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Whitehead groups of localizations and the endomorphism class group

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Abstract

We compute the Whitehead groups of the associative rings in a class which includes (twisted) formal power series rings and the augmentation localizations of group rings and polynomial rings. For any associative ring A, we obtain an invariant of a pair (P, α), where P is a finitely generated projective A-module and $\alpha : P \to P$ is an endomorphism. This invariant determines (P, α) up to extensions, yielding a computation of the (reduced) endomorphism class group $\widetilde{\mathrm{End}}_0(A)$. We also refine the analysis by Pajitnov and Ranicki of the Whitehead group of the Novikov ring, a computation which Pajitnov has used in work on circle-valued Morse theory.

1. Introduction

1.1. Endomorphisms

The characteristic polynomial of an endomorphism of a vector space determines the endomorphism uniquely 'up to choices of extension.' To make such a statement precise, one makes the following definition (Almkvist [1–3], Grayson [12]) which we discussed in [31].

Definition 1.1. Let A be an associative ring. The reduced endomorphism class group $\widetilde{\mathrm{End}}_0(A)$ is the abelian group with one generator for each isomorphism class of pairs $[A^n, \alpha]$ and relations:

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- $[A^n, \alpha] + [A^{n''}, \alpha''] = [A^{n'}, \alpha']$ if there is an exact sequence

$$0 \to (A^n, \alpha) \to \big(A^{n'}, \alpha'\big) \to \big(A^{n''}, \alpha''\big) \to 0.$$

- $[A, 0] = 0$.

Almkvist proved [2] that if A is commutative then the characteristic polynomial induces an isomorphism

$$\begin{aligned} \widetilde{\mathrm{End}}_0(A) &\cong \left\{ \frac{1 + a_1 x + \cdots + a_n x^n}{1 + b_1 x + \cdots + b_m x^m} \,\middle|\, a_i, b_i \in A \right\}, \\ [A^n, \alpha] &\mapsto \det(1 - \alpha x). \end{aligned} \tag{1}$$

A goal of the present paper is to obtain an analogous statement for arbitrary associative rings.

We first reformulate the right-hand side of (1). Let $\epsilon : A[x] \to A$; $x \mapsto 0$ denote augmentation and let $P \subset A[x]$ denote the set of polynomials p such that $\epsilon(p)$ is invertible. There is a canonical factorization of ϵ through the ring of fractions $P^{-1}A[x]$

$$A[x] \rightarrowtail P^{-1}A[x] \xrightarrow{\epsilon_P} A$$

and Almkvist's result says that for any commutative ring A there is an isomorphism $\widetilde{\mathrm{End}}_0(A) \cong \epsilon_P^{-1}(1)$.

Dropping the assumption that A is commutative, let $A[x]$ denote the polynomial ring in a central indeterminate x. It is advantageous to make invertible not only elements but matrices in $A[x]$. Let Σ denote the set of matrices σ with entries in $A[x]$ such that $\epsilon(\sigma)$ is invertible. There exists a (formal) localization $A[x] \to \Sigma^{-1}A[x]$ (Cohn [8], Schofield [30, Chapter 4]), which is not in general a ring of fractions but has the following properties:

1. Every matrix $\sigma \in \Sigma$ is invertible over $\Sigma^{-1}A[x]$.
2. The map $A[x] \to \Sigma^{-1}A[x]$ is universal with respect to property 1. In other words, any map $A[x] \to B$ which makes all the matrices in Σ invertible factors uniquely through $\Sigma^{-1}A[x]$.

In particular $\epsilon : A[x] \to A$ is the composite $A[x] \rightarrowtail \Sigma^{-1}A[x] \xrightarrow{\epsilon_\Sigma} A$.

Theorem A. *Let A be an associative ring. There is an isomorphism*

$$\widetilde{\mathrm{End}}_0(A) \cong \epsilon_\Sigma^{-1}(1)/C; \quad [A^n, \alpha] \mapsto D(1 - \alpha x),$$

where C is the subgroup generated by commutators:

$$\big\{(1 + ab)(1 + ba)^{-1} \mid a, b \in A[x],\ \epsilon(ab) = \epsilon(ba) = 0\big\}.$$

The symbol D is defined in Section 4 (Definition 4.3) and is analogous to the Dieudonné determinant. In the commutative case, a matrix in Σ is invertible if and only if its determinant is invertible, so $\Sigma^{-1}A[x] = P^{-1}A[x]$. Moreover, C is trivial and D is the traditional determinant so Theorem A is a generalization of Almkvist's identity (1) above.

Suppose A is non-commutative. Now

$$\left[\epsilon_\Sigma^{-1}(1), \epsilon_\Sigma^{-1}(1)\right] \subset \left[\left(\Sigma^{-1}A[x]\right)^\bullet, \left(\Sigma^{-1}A[x]\right)^\bullet\right] \cap \epsilon_\Sigma^{-1}(1) \subset C,$$

where $(\Sigma^{-1}A[x])^\bullet$ denotes the group of units in $\Sigma^{-1}A[x]$, but neither of these inclusions is an equality in general and $[\epsilon_\Sigma^{-1}(1), \epsilon_\Sigma^{-1}(1)] \subset C$ is never an equality (see Section 3).

Our proof of Theorem A uses a result of A. Ranicki, which we state next. Recall that for an arbitrary ring A, the group $K_1(A) = \mathrm{GL}(A)^{\mathrm{ab}}$ is the abelianization of the group of invertible square matrices of arbitrary size. Ranicki established [26, Proposition 10.21] an isomorphism

$$K_1(A) \oplus \widetilde{\mathrm{End}}_0(A) \xrightarrow{\cong} K_1\left(\Sigma^{-1}A[x]\right),$$

which is the canonical inclusion of $K_1(A)$ and sends the element $[A^n, \alpha] \in \widetilde{\mathrm{End}}_0(A)$ to $[1 - \alpha x] \in K_1(\Sigma^{-1}A[x])$. We prove here that there is an isomorphism D between the image of $\widetilde{\mathrm{End}}_0(A)$ in $K_1(\Sigma^{-1}A[x])$ and $\epsilon_\Sigma^{-1}(1)/C$.

1.2. Local augmentations

Our main result is more general and concerns a class of homomorphisms $\epsilon : B \to A$ which we call 'local augmentations.' The word 'augmentation' just means split surjection or in other words retraction in the category of rings. By local we mean that a square matrix α with entries in B is invertible if $\epsilon(\alpha)$ is invertible.

Any augmentation $\epsilon : B \to A$ can be made a local augmentation $\Sigma^{-1}B \to A$ by adjoining a formal inverse to every square matrix σ with entries in B such that $\epsilon(\sigma)$ is invertible (see [31, Lemma 3.1] and Lemmas 2.4 and 2.9 below). In particular, the map $\epsilon_\Sigma : \Sigma^{-1}A[x] \to A$ above is a local augmentation. To give another example, if $A[\![x]\!]$ denotes the ring of formal power series in a central indeterminate x with coefficients in A then $A[\![x]\!] \to A$; $x \to 0$ is a local augmentation.

Theorem B (Main theorem). *If $\epsilon : B \to A$ is a local augmentation then there is a canonical isomorphism*

$$K_1(B) \cong K_1(A) \oplus \frac{\epsilon^{-1}(1)}{C},$$

where C is the subgroup generated by commutators:

$$\left\{(1 + ab)(1 + ba)^{-1} \mid a, b \in B,\ \epsilon(ab) = \epsilon(ba) = 0\right\}.$$

In particular, we may apply Theorem B to augmentation localizations of group rings. Suppose π is a group and $A\pi = A \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ is the corresponding group ring with coefficients in an associative ring A. Let $\epsilon : A\pi \to A$ be the augmentation, defined by $\epsilon(g) = 1$ for $g \in \pi$ and $\epsilon(a) = a$ for $a \in A$. Let Σ denote the set of matrices σ with entries in $A\pi$ such that $\epsilon(\sigma)$ is invertible; ϵ can be written as the composite $A\pi \to \Sigma^{-1}A\pi \xrightarrow{\epsilon_\Sigma} A$.

Corollary 1.2. $K_1(\Sigma^{-1}A\pi) \cong K_1(A) \oplus \epsilon_\Sigma^{-1}(1)/C$.

Corollary 1.2 will be applied in a subsequent paper, with $A = \mathbb{Z}$, to study Reidemeister torsion of *homology* equivalences between finite CW-complexes.

We may also apply Theorem B to the ring of formal power series in a central indeterminate:

$$K_1\big(A[\![x]\!]\big) \cong K_1(A) \oplus \frac{1 + A[\![x]\!]x}{C}. \tag{2}$$

It may be useful, if one is studying endomorphisms via Theorem A, to pass from the localization $\Sigma^{-1}A[x]$ to the completion $A[\![x]\!]$. The universal property of localization provides a canonical map $\gamma : \Sigma^{-1}A[x] \to A[\![x]\!]$—see Lemma 2.9(d)—but neither γ nor the induced map

$$\frac{\epsilon_\Sigma^{-1}(1)}{C} \to \frac{1 + A[\![x]\!]x}{C}$$

is an injection in general [31].

Generalizing (2), suppose A is an associative ring and $\xi : X \to \mathrm{Aut}(A)$ assigns a ring automorphism to each element of a set X of indeterminates. Let $A_\xi\langle\!\langle X\rangle\!\rangle$ denote the (twisted) power series ring whose elements are infinite formal sums, with one term for each word in the alphabet X. One may also impose relations, such as commutativity of the indeterminates, if compatible with ξ; see Example 2.10 below. □

Corollary 1.3. $K_1(A_\xi\langle\!\langle X\rangle\!\rangle) \cong K_1(A) \oplus (1 + A_\xi\langle\!\langle X\rangle\!\rangle X)/C$.

We remark that commutators of the form $(1 + ab)(1 + ba)^{-1}$ have appeared in earlier computations related to Whitehead groups (e.g., [6, p. 269], [16,32]). Whitehead groups of universal localizations appear in the K-theory exact sequences of Schofield [30] and Neeman and Ranicki [19] and certain Whitehead groups of localizations are computed in papers of Revesz [28], Ara, Goodearl, and Pardo [5], and Ara [4].

1.3. The Novikov ring

Corollary 1.3 refines a computation by Pajitnov and Ranicki [22], which was motivated by work of Pajitnov [25] on circle-valued Morse theory and gradient flow on manifolds. We briefly outline this application; background references include [7,17,18,20,21,23,24,27].

Let $A_\xi(\!(z)\!) = A_\xi[\![z]\!][z^{-1}]$ denote the (twisted) Novikov ring whose elements are power series with finitely many negative powers of z but, in general, infinitely many positive powers of z. Suppose M is a (closed compact) manifold, $f: M \to S^1$ is a Morse map and $v: M \to \tau_M$ is a generic choice of vector field, which is 'gradient-like' for f. One would like to describe the closed orbits of the associated flow on M.

Now if f induces a surjection $\pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ then

$$\pi_1(M) \cong \pi \rtimes_\xi \mathbb{Z} = \pi \rtimes_\xi \{z^n \mid n \in \mathbb{Z}\}$$

for some homomorphism $\xi: \mathbb{Z} \to \mathrm{Aut}(\pi)$ and the Novikov ring $\mathbb{Z}\pi_\xi(\!(z)\!)$ is a completion of the group ring $\mathbb{Z}\pi_1(M)$. Information about the closed orbits of the gradient flow v is encoded in the torsion

$$\tau(\phi_v) \in \frac{K_1(\mathbb{Z}\pi_\xi(\!(z)\!))}{\mathrm{Image}(\pm\pi_1 M)}$$

of a canonical chain equivalence (Novikov [20,21], Pajitnov [23–25])

$$\phi_v : \mathbb{Z}\pi_\xi(\!(z)\!) \otimes_{\mathbb{Z}\pi_1(M)} C^\Delta \to C^{\mathrm{Nov}}(v).$$

The symbol C^Δ denotes the chain complex over $\mathbb{Z}\pi_1(M)$ associated to a smooth triangulation of M. The 'Novikov complex' $C^{\mathrm{Nov}}(v)$ is a finitely generated free chain complex over $\mathbb{Z}\pi_\xi(\!(z)\!)$ with one basis element in C_i^{Nov} for each critical point of f of index i. The differential $d: C^{\mathrm{Nov}}(v)_i \to C^{\mathrm{Nov}}(v)_{i-1}$ counts (with signs) the number of flow lines from each critical point of index i to each critical point of index $i-1$ (in the universal cover of M). In contrast to real-valued Morse theory, the image in S^1 of a flow-line may 'wrap around the circle' any number of times, so the differentials in $C^{\mathrm{Nov}}(v)$ are formal power series in z.

Now the torsion $\tau(\phi_v)$ lies in the image of the canonical map $1 + \mathbb{Z}\pi_\xi[\![z]\!]z \to K_1(\mathbb{Z}\pi_\xi(\!(z)\!))$. For any A, let

$$W_1(A,\xi) = \mathrm{Image}\big(1 + A_\xi[\![z]\!]z \to K_1\big(A_\xi(\!(z)\!)\big)\big).$$

Pajitnov and Ranicki showed [22] that $W_1(A,\xi)$ is a summand of $K_1(A_\xi(\!(z)\!))$ and is naturally isomorphic to the image of $1 + A_\xi[\![z]\!]z$ in $K_1(A_\xi[\![z]\!])$. □

In the light of Corollary 1.3, we have:

Corollary 1.4. $W_1(A,\xi) \cong (1 + A_\xi[\![z]\!]z)/C$.

Pajitnov defined [25] a logarithm based on the standard formula

$$\mathcal{L}(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} \cdots,$$

which sends each element of $W_1(\mathbb{Z}\pi,\xi)$ to a formal power series with one term for each conjugacy class β of $\pi_1(M)$ such that $f(\beta) \geqslant 0$. He showed that the coefficient of β in $\mathcal{L}(\tau(\phi_v))$ is the number of closed orbits of the flow, counted with signs, in the class β.

This paper is structured as follows: In Section 2 we discuss local homomorphisms and augmentations and we define $K_1(A)$ and $\widetilde{\mathrm{End}}_0(A)$. Section 3 concerns the group C of commutators, which appears in Theorems A and B. Section 4 is devoted to the proofs of Theorems A and B. $\square$

2. Definitions and examples

Rings will be assumed associative with multiplicative unit. The set $\{0\}$ will be considered a ring, in which $1 = 0$, but will not be considered a field.

2.1. Local homomorphisms

Definition 2.1. A ring homomorphism $f : B \to A$ will be called *local* if every square matrix α with entries in B has the following property:

$$\text{If} \quad f(\alpha) \text{ is invertible} \quad \text{then} \quad \alpha \text{ is invertible.}$$

Equivalently, f is local if it has the property that if $\alpha : P \to P$ is an endomorphism of a finitely generated (f.g.) projective B-module and $1 \otimes \alpha : A \otimes_B P \to A \otimes_B P$ is invertible then $\alpha : P \to P$ is invertible.

For example, for any ring B the surjection $B \to B/\mathrm{rad}(B)$ is a local homomorphism (e.g., Lam [15, Proposition 4.8 and (7) on p. 57]).

Formal power series provide another example. If A is any associative ring let $A[x]$ denote the polynomial ring in a central indeterminate x. The augmentation $\epsilon : A[x] \to A$ given by $\epsilon|_A = \mathrm{id}_A$ and $\epsilon(x) = 0$ induces local homomorphisms $A[x]/(x^n) \to A$ and a local homomorphism $\hat{\epsilon} : A[\![x]\!] \to A$; see Lemma 2.5 below.

It is easy to check that a composite of two local homomorphisms is again local and that if a composite $C \xrightarrow{g} B \xrightarrow{f} A$ is local then g is local.

Although neither A nor B are local rings in most of our examples, we consider homomorphisms between local rings in our first two lemmas to prove that Definition 2.1 is consistent with terminology used in algebraic geometry (see, for example, Hartshorne [13, p. 73]) and by Cohn [9, p. 388]. Indeed, Lemmas 2.2 and 2.3 together imply that a homomorphism $f : B \to A$ between local rings is a local homomorphism if and only if $f^{-1}(\mathrm{rad}(A)) = \mathrm{rad}(B)$.

Lemma 2.2. *Suppose $f : B \to A$ is a ring homomorphism. If B is a local ring, $A \neq 0$, and $f^{-1}(\mathrm{rad}(A)) = \mathrm{rad}(B)$ then f is a local homomorphism.*

The hypotheses of the lemma are not redundant. For example, if f is the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ then $f^{-1}(\mathrm{rad}(\mathbb{Q})) = 0 = \mathrm{rad}(\mathbb{Z})$ but f is not a local homomorphism. On the other hand, if $\mathbb{Z}_{(p)}$ denotes the local ring obtained from $\mathbb{Z}$ by making invertible all the integers not divisible by the prime p and f is the inclusion of $\mathbb{Z}_{(p)}$ in $\mathbb{Q}$ then $f^{-1}(\mathrm{rad}(\mathbb{Q})) \neq \mathrm{rad}(\mathbb{Z}_{(p)})$ and f is not a local homomorphism. The following proof was pointed out to me by P. Ara.

Proof of Lemma 2.2. Suppose first that B is a division ring so the homomorphism $f: B \to A$ is an injection (since $A \neq 0$). Every homomorphism $B^n \to B^n$ is either an isomorphism or has non-zero kernel, so every square matrix α with entries in B is either invertible or a zero-divisor. It follows that α is invertible in A if and only if α is invertible in B and hence f is local.

Now if B is a local ring consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ \frac{B}{\mathrm{rad}(B)} \arrow[r] & \frac{A}{\mathrm{rad}(A)}. \end{tikzcd}$$

Since $B/\mathrm{rad}(B)$ is a division ring, the lower horizontal arrow is a local homomorphism by the argument above. The vertical arrows are local homomorphisms, so it follows that f is a local homomorphism. □

Lemma 2.3. *If $f: B \to A$ is a local homomorphism then $f^{-1}(\mathrm{rad}(A)) \subset \mathrm{rad}(B)$. If, in addition, A is a local ring then $f^{-1}(\mathrm{rad}(A)) = \mathrm{rad}(B)$ and B is a local ring.*

We remark that not every local homomorphism f has $f^{-1}(\mathrm{rad}(A)) = \mathrm{rad}(B)$. For example, if k is a field, the inclusion of the formal power series ring $k[\![x]\!]$ in the polynomial extension $k[\![x]\!][y]$ is a local homomorphism but

$$f^{-1}\big(\mathrm{rad}\big(k[\![x]\!][y]\big)\big) = f^{-1}(0) = 0 \neq xk[\![x]\!] = \mathrm{rad}\big(k[\![x]\!]\big).$$

Proof of Lemma 2.3. An element $x \in B$ lies in $\mathrm{rad}(B)$ if and only if $1 + bxb'$ is invertible for all $b, b' \in B$ (e.g., Lam [15, Lemma 4.3]). If $x \in f^{-1}(\mathrm{rad}(A))$ then $f(x) \in \mathrm{rad}(A)$, so $f(1 + bxb') = 1 + f(b)f(x)f(b')$ is invertible. Since f is local, it follows that $1 + bxb'$ is invertible for all $b, b' \in B$ and hence that $x \in \mathrm{rad}(B)$. Thus $f^{-1}(\mathrm{rad}(A)) \subset \mathrm{rad}(B)$.

If A is a local ring and f is a local homomorphism then $x \in B$ is invertible if and only if $x \notin f^{-1}(\mathrm{rad}(A))$, so B is a local ring and $f^{-1}(\mathrm{rad}(A)) = \mathrm{rad}(B)$. □

The next lemma says that any ring homomorphism can be made local in a universal way:

Lemma 2.4. *Suppose $f: B \to A$ is a ring homomorphism. There is an initial object*

$$B \xrightarrow{i_\Sigma} \Sigma^{-1}B \xrightarrow{f_\Sigma} A \tag{3}$$

in the category of diagrams $B \xrightarrow{i} B' \xrightarrow{f'} A$ such that $f'i = f$ and f' is local.

In other words, if $f : B \to A$ is the composite $B \xrightarrow{i} B' \xrightarrow{f'} A$ and f' is local then there is a unique commutative diagram

$$\begin{tikzcd} & \Sigma^{-1}B \arrow[dd, "\gamma"] \arrow[dr, "f_\Sigma"] & \\ B \arrow[ur, "i_\Sigma"] \arrow[dr, "i"'] & & A. \\ & B' \arrow[ur, "f'"'] & \end{tikzcd} \tag{4}$$

Since it is an initial object, (3) is certainly unique. The existence of (3) follows from the universal localization of rings: given any set Σ of matrices with entries in B, one can adjoin formal inverses to each matrix in Σ to obtain a map $i_\Sigma : B \to \Sigma^{-1}B$ (see Cohn [8, Chapter 7] and Schofield [30, Chapter 4]). Now i_Σ is Σ-inverting in the sense that, for every matrix $\sigma \in \Sigma$, the image $i_\Sigma(\sigma)$ is invertible. Moreover, i_Σ is characterized as the initial object in the category of Σ-inverting homomorphisms $B \to B'$. In other words, every Σ-inverting homomorphism $B \to B'$ factors through i_Σ in a unique way. We remark that i_Σ is an epimorphism, i.e. $f i_\Sigma = g i_\Sigma$ implies $f = g$.

Proof of Lemma 2.4. Given a ring homomorphism $f : B \to A$ let Σ be the set of A-invertible matrices with entries in B. In other words, let $\sigma \in \Sigma$ if and only if $f(\sigma)$ is invertible. Now f is Σ-inverting, and hence factors uniquely through $\Sigma^{-1}B$, i.e. $f = f_\Sigma i_\Sigma$, where

$$B \xrightarrow{i_\Sigma} \Sigma^{-1}B \xrightarrow{f_\Sigma} A. \tag{5}$$

It is proved in [31, Lemma 3.1] that f_Σ is a local homomorphism.

To see that (5) has the universal property illustrated in (4), suppose $f = f'i$, where $f' : B' \to A$ is local and $i : B \to B'$. If $\sigma \in \Sigma$ then $f(\sigma)$ is invertible and, because f' is local, $i(\sigma)$ is invertible. It follows that there is a unique map $\gamma : \Sigma^{-1}B \to B'$ such that $i = \gamma i_\Sigma$. Now $f_\Sigma i_\Sigma = f'i = f'\gamma i_\Sigma$ and i_Σ is an epimorphism, so we also have $f_\Sigma = f'\gamma$. $\square$

One can obtain further examples of local homomorphisms by limit constructions. For example, a product of local homomorphisms or an inverse limit of local homomorphisms is again a local homomorphism. To make a general statement, Lemma 2.5 below, let us briefly recall the notion of limit in category theory. Suppose $F : J \to \mathcal{C}$ is a functor from a small category J to a category $\mathcal{C}$. If M is an object of $\mathcal{C}$ let $c_M : J \to \mathcal{C}$ denote the constant functor which sends every object of J to M and every morphism to id_M. By definition, a limit of F is a final object in the category of pairs (M, θ), where M is an object of $\mathcal{C}$ and θ is a natural transformation from c_M to F. The category $\mathcal{C}$ is said to be *complete* if every functor $F : J \to \mathcal{C}$, where J is small, has a limit.

The category of rings, for example, is complete (recall that we consider {0} a ring). Of interest here is the category in which an object is a ring homomorphism and a morphism from $f: B \to A$ to $f': B' \to A'$ is a commutative square:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ B' \arrow[r, "f'"'] & A'. \end{tikzcd}$$

This homomorphism category is also complete. Although the category of local rings is not complete, one has

Lemma 2.5. *The category of local homomorphisms is complete.*

The proof of Lemma 2.5 is not difficult; it suffices to check that equalizers and arbitrary products of local homomorphisms are again local. The details are left to the reader.

Dually, one can attempt to construct examples of local homomorphisms by colimit constructions. However, the category of local homomorphisms is *not* cocomplete. For example, the coproduct of two copies of the local homomorphism $\mathbb{Z}[x]/(x^2) \to \mathbb{Z}$; $x \mapsto 0$ is not local. On the positive side, the reader can check that a direct colimit (often called a direct limit) of local homomorphisms is a local homomorphism:

Lemma 2.6. *Let J be a directed set. If $(\{f_j : B_j \to A_j\}_{j\in J}, \{\theta_j^k : f_j \to f_k\}_{j\leqslant k})$ is a direct system of local homomorphisms then the colimit*

$$\varinjlim f_j : \varinjlim B_j \to \varinjlim A_j$$

is a local homomorphism.

2.2. Local augmentations

The word ‘augmentation’ is synonymous with ‘split surjection’ or ‘retraction:’ □

Definition 2.7. An *augmentation* (ϵ, j) is a pair of ring homomorphisms $\epsilon : B \to A$ and $j : A \to B$ such that $\epsilon j = \mathrm{id}_A$.

The equation $\epsilon j = \mathrm{id}_A$ implies that B can be expressed as a direct sum $j(A) \oplus \mathrm{Ker}(\epsilon)$ of (A, A)-bimodules. We shall usually suppress j, regarding A as a subset of B. Note that the category of augmentations is both complete and cocomplete. In particular, a direct or inverse limit of augmentations is again an augmentation. A ring homomorphism which is both local and an augmentation will be called a local augmentation.

Lemma 2.8. *If $\epsilon : B \to A$ is an augmentation and $I = \mathrm{Ker}(\epsilon)$ satisfies $I^n = 0$ for some $n \in \mathbb{N}$ then ϵ is a local augmentation.*

Proof. Suppose α is a square matrix with entries in B and $\epsilon(\alpha)$ is invertible. The matrix $\alpha_0 = \epsilon(\alpha)^{-1}\alpha - 1$ has entries in I so $1+\alpha_0$ has inverse $1-\alpha_0+\alpha_0^2-\cdots+(-1)^{n-1}\alpha_0^{n-1}$. Thus α is invertible and

$$\alpha^{-1} = \left(1-\alpha_0+\alpha_0^2-\cdots+(-1)^{n-1}\alpha_0^{n-1}\right)\epsilon(\alpha)^{-1}. \qquad \square$$

Lemma 2.9 (Localization and completion). *Suppose $\epsilon : B \to A$ is an augmentation with $\epsilon j = \mathrm{id}_A$.*

(a) *The universal localization $\epsilon_\Sigma : \Sigma^{-1}B \to A$ is a local augmentation.*
(b) *Let $I = \mathrm{Ker}(\epsilon)$ and let $\widehat{B} = \varprojlim B/I^n$ denote the I-adic completion of B. For each n, the induced map $\epsilon_n : B/I^n \to A$ is a local augmentation. The inverse limit $\widehat{\epsilon} : \widehat{B} \to A$ is also a local augmentation.*
(c) *There are natural isomorphisms $\Sigma^{-1}(\Sigma^{-1}B) \cong \Sigma^{-1}B$; $\Sigma^{-1}\widehat{B} \cong \widehat{B}$; $\widehat{\widehat{B}} \cong \widehat{B}$; $\widehat{\Sigma^{-1}B} \cong \widehat{B}$.*
(d) *There is a natural homomorphism $\gamma : \Sigma^{-1}B \to \widehat{B}$ such that $\widehat{\epsilon}\gamma = \epsilon_\Sigma$.*

Proof. (a) By [31, Lemma 3.1], we need only check that ϵ_Σ is an augmentation. Indeed, $i_\Sigma j : A \to \Sigma^{-1}B$ has the property $\epsilon_\Sigma i_\Sigma j = \epsilon j = \mathrm{id}_A$.

(b) Since an inverse limit of local augmentations is a local augmentation, it suffices to show that $\epsilon_n : B/I^n \to A$ is a local augmentation. The composite $A \xrightarrow{j} B \to B/I^n \xrightarrow{\epsilon_n} A$ is the identity morphism, so ϵ_n is an augmentation. By Lemma 2.8, ϵ_n is a local homomorphism.

(c) The first two identities merely reassert that ϵ_Σ and $\hat{\epsilon}$ are local. Now define $\hat{I} = \mathrm{Ker}(\hat{\epsilon} : \widehat{B} \to A)$ and let $I_\Sigma = \mathrm{Ker}(\epsilon_\Sigma : \Sigma^{-1}B \to A)$. The last two identities follow respectively from natural isomorphisms $\widehat{B}/\hat{I}^n \cong B/I^n$ and $\Sigma^{-1}B/I_\Sigma^n \cong B/I^n$.

(d) There are natural maps $\Sigma^{-1}B \to \widehat{\Sigma^{-1}B} \to \widehat{B}$. $\square$

Example 2.10. Let X be any set and let X^* denote the free monoid of words in the alphabet X. Suppose $\xi : X \to \mathrm{Aut}(A)$; $x \mapsto \xi_x$ is any function which takes values in the group of ring automorphisms of A. Let $A_\xi\langle X\rangle = A_\xi[X^*]$ denote the twisted monoid ring of finite formal sums $\sum_{w\in X^*} a_w w$ with each $a_w \in A$. Multiplication is defined by concatenation of words and by the relations $ax = x\xi_x(a)$ for $a \in A$ and $x \in X$. Let $\epsilon : A_\xi\langle X\rangle \to A$ be the augmentation, which sends every letter in X to zero. By Lemma 2.9(b), there is a local augmentation $\hat{\epsilon} : A_\xi\langle\langle X\rangle\rangle \to A$, where $A_\xi\langle\langle X\rangle\rangle$ is the formal power series ring whose elements are infinite (formal) sums $\sum_{w\in X^*} a_w w$. More generally, if $H \triangleleft A_\xi\langle X\rangle$ is a (two-sided) ideal and $H \subset I = \mathrm{Ker}(\epsilon)$ then there is a local augmentation

$$\varprojlim_n \frac{A_\xi\langle X\rangle}{I^n + H} \to A.$$

2.3. $\widetilde{\mathrm{End}}_0(A)$ *and* $K_1(A)$

Let A be any associative ring. Let $\underline{\mathrm{End}}(A)$ denote the category whose objects are pairs (P, α), where P is a finitely generated (f.g.) projective left A-module and $\alpha : P \to P$ is an endomorphism. A morphism $(P,\alpha)\to(Q,\beta)$ in $\underline{\mathrm{End}}(A)$ is a map $f:P\to Q$ such that $\beta f=f\alpha$.

Definition 2.11. Let $\widetilde{\mathrm{End}}_0(A)$ be the abelian group with one generator $[P,\alpha]$ for each isomorphism class of objects (P,α) in $\underline{\mathrm{End}}(A)$ and relations:

(1) $[P',\alpha']=[P,\alpha]+[P'',\alpha'']$ if there is an exact sequence $0\to(P,\alpha)\to(P',\alpha')\to(P'',\alpha'')\to 0$ in $\underline{\mathrm{End}}(A)$,
(2) $[P,0]=0$ for all f.g. projective modules P.

Definition 2.11 is consistent with Definition 1.1; see [31, Lemma 2.4]. To define $K_1(A)$, let $\underline{\mathrm{Aut}}(A)$ denote the full subcategory of $\underline{\mathrm{End}}(A)$ whose objects are pairs (P,α) such that α is an automorphism.

Definition 2.12. The abelian group $K_1(A)$ is generated by the isomorphism classes $[P,\alpha]$ in $\underline{\mathrm{Aut}}(A)$ subject to relations:

(1) $[P\oplus P',\alpha\oplus\beta]=[P,\alpha]+[P,\beta]$,
(2) $[P,\alpha\beta]=[P,\alpha]+[P,\beta]$.

The group $K_1(A)$ is also unchanged if one replaces the projective modules in Definition 2.12 by free modules throughout. Indeed, relation (2) implies that $[P,1_P]=[P,1_P]+[P,1_P]$ and hence $[P,1_P]=0\in K_1(A)$ for all P. It follows that $[P,\alpha]$ can be identified with $[P\oplus Q,\alpha\oplus 1_Q]$, where $P\oplus Q$ is free.

There is a natural isomorphism between $K_1(A)$ and $\mathrm{GL}(A)^{\mathrm{ab}}$, the abelianization of the direct limit $\mathrm{GL}(A)=\varinjlim \mathrm{GL}_n(A)$ of general linear groups over A (e.g., [32, p. 109], [29, Theorem 3.1.7]). The commutator subgroup $[\mathrm{GL}(A),\mathrm{GL}(A)]$ is generated by the 'elementary' matrices $b_{ij}(a)=[b_{ik}(a),b_{kj}(1)]$, where $b_{ij}(a)$ is the matrix whose ijth entry is a, whose diagonal entries are 1 and whose other entries are 0.

3. The commutator group C

Suppose $\epsilon:B\to A$ is a local augmentation. In this section, we study the group

$$C=\big\langle(1+ab)(1+ba)^{-1}\;\big|\;\epsilon(ab)=\epsilon(ba)=0\big\rangle\subset\epsilon^{-1}(1).$$

A consequence of Proposition 3.4 is that $[\epsilon^{-1}(1),\epsilon^{-1}(1)]\subset C$, a fact we shall need in the proof of Theorem B. Although $[\epsilon^{-1}(1),\epsilon^{-1}(1)]\neq C$ in general, the following lemma says that the image of C in $K_1(B)$ is trivial.

Lemma 3.1. *Suppose B is any ring and $a,b\in B$. The element $1+ab$ is invertible if and only if $1+ba$ is invertible. Moreover, if $1+ab$ and $1+ba$ are invertible then $[1+ab]=[1+ba]\in K_1(B)$.*

Proof. One can check that $\begin{pmatrix}1&-a\\0&1\end{pmatrix}\begin{pmatrix}1+ab&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\b&1\end{pmatrix}=\begin{pmatrix}1&0\\b&1\end{pmatrix}\begin{pmatrix}1&0\\0&1+ba\end{pmatrix}\begin{pmatrix}1&-a\\0&1\end{pmatrix}$. $\square$

Notation 3.2. Let $B^\bullet$ and $A^\bullet$ denote the groups of units in B and A, respectively. Suppose S is a subset of $B \times B$ with the property that $1 + ba \in B^\bullet$ for all $(a, b) \in S$. Let $C(S)$ denote the intersection of $\epsilon^{-1}(1)$ with the group generated by $\{(1 + ab)(1 + ba)^{-1} \mid (a, b) \in S\}$. In symbols:

$$C(S) = \epsilon^{-1}(1) \cap \big\langle (1 + ab)(1 + ba)^{-1} \mid (a, b) \in S \big\rangle.$$

We usually describe S in terms of equations or conditions on a and b. For example, the commutator group in Theorem B is $C = C(\epsilon(ab) = \epsilon(ba) = 0)$. Note that if $S^{\mathrm{T}} = \{(a, b) \mid (b, a) \in S\}$ then $C(S) = C(S^{\mathrm{T}}) = C(S \cup S^{\mathrm{T}})$, because $(1 + ab)(1 + ba)^{-1} = ((1 + ba)(1 + ab)^{-1})^{-1}$. The following fact was attributed to L. Vaserstein by V. Srinivas [33, p. 5].

Lemma 3.3. *If $1 + ac$ and $1 + ba$ are invertible and $ac = ca$ then $(1 + ab)(1 + ba)^{-1} = (1 + a(b + c + bac))(1 + (b + c + bac)a)^{-1}$.*

Proof. Observe that $(1 + ab)(1 + ac) = 1 + a(b + c + bac)$ and $(1 + ba)(1 + ca) = 1 + (b + c + bac)a$. $\square$

Proposition 3.4. *Let $\epsilon : B \to A$ be a local augmentation.*

(1) $C(\epsilon(a) = 0) = C(\epsilon(ab) = \epsilon(ba) = 0) = C(\epsilon(ba) = 0) = C(1 + ba \in B^\bullet)$.
(2) $[B^\bullet, B^\bullet] \cap \epsilon^{-1}(1) = C(b \in B^\bullet; 1 + ba \in B^\bullet)$.
(3) $[\epsilon^{-1}(1), \epsilon^{-1}(1)] = C(\epsilon(a) = 0; \epsilon(b) = \zeta)$ *for any $\zeta \in A$ which commutes with every element of $I = \mathrm{Ker}(\epsilon)$.*

In particular,

$$\big[\epsilon^{-1}(1), \epsilon^{-1}(1)\big] \subset [B^\bullet, B^\bullet] \cap \epsilon^{-1}(1) \subset C = C\big(\epsilon(ab) = \epsilon(ba) = 0\big) \tag{6}$$

and C is a normal subgroup of $\epsilon^{-1}(1)$ with abelian quotient.

Statement (1) will follow from the proof of Theorem B in Section 4. We do not use (1) in the proof of (2) or (3).

Proof of statement (2). (Compare Silvester [32, p. 135].) Suppose $\alpha, \beta \in B^\bullet$. Define $a = \alpha\beta - \alpha$ and $b = \alpha^{-1}$ so that $(1 + ab)(1 + ba)^{-1} = \alpha\beta\alpha^{-1}\beta^{-1}$. Now $b \in B^\bullet$ and $1 + ba = \beta \in B^\bullet$, so $[B^\bullet, B^\bullet] \cap \epsilon^{-1}(1) \subset C(b \in B^\bullet; 1 + ba \in B^\bullet)$.

Inversely, if $a, b \in B$ with $b \in B^\bullet$ and $1 + ba \in B^\bullet$, one can set $\alpha = b^{-1}$ and $\beta = 1 + ba$ to obtain $\alpha\beta\alpha^{-1}\beta^{-1} = (1 + ab)(1 + ba)^{-1}$. $\square$

Proof of statement (3). We continue to use the notation of the preceding proof. Observe that $\epsilon(\alpha) = \epsilon(\beta) = 1$ if and only if $\epsilon(a) = 0$ and $\epsilon(b) = 1$. Thus $[\epsilon^{-1}(1), \epsilon^{-1}(1)] = C(\epsilon(a) = 0; \epsilon(b) = 1)$. It remains to prove that

$$C\big(\epsilon(a) = 0;\ \epsilon(b) = 1\big) = C\big(\epsilon(a) = 0;\ \epsilon(b) = \zeta\big).$$

If $\epsilon(a)=0$ and $\epsilon(b)=1$, put $c=\zeta-1$ so that $\epsilon(b+c+bac)=\zeta$. Lemma 3.3 implies that $C(\epsilon(a)=0;\epsilon(b)=1)\subset C(\epsilon(a)=0;\epsilon(b)=\zeta)$. Conversely, if $\epsilon(a)=0$ and $\epsilon(b)=\zeta$, put $c=1-\zeta$ so that $\epsilon(b+c+bac)=1$ and Lemma 3.3 implies that $C(\epsilon(a)=0;\epsilon(b)=\zeta)\subset C(\epsilon(a)=0;\epsilon(b)=1)$. □

Using Lemma 3.3, one can show that the second inclusion of (6) is an equality in certain cases of interest (e.g. if A is a local ring). However, neither inclusion is an equality in general as the following example illustrates.

Example 3.5. Suppose $B=A[\![x]\!]$, where x is a central indeterminate, and $\epsilon: A[\![x]\!]\to A$; $x\mapsto 0$. Pajitnov and Ranicki observed [22, §3.2] that the canonical homomorphism $\epsilon^{-1}(1)^{\mathrm{ab}}\to K_1(B)$ need not be injective. In fact, whenever there exist elements $a,b\in A$ such that $ab\neq ba$ one finds that

$$\bigl(1+a(bx)\bigr)\bigl(1+(bx)a\bigr)^{-1}=1+(ab-ba)x+\cdots$$

is an element of C but is not in $[\epsilon^{-1}(1),\epsilon^{-1}(1)]$. Furthermore, if one can choose $a\in A^\bullet$ and $b\in B$ with $ab\neq ba$ then

$$a(1+bx)a^{-1}(1+bx)^{-1}=1+\bigl(aba^{-1}-b\bigr)x+\cdots$$

lies in $[B^\bullet,B^\bullet]\cap\epsilon^{-1}(1)$, but not in $[\epsilon^{-1}(1),\epsilon^{-1}(1)]$. On the other hand, if say $A=\mathbb{Z}\langle y,z\rangle$ is the free associative ring on two generators then $A^\bullet=\{\pm 1\}$, so $B^\bullet=\pm(1+A[\![x]\!]x)$ and $[B^\bullet,B^\bullet]\subset 1+A[\![x]\!]x^2$. Putting $a=xz$ and $b=y$ we have

$$(1+ab)(1+ba)^{-1}=1+(zy-yz)x+\cdots,$$

so $(1+ab)(1+ba)^{-1}$ lies in C, but not in $[B^\bullet,B^\bullet]$.

It follows from Lemma 2.9(d), Proposition 3.4, and Example 3.5 that, in the context of Theorem A, we have $[\epsilon_\Sigma^{-1}(1),\epsilon_\Sigma^{-1}(1)]\subsetneq C$ in $\Sigma^{-1}A[x]$ whenever A is non-commutative.

4. Proof of Theorems A and B

Suppose $\epsilon: B\to A$ is a local homomorphism and $j: A\to B$ satisfies $\epsilon j=\mathrm{id}_A$. We use the same symbols ϵ and j to denote the functors $A\otimes_B \_$ and $B\otimes_A \_$ and the induced maps $K_1(B)\to K_1(A)$ and $K_1(A)\to K_1(B)$. Since $\epsilon j=\mathrm{id}_A$ induces the identity on $K_1(A)$, we have a decomposition

$$K_1(B)=K_1(A)\oplus\widetilde{K}_1(B),$$

where, by definition, $\widetilde{K}_1(B) = \mathrm{Ker}(\epsilon : K_1(B) \to K_1(A))$. To prove Theorem B, we must show that $\widetilde{K}_1(B)$ is isomorphic to $\epsilon^{-1}(1)/C$, where C is the subgroup of $\epsilon^{-1}(1)$ generated by the subset

$$\{(1+ab)(1+ba)^{-1} \mid a, b \in B,\ \epsilon(ab) = \epsilon(ba) = 0\}. \tag{7}$$

We shall continue to write the group operation multiplicatively in $\epsilon^{-1}(1)/C$, but additively in $K_1(B)$.

We first deduce Theorem A from Theorem B.

Proof of Theorem A. Recall that $\epsilon : A[x] \to A$; $x \mapsto 0$ and Σ denotes the set of matrices σ with entries in $A[x]$ such that $\epsilon(\sigma)$ is invertible. The universal localization $\epsilon_\Sigma : \Sigma^{-1}A[x] \to A$ is a local augmentation by Lemma 2.9(a). Ranicki showed [26, Proposition 10.21] that there is an isomorphism

$$\widetilde{\mathrm{End}}_0(A) \cong \widetilde{K}_1\bigl(\Sigma^{-1}A[x]\bigr); \quad [P, \alpha] \mapsto [1 - \alpha x].$$

By Theorem B, the map $\widetilde{\mathrm{End}}_0(A) \to \epsilon_\Sigma^{-1}(1)/C$; $[P, \alpha] \mapsto D(1 - \alpha x)$ is also an isomorphism. □

Outline proof of Theorem B. The proof of Theorem B is analogous to the proof of the identity $K_1(k) \cong k^\bullet$, where k is a (commutative) field. The latter is proved by observing that the determinant

$$\mathrm{GL}(k) \to k^\bullet, \quad \alpha \mapsto \det(\alpha)$$

induces a map $\det : K_1(k) \to k^\bullet$, which is inverse to the canonical map from $k^\bullet$ to $K_1(k)$.

In a non-commutative setting, this traditional determinant is not defined. By Lemma 3.1 there is a canonical map $\epsilon^{-1}(1)/C \to \widetilde{K}_1(B)$; we shall construct an inverse

$$D : \widetilde{K}_1(B) \to \epsilon^{-1}(1)/C,$$

which is a version of the Dieudonné determinant. The idea is that the class in $K_1(B)$ represented by a given invertible matrix is unchanged when one performs elementary operations, adding a (left) multiple of one row to another row, or adding a (right) multiple of one column to another column. If $[\alpha] \in \widetilde{K}_1(B)$ then α can be reduced to a diagonal matrix by these row and column operations. One can then define $D(\alpha)$ to be the product of the diagonal entries in this diagonal matrix. Modulo appropriate relations, which turn out to be C, the determinant D is well-defined. □

We treat the preceding proof in more detail below showing in fact that D is well-defined modulo $C_0 = C(\epsilon(a) = 0) = \langle (1+ab)(1+ba)^{-1} \mid \epsilon(a) = 0\rangle$. Let us now deduce that the various definitions of C in Proposition 3.4(1) are identical:

Proof of Proposition 3.4(1). It is immediate that

$$C_0 = C\big(\epsilon(a)=0\big) \subset C = C\big(\epsilon(ab)=\epsilon(ba)=0\big) \subset C\big(\epsilon(ba)=0\big) \subset C(1+ba \in B^\bullet). \quad (8)$$

By Lemma 3.1, $C(1+ba \in B^\bullet)$ vanishes in $\widetilde{K}_1(B)$, so the composite

$$\frac{\epsilon^{-1}(1)}{C_0} \twoheadrightarrow \frac{\epsilon^{-1}(1)}{C(1+ba \in B^\bullet)} \to \widetilde{K}_1(B) \to \frac{\epsilon^{-1}(1)}{C_0}$$

is the identity. Hence $C_0 = C(1+ba \in B^\bullet)$ and all the inclusions in (8) are equalities. In particular, $C_0 = C$. $\square$

We begin now a detailed proof that $D: \widetilde{K}_1(B) \to \epsilon^{-1}(1)/C_0$ is well-defined. Let $\widetilde{\underline{\mathrm{Aut}}}(B)$ denote the full subcategory of $\underline{\mathrm{End}}(B)$, whose objects are pairs (P,α) such that $\alpha : P \to P$ is an automorphism and $\epsilon(\alpha) = 1$.

Lemma 4.1. *$\widetilde{K}_1(B)$ is (isomorphic to) the abelian group generated by isomorphism classes $[P,\alpha]$ of automorphisms $(P,\alpha) \in \widetilde{\underline{\mathrm{Aut}}}(B)$, subject to the following relations*:

(1) $[P \oplus P', \alpha \oplus \beta] = [P,\alpha] + [P',\beta]$.
(2) $[P, \alpha\beta] = [P,\alpha] + [P,\beta]$.
(3) *If $(P,c) = j(P_0,c_0) \in \underline{\mathrm{Aut}}(B)$ and $(P,\alpha) \in \widetilde{\underline{\mathrm{Aut}}}(B)$ then $[P, c\alpha c^{-1}] = [P,\alpha]$.*

In Lemma 4.1, just as in Definition 2.12, one may replace f.g. projective modules by f.g. free modules throughout. We usually abbreviate $[P,\alpha]$ to $[\alpha]$.

Proof of Lemma 4.1. Let T denote the abelian group with the generators and relations given in Lemma 4.1. There is an obvious homomorphism $T \to K_1(B)$ and the composite $T \to K_1(B) \xrightarrow{\epsilon} K_1(A)$ is zero, so the image of T lies in $\widetilde{K}_1(B)$. Conversely, given a generator $[\alpha] \in K_1(B)$, we may write $\alpha = c\tilde{\alpha}$, where $c = j\epsilon(\alpha)$ and $\tilde{\alpha} = c^{-1}\alpha$ has the property $\epsilon(\tilde{\alpha}) = 1$. We can define a map

$$\theta : K_1(B) \to T; \quad [\alpha] \mapsto [\tilde{\alpha}] = \big[c^{-1}\alpha\big].$$

Plainly $\theta[\alpha \oplus \beta] = \theta[\alpha] + \theta[\beta]$. We must check also that $\theta[\alpha\beta] = \theta[\alpha] + \theta[\beta]$. Indeed, if $c = \epsilon(\alpha)$ and $d = \epsilon(\beta)$ then $\theta[\alpha\beta] = \theta[c\tilde{\alpha}d\tilde{\beta}] = \theta[(cd)d^{-1}\tilde{\alpha}d\tilde{\beta}] = [d^{-1}\tilde{\alpha}d\tilde{\beta}] = [d^{-1}\tilde{\alpha}d] + [\tilde{\beta}]$, so by relation (3) we have $\theta[\alpha\beta] = [\tilde{\alpha}] + [\tilde{\beta}] = \theta[\alpha] + \theta[\beta]$. Plainly the restriction of θ to $\widetilde{K}_1(B)$ is inverse to the canonical map $T \to \widetilde{K}_1(B)$. $\square$

It is easy to define a map $i : \epsilon^{-1}(1) \to \widetilde{K}_1(B)$. Indeed, a unit $\alpha \in B^\bullet$ determines an automorphism $B \to B$; $x \mapsto x\alpha$ of the free B-module on one generator. Moreover, if $\epsilon(\alpha) = 1$ then $[\alpha] \in \widetilde{K}_1(B) \subset K_1(B)$.

We prove next that i is surjective, the idea being to reduce an automorphism of B^n, 'by row and column operations,' to a direct sum of automorphisms of smaller modules. If $(B,\alpha) \in \widetilde{\underline{\mathrm{Aut}}}(B)$ then $[\alpha]$ is certainly in the image of i.

Lemma 4.2. *If α is an automorphism of B^n with $n \geqslant 2$ and $\epsilon(\alpha) = 1$ then there is a unique expression*

$$\alpha = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{9}$$

where $l : B \to B^{n-1}$, $u : B^{n-1} \to B$, $d_1 : B \to B$, and $d_2 : B^{n-1} \to B^{n-1}$. This expression has the properties $\epsilon(d_1) = 1_A$, $\epsilon(d_2) = 1_{A^{n-1}}$, $\epsilon(u) = 0$, and $\epsilon(l) = 0$.

Proof of Lemma 4.2. We may write $\alpha = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$, where $\epsilon(\alpha_{11}) = 1$, $\epsilon(\alpha_{22}) = 1$, $\epsilon(\alpha_{12}) = 0$, and $\epsilon(\alpha_{21}) = 0$. Now α_{11} is invertible since ϵ is a local homomorphism, so

$$\alpha = \begin{pmatrix} 1 & 0 \\ \alpha_{21}\alpha_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12} \end{pmatrix} \begin{pmatrix} 1 & \alpha_{11}^{-1}\alpha_{12} \\ 0 & 1 \end{pmatrix}.$$

This equation proves existence of the expression (9) and the properties in the last sentence of the lemma follow immediately. To show uniqueness, suppose

$$\begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l' & 1 \end{pmatrix} \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 0 \\ -l' + l & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' - u \\ 0 & 1 \end{pmatrix},$$

which implies first that $d_1 = d_1'$ and $d_2 = d_2'$ and second, since d_1 and d_2 are invertible, that $u = u'$ and $l = l'$. $\square$

Since

$$\begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = 0 \in \widetilde{K}_1(B),$$

the 'existence' part of Lemma 4.2 implies, by induction on the rank n, that the natural map $\epsilon^{-1}(1) \to \widetilde{K}_1(B)$ is surjective.

The 'uniqueness' part of Lemma 4.2 leads to a map from the objects in $\widetilde{\underline{\mathrm{Aut}}}(B)$ to $\epsilon^{-1}(1)/C_0$, defined recursively. This map is analogous to the Dieudonné determinant [10] (compare also Klingenberg [14], Draxl [11, Chapter 20], Silvester [32, pp. 122–140], Srinivas [33, Example 1.6]). We continue to use the notation of Lemma 4.2.

Definition 4.3. Suppose $\alpha : B^n \to B^n$ is an automorphism. Define $D(\alpha) = \alpha \in \epsilon^{-1}(1)/C_0$ if $n = 1$. If $n \geqslant 2$ then define

$$D(\alpha) = d_1 D(d_2) = \alpha_{11} D\big(\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}\big) \in \frac{\epsilon^{-1}(1)}{C_0}.$$

It is easy to see that if $\alpha = \mathrm{id}: B^n \to B^n$ then $D(\alpha) = 1$. One can extend the definition of D to automorphisms of f.g. projective modules by $D(\alpha : P \to P) = D(\alpha \oplus 1_Q)$, where $P \oplus Q$ is f.g. free.

Now, if we can show that D respects the relations (1)–(3) of Lemma 4.1 then D will induce a map $\widetilde{K}_1(B) \to \epsilon^{-1}(1)/C_0$, which is plainly inverse to $i: \epsilon^{-1}(1)/C_0 \to \widetilde{K}_1(B)$. It suffices to consider automorphisms of free modules; we shall check the relations by induction starting with relation (1).

Lemma 4.4. *If $\alpha : B^n \to B^n$ and $\beta : B^m \to B^m$ then $D(\alpha \oplus \beta) = D(\alpha)D(\beta)$.*

Proof. We perform induction on n. The case $n = 1$ is trivial so let us suppose $n \geqslant 2$. Writing $\alpha = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$ with the α_{ij} defined as in the proof of Lemma 4.2, we have

$$
\begin{aligned}
D(\alpha \oplus \beta) = D\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \beta \end{pmatrix} &= \alpha_{11} D\left(\begin{pmatrix} \alpha_{22} & 0 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \alpha_{21} \\ 0 \end{pmatrix} \alpha_{11}^{-1} \begin{pmatrix} \alpha_{12} & 0 \end{pmatrix} \right) \\
&= \alpha_{11} D \begin{pmatrix} \alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12} & 0 \\ 0 & \beta \end{pmatrix} \\
&= \alpha_{11} D\big(\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}\big) D(\beta) \quad \text{(by the inductive hypothesis)} \\
&= D(\alpha)D(\beta). \qquad \square
\end{aligned}
$$

We treat relations (2) and (3) together:

Proposition 4.5. (a) *If $\alpha : B^n \to B^n$ and $\beta : B^n \to B^n$ are automorphisms and $\epsilon(\alpha) = \epsilon(\beta) = 1 : A^n \to A^n$ then $D(\alpha\beta) = D(\alpha)D(\beta) \in \epsilon^{-1}(1)/C_0$.*

(b) *Suppose $\alpha : B^n \to B^m$, $\beta : B^m \to B^n$, and either $\epsilon(\alpha) = 0$ or $\epsilon(\beta) = 0$ (or both). Then $D(1 + \alpha\beta) = D(1 + \beta\alpha) \in \epsilon^{-1}(1)/C_0$.*

Part (a) of Proposition 4.5 says that D respects relation (2) of Lemma 4.1. Part (b) implies that D respects relation (3) for, if $\alpha = 1 + \alpha_0$ and $\epsilon(\alpha_0) = 0$ then $D(c\alpha c^{-1}) = D(1 + c\alpha_0 c^{-1}) = D(1 + \alpha_0 c^{-1} c) = D(\alpha)$.

Proof of Proposition 4.5. We shall prove parts (a) and (b) at the same time, by induction. In part (a) there is one statement, denoted $\mathfrak{a}(n)$, for each positive integer n. In part (b) there is one statement $\mathfrak{b}(m, n)$ for each pair (m, n) of positive integers and we have $\mathfrak{b}(m, n) \Leftrightarrow \mathfrak{b}(n, m)$.

Plainly $\mathfrak{a}(1)$ and $\mathfrak{b}(1, 1)$ are true. We shall establish the following inductive steps:

$$\mathfrak{b}(1, n-1) \wedge \mathfrak{a}(n-1) \quad \Rightarrow \quad \mathfrak{a}(n) \quad \text{for all } n \geqslant 2, \tag{10}$$

$$\mathfrak{a}(m) \wedge \mathfrak{b}(1, m) \wedge \mathfrak{b}(m, n-1) \quad \Rightarrow \quad \mathfrak{b}(m, n) \quad \text{for } m \geqslant 1, n \geqslant 2. \tag{11}$$

These are sufficient to prove $\mathfrak{a}(n)$ and $\mathfrak{b}(m, n)$ for all m and n. Indeed, a special case of (11) reads $\mathfrak{a}(1) \wedge \mathfrak{b}(1, 1) \wedge \mathfrak{b}(1, n-1) \Rightarrow \mathfrak{b}(1, n)$, so $\mathfrak{b}(1, n)$ holds for all $n \geqslant 1$. It then follows from (10) that $\mathfrak{a}(n)$ holds for all $n \geqslant 1$. Finally, (11) shows by induction on n that $\mathfrak{b}(m,n)$ holds for all $m, n \geqslant 1$. $\square$

Proof of (10). Suppose $\alpha, \beta : B^n \to B^n$. By Lemma 4.2, it suffices to show $D(\alpha\beta) = D(\alpha)D(\beta)$ in the cases

$$\text{(i)}\quad \alpha = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}, \qquad \text{(ii)}\quad \alpha = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \text{and} \quad \text{(iii)}\quad \alpha = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Here $l : B \to B^{n-1}$, $d_1 : B \to B$, $d_2 : B^{n-1} \to B^{n-1}$, and $u : B^{n-1} \to B$.

(i) Let us write

$$\beta = \begin{pmatrix} 1 & 0 \\ l' & 1 \end{pmatrix} \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}.$$

Now

$$D(\alpha\beta) = D\begin{pmatrix} 1 & 0 \\ l+l' & 1 \end{pmatrix} \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} = d_1' D(d_2') = D(\beta) = D(\alpha)D(\beta).$$

$$\begin{aligned} \text{(ii)}\quad D(\alpha\beta) &= D\left(\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l' & 1 \end{pmatrix} \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\right) \\ &= D\left(\begin{pmatrix} 1 & 0 \\ d_2 l' d_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} d_1 d_1' & 0 \\ 0 & d_2 d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\right) = d_1 d_1' D(d_2 d_2'). \end{aligned}$$

Now $\epsilon^{-1}(1)/C_0$ is abelian by Proposition 3.4(3), so by $\mathfrak{a}(n-1)$ we have $D(\alpha\beta) = d_1 d_1' D(d_2) D(d_2') = d_1 D(d_2) d_1' D(d_2') = D(\alpha)D(\beta)$. This completes part (ii).

In the proofs of (iii) and (11), we use the following identity, which can be verified by direct calculation:

Lemma 4.6. *If $1 + \alpha\beta$ is invertible then $1 - \beta(1+\alpha\beta)^{-1}\alpha = (1+\beta\alpha)^{-1}$.*

(iii) If $\alpha = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ then setting $\gamma = 1 + ul'$, we have

$$\begin{aligned} D(\alpha\beta) &= D\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l' & 1 \end{pmatrix} \begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\right) \\ &= D\left(\begin{pmatrix} 1 & 0 \\ l'\gamma^{-1} & 1 \end{pmatrix} \begin{pmatrix} \gamma d_1' & 0 \\ 0 & (1 - l'\gamma^{-1}u) d_2' \end{pmatrix} \begin{pmatrix} 1 & (\gamma d_1')^{-1} u d_2' + u' \\ 0 & 1 \end{pmatrix}\right) \\ &= \gamma d_1' D\big((1 - l'\gamma^{-1}u) d_2'\big) \\ &= (1 + ul') d_1' D\big((1 + l'u)^{-1} d_2'\big) \quad \text{(by Lemma 4.6)}. \end{aligned}$$

It follows by $\mathfrak{a}(n-1)$ and $\mathfrak{b}(1, n-1)$ that

$$D(\alpha\beta) = (1+ul')d_1'(1+ul')^{-1}D(d_2') = d_1'D(d_2') = D(\alpha)D(\beta).$$

This completes the proof of (10). $\square$

Proof of (11). Suppose $\alpha : B^n \to B^m$, $\beta : B^m \to B^n$, and either $\epsilon(\alpha) = 0$ or $\epsilon(\beta) = 0$. Write $\alpha = (\alpha_1 \quad \alpha_2)$, where $\alpha_1 : B \to B^m$ and $\alpha_2 : B^{n-1} \to B^m$, and write $\beta = \left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)$, where $\beta_1 : B^m \to B$ and $\beta_2 : B^m \to B^{n-1}$. Now

$$1 + \beta\alpha = 1 + \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}(\alpha_1 \quad \alpha_2) = \begin{pmatrix}1+\beta_1\alpha_1 & \beta_1\alpha_2\\ \beta_2\alpha_1 & 1+\beta_2\alpha_2\end{pmatrix},$$

so by Definition 4.3 we have

$$\begin{aligned}
D(1+\beta\alpha) &= (1+\beta_1\alpha_1)D\big(1+\beta_2\alpha_2 - \beta_2\alpha_1(1+\beta_1\alpha_1)^{-1}\beta_1\alpha_2\big)\\
&= (1+\beta_1\alpha_1)D\big(1+\beta_2\big(1-\alpha_1(1+\beta_1\alpha_1)^{-1}\beta_1\big)\alpha_2\big)\\
&= (1+\beta_1\alpha_1)D\big(1+\beta_2(1+\alpha_1\beta_1)^{-1}\alpha_2\big) \quad \text{(by Lemma 4.6)}\\
&= D(1+\alpha_1\beta_1)D\big(1+(1+\alpha_1\beta_1)^{-1}\alpha_2\beta_2\big) \quad \text{(using } \mathfrak{b}(1,m) \text{ and } \mathfrak{b}(n-1,m))\\
&= D\big((1+\alpha_1\beta_1)\big(1+(1+\alpha_1\beta_1)^{-1}\alpha_2\beta_2\big)\big) \quad \text{(by } \mathfrak{a}(m))\\
&= D(1+\alpha\beta).
\end{aligned}$$

This completes the proof of Proposition 4.5. $\square$

Thus $D : \widetilde{K}_1(B) \to \epsilon^{-1}(1)/C_0$ is well-defined and the proof of Theorem B is also complete.

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