun knot, let

$$Y = S^n - (S^{n-2} \cup \mathrm{int}(M^k \times D^m)),$$
$$Z = M^k \times (D^m_+ - D^{m-2}_+),$$

so that

$$Y \cap Z = M^k \times (S^{m-1} - S^{m-3}),$$
$$Y \cup Z = S^n - S^{n-2}.$$

Then we can employ the Mayer Vietoris sequence

$$(4.27) \quad \to H_j(Y \cap Z; \mathit{\Gamma}|_{Y \cap Z}) \xrightarrow{i_*} H_j(Y; \mathit{\Gamma}|_Y) \oplus H_j(Z; \mathit{\Gamma}|_Z) \to H_j(Y \cup Z; \mathit{\Gamma}) \to$$

to computer the Alexander modules.

We now examine the terms and maps of this sequence.

First, we observe that $Y \sim_{h.e.} D^{n-1} \times S^1 \sim_{h.e.} S^1$, just as it is for the corresponding piece of the exterior of the non-twist frame-spun knot in the previous section. The S^1 here can be viewed as a meridian of the knot outside of a neighborhood of the surgery. Therefore, $H_j(Y;\Gamma|_Y) \cong H_j(\tilde{Y};\mathbb{Q}) \cong H_j(\mathbb{R};\mathbb{Q})$, so

$$H_j(Y;\Gamma|_Y) = \begin{cases} \mathbb{Q} \cong \frac{\Gamma}{t-1}, & j = 0, \\ 0, & j \neq 0. \end{cases}$$

For the Z component, we need to investigate the coefficient system $\Gamma|_Z$. Since Z is a product space, its fundamental group is $\pi_1(M^k \times (D_+^m - D_+^{m-2})) = \pi_1(M^k) \times \pi_1(D_+^m - D_+^{m-2})$. Therefore, the action of an element $\alpha \times \beta = (\alpha \times 1) \cdot (1 \times \beta) = (1 \times \beta) \cdot (\alpha \times 1)$ of the fundamental group on the stalk Γ over the basepoint is determined by the product of the actions of α and β, which we can take to be loops in $M^k \times *$ and $* \times (D_+^m - D_+^{m-2})$. But this means that $\Gamma|_Z$ is equivalent to the product system $\Gamma|_{M^k \times *} \boxtimes \Gamma|_{* \times (D_+^m - D_+^{m-2})}$. Therefore, we can compute $H_j(Z;\Gamma|_Z)$ via the Künneth theorem (see [34]) to be

$$\begin{aligned} H_j(Z;\Gamma|_Z) \cong & \bigoplus_{r+s=j} H_r(M^k;\Gamma|_{M^k}) \otimes H_s(D_+^m - D_+^{m-2};\Gamma|_{D_+^m - D_+^{m-2}}) \\ & \oplus \bigoplus_{r+s=j-1} H_r(M^k;\Gamma|_{M^k}) * H_s(D_+^m - D_+^{m-2};\Gamma|_{D_+^m - D_+^{m-2}}), \end{aligned}$$

where we have written $\Gamma|_{M^k}$ to mean $\Gamma|_{M^k \times *}$ and similarly for the other term.

We may observe that the terms $H_s(D_+^m - D_+^{m-2};\Gamma|_{D_+^m - D_+^{m-2}})$ are none other than the Alexander modules for the knot K. To see this, we need only show that the action of an element $\alpha \in \pi_1(D_+^m - D_+^{m-2})$ on the stalk Γ is given by multiplication by $t^{\ell_K(\alpha)}$, $\ell_K(\alpha)$ being the linking number of α with the knot K. But $\Gamma|_{D_+^m - D_+^{m-2}}$ is the restriction of the system Γ on $S^n - S^{n-2}$, so the action on Γ of a curve representing α is multiplication by $t^{\ell_{\sigma(K)}(\alpha)}$, where the exponent is the linking numer of the loop $\alpha \subset D_+^m - D_+^{m-2} \subset S^n - S^{n-2}$ with the spun knot. So we need only show that the two linking numbers are equivalent. As an element of $H_1(D_+^m - D_+^{m-2})$ (or $H_1(S^n - S^{n-2})$) under the Hurewicz map, α bounds in $D_+^m \subset S^n$. If $\alpha = \partial c$ and we use $a \cap b$ to denote the intersection number of the chains a and b, then

$$\ell_K(\alpha) = c \cap D_+^{m-2} = c \cap S^{n-2} = \ell_{\sigma(K)}(\alpha),$$

where the leftmost and rightmost equalities are taken from the definitions of linking and intersection numbers and the central equality is due to $D_+^{m-2} = S^{n-2} \cap D_+^m$.

Observe that, because the knot modules are all torsion Γ-modules, then $H_j(Z;\Gamma|_Z)$ will also be a torsion Γ-module.

The homology modules $H_r(M^k;\Gamma|_{M^k})$ depend, of course, on M so that we cannot give a general formula for their structure. However, we can obtain a little more information about the structure of the coefficient system $\Gamma|_{M^k}$. In fact, we claim that the action of $\alpha \in \pi_1(M)$ on the stalk Γ is given by multiplication by $t^{\deg(\tau(\alpha))}$, where $\deg(\tau(\alpha))$ is the degree of the map $S^1 \to S^1$ given by the image of the loop α under τ. To see this, it is simplest if we choose basepoints so that $M \times * \subset M \times (D^m_+ - D^{m-2}_+)$ lies in the boundary $M \times (S^{m-1} - S^{m-3})$. This allows us to consider the loop which represents $\alpha \in \pi_1(M \times *)$ as lying in the component Y via the attaching homeomorphism f. We need to compute the linking number of α with the spun knot. As remarked, $Y \sim_{h.e.} D^{n-1} \times S^1 \sim_{h.e.} S^1$, where S^1 gives a meridian of the knot. So if we let $h : Y \to S^1$ be the homotopy equivalence, we need only compute the degree of $h \circ f(\alpha)$. But by considering the construction, we can choose h so that its restriction to $M \times (S^{m-1} - S^{m-3}) \subset Y$ is the projection to the third coordinate in the (x, z, θ) coordinate system. In other words, the map is given by projection to the longitude coordinate. So if $M \times * = M \times (0,0)$ in the coordinate system, then it is clear from the above description of f that $h \circ f \circ \alpha(t) = \tau(\alpha(t))$, so the degree of $h \circ f(\alpha)$ is equal to the degree of $\tau(\alpha)$.

The homology of $Y \cap Z \cong M^k \times (S^{m-1} - S^{m-3})$ can also be computed by the Künneth theorem, but here the result is much simpler becuse $S^{m-1} - S^{m-3}$ is an unknotted sphere pair. Since m must be ≥ 3, the same linking number argument applies to show that the homology modules of $S^{m-1} - S^{m-3}$ with coefficient system $\Gamma|_{S^{m-1}-S^{m-3}}$ are the Alexander modules of a trivial knot. In other words, $H_0(S^{m-1} - S^{m-3};\Gamma|_{S^{m-1}-S^{m-3}}) \cong \Gamma/(t-1)$, and the homology is trivial in all other dimensions. Thus,

$$\begin{aligned} &H_j(Y \cap Z;\Gamma|_{Y\cap Z}) \\ &\qquad \cong \left(H_j(M^k;\Gamma|_{M^k}) \otimes \frac{\Gamma}{t-1}\right) \oplus \left(H_{j-1}(M^k;\Gamma|_{M^k}) * \frac{\Gamma}{t-1}\right). \end{aligned}$$

Notice that $t-1$ is prime in Γ, so for any $p \in \Gamma$, $d(t-1, p)$ is $(t-1)$ or 1. Therefore, $H_j(Y \cap Z;\Gamma|_{Y\cap Z})$ is a direct sum of $\Gamma/(t-1)$'s.

Next, we claim that the map i_* of the Mayer-Vietoris sequence (4.27) is injective. We have computed that all the terms of the sequence are torsion Γ-modules except for the $H_j(S^n - S^{n-2};\Gamma)$, but these must also be torsion modules because the other terms are or simply because we know that these are knot modules. We know that exact sequences of Γ-torsion modules can be broken up into the direct sum of the exact sequences of their p-primary components (see the proof of Proposition 2.1). We also know that $H_j(S^n - S^{n-2};\Gamma)$ has no $(t-1)$-primary component for $j > 0$ because $t-1$ does not divide the Alexander polynomials (which we know up to similarity must evaluate to ± 1 at 1). Therefore, on the exact sequence summand corresponding to the $(t-1)$-primary components, the $H_j(S^n - S^{n-2};\Gamma)$ terms are 0, and i_* must be injective. But $H_j(Y\cap Z;\Gamma|_{Y\cap Z})$ consists entirely of its $t-1$ primary component as noted in the previous paragraph. Therefore, i_* is injective for all $j>0$. It is also injective for $j=0$ by standard arguments.

Therefore, we obtain short exact sequences

$$0 \longrightarrow H_j(Y\cap Z;\Gamma|_{Y\cap Z}) \xrightarrow{i_*} H_j(Y;\Gamma|_Y)\oplus H_j(Z;\Gamma|_Z) \longrightarrow H_j(Y\cup Z;\Gamma) \longrightarrow 0,$$

and based upon the previous calculations, we can compute

$$H_j(Y\cup Z;\Gamma)\cong H_j(S^n - S^{n-2};\Gamma)$$

to be

$$\begin{aligned} H_j(S^n - S^{n-2};\Gamma) \cong & \bigoplus_{\substack{r+s=j\\ s>0}} H_r(M^k;\Gamma|_{M^k})\otimes H_s(D^m_+ - D^{m-2}_+;\Gamma|_{D^m_+ - D^{m-2}_+}) \\ & \oplus \bigoplus_{\substack{r+s=j-1\\ s>0}} H_r(M^k;\Gamma|_{M^k}) * H_s(D^m_+ - D^{m-2}_+;\Gamma|_{D^m_+ - D^{m-2}_+}) \end{aligned}$$

for $j>0$.

Supposing that the Alexander modules of the knot K are given as

$$H_j(D^m_+ - D^{m-2}_+;\Gamma)\cong \bigoplus_{\ell}\frac{\Gamma}{(\lambda_{j\ell})},$$

$$H_j(M^k;\Gamma|_{M^k})\cong \Gamma^{\mathfrak{B}_j}\oplus\bigoplus_{\ell}\frac{\Gamma}{(\zeta_{j\ell})},$$

we can then compute the Alexander polynomial $\lambda^\tau_j(K)$, $j>0$, of the frame twist-spun knot to be

$$\lambda^\tau_j(t) = \prod_{\substack{r+s=j\\ s>0}}\Big(\prod_{\ell}\lambda_{sl}^{\mathfrak{B}_r}\cdot\prod_{i,\ell} d(\zeta_{ri},\lambda_{sl})\Big)\cdot\prod_{\substack{r+s=j-1\\ s>0}}\Big(\prod_{i,\ell} d(\zeta_{ri},\lambda_{sl})\Big).$$

We next calculate the "relative" and "boundary" polynomials $\mu^\tau_j(t)$ and $\nu^\tau_j(t)$ of the spun knot $\sigma^{\phi,\tau}_M(K)$. Let $\bar p(t)=p(t^{-1})$ for any $p\in\Gamma$, suppose $\mathfrak{B}_i$ continues to denote the rank of the free Γ component of $H_i(M;\Gamma|_M)$, and let $\tilde\beta_i$ denote the i^{th} reduced Betti number of $M\times\Sigma$. Then, for $j>0$, we can calculate $\mu^\tau_j(t)$

using the duality of Alexander polynomials:

$$
\begin{aligned}
\mu_j^\tau(t) &\sim (t-1)^{\bar\beta_{j-1}} \bar\lambda^\tau_{n-1-i} \\
&= (t-1)^{\bar\beta_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} \Big(\prod_\ell \bar\lambda_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} \overline{d(\zeta_{ri},\lambda_{sl})}\Big) \cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \Big(\prod_{i,\ell} \overline{d(\zeta_{ri},\lambda_{sl})}\Big) \\
&= (t-1)^{\bar\beta_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} \Big(\prod_\ell \bar\lambda_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\bar\zeta_{ri},\bar\lambda_{sl})\Big) \cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \Big(\prod_{i,\ell} d(\bar\zeta_{ri},\bar\lambda_{sl})\Big).
\end{aligned}
$$

One can also go a step further and calculate μ_i^τ in terms of the μ_{sl}, which we define as follows: Let $X(K)$ denote the exterior of the knot K, and let $L(K)$ represent intersection of the knot exterior $X(K)$ with the closed neighborhood of the singularity $\overline{N(\Sigma)}$ (i.e. the "link exterior"). Then we know that $H_i(X(K),L(K);\Gamma)$ has the form $T_i \oplus_\ell \Gamma/(\mu_{i\ell})$, where T_i is the $(t-1)$-primary summand of $H_i(X(K),L(K);\Gamma)$, $t-1 \nmid \mu_{i\ell}$, and $\mu_{i\ell} \neq 0$. Applying Theorem 4.3, we may assume that each $\mu_{i\ell} = \bar\lambda_{m-i-1,\ell}$. Thus

$$
\begin{aligned}
\mu_j^\tau(t) \sim (t-1)^{\bar\beta_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} &\Big(\prod_\ell \mu_{m-s-1,\ell}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\bar\zeta_{ri},\mu_{m-s-1,\ell})\Big) \\
&\cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \Big(\prod_{i,\ell} d(\bar\zeta_{ri},\mu_{m-s-1,\ell})\Big).
\end{aligned}
$$

Lastly, to calculate $\nu_j^\tau(t)$, we can once again employ the Künneth theorem since $L(\sigma_M^{\phi,\tau}(K)) = M \times L(K)$. We have

$$
\begin{aligned}
H_j(L(\sigma_M^{\phi,\tau}(K));\Gamma) &\cong H_j(M\times L(K);\Gamma) \\
&\cong \bigoplus_{r+s=j} H_r(M^k;\Gamma|_{M^k}) \otimes H_s(L(K);\Gamma|_{L(K)}) \\
&\quad \oplus \bigoplus_{r+s=j-1} H_r(M^k;\Gamma|_{M^k}) * H_s(L(K);\Gamma|_{L(K)}).
\end{aligned}
$$

Based on our previous calculations in Section ***4.2.3***, we know that if we let $\mathfrak{b}_i$ stand for the i^{th} Betti number of Σ, the singular set of K, then the $(t-1)$-primary summand, T_j, of $H_j(L(K);\Gamma)$ will be isomorphic to $[\Gamma/(t-1)]^{\mathfrak{b}_j}$. (For $j>0$, we showed that it was $[\Gamma/(t-1)]^{\tilde{\mathfrak{b}}_j}$ for reduced Betti number $\tilde{\mathfrak{b}}_j$, but $\tilde{\mathfrak{b}}_j = \mathfrak{b}_j$ in this range and clearly $H_0(\widetilde{L(K)};\mathbb{Q}) \cong [\Gamma/(t-1)]^{\mathfrak{b}_0} \cong \mathbb{Q}^{\mathfrak{b}_0}$.) So we can set $H_j(L(K);\Gamma|_{L(K)}) \cong T_j \oplus_\ell \Gamma/(\nu_{j\ell})$, where $T_j \cong [\Gamma/(t-1)]^{\mathfrak{b}_j}$, $t-1 \nmid \nu_{j\ell}$, and $\nu_{j\ell} \neq$

0. Then we can use the above equation to calculate the Alexander polynomial of $H_j(L(\sigma_M^{\phi,\tau}(K));\Gamma)$:

$$\begin{aligned}\nu_j^\tau(t) \sim \prod_{r+s=j} \Big((t-1)^{\mathfrak{B}_r\cdot\mathfrak{b}_s}\prod_\ell \nu_{sl}^{\mathfrak{B}_r}\cdot\prod_i d(\zeta_{ri},t-1)^{\mathfrak{b}_s}\cdot\prod_{i,\ell} d(\zeta_{ri},\nu_{sl})\Big)\\ \cdot\prod_{r+s=j-1}\Big(\prod_i d(\zeta_{ri},t-1)^{\mathfrak{b}_s}\cdot\prod_{i,\ell} d(\zeta_{ri},\nu_{sl})\Big)\\ \sim (t-1)^{\beta_j}\prod_{r+s=j}\Big(\prod_\ell \nu_{sl}^{\mathfrak{B}_r}\cdot\prod_{i,\ell} d(\zeta_{ri},\nu_{sl})\Big)\cdot\prod_{r+s=j-1}\Big(\prod_{i,\ell} d(\zeta_{ri},\nu_{sl})\Big),\end{aligned}$$

where, for the last line, we have used our knowledge of to what power the $t-1$ factor should occur, based upon some polynomial algebra and our calculations for λ_j^τ and μ_j^τ

Remark 4.25. As a special case, we can take $M = S^1$ with the standard trivialization and $\tau : S^1 \longrightarrow S^1$ to be a map of degree $k \neq 0$. Then $\sigma_{S^1}^{\phi,\tau}(K)$ is the Zeeman k-twist spin of K. Since the action of a generator of $\alpha \in \pi_1(S^1)$ on the stalk Γ is multiplication by t^k, we have

$$H_i(S^1;\Gamma|_{S^1}) \cong \begin{cases}\dfrac{\Gamma}{t^k-1}, & i=0,\\ 0, & i\neq 0.\end{cases}$$

Therefore, $\mathfrak{B}_r = 0$ for all r, $\zeta_{0,1} = t^k - 1$, and all other torsion invariants ζ_{ri} are trivially equal to 1. Thus, for $j > 0$, we get the polynomials:

$$\begin{aligned}\lambda_j^\tau(t) &\sim \prod_\ell d(t^k-1,\lambda_{j\ell})\cdot\prod_\ell d(t^k-1,\lambda_{j-1,\ell}),\\ \mu_j^\tau(t) &\sim (t-1)^{\tilde\beta_{j-1}}\prod_\ell d(t^{-k}-1,\mu_{m-n+j,\ell})\cdot\prod_\ell d(t^{-k}-1,\mu_{m-n+j+1,\ell}),\\ \nu_j^\tau(t) &\sim (t-1)^{\mathfrak{b}_j+\mathfrak{b}_{j-1}}\prod_\ell d(t^k-1,\nu_{j\ell})\cdot\prod_\ell d(t^k-1,\nu_{j-1,\ell}).\end{aligned}$$

Remark 4.26. If $k = 0$, we can check that we obtain the polynomials of the non-twist frame-spun knots as in the last section. For $k = 1$, note that all of the λ_i^τ, $0 < i < n-1$, are trivial (i.e. similar to 1), while the μ_i^τ and ν_i^τ are all powers of $t-1$.

As for the subpolynomials $a_i^\tau(t)$, $b_i^\tau(t)$, and $c_i^\tau(t)$, the existence of Γ-torsion terms in $H_i(M;\Gamma|_M)$, the lack of naturality in the splitting of the Künneth theorem, and the lack of exactness of the tensor and torsion products make it impossible to derive simple formulae in terms of the subpolynomials of the knot being spun as we did in Section 4.3.1 for frame spun knots. This is not a great loss, however, since we can always calculate the subpolynomials from $\lambda_i^\tau(t)$, $\mu_i^\tau(t)$, and $\nu_i^\tau(t)$ by "dividing in" from the outside of the exact sequence. In other words, recall that we can calculate $a_i^\tau(t)$, $b_i^\tau(t)$, and $c_i^\tau(t)$ by $c_{n-2}^\tau(t) = \lambda_{n-2}^\tau(t)$ and then

$$\begin{aligned} a_{n-3}^\tau(t) &= \frac{\mu_{n-2}^\tau(t)}{c_{n-2}^\tau(t)}, \\ b_{n-3}^\tau(t) &= \frac{\nu_{n-3}^\tau(t)}{a_{n-3}^\tau(t)}, \\ c_{n-3}^\tau(t) &= \frac{\lambda_{n-3}^\tau(t)}{b_{n-3}^\tau(t)}, \\ &\vdots \end{aligned}.$$

Of course we could also begin from the other side with $c_1^\tau(t)$ equal to $\mu_1^\tau(t)$ divided by its $t-1$ terms, and so on.

Lastly, we summarize the above calculations as the following realization theorem:

Theorem 4.27. *Let M^k, $n-k>3$, be a manifold which embeds in S^{n-2} with trivial normal bundle with framing ϕ. Given a map $\tau : M \longrightarrow S^1$, let $\mathfrak{B}_i$ be the rank of the free part and $\zeta_{i\ell}$ be the torsion invariants of the Γ-modules $H_i(M;\Gamma|_M)$, where the coefficient system is given as above. (Note that these modules are independent of the knot being spun in the construction.) Then, if K is a knot $S^{m-2} \subset S^m$ with Alexander invariants $\lambda_{i\ell}$, $\mu_{i\ell}$, and $\nu_{i\ell}$ and with singular set Σ with reduced Betti numbers $\tilde{\mathfrak{b}}_i$, then there exists a frame twist-spun knot $\sigma_M^{\phi,\tau}(K)$ with singular set $M\times\Sigma$ (whose reduced Betti numbers we denote $\tilde{\beta}_i$) and with Alexander polynomials given for $j>0$ by:*

$$\lambda_j^\tau(K) \sim \prod_{\substack{r+s=j\\ s>0}} \Big((\prod_\ell \lambda_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri},\lambda_{sl}) \Big) \cdot \prod_{\substack{r+s=j-1\\ s>0}} \Big(\prod_{i,\ell} d(\zeta_{ri},\lambda_{sl}) \Big),$$

$$\begin{aligned} \mu_j^\tau(K) \sim (t-1)^{\tilde{\beta}_{j-1}} \prod_{\substack{r+s=n-j-1\\ s>0}} & \Big((\prod_\ell \mu_{m-s-1,\ell}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\bar{\zeta}_{ri},\mu_{m-s-1,\ell}) \Big) \\ & \cdot \prod_{\substack{r+s=n-j-2\\ s>0}} \Big(\prod_{i,\ell} d(\bar{\zeta}_{ri},\mu_{m-s-1,\ell}) \Big), \end{aligned}$$

$$\nu_j^\tau(t) \sim (t-1)^{\tilde{\beta}_j} \prod_{r+s=j} \Big(\prod_\ell \nu_{sl}^{\mathfrak{B}_r} \cdot \prod_{i,\ell} d(\zeta_{ri},\nu_{sl}) \Big) \cdot \prod_{r+s=j-1} \Big(\prod_{i,\ell} d(\zeta_{ri},\nu_{sl}) \Big).$$

In particular, by frame twist-spinning knots with a single point as their singular set, we obtain knots with M as their singular sets.

Remark 4.28. Although we have focused on realizing given Alexander polynomials in our previous constructions of knots with point singularities, observe that the methods of proof actually allow us to create knots with given *invariants*. In fact, we can create disk knots with specific single invariants and string these together using the disk knot sum. Then coning on the boundary gives us a knot with the same invariants and a point singularity. Putting this together with the above theorem, we know exactly what kinds of polynomials can be realized as those of frame twist-spun knots with singular set M, modulo our ability to compute the homology $H_j(M;\Gamma|_M)$ and our previous difficulty with the polynomial $c_2(t)$ of a disk knot $D^3 \subset D^5$.

Remark 4.29. We can, of course, further enrich the class of polynomials we can realize as polynomials of a knot with singular set M by attaching locally-flat knots to our frame twist-spun knots using ordinary knot sums away from the singularities.

4.3.3. *Suspensions.* Another method for obtaining new knots from old ones is by suspension, which, in some sense, constitutes the extreme case, as the number of singular strata will always increase. In particular, if we begin with the knotted sphere pair $S^{n-3} \subset S^{n-1}$ with singular set Σ, filtered by the nested subsets Σ_i and with "pure strata" $U_k = \Sigma_{n-k+1} - \Sigma_{n-k}$, then the suspension, thought of as $(S^{n-1}, S^{n-3}) \times I/\{x \times 0 \sim *_-, x \times 1 \sim *_+\}$, is a sphere pair $S^{n-2} \subset S^n$. Its singular set is given by the suspension of Σ, and it is filtered by the suspensions points, $\{*_\pm\}$, and the suspensions of the Σ_i.

We will employ the italicized $\mathit{\Sigma}$ to denote suspensions. Thus, the suspension of the knot K with Alexander polynomials $\lambda_i \sim b_i c_i$, $\mu_i \sim c_i a_{i-1}$, and $\nu_i \sim a_i b_i$ will be denoted by $\mathit{\Sigma} K$ with polynomials $\lambda_i^{\mathit{\Sigma}}$, $\mu_i^{\mathit{\Sigma}}$, and $\nu_i^{\mathit{\Sigma}}$. We will compute these polynomials.

Proposition 4.30. *With the notation as above,*

1. $\lambda_i^{\mathit{\Sigma}} \sim \lambda_i \sim b_i c_i$
2. $\mu_i^{\mathit{\Sigma}} \sim \mu_{i-1} \sim c_{i-1} a_{i-2}$
3. $\nu_i^{\mathit{\Sigma}} \sim a_{i-1} b_i c_i^2$.

Proof. We first observe that $\lambda_i^{\mathit{\Sigma}} = \lambda_i$. This follows immediately from the fact that the suspension points $*_\pm$ lie in the knot $\mathit{\Sigma} K$ so that $S^n - \mathit{\Sigma} K \cong (S^{n-1} - K) \times (0,1)$. Therefore, $S^n - \mathit{\Sigma} K \sim_{h.e.} S^{n-1} - K$, and $H_i(S^n - \mathit{\Sigma} K; \Gamma) \cong H_i(S^{n-1} - K; \Gamma)$. The claim follows because $\lambda_i^{\mathit{\Sigma}}$ and λ_i are the polynomials associated to these modules, respectively. (Note that the local coefficient system, Γ, on $S^n - \mathit{\Sigma} K$ is simply that induced by the homotopy equivalence with $S^{n-1} - K$).

We next turn to the computation of $\nu_i^{\mathit{\Sigma}}$. We will use N to denote open regular neighborhoods and $\bar{N}$ to denote closed regular neighborhoods, letting the context in each case determine the ambient space. Then $\nu_i^{\mathit{\Sigma}}$ will be the polynomial associated to the homology of the "link compliment" $\partial \bar{N}(\mathit{\Sigma}(\Sigma)) - \mathit{\Sigma} K \cap \partial \bar{N}(\mathit{\Sigma}(\Sigma))$ or, equivalently, the "link exterior" $\partial \bar{N}(\mathit{\Sigma}(\Sigma)) - N(\mathit{\Sigma} K \cap \partial \bar{N}(\mathit{\Sigma}(\Sigma)))$.

For the current argument, it is simplest if we think of the suspended knot pair as $[(S^{n-1}, S^{n-3}) \times I] \cup \amalg_\pm \bar{c}_\pm(S^{n-1}, S^{n-3})$, where $\amalg_\pm \bar{c}_\pm(S^{n-1}, S^{n-3})$ indicates the disjoint union of the "northern" $(+)$ and "southern" $(-)$ closed cones on the original knot pair. The cones attach to the product with the unit interval in the obvious manner. In this case, it is clear that $\bar{N}(\Sigma(\Sigma)) \cong \bar{N}(\Sigma) \times I \cup \amalg_\pm \bar{c}_\pm S^{n-1}$ and $\partial\bar{N}(\Sigma(\Sigma)) \cong (\partial\bar{N}(\Sigma) \times I) \cup \amalg_\pm (S^{n-1} - N(\Sigma))$. Finally, since $N(\Sigma) \subset N(K) \subset S^{n-1}$, when we remove the neighborhood around ΣK in $\partial\bar{N}(\Sigma(\Sigma))$, we see that

$$\partial\bar{N}(\Sigma(\Sigma)) - N(\Sigma K \cap \partial\bar{N}(\Sigma(\Sigma))) \cong \left[(\partial\bar{N}(\Sigma) - N(K \cap \partial\bar{N}(\Sigma))) \times I\right] \cup \amalg_\pm (S^{n-1}_\pm - N(K_\pm))$$

glued together at $(\partial\bar{N}(\Sigma) - N(K \cap \partial\bar{N}(\Sigma))) \times \{0\}$ and $(\partial\bar{N}(\Sigma) - N(K \cap \partial\bar{N}(\Sigma))) \times \{1\}$.

For simplicity, let us readopt some of the notation of Section 3. Let us set $X = \partial\bar{N}(\Sigma) - N(K \cap \partial\bar{N}(\Sigma))$, the link complement of K, and $C = S^{n-1} - N(K)$, the knot complement of K. Let us also define $X_\Sigma = \partial\bar{N}(\Sigma(\Sigma)) - N(\Sigma K \cap \partial\bar{N}(\Sigma(\Sigma)))$, the link complement of ΣK, and $C_\Sigma = S^n - N(\Sigma K)$, the knot complement of ΣK. We will also continue to use $+$ and $-$ in the subscript as indicators in the cases where there are multiple copies. From the preceding paragraph, we can form a Mayer-Vietoris sequence (in which the coefficient system Γ or its restriction is implied):

$$\longrightarrow H_i(X_+) \oplus H_i(X_-) \xrightarrow{i_*} H_i(X \times I) \oplus H_i(C_+) \oplus H_i(C_-) \longrightarrow H_i(X_\Sigma) \longrightarrow .$$

Now, to study the polynomials, we already know that each $H_i(X)$ has associated polynomial $\nu_i \sim a_i b_i$ and each $H_i(C)$ has associated polynomial $\lambda_i \sim b_i c_i$. So, according to the results of Section 2, we can determine the polynomial associated to $H_i(X_\Sigma)$ by determining the polynomial associated to the kernel of i_*. But, by the definition of the Mayer-Vietoris sequence, the map i_* is induced by inclusion so that each induced map $H_i(X_\pm) \to H_i(X \times I)$ is the identity and each map $H_i(X_\pm) \to H_i(C_\pm)$ is the standard map, say j_*, induced by inclusion. Form this, and identifying $H_i(X_+) \cong H_i(X_-) \cong H_i(X \times I)$ and $H_i(C_+) \cong H_i(C_-)$, we can conclude that the kernel of i_* consists of pairs $(\alpha, -\alpha)$, where $\alpha \in H_i(X)$ and furthermore $\alpha \in \ker(j_*)$. This implies that $\ker(i_*) \cong \ker(j_*)$ and that the polynomial associated to the kernel of i_* is a_i, as, by definition, this is the polynomial of the kernel of j_*. Hence, in the exact sequence, the natural factorization of the polynomial associated to $H_i(X_+) \oplus H_i(X_-)$ is as a_i times $a_i b_i^2$, and the natural factorization of the polynomial associated to $H_i(X \times I) \oplus H_i(C_+) \oplus H_i(C_-)$ is as $a_i b_i^2$ times $b_i c_i^2$. Applying this argument in all dimensions, we see then that the polynomial ν_i^Σ associated to $H_i(X_\Sigma)$ must be $a_{i-1} b_i c_i^2$.

Finally, note that in the long exact sequence of the pair $(C_\Sigma, X_\Sigma) \cong (C \times I, X_\Sigma)$,

$$\longrightarrow H_i(X_\Sigma) \xrightarrow{j_*} H_i(C \times I) \longrightarrow H_i(C \times I, X_\Sigma) \longrightarrow ,$$

the map $j_* : H_i(X_\Sigma) \to H_i(C \times I)$ is an epimorphism. This is due to the fact that any cycle in $C \times I$ can be homotoped to a cycle in $C \times [0] \subset X_\Sigma$. Since the polynomial of $H_{i-1}(C \times I)$ is $\lambda_{i-1} \sim \lambda^\Sigma_{i-1} \sim b_{i-1}c_{i-1}$ and the polynomial of $H_{i-1}(X_\Sigma)$ is $a_{i-2}b_{i-1}c^2_{i-1}$, this implies that the polynomial of $H_i(C \times I, X_\Sigma)$ is $\mu^\Sigma_i \sim a_{i-2}c_{i-1}$. $\square$

REFERENCES

[1] R.C. BLANCHFIELD, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. **65** (1957), 340–356.

[2] GLEN BREDON, *Topology and Geometry*, Springer-Verlag, New York, 1993.

[3] WILLIAM BROWDER, *Surgery on simply-connected manifolds*, Springer-Verlag, New York-Heidelberg-Berlin, 1972.

[4] SYLVAIN CAPPELL, *Superspinning and knot complements*, Topology of Manifolds (Proc. Inst. Univ. of Georgia, Athens, Ga, 1969), Markham, Chicago, 1970, pp. 358–383.

[5] SYLVAIN CAPPELL and JULIUS SHANESON, *Piecewise linear embeddings and their singularities*, Ann. of Math. **103** (1976), 163–228.

[6] ______, *Singularities and immersions*, Ann. of Math. **105** (1977), 539–552.

[7] R. CROWELL and R. FOX, *Introduction to Knot Theory*, Ginn, Boston, 1963.

[8] GREG FRIEDMAN, *Intersection Alexander polynomials*, Topology **43** (January 2004), 71–117.

[9] ______, *Polynomial Invariants of Non-locally-flat Knots*, Ph.D. Thesis, New York University, New York, 2001.

[10] ALLEN HATCHER, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[11] I.N. HERSTEIN, *Topics in Algebra*, 2nd Edition, John Wiley, New York, 1975.

[12] WILLIAM D. HOMER, *Equivariant PL embeddings of spheres*, Topology **19** (1980), 51–63.

[13] THOMAS W. HUNGERFORD, *Algebra*, Holt, Rinehart and Winston, New York, 1974.

[14] BIRGER IVERSEN, *Cohomology of Sheaves*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986.

[15] S. KINOSHITA, *On the Alexander polynomial of 2-spheres in a 4-sphere*, Annals of Math. **74** (1961), 518–531.

[16] JOHN R. KLEIN and ALEXANDER I. SUCIU, *Inequivalent fibred knots whose homotopy Seifert pairings are isometric*, Math. Ann. **289** (1991), 683–701.

[17] JEROME LEVINE, *A characterization of knot polynomials*, Topology **4** (1965), 135–41.

[18] ______, *Unkotting spheres in codimension two*, Topology **4** (1965), 9–16.

[19] ______, *Polynomial invariants of knots of codimension two*, Ann. of Math **84** (1966), 537–554.

[20] ______, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[21] ______, *Knot modules I*, Trans. Amer. Math. Soc. **229** (1977), 1–50.

[22] A. LIBGOBER, *Alexander invariants of plane algebraic curves*, Proc. Singularities, Part 2 (Providence, RI), vol. 40, Amer. Math. Soc., 1983, pp. 135–143.

[23] ______, *Homotopy groups of the complements of singular hypersurfaces*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), 49–52.

[24] A. LIBGOBER, *Homotopy groups of the complements of singular hypersurfaces, II*, Ann. of Math. **139** (1994), 117–144.

[25] JOHN MILNOR, *Infinite cyclic coverings* (J. G. Hocking, ed.), Proc. Conference on the Topology of Manifolds (Boston), PWS, 1968, pp. 115–133.

[26] JOHN MILNOR and R.H. Fox, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267.
[27] JOHN MILNOR, *Singular Points of Complex Hypersurfaces*, Princeton University Press, Princeton, N.J., 1968.
[28] JAMES R. MUNKRES, *Elements of Algebraic Topology*, Addison-Wesley, Reading, MA, 1984.
[29] DALE ROLFSEN, *Knots and Links*, Publish or Perish, Berkeley, CA, 1976.
[30] DENNIS ROSEMAN, *Spinning knots about submanifolds; spinning knots about projections of knots*, Topology and Appl. **31** (1989), 225–241.
[31] C.P. ROURKE and B.J. SANDERSON, *Introduction to Piecewise-linear Topology*, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
[32] H. SEIFERT, *Über das geschlecht von knoten*, Math. Ann. **110** (1934), 571–92.
[33] S. SMALE, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.
[34] EDWIN H. SPANIER, *Algebraic Topology*, Springer-Verlag, New York, 1966.
[35] DAVID A. STONE, *Stratified Polyhedra*, Lecture Note in Mathematics, vol. 252, Spring-Verlag, Berlin-Heidelberg-New York, 1972.
[36] ALEXANDER I. SUCIU, *Inequivalent frame-spun knots with the same complement*, Comment. Math. Helvetici **67** (1992), 47–63.
[37] D.W. SUMNERS, *Higher-dimensional Slice Knots*, Ph.D. Thesis, Cambridge University, 1967.
[38] ———, *Homotopy torsion in codimension two knots*, Proc. Amer. Math. Soc. **24** (1970), 229–240.
[39] H.F. TROTTER, *On S-equivalence of Seifert matrices*, Inventiones Math. **20** (1973), 173–207.
[40] E.C. ZEEMAN, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

Department of Mathematics
Yale University
10 Hillhouse Avenue
P.O. Box 208283
New Haven, CT 06520, U.S.A.
E-MAIL: `friedman@math.yale.edu`
ACKNOWLEDGMENT: The author gratefully akcnowledges Joan Doran, for making the hand drawings.

KEY WORDS AND PHRASES: knot; singular knot; Alexander polynomial; Blanchfield pairing; stratified space; knot spinning.

2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary 57Q45; Secondary 57Q35; 57N80.

Received: *October 7th, 2002.*