CHAPTER 10

Classifications and Embeddings

Technically, the main result of the section is the π_1-negligible embedding theorem 10.6, which gives existence and uniqueness criteria for codimension zero embeddings of 4-manifolds with boundary. From this we deduce a criterion for a manifold to decompose as a connected sum. Perhaps the most basic result, the classification of closed simply-connected 4-manifolds, is obtained as a corollary of the sum theorem.

These results are presented in the reverse of the logical order because the more general ones are more complicated and technical. The chapter begins, then, with the classification theorem in 10.1. The invariants used in the classification are discussed in 10.2. Connected sums are described in 10.3, and the characterization theorem is deduced from this. π_1-negligible embeddings are discussed in 10.5, and using this the connected sum theorem is deduced in 10.6. A classification of manifolds with infinite cyclic fundamental group is given as an exercise in section 10.7. Finally the embedding theorem is proved; the existence part in 10.8, and the uniqueness using embeddings in 5-manifolds in 10.9. The technically inclined reader may prefer to see these in logical order, beginning with 10.5.

10.1 Classification of closed 1-connected 4-manifolds

For the classification theorem we recall that if M is a compact oriented manifold then intersection numbers define a bilinear form $\lambda\colon H_2M \otimes H_2M \longrightarrow \mathbf{Z}$. This is symmetric, and if ∂M is empty it is nonsingular (the adjoint $H_2M \longrightarrow (H_2M)^*$ is the Poincaré duality isomorphism). The Kirby-Siebenmann invariant $\mathrm{ks}M \in \mathbf{Z}/2$ is the obstruction to the existence of a smooth structure on $M \times \mathbf{R}$; see 8.3D and10.2B below.

Theorem.

(1) **Existence:** *Suppose (H, λ) is a nonsingular symmetric form on a finitely generated free Z-module, $k \in Z/2$, and if λ is even then we assume $k \equiv (\text{signature}\lambda)/8$, mod 2. Then there is a closed oriented 1-connected manifold with form λ and Kirby-Siebenmann invariant k.*

(2) **Uniqueness:** *Suppose M, N are closed and 1-connected, $h: H_2 \to H_2N$ is an isomorphism which preserves intersection forms, and* $\mathrm{ks}M = \mathrm{ks}N$. *Then there is a homeomorphism $f: M \to N$, unique up to isotopy, such that $f_* = h$.*

For example, S^4 and $S^2 \times S^2$ have even forms, so are uniquely determined by their forms. CP^2 has an odd form, so there is another manifold with this form but which is not stably smoothable.

Note that by setting $M = N$ in the uniqueness statement we see that taking a homeomorphism to its induced isometry of the form gives an isomorphism $\pi_0\mathrm{TOP}(M) \to \mathrm{ISO}(H_2M, \lambda)$. $\pi_0\mathrm{TOP}(M)$ is the group of path components of the homeomorphism group of M, which is the same as isotopy classes of homeomorphisms. A homotopy equivalence also induces an isometry of the form, but the analogous statement for these is false; there are often many non-homotopic homotopy equivalences inducing each isometry (see Wall [**1**, §16], Quinn [**8**], and Cochran and Habegger [**1**]).

Some notation, suggested by the theorem, will be useful. If λ is a nonsingular form, denote by $\|\lambda\|$ the manifold with this form, and such that $\mathrm{ks}(\|\lambda\|) = 0$ if the form is not even. So for example $CP^2 = \|[1]\|$, and $S^2 \times S^2 = \|\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\|$. Part (1) of the theorem implies $\|\lambda\|$ exists, and (2) implies it is well-defined up to homeomorphism.

Suppose M is closed and 1-connected. Define $*M$ to be M if the form is even, and the manifold with the same form and opposite Kirby-Siebenmann invariant if the form is odd. Again the theorem implies this exists and is well-defined, and every closed 1-connected manifold is either $\|\lambda\|$ or $*\|\lambda\|$ for some appropriate λ. This operation will be defined (independently of the classification theorem) for a wider class of manifolds in 10.4.

10.2 The invariants

We briefly describe the invariants used in the theorem. For further information about forms over the integers see Milnor-Husemoller [**1**, Chapter II]. Hirzebruch and Neumann [1] describe relations between these invariants and the topology of manifolds. For information about the Kirby-Siebenmann invariant see Kirby-Siebenmann [**1**]. For a geometric point of view on the algebraic invariants of 4-manifolds see Kirby [**2**].

10.2A Symmetric forms over the integers. Important invariants of these are rank, signature (or "index"), type, and whether or not the form is definite.

The *rank* is the rank of the group on which the form is defined. The *type* is "even" if $\lambda(x, x)$ is even for all x, and is "odd" otherwise. The form is *definite* if $\lambda(x,x)$ is always nonnegative, or always nonpositive (and then is positive definite, or negative definite). Finally when the form is tensored with the real numbers it can be divided into positive and negative eigenspaces. The *signature* of the form is the dimension of the positive eigenspace minus the dimension of the negative eigenspace.

For example $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, the form of $S^2 \times S^2$, has rank 2, is even, is indefinite, and has signature 0. The form of CP^2 is [1], which has rank 1, is odd, positive definite, and has signature 1.

There are some simple relations among these invariants. The sum of the dimensions used to define the signature gives the rank. Therefore $|\text{signature}(\lambda)| \le \text{rank}(\lambda)$, and $\text{signature}(\lambda) \equiv \text{rank}(\lambda) \bmod 2$. Similarly the form is definite if and only if the real form is definite, so if and only if $|\text{signature}(\lambda)| = \text{rank}(\lambda)$. A deeper fact is that if the form is even then the signature is divisible by 8.

Indefinite forms are determined up to isometry by the rank, signature, and type. If the form is odd, then it is isomorphic to $j[1] \oplus k[-1]$, where $j+k = \text{rank}$ and $j-k = \text{signature}$. If the form is even then it is isomorphic to $j(\pm E_8) \oplus k\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, where $8j+2k = \text{rank}$, $8j = |\text{signature}|$, and E_8 is the even definite form of rank 8 represented by the following matrix (blank entries are 0);

$$\begin{pmatrix}
2 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
 & 1 & 2 & 1 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{pmatrix}$$

An interesting family of examples is provided by the algebraic surfaces $Z_d \subset CP^3$ obtained from the zeros of a generic homogeneous complex polynomial of degree d. $H_2(Z_d)$ has rank $d^3 - 4d^2 + 6d - 2$, the form has signature $-\frac{1}{3}(d^3 - 4d)$, and is even if and only if d is even. According to the classification described above, this information is sufficient to identify these forms up to isomorphism. For example the Kummer surface ($d = 4$) has rank 22, and even form with signature -16. Therefore the form is $2(-E_8) \oplus 3\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$.

Definite forms are much more complicated. Unlike the indefinite forms they have unique decompositions into sums of indecomposibles (Eichler's theorem, Milnor-Husemoller [1, Theorem II6.4]). Unfortunately there are over 10^{51} different indecomposable even definite forms of rank 40, and this number grows rapidly with increasing rank. We know nothing useful about them. None of this host of indecomposible definite forms occurs as the form of a smooth 4-manifold, however (see 8.4A).

If λ is the intersection form of a 4-manifold, then $\omega_2(x) \equiv \lambda(x,x)$, mod 2. Therefore evenness of the form is equivalent to vanishing of ω_2 on the integral homology $H_2(M;\mathbf{Z})$. If there is no 2-torsion in $H_1(M;\mathbf{Z})$ then this also implies ω_2 vanishes on mod 2 homology $H_2(M;\mathbf{Z}/2)$, which is equivalent to the existence of a spin structure. (For the significance of spin structures, see below.) We caution that without the H_1 condition the connection with spin structures can fail: there are non-spin manifolds with even forms on integral homology (Habegger [**1**]).

The form of a connected sum is the direct sum of the forms; $(H_2(M_1 \# M_2), \lambda) = (H_2(M_1), \lambda_1) \oplus (H_2(M_2), \lambda_2)$. The characterization theorem implies a converse; a direct sum decomposition of the form comes from a (topological) connected sum decomposition of the manifold. For example the Kummer surface decomposes as $2\| - E_8 \| \# 3S^2 \times S^2$. It does not decompose smoothly, however (see 8.4A).

If M is a compact manifold with boundary there is still an intersection form defined on $H_2(M;\mathbf{Z})$, but it usually is singular. All the invariants can be defined for singular forms, but only the signature seems to be useful. If there is an isomorphism of boundaries $\partial M \simeq \partial N$ then "Novikov additivity" asserts that $\text{signature}(M \cup_\partial N) = \text{signature}(M) + \text{signature}(N)$, where the right side of the equation uses the signature of the singular forms. This generalizes the additivity for connected sums, where the boundary is S^3.

10.2B The Kirby-Siebenmann invariant. Suppose M is a compact topological 4-manifold. The boundary has a unique smooth structure, and there is an obstruction in $H^4(M, \partial M; \mathbf{Z}/2)$ to extension of this to a smooth structure on $M \times \mathbf{R}$, or $M \# kS^2 \times S^2$ (see 8.3D and 8.7). Each component of M has $H^4(-, \partial; \mathbf{Z}/2) \simeq \mathbf{Z}/2$. Define $\text{ks}(M)$ to be the sum over all components, of these invariants in $\mathbf{Z}/2$.

If M is connected then $\text{ks}(M)$ is the stable smoothing obstruction. In general it is the number of components mod 2 which are not stably smoothable.

This number is an unoriented bordism invariant (i.e. if $M \cup N = \partial W$ then $\text{ks}(M) = \text{ks}(N)$), and is additive in a very strong sense. If M is a union of manifolds $N_0 \cup N_1$ with $N_0 \cap N_1$ in the boundary of each N_i, then $\text{ks}(M) = \text{ks}(N_0) + \text{ks}(N_1)$. This is a consequence of the uniqueness of smooth structures on 3-manifolds.

In general this invariant is somewhat distantly related to other characteristic classes, but there is a direct relation for spin manifolds. Let

SPTOP denote the universal cover of the identity component of the stable homeomorphism group TOP. (This is a 2-fold cover; π_0 and π_1 of TOP are the same as those of the orthogonal group O, and are both $\mathbf{Z}/2$). A *spin structure* is a lifting of the classifying map for the stable tangent bundle from B_{TOP} to B_{SPTOP}. A spin structure exists if $\omega_1 M = \omega_2 M = 0$ on $H_2(M;\mathbf{Z}/2)$. When one such structure exists others are classified by $H^1(M;\mathbf{Z}/2)$.

The basis for the relation is the theorem of Rochlin that a closed smooth spin 4-manifold has signature divisible by 16. Divisibility by 8 follows from the evenness of the form, so the force of the theorem is that there is another factor of 2. For closed topological spin manifolds the result is that $\mathrm{signature}(M)/8 = \mathrm{ks}(M) \bmod 2$. Note this requires only the existence of a spin structure, not a particular choice.

Now we consider manifolds with boundary. Suppose N is a closed 3-manifold, with a spin structure denoted by τ. The 3-dimensional smooth spin bordism group is trivial, so (N,τ) is the boundary of a smooth spin 4-manifold (W,η). The *Rochlin invariant* $\mathrm{roc}(N,\tau)$ is defined to be the signature of W, mod 16. This is well defined: Two such bounding manifolds glue together to give a closed smooth spin 4-manifold, whose signature is the difference of the signatures of the pieces. According to Rochlin's theorem for closed manifolds this difference is divisible by 16.

This invariant may depend on the spin structure. For example the 3-torus has $H^1(M;\mathbf{Z}/2)$ of order 8, so has 8 spin structures. 7 of these have invariant 0, and one (corresponding to the Lie group framing) has invariant 8 (see Kirby [**2**]). When there is a unique spin structure then of course the invariant is well defined.

This occurs when $H^1(N;\mathbf{Z}/2) = 0$, i.e. exactly when components of N are $\mathbf{Z}/2$ homology spheres.

Proposition. *Suppose (M,η) is a compact oriented 4-dimensional topological spin manifold. Then* $8\mathrm{ks}(M) \equiv \mathrm{signature}(M) + \mathrm{roc}(\partial M, \partial\eta)$, *mod 16.*

This follows easily from the additivity of the signature and ks, and the closed case. When M is not spin there is a more complicated formula involving the Brown invariant of a linking form on a surface dual to ω_2, see Guillou and Marin [**1**], and Kirby [**2**].

10.3 Connected sum decompositions

The objective is to determine when a manifold W can be expressed as a connected sum $M \# W_1$, where M is a closed simply connected 4-manifold. The hypotheses are in terms of intersection and selfintersection forms on $\pi_2 W$.

If W is a connected sum $M \# W_1$, with M closed and 1-connected, then $\pi_2 W \simeq (\pi_2 M \otimes \mathbf{Z}\pi_1 W) \oplus \pi_2 W_1$. Intersection numbers on the M summand are given by: if $x \otimes \alpha, y \otimes \beta \in \pi_2 M \otimes \mathbf{Z}\pi_1 W$, then $\lambda(x \otimes \alpha, y \otimes \beta) = \lambda(x,y)\alpha\bar{\omega}(\beta)$. Similarly $\tilde{\mu}(x \otimes \alpha) = \tilde{\mu}(x)\alpha\bar{\omega}(\alpha)$.

Abstracting this, we say a $\mathbf{Z}\pi_1 W$ homomorphism $(\pi_2 M \otimes \mathbf{Z}\pi_1 W) \longrightarrow \pi_2 W$ "preserves λ and $\tilde{\mu}$" if intersection numbers of images are given by these expressions. Since λ in M is nonsingular this implies the homomorphism is an injection onto a direct summand of $\pi_2 W$.

Theorem. *Suppose M is a closed 1-connected 4-manifold, and W has good fundamental group.*

(1) *Let $(\pi_2 M) \otimes \mathbf{Z}\pi_1 W \longrightarrow \pi_2 W$ be a $\pi_1 W$ monomorphism which preserves λ and $\tilde{\mu}$. If either $\omega_2 = 0$ on $\pi_2 W$ or ω_2 does not vanish on the subspace of $\pi_2 W$ perpendicular to the image, then there is a decomposition $W \simeq M \# W'$ inducing the given decomposition of π_2. If $\omega_2 \neq 0$ does vanish on the perpendicular then exactly one of W or $*W$ decomposes (see 10.4 below for $*W$).*

(2) *Suppose $h_1 \colon W \simeq M \# W_1$ and $h_2 \colon W \simeq M \# W_2$ are two decompositions inducing the same decomposition of π_2. If $\omega_1 \colon \pi_1 W \longrightarrow \mathbf{Z}/2$ is injective on elements of order 2, then the decompositions are pseudoisotopic. If the form of M is even the canonical homotopy equivalence $W_1 \longrightarrow W_2$ is homotopic to a homeomorphism (regardless of ω_1).*

Two decompositions are isotopic if there is an isotopy from the identity to a homeomorphism $g \colon W \longrightarrow W$ so that $h_2 g (h_1)^{-1}$ is the identity on M_0. Similarly they are pseudoisotopic if there is a pseudoisotopy to such a g; a homeomorphism $W \times I \simeq W \times I$ which is the identity on one end and g on the other. Pseudoisotopy implies isotopy if $\pi_1 W = 0$ (Quinn [8]).

We remark on the ω_2 hypothesis. If $J \subset \pi_2 W$ is a subgroup then a is "perpendicular" to J if $\lambda(a,b) = 0$ for all $b \in J$. Therefore the phrase "ω_2 vanishes on the subspace perpendicular to J" means that if $\lambda(a,b) = 0$ for all $b \in J$ then $\omega_2(a) = 0$. If the form λ is nonsingular on π_2, for example if W is closed, then this condition is equivalent to the existence of an element $b \in J$ dual to ω_2 in the sense that $\lambda(a,b) \equiv \omega_2(a)$, mod 2.

To complete the statement a definition of the $*$ operation used in the second part of (1) is needed. This was described in a special case in the discussion after the characterization theorem.

10.4 Definition of $*W$

Suppose W is a 4-manifold. If $\omega_2 \colon \pi_2 W \longrightarrow \mathbf{Z}/2$ is trivial, define $*W =$

W. If ω_2 is nontrivial define $*W$ to be a manifold with a homeomorphism $(*W) \# CP^2 \simeq W \# (*CP^2)$ which preserves the decompositions of π_2, where $*CP^2$ is a manifold with form [1] and ks = 1.

Note that if $*W$ is defined and different from W, then it has the opposite Kirby-Siebenmann invariant. Also there is a canonical homotopy equivalence $*W \longrightarrow W$ which is an isomorphism on the boundary.

The manifold $*CP^2$ is shown to exist and be unique in the beginning of the proof of 10.1, below. If the fundamental group of W is good then the first part of 10.3(1) implies that a manifold $*M$ exists: first note that $\pi_2 W$ is the subspace of $\pi_2(W \# *CP^2)$ perpendicular to $\pi_2(*CP^2) \otimes \mathbf{Z}\pi_1 W$. Therefore if $\omega_2 \neq 0$ on $\pi_2 W$, part (1) of the theorem implies the injection of $\pi_2(*CP^2) \otimes \mathbf{Z}\pi_1 W$ corresponds to a decomposition as a connected sum with CP^2. The complementary piece of the decomposition is—by definition—$*W$.

Similarly (2) shows that if $\pi_1 W$ is good and $\omega_1 \colon \pi_1 W \longrightarrow \mathbf{Z}/2$ is injective on elements of order 2, then $*W$ is well-defined up to homeomorphism. This applies for example when $\pi_1 W$ has no 2-torsion, or is RP^4. We do not know if $*W$ is well-defined when the ω_1 condition fails, for example if W is $RP^3 \times S^1$ or $RP^2 \times RP^2$. ∎

Proof of 10.1: We deduce the Characterization Theorem from the sum theorem, and the fact that a homology 3-sphere bounds a contractible topological 4-manifold.

The first step is to realize the matrix E_8. This can be realized as the intersection matrix of a simply connected manifold with boundary, by plumbing together 8 copies of the D^2 bundle over S^2 whose core 2-sphere has selfintersection 2 (this is the tangent disk bundle of S^2). Specifically, form a linear chain of 7 copies by introducing single intersection points between their core spheres. Then introduce an intersection between the remaining one, and the third from one end in the chain.

It follows from the fact that the intersection matrix is nonsingular that the boundary of this manifold is a homology sphere (see Browder [**1**, V.2.6]; in fact it is the famous Poincaré homology sphere). According to 9.3C a homology sphere bounds a contractible manifold. The union of the plumbing manifold and the contractible one gives a closed 1-connected manifold which we denote by $\|E_8\|$.

Next we construct $*CP^2$, with form [1] but ks = 1. According to 10.2B, since E_8 is even ks$\|E_8\| = 1$, and by additivity ks$(\|E_8\| \# (-CP^2)) = 1$. However according to the classification of indefinite forms, the form of this manifold is isomorphic to $8[1] \oplus [-1]$, which is the form of $8CP^2 \# (-CP^2)$. Restrict the isomorphism to get an injection of the form of $7CP^2 \# (-CP^2)$ to a direct summand. The perpendicular subspace is the homology of the remaining copy of CP^2, so the form is not even on it, and ω_2 does not vanish on it. The existence part of the decomposition theorem therefore applies to show there is a decomposition $(\|E_8\| \# (-CP^2)) \simeq 7CP^2 \# (-CP^2) \# N$, for some manifold N. This manifold has form [1], and additivity of ks shows ks$N = 1$. It therefore satisfies the conditions required of $*CP^2$.

Next we show there is a manifold realizing an arbitrary form λ. Since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is indefinite and odd, $\lambda \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is also. By the classification of forms it is isomorphic to $j[1] \oplus k[-1]$ for some j, k. This can be realized as the form of the manifold $jCP^2 \# k(-CP^2)$. The isomorphism gives an injection of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ into this form. Since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the form of $CP^2 \# (-CP^2)$, part (1) of the theorem gives a decomposition $jCP^2 \# k(-CP^2) \simeq CP^2 \# (-CP^2) \# N$, for some manifold N with form λ.

We now have all the manifolds required for the existence part of the characterization theorem; the paragraph above gives at least one with any given form, and the operation $*$ defined in 10.4 reverses the Kirby-Siebenmann invariant when the form is not even.

For the uniqueness part suppose M and N are closed and 1-connected, have the same Kirby-Siebenmann invariant, and $h\colon H_2M \to H_2N$ is an isomorphism which preserves forms. Regarding an isomorphism as an injection, the theorem asserts that there is a decomposition, either $N \simeq M \# P$, or $*N \simeq M \# P$, realizing this injection. However P must be a simply connected homology sphere, so is homeomorphic to S^4, by the Poincaré conjecture Corollary 7.1B. This means $N \simeq M$ or $*N \simeq M$. Since $*$ changes the Kirby-Siebenmann invariant if it changes the manifold, we conclude there is a homeomorphism $N \simeq N$ realizing the isomorphism of forms.

Finally we show that the homeomorphism is determined up to isotopy. Suppose $h_i\colon M \to N$ are homeomorphisms, $i = 1, 2$, which induce the same homomorphism on H_2. Regard N as $N \# S^4$, then according to the uniqueness part of the decomposition theorem there is a pseudoisotopy of the identity of M to a homeomorphism g so that $(h_1 g)(h_2^{-1})$ is the identity on N_0. Since M is 1-connected, pseudoisotopy implies isotopy (Quinn, [8]). The complement of N_0 is a disk, and $h_1 g(h_2^{-1})$ is the identity on the boundary. Therefore Alexander's isotopy ("squeeze toward the middle") on the disk gives an isotopy of $(h_1 g)(h_2^{-1})$ to the identity. The Alexander isotopy and the isotopy of g therefore give an isotopy of h_1 to a map which when composed with (h_2^{-1}) gives the identity. This characterizes h_2, so h_1 is isotopic to h_2, as required for the theorem. ∎

10.5 π_1-negligible embeddings

An embedding $(V, \partial_0 V) \to (W, \partial W)$ is π_1-negligible if $\pi_1(W - V) \to \pi_1(W)$ is an isomorphism. We also assume it is proper in the sense that there is a collar on ∂W which intersects V in a collar on $\partial_0 V$. The disk embedding theorems of chapter 5 provide π_1-negligible embeddings of unions of 2-handles $(D^2 \times D^2, S^1 \times D^2)$ extending a given embedding of $S^1 \times D^2$ in the boundary. Here this is generalized to embeddings of 4-manifold pairs $(V, \partial_0 V)$, extending given embeddings of $\partial_0 V$ in ∂W.

10.5A Embedding theorem. *Let $(V; \partial_0 V, \partial_1 V)$ be a compact 4-manifold triad so that $\pi_1(V, \partial_0 V) = \{1\} = \pi_1(V, \partial_1 V)$ (all basepoints), each component has nonempty intersection with $\partial_1 V$, and components disjoint from $\partial_0 V$ are 1-connected. Suppose $h: V \longrightarrow W$ is a map which restricts to an embedding of $\partial_0 V$ in ∂W.*

(1) **Existence:** *Suppose $\pi_1 W$ is good, $H^3_f(W_h, V \cup \partial W; \mathbf{Z}\pi_1 W) = 0$, and h "preserves relative intersection and selfintersection numbers." If ω_2 is trivial on $\pi_2 W$, or does not vanish on the subspace of $\pi_2 W$ perpendicular to $H_2(V, \partial_0 V; \mathbf{Z}\pi_1 W)$, then h is homotopic rel $\partial_0 V$ to a π_1-negligible embedding. If $\omega_2 \neq 0$ does vanish on the perpendicular, then h is homotopic to such an embedding in exactly one of W or $*W$.*

(2) **Uniqueness:** *Suppose $h_0, h_1: V \times I \longrightarrow W$ are π_1-negligible embeddings, homotopic rel $\partial_0 V$. Then there is an obstruction in $H^2(V, \partial_0 V; (\mathbf{Z}/2)[T_+])$, where T_+ is the set of elements in* $\ker \omega_2 \subset \pi_1 W$ *of order exactly two. If this vanishes then h_0 is π_1-negligibly concordant to h_1.*

A concordance of embeddings, as in (2), is an embedding $V \times I \longrightarrow W \times I$ which restricts to the given embeddings on $V \times \{i\} \longrightarrow W \times \{i\}$, $i = 0, 1$. The π_1-negligibility and duality imply that the complement of the interior of $F(V \times I)$ is an s-cobordism. If the fundamental group of W is good then the s-cobordism theorem implies this has a product structure, so the two embeddings are "pseudoisotopic." Finally if W is 1-connected the pseudoisotopy theorem of Quinn [**8**] implies the embeddings are isotopic. We caution that the concordance produced by the theorem may not be homotopic to the original homotopy.

The proof of the existence part of the theorem is given in 10.8, and the uniqueness in 10.9.

We now discuss the hypotheses. H^3_f denotes the finite cochain cohomology, which on compact spaces is the same as ordinary cohomology. For manifolds it is Poincaré dual to ordinary homology even when the manifold is noncompact. W_h denotes the mapping cylinder of h. As in 10.3 above, the subspace of $\pi_2 W$ perpendicular to $H_2(V, \partial_0 V)$ is the set of elements $a \in \pi_2 W$ with $\lambda(a, b) = 0$ for all $b \in H_2(V, \partial_0 V; \mathbf{Z}\pi_1 W)$.

Next "preserves relative intersection numbers" will be defined. The term is supposed to suggest that relative classes in $(V, \partial_0 V)$ have the same algebraic intersection structure as the images in $(W, \partial_0 W)$. This cannot be defined the way it sounds because intersection numbers are not well defined for relative classes. Instead we form absolute classes from the differences $a - h_* a$ in $V \cup W$, and say that "relative intersections" are the same if the absolute intersections of these differences are trivial.

Let $d_h\colon H_i(V, \partial_0 V; \mathbf{Z}\pi) \to H_i(V \cup_{\partial_0 V} W; \mathbf{Z}\pi)$ be induced by the chain map which takes a relative chain x to $x - h_* x$. Define $\lambda_h(x, y)$ to be the intersection number $\lambda(d_h(x), d_h(y)) \in \mathbf{Z}\pi$, and similarly $\tilde{\mu}_h(x) = \tilde{\mu}(d_h(x))$. Then "$h$ preserves relative intersection numbers" if λ_h and μ_h vanish on $H_i(V, \partial_0 V; \mathbf{Z}\pi)$.

Sometimes the selfintersection part of this definition can be omitted. Recall that intersection numbers (in $4n$-manifolds) determine selfintersections except for coefficients of elements $g \in \pi_1 W$ with $g = g^{-1}$ and $\omega_1(g) = 1$. Therefore the selfintersection condition is unnecessary if ω_1 is trivial on 2-torsion in $\pi_1 W$. In general if the selfintersection hypothesis is dropped then the undetermined part can be organized to give an obstruction: denote the elements of order 2 and $\omega_1 = 1$ by T_-, then we get an obstruction in $H^2(V, \partial_0 V; \mathbf{Z}/2[T_-])$.

Analogously to this the $\mathbf{Z}/2[T_+]$ obstruction encountered in the uniqueness part of the theorem also comes from a selfintersection problem. The signs change because this problem occurs in a 6-manifold.

Next we show that many of the hypotheses are necessary as well as sufficient.

10.5B Proposition. *Suppose $h\colon V \to W$, as above, is homotopic rel $\partial_0 V$ to a π_1-negligible embedding, either in W or $*W$. Then $\pi_1(V, \partial_1 V) = \{1\}$ (for all basepoints), h "preserves relative intersection and selfintersection numbers," and $H^3_f(W_h, V \cup \partial W; \mathbf{Z}\pi_1 W) = 0$.*

Basically, the only condition in 10.5A that is not always necessary is the requirement that $\pi_1(V, \partial_0 V) = \{1\}$.

Proof: If V is embedded, the Seifert-Van Kampen theorem identifies the fundamental group of W as the free product of $\pi_1 V$ and π_1 of the complement, amalgamated over $\pi_1 \partial_1 V$. If the complement has the same fundamental group as W then $\partial_1 V$ must be connected and $\pi_1 \partial_1 V \to \pi_1 V$ must be onto. The long exact sequence of the pair shows these to be equivalent to the vanishing of $\pi_1(V, \partial_1 V)$.

An embedding does not change intersections, so preserves intersection numbers. Finally if h is an embedding then by excision $H^3_f(W_h, V \cup \partial W; \mathbf{Z}\pi_1 W) = H^3_f(W - \operatorname{int} h(V), \partial(W - \operatorname{int} h(V)))$. This is dual to $H_1(W - \operatorname{int} h(V); \mathbf{Z}\pi_1 W)$, which is H_1 of the covering space associated to the homomorphism to $\pi_1 W$. The π_1-negligible condition implies this covering space is 1-connected, so H_1 vanishes. ∎

10.6 Proof of 10.3

We show that the connected sum decomposition theorem follows from the π_1-negligible embedding theorem.

Let M be a closed simply-connected manifold, and denote—as above—the complement of a flat open 4-ball by M_0. Consider this as a triad with $\partial_0 M_0$ empty, then a decompostion of W as a connected sum with M corresponds exactly to an embedding of this triad in W.

M_0 is homotopy equivalent to a wedge of 2-spheres, so $\pi_2 M \otimes \mathbf{Z}\pi_1 W$ is a free $\mathbf{Z}\pi_1 W$-module. Given a $\mathbf{Z}\pi_1 W$ homomorphism $(\pi_2 M \otimes \mathbf{Z}\pi_1 W) \longrightarrow \pi_2 W$ there is a map (unique up to homotopy) $M_0 \longrightarrow W$ inducing it. This map preserves intersection and selfintersection numbers in the sense of 10.5 if and only if the homomorphism preserves λ and μ in the sense of 10.3. Finally the ω_2 on perpendicular subspaces is the same in the two statements. Thus 10.5(1) implies 10.3(1).

For the uniqueness part, assume that two connected sum decompositions of W are given. Since the homomorphism on π_2 determines the map $M_0 \longrightarrow W$ up to homotopy, if the decompositions induce the same decomposition of $\pi_2 W$ the embeddings are homotopic. Therefore 10.5(2) provides an obstruction in $\mathbf{Z}/2[T_+]$ whose vanishing implies pseudoisotopy of the embeddings. But T_+ is the set of elements in the kernel of $\omega_1\colon \pi_1 W \longrightarrow \mathbf{Z}/2$ of exponent exactly 2, so the injectivity assumption in 10.3(2) is equivalent to T_+ being empty. Thus 10.5(2) implies the pseudoisotopy part of 10.3(2).

To complete the proof of 10.3 we need uniqueness of the complement when the form of M is even. This is proved independently of 10.5, and in fact will be used in its proof.

We suppose $W \simeq M \# W_1 \simeq M \# W_2$. Define $\hat{W}$ by connected sum of with a copy of $-M$, so $\hat{W} \simeq \hat{M} \# W_1 \simeq \hat{M} \# W_2$, with $\hat{M} = M \# (-M)$. The form of $\hat{M}$ is even, indefinite, and has signature 0, so it is isomorphic to $k\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ for some k. According to the characterization theorem the manifold is supposed to be $kS^2 \times S^2$. We give a direct proof of this by finding the core spheres using the disk embedding theorem.

Choose a basis for $\pi_2 \hat{M}$, $\{a_i, b_i\}$ for $i = 1, \ldots, n$ so that all intersections and selfintersections are trivial except $\lambda(a_i, b_i) = 1$. Represent the $\{a_i\}$ by framed immersions $A_i\colon S^2 \longrightarrow \hat{M}$, and choose a 4-ball D which intersects the A_i in mutually disjoint embedded disks. Delete the interior of D, then the result is a collection of framed immersed disks with algebraically transverse spheres (the b_i). $\hat{M}$ is simply connected, so the corollary to 5.1 applies to give a regular homotopy rel boundary of $\bigcup_i A_i$ to an embedding. Replacing D gives disjoint framed embedded spheres representing $\{a_i\}$.

Next find disjoint framed embedded spheres $\{B_i\}$ representing $\{b_i\}$, and so that the only intersections with the A_* is a single point in each $A_i \cap B_i$. This is done either by deleting a neighborhood of $\bigcup_i A_i$ and applying corollary 5.1 again, or much more directly by using the spheres A_i as in the exercise in 1.9.

Now let $h_j \colon \hat{W} \simeq \hat{M} \# W_j$ denote the given homeomorphism for $j = 1, 2$, so that $h_1 h_2^{-1}$ preserves the decompositions of the forms and is the identity on the part coming from $\hat{M}$. Construct a 5-manifold Z by starting with $\hat{W} \times I$, adding 3-handles on the spheres $h_1^{-1} A_i \subset \hat{W} \times \{0\}$, and 3-handles on the spheres $h_2^{-1} B_i \subset \hat{W} \times \{1\}$. The boundary of Z is the union of three pieces; W_1 connected sum the manifold obtained by surgery on $\bigcup_i A_i \subset \hat{M}$, W_2 connected sum the manifold obtained by surgery on $\bigcup_i B_i \subset \hat{M}$, and $(\partial W) \times I$. Since $\hat{M}$ is reduced to a homotopy sphere (thus a sphere) by surgery on either the A_i or the B_i, the first two pieces are W_1 and W_2.

The basic idea is that Z is an s-cobordism rel boundary from W_1 to W_2, so since the fundamental group is good the s-cobordism theorem implies the ends are homeomorphic. We have not assumed W is compact, so we use the "enlargement" version 7.1C of the s-cobordism theorem rather than 7.1A. There is a handlebody structure for (Z, W_1) with 2-handles the duals of the handles added to the $h_1^{-1} A_i$, and with 3-handles the handles added to the $h_2^{-1} B_i$. $\hat{W}$ is the level manifold between the 2- and 3-handles. Since $h_1 h_2^{-1}$ is the identity on the part of the forms coming from $\hat{M}$, the matrix of intersection between the $h_1^{-1} A_*$ and the $h_2^{-1} B_*$ is the identity.

Choose nullhomotopies for Whitney circles for all excess intersections and intersections, and let X be a compact submanifold of $\hat{W}$ containing all the spheres and nullhomotopies. Let Y be the submanifold of Z obtained by attaching the handles, then $(Y, Y \cap W_1)$ is an s-cobordism and Z has a product structure in the complement of Y. Since $\pi_1 Z = \pi_1 W$ is good, version 7.1C of the h-cobordism theorem implies that Z has a product structure. In particular there is a homeomorphism $W_1 \simeq W_2$, as required. ∎

10.7 Manifolds with infinite cyclic fundamental group

Analogs of the characterization and connected sum theorems are stated without formal proof, basically as an extended exercise. We denote the infinite cyclic group by J, to distinguish it from the coefficient ring $\mathbf{Z}$.

Suppose M is a closed manifold with $\pi_1 M \simeq J$. Then $H_2(M; \mathbf{Z}[J])$ is a free $\mathbf{Z}[J]$ module; the fact that it is stably free is a byproduct of the proofs of the results below, and for this ring stably free implies free.

When M is orientable the intersection form is a nonsingular hermitian form on this free module. Generally the form is defined using an orientation at the basepoint, and is $\bar{\omega}_1$-hermitian.

10.7A Classification Theorem.

(1) **Existence:** *Suppose (H, λ) is a nonsingular hermitian form on a finitely generated free $Z[J]$-module, $k \in Z/2$, and if λ is even then we assume $k \equiv$ (signatureλ)/8, mod 2. Then there is a oriented closed manifold with $\pi_1 = J$, intersection form λ and Kirby-Siebenmann invariant k.*

(2) **Uniqueness:** *Suppose M, N are closed and oriented with $\pi_1 = J$, $h: H_2(M; \mathbf{Z}[J]) \to H_2(N; \mathbf{Z}[J])$ is a $\mathbf{Z}[J]$ isomorphism which preserves intersection forms, and* ks$M$ = ks$N$. *Then there is a homeomorphism $f: M \to N$, unique up to pseudoisotopy, which induces the given identification of fundamental groups, preserves orientation and such that $f_* = h$.*

This is the analog of 10.1, except the uniqueness is only up to pseudoisotopy. We caution that the situation with other fundamental groups is much more complicated (see 10.10).

To state the analog of the sum theorem we need an S^1 analog of connected sums. A map $S^1 \to M^4$ is homotopic to an embedding with neighborhood a D^3 bundle over S^1. This bundle is either $S^1 \times D^3$ or a Mobius band times D^2, depending on the value of ω_1 on the homology class. Now suppose there are embeddings of S^1 in both M and W, on which the respective ω_1 take the same value. Define $M \#_{S^1} W$ by deleting the interiors of disk bundle neighborhoods and identifying the boundaries. More precisely if local orientations are given then this can be done so the result has a local orientation compatible with those of the pieces, and when normalized in this way the operation is well-defined.

Continuing with the same notation, suppose M has fundamental group J and $S^1 \subset M$ represents the generator. Suppose also the inclusion $S^1 \subset N$ is injective on π_1. Then $\pi_1(M \#_{S^1} N) \simeq \pi_1 N$. Denoting this group by π, it is also true that $H_2(M \#_{S^1} N; \mathbf{Z}[\pi]) = (H_2(M; \mathbf{Z}[J]) \otimes_{\mathbf{Z}J} \mathbf{Z}[\pi]) \oplus H_2(N; \mathbf{Z}[\pi])$. Similarly the intersection form of the sum is the form on M tensored up to $\mathbf{Z}[\pi]$, plus the form on N.

10.7B Theorem. *Suppose M is a closed locally oriented 4-manifold with fundamental group J, W has good fundamental group π, $J \to \pi$ is injective, and ω_1 of the two manifolds agree on J.*

(1) *Suppose $H_2(M; \mathbf{Z}[J]) \otimes_{\mathbf{Z}J} \mathbf{Z}[\pi] \to H_2 W$ is a $\mathbf{Z}\pi$ homomorphism which preserves λ and $\tilde{\mu}$, and either $\omega_2 = 0$ on $\pi_2 W$, or ω_2 does not vanish on the subspace perpendicular to the image. Then*

there is a decomposition $W \simeq M \#_{S^1} W'$ inducing the given homomorphism to $\pi_2 W$. If $\omega_2 \neq 0$ does vanish on the perpendicular, then exactly one of W or $*W$ decomposes.

(2) *Suppose $h_1\colon W \simeq M \#_{S^1} W_1$ and $h_2\colon W \simeq M \#_{S^1} W_2$ are two decompositions inducing the same decomposition of π_2. If $\omega_1\colon \pi_1 W \longrightarrow \mathbf{Z}/2$ is injective on elements of order 2, then the decompositions are pseudoisotopic.*

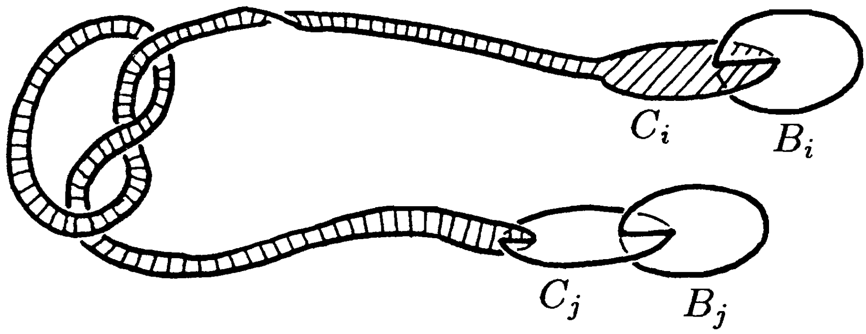
Note that, unlike the previous theorem, the manifolds are not assumed orientable.

This result is proved from the π_1-negligible embedding theorem. Denote by M_S the complement of the open disk bundle of the embedding $S^1 \subset M$, then the data of (1) defines a map $M_S \longrightarrow W$. Let $S^1 \subset M_S$ be another generating circle, with disk bundle neighborhood E. Show that the map $M_S \longrightarrow W$ is homotopic to one which is a homeomorphism on E and takes $M_S - E$ into the complement of the image of E. Then 10.5 applies to the map $(M_S - \mathrm{int}E, \partial E) \longrightarrow (W - \mathrm{im}E, \mathrm{im}\partial E)$.

It may be helpful to note that after addition of a closed 1-connected manifold with even form, $(M_S - \mathrm{int}E, \partial E)$ has a handlebody structure with only 2-handles (see the next section).

The characterization theorem is proved, as in the 1-connected case, from the sum theorem and a few additional facts. The first fact is a stable classification of nonsingular hermitian forms over $\mathbf{Z}[J]$. Given (H, λ) there is an $n \times n$ hermitian $\mathbf{Z}[J]$-matrix A (possibly singular) such that $(H, \lambda) \oplus (\mathbf{Z}[J]^{2n}, \left[\begin{smallmatrix} 0 & 1 \\ 1 & A \end{smallmatrix}\right])$ is isomorphic to a sum of copies of $[1]$ and $[-1]$. This is the statement that the group of stable equivalence classes of such forms is generated by the form $[1]$, see Ranicki [**1**, §10].

To remove summands $\left[\begin{smallmatrix} 0 & 1 \\ 1 & A \end{smallmatrix}\right]$ as in the proof of the 1-connected version, we need to construct manifolds with these forms. Begin with $E = S^1 \times D^3$. Choose n pairs of embedded disks B_i, C_i in ∂E which are mutually disjoint except for a clasp of intersections in $B_i \cap C_i$. Adding 2-handles to E on the standard framings of ∂B_i, ∂C_i gives $E \# n(S^2 \times S^2)$, with form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Now modify the C_i: push a piece of C_i around a loop g and introduce a clasp with C_j.



This changes the intersection matrix by the symmetrization of the matrix with $\pm g$ in the i, j place. Any symmetric matrix A over $\mathbf{Z}[J]$ can be realized by this operation and changing the framing on the ∂C_i. Attaching handles to these new framed circles gives a manifold with boundary, and form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & A \end{smallmatrix}\right]$.

The nonsingularity of the form implies the boundary of this manifold has the $\mathbf{Z}[J]$ homology of ∂E. However (according to surgery, in 11.6) any such manifold bounds a topological manifold with the homotopy type of S^1. Attach such a bounding manifold to the construction to obtain a closed manifold with the desired form.

The uniqueness statements require an additional fact from surgery. Let E be a D^j bundle over S^1. If $N \to E$ is a homotopy equivalence which is a homeomorphism on the boundary, then it is homotopic rel boundary to a homeomorphism (see 11.5). This is used in the cases $j = 3$ and $j = 4$.

10.8 Proof of 10.5(1)

In outline, the theorem is first reduced to the case where $(V, \partial_0 V)$ is a handlebody with only 2-handles. The 2-handles are then embedded using the embedding theorem for disks. When $\omega_2 \neq 0$ on π_2 but vanishes on the subspace perpendicular to $H_2(V, \partial_0 V; \mathbf{Z}\pi_1 W)$ then complications arise because we cannot always find framed immersed dual classes for the 2-handles.

Much of the chapter to this point has consisted of deductions from the result now being proved. However the uniqueness of cancellation for manifolds with even forms, and the existence of the manifold $\|E_8\|$ were established independently, in 10.6 and the proof of 10.1, respectively. We will use these facts here.

Suppose M is a closed 1-connected manifold with even form, and suppose the map obtained by connected sum with the identity $M \# V \to M \# W$ is homotopic rel $\partial_0 V$ to a π_1-null embedding. This gives a second embedding of M_0 in $M \# W$, and therefore a decomposition $M \# W \simeq M \# W'$, with the two copies of M representing the same summand of the form. The uniqueness of cancellation shows the canonical homotopy equivalence $W \simeq W'$ is homotopic rel ∂W to a homeomorphism. But W' has V embedded π_1-negligibly in it. Therefore we get an embedding of V in W.

This shows it is sufficient to embed $M \# V$ in $M \# W$, where M is any closed manifold with even form and 1-connected components.

As a first consequence we note we can assume V is smoothable. Add a copy of the E_8 manifold to each component with nontrivial Kirby-Siebenmann invariant to change it to be zero. According to 8.7(1), a manifold with vanishing Kirby-Siebenmann invariant can be smoothed after connected sum with some number of copies of $S^2 \times S^2$. Therefore $V \# j\|E_8\| \# k(S^2 \times S^2)$ is smoothable, and to embed V it is sufficient to embed this.

The second consequence of cancellation is that we can assume $(V, \partial_0 V)$ has a handlebody structure with only 2-handles. We first arrange that $\partial_0 V$ intersects each component nontrivially; in each component where this fails make h a homeomorphism on a ball, delete the interior from both V and W, and add the boundary to $\partial_0 V$. Now since V is smoothable it has some handlebody structure. Each component intersects $\partial_0 V$ and $\partial_1 V$ nontrivially, so there is a handlebody structure with no 0- or 4-handles. 1- and 3-handles will be eliminated by showing they can be changed into 2-handles by connected sums with $S^2 \times S^2$.

Consider a 1-handle, attached to $\partial_0 V$. The connectivity assumptions imply that $\pi_1(V, \partial_0 V) = 0$, so the core of this 1-handle is homotopic relative to the ends into $\partial_0 V$.

Approximate the homotopy by a framed immersed 2-disk which is standard near the core of the handle, and push selfintersections off by finger moves (see 1.5). This gives a framed embedded disk whose boundary is the union of the handle core and an arc on $\partial_0 V$. Let B denote a 3-ball neighborhood of the arc in $\partial_0 V$, then a collar on B union with the 1-handle is isomorphic to $S^1 \times D^3$.

The connected sum operation $\# S^2 \times S^2$ removes a 4-ball and replaces it with $S^2 \times S^2 - D^2_- \times D^2_-$, where $S^2 = D^2_- \cup D^2_+$. Identify $D^2_- \times D^2_+$ in this with a 1-handle in the original 4-ball. The complement of this 1-handle is isomorphic to $S^1 \times D^3$, so the operation becomes: remove a copy of $S^1 \times D^3$ (whose core bounds an embedded D^2), and replace it with $D^2 \times S^2$. (This is a "surgery" on the S^1.) Therefore by such a connected sum we can replace the 1-handle on $B \times I$ with a copy of $D^2 \times S^2$. This can be regarded as a 2-handle added on $B \times I$, so the original 1-handle has been replaced by a 2-handle.

A 3-handle can be considered dually as a 1-handle attached to $\partial_1 V$, so the same argument can be used to convert all 3-handles into 2-handles.

We now assume $(V, \partial_0 V)$ has only 2-handles. Denote the core 2-disks of these handles by $\{D_i\}$. The first step toward applying the disk embedding theorems is to find an appropriate immersion homotopic to h.

According to the immersion lemma h is homotopic to an immersion which differs by rotations from the given one on the attaching region of the handles. Let χ_i denote the rotation on $\partial_0 D_i$. $A_i = D_i \cup h(D_i)$ defines immersed 2-spheres with possibly nontrivial normal bundles. The 1 coefficient (in $\mathbf{Z}\pi_1 W$) of the selfintersection forms of these immersions satisfy $2\mu_1(A_i) + \chi_i = \lambda_1(A_i, A_i)$.

The hypothesis that h "preserves relative intersection numbers" implies $\lambda_1(A_i, A_i) = 0$, so χ_i is even. According to 1.3 it can be changed by any even number by twisting inside $h(D_i)$, so we can arrange for it to be zero. This gives an immersion of V extending the embedding of $\partial_0 V$, and the intersection and selfintersection forms of the core disks are trivial. Note that according to Proposition 1.7 this immersion is well-defined up to regular homotopy; different regular homotopy classes rel boundary of immersions of disks can be distinguished by selfintersection numbers.

Next we locate homologically transverse elements for the $h(D_i)$, in $\pi_2 W$.

Since the cohomology group $H^3_f(W_h, \partial W \cup V)$ ($\mathbf{Z}\pi_1 W$ coefficients) is assumed trivial, the long exact sequence of the triple $(W_h, \partial W \cup V, \partial W)$ shows that $H^2_f(W, \partial W) \to H^2(V, \partial_0 V)$ is onto. The first group is Poincaré dual to $H_2(W; \mathbf{Z}\pi_1 W) \simeq \pi_2 W$. Since $(V, \partial_0 V)$ has only 2-handles the cohomology group is isomorphic to the dual of the homology; $(H_2(V, \partial_0 V))^*$. The resulting homomorphism $H_2(W; \mathbf{Z}\pi_1 W) \to (H_2(V, \partial_0 V))^*$ is the adjoint of the intersection form $\lambda\colon H_2(W; \mathbf{Z}\pi_1 W) \times H_2(V, \partial_0 V) \to \mathbf{Z}\pi_1 W$. Since the homomorphism is onto we conclude that there are elements $\alpha_i \in \pi_2 W$ such that $\lambda(D_i, \alpha_j) = 0$ if $i \neq j$, and $= 1$ if $i = j$.

To apply the embedding theorem we need such classes α_i represented by framed immersions. According to 1.3 this will be the case if $\omega_2(\alpha_i) = 0$. If $\omega_2 = 0$ on $\pi_2 W$ this holds automatically. If there is an element $\gamma \in \pi_2 W$ so that $\lambda(D_i, \gamma) = 0$ for all i and $\omega_2(\gamma) \neq 0$, then we can add γ to the α_i with nontrivial ω_2, to obtain new α_i on which ω_2 vanishes. Therefore framed immersions can be obtained unless ω_2 is nontrivial on $\pi_2 W$, but vanishes on the subspace perpendicular to $H_2(V, \partial_0 V)$.

When there are framed immersed α_i, Corollary 5.1B implies that h is regularly homotopic rel $\partial_0 V$ to a π_1-negligible embedding. This proves theorem 10.5(1) unless $\omega_2 \neq 0$ on $\pi_2 W$, but vanishes on the perpendicular to $H_2(V, \partial_0 V)$.

To analyse the situation when the α_i cannot be chosen to be framed, we recall the proof of the appropriate embedding theorem (the proof of Corollary 5.1B from Theorem 5.1A). First immersed Whitney moves are used to change h by regular homotopy so that the α_i are represented by framed immersions which are transverse spheres for the $h(D_i)$. Then immersed Whitney disks are chosen for all intersections among the $h(D_i)$. These are made disjoint from the $h(D_i)$ by adding copies of the α_i. Framed immersed transverse spheres for the Whitney disks are constructed by contraction from the linking tori and caps, after the caps are made disjoint from $h(D_*)$ using the α_i. Theorem 5.1 then applies to the Whitney disks; they can be replaced by embeddings with the same framed boundaries. Whitney isotopies using these embedded disks give a regular homotopy of h to the desired embedding.

The only step which requires the framing is the use of sums with the α_i to make the Whitney disks disjoint from $h(D_i)$. For example, in the last step (constructing transverse spheres for the Whitney disks) each α_i enters algebraically zero times, because it is added to caps in a capped surface which is then contracted. This implies twists in the normal bundle cancel out. Since the problem comes from these Whitney disk–$h(D_i)$ intersections, we use them to define an invariant.

10.8A Definition. In the situation above suppose B is an immersed Whitney disk for intersections among the $h(D_*)$, with boundary arcs on $h(D_i)$ and $h(D_j)$. Define $\mathrm{km}(B) = \omega_2(\alpha_i)\omega_2(\alpha_j)\sum_k |B \cap h(D_k)|\omega_2(\alpha_k)$, in $\mathbf{Z}/2$. Here $|B \cap h(D_i)|$ denotes the number of intersection points, in general position, mod 2. If $\{B_j\}$ is a complete set of Whitney disks (with disjoint boundaries) for all the intersections, define $\mathrm{km}(h) = \sum_j \mathrm{km}(B_j)$. Finally if h_0 is a map as originally specified in the theorem, and h is a map of a 2-handlebody obtained from it as above, then define $\mathrm{km}(h_0) = \mathrm{km}(h)$.

This invariant extends an embedding obstruction for 2-spheres discovered by Kervaire and Milnor [**1**], and the notation is intended to reflect this. The following lemma completes the proof of part (1) of the theorem.

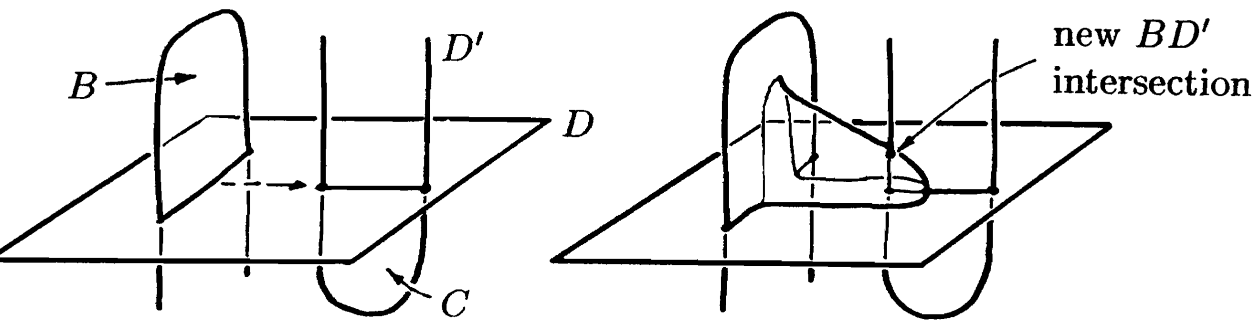
10.8B Lemma. *Suppose h satisfies the hypotheses of 10.5(1), and determines ω_2 on $\pi_2 W$. Then* $\mathrm{km}(h)$ *is well defined, and h is homotopic rel $\partial_0 V$ to an embedding in W, or $*W$, if and only if* $\mathrm{km}(h) = 0$, *or* $\mathrm{km}(h) = 1$ *respectively.*

Proof: We claim it is sufficient to show the invariant is well-defined, and that if it vanishes then h is homotopic to an embedding. Since an embedding trivially has vanishing invariant these assertions directly imply that h is homotopic to an embedding if and only if the invariant vanishes. We show next the assertions also imply that if $\mathrm{km}(h) = 1$ then h is homotopic to an embedding in $*W$. Together these statements imply the lemma.

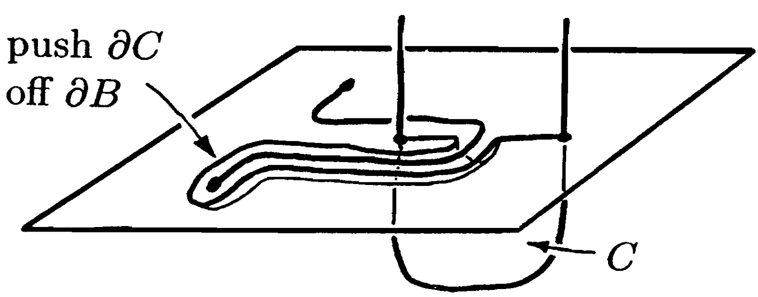
Let $f\colon CP^2 \to *CP^2$ denote the canonical homotopy equivalence, then $f_0\colon CP^2_0 \to *CP^2$ satisfies the hypotheses of the theorem, but cannot be homotopic to an embedding. Therefore $\mathrm{km}(f_0) = 1$. Consider $h \cup f_0\colon V \cup CP^2_0 \to W \# (*CP^2)$. If $\mathrm{km}(h) = 1$ then $\mathrm{km}(h \cup f_0) = 0$, so is homotopic to an embedding. This gives an embedding of V in the complement of $CP^2_0 \subset W \# (*CP^2)$, which is by definition $(*W)_0$. Therefore h is homotopic to an embedding in $*W$.

We now show that if V has a handlebody structure with only 2-handles such that $\mathrm{km}(h)$ (defined with respect to this structure) vanishes, then h is homotopic to an embedding.

The first step is an operation which changes intersections of Whitney disks. Suppose D is an immersed disk, and B, C are Whitney disks with disjoint boundaries, for intersections of D with a surface D'. Push a piece of the boundary arc of B through one of the C intersection points, to introduce a new BD' intersection and a point of intersection of the Whitney arcs.



Now push the C boundary arc along the B arc through one of the B intersection points.



This makes the boundary arcs disjoint once again, and introduces a CD' intersection. If km is defined using intersections with D' then this procedure changes both km(B) and km(C).

Now suppose h is an immersion of a 2-handlebody, with Whitney disks $\{B_j\}$ and km$(h) = 0$. We change this so that each Whitney disk has km $= 0$. If there is a Whitney disk with km $\neq 0$ then there must be two since the total is 0 mod 2. Suppose they are B_1 and B_2, with a boundary arc on $h(D_1)$, $h(D_2)$ respectively. Use a finger move to introduce a new pair of intersections between $h(D_1)$ and $h(D_2)$, with embedded Whitney disk C with boundary arcs on both $h(D_1)$ and $h(D_2)$. Apply the procedure above to the pair C, B_1, and to C, B_2. A single new $B \cap h(D)$ intersection is introduced into each of B_1 and B_2, so they now have km $= 0$. Two intersection points are introduced into C, so it also has km $= 0$. This reduces the number of km $\neq 0$ Whitney disks by 2, so by induction they can all be converted.

Thus we may assume all Whitney disks have km $= 0$. Use connected sums with the (possibly unframed) transverse spheres to make them disjoint from the $h(D_*)$. If a disk B_j is changed by an even number of sums with spheres with $\omega_2 \neq 0$ then the normal bundle of the result differs on the boundary by an even number of twists from the Whitney framing. Interior twists as in 1.3 can be used to correct this. If there are an odd number of such spheres then by definition of km(B_j) one of the boundary arcs must be on a disk with framed transverse sphere, say $h(D_i)$. A boundary twist about this arc corrects the framing on the Whitney disk, but introduces a new intersection with $h(D_i)$. Since $h(D_i)$ has framed transverse sphere this intersection can be removed without disturbing the framing.

Therefore we can arrange that there are Whitney disks with interiors disjoint from the $h(D_*)$. As noted above the rest of the proof of the embedding theorem does not require framings on the transverse spherse of the $h(D_*)$, so (if the fundamental group is good) there are embedded Whitney disks, and h is homotopic to an embedding.

The lemma will be completed by showing that km(h) is well-defined. Note that if it is well-defined with respect to a fixed handlebody structure then its vanishing is equivalent to embedibility of the manifold. But embedibility is independent of the handlebody structure, and independent of the connected sums used to get such a structure. Therefore it is a well-defined invariant of the original map. This reduces the problem to showing it is well defined with respect to a fixed handlebody structure.

Fix a handlebody structure for $(V, \partial_0 V)$ with only 2-handles. Recall that the immersion homotopic to h is well-defined up to regular homotopy by the intersection number conditions. A regular homotopy is a sequence of Whitney moves, finger moves, and isotopies. Since a Whitney move is inverse to a finger move, it is sufficient to see that km is invariant under finger moves.

A finger move introduces a single embedded Whitney disk disjoint from everything else, so adding it to a collection of Whitney disks gives a new collection with the same invariant. This reduces the proof of the lemma to showing the invariant is independent of the choice of Whitney disks, for a fixed immersion in general position.

The first step is to generalize the definition. Define "weak" Whitney disks to be disks satisfying all the conditions on Whitney disks, except the boundary circles are allowed to be immersed arcs in $h(D_*)$. (Usually they are required to be disjointly embedded.) Now suppose $\{B_k\}$ is a complete collection of weak Whitney disks for the intersections of $h(D_*)$. Define $\mathrm{km}(B_*)$ to be the sum over all intersections $\mathrm{int} B_* \cap h(D_*)$ and selfintersections of ∂B_* of products $\omega_2(\alpha_i)\omega_2(\alpha_j)\omega_2(\alpha_k)$ in $\mathbf{Z}/2$. For a BD intersection, i is the index of the D_*, and j, k are the indices of the $h(D_*)$ containing the boundary of B. For a ∂B intersection between boundaries of distince B, i is the index of the D_* on which the intersection takes place, and j, k are the indices of D_* containing the other boundary arcs of the Bs. For a ∂B selfintersection, i is the index of the D_* on which the intersection takes place, and $j = k$ is the index of the D_* containing the other boundary arc of B.

If the Whitney disks are standard, so there are no boundary intersections, this reduces to the previous definition.

The new definition of km is invariant under regular homotopy of the Whitney disks. During a regular homotopy intersection points appear and disappear in pairs, leaving the sum in $\mathbf{Z}/2$ unchanged, except when one boundary arc of a Whitney disk passes over an endpoint of another boundary arc. This changes the number of boundary intersections by one. Since endpoints are $h(D_*)$ intersection points, it also changes the interior intersections by one, as in the pictures above. The $\mathbf{Z}/2$ elements assigned to the two types of intersections are the same, so the sum is invariant in this case also.

Next we show the invariant depends only on the Whitney circles, not on the choice of disks. Suppose B and B' are Whitney disks with the same boundary circle, on disks $h(D_i) \cup h(D_j)$. Both B and B' define framings of the normal bundle of the boundary circle. Using the uniqueness of the normal bundle of the circle, we may assume after isotopy that B' intersects the B framing in a subbundle, which may be twisted. Suppose there are r twists along the arc on $h(D_i)$ and s twists on $h(D_j)$.

Applying the boundary twisting operation of 1.3 to B' straightens out

B', but introduces rotations in its normal bundle. Once B and B' are aligned near the boundary they can be glued together a short distance out from the boundary curve to give an immersed sphere, C. Because of the rotations introduced by the twisting operation $\omega_2(C) = r + s \mod 2$.

Note that $\omega_2(C)$ is also given by $\sum_k |C \cap h(D_k)|\omega_2(\alpha_k)$. This is because C can be expressed (in $\pi_2 W$) as a linear combination of the α_*, plus an element perpendicular to $h(D_*)$. Since ω_2 vanishes on the perpendicular subspace, this last has $\omega_2 = 0$. The number of copies of α_k in the sum is $|C \cap h(D_k)|$, so the formula follows from additivity of ω_2.

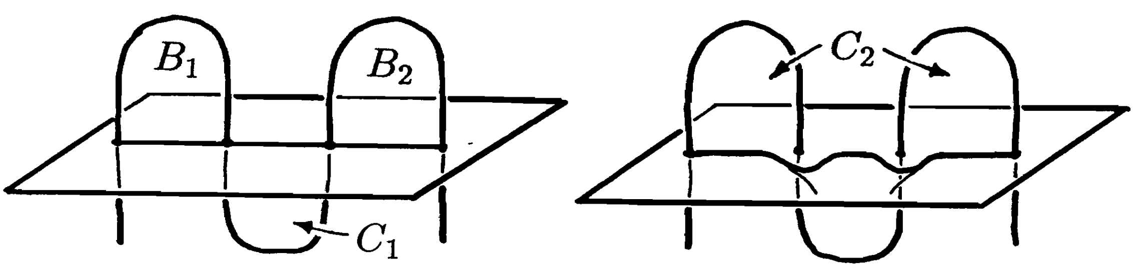
This expression for $\omega_2(C)$ can be related to the km contributions of B and B'. C has the same intersections with the $h(D_*)$ as $\text{int}B \cup \text{int}B'$, plus some introduced by the twisting operation; r with $h(D_i)$ and s with $h(D_j)$. Since we have assumed $\omega_2(\alpha_i)$ and $\omega_2(\alpha_j)$ are nontrivial, the intersection expression gives $\omega_2(C) = \sum_k |(\text{int}B) \cap h(D_k)|\omega_2(\alpha_k) + \sum_k |(\text{int}B') \cap h(D_k)|\omega_2(\alpha_k) + r + s$.

Subtracting $\omega_2(C) = r + s$ and multiplying by $\omega_2(\alpha_i)\omega_2(\alpha_j)$ shows that the interior contributions of B and B' to km are equal. Therefore $\text{km}(B_*)$ depends only on ∂B_*.

Now we show it does not depend on ∂B either. Note that two collections of immersed arcs in a 2-disk, with the same endpoints, are regularly homotopic rel endpoints. Regular homotopies of the boundary curves can be extended to Whitney disks, and km is invariant under regular homotopy. Therefore km depends only on the pairing of intersection points, not on the choice of arcs.

A pairing of intersection points can be deformed to any other pairing by a sequence of moves, each of which changes two pairs. It is therefore sufficient to see that such a move does not change the invariant.

Suppose a_1, a_2 are positive and b_1, b_2 are negative intersection points between $h(D_i)$ and $h(D_j)$, all with the same associated element of the fundamental group. Let B_1, B_2 be (weak) Whitney disks for the pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Suppose C_1 is a Whitney disk for $\{a_1, b_2\}$. Then a Whitney disk C_2 for $\{a_2, b_1\}$ can be constructed from parallel copies of B_1, B_2, and C_1, and small twists near $\{a_1, b_2\}$.



C_2 has the same intersections as the union of the other disks, so $C_1 \cup C_2$ has the same intersections mod 2 as $B_1 \cup B_2$. Therefore km is invariant under replacement of $C_1 \cup C_2$ by $B_1 \cup B_2$.

This completes the proof of the lemma, and therefore of the existence part of the π_1-null embedding theorem. ∎

10.9 5-dimensional embeddings

The uniqueness part of the embedding theorem will be deduced from an analogous embedding theorem for 5-manifolds. Specifically, suppose $g\colon (V \times I, V \times \{0,1\} \cup \partial_0 V \times I) \to W \times I$ is a homotopy rel boundary between embeddings of V. Set $(X, \partial_0 X) = (V \times I, V \times \{0,1\} \cup \partial_0 V \times I)$, and $Y = W \times I$, then g is a map which is an embedding on $\partial_0 X$. A π_1-negligible embedding which agrees with g on $\partial_0 X$ is exactly the "concordance" required for the uniqueness statement.

Theorem. *Suppose $(X; \partial_0 X, \partial_1 X)$ is a compact 5-manifold triad with $(X, \partial_0 X)$ 2-connected and $(X, \partial_1 X)$ 1-connected. Suppose $g\colon X^5 \to Y^5$ is a map which embeds $\partial_0 X$ in ∂Y, and $H^4_f(Y_g, X \cup \partial Y; \mathbf{Z}\pi_1 Y) = 0$. Then there is an obstruction $\tilde{\mu} d_g \in H^3(X, \partial_0 X; (\mathbf{Z}/2)[T_+])$ which vanishes if and only if there is a π_1-negligible embedding $g'\colon X \to Y$ equal to g on $\partial_0 X$ and inducing the same homomorphism $H_3(X, \partial_0 X; \mathbf{Z}\pi_1 Y) \to H_3(Y, \partial_0 X; \mathbf{Z}\pi_1 Y)$.*

H_f denotes, as in 10.5, the finite cocycle cohomology, and Y_g the mapping cylinder of g. T_+ is the subset of $\pi_1 Y$ of elements of exponent exactly 2 on which ω_1 is trivial, as in 10.5. In particular if $\omega_1\colon \pi_1 Y \to \mathbf{Z}/2$ is injective on elements of order 2 then the obstruction group is trivial, and embeddings exist.

As explained above, the uniqueness in 10.5 follows by applying this to a homotopy $g\colon (V \times I, V \times \{0,1\} \cup \partial_0 V \times I) \to W \times I$. Delete a ball from each component of V which does not intersect $\partial_0 V$, to get the proper connectivity.

Proof: The proof is rather lengthy, so we begin with an outline. The first step is the definition of the obstruction, in terms of selfintersections in a 6-manifold. Then there is a reduction to the case where $(X, \partial_0 X)$ has a handlebody structure with only 3-handles. The images of these handles have 1-dimensional intersections and selfintersections. The invariant is described in terms of these intersections, and in fact only involves circles with connected preimages. Then the process of simplifying intersections begins, with the elimination of arcs joining singular points. Next is a version of the Whitney move, which joins intersection circles. When the invariant vanishes this is used to eliminate intersection circles with connected preimage. This leaves circles of intersection whose preimages are two circles. These are eliminated (after some preparation) by a surgery procedure, which may change the homotopy class of the map. The result is a map with no intersections or selfintersections among the handles, so an embedding.

We begin by describing the obstruction. The connectivity hypotheses imply that the relative homology and cohomology of $(X, \partial_0 X)$ is trivial except in dimension 3. Therefore the obstruction group is given by $H^3(X, \partial_0 X; (\mathbf{Z}/2)[T_+]) = \hom_\pi(H_3(X, \partial_0 X; \mathbf{Z}\pi), (\mathbf{Z}/2)[T_+])$, where π denotes $\pi_1 Y$. The obstruction will be described as a homomorphism on H_3.

The homomorphism $d_g\colon H_3(X, \partial_0 X) \to H_3(X \cup Y)$ is induced by the chain map $1 - g_*$, as in 10.5. Multiplying by $\mathbf{R}$ takes this to $H_3(X \cup Y \times \mathbf{R})$, where intersection and selfintersection forms are defined. The intersection form vanishes since homotopy classes can be separated using the $\mathbf{R}$ coordinate. The selfintersection form takes values in $\mathbf{Z}\pi/((1+\bar{\omega})\mathbf{Z}\pi)$. Recall that in an $2n$-manifold λ satisfies $\lambda(x,y) = (-1)^n \bar{\omega}\lambda(y,x)$, and $\mu(x)$ is defined modulo $((1+(-1)^{n+1}\bar{\omega})\mathbf{Z}\pi)$. Before now we have been working in dimension 4. Here the dimension is 6, so the signs have changed. Except for the coefficient on the identity element in $\mathbf{Z}\pi$ these satisfy $\lambda(x,x) = \mu(x) - \bar{\omega}\mu(x)$ (see 1.7), so the reduced selfintersection takes values in the kernel of this symmetrization homomorphism. This kernel can be naturally identified with $(\mathbf{Z}/2)[T_+]$. Assembling these gives $\tilde{\mu}(d_g \times \mathbf{R})\colon H_3(X, \partial_0 X) \to (\mathbf{Z}/2)[T_+]$, which will be the obstruction to embedding.

The first objective in the proof is to arrange $(X, \partial_0 X)$ to have a handlebody structure with only 3-handles. This is done in several steps, each of which involves "stabilizing" the problem. To stabilize, suppose g is an embedding on a collar of $\partial_0 X$, and consider the collar as a handlebody with a cancelling pair of 2- and 3-handles. Delete the 2-handle from both X and Y to obtain $g'\colon X' \to Y'$. This map also satisfies the connectivity and homological hypotheses of the theorem. If g' can be replaced by an embedding agreeing with g' on $\partial_0 X'$, then an embedding of X is obtained by replacing the 2-handle. It is therefore sufficient to consider the stabilized problem.

X is changed by the addition of the 3-handle. Specifically X' is the boundary connected sum $X \#_{\partial_0} S^2 \times D^3$. This new 3-handle is embedded in both X' and Y', so the obstruction homomorphism vanishes on it. This shows the obstruction for g' is the image of the obstruction for g. $\partial_0 X$ is changed by connected sum with $S^2 \times S^2$, and $\partial_1 X$ is unchanged. The homotopy type of X' is the 1-point union $X \vee S^2$.

Choose a handlebody structure on $(X, \partial_0 X)$ (one exists since X is a

5-manifold; see 9.1). According to the "geometrical connectivity" theorem (Wall [2]) a 1-connected 5-manifold pair has a handlebody structure without 0- and 1-handles. More precisely these handles can be eliminated from a given structure at the expense of introduction of 2- and 3-handles. Applying this to both $(X, \partial_0 X)$ and $(X, \partial_1 X)$ gives a structure with only 2- and 3-handles. Basically we would like to make g an embedding on the 2-handles, delete them as in stabilization, and concentrate on the 3-handles. It is a bit difficult to get the 3-handles disjoint from the 2-handles, and to relate the properties of the resulting g' to those of the original g, so we use a less direct approach. We show that after a suitable number of stabilizations the resulting X' has a handlebody structure with only 3-handles.

$(X, \partial_1 X)$ has a handle structure dual to the one on $(X, \partial_0 X)$; let Z denote $\partial_1 X$ union with the 2-handles in this dual structure. Obstructions to factoring the identity of X (up to homotopy, rel $\partial_1 X$) through the inclusion $i\colon Z \to X$ lie in $H^i(X, \partial_1 X; \pi_i(X, Z))$. These groups vanish, since Z is a 2-skeleton if $i \leq 2$, and by duality using the 2-connectivity of $(X, \partial_0 X)$, if $i > 2$. Therefore there is $j\colon X \to Z$, with ij homotopic to the identity.

This factorization gives a decomposition $H_i(Z) = H_i(X) \oplus H_{i+1}(X, Z)$ ($\mathbf{Z}\pi_1 X$ coefficients). But $H_{i+1}(X, Z)$ is concentrated in dimension 3, and is free (generated by the 3-handles). $H_2(Z) = \pi_2(Z)$, so choose maps $S^2 \to Z$ representing the generators of $H_3(X, Z)$. Together with j these define $X \vee kS^2 \to Z$ which induces a simple equivalence of chain complexes with $\mathbf{Z}\pi_1 X$ coefficients. Since both complexes have fundamental group $\pi_1 X$, the map is a simple homotopy equivalence.

The conclusion is that the manifold X' obtained by k-fold stabilization is simple homotopy equivalent rel $\partial_1 X$ to a handlebody Z (on $\partial_1 X$) with only 2-handles. Approximate the equivalence rel $\partial_1 X$ by an embedding $Z \to X$. This is be done using the 5-dimensional analog of the immersion lemma, and might involve changing the attaching maps of the 2-handles by rotations. (Although it is easy to see that for the particular Z constructed above no rotations are necessary.) Let W denote the closure of the complement of Z in X.

W is an s-cobordism from $\partial_0 X$ to the rest of its boundary. The fact that the inclusion is a simple equivalence implies by excision and duality that W is a simple $\mathbf{Z}\pi_1 X$ coefficient homology H-cobordism, so it is sufficient to see the boundary pieces and W all have the same fundamental group as X. $\partial_0 X$ does by the 2-connected hypothesis. W does since it is obtained by deleting 2-handles from X, and X is 5-dimensional. Similarly $\partial_1 W$ is obtained by deleting 2-handles from Z, so has the same fundamental group as Z.

Further stabilization of the problem can be arranged to change W by connected sum along an arc with $S^2 \times S^2 \times I$, as in the exercise preceeding 7.3. To see this choose the cancelling 2- and 3-handle pair to lie in a neighborhood of a collar arc going from $\partial_0 X$ to $\partial_1 X$ (missing the handles). Delete a neighborhood of the 2-handle to get X', and add the dual of the 3-handle to Z. Then the dual also gets deleted to form W', leaving the desired arc sum.

Finally apply the stable 5-dimensional s-cobordism theorem (in the exercise preceeding 7.3) to conclude that after some number of stabilizations the associated manifold W is a product. This means X is homeomorphic to Z, so $(X, \partial_1 X)$ has only 2-handles. Dually $(X, \partial_0 X)$ has only 3-handles, as desired.

Now we construct "transverse spheres" for the images of the 3-handles in Y. If the core disks of the handles are denoted D_i, then the objective is a collection of embeddings $\alpha_i \colon S^2 \to Y$ so that $g(D_i) \cap \alpha_j$ is empty unless $i = j$, and then is a single (transverse) point. These embeddings are not required to be framed.

As in the proof of 10.5(1) in 10.8 the vanishing of the cohomology group $H^4_f(Y_g, X \cup \partial Y; \mathbf{Z}\pi_1 Y)$ implies there are elements $\alpha_i \in \pi_2(Y)$ with intersection numbers $\lambda(g(D_i), \alpha_j) = 0$ if $i \neq j$, and $= 1$ if $i = j$. Since $\dim(Y) = 5$ these can be approximated by embeddings. The extra intersections between these and the $g(D_i)$ can be arranged in pairs with Whitney disks. Again since $\dim(Y) = 5$ the Whitney disks can be chosen to be embedded, although their interiors may intersect the $g(D_i)$. Push the $g(D_i)$ across these Whitney disks. This may introduce new intersections among the $g(D_i)$, but reduces intersections with the α_j to single points.

The rest of the proof involves studying the intersections among the images of the 3-disks $g(D_i)$. Approximate them to be in general position and transverse to each other, then the intersections are circles and arcs of double points. These are disjoint from the boundaries of the disks since g is an embedding there. g is an immersion except at isolated "cusps" ocurring at the ends of arcs of double points. The preimage in $\cup D_i$ is a union of circles. Over a circle of intersections the preimage is a 2-fold covering space, so either two circles each going by homeomorphism, or a single circle going around twice. There is a single circle over an intersection arc, a 2-fold "branched cover" branched over the endpoints.

We describe the obstruction in terms of these intersections. For this it is sufficient to determine the value of $\tilde{\mu}(d_g \times \mathbf{R})$ on generators; the classes represented by the handle cores D_j. The images in $H_3(X \cup Y)$ are represented by $D_j \cup g(D_j)$. Let $f_j \colon S^3 \to \mathbf{R}$, then the graphs of these in $(X \cup Y) \times \mathbf{R}$ represent $(d_g \times \mathbf{R})([D_j])$.

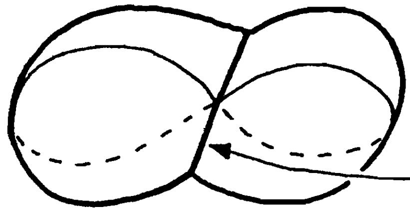
Intersections among these occur at points identified both by g and f_j. Consider a circle of intersections of g which is covered by two circles in D_*. The function f can be arranged to have values on one component strictly greater than on the other, so this circle produces no intersections in $(X \cup Y) \times \mathbf{R}$. The intermediate value theorem implies that arcs, and circles covered by a single circle, must have points for which the two points in the preimage are taken to the same value by f. But f can be arranged so there is exactly one such point on each such arc or circle, so these produce exactly one selfintersection point each. $\tilde{\mu} \in (\mathbf{Z}/2)[T_+])$ therefore counts these mod 2; if $\gamma \in T_+$ then the coefficient on γ in $\tilde{\mu}(D_j \cup g(D_j))$ is the number mod 2 of intersection circles covered by a single circle, and with associated π_1 element γ. The arcs do not show up since they have associated element $1 \in \pi$, and 1 is omitted from $\tilde{\mu}$ and T_+.

A selfintersection circle covered by a single circle is itself a representative for its associated element in $\pi_1 Y$. Since the double cover is nullhomotopic, it follows that it is an element of exponent 2. The fact that D is self-transverse at this circle divides the normal bundle of the circle locally into two 2-dimensional subbundles. The double cover splits globally as a sum of orientable subbundles. This identifies the normal bundle of the circle in Y as being obtained from the bundle $\mathbf{R}^2 \times \mathbf{R}^2 \times I$ over I by identifying the ends via the matrix $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ This is orientable, so as a π_1 element it lies in the kernel of ω_1. Therefore the elements which can occur this way are $T_+ \cup \{1\}$. We conclude that the obstruction $\tilde{\mu}(d_g \times \mathbf{R})$ vanishes if and only if: for each j and $\gamma \neq 1$ in $\pi_1 Y$ there are an even number of selfintersection circles of $g(D_j)$ covered by a single circle and with associated group element γ.

We begin the modification of intersections by first eliminating arcs. The map has no arcs if and only it is an immersion, so the fastest way to do this is to use the immersion theorem and the fact that $\pi_3(B_{TOP}) = 0$. A geometric argument will be given because it gives additional information about circles double covered by a circle.

At each end of an arc of intersections is a "cusp" as pictured in 1.6 (but multiplied by $(\mathbf{R}^2, \mathbf{R})$ to make it 5-dimensional; see Whitney [**2**]). The basic idea is that the cusps on the ends of an arc can be joined together to give an intersection circle. After this is done the map is an immersion.

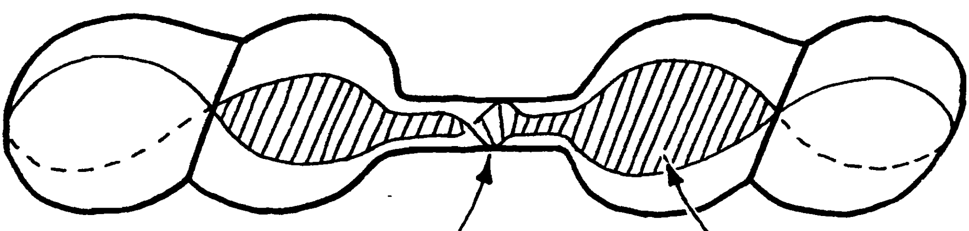
In detail, a small ball about the singular point intersects the image $g(D_j)$ in the cone on a standard immersion of S^2 in S^4, with a single selfintersection. This immersion is obtained from a 2-sphere in 3-space with a line of intersections by pushing one sheet into the future on one side of the midpoint of this line, and into the past on the other side.



There are two of these, depending on which side of the midpoint is pushed into the future, and these are distinguished by the sign of the intersection point. In a 5-manifold these signs depend—as do intersection numbers—on choices of paths from the handle images to the basepoint, and an orientation at the basepoint. Transport the orientation along the path to get an orientation at the singular point, and use the induced orientation on the boundary of a ball about that point. Cusps on opposite ends of an arc of intersections have opposite signs.

Cusps of opposite sign can be "cancelled." Choose an arc between them disjoint from the double curves. A neighborhood of this arc can be thought of as a boundary connected sum of ball neighborhoods of the endpoints. The intersection of the handle image with the boundary of this neighborhood is the connected sum of two standard immersed spheres. If the intersection points have opposite sign then this is also the boundary of the product with I of a standard immersion of D^2 in D^4 with a single selfintersection. Replacing the original intersection with this product removes the cusps and joins the ends of the original double point arcs. As observed above joining the ends of all double arcs in this way gives an immersion with only double circles.

When ends of a double arc are joined in this way there is a choice which determines whether the resulting circle is covered by a single circle or by two circles. As above consider the intersection of the handle image with the boundary of the arc neighborhood as a sum of two immersed 2-spheres. The resulting sphere has two intersections of opposite sign, and there are two Whitney disks, differing by a twist, which can be used to remove the intersections leaving a standard (unknotted) 2-sphere.

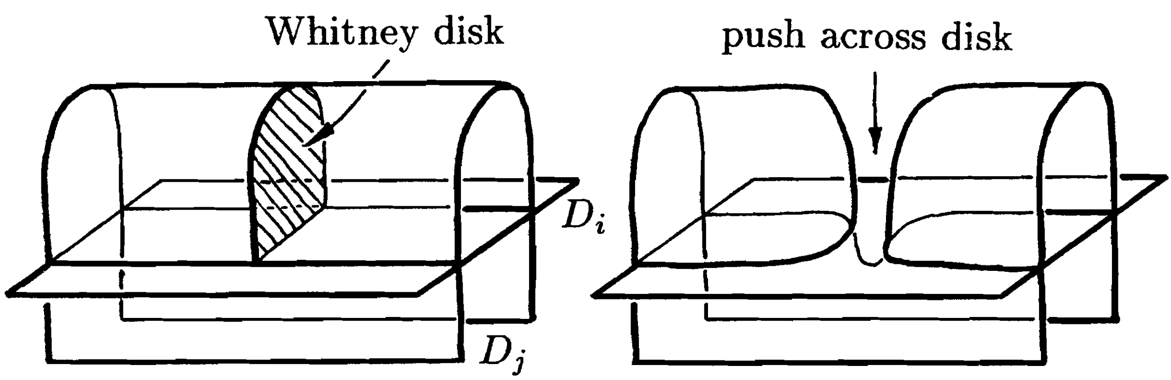


Put the trace of the Whitney move in a collar of the boundary S^4, and then fill in with a standard embedded $D^3 \subset D^5$. The track of the intersection points in the Whitney move is the new double arc. By following sheets near this arc the structure of the double cover can be deduced, and in particular single twists in the sum tube switch the double cover between a single circle and two circles.

The intersection points in circles generated from arcs have trivial associated element in the fundamental group. This is because near the singularity there are loops going from one sheet to the other inside a ball. The discussion above implies that we can introduce an arbitrary number of circles doubly covered by a single circle, and with trivial associated element in π_1. To do this introduce folds with arcs of double points, then join the ends in such a way to get a connected double cover.

We now describe a version of the Whitney isotopy, which will be used to eliminate circles with connected double covers.

Suppose x and y are points of intersection between $g(D_i)$ and $g(D_j)$, whose associated group elements are equal. Join the preimages by embedded arcs in both D_i and D_j, then the resulting loop in Y is contractible. Approximate a nullhomotopy by an embedded disk. The interior of this disk may have point intersections with the $g(D_*)$, which can be removed using connected sums with the transverse spheres α_*. A neighborhood of this disk can be parameterized by the model shown below, modified to be 5-dimensional; multiply by $I \times I$, and multiply D_i, D_j by the first and second coordinates respectively.



the model after the move

The reason this is always possible, and there is not an "opposite sign" condition as with isolated intersection points is that there is an additional degree of freedom: the orientations of the intersection curves can be changed. The picture on the right shows D_i, D_j after a Whitney isotopy across the disk. The two intersections circles are modified by a connected sum operation at the points x and y.

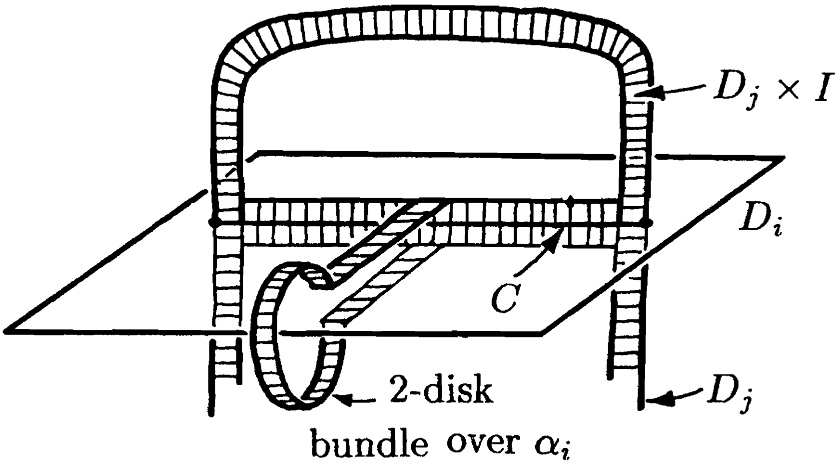
For the first application of this Whitney move we observe that if x

and y lie on different circles, both covered by a single circle, then the operation combines the circles to give a single circle covered by two circles. This implies that if there are an even number of circles with connected double cover and some fixed associated $\gamma \in \pi_1$ then they can be joined in pairs and all converted to trivially covered circles.

We conclude from this that if the obstruction $\tilde{\mu}(d_g)$ vanishes g can be modified to have no circles with connected double cover. First, vanishing means that for any nontrivial element in $\pi_1 Y$ there are an even number of circles in each D_j with connected cover and that particular element. Thus all these can be combined, leaving only circles with trivial associated π_1 element (or disconnected cover). If the number of these in some D_j is not even then according to the analysis of arcs we can introduce a single new one, and then eliminate these also.

Now we describe a type of "surgery" on $g(D_*)$, which will be the main technique for removing intersection circles. Suppose a circle of intersections has preimage two circles $r \subset D_i$ and $r' \subset D_j$. Begin with the simplest case where r is unknotted and unlinked, so suppose $C \subset D_i$ is an embedded 2-disk with boundary r and interior disjoint from the double points of g. This data will be used to construct a 4-dimensional handlebody with D_j at one end, and a map into Y extending g.

Start with $D_j \times I$, and add a 2-handle on $r' \times \{1\} \subset D_j \times \{1\}$. Map this to the normal disk bundle of the image of D_i, restricted to the 2-disk C (note that requiring this map to be defined singles out a specific framing of $r' \times \{1\}$ on which to attach the handle). To construct a second 2-handle begin with the core of the dual of the first handle (a fiber of the disk bundle over C). Perturb the interior to be disjoint from the construction to date. There is a single intersection point of this disk with D_i, remove this by connected sum with a copy of the transverse 2-sphere α_i. This gives an embedded 2-disk in Y. To get a 2-handle we need a 2-disk bundle over this extending the normal bundle of the boundary circle in the boundary of the handlebody. The normal bundle in Y is 3-dimensional, and the complement of the 2-dimensional bundle over the boundary is trivial, so according to the stability of bundles it is possible to extend the splitting over the 2-disk. This splitting is not unique, and we discuss the effect of different choices below.



The handlebody gives a bordism from g on D_j to a map on some 3-manifold, which has the same selfintersections and intersections with the other D_* as D_j, except that the image of $r \cup r'$ has been eliminated. The bordism shows that this manifold and D_j represent the same relative homology class, so we can replace D_j with it (thereby simplifying the situation) provided the new manifold is a 3-disk. We describe two situations in which this is the case.

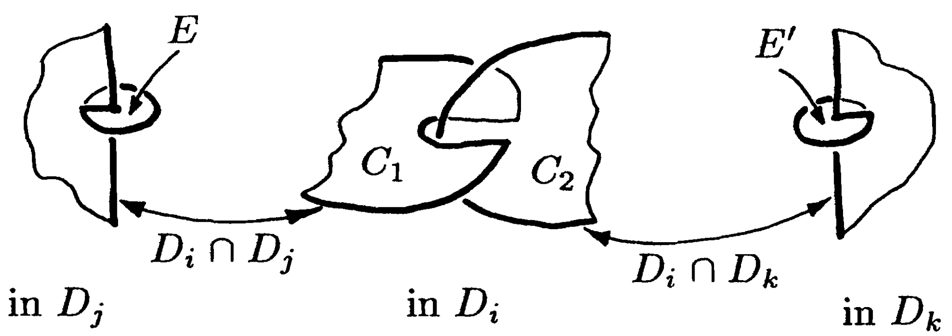
First, if the sphere α_i can be framed then the manifold can be arranged to be a disk. If α_i can be framed (which is equivalent to $\omega_2(\alpha_i) = 0$) then there is a trivial 2-disk subbundle of its normal bundle. Connect sum with the trivial bundle does not change framings on the boundary, so in this case the second 2-handle is attached on the same framing of the boundary S^1 as the framing coming from the dual disk of the first 2-handle. The effect on the boundary is the same as deleting a neighborhood of the dual disk. However this operation undoes the first handle attachment, and gives D^3.

For the other case suppose that the circle $r' \subset D_j$ bounds an embedded disk (not necessarily disjoint from the other double points). This disk defines a trivialization of the normal bundle, which differs by rotations (ie. by an element of $\pi_1(\mathrm{O}(2)) \simeq \mathbf{Z}$) from the framing used to attach the first 2-handle. If these framings agree then the new manifold is a 3-disk. This is because the disk and the core of the first 2-handle fit together to give an embedding of $S^2 \times D^2$ in the handlebody. Replacing this with $D^3 \times S^1$ gives a new handlebody with a 1-handle and a 2-handle (the second 2-handle in the original). The 2-handle is attached on a circle which intersects the dual sphere of the 1-handle in exactly one point, so the two handles can be cancelled. Again the modified handlebody is a collar on D_j and the complementary piece of the boundary is a 3-disk.

The next step is to obtain embedded disks bounding preimage circles, so the surgery procedure above can be applied. To begin with there are immersed disks bounding the circles, whose intersections are all "clasps." To get these choose nullhomotopies in general position, so immersed except at isolated cusps. Push cusps across the boundary (this introduces twists about the boundary circle which change the framing). If there are circles of intersection, push one disk across the edge of the other to convert them to arcs. Then triple points can be pushed along a double arc across the boundary, and eliminated. Finally if there is an arc lying entirely in the interior of a disk, a piece of its interior can be pushed across the boundary to convert it into two clasps. None of these modifications involve changing the boundary circles.

We now associate a "rotation number" to preimage circles. If $r \subset D_i$ is the preimage of $g(D_j)$ then the normal bundle of r in D_i is (by transversality) the restriction of the normal bundle of $g(D_j)$ in Y. The contractibility of D_j defines a trivialization of this bundle. But r is also the boundary of an immersed disk in D_i, which also gives a trivialization of the bundle. These trivializations differ by a rotation in $\pi_1(\mathrm{O}(2)) \simeq \mathbf{Z}$, and this is defined to be the rotation number of r.

Now clasps will be removed. Suppose C_1 and C_2 are disks in D_i with boundaries preimages of D_j and D_k respectively. Suppose these intersect in a clasp arc. A finger move of D_j along this arc gives an immersion in which the clasp is removed, but introduces a new circle of intersections between D_j and D_k. There are standard embedded 2-disks, say E and E', bounding the preimage circles, and there are clasps between these and disks in D_j and D_k.



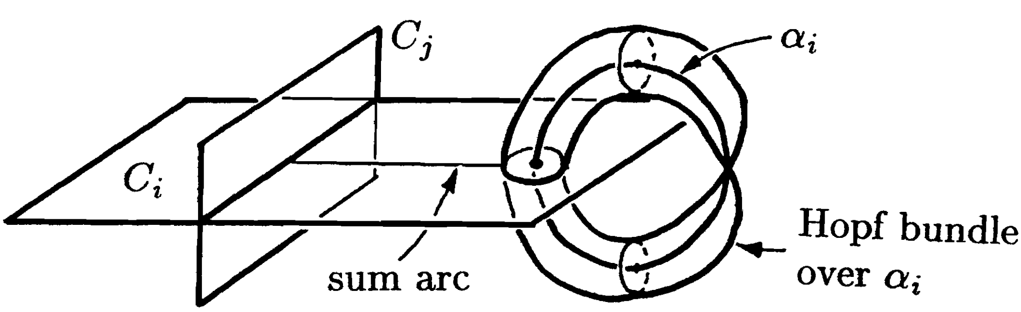
These new intersection circles have rotation number 0 since they arise from finger moves. We repeat the move, pushing D_i along the clasp in E. This introduces new D_i, D_k intersections with disks F, F' which clasp E' and C_1 respectively. But E is now embedded disjointly from other double points, and has rotation number 0, so the surgery procedure can be used to change the immersion to eliminate it. The disk E' is also removed, so after this the disk F is disjointly embedded. It also has rotation number 0, so can be removed by surgery. After doing this we have an immersion with all the same data as the original except that the clasp in $C_1 \cap C_2$ has been removed.

Applying this operation at each clasp yields an immersion whose preimage circles bound disjointly embedded disks.

At this point we can use the surgery procedure to eliminate a great many circles: any (trivially covered) circle involving a D_j with $\omega_2(\alpha_j) = 0$, or with trivial rotation number. In fact we can complete the proof of the theorem if ω_2 does not vanish on the subspace of $\pi_2 Y$ perpendicular to the image of $H_3(X, \partial_0 X; \mathbf{Z}\pi_1 Y)$ in a sense analogous to the condition in 10.5. This condition is equivalent to the existence of a 2-sphere disjoint from the image of g, with nontrivial ω_2. Using connected sums with this sphere we can arrange that all α_j can be framed. This means all trivially covered circles can be eliminated. Since vanishing of the $\tilde{\mu}(d_g)$ obstruction implies the nontrivially covered circles can be eliminated, we can get an embedding.

To complete the proof in the remaining cases we show how to change the rotation numbers of circles in D_i when $\omega_2(\alpha_i) \neq 0$. If g can be modified so all the rotation numbers are zero, then circles can be eliminated using the surgery procedure independently of $\omega_2(\alpha_*)$.

Suppose $\omega_2(\alpha_i) \neq 0$. Then the normal bundle of α_i in Y is the sum of the 2-dimensional bundle with Euler number 1 (see 1.7) and a trivial line bundle. The sphere bundle of this bundle is the Hopf fibration $S^3 \to S^2$. Let $E\colon S^3 \to Y$ denote the embedding of this sphere bundle, then $E \cap g(D_i)$ is a circle; a fiber of the Hopf fibration.



The preimages of this circle bound embedded disks in both S^3 and D_i. The rotation number of the circle in S^3 is 1: the preimage of the normal bundle of $g(D_i)$ is the same as the preimage under the Hopf map of the normal bundle of a point in S^2. That this bundle has rotation number 1 can be seen from the fact that distinct fibers of the Hopf map have linking number 1: linking numbers are defined in terms of intersections with bounding disks, so in the case of images of standard sections of a framing linking is the same as the rotation number.

Now suppose $x \in g(D_i) \cap g(D_j)$. Choose an arc in D_i from the preimage of x to a point in $E \cap g(D_i)$. Denote by g' the map of $D_j = D_j \# S^3$ obtained by connected sum of $g(D_j)$ with E. This map induces the same homomorphism on homology as g because E bounds the disk bundle, so is trivial in $H_3(Y)$. All intersection structure is the same except that the circle containing x is replaced by its connected sum with $E \cap g(D_i)$. Rotation numbers are additive under connected sum, so the rotation number of the new circle is greater by 1 than that of the old. Alternatively, reversing the orientation of E gives a sphere with intersection circle with rotation number -1, and connected sum with this reduces rotation numbers by 1.

All rotation numbers can be changed to be 0 by adding such spheres repeatedly. This gives intersection circles which can be eliminated by surgery, so completes the proof of 10.9. ∎

10.10 References

The classification theorem 10.1 comes from Freedman [**2**]. The characterization of connected sums in 10.3, the definition of $*W$ in 10.4, and the π_1-negligible embedding theorem of 10.5 all appear here for the first time.

The characterization theorem has been extended in several ways. Simply connected 4-manifolds with boundary have been described by Vogel [**1**], and Boyer [**1**]. The surgery approach to classification has been pushed through for certain classes of finite fundamental groups by Hambleton and Kreck [**1**]. The results are quite a bit more complicated than the trivial and infinite cyclic cases treated here.