

# MAYER-VIETORIS PRESENTATIONS OVER COLIMITS OF RINGS

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## 1. Summary

If  $K \begin{array}{c} \nearrow T \\ \searrow S \end{array} \rightarrow R$  is a pushout in the category of rings (that is,  $R$  is the coproduct of  $T$  and  $S$  amalgamating  $K$ ) then

$$R \otimes_K R \begin{array}{c} \nearrow R \otimes_T R \\ \searrow R \otimes_S R \end{array} \rightarrow R$$

is a pushout *and pullback* in the category of  $(R, R)$  bimodules; equivalently,

$$(1) \quad 0 \longrightarrow R \otimes_K R \xrightarrow{(+, -)} R \otimes_S R \amalg R \otimes_T R \xrightarrow{\begin{pmatrix} + \\ + \end{pmatrix}} R \longrightarrow 0$$

is a short exact sequence of  $(R, R)$  bimodules.

The situation  $K \begin{array}{c} \nearrow T \\ \searrow S \end{array}$  can be conveniently described by an 'edge'  $K$

with two 'vertices'  $T, S$ , depicted  $T \xleftarrow{K} S$ . The choice of sign in (1) then amounts to an orientation of this edge.

We can extend the above, as follows. Informally, a 'graph of rings' is a graph whose vertices and edges are labelled with rings, and for each edge  $e$ , and any vertex  $v$  of  $e$ , there is a homomorphism from the ring  $R(e)$  corresponding to  $e$ , to the ring  $R(v)$  corresponding to  $v$ . By generators-and-relations arguments, this diagram of ring homomorphisms has a colimit,  $R$  say. Given any orientation of the graph (that is, of each edge) we can, as above, construct a complex

$$(2) \quad 0 \rightarrow \amalg_e R \otimes_{R(e)} R \rightarrow \amalg_v R \otimes_{R(v)} R \rightarrow R \rightarrow 0$$

of  $(R, R)$  bimodules. Our main result, Theorem 2, says that the complex (2) is exact if and only if the graph is a tree or  $R = 0$ .

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Under certain flatness assumptions it is straightforward to use the exactness of (2) to obtain Mayer–Vietoris exact triangles, and hence deduce an upper bound for the global dimension of the colimit of a tree of rings. In more restricted situations there is even a formula for the Euler–Poincaré characteristic.

Some of the applications require the author’s results on the module structure of the coproduct of two algebras. G. M. Bergman has provided elegant proofs of those structure theorems in a self-contained appendix.

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## 2. The Mayer–Vietoris presentation

By an (oriented) *graph*  $\Gamma$  we mean a set  $\Gamma^0$ , whose elements are called *vertices*, and a set  $\Gamma^1$  whose elements are called *edges* together with two maps  $\iota, \tau: \Gamma^1 \rightarrow \Gamma^0$ . For  $e \in \Gamma^1$  we think of  $\iota e$  and  $\tau e$  as the initial and terminal vertices of  $e$  respectively; although we are glossing over the situation where  $\iota e = \tau e$ , since we are leaving such edges unoriented. We may view  $\Gamma$  as a small category which has set of objects  $|\Gamma| = \Gamma^0 \cup \Gamma^1$  and has a morphism from each edge  $e$  to each of the edge’s vertices  $\iota e$  and  $\tau e$  (together with the identity morphisms).

By analogy with Serre’s graphs of groups [16], a *graph of rings*  $\mathbf{R}: \Gamma \rightarrow \mathit{Rings}$  is a functor from a graph  $\Gamma$  (viewed as a category) to the category of (unitary associative) rings and (unity preserving) homomorphisms.

Recall from [9] that any functor  $\mathbf{R}: \mathfrak{C} \rightarrow \mathit{Rings}$  from a small category  $\mathfrak{C}$ , has a colimit  $(R, \varphi(c): \mathbf{R}(c) \rightarrow R)_{c \in |\mathfrak{C}|}$ , written  $\text{colim } \mathbf{R} = R$ , and a limit  $(Q, \psi(c): Q \rightarrow \mathbf{R}(c))_{c \in |\mathfrak{C}|}$ .

We now fix a graph of rings  $\mathbf{R}: \Gamma \rightarrow \mathit{Rings}$  and denote its colimit by  $R$ . To avoid trivialities we assume  $R$  is non-zero. By abuse of notation we shall, for each  $\gamma \in |\Gamma|$ , also write  $\mathbf{R}(\gamma)$  for the image of  $\mathbf{R}(\gamma)$  in  $R$ , and we shall use the same symbol for an element of  $\mathbf{R}(\gamma)$  and its image in  $R$ , though none of the maps is necessarily injective. For convenience we abbreviate  $\otimes_{\mathbf{R}(\gamma)}$  to  $\otimes_\gamma$  for all  $\gamma \in |\Gamma|$ .

We say an element  $m$  of an  $(R, R)$  bimodule  $M$  commutes with an element  $x$  of  $R$  if the commutator  $[m, x] = mx - xm$  is zero.

We wish to construct the presentation (2) of the  $(R, R)$  bimodule  $R$ . Since  $R$  is generated as a ring by the  $\mathbf{R}(v)$ , where  $v \in \Gamma^0$ , an element of an  $(R, R)$  bimodule will commute with every element of  $R$  if and only if it commutes with all elements of each of the  $\mathbf{R}(v)$ . It follows that we can present  $R$  as an  $(R, R)$  bimodule with

- (i) generators  $\bar{v}$ , where  $v \in \Gamma^0$ ,
- (ii) relations saying that for all  $v \in \Gamma^0$ ,  $\bar{v}$  commutes with all elements of  $\mathbf{R}(v)$ ,
- (iii) relations saying that all the  $\bar{v}$  are equal.

Now clearly (i) and (ii) by themselves present the  $(R, R)$  bimodule

$$(3) \quad A = \coprod_{v \in \Gamma^0} R\bar{v}R = \coprod_{v \in \Gamma^0} R \otimes_v R$$

which maps onto  $R$  by sending  $\bar{v} = 1 \otimes_v 1$  to 1. We shall attempt to construct the kernel of this map. It is determined by (iii), so is generated by  $\{\bar{\iota e} - \bar{\tau e} \mid e \in \Gamma^1\}$  if and only if  $\Gamma$  is connected; we shall henceforth make this assumption on  $\Gamma$ . (In situations where one has started with a disconnected graph of rings, it is possible to connect it by adding edges and associating to them rings, for example  $\mathbf{Z}$ , without affecting the colimit.)

Recall that a (reduced)  $\Gamma$  path  $P$  consists of a finite sequence  $e_1, e_2, \dots, e_n$  of edges, with no repetitions, such that there is a sequence of vertices  $v_1, v_2, \dots, v_{n+1}$  and  $e_j$  connects  $v_j$  and  $v_{j+1}$ . When  $v_{n+1} = v_1$  the path is called a *cycle*. To describe the orientation of an edge  $e$  in the path  $P$  we define the *sign of  $e$  in  $P$*  to be

$$\sigma(e, P) = \begin{cases} 0 & \text{if } e \text{ does not occur as some } e_j, \\ +1 & \text{if } e \text{ occurs as some } e_j \text{ and } \iota e_j = v_j, \\ -1 & \text{if } e \text{ occurs as some } e_j \text{ and } \iota e_j \neq v_j. \end{cases}$$

We remark that if  $e$  occurs as some  $e_j$  and  $\iota e_j \neq v_j$  then  $\tau e_j = v_j$  and  $\iota e_j = v_{j+1}$ .

Now, our candidate  $B$  for the kernel of the above map is the  $(R, R)$  bimodule presented with

- (i') generators  $\bar{e}$ , where  $e \in \Gamma^1$ ,
- (ii') relations saying that for all  $e \in \Gamma^1$ ,  $\bar{e}$  commutes with all elements of  $\mathbf{R}(e)$ ,
- (iii') relations saying that  $\sum_{e \in \Gamma^1} \sigma(e, c)\bar{e} = 0$  for all cycles  $c$  in  $\Gamma$ .

It is readily verified that the  $(R, R)$  map  $B \rightarrow A$  sending  $\bar{e}$  to  $\bar{\iota e} - \bar{\tau e}$  is well defined, and it clearly maps onto the kernel of the map  $A \rightarrow R$  (since  $\Gamma$  is assumed connected) so we have a presentation  $B \rightarrow A \rightarrow R \rightarrow 0$ . We wish to prove that the left-hand map is injective, i.e. that

$$(4) \quad 0 \rightarrow B \rightarrow A \rightarrow R \rightarrow 0$$

is exact. Before we can do this we need a presentation of  $B$ . Let  $\Gamma^2$  denote the set of cycles of  $\Gamma$  and for  $c \in \Gamma^2$  let  $\mathbf{R}(c)$  denote the limit of the graph of rings obtained by restricting  $\mathbf{R}$  to  $c$ . (This is by analogy with  $\mathbf{R}(e)$  and  $\mathbf{R}(v)$  since it is obvious these are also limits.) Now (i') and (ii')



as dividing out by  $A'$ . Similarly  $\coprod_{\Gamma^1} \Omega_e$ , and hence  $B'$ , is generated by  $\{[\bar{e}, x] \mid e \in \Gamma^1, x \in \mathbf{R}(v), v \in \Gamma^0\}$ .

Now every  $[\bar{v}, x]$ , with  $x \in \mathbf{R}(v')$  and  $v' \in \Gamma^0$ , comes from an element of  $B'$  of the form  $\sum_{e \in \Gamma^1} \sigma(e, P)[\bar{e}, x]$ , where  $P$  is a path from  $v$  to  $v'$ . Moreover the choice of path  $P$  does not affect the element of  $B'$ . This suggests constructing an  $(R, R)$  map  $A' \rightarrow B'$  which sends  $[\bar{v}, x]$  to  $\sum_e \sigma(e, P)[\bar{e}, x]$ , where  $P$  is a path from  $v$  to  $v'$  and  $x \in \mathbf{R}(v')$ . We will devote the next section to proving that there is such a map. (It is clearly unique since it is defined on a set of generators.)

For the sake of continuity, let us assume, throughout the remainder of this section, the truth of

LEMMA 1. *There is an  $(R, R)$  linear map  $A' \rightarrow B'$  which sends  $[\bar{v}, x]$  to  $\sum_e \sigma(e, P)[\bar{e}, x]$ , whenever  $P$  is a path from  $v$  to  $v'$  and  $x \in \mathbf{R}(v')$ .*

*Proof.* See §3.

From this it follows that  $B' \rightarrow A'$  is an isomorphism, since for all  $v \in \Gamma^0, e \in \Gamma^1, x \in \mathbf{R}(v)$  the composite  $B' \rightarrow A' \rightarrow B'$  sends  $[\bar{e}, x]$  to  $[\bar{ie} - \bar{re}, x]$  to  $\sum_{e'} \sigma(e', P)[\bar{e}', x] - \sum_{e'} \sigma(e', Q)[\bar{e}', x]$ , where  $P$  is a path from  $ie$  to  $v$  and  $Q$  is a path from  $re$  to  $v$ . Since  $e, Q$  is a path from  $ie$  to  $v$  we may assume that  $P$  is  $e, Q$  and therefore  $[\bar{e}, x]$  is the difference, that is, the composite sends  $[\bar{e}, x]$  to  $[\bar{e}, x]$ . Since the composite fixes a generating set it must be the identity map. Hence  $B' \rightarrow A'$  is an isomorphism, so the top row of (6) is exact, and hence (4) is exact. From (5) we have the following result.

THEOREM 2. *Let  $\Gamma$  be an oriented connected graph, and let  $\mathbf{R}: \Gamma \rightarrow \text{Rings}$  be a graph of rings with colimit  $R$  say. Then there is an exact sequence of  $(R, R)$  bimodules*

$$(7) \quad \coprod_{c \in \Gamma^2} R \otimes_c R \rightarrow \coprod_{e \in \Gamma^1} R \otimes_e R \rightarrow \coprod_{v \in \Gamma^0} R \otimes_v R \rightarrow R \rightarrow 0$$

where  $1 \otimes_c 1 \mapsto \sum_e \sigma(e, c) 1 \otimes_e 1, 1 \otimes_e 1 \mapsto 1 \otimes_{ie} 1 - 1 \otimes_{re} 1, 1 \otimes_v 1 \mapsto 1$ .

APPLICATION. Let  $K$  be a ring. Recall that a  $K$ -ring is a ring  $R$  given with a homomorphism  $K \rightarrow R$ . Let  $\{R_v \mid v \in \Lambda\}$  be a set of  $K$ -rings, and let  $R$  denote the coproduct as  $K$ -rings (i.e. amalgamating  $K$ ). We can

construct a graph of rings whose edges are of the form  $K \xrightarrow{K} R_v, v \in \Lambda$ .

If for some  $\kappa \in \Lambda$  we collapse the edge  $K \xrightarrow{K} R_\kappa$  to  $R_\kappa$ , we get a new graph of rings (with the same colimit  $R$ ) whose edges are of the form

$R_\kappa \xrightarrow{K} R_v, \text{ where } v \neq \kappa$ . Since our new graph is connected at  $\kappa$ , (7) in

this case is

$$(8) \quad 0 \rightarrow \coprod_{\Lambda \rightarrow (\kappa)} R \otimes_K R \rightarrow \coprod_{\Lambda} R \otimes_{R_v} R \rightarrow R \rightarrow 0.$$

REMARKS. (i) The above application demonstrates the general principle for constructing a graph of rings from a given diagram of ring homomorphisms. Let  $\mathbf{R}: \mathfrak{C} \rightarrow \mathcal{Rings}$  be a functor from a small category. Let  $\Gamma(\mathfrak{C})$  be the (oriented) graph whose vertices are the objects (or identity morphisms) of  $\mathfrak{C}$  and whose edges are the non-identity morphisms of  $\mathfrak{C}$ . For such an edge  $e$  we understand  $\iota e = \text{domain } e$  and  $\tau e = \text{codomain } e$ . Define  $\mathbf{R}': \Gamma(\mathfrak{C}) \rightarrow \mathcal{Rings}$  as the functor given by  $\mathbf{R}'(v) = \mathbf{R}(v)$  for all  $v \in |\mathfrak{C}|$ ,  $\mathbf{R}'(e) = \mathbf{R}(\text{domain } e)$  for all  $e \in \mathfrak{C}$ , and  $\mathbf{R}'(e \rightarrow v)$  as the obvious map, namely either  $\mathbf{R}(e)$  or the identity on  $\mathbf{R}(\text{domain } e)$ . In a sense  $\mathbf{R}'$  is a 'barycentric subdivision' of  $\mathbf{R}$ .

(ii) There seems little hope of extending (7) further to the left in any generality, since the kernel of the left-hand map in (7) may contain elements that depend heavily on the choice of functor  $\mathbf{R}$ . For example, let  $\Gamma$  consist of two vertices  $v_1$  and  $v_2$ , and two edges  $e_1$  and  $e_2$  both connecting the vertices. Let  $K$  be a non-zero ring, and let  $x_1$  and  $x_2$  be commuting indeterminates over  $K$ . Define  $\mathbf{R}(v_i) = K[x_1, x_2]$  and  $\mathbf{R}(e_i) = K[x_i]$ , for  $i = 1, 2$ , where the corresponding maps are the obvious ones. Then  $\text{colim } \mathbf{R} = R$  is  $K[x_1, x_2]$ , and, for  $c = e_1, e_2 \in \Gamma^2$ ,  $\mathbf{R}(c) = K$ . But in  $R \otimes_c R$ ,  $x_1 x_2 \otimes 1 - x_1 \otimes x_2 - x_2 \otimes x_1 + 1 \otimes x_1 x_2$  is a non-zero element of the kernel.

### 3. Universal derivations and a proof of Lemma 1

If  $R$  is a ring and  $B$  is an  $(R, R)$  bimodule, a *derivation* of  $R$  into  $B$  is generally defined as a map  $\delta: R \rightarrow B$  satisfying

$$(9) \quad \delta(x + y) = \delta(x) + \delta(y), \quad \delta(xy) = \delta(x)y + x\delta(y).$$

For example, if  $b \in B$  then  $x \mapsto [x, b]$  is a derivation. This fact, together with the theory of derivations, will provide the key to verifying Lemma 1.

If we form the matrix ring  $\begin{pmatrix} R & B \\ 0 & R \end{pmatrix}$  then (9) is equivalent to the condition that the map

$$(10) \quad R \rightarrow \begin{pmatrix} R & B \\ 0 & R \end{pmatrix}, \quad x \mapsto \begin{pmatrix} x & \delta x \\ 0 & x \end{pmatrix},$$

is a homomorphism of rings. More generally, if  $R$  is a  $K$ -ring for some ring  $K$ , and we make our matrix ring into a  $K$ -ring by the diagonal map, then the condition that (10) be a homomorphism of  $K$ -rings is equivalent to (9) and  $\delta(K) = 0$ . Such a map is called a *K-derivation*.

The ring homomorphism form of the concept of a derivation makes it easy to relate it to that of a colimit.

LEMMA 3. Let  $R$  be the colimit of a functor  $\mathbf{R}: \mathfrak{C} \rightarrow \mathcal{R}ings$ , where  $\mathfrak{C}$  is a small category. Let  $B$  be an  $(R, R)$  bimodule, and let us be given a set  $\{\delta_c: \mathbf{R}(c) \rightarrow B \mid c \in |\mathfrak{C}|\}$  of derivations such that for any morphism  $c \rightarrow c'$  in  $\mathfrak{C}$ , the restriction of  $\delta_{c'}$  to  $\mathbf{R}(c)$  is  $\delta_c$ . Then there is a unique derivation  $\delta: R \rightarrow B$  such that for any object  $c$ , the restriction of  $\delta$  to  $\mathbf{R}(c)$  is  $\delta_c$ .

We say that the  $\delta$  so constructed is 'patched together' from the  $\delta_c$ .

*Proof of Lemma 3.* We are given a set  $\left\{ \mathbf{R}(c) \rightarrow \begin{pmatrix} R & B \\ 0 & R \end{pmatrix} \mid c \in |\mathfrak{C}| \right\}$  of homomorphisms, which by the universal property of the colimit induces a unique homomorphism  $R \rightarrow \begin{pmatrix} R & B \\ 0 & R \end{pmatrix}$ . It is clear that this gives the desired derivation.

COROLLARY 4. Let  $R$  and  $B$  be as above, and let  $\{b_c \mid c \in |\mathfrak{C}|\}$  be a subset of  $B$  such that, for any morphism  $c \rightarrow c'$  in  $\mathfrak{C}$ ,  $b_{c'} - b_c$  commutes with all elements of  $\mathbf{R}(c)$ . Then there exists a unique derivation  $\delta: R \rightarrow B$  whose restriction to  $\mathbf{R}(c)$  sends  $x$  to  $[b_c, x]$ .

*Proof.* Patch together the derivations  $\delta_c: \mathbf{R}(c) \rightarrow B$  which send  $x$  to  $[b_c, x]$ .

We need one more preliminary result, this time from [6, IX.3] (cf. also [5]).

LEMMA 5. Let  $K \rightarrow R$  be a ring homomorphism. Then the map  $d_K: R \rightarrow \Omega_K, x \mapsto 1 \otimes x - x \otimes 1$ , is a universal  $K$ -derivation in the sense that any  $K$ -derivation  $R \rightarrow B$  comes from a unique  $(R, R)$  linear map  $\Omega_K \rightarrow B$  composed with  $d_K$ .

*Proof.* [6, Proposition IX.3.2.]

*Proof of Lemma 1.* In the situation of §2, let  $v_0 \in \Gamma^0$ . For any  $v \in \Gamma^0$  there is a derivation

$$(12) \quad \mathbf{R}(v) \rightarrow B', \quad x \mapsto [\sum_{\sigma} \sigma(e, P)\tilde{e}, x],$$

where  $P$  is a path from  $v_0$  to  $v$ , and this derivation is independent of the choice of  $P$ . We claim that the derivations (12) patch together to give a derivation  $d_{v_0}: R \rightarrow B'$ . By Lemma 3 we need only show that for any  $e_0 \in \Gamma^1$  the diagram

$$\begin{array}{ccc} \mathbf{R}(e_0) & \longrightarrow & \mathbf{R}(\tau e_0) \\ \downarrow & & \downarrow \\ \mathbf{R}(\iota e_0) & \longrightarrow & B' \end{array}$$

commutes, i.e. that for all  $x \in \mathbf{R}(e_0)$

$$(13) \quad [\sum_e \sigma(e, P)\bar{e}, x] = [\sum_e \sigma(e, Q)\bar{e}, x],$$

where  $P$  is a path from  $v_0$  to  $\iota e_0$  and  $Q$  is a path from  $v_0$  to  $\tau e_0$ . But  $P, e_0$  is also a path from  $v_0$  to  $\tau e_0$ , so we may assume that  $Q = P, e_0$ . Hence  $\sum_e \sigma(e, Q)\bar{e} = \sum_e \sigma(e, P)\bar{e} + \bar{e}_0$ . This proves (13) since  $[\bar{e}_0, x] = 0$ . Hence the derivations (12) patch together to give a derivation  $d_{v_0}: R \rightarrow B'$ . Moreover, since the path from  $v_0$  to  $v_0$  is empty, we see from (12) that  $d_{v_0}$  vanishes on  $\mathbf{R}(v_0)$ , i.e. that  $d_{v_0}$  is an  $\mathbf{R}(v_0)$  derivation. Let us write  $v$  for  $v_0$ .

Now by Lemma 5, there is a unique  $(R, R)$  linear map

$$(14) \quad \alpha_v: \Omega_v \rightarrow B'$$

sending  $[\bar{v}, x] = 1 \otimes x - x \otimes 1$  to  $d_v(x)$  for all  $x \in R$ . Since  $A' = \coprod_{v \in \Gamma^0} \Omega_v$ , there is an  $(R, R)$  map  $\coprod \alpha_v: A' \rightarrow B'$ . By (14) and (12) this map has the property described in the statement of Lemma 1.

4. Mayer-Vietoris triangles

We turn now to the situation where  $\Gamma$  is an oriented connected graph with no cycles, commonly called a *tree*. In this case a graph of rings  $\mathbf{R}: \Gamma \rightarrow \mathbf{Rings}$  is called a *tree of rings*. Throughout this section we fix a tree of rings  $\mathbf{R}$ , and denote its colimit by  $R$ .

Trees are of interest because they are precisely the graphs for which (7) becomes a short exact sequence

$$(15) \quad 0 \rightarrow \coprod_{\Gamma^1} R \otimes_e R \rightarrow \coprod_{\Gamma^0} R \otimes_v R \rightarrow R \rightarrow 0$$

of  $(R, R)$  bimodules.

Since the right-hand term of (15) is flat (even free) as left  $R$ -module, we may tensor this sequence over  $R$  with any (unital) right  $R$ -module  $M$  to get, in the language of [18], a *Mayer-Vietoris presentation* of  $M$ :

$$(16) \quad 0 \rightarrow \coprod_{\Gamma^1} M \otimes_e R \rightarrow \coprod_{\Gamma^0} M \otimes_v R \rightarrow M \rightarrow 0.$$

Notice that the first two terms are constructed from the  $\mathbf{R}(\gamma)$  module structure of  $M$  for  $\gamma \in \Gamma^1, \Gamma^0$  respectively. Hence in good cases this presentation allows us to study the module theory of  $R$  in terms of the module theories of the  $\mathbf{R}(\gamma), \gamma \in |\Gamma|$ . For any right  $R$ -module  $N$ ,  $\text{Ext}_R(-, N)$  applied to (16) gives an exact triangle

$$(17) \quad \begin{array}{ccc} \text{Ext}_R(M, N) & \longrightarrow & \text{Ext}_R(\coprod_{\Gamma^0} M \otimes_v R, N) \\ & \searrow \delta & \swarrow v \\ & \text{Ext}_R(\coprod_{\Gamma^1} M \otimes_e R, N) & \end{array}$$



where  $\delta$  is of degree 1. Since  $\text{Ext}_R(\prod -, -) = \prod \text{Ext}_R(-, -)$  (cf. [6, V.9]), (17) can be written

$$(18) \quad \begin{array}{ccc} \text{Ext}_R(M, N) & \longrightarrow & \prod_v \text{Ext}_R(M \otimes_v R, N) \\ & \swarrow & \searrow \\ & \prod_e \text{Ext}_R(M \otimes_e R, N) & \end{array}$$

We have not yet reached the true Mayer-Vietoris triangle, which should look like

$$(19) \quad \begin{array}{ccc} \text{Ext}_R(M, N) & \longrightarrow & \prod_v \text{Ext}_{\mathbf{R}(v)}(M, N) \\ & \swarrow & \searrow \\ & \prod_e \text{Ext}_{\mathbf{R}(e)}(M, N) & \end{array}$$

In some situations (cf. the example in § 5) there is no exact triangle (19); but at least in certain cases (18) is already in the form (19), namely if for every  $\gamma \in |\Gamma|$  the canonical map

$$(20) \quad \text{Ext}_R(M \otimes_\gamma R, N) \rightarrow \text{Ext}_{\mathbf{R}(\gamma)}(M, N)$$

is an isomorphism. Let us recall a situation where this will occur. Let

$$(21) \quad \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

be a projective  $\mathbf{R}(\gamma)$  resolution of  $M$ , and apply  $-\otimes_\gamma R$  to it. If the new sequence

$$\dots \rightarrow P_1 \otimes_\gamma R \rightarrow P_0 \otimes_\gamma R \rightarrow M \otimes_\gamma R \rightarrow 0$$

is exact we will have a projective  $R$ -resolution of  $M \otimes_\gamma R$ , and applying  $\text{Hom}_R(-, N)$  to it we get the same complex as we would by applying  $\text{Hom}_{\mathbf{R}(\gamma)}(-, N)$  to (21); so the homology groups are naturally isomorphic, that is,  $\text{Ext}_R(M \otimes_\gamma R, N) = \text{Ext}_{\mathbf{R}(\gamma)}(M, N)$ , which means that (20) is an isomorphism of graded modules. To get this we only need projective  $\mathbf{R}(\gamma)$ -resolutions (21) of  $M$  to 'lift' under  $-\otimes_\gamma R$  to  $R$ -resolutions of  $M \otimes_\gamma R$ . The precise requirement for this to hold is clearly

$$(22) \quad \text{Tor}_n^{\mathbf{R}(\gamma)}(M, R) = 0 \quad \text{for all } n > 0.$$

DEFINITION. We call a homomorphism  $K \rightarrow R$  of rings *lifting*, and call  $R$  *K-lifting*, if for all  $n > 0$ ,  $\text{Tor}_n^K(M, R) = 0$  for all right  $R$ -modules  $M$ .

By what has been said, we get an exact Mayer-Vietoris triangle (19) if  $R$  is  $\mathbf{R}(\gamma)$ -lifting for all  $\gamma \in |\Gamma|$ . We will be more interested in the stronger requirement that  $R$  be *flat* as left  $\mathbf{R}(\gamma)$ -module for all  $\gamma \in |\Gamma|$ ,

but we postpone a discussion of this situation to §5 and here summarize the above information and its counterpart for homology.

**THEOREM 6.** *Let  $\mathbf{R}: \Gamma \rightarrow \text{Rings}$  be a tree of rings, and denote its colimit by  $R$ . If  $R$  is  $\mathbf{R}(\gamma)$ -lifting (for example, left flat) for all  $\gamma \in |\Gamma|$  then there is an exact Mayer–Vietoris triangle*

$$(23) \quad \begin{array}{ccc} \text{Ext}_R & \longrightarrow & \coprod_v \text{Ext}_{\mathbf{R}(v)} \\ & \swarrow & \searrow \\ & \coprod_e \text{Ext}_{\mathbf{R}(e)} & \end{array}$$

of bifunctors on right  $R$ -modules, and there is an exact Mayer–Vietoris triangle

$$(24) \quad \begin{array}{ccc} \text{Tor}^R & \longleftarrow & \coprod_v \text{Tor}^{\mathbf{R}(v)} \\ & \searrow & \swarrow \\ & \coprod_e \text{Tor}^{\mathbf{R}(e)} & \end{array}$$

of bifunctors on  $R$ -modules (right modules in the first variable and left modules in the second variable).

*Proof.* We have already proved (23) by applying the trifunctor  $\text{Ext}_R(- \otimes_R -, -)$  to (15); and (24) arises similarly under our hypotheses by applying  $\text{Tor}^R(- \otimes_R -, -)$  to (15).

**REMARKS.** (i) In the situation where  $R$  is the group ring of the co-product of two groups amalgamating a subgroup and  $M = \mathbf{Z}$ , (16) as derived from (8) is due to Lyndon [12] and was made explicit by Swan [17] in his construction of the consequent Mayer–Vietoris triangle; cf. also [18].

(ii) It is possible to derive results analogous to Theorem 6 for Hochschild’s relative cohomology, as follows. Observe that for any  $v_0 \in \Gamma^0$ , (16) is split as a sequence of right  $\mathbf{R}(v_0)$ -modules, with splitting  $M \rightarrow \coprod M \otimes_{v_0} R$ ,  $m \mapsto m \otimes_{v_0} 1$ . In particular (16) is split as a sequence of right  $Q$ -modules, where  $Q$  is the limit of all the natural maps  $\lim \mathbf{R} \rightarrow R$  (that is,  $Q$  is the limit of the system obtained by augmenting  $\mathbf{R}$  with  $R$ ). In the language of [10], (16) is  $(R, Q)$  exact. Thus we can replace  $\text{Ext}_-$  in Theorem 6 with  $\text{Ext}_{(-, Q)}$  and the result will still hold. For coproducts amalgamating a ring we have  $Q = \mathbf{R}(e)$  for all  $e \in \Gamma^1$ , and since  $\text{Ext}_{(Q, Q)}$  is trivial in degrees

greater than 0, (23) gives rise to an isomorphism  $\text{Ext}_{(R,Q)} \simeq \prod_v \text{Ext}_{\mathbf{R}(v),Q}$  in degrees greater than 1; cf. [1, Theorem 4.1].

(iii) We observed that (23) arises by applying  $\text{Ext}_R(- \otimes_R -, -)$  to (15). Alternatively, we can apply  $\text{Ext}_R(-, \text{Hom}_R(-, -))$  to (15) (still as the second variable), and if  $R$  is projective as right  $\mathbf{R}(\gamma)$ -module for all  $\gamma \in |\Gamma|$  we again derive (23). However, situations where  $R$  is known to be right projective, but not known to be left flat over  $\mathbf{R}(\gamma)$ , are rare indeed, so we will say no more of this case. Similarly, (24) can be derived by applying  $\text{Tor}^R(-, - \otimes_R -)$  to (15) and assuming that  $R$  is right flat over each  $\mathbf{R}(\gamma)$ .

In any category a Mayer-Vietoris triangle will have automatic consequences for 'dimension'. Thus we confirm one of Bergman's suspicions ([3], and correspondence):

**COROLLARY 7.** *Let  $\mathbf{R}: \Gamma \rightarrow \mathcal{R}\text{ings}$  be a tree of rings with colimit  $R$  say. If  $R$  is flat as left  $\mathbf{R}(\gamma)$ -module for all  $\gamma \in |\Gamma|$ , and  $\sup\{\text{rt.gl.dim } \mathbf{R}(v) \mid v \in \Gamma^0\}$  is finite, say  $n$ , then*

$$\text{rt.gl.dim } R \leq n + 1;$$

and equality can only be achieved if there is an edge  $e$  such that

$$\text{rt.gl.dim } \mathbf{R}(e) \geq \text{rt.gl.dim } \mathbf{R}(te) = \text{rt.gl.dim } \mathbf{R}(re) = n.$$

*Proof.* For any  $e \in \Gamma^1$  and right  $R$ -modules  $M, N$  we have

$$\begin{aligned} \text{Ext}_{\mathbf{R}(e)}(M, N) &= \text{Ext}_R(M \otimes_{\mathbf{R}(e)} R, N) \\ &= \text{Ext}_R(M \otimes_{\mathbf{R}(e)} \mathbf{R}(te) \otimes_{\mathbf{R}(te)} R, N) \\ &= \text{Ext}_{\mathbf{R}(te)}(M \otimes_{\mathbf{R}(e)} \mathbf{R}(te), N). \end{aligned}$$

Since  $\text{Ext}_{\mathbf{R}(te)}^{n+1} = 0$ , it follows that  $\prod_{e \in \Gamma^1} \text{Ext}_{\mathbf{R}(e)}^{n+1} = 0$  on right  $R$ -modules (and it is clear that  $\prod_{v \in \Gamma^0} \text{Ext}_{\mathbf{R}(v)}^{n+2} = 0$ ), so by the exactness of (23),  $\text{Ext}_R^{n+2} = 0$  and  $\text{rt.gl.dim } R \leq n + 1$ . If in fact equality holds then, from the vanishing of  $\prod \text{Ext}_{\mathbf{R}(v)}^{n+1}$  and the exactness of (23), we see there is some edge  $e$  such that  $\text{Ext}_{\mathbf{R}(e)}^n$  is not the zero bifunctor on right  $R$ -modules. Hence, as in the first sentence of this proof, the right global dimension of each of the rings  $\mathbf{R}(e), \mathbf{R}(te), \mathbf{R}(re)$  is at least  $n$ .

**REMARKS.** (i) The analogous result holds for weak global dimension.

(ii) For the coproduct  $R$  of a set  $\{R_v \mid v \in \Lambda\}$  of  $K$ -rings, if we choose  $\kappa$  in (8) carefully, Corollary 7 implies that if  $R$  is flat as left module over  $K$  and all the  $R_v$ , and  $\sup\{\text{rt.gl.dim } R_v \mid v \in \Lambda\} = n$  is finite, then

$$(25) \quad \text{rt.gl.dim } R \leq \begin{cases} n + 1 & \text{if } \text{rt.gl.dim } K \geq n = \text{rt.gl.dim } R_v \text{ for all } v \in \Lambda, \\ n & \text{in all other cases.} \end{cases}$$

In [3], Bergman shows that if  $K$  is a semisimple (that is, global dimension zero) subring of the  $R_n$  then equality holds in the second case of (25).

(iii) Bergman has made the following observations. For the direct limit (that is, colimit)  $R$  of a countable directed system  $R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow \dots$  of rings, Corollary 7 has conclusion

$$(26) \quad \text{rt.gl.dim } R \leq 1 + \liminf\{\text{rt.gl.dim } R_n\},$$

a result that is already known without any lifting assumptions [2]. (To see ‘lim inf’ take an appropriate cofinal subsystem.) Observe that in this case the exactness of (16), which is used in [2], is very easy to see. It is not surprising that (26) is inappropriate for uncountable directed systems (see [13]) since an uncountable directed system is clearly not a tree.

Another application of (16) is the derivation of a formula for the Euler-Poincaré characteristic. In the situation studied by Lyndon and Swan, this formula was observed by Serre [15].

DEFINITION. If  $R$  is a ring and  $M$  is a right  $R$  module, we say  $M$  has  $R$ -characteristic if there is a finite  $R$ -resolution

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where all the  $P_i$  are finitely generated projective  $R$ -modules. In this event, there is a well-known (cf. [14]) invariant of  $M$ , the  $R$ -characteristic  $\chi_R(M) = \sum_{i=0}^n (-1)^i [P_i]$ . Here  $[P]$  denotes the class of  $P$  in the Grothendieck group  $K(R)$  of the semigroup of finitely generated projective  $R$ -modules, where addition is given by  $\amalg$ .

THEOREM 8. Let  $\Gamma$  be a finite tree, and let  $\mathbf{R}: \Gamma \rightarrow \mathcal{R}\text{ings}$  be a tree of rings, and denote its colimit by  $R$ . If  $R$  is  $\mathbf{R}(\gamma)$ -lifting for all  $\gamma \in |\Gamma|$  then any right  $R$ -module  $M$  which has  $\mathbf{R}(\gamma)$ -characteristic for all  $\gamma \in |\Gamma|$  has  $R$ -characteristic, and

$$\chi_R(M) = \sum_{\Gamma^0} \chi_{\mathbf{R}(v)}(M) - \sum_{\Gamma^1} \chi_{\mathbf{R}(e)}(M).$$

Here, for simplicity, we have identified  $K(\mathbf{R}(\gamma))$  with its image in  $K(R)$  under the canonical map sending  $[P]$  to  $[P \otimes_\gamma R]$ . We are not assuming this map to be injective.

Proof. The lifting assumption guarantees that projective  $\mathbf{R}(\gamma)$ -resolutions lift, so  $\chi_R(M \otimes_\gamma R) = \chi_{\mathbf{R}(\gamma)}(M)$ . Thus  $\chi_R(\amalg_{\Gamma^0} M \otimes_v R) = \sum_{\Gamma^0} \chi_{\mathbf{R}(v)}(M)$ , and similarly for  $\Gamma^1$ . But for any short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , if any two terms have  $R$ -characteristic then so does the third and  $\chi(A) - \chi(B) + \chi(C) = 0$ . The assertion therefore follows from the exactness of (16).

### 5. When is the colimit flat?

Recall that  $R$  is the colimit of a tree of rings  $\mathbf{R}: \Gamma \rightarrow \mathcal{Rings}$ . We want to know when  $R$  is flat as left  $\mathbf{R}(\gamma)$ -module for all  $\gamma \in |\Gamma|$ . The minimal assumption we could hope to get away with is that each  $\mathbf{R}(v)$  is flat as left  $\mathbf{R}(e)$  module for each vertex  $v$  of each edge  $e$  (but we will see that this is not sufficient). By transitivity of flatness, we would then only need that  $R$  is flat as left  $\mathbf{R}(v)$ -module for all vertices  $v$ .

Let us fix a vertex  $v$ . Then  $R$  as  $\mathbf{R}(v)$ -ring is the direct limit of the following directed system. For each finite subtree  $T$  of  $\Gamma$  containing  $v$ , restrict  $\mathbf{R}$  to  $T$  to get a 'subtree of rings' and form its colimit. It is easy to see that this is a directed system of  $\mathbf{R}(v)$ -rings and that its direct limit (in the category of rings or the category of left  $\mathbf{R}(v)$ -modules) is  $R$ . Since a direct limit of flat modules is flat, it suffices therefore to consider the case where  $\Gamma$  is finite. Here the colimit is obtained by inductively forming the colimit of two-vertex trees. Thus it should suffice to have enough information about the colimit in the two-vertex case, that is, the coproduct with amalgamation.

To facilitate a summary of what is known in this area, we introduce a convention and recall some definitions. Let  $K \rightarrow R$  be a ring homomorphism. We will say  $R$  is *flat* as  $K$ -ring if  $R$  is flat as left  $K$ -module, and similarly for other module properties. A left  $K$ -module  $M$  is *faithfully flat* if the functor  $-\otimes_K M$  carries exact sequences of right  $K$ -modules to exact sequences of abelian groups, and carries non-exact sequences of right  $K$ -modules to non-exact sequences of abelian groups. If  $K$  is commutative, a  $K$ -algebra is a  $K$ -ring  $R$  such that the image of  $K$  lies in the centre of  $R$ . An *epic*  $K$ -ring  $R$  is such that  $K \rightarrow R$  is an epimorphism in the category of rings.

**THEOREM 9.** *Let  $K$  be a ring, let  $S$  and  $T$  be  $K$ -rings, and let  $R$  be the coproduct amalgamating  $K$ .*

(i) (Cohn) *If  $S$  and  $T$  are faithfully flat  $K$ -rings then  $R$  is a faithfully flat  $S$ -ring.*

(ii) (Knight) *If  $S$  and  $T$  are flat epic  $K$ -rings then  $R$  is a flat epic  $S$ -ring.*

(iii) *If  $S$  and  $T$  are flat  $K$ -algebras then  $R$  is a flat  $S$ -ring.*

*Proof.* (i) [7, Theorem 4.4.]

(ii) [11, Proposition 2.]

(iii) Corollary A3 of our appendix.

**COROLLARY 10.** *Let  $\mathbf{R}: \Gamma \rightarrow \mathcal{Rings}$  be a tree of rings. The colimit  $R$  is flat as left  $\mathbf{R}(\gamma)$ -module for all  $\gamma \in |\Gamma|$  if any of the following hold:*

(i)  $\mathbf{R}(v)$  is a faithfully flat  $\mathbf{R}(e)$ -ring for each vertex  $v$  of each edge  $e$ ;

- (ii)  $\mathbf{R}(v)$  is a flat epic  $\mathbf{R}(e)$ -ring for each vertex  $v$  of each edge  $e$ ;
- (iii) (for coproducts of  $K$ -algebras, where  $K$  is a ring)  $\mathbf{R}(v)$  is a flat  $K$ -algebra for all vertices  $v$ , and  $\mathbf{R}(e) = K$  for all edges  $e$ .

*Proof.* As in the argument at the beginning of this section, we may as well assume that  $\Gamma$  is finite. We can now proceed by induction on the number of vertices of  $\Gamma$ . If this is 1, the result is obvious. Otherwise, we form the colimit of a two-vertex subtree of rings, and easily form a new tree of rings with fewer vertices, while preserving all the induction hypotheses. The result now follows.

**EXAMPLE.** Let  $k$  be a commutative field, and let  $x, y, e$  be indeterminates, and suppose  $e^2 = e$  so  $k[e] \simeq k \times k$ . Then the tree

$$k[e] \xrightarrow{k} k[x] \xrightarrow{k} k[y]$$

has colimit  $k\langle x, y, e \mid e^2 = e \rangle$ , and by Corollary 7 the global dimension does not exceed 1. Now consider the tree

$$k\langle x, y, e, t = xey \mid e^2 = e \rangle \xrightarrow{k[t]} k[z, t = z^2].$$

Both extensions are left (and right) free on monomial bases. So again by Corollary 7, the global dimension of the colimit  $k\langle x, y, e, z \mid xey = z^2 \rangle$  does not exceed 2. Finally consider the tree

$$k\langle x, y, e, z \mid xey = z^2 \rangle \xrightarrow{k[e]} k[e]/(e) = k.$$

By rights this should be a good tree: we are amalgamating a semisimple ring, the left extension is free (so faithfully flat) and the right extension is a flat epic algebra. Nonetheless, the colimit  $k\langle x, y, z \mid z^2 = 0 \rangle$  is well known to have infinite global dimension (one implication of which is that there can be no exact triangle (23) for this tree). This demonstrates that the various conditions of Theorem 9 cannot be readily weakened. In [11], Knight remarks that (i) and (ii) of Theorem 9 are complementary since a faithfully flat epimorphism is an isomorphism. We now see further that there can be no nice common generalization.

(Added 28 May 1975.) Another version of Theorem 2 is suggested by recent results of Chiswell [19]. Let us pick a maximal subtree  $T$  of the connected graph  $\Gamma$  and construct a ring  $R_T$  universal with respect to the following properties: for each  $v \in \Gamma^0$  there is a homomorphism  $\mathbf{R}(v) \rightarrow R_T$ ; for each  $e \in \Gamma^1$  the two induced homomorphisms  $\iota_e, \tau_e: \mathbf{R}(e) \rightarrow R_T$  are 'conjugate' by a distinguished element  $t_e \in R_T$  such that  $t_e \iota_e = \tau_e t_e$  for all  $x \in \mathbf{R}(e)$ ; and for all  $e \in T$ ,  $t_e = 1$ . (Thus  $\iota_e$  and  $\tau_e$  agree for  $e \in T$ .) We view  $R_T$  as left and right  $\mathbf{R}(e)$ -module via  $\iota_e$  and  $\tau_e$  respectively.

By our methods it is possible to show that there is a short exact sequence  $0 \rightarrow \coprod_{\Gamma^1} R_T \otimes_e R_T \rightarrow \coprod_{\Gamma^0} R_T \otimes_v R_T \rightarrow R_T \rightarrow 0$  where the first map is given by  $x \otimes_e y \mapsto xt_e \otimes_{ie} y - x \otimes_{\tau_e} t_e y$  (cf. [19, Theorem 1]). We could use this to generalize Theorem 6. Unfortunately examples abound where the isomorphism class of  $R_T$  depends on  $T$ . We can overcome this by insisting that all the  $t_e$  be units in  $R_T$ , which does not affect the (proof of) exactness of the above sequence. Since the analogue of Corollary 10 (i) holds, we have a useful generalization of Theorem 6 to arbitrary graphs (cf. [19, Theorem 2]). The reader can verify the analogue of Corollary 10 (i) by a familiar argument which we sketch. Suppose the  $\mathbf{R}(ie)$  and  $\mathbf{R}(\tau e)$  are all left faithfully flat as  $\mathbf{R}(e)$ -rings. We wish to show that  $R_T$  is left faithfully flat over the  $\mathbf{R}(\gamma)$ . Since left faithfully flat is a local condition for ring extensions (but not for modules!) it is preserved by direct limits. Thus we may assume  $\Gamma$  is finite. By first forming the colimit of the restriction of  $\mathbf{R}$  to  $T$  (and knowing Theorem 9 (i)) we may assume that  $T$  consists of one point. By induction we may assume that  $\Gamma$  has one edge,  $e$ . Consider the infinite tree

$$\dots \mathbf{R}(ie) \xrightarrow{\mathbf{R}(e)} \mathbf{R}(\tau e) = \mathbf{R}(ie) \xleftarrow{\mathbf{R}(e)} \dots$$

and form its colimit  $S$  say. We can make the shift automorphism  $\alpha$  inner via an indeterminate  $t_e$  and its inverse. Then  $R_T = S[t_e, t_e^{-1}, \alpha]$  and is clearly left faithfully flat over  $S$  and hence over  $\mathbf{R}(e)$ ,  $\mathbf{R}(ie)$ , and  $\mathbf{R}(\tau e)$ , as desired.

**Appendix. The module structure of the coproduct of two algebras**

In the following we present G. M. Bergman’s simplified proofs and generalized statements of results that the author discovered in the course of proving Theorem 9 (iii). This account is taken verbatim from a written communication from Bergman.

Let  $K$  be a commutative ring, let  $S, T$  be  $K$ -algebras, and let  $R$  denote their coproduct as  $K$ -algebras. By abuse of notation, we shall also write  $K$  for the image of  $K$  in  $S$  or  $T$ , and we shall use the same symbol for an element of  $S$  or  $T$  and its image in  $R$  (though none of these maps is necessarily injective).

Let  $M$  denote the  $K$ -module  $(S/K) \otimes_K (T/K)$ . Clearly, we may define a  $K$ -linear map  $M \rightarrow R$  by  $(s+K) \otimes (t+K) \mapsto [s, t] = st - ts$ ; we shall write this map  $m \mapsto \tilde{m}$  (where  $m \in M$ ). For suggestiveness, we shall denote the generators  $(s+K) \otimes (t+K)$  of  $M$  by  $\{s, t\}$  (where  $s \in S, t \in T$ ).

Now let  $K\langle M \rangle$  denote the tensor algebra on the  $K$ -module  $M$ . By the universal property of tensor algebras,  $\sim$  extends to a  $K$ -algebra

homomorphism  $K\langle M \rangle \rightarrow R$ , which we shall again denote  $p \mapsto \tilde{p}$ . We can now prove

**PROPOSITION A1.** *The map  $\alpha: S \otimes_{\mathbb{K}} T \otimes_{\mathbb{K}} K\langle M \rangle \rightarrow R$  given by  $\alpha(s \otimes t \otimes p) = s\tilde{p}$  is an isomorphism of left  $S$ -modules.*

*Proof.* Consider also the  $T$ -module  $T \otimes_{\mathbb{K}} S \otimes_{\mathbb{K}} K\langle M \rangle$  and the  $T$ -module homomorphism of this into  $R$  given by  $\alpha'(t \otimes s \otimes p) = t\tilde{s}$ . We claim these two modules may be joined by  $K$ -module isomorphisms to get a commuting diagram

$$(A1) \quad \begin{array}{ccc} S \otimes T \otimes K\langle M \rangle & & \\ \downarrow \iota & \uparrow \iota' & \searrow \alpha \\ T \otimes S \otimes K\langle M \rangle & & R \\ & & \nearrow \alpha' \end{array}$$

Indeed, the natural choices for  $\iota$  and  $\iota'$  are given by

$$(A2) \quad \begin{cases} \iota(s \otimes t \otimes p) = t \otimes s \otimes p + 1_T \otimes 1_S \otimes \{s, t\}p, \\ \iota'(t \otimes s \otimes p) = s \otimes t \otimes p - 1_S \otimes 1_T \otimes \{s, t\}p, \end{cases}$$

and one easily verifies that these are inverse to each other and make (A1) commute. We can use the isomorphism  $\iota$  to carry the natural left  $T$ -module structure of  $T \otimes S \otimes K\langle M \rangle$  over to  $S \otimes T \otimes K\langle M \rangle$ . Having both  $S$  and  $T$  module structures which agree on  $K$ , it becomes a left  $R$ -module. Let us record the formulae defining left multiplication by elements of  $S$  and  $T$  (the latter obtained using (A2)):

$$(A3) \quad \begin{cases} s'(s \otimes t \otimes p) = s's \otimes t \otimes p, \\ t'(s \otimes t \otimes p) = s \otimes t't \otimes p - 1_S \otimes 1_T \otimes \{s, t't\}p + 1_S \otimes t' \otimes \{s, t\}p. \end{cases}$$

*(Remark.* We could have defined  $S$  and  $T$  module structures on  $S \otimes T \otimes K\langle M \rangle$  directly by (A3), without introducing  $T \otimes S \otimes K\langle M \rangle$  and  $\iota$ . However, then we would have had to verify  $t''(t'x) = (t''t')x$  and  $\alpha(t'x) = t'\alpha(x)$  (where  $t', t'' \in T$  and  $x \in S \otimes T \otimes K\langle M \rangle$ ) which would have been unmotivated and moderately tedious.)

In particular we have

$$(A4) \quad \begin{cases} t(1 \otimes 1 \otimes p) = 1 \otimes t \otimes p, \\ s(1 \otimes t \otimes p) = s \otimes t \otimes p, \\ s(t(1 \otimes 1 \otimes p)) - t(s(1 \otimes 1 \otimes p)) = 1 \otimes 1 \otimes \{s, t\}p. \end{cases}$$



From this it follows that  $S \otimes T \otimes K\langle M \rangle$  is generated as an  $R$ -module by  $1 \otimes 1 \otimes 1$ . (The last formula shows that the  $R$ -submodule generated by this element contains all of  $1 \otimes 1 \otimes K\langle M \rangle$ ; the first two formulae then give us everything.) Now define a right  $R$ -module homomorphism  $\beta: R \rightarrow S \otimes T \otimes K\langle M \rangle$  by  $\beta(1) = 1 \otimes 1 \otimes 1$ . We see that  $\alpha\beta$  and  $\beta\alpha$  fix the generators  $1$  and  $1 \otimes 1 \otimes 1$ , hence are the identity maps, and hence  $\alpha$  is an isomorphism as claimed.

In particular, the above result shows that the  $K$ -module structure of  $R$  is uniquely determined by the  $K$ -module structure of  $S, T, S/K, T/K$ . Now, in general, even when a  $K$ -ring  $S$  has 'nice'  $K$ -module structure, the  $K$ -module  $S/K$  may not be 'nice', for example,  $\mathbf{Q}$  is flat over  $\mathbf{Z}$ , but  $\mathbf{Q}/\mathbf{Z}$  is not. Nevertheless, the next result will show that the kind of tensor product involved in the above description *will* have 'nice' module structure over  $K$ , and even over  $S$ .

LEMMA A2. *Let us be given the following commutative diagram of associative rings*

$$\begin{array}{ccccccc}
 K_1 & \longrightarrow & K_2 & \longrightarrow & \dots & \longrightarrow & K_n \\
 \downarrow & & \downarrow & & & & \downarrow \\
 S_1 & \longrightarrow & S_2 & \longrightarrow & \dots & \longrightarrow & S_n \longrightarrow S_{n+1} \quad (n \geq 1).
 \end{array}$$

Then the natural map

$$S_1 \otimes_{K_1} S_2 \otimes_{K_2} \dots \otimes_{K_n} S_{n+1} \rightarrow (S_1/K_1) \otimes_{K_1} \dots \otimes_{K_n} (S_n/K_n) \otimes_{K_n} S_{n+1}$$

of  $(K_i, S_{n+1})$  bimodules, splits.

*Proof.* Consider the case where  $n = 1$ ; there the splitting of the map  $s_1 \otimes s_2 \mapsto (s_1 + K) \otimes s_2$  may be given by  $(s_1 + K) \otimes s_2 \mapsto s_1 \otimes s_2 - 1 \otimes s_1 s_2$ . The generalization of this formula to arbitrary  $n$  involves  $2^n$  terms, but it may be formally abbreviated

$$\begin{aligned}
 & (s_1 + K) \otimes (s_2 + K) \otimes \dots \otimes (s_n + K) \otimes s_{n+1} \\
 & \mapsto (s_1 \otimes 1 - 1 \otimes s_1)(s_2 \otimes 1 - 1 \otimes s_2) \dots (s_n \otimes 1 - 1 \otimes s_n) s_{n+1}.
 \end{aligned}$$

Indeed this is easily seen to be well defined and to split the given map.

Now the left  $S$ -module  $S \otimes T \otimes K\langle M \rangle$  of Proposition A1 will be a direct sum over  $m \geq 0$  of  $S \otimes T \otimes M^{\otimes m}$  (all tensor products over  $K$ ,  $(-)^{\otimes m}$  denoting an  $m$ -fold tensor product of copies). Applying the definition of  $M$ , we see that this becomes  $S \otimes T \otimes [(S/K) \otimes (T/K)]^{\otimes m}$ , which is isomorphic to  $[S \otimes (S/K)^{\otimes m}] \otimes [T \otimes (T/K)^{\otimes m}]$ . By Lemma A2

these two factors become direct summands in  $S^{\otimes m+1}$  and  $T^{\otimes m+1}$  respectively. Combining Lemma A2 and Proposition A1 we get

**COROLLARY A3.** *If  $K$  is a commutative ring, and  $S, T$  are  $K$ -algebras, then the map of left  $S$ -modules  $\coprod_{n \geq 1} S^{\otimes n} \otimes T^{\otimes n} \rightarrow S \amalg_K T$  given by  $s_1 \otimes \dots \otimes s_n \otimes t_1 \otimes \dots \otimes t_n \mapsto s_1 t_1 [s_2, t_2] \dots [s_n, t_n]$  splits. Hence if  $S, T$  are flat (respectively projective) as  $K$ -modules,  $S \amalg_K T$  will be flat (respectively projective) as a left  $S$ -module.*

(Remark. To get the above corollary, we did not need to know that the map  $\alpha$  in Proposition A1 was an isomorphism, only that it split as a map of left  $S$ -modules. This followed as soon as we had a left  $R$ -module structure on  $S \otimes T \otimes K\langle M \rangle$  such that  $\alpha$  was an  $R$ -module homomorphism. Thus so far as Corollary A3 is concerned we could have ended the proof of Proposition A1 before (A3).)

There are variants to the description of  $R$  as an  $S$ -module obtained in Proposition A1. For instance, we may take  $S \otimes K\langle M \rangle \otimes T$  and map it to  $R$  by  $s \otimes p \otimes t \mapsto s \tilde{p} t$  and show this to be an isomorphism of  $(S, T)$  bi-modules. It is clearly an  $(S, T)$  linear map so it suffices to show it is an isomorphism of left  $S$ -modules. This is quite messy to prove. Essentially we must again find the left  $T$ -module structure on our tensor product. This turns out to be given by the formula

$$\begin{aligned}
 \text{(A5)} \quad & t_0(s_0 \otimes \{s_1, t_1\} \dots \{s_n, t_n\} \otimes t_{n+1}) \\
 &= -1_S \otimes \{s_0, t_0\} \dots \{s_n, t_n\} \otimes t_{n+1} \\
 &\quad + s_0 \otimes \{s_1, t_0 t_1\} \{s_2, t_2\} \dots \{s_n, t_n\} \otimes t_{n+1} \\
 &\quad \cdot \quad \cdot \quad \cdot \\
 &\quad \pm s_0 \otimes \{s_1, t_0\} \dots \{s_{n-1}, t_{n-2}\} \{s_n, t_{n-1} t_n\} \otimes t_{n+1} \\
 &\quad \mp s_0 \otimes \{s_1, t_0\} \dots \{s_n, t_{n-1}\} \otimes t_n t_{n+1}.
 \end{aligned}$$

However, there is a less computational way to get this result from the one already proved. We first note that the idea behind (A5) is simply that in  $R$ , elements of  $T$  can be ‘moved past’ elements of  $[S, T]$ ; the explicit formula is

$$\text{(A6)} \quad t[s, t'] = [s, tt'] - [s, t]t',$$

which simply says that  $[s, -]$  is a derivation.

We now use (A6) to define a  $K$ -module isomorphism  $T \otimes_K M \simeq M \otimes_K T$ :

$$\begin{aligned}
 t \otimes \{s, t'\} &\mapsto \{s, tt'\} \otimes 1_T - \{s, t\} \otimes t', \\
 1_T \otimes \{s, tt'\} - t \otimes \{s, t'\} &\leftarrow \{s, t\} \otimes t'.
 \end{aligned}$$

Again the verification that these maps are well defined and mutually inverse is immediate. Thus,  $M \otimes_K T \simeq T \otimes_K M$  becomes a  $(T, T)$  bimodule which we name  $M_T$ . We then form the tensor algebra  $T\langle M_T \rangle$ ; because of (A6) this will have a  $T$ -ring homomorphism into  $R$  taking  $t$  to  $t$ , and  $\{s, t\}$  to  $[s, t]$ .

The general homogeneous component of this tensor algebra will have the form  $M_T \otimes_T \dots \otimes_T M_T$ . Now consider for simplicity the second degree term

$$\begin{aligned} (M \otimes_K T) \otimes_T (M \otimes_K T) &\simeq M \otimes_K (T \otimes_T (M \otimes_K T)) \\ &\simeq M \otimes_K (M \otimes_K T) \\ &\simeq M^{\otimes 2} \otimes_K T. \end{aligned}$$

(Caution: the parentheses are essential since  $M$  is not a left  $T$ -module.) Treating the  $n$ th degree term analogously we see that

$$T\langle M_T \rangle \simeq K\langle M \rangle \otimes_K T$$

as right  $T$ -modules (and this isomorphism respects our maps into  $R$ ). But likewise, using the identification  $M_T = T \otimes_K M$ , we get

$$T\langle M_T \rangle \simeq T \otimes_K K\langle M \rangle.$$

Hence

$$S \coprod_K T \simeq S \otimes T \otimes K\langle M \rangle \simeq S \otimes T\langle M_T \rangle \simeq S \otimes K\langle M \rangle \otimes T$$

as left  $S$ -modules, as desired. Thus  $S \coprod_K T \simeq S \otimes K\langle M \rangle \otimes T$  as  $(S, T)$  bimodules.

(*Exercise.* To indicate an opportunity to apply the results of this section we remark that recently W. Stephenson orally raised the question "If  $K$  is a commutative ring and  $R$  is a  $K$ -algebra, what is the centre of  $R \coprod_K K[X]$ ?" It is straightforward using (A5) to show that the centralizer of  $X$  is  $D[X]$  where  $D = \{r \in R \mid r \otimes 1 = 1 \otimes r \text{ in } R \otimes_K R\}$  (that is, the kernel of the universal  $K$ -derivation on  $R$ ), and hence that the centre is  $D + c[X]$ , where  $c = \{r \in R \mid rR \subseteq D\}$ . For example, for  $K = \mathbf{Z}$  and  $R = \mathbf{Z}[t, y]/(4t - 2, ty)$ , we have  $D = \mathbf{Z} + 2R, c = 2R$ .)

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