Finiteness conditions for CW complexes. II†

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A CW complex is a topological space which is built up in an inductive way by a process of attaching cells. Spaces homotopy equivalent to CW complexes play a fundamental role in topology. In the previous paper with the same title we gave criteria (in terms of more-or-less standard invariants of the space) for a CW complex to be homotopy equivalent to one of finite dimension, or to one with a finite number of cells in each dimension, or to a finite complex. This paper contains some simplification of these results. In addition, algebraic machinery is developed which provides a rough classification of CW complexes homotopy equivalent to a given one (the existence clause of the classification is the interesting one). The results would take a particularly simple form if a certain (rather implausible) conjecture could be established.

It has been pointed out by several people that many of the arguments of my previous paper (I)† can be presented more clearly by using a more algebraic setting. In this paper, I shall give such a reformulation, and then go on to some minor improvements of the original results. These improvements were all motivated by a study of the Poincaré duality theorem, and of so-called 'Poincaré complexes' which provide a homotopy framework for the theorem. This study will appear in a subsequent paper.

We shall begin by discussing how to associate a chain complex to a topological space, noting in particular that for a CW complex, singular and cellular chains give equivalent results. For this we need a brief summary of useful facts about chain complexes, and we introduce the 'derived category'. A version of the theorems of I will then be given in our new framework; for this we use a mild improvement of our old theorem G, which thus now assumes a central role. The improvement also gives this result a much wider applicability. Then we give a simplified form of the finite-dimensionality criterion $D(n)$. We conclude by discussing the relativization of our finiteness obstruction.

Let X be a path-connected topological space, $*$ a base point in it, $\pi = \pi_1(X, *)$ and $\Lambda = \mathbf{Z}[\pi]$ the integral group ring of π. We suppose that X has a universal covering space $\tilde{X}$ (in fact we are only interested in the case when X has the homotopy type of a CW complex, when this is certainly the case). Then $\tilde{X}$ is simply connected, π acts freely on $\tilde{X}$ with orbit space X, and the projection $\tilde{X} \to X$ is a covering map.

Consider the singular complex of $\tilde{X}$: this is the chain complex, in which the group of n-chains is the free abelian group on the set of singular n-simplices (i.e. maps from the standard n-simplex to $\tilde{X}$) for each $n \geqslant 0$, with the boundary defined by the

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usual formula. Since π acts on $\tilde{X}$, it acts also on the set of singular n-simplices, and this action is free since the first is. Thus the group of n-chains has a natural Λ-module structure, and is a free module. We shall write $C_*(X)$ for the chain complex, considered as a complex of Λ-modules. We often refer to the chains determined by the simplices as basis elements.

We will be studying positive chain complexes of projective Λ-modules. We form these into a category—the so-called derived category—by defining morphisms to be chain homotopy classes of chain maps. Since our complexes are positive and projective, chain maps are chain-homotopic if and only if they induce the same homology map. The derived category is used extensively in the study by Grothendieck & Verdier (unpublished) of duality theorems for Grothendieck cohomology of schemes. We shall not use it extensively, but will need the notion of equivalence of chain complexes in the derived category.

Any continuous map $f: X \to Y$ of path-connected spaces, which induces an isomorphism of fundamental groups, induces in the obvious way a chain map of $C_*(X)$ to $C_*(Y)$. A homotopy of maps f gives rise to a chain homotopy of the induced chain maps. Thus a homotopy class of maps $X \to Y$ induces a morphism of $C_*(X)$ to $C_*(Y)$: C_* is a functor on the category of spaces X with a universal covering and isomorphism of the fundamental group to π, and homotopy classes of maps inducing the identity on fundamental groups. In particular, if X and Y are homotopy equivalent, $C_*(X)$ and $C_*(Y)$ are equivalent.

Now let X be a CW complex, with skeletons X^p. Then we filter $C_* X$, defining $F^pC_*(X)$ as the subgroup generated by singular simplices of $\tilde{X}$ which project to $X^p \subset X$. It is well known that the homology modules of $F^pC_*(X)/F^{p-1}C_*(X)$ vanish except in dimension p, where we have a free module $C^c_p(X)$ with basis determined (up to operation by elements of π: also (except in dimension 0) up to sign) by the p-cells of X. We define $d: C^c_p(X) \to C^c_{p-1}(X)$ as the boundary operator in the exact homology sequence of the triple $(F^pC_*(X), F^{p-1}C_*(X), F^{p-2}C_*(X))$. We have thus defined the cellular chain complex of X, $C^c_*(X)$, which is another object of the derived category.

The following is well known, but a careful treatment seems not out of place here.

Lemma 1. *Let X be a connected CW complex. Then $C_*(X)$ is equivalent to $C^c_*(X)$.*

It is, of course, standard that both have isomorphic homology modules: here we must find a chain map inducing the isomorphism.

Proof. Let $D_p(X)$ be the kernel of

$$d: F^pC_p(X) \to F^pC_{p-1}(X)/F^{p-1}C_{p-1}(X).$$

Then $dD_p(X) \subset D_{p-1}(X)$, so we have a subcomplex $D_*(X)$ of $C_*(X)$, and we observe that $C^c_*(X)$ is a quotient complex of $D_*(X)$.

Since $F^pC_*(X)/F^{p-1}C_*(X)$ only has homology in dimension p, a simple argument using induction and exact sequences shows that (for $q < p$) $F^pC_*(X)/F^qC_*(X)$ has no homology in dimensions $\leqslant q$ or $> p$. Also these are positive complexes, and if B_*

is a positive complex with no homology below dimension p, we see by induction on $i \leqslant p$ that the sequence

$$0 \to B_i \cap d^{-1}0 \to B_i \to dB_i = B_{i-1} \cap d^{-1}0 \to 0$$

splits, so has all terms projective.

Thus if K_p is the module of p-cycles of $F^pC_*(X)/F^{p-1}C_*(X)$, K_p is projective; the exact sequence

$$0 \to F^{p-1}C_p(X) \to D_p \to K_p \to 0$$

now shows that D_p also is projective. Hence it remains to prove that the inclusion $D_* \subset C_*(X)$ and the projection $D_* \to C_*^c(X)$ both induce homology isomorphisms, or equivalently, that the cokernel and kernel are homologically trivial.

A cycle of $C_*(X)/D_*$ is represented by $c \in C_p(X)$ such that $dc \in D_{p-1} \subset F^{p-1}C_{p-1}(X)$. Since X is a CW complex, any singular simplex lies in a finite subcomplex, so $c \in F^qC_p(X)$ for some integer $q \geqslant p$. The class of c in $F^qC_*(X)/F^pC_*(X)$ is a p-cycle, hence a p-boundary, so for some $b \in F^qC_{p+1}(X)$ we have $c' = c - db \in F^pC_p(X)$. As $dc' = dc \in F^{p-1}C_{p-1}(X)$, we have $c' \in D_p$. So c bounds b modulo D_*, as required.

Now let E_* be the kernel of the projection $D_* \to C_*^c(X)$. Then

$$E_p = F^{p-1}C_p + dF^pC_{p+1}.$$

Thus a p-cycle of E_* has the form $z_1 + z_2$, where $z_2 \in dF^pC_{p+1} = dE_{p+1}$. But the pth homology of $F^{p-1}C_*$ vanishes, so z_1 bounds an element of $F^{p-1}C_{p+1} \subset E_{p+1}$ also. This completes the proof of the lemma.

We thus see that the equivalence class of $C_*(X)$ can be calculated from a cell structure on X. Our main interest, however, is to go in the opposite direction. We will use a modified form of theorem G of I. In order to state this, we need extra conditions in low dimensions. Let $\Lambda = \mathbf{Z}[\pi]$, and let A be a positive chain complex of free Λ-modules (with bases chosen). We call A *admissible* if there exists a connected CW complex K of dimension 2, an isomorphism of $\pi_1(K)$ on π, and a compatible isomorphism of $C_*^c(K)$ on the subcomplex

$$A_2 \to A_1 \to A_0 \to 0.$$

Admissibility is not a void condition; since $\tilde{K}$ is connected and simply connected it implies, for example, that $H_0(A_*) \simeq \mathbf{Z}$ and that $H_1(A_*) = 0$. Also we see, by considering the boundary of a 1-cell, that for each basis element α of A_1 we have $d\alpha = g_1P_1 - g_0P_0$, where P_0 and P_1 are basis elements of A_0, and g_0, $g_1 \in \pi$. This condition (unlike the first) fails to be invariant under equivalence. I do not know any examples of complexes satisfying these conditions which fail to be admissible, although I think it more likely than not that such exist. It would be of considerable interest to have this problem resolved, particularly if the following were true.

Conjecture. *Every complex satisfying the above conditions is admissible.* We shall discuss this further after theorem 4 and shall see, in particular, that it reduces to a problem about relations between generators of a finitely presentable group. At present the author has only established it for the trivial group: the proof uses the fact that the group ring is a principal ideal ring, so is of no use in considering the conjecture in general.

A chain map of admissible chain complexes is 0-*admissible* if it carries each basis element in dimension 0 to another basis element.

The following is our main construction technique.

THEOREM 2. *Let X be a connected CW complex. Write $\pi = \pi_1(X)$, $\Lambda = \mathbf{Z}[\pi]$, and $\tilde{X}$ for the universal covering of X. Suppose given an admissible positive chain complex A_* of free Λ-modules, and a 0-admissible chain map $f\colon A_* \to C^c_*(X)$ defining an equivalence. Then we can construct a CW complex Y, and cellular homotopy equivalence $g\colon Y \to X$, such that $C^c_*(Y) = A_*$, and f is the chain map induced by g.*

Note that the assumed conditions of admissibility are evidently necessary for the conclusion to hold.

Proof. We construct Y and h by induction on skeletons. Since A is admissible, there is a 2-complex K with $\pi = \pi_1(K)$ and with $C^c_*(K)$ equal to the part of A in dimensions $\leqslant 2$: we choose $Y^2 = K$ to start the induction. (A special argument is needed at this point to get the fundamental group right: the fundamental group of Y^1 is necessarily distinct from π, unless π is free.) Since f is 0-admissible, it maps each 0-cell of $\tilde{Y}$ (= basis element of A_0) to a 0-cell of $\tilde{X}$ (= basis element of $C^c_0(X)$): we thus define $\tilde{g}$ by f on $\tilde{Y}^0$ to start the inductive construction of g.

The rest of the proof follows that of (I, theorem G): we repeat it here for the reader's convenience, and also to set out the extra points arising in defining g on Y^2.

Suppose inductively Y^r and $g_r\colon Y^r \to X^{r+1}$ constructed, g_r cellular, r-connected, and lifted to $\tilde{g}_r\colon \tilde{Y}^r \to \tilde{X}^{r+1}$ (where $\tilde{Y}^r$ is the covering of Y^r induced by g_r from $\tilde{X}$); and that $C^c_*(Y^r)$ is the part of A in dimensions $\leqslant r$, and in these dimensions $\tilde{g}_r$ induces the chain map f. Our first step is to compute $H_{r+1}(\tilde{g}_r)$ which is, by definition, the homology group of the algebraic mapping cone, and hence the kernel of

$$C^c_{r+1}(X) \oplus A_r \xrightarrow[\begin{pmatrix} d & (-)^r f \\ 0 & d \end{pmatrix}]{} C^c_r(X) \oplus A_{r+1}.$$

But f is an equivalence, so its mapping cone gives an exact sequence

$$C^c_{r+2}(X) \oplus A_{r+1} \xrightarrow{\begin{pmatrix} d & (-)^{r+1} f \\ 0 & d \end{pmatrix}} C^c_{r+1}(X) \oplus A_r \xrightarrow{\begin{pmatrix} d & (-)^r f \\ 0 & d \end{pmatrix}} C^c_r(X) \oplus A_{r-1}.$$

Thus $((-)^{r+1}f)$ gives a map $A_{r+1} \to H_{r+1}(\tilde{g}_r)$.

For each basis element x of A_{r+1}, dx is an r-cycle of $A_r = C^c_*(Y^r)$, which we claim is spherical. For $r \leqslant 1$, this is clear, since Y^2 is provided by the admissibility hypothesis (and in fact if $r = 1$, every cycle is spherical). But if $r \geqslant 2$, the Hurewicz theorem gives an isomorphism $\pi_{r+1}(g_r) = \pi_{r+1}(\tilde{g}_r) = H_{r+1}(\tilde{g}_r)$ since g_r is then r-connected, and induces an isomorphism of fundamental groups. Then dx is in the image of $\pi_r(Y) \to H_r(\tilde{Y})$, so is spherical, as claimed. For each element x of our free basis of A_{r+1} we choose a map $S^r \to \tilde{Y}^r$ with image cycle dx. These maps are necessarily cellular, and we attach corresponding $(r+1)$-cells to Y^r to form Y^{r+1}, with covering $\tilde{Y}^{r+1}$. (For $r = 0$, 1 this is consistent with the definition of Y^2 already given.)

Now if x determines an element of $\pi_{r+1}(\tilde{g}_r)$, the component map $D^{r+1} \to \tilde{X}^{r+1}$ tells us how to map the attached cell, and thus extend g_r over Y^{r+1}. The extension g_{r+1} has chain map f (this follows from our choices). And g_{r+1} is $(r+1)$-connected: if $r = 0$, since $\pi_1(Y^1)$ maps onto $\pi_1(Y^2) = \pi = \pi_1(X^2)$; if $r \geqslant 1$, since

$\pi_1(Y^{r+1}) = \pi = \pi_1(X^{r+2})$ (here is the point where we make full use of admissibility of A_*) and $H_i(\tilde{g}_{r+1}) = 0$ for $i \leqslant r+1$, as g_{r+1} induces f. The induction is thus complete.

Now for $r \geqslant 2$, we have already checked that an element of A_{r+1} induces an element of $H_{r+1}(\tilde{g}_r) = \pi_{r+1}(\tilde{g}_r)$. For $r = 1$ we use the commutative and exact diagram

$$\begin{tikzcd} & \pi_2(\tilde{X}^2) \arrow[r] \arrow[d, "1|2"] & \pi_2(\tilde{g}_1) \arrow[r] \arrow[d] & \pi_1(\tilde{Y}^1) \arrow[r] \arrow[d] & \pi_1(\tilde{X}^2) = 0 \\ 0 = H_2(\tilde{Y}^1) \arrow[r] & H_2(\tilde{X}^2) \arrow[r] & H_2(\tilde{g}_1) \arrow[r] & H_1(\tilde{Y}^1) & \end{tikzcd}$$

(the coverings are those induced from $\tilde{X}$), and note that each basis element x of A_2 determines not only an element y_1 of $H_2(\tilde{g}_1)$ but also the attaching map of the corresponding cell of Y^2, an element y_2 of $\pi_1(\tilde{Y}^1)$, and both have the same image in $H_1(\tilde{Y}^1)$ since $C^c_*(Y^2)$ is essentially A_*. A short diagram chase now shows that there is a unique element $y \in \pi_2(\tilde{g}_1)$ inducing y_1 and y_2. The same argument (except for uniqueness of y) works for $r = 0$ using the diagram

$$\begin{tikzcd} \pi_1(\tilde{X}^1) \arrow[r] \arrow[d, "Epi"] & \pi_1(\tilde{g}_0) \arrow[r] \arrow[d] & \pi_0(\tilde{Y}^0) \arrow[r] \arrow[d] & \pi_0(\tilde{Y}^1) = 0 \\ H_1(\tilde{X}^1) \arrow[r] & H_1(\tilde{g}_0) \arrow[r] & H_0(\tilde{Y}^0) & \end{tikzcd}$$

and using the action of $\pi_1(\tilde{X}^1)$ on $\pi_1(\tilde{g}_0)$. (Note that the choices here are made in $\tilde{X}$, so do not affect the ultimate map of fundamental groups.)

Corollary 2·1. *Let X be a path connected space with fundamental group π and a universal covering $\tilde{X}$. Let A_* be an admissible positive chain complex of free Λ-modules, $f: A_* \to C_*(X)$ a 0-admissible chain map defining an equivalence. Then there exist a CW-complex Y with $A = C^c_*(Y)$, and a weak homotopy equivalence $h: Y \to X$ inducing the same morphism $A_* \to C_*(X)$ as does f.*

For let SX be the (unnormalized) singular complex of X, geometrically realized. Then SX is a CW complex, and $C^c_*(SX) = C_*(X)$; also the natural projection $p: SX \to X$ is a weak homotopy equivalence. Apply the theorem to obtain $g: Y \to SX$, and set $h = p \circ g$.

Corollary 2.2. *Let X have the homotopy type of a connected CW complex, and suppose $f: A_* \to C_*(X)$ a 0-admissible chain map defining an equivalence of the admissible Λ-free complex A_* with $C_*(X)$. Suppose that A_* satisfies any combination of the conditions*

(i) *A_i is finitely generated for $i \leqslant m_1$;*

(ii) *A_i is countably generated for $i \leqslant m_2$;*

(iii) *$A_i = 0$ for $i > m_3$.*

Then X is homotopy equivalent to a CW complex Y satisfying the corresponding conditions (i) *Y^{m_1} is finite,* (ii) *Y^{m_2} is countable,* (iii) *dim $Y \leqslant m_3$.*

This is now immediate. However, the algebraic conditions are more natural algebraically if the admissibility restrictions are omitted. We thus need to be able to replace a complex by an admissible one with as little alteration as possible.

Lemma 3A. *Let C_*, D_* (with differentials c, d) be free positive chain complexes, and $f: D_* \to C_*$ a chain homotopy equivalence. Then there is another equivalent free positive* *chain complex* D'_*, *with* $D'_i = D_i$, $d'_i = d_i$ *for* $i \geqslant 4$, $D'_3 = D_3 \oplus E_3$, $D'_2 = C_2 \oplus E_2$, $D'_1 = C_1, D'_0 = C_0$ *and* $d'_2 E_2 = 0$, $d'_2 | C_2 = c_2$, $d'_1 = c_1$.

Proof. The idea of the proof is to use J. H. C. Whitehead's 'folding' process, which goes roughly as follows. First add C_0 to D_1 and D_0, then manipulate till

$$f_0 \colon D_0 \oplus C_0 \to C_0$$

is the projection. Next add the old D_0 to D_1 and D_2 and manipulate again; finally cancel the old D_0 from D_0 and D_1. We will just give the complex obtained by this process.

Let $g \colon C_* \to D_*$ be homotopy inverse to f, and $s \colon fg \simeq 1$, $t \colon gf \simeq 1$ chain homotopies. Maps α and β of chain complexes are defined by:

$$\begin{tikzcd}
\ldots D_4 \arrow[r, "d_4"] \arrow[d, "1", "\alpha"'] & D_3 \arrow[r, "d_3"] \arrow[d, "1"] & D_2 \arrow[r, "d_2"] \arrow[d, "\binom{1}{0}"] & D_1 \arrow[r, "d_1"] \arrow[d, "\binom{1}{0}"] & D_0 \arrow[d, "f_0"] \\
\ldots D_4 \arrow[r, "d_4"] \arrow[d, "1", "\beta"'] & D_3 \arrow[r, "\binom{d_3}{0}"] \arrow[d, "1"] & D_2 \oplus D_0 \arrow[r, "\begin{pmatrix} d_2 & -t_0 \\ 0 & f_0 \end{pmatrix}"] \arrow[d, "{(1,0)}"] & D_1 \oplus C_0 \arrow[r, "{(f_0 d_1, c_1 s_0)}"] \arrow[d, "{(1+t_0 d_1, t_0 g_0)}"] & C_0 \arrow[d, "g_0"] \\
\ldots D_4 \arrow[r, "d_4"] & D_3 \arrow[r, "d_3"] & D_2 \arrow[r, "d_2"] & D_1 \arrow[r, "d_1"] & D_0
\end{tikzcd}$$

We have $\sigma \colon \beta\alpha \simeq 1$ and $\tau \colon \alpha\beta \simeq 1$, where the only non-zero components of σ and τ are $\sigma_0 = t_0$, $\tau_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\tau_1 = \begin{pmatrix} 0 & 0 \\ -d_1, & -g_0 \end{pmatrix}$. Also, $f\beta$ is a homotopy equivalence of the constructed complex with C_*; this induces $f_0 g_0$ in dimension 0, but if we subject it to the chain homotopy whose only non-zero component is s_0 in dimension 0 this changes the chain map in dimension 0 to the identity.

The resulting chain homotopy equivalence, since it induces the identity map (on C_0) in dimension 0, remains a homotopy equivalence if the terms in dimension 0 are deleted. We can thus iterate the argument. and then put C_0 back again. We need only one iteration, giving a chain homotopy equivalence:

$$\begin{tikzcd}
\ldots D_4 \arrow[r, "\binom{d_4}{0}", "(0)"'] \arrow[d, "f_4"] & D_3 \oplus D_1 \oplus C_0 \arrow[r, "\delta_3"] \arrow[d, "{(f_3,0,0)}", "\phi_3="'] & D_2 \oplus D_0 \oplus C_1 \arrow[r, "\delta_2"] \arrow[d, "\phi_2"] & C_1 \arrow[r, "c_1"] \arrow[d, "1"] & C_0 \arrow[d, "1"] \\
\ldots C_4 \arrow[r, "c_4"] & C_3 \arrow[r, "c_3"] & C_2 \arrow[r, "c_2"] & C_1 \arrow[r, "c_1"] & C_0.
\end{tikzcd}$$

We also perform the first part of the next step: add C_2 in dimensions 2 and 3. Setting $E_3 = D_1 \oplus C_2$, $E_2 = D_2 \oplus D_0 \oplus C_1$, all is as required except for d'_2.

Since the above gives a homology isomorphism in dimension 2, the map (ϕ_2, c_3) is onto; since C_2 is free, it has a right inverse $\begin{pmatrix} u \\ v \end{pmatrix}$. We replace the above complex by adding a trivial complex $C_2 \xrightarrow{1} C_2$, so making the boundary in dimension 3 the direct sum of δ_3 and $1 \colon D_3 \oplus E_3 \xrightarrow{\delta'_3} C_3 \oplus E_3$, and the two corresponding chain maps (ϕ_3, v) and $(c_3 v, \phi_2)$. Then subject $C_2 \oplus E_2$ to the automorphisms $\begin{pmatrix} 1 & -\phi_2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$.

Since $\phi_2 u + c_3 v = 1$, we have

$$(c_3 v, \phi_2)\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\begin{pmatrix}1 & -\phi_2\\ 0 & 1\end{pmatrix} = (c_3 v + \phi_2 u, -c_3 v\phi_2 + \phi_2 - \phi_2 u \phi_2)$$
$$= (1,0)$$

so the chain map in dimension 2 has become $C_2 \oplus E_2 \xrightarrow{(1,0)} C_2$. Commutativity shows that the boundary map is $(c_2, 0)$, which completes the proof of the lemma.

We also need a variant of this result.

Lemma 3B. *Let A_* be a positive free chain complex, B_* another, of dimension 2, and $g\colon B_* \to A_*$ induce isomorphisms of H_0 and H_1. Then A_* is equivalent to a complex of the form*

$$\dots A_4 \xrightarrow{\binom{a_4}{0}} A_3 \oplus E_3 \to B_2 \oplus E_2 \xrightarrow{(b_2,0)} B_1 \xrightarrow{b_1} B_0 \to 0,$$

where $E_3 = B_2 \oplus A_1 \oplus B_0$, $E_2 = A_2 \oplus B_1 \oplus A_0$.

Proof. As in lemma 3A, we find a complex equivalent to A_*, exactly as above except for the shape of the second differential, which has the form $(0, e_2)$. Since the 1-dimensional homology groups are isomorphic, $b_2(B_2) = e_2(E_2)$. But B_2 and E_2 are free, so there are homomorphisms

$$x\colon B_2 \to E_2 \quad y\colon E_2 \to B_2,$$

with $e_2 \circ x = b_2$ and $b_2 \circ y = e_2$. Now we have

$$(0, e_2)\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\begin{pmatrix}1 & -y\\ 0 & 1\end{pmatrix} = (b_2, 0),$$

so the differential can be adjusted by subjecting $B_2 \oplus E_2$ to the automorphisms indicated.

Theorem 4. *Let X be a connected CW complex, A_* a positive free chain complex equivalent to $C^c_*(X)$, K^2 another connected CW-complex with fundamental group π. Then there exist a CW complex Y and a homotopy equivalence $h\colon Y \to X$, such that Y is constructed from K^2 by adding 2-cells (at the base point) and 3-cells to obtain a complex Y_0 satisfying D2, and then further cells so that $C^c_*(Y, Y_0)$ is the part of A_* in dimensions $\geqslant 3$.*

If the symbol α_i denotes number of i-cells, or of generators in dimension i,

$$\alpha_2(Y_0 - K^2) = \alpha_2(A) + \alpha_1(K) + \alpha_0(A),$$
$$\alpha_3(Y_0 - K^2) = \alpha_2(K) + \alpha_1(A) + \alpha_0(K).$$

The condition $D\,2$ is defined on [I, p. 62] and will be further discussed below.

Proof. Write $B_* = C^c_*(K)$. We can define a map from K to X inducing the identity of π by mapping all vertices of K to the base point of X, the 1-cells of K to loops representing appropriate elements of π: easiest is first to collapse a maximal tree in K to a point, then each remaining 1-cell represents an element of π. The 2-cells of K are then attached by maps whose images in X are nullhomotopic, so we can extend the map over them.

Apply lemma 3B to construct a complex: the result is admissible since we can realize the part of dimension $\leqslant 2$ as the chain complex of K^2, with 2-cells attached trivially corresponding to generators of E_2. Also, by construction, the equivalence with $C^c_*(X)$ can be supposed 0-admissible. Now by theorem 2 we can construct Y and h: the assertions about the chain complex of Y are verified by looking at the complex of lemma 3B. Finally, Y_0 satisfies $D2$ since its chain complex is

$$0 \to E_3 \to B_2 \oplus E_2 \to B_1 \to B_0 \to 0,$$

which the proof of lemma 3B shows to be equivalent to

$$0 \to A_2 \to A_1 \to A_0 \to 0.$$

By using theorem 4 in place of theorem 2, we can improve corollary 2·2 as follows. All admissibility hypotheses can be deleted: in place of them use

$$m = 1: \pi \text{ is finitely generated,}$$

$$m_1 \geqslant 2: \pi \text{ is finitely presented,}$$

$$m_2 \geqslant 1: \pi \text{ is countable,}$$

$$m_3 \geqslant 3: \text{no condition on } \pi.$$

However, no theorem is obtained for $m_3 \leqslant 2$. Note that this result is very closely related to theorems A, C and E of I: it does not seem worth while here pursuing the analogy more deeply (we leave to the reader as exercises—if he so wishes—to give an algebraic analogue of Fn, and to show that a chain complex dominated by one of finite type is equivalent to one of finite type). We will, however, give an improvement of Dn below (theorem 5).

Theorem 4 gives the best method known to the author of making complexes admissible. It is thus clear that all hinges on the construction of the complex K with fundamental group π. We may suppose that K only has one 0-cell (otherwise, as above, collapse a maximal tree—thereby decreasing numbers of cells throughout). Then the 1-cells of K represent generators of π, and any set of generators can so appear. Similarly, the 2-cells give relations between the chosen set of generators, and any set of defining relations can be used.

Let e_0 be the 0-cell of K, e^i_1 the 1-cells, corresponding to $g_i \in \pi$, and e^j_2 the 2-cells. Then $de^i_1 = g_i e_0 - e_0$. Since $\tilde{K}$ is connected, $H_0(B_*) = H_0(\tilde{K}) \simeq \mathbf{Z}$. We claim that if the chain complex with free generators e_0, e^i_1 as above has 0th homology group infinite cyclic, then the g_i must generate π. For the augmentation $\epsilon: \Lambda \to \mathbf{Z}$ (induced by $\pi \to 1$) must induce the isomorphism since $\operatorname{im} d \subset (\ker \epsilon)\, e_0$. Thus for $g \in \pi$, $ge_0 - e_0 \in \operatorname{im} d$, hence for suitable $\lambda_i \in \Lambda$

$$ge_0 - e_0 = \sum_i \lambda_i g_i e_0 - \lambda_i e_0.$$

We express λ_i as a sum of elements of π with signs attached: $\lambda_i = \Sigma_j \pm g_{ij}$. The right-hand side becomes a sum of terms $g_{ij} g_i e_0 - g_{ij} e_0$ or $g_{ij} e_0 - g_{ij} g_i e_0$. All must cancel out in pairs except $ge_0 - e_0$, so we obtain a chain of terms $ge_0 - h_1 e_0, h_1 e_0 - h_2 e_0$, $\ldots, h_{n-1} e_0 - h_n e_0, h_n e_0 - e_0$ on the right-hand side. Then each of gh_1^{-1}, $h_r h_{r+1}^{-1}$ and h_n is one of the g_i or their inverses, so g is a product of them.

Now $H_1(\tilde{K}) = 0$, as $\tilde{K}$ is simply connected. Conversely, suppose given a complex

$$B_2 \to B_1 \to B_0 \to 0$$

with B_0 generated by e_0, B_1 by $\{e_1^i\}$, B_2 by $\{e_2^i\}$, $de_1^i = g_i e_0 - e_0$, and $H_1(B) = 0$, $H_0(B) \simeq \mathbf{Z}$. We know that the g_i generate π: what is not clear is whether the de_2^i arise from a set of defining relations. But this last is necessary and sufficient for our complex to be admissible.

We now consider Dn (I, p. 62): in the case $n = 1$, we no longer tolerate non-abelian coefficients. By corollary 2·2, and the improvement given after theorem 4, X satisfies Dn ($n \geqslant 3$) if and only if $C_*^c(X)$ is equivalent to an n-dimensional complex.

THEOREM 5. *Let X be a connected CW complex satisfying Dn. Then X satisfies $D(n-1)$ if and only if for every free Λ-module F, $H^n(X; F) = 0$.*

Proof. Certainly $D(n-1)$ implies that $H^n(X; F)$ vanishes. Conversely, let X satisfy $D(n)$ and all $H^n(X; F) = 0$. By the remark above, if $n \geqslant 3$ $C_*^c(X)$ is equivalent to an n-dimensional complex A_*. Applying the assumption with $F = A_n$ shows that

$$\hom_\Lambda(A_{n-1}, A_n) \to \hom_\Lambda(A_n, A_n) \to 0$$

is exact, so that $d_n\colon A_n \to A_{n-1}$ has a left inverse s. But now A_* is equivalent to the $(n-1)$-dimensional complex

$$0 \to \ker s \to A_{n-2} \to A_{n-3} \to \ldots,$$

and *a fortiori* X satisfies $D(n-1)$.

This also shows that if X satisfies $D2$, $C_*^c(X)$ is equivalent to a 2-dimensional complex: we can now repeat the argument with $n = 2$, and, in particular, deduce that if X satisfies $D1$, $C_*^c(X)$ is equivalent to a 1-dimensional complex: finally (though not usefully) the argument works again if $n = 1$.

COROLLARY 5·1. *If X is a finite connected CW complex (resp. finite dimensional and dominated by a finite complex) and $H^i(X; \Lambda) = 0$ for all $i > n \geqslant 3$, then X is homotopy equivalent to an n-dimensional complex which is finite (resp. dominated by a finite complex).*

The finiteness hypothesis shows that we can suppose A_* in the proof above finitely generated. Then A_r is a direct summand of a finite direct sum of copies of Λ, so $H^r(X; \Lambda) = 0$ implies $H^r(X; A_r) = 0$. The theorem now shows that X satisfies Dn, and the corollary follows from results of I, or equivalently, from ones above.

I had planned to prove theorem 5 with this simplified hypothesis, but my original proof was in error. I do not know whether it is possible to replace $H^n(X; F)$ by $H^n(X; \Lambda)$ in the theorem or not.

Having now algebraized Dn, we have the following:

THEOREM 6. *A projective positive chain complex A_* is equivalent to an n-dimensional complex if and only if $H_i(A_*) = 0$ for $i > n$ and the image of $d\colon A_{n+1} \to A_n$ is a direct summand (hence projective).*

Proof. If the conditions hold, and B_n is a complement to dA_{n+1}, then A_* is equivalent to $0 \to B_n \to A_{n-1} \to A_{n-2} \ldots$ Conversely, if A_* is equivalent to an n-dimensional complex, it is clear that $H_i(A_*) = 0$ for $i > n$. Also,

$$H^{n+1}(\hom_\Lambda(A_*, dA_{n+1})) = 0,$$

and $d: A_{n+1} \to dA_{n+1}$ gives an $(n+1)$ cocycle ($d^2 = 0$), which is thus a coboundary, so factors through A_n, giving a retraction of A_n on dA_{n+1}. (For this proof, cf. I, lemma 2·1).

We now give a reformulation (due to Gersten† and Milnor) of our finiteness obstruction. If X satisfies F, $C^c_*(X)$ is equivalent to a complex A_* of free finitely generated modules. If also X satisfies Dn, we can truncate at A_n, replacing A_n by a direct summand as above.

Now if A_* and B_* are equivalent n-dimensional chain complexes, it is easily seen by an argument of J. H. C. Whitehead (cf. proof of lemma 3) that

$$\oplus A_{2i} \oplus B_{2i+1} \simeq \oplus B_{2i} \oplus A_{2i+1}.$$

Denote by braces the class in $K_0(\Lambda)$ of a finitely generated projective module: then it follows that $\Sigma(-1)^i\{A_i\}$ depends only on the chain homotopy class of A_*. Thus for X satisfying Fn and Dn we have defined a homotopy type invariant $\sigma(X) \in K_0(\Lambda)$. The image of $\sigma(X)$ in the projective class group $\tilde{K}_0(\Lambda)$ differs in sign from the obstruction $\theta(X)$ of I.

This definition has several advantages over our former one. First, it is defined for chain complexes and not spaces, which gives more freedom in use. For we can generalize to the case where $C^c_*(X)$ is not equivalent to a complex of finite type, but for some ring homomorphism $\Lambda \to R$, the complex $C^c_*(X) \otimes_\Lambda R$ of free R-modules satisfies this condition. Probably more interesting, we also have a natural definition of a relative invariant $\sigma(X, Y)$, by considering the complex $C^c_*(X, Y)$.

LEMMA 7. *Suppose* $0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$ *an exact sequence of positive free chain complexes, two of which are equivalent to finitely generated projective complexes of finite dimension. Then so is the third, and* $\sigma(B_*) = \sigma(A_*) + \sigma(C_*)$.

Proof. Let M_*, N_* be the mapping cylinder and mapping cone respectively of i. Then we have a commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "i'"] \arrow[d, equal] & M_* \arrow[r, "j'"] \arrow[d, "p_1"] & N_* \arrow[r] \arrow[d, "p_2"] & 0 \\ 0 \arrow[r] & A_* \arrow[r, "i"] & B_* \arrow[r, "j"] & C_* \arrow[r] & 0. \end{tikzcd}$$

The central vertical map is an equivalence: applying the Five lemma to the induced map of homology sequences, we deduce that $N_* \to C_*$ also is. Now there is another exact sequence

$$0 \to B_* \xrightarrow{k} N_* \xrightarrow{q} A_* \to 0,$$

with $p_2 \circ k = j$ and q of degree -1, so, apart from the dimension shift, A_* is equivalent to the mapping cone of k, or of j. Iterating the argument, we find B_* equivalent to the mapping cone of q. Now given a chain map of finitely generated

† Gersten, S. M. 1966. A product formula for Wall's obstruction. *Amer. J. Math.* 88, 337–46.

projective complexes, its mapping cone satisfies the same condition, which proves the first assertion. We can now suppose that each of A_*, B_* and C_* is finitely generated and projective. Then $B_i \simeq A_i \oplus C_i$, for as C_i is projective, the sequences split. It follows that $\Sigma(-)^i\{B_i\} = \Sigma(-)^i\{A_i\}+\Sigma(-)^i\{C_i\}$, as asserted.

The above lemma is closely related to a result of L. Siebenmann (Princeton thesis, 1965) giving a 'sum theorem' for the finiteness obstruction. Siebenmann obtains also a 'product theorem' which we will not discuss here. A special case of the following is also due to Siebenmann.

THEOREM 8. *Let ϕ: $X \to Y$ be a map of CW complexes, inducing an isomorphism of fundamental groups π, and such that the induced map of universal covers $\tilde{\phi}$: $\tilde{X} \to \tilde{Y}$ has only one non-vanishing homology group $P = H_k(\tilde{\phi})$. Then*

(*i*) *if P is finitely generated and projective, and one of X, Y is dominated by a finite complex, so is the other and*

$$\sigma(Y) = \sigma(X)+(-1)^k\{P\}.$$

(*ii*) *If each of X, Y has finite dimension, P has finite projective dimension; if also π is finite and P torsion-free, P is projective.*

(*iii*) *If $H^{k+1}(\varphi; M) = 0$ for all coefficient modules M, then P is projective.*

Proof. When P is projective, we may consider it as a complex, and can then find A_* equivalent to $C^c_*(X)$, and B_* to $C^c_*(Y)$ with an exact sequence

$$0 \to A_* \to B_* \to P \to 0.$$

The result (i) now follows from lemma 7.

In (ii), we may suppose ϕ an inclusion; then $C^c_*(X, Y)$ gives a finite dimensional free resolution of P (note that d_k has a right inverse, and discard everything below dimension k: cf. theorem 6). The last result is due to Nakayama: see J.-P. Serre, *Corps Locaux* (Hermann 1962), p. 151.

The third part follows by the argument of I, lemma 2.1.

For this last result, see also the paper (which overlaps our own results): C. B. de Lyra 1965. On a conjecture in homotopy. *Anais Acad. Bras. Ciênc.* **37**, 167–84.