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## On The Algebraic $L$ -theory of $\Delta$ -sets

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**Abstract:** The algebraic  $L$ -groups  $L_*(\mathbb{A}, X)$  are defined for an additive category  $\mathbb{A}$  with chain duality and a  $\Delta$ -set  $X$ , and identified with the generalized homology groups  $H_*(X; \mathbb{L}_\bullet(\mathbb{A}))$  of  $X$  with coefficients in the algebraic  $L$ -spectrum  $\mathbb{L}_\bullet(\mathbb{A})$ . Previously such groups had only been defined for simplicial complexes  $X$ .

**Keywords:** Surgery theory,  $\Delta$ -set,  $L$ -groups.

### INTRODUCTION

A ‘ $\Delta$ -set’  $X$  in the sense of Rourke and Sanderson [9] is a simplicial set without degeneracies. A simplicial complex is a  $\Delta$ -set; conversely, the second barycentric (aka derived) subdivision of a  $\Delta$ -set is a simplicial complex, and the homotopy theory of  $\Delta$ -sets is the same as the homotopy theory of simplicial complexes. However,  $\Delta$ -sets are sometimes more convenient than simplicial complexes: they are generally smaller, and the quotient of a  $\Delta$ -set by a group action is again a  $\Delta$ -set. In this paper we extend the algebraic  $L$ -theory of simplicial complexes of Ranicki [6] to  $\Delta$ -sets.

In the original formulation of Wall [10] the surgery obstruction theory of high-dimensional manifolds involved the algebraic  $L$ -groups  $L_*(R)$  of a ring with involution  $R$ , which are the Witt groups of quadratic forms over  $R$  and their automorphisms. The subsequent development of the theory in [6] viewed  $L_*(R)$  as the cobordism groups of  $R$ -module chain complexes with quadratic Poincaré duality, constructed a spectrum  $\mathbb{L}_\bullet(R)$  with homotopy groups  $L_*(R)$ , and also introduced the algebraic  $L$ -groups  $L_*(R, X)$  of a

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simplicial complex  $X$ . An element of  $L_n(R, X)$  is a cobordism class of directed systems over  $X$  of  $R$ -module chain complexes with an  $n$ -dimensional quadratic Verdier-type duality. The groups  $L_*(R, X)$  were identified with the generalized homology groups  $H_*(X; \mathbb{L}_\bullet(R))$ , and the algebraic  $L$ -theory assembly map  $A : L_*(R, X) \rightarrow L_*(R[\pi_1(X)])$  was defined and extended to the algebraic surgery exact sequence

$$\cdots \longrightarrow L_n(R, X) \xrightarrow{A} L_n(R[\pi_1(X)]) \longrightarrow \mathcal{S}_n(R, X) \longrightarrow L_{n-1}(R, X) \longrightarrow \cdots$$

with  $\mathcal{S}_n(R, X)$  the cobordism groups of the  $R[\pi_1(X)]$ -contractible directed systems. In particular, the 1-connective version gave an algebraic interpretation of the exact sequence of the topological version of the Browder-Novikov-Sullivan-Wall surgery theory: if the polyhedron  $\|X\|$  of a finite simplicial complex  $X$  has the homotopy type of a closed  $n$ -dimensional topological manifold then  $\mathcal{S}_{n+1}(\mathbb{Z}, X)$  is the structure set of closed  $n$ -dimensional topological manifolds  $M$  with a homotopy equivalence  $M \simeq \|X\|$ .

The Verdier-type duality of [6] used the dual cells in the barycentric subdivision of a simplicial complex  $X$  to define the dual of a directed system over  $X$  of  $R$ -modules to be a directed system over  $X$  of  $R$ -module chain complexes. The  $\Delta$ -set analogues of dual cells introduced by us in Ranicki and Weiss [8] are used here to define a Verdier-type duality for directed systems of  $R$ -modules over a  $\Delta$ -set  $X$ , which is used to define the generalized homology groups  $L_*(R, X) = H_*(X; \mathbb{L}_\bullet(R))$  and an algebraic surgery exact sequence as in the simplicial complex case.

The algebraic  $L$ -theory of  $\Delta$ -sets is used in Macko and Weiss [5], and its multiplicative properties are investigated in Laures and McClure [3].

## 1. FUNCTOR CATEGORIES

In this section,  $X$  denotes a category with the following property. For every object  $x$ , the set of morphisms to  $x$  (with unspecified source) is finite; moreover, given morphisms  $f : y \rightarrow x$  and  $g : z \rightarrow x$  in  $X$ , there exists at most one morphism  $h : y \rightarrow z$  such that  $gh = f$ .

Let  $\mathbb{A}$  be an additive category with zero object  $0 \in \text{Ob}(\mathbb{A})$ .

**Definition 1.1.** (i) A function

$$M : \text{Ob}(X) \rightarrow \text{Ob}(\mathbb{A}) ; x \mapsto M(x)$$

is *finite* if  $M(x) = 0$  for all but a finite number of objects  $x$  in  $\mathbb{A}$ .

The direct sum  $\sum_{x \in \text{Ob}(X)} M(x)$  will be written as  $\sum_{x \in X} M(x)$ .

(ii) A functor  $F : X \rightarrow \mathbb{A}$  is *finite* if the function  $F : \text{Ob}(X) \rightarrow \text{Ob}(\mathbb{A})$  is finite.  $\square$

**Definition 1.2.** (i) The *contravariant functor category*  $\mathbb{A}_*[X]$  is the additive category of finite contravariant functors  $F : X \rightarrow \mathbb{A}$ . The morphisms in  $\mathbb{A}_*[X]$  are the natural transformations.

(ii) The *covariant functor category*  $\mathbb{A}^*[X]$  is the additive category of covariant functors  $F : X \rightarrow \mathbb{A}$ . The morphisms in  $\mathbb{A}^*[X]$  are the natural transformations. We write  $\mathbb{A}_f^*[X]$  for the full subcategory whose objects are the finite functors in  $\mathbb{A}^*[X]$ .  $\square$

**Remark 1.3.** We use the terminology  $\mathbb{A}^*[X]$  for the *covariant* functor category because it behaves contravariantly in the variable  $X$ . Indeed a functor  $g : X \rightarrow Y$  induces a functor  $\mathbb{A}^*[Y] \rightarrow \mathbb{A}^*[X]$  by composition with  $g$ . Our reasons for using the terminology  $\mathbb{A}_*[X]$  for the *contravariant* functor category are similar, but more complicated. Below we introduce a variation denoted  $\mathbb{A}_*(X)$  which behaves covariantly in  $X$ .  $\square$

For the remainder of this section we shall only consider the contravariant functor category  $\mathbb{A}_*[X]$ , but every result also has a version for the covariant functor category  $\mathbb{A}^*[X]$  (or  $\mathbb{A}_f^*[X]$  in some cases).

**Definition 1.4.** (i) A chain complex in an additive category  $\mathbb{A}$

$$C : \dots \longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \dots \quad (d^2 = 0)$$

is *finite* if  $C_n = 0$  for all but a finite number of  $n \in \mathbb{Z}$ .

(ii) Let  $\mathbb{B}(\mathbb{A})$  be the additive category of finite chain complexes in  $\mathbb{A}$  and chain maps.  $\square$

A finite chain complex  $C$  in  $\mathbb{A}_*[X]$  is just an object in  $\mathbb{B}(\mathbb{A})_*[X]$ , and likewise for chain maps, so that

$$\mathbb{B}(\mathbb{A}_*[X]) = \mathbb{B}(\mathbb{A})_*[X] .$$

**Definition 1.5.** A chain map  $f : C \rightarrow D$  of chain complexes in  $\mathbb{A}_*[X]$  is a *weak equivalence* if each

$$f[x] : C[x] \rightarrow D[x] \quad (x \in X)$$

is a chain equivalence in  $\mathbb{A}$ .  $\square$

A morphism  $f : C \rightarrow D$  in  $\mathbb{B}(\mathbb{A}_*[X])$  which is a chain equivalence is also a weak equivalence, but in general a weak equivalence need not be a chain equivalence – see 1.11 for a more detailed discussion.

**Definition 1.6.** Let  $x$  be an object in  $X$ .

(i) The *under category*  $x/X$  is the category with objects the morphisms  $f : x \rightarrow y$  in  $X$ , and morphisms  $g : f \rightarrow f'$  the morphisms  $g : y \rightarrow y'$  in  $X$

such that  $gf = f'$

$$\begin{array}{ccc} & x & \\ f \swarrow & & \searrow f' \\ y & \xrightarrow{g} & y' \end{array}$$

The *open star* of  $x$  is the set of objects in  $x/X$

$$\text{st}(x) = \text{Ob}(x/X) = \{x \rightarrow y\} .$$

(ii) The *over category*  $X/x$  is the category with morphisms  $f : y \rightarrow x$  in  $X$  as its objects, and so that morphisms  $g : f \rightarrow f'$  are the morphisms  $g : y \rightarrow y'$  in  $X$  such that  $f = f'g$

$$\begin{array}{ccc} y & \xrightarrow{g} & y' \\ f \searrow & & \swarrow f' \\ & x & \end{array}$$

The *closure* of  $x$  is the set of objects in  $X/x$

$$\text{cl}(x) = \text{Ob}(X/x) = \{y \rightarrow x\} .$$

Because of our standing assumptions on  $X$ , the over category  $X/x$  is isomorphic to a finite poset.  $\square$

In the applications of the contravariant functor category  $\mathbb{A}_*[X]$  to topology we shall be particularly concerned with the subcategory of functors satisfying the following property.

**Definition 1.7.** A contravariant functor

$$F : X \rightarrow \mathbb{A} ; x \mapsto F[x]$$

in  $\mathbb{A}_*[X]$  is *induced* if there exists a finite function  $x \mapsto F(x) \in \text{Ob}(\mathbb{A})$  and a natural isomorphism

$$F[x] \cong \bigoplus_{x \rightarrow y} F(y) .$$

The sum ranges over  $\text{st}(x)$ , and since the function  $x \mapsto F(x)$  is finite,  $F[x]$  is only a sum of a finite number of non-zero objects in  $\mathbb{A}$ .

Similarly a covariant functor

$$F : X \rightarrow \mathbb{A} ; x \mapsto F[x]$$

in  $\mathbb{A}^*[X]$  is *induced* if there exists a function  $x \mapsto F(x) \in \text{Ob}(\mathbb{A})$  and a natural isomorphism

$$F[x] \cong \bigoplus_{y \rightarrow x} F(y) .$$

The full subcategories of the functor categories  $\mathbb{A}_*[X]$ , respectively  $\mathbb{A}^*[X]$ , with objects the induced functors  $F : X \rightarrow \mathbb{A}$  are equivalent, as we shall prove below, to the following categories.

**Definition 1.8.** Let  $\mathbb{A}_*(X)$  be the additive category whose objects are functions  $x \mapsto F(x)$  such that  $F(x) = 0$  for all but a finite number of objects  $x$ . A morphism  $f : E \rightarrow F$  in  $\mathbb{A}_*(X)$  is a collection of morphisms  $f(\phi) : E(x) \rightarrow F(y)$  in  $\mathbb{A}$ , one for each morphism  $\phi : x \rightarrow y$  in  $X$ . The composite of the morphisms

$$f = \{f(\phi)\} : M \rightarrow N, \quad g = \{g(\theta)\} : N \rightarrow P$$

is the morphism

$$gf = \{gf(\psi)\} : M \rightarrow P$$

with

$$gf(\psi : x \rightarrow z) = \sum_{\phi : x \rightarrow y, \theta : y \rightarrow z, \theta\phi = \psi} g(\theta)f(\phi) : M(x) \rightarrow P(z).$$

We can view an object  $F$  of  $\mathbb{A}_*(X)$  as an object in  $\mathbb{A}_*[X]$  by writing

$$F[x] = \bigoplus_{x \rightarrow y} F(y).$$

A morphism  $\theta : w \rightarrow x$  in  $X$  induces a morphism  $F[x] \rightarrow F[w]$  in  $\mathbb{A}$  which maps the summand  $F(y)$  corresponding to some  $\phi : x \rightarrow y$  identically to the summand  $F(y)$  corresponding to the composition  $\phi\theta : w \rightarrow y$ .

Let  $\mathbb{A}^*(X)$  be the additive category whose objects are functions  $x \mapsto F(x)$ . A morphism  $f : E \rightarrow F$  in  $\mathbb{A}^*(X)$  is a collection of morphisms  $f(\phi) : E(y) \rightarrow F(x)$  in  $\mathbb{A}$ , one for each morphism  $\phi : x \rightarrow y$  in  $X$ . Again we can view an object  $F$  of  $\mathbb{A}^*(X)$  as an object in  $\mathbb{A}^*[X]$  by writing

$$F[x] = \bigoplus_{y \rightarrow x} F(y).$$

**Proposition 1.9.** (i) For any object  $M$  in  $\mathbb{A}_*(X)$  and any object  $N$  in  $\mathbb{A}_*[X]$

$$\text{Hom}_{\mathbb{A}_*[X]}(M, N) = \sum_{x \in X} \text{Hom}_{\mathbb{A}}(M(x), N[x]).$$

(ii) For any objects  $L, M$  in  $\mathbb{A}_*(X)$

$$\text{Hom}_{\mathbb{A}_*[X]}(L, M) = \sum_{x \rightarrow y} \text{Hom}_{\mathbb{A}}(L(x), M(y)).$$

(iii) The additive category  $\mathbb{A}_*(X)$  is equivalent to the full subcategory of the contravariant functor category  $\mathbb{A}_*[X]$  with objects the induced functors.

*Proof.* (i) A morphism  $f : M \rightarrow N$  in  $\mathbb{A}_*[X]$  is determined by the composite morphisms in  $\mathbb{A}$

$$M(x) \xrightarrow{\text{inclusion}} M[x] \xrightarrow{f[x]} N[x] \quad (x \in X)$$

(ii) By (i), a morphism  $f : L \rightarrow M$  in  $\mathbb{A}_*[X]$  is determined by the composite morphisms in  $\mathbb{A}$

$$L(x) \xrightarrow{\text{inclusion}} L[x] \xrightarrow{f[x]} M[x] = \sum_{x \rightarrow y} M(y) \quad (x \in X) .$$

(iii) Every object  $M$  in  $\mathbb{A}_*(X)$  determines an induced contravariant functor

$$X \rightarrow \mathbb{A} ; x \mapsto M[x] = \sum_{x \rightarrow y} M(y) ,$$

i.e. an object in  $\mathbb{A}_*[X]$ , and every induced functor is naturally equivalent to one of this type.  $\square$

**Proposition 1.10.** *The following conditions on a chain map  $f : C \rightarrow D$  in  $\mathbb{A}_*(X)$  are equivalent:*

- (a)  $f$  is a chain equivalence,
- (b) each of the component chain maps in  $\mathbb{A}$

$$f(1_x) : C(x) \rightarrow D(x) \quad (x \in X)$$

*is a chain equivalence,*

- (c)  $f : C \rightarrow D$  is a weak equivalence in  $\mathbb{A}_*[X]$ , that is,  $C[x] \rightarrow D[x]$  is a chain equivalence for all  $x$ .

*Proof.* The proof given in Proposition 2.7 of Ranicki and Weiss [8] in the case when  $\mathbb{A}$  is the additive category of  $R$ -modules (for some ring  $R$ ) works for an arbitrary additive category.  $\square$

**Remark 1.11.** Every chain equivalence of chain complexes in  $\mathbb{A}_*[X]$  is a weak equivalence. By 1.10 every weak equivalence of degreewise induced finite chain complexes in  $\mathbb{A}_*[X]$  is a chain equivalence. See Ranicki and Weiss [8, 1.13] for an explicit example of a weak equivalence of finite chain complexes in  $\mathbb{A}_*[X]$  which is not a chain equivalence. It is proved in [8, 2.9] that every finite chain complex  $C$  in  $\mathbb{A}_*[X]$  is weakly equivalent to one in  $\mathbb{A}_*(X)$ .  $\square$

## 2. $\Delta$ -SETS

Let  $\Delta$  be the category with objects the sets

$$[n] = \{0, 1, \dots, n\} \quad (n \geq 0)$$

and morphisms  $[m] \rightarrow [n]$  order-preserving injections. Every such morphism has a unique factorization as the composite of the order-preserving injections

$$\partial_i : [k-1] \rightarrow [k] ; j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i . \end{cases}$$

**Definition 2.1.** (Rourke and Sanderson [9]) A  $\Delta$ -set is a contravariant functor

$$X : \Delta \rightarrow \{\text{sets and functions}\} ; [n] \mapsto X^{(n)} .$$

□

Equivalently, a  $\Delta$ -set  $X$  can be regarded as a sequence  $X^{(n)}$  ( $n \geq 0$ ) of sets, together with face maps

$$\partial_i : X^{(n)} \rightarrow X^{(n-1)} \quad (0 \leq i \leq n)$$

such that

$$\partial_i \partial_j = \partial_{j-1} \partial_i \text{ for } i < j .$$

The elements  $x \in X^{(n)}$  are the  $n$ -simplices of  $X$ .

**Definition 2.2.** (Rourke and Sanderson [9])

(i) The *realization* of a  $\Delta$ -set  $X$  is the  $CW$  complex

$$\|X\| = \prod_{n=0}^{\infty} (X^{(n)} \times \Delta^n) / \sim$$

with

$$\begin{aligned} \Delta^n &= \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \leq s_i \leq 1, \sum_{i=0}^n s_i = 1\} , \\ \partial_i : \Delta^{n-1} &\hookrightarrow \Delta^n ; (s_0, s_1, \dots, s_{n-1}) \mapsto (s_0, s_1, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_n) , \\ (x, \partial_i s) &\sim (\partial_i x, s) \quad (x \in X^{(n)}, s \in |\Delta^{n-1}|) . \end{aligned}$$

(ii) There is one  $n$ -cell  $x(\Delta^n) \subseteq \|X\|$  for each  $n$ -simplex  $x \in X$ , with *characteristic map*

$$x : \Delta^n \rightarrow \|X\| ; (s_0, s_1, \dots, s_n) \mapsto (x, (s_0, s_1, \dots, s_n)) .$$

The *boundary*  $x(\partial\Delta^n) \subseteq \|X\|$  is the image of

$$\begin{aligned} \partial\Delta^n &= \bigcup_{i=0}^n \partial_i |\Delta^{n-1}| \\ &= \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \leq s_i \leq 1, \sum_{i=0}^n s_i = 1, s_i = 0 \text{ for some } i\} \end{aligned}$$

and the *interior*  $x(\overset{\circ}{\Delta}^n) \subseteq \|X\|$  is the image of

$$\begin{aligned} \overset{\circ}{\Delta}^n &= \Delta^n \setminus \partial\Delta^n \\ &= \{(s_0, s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 < s_i \leq 1, \sum_{i=0}^n s_i = 1\} \subseteq \Delta^n . \end{aligned}$$

The characteristic map  $x : \Delta^n \rightarrow \|X\|$  is injective on  $\overset{\circ}{\Delta}^n \subseteq \Delta^n$ .

□

**Example 2.3.** Let  $\Delta^n$  be the  $\Delta$ -set with

$$(\Delta^n)^{(m)} = \{\text{morphisms } [m] \rightarrow [n] \text{ in } \Delta\} \quad (0 \leq m \leq n) .$$

The realization  $\|\Delta^n\|$  is the geometric  $n$ -simplex  $\Delta^n$  (as in the above definition). It should be clear from the context whether  $\Delta^n$  refers to the  $\Delta$ -set or the geometric realization.  $\square$

We regard a  $\Delta$ -set  $X$  as a category, whose objects are the simplices, writing the dimension of an object  $x \in X$  as  $|x|$ , i.e.  $|x| = m$  for  $x \in X^{(m)}$ . A morphism  $f : x \rightarrow y$  from an  $m$ -simplex  $x$  to an  $n$ -simplex  $y$  is a morphism  $f : [m] \rightarrow [n]$  in  $\Delta$  such that

$$f^*(y) = x \in X^{(m)} .$$

In particular, for any  $x \in X^{(m)}$  with  $m \geq 1$  there are defined  $m + 1$  distinct morphisms in  $X$

$$\partial_i : \partial_i x \rightarrow x \quad (0 \leq i \leq m) .$$

**Example 2.4.** (i) Let  $X$  be a  $\Delta$ -set. An object  $M$  of  $\mathbb{A}_*(X)$  is just an object  $M$  of  $\mathbb{A}$  with a direct sum decomposition  $M = \bigoplus_{x \in X} M(x)$ . A morphism  $f : M \rightarrow N$  in  $\mathbb{A}_*(X)$  is a collection of morphisms  $f_{xy,\lambda} : M(x) \rightarrow N(y)$ , one such for every pair of simplices  $x, y$  and face operator  $\lambda$  such that  $\lambda^*y = x$ . We like to think of a morphism  $f : M \rightarrow N$  in  $\mathbb{A}_*(X)$  as a morphism in  $\mathbb{A}$  with additional structure. Source and target of that morphism in  $\mathbb{A}$  are  $M(X) = \bigoplus_x M(x)$  and  $N(X) = \bigoplus_x N(x)$ , respectively. For simplices  $x$  and  $y$ , the  $xy$ -component of the morphism  $M(X) \rightarrow N(X)$  determined by  $f$  is

$$\sum_{\lambda} f_{xy,\lambda}$$

where the sum runs over all  $\lambda$  such that  $\lambda^*y = x$ .

(ii) If  $X$  is a simplicial complex then a morphism in  $\mathbb{A}_*(X)$  is just a morphism  $f : M \rightarrow N$  in  $\mathbb{A}$  between objects with finite direct sum decompositions

$$M = \sum_{x \in X} M(x) , \quad N = \sum_{y \in X} N(y)$$

such that the components  $f(x, y) : M(x) \rightarrow N(y)$  are 0 unless  $x \leq y$ .

(iii) The description of  $\mathbb{A}_*(X)$  in (ii) also applies in the case of a  $\Delta$ -set  $X$  where, for any two simplices  $x$  and  $y$ , there is at most one morphism from  $x$  to  $y$ . In particular it applies when  $X = Y'$  is the barycentric subdivision of another  $\Delta$ -set  $Y$ , to be defined in the next section.  $\square$

**Definition 2.5.** Let  $X$  be a  $\Delta$ -set, and let  $R$  be a ring.

(i) The  $R$ -coefficient simplicial chain complex of  $X$  is the free (left)  $R$ -module chain complex  $\Delta(X; R)$  with

$$d = \sum_{i=0}^n (-)^i \partial_i : \Delta(X; R)_n = R[X^{(n)}] \rightarrow \Delta(X; R)_{n-1} = R[X^{(n-1)}] .$$

The  $R$ -coefficient homology of  $X$  is the homology of  $\Delta(X; R)$

$$H_*(X; R) = H_*(\Delta(X; R)) = H_*(\|X\|; R) ,$$

noting that  $\Delta(X; R)$  is the  $R$ -coefficient cellular chain complex of  $\|X\|$ .

(ii) Suppose that  $R$  is equipped with an involution

$$R \rightarrow R ; r \mapsto \bar{r}$$

(e.g. the identity for a commutative ring), allowing the definition of the *dual* of an  $R$ -module  $M$  to be the  $R$ -module

$$M^* = \text{Mod}_R(M, R) , R \times M^* \rightarrow M^* ; (r, f) \mapsto (x \mapsto f(x)\bar{r}) .$$

The  $R$ -coefficient simplicial cochain complex of  $X$

$$\Delta(X; R)^* = \text{Hom}_R(\Delta(X; R), R)$$

is the  $R$ -module cochain complex with

$$d^* = \sum_{i=0}^{n+1} (-)^i \partial_i^* : \Delta(X; R)^n = R[X^{(n)}]^* \rightarrow \Delta(X; R)^{n+1} = R[X^{(n+1)}]^* ,$$

The  $R$ -coefficient cohomology of  $X$  is the cohomology of  $\Delta(X; R)^*$

$$H^*(X; R) = H^*(\Delta(X; R)^*) = H^*(\|X\|; R) ,$$

noting that  $\Delta(X; R)^*$  is the  $R$ -coefficient cellular cochain complex of  $\|X\|$ .  $\square$

A simplicial complex  $X$  is *ordered* if the vertices in any simplex are ordered, with faces having compatible orderings. From now on, in dealing with simplicial complexes we shall always assume an ordering.

**Example 2.6.** A simplicial complex  $X$  can be regarded as a  $\Delta$ -set, with  $X^{(n)}$  the set of  $n$ -simplices and

$$\partial_i : X^{(n)} \rightarrow X^{(n-1)} ; (v_0 v_1 \dots v_n) \mapsto (v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_n) .$$

There is one morphism  $x \rightarrow y$  in  $X$  for each face inclusion  $x \leq y$ . The realization  $\|X\|$  of  $X$  regarded as a  $\Delta$ -set is the polyhedron of the simplicial complex  $X$ , with the characteristic maps  $x : \Delta^{|x|} \rightarrow \|X\|$  ( $x \in X$ ) injections. The simplicial chain complex  $\Delta(X; R)$  is just the usual  $R$ -coefficient simplicial chain complex of  $X$ , and  $\Delta(X; R)^*$  is the  $R$ -coefficient simplicial cochain complex of  $X$ .  $\square$

**Example 2.7.** Let  $X$  be a  $\Delta$ -set, and let  $x \in X$  be a simplex.

(i) In general, the canonical map

$$\text{Ob}(x/X) = \text{st}(x) \rightarrow \text{Ob}(X) ; (x \rightarrow y) \mapsto y$$

is not injective. The simplices  $y \in \text{Ob}(X) \setminus \text{im}(\text{st}(x))$  are the objects of a sub- $\Delta$ -set  $X \setminus \text{im}(\text{st}(x)) \subset X$ . If  $X$  is a simplicial complex then  $\text{st}(x) \rightarrow \text{Ob}(X)$  is injective, and  $X \setminus \text{st}(x) \subset X$  is the subcomplex with simplices  $y \in X$  such

that  $x \not\leq y$ .

(ii) The over category  $X/x = \{y \rightarrow x\}$  (1.6) is a  $\Delta$ -set with

$$(X/x)^{(n)} = \{y \rightarrow x \mid y \in X^{(n)}\} \quad (n \geq 0) .$$

It is isomorphic as a  $\Delta$ -set to  $\Delta^{|x|}$ . The forgetful functor

$$X/x \rightarrow X ; (y \rightarrow x) \mapsto y$$

is a  $\Delta$ -map, inducing the characteristic map  $\Delta^{|x|} \rightarrow \|X\|$ . If  $X$  is a simplicial complex then  $X/x \rightarrow X$  is injective, and so is the induced characteristic map.  $\square$

**Example 2.8.** (i) If a group  $G$  acts on a  $\Delta$ -set  $X$  the quotient  $X/G$  is again a  $\Delta$ -set, with realization  $\|X/G\| = \|X\|/G$ . However, if  $X$  is a simplicial complex and  $G$  acts on  $X$ , then  $X/G$  is not in general a simplicial complex. See (ii) for an example.

(ii) Suppose  $X = \mathbb{R}$ , the  $\Delta$ -set with

$$X^{(0)} = X^{(1)} = \mathbb{Z} , \quad \partial_0(n) = n , \quad \partial_1(n) = n + 1 ,$$

and let the infinite cyclic group  $G = \mathbb{Z} = \{t\}$  act on  $X$  by  $tn = n + 1$ . The quotient  $\Delta$ -set  $S^1 = \mathbb{R}/\mathbb{Z}$  is the circle, with one 0-simplex  $x_0$  and one 1-simplex  $x_1$

$$(S^1)^{(0)} = \{x_0\} , \quad (S^1)^{(1)} = \{x_1\} , \quad \partial_0(x_1) = \partial_1(x_1) = x_0 .$$

$\square$

**Example 2.9.** For any space  $M$  use the standard  $n$ -simplices  $\Delta^n$  and face inclusions  $\partial_i : \Delta^{n-1} \hookrightarrow \Delta^n$  to define the *singular  $\Delta$ -set*  $X = M^\Delta$  by

$$X^{(n)} = M^{\Delta^n} , \quad \partial_i : X^{(n)} \rightarrow X^{(n-1)} ; x \mapsto x \circ \partial_i .$$

We shall say that a singular simplex  $x : \Delta^n \rightarrow X$  is a face of a singular simplex  $y : \Delta^m \rightarrow X$  if  $x = y \circ \partial_{i_1} \circ \cdots \circ \partial_{i_{m-n}}$  for a given face inclusion

$$\partial_{i_1} \circ \cdots \circ \partial_{i_{m-n}} : \Delta^n \hookrightarrow \Delta^m ,$$

writing  $x \leq y$  (and  $x < y$  if  $x \neq y$ ). The simplicial chain complex  $\Delta(X; R) = S(M; R)$  is just the usual  $R$ -coefficient singular chain complex of  $M$ , so that

$$H_*(\|X\|; R) = H_*(X; R) = H_*(M; R) .$$

Also  $\Delta(X; R)^* = S(M; R)^*$  is the  $R$ -coefficient singular cochain complex of  $M$ , and

$$H^*(\|X\|; R) = H^*(X; R) = H^*(M; R) .$$

$\square$

3. THE BARYCENTRIC SUBDIVISION

The  $\Delta$ -set analogue of the barycentric subdivision  $X'$  of a simplicial complex  $X$  and the dual cells  $D(x, X) \subset X'$  ( $x \in X$ ) makes use of the following standard categorical construction.

**Definition 3.1.** (i) The *nerve* of a category  $\mathcal{C}$  is the simplicial set with one  $n$ -simplex for each string  $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$  of morphisms in  $\mathcal{C}$ , with

$$\partial_i(x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n) = (x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{i-1} \rightarrow x_{i+1} \rightarrow \cdots \rightarrow x_n) .$$

(ii) An  $n$ -simplex  $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$  in the nerve is *non-degenerate* if none of the morphisms  $x_i \rightarrow x_{i+1}$  is the identity.  $\square$

If the category  $\mathcal{C}$  has the property that the composite of non-identity morphisms is a non-identity, then the non-degenerate simplices in the nerve define a  $\Delta$ -set, which we shall also call the nerve and denote by  $\mathcal{C}$ .

**Definition 3.2.** (Rourke and Sanderson [9, §4], Ranicki and Weiss [8, 1.6, 1.7]) Let  $X$  be a  $\Delta$ -set.

(i) The *barycentric subdivision* of  $X$  is the  $\Delta$ -set  $X'$  defined by the nerve of the category  $X$ .

(ii) The *dual*  $x^\perp$  of a simplex  $x \in X$  is the nerve of the under category  $x/X$  (1.6). An  $n$ -simplex in the  $\Delta$ -set  $x^\perp$  is thus a sequence of morphisms in  $X$

$$x \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$$

such that  $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$  is non-degenerate. In particular

$$(x^\perp)^{(0)} = \{x \rightarrow x_0\} = \text{st}(x) .$$

(iii) The *boundary of the dual*  $\partial x^\perp$  is the sub- $\Delta$ -set of  $x^\perp$  consisting of the  $n$ -simplices  $x \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$  such that  $x \rightarrow x_0$  is not the identity.  $\square$

The under category  $x/X$  has an initial object, so that the nerve  $x^\perp$  is contractible. The rule  $x \rightarrow x^\perp$  is contravariant, i.e. every morphism  $x \rightarrow y$  induces a  $\Delta$ -map  $y^\perp \rightarrow x^\perp$ .

**Lemma 3.3.** *The realizations  $\|X\|$ ,  $\|X'\|$  of a  $\Delta$ -set  $X$  and its barycentric subdivision  $X'$  are homeomorphic, via a homeomorphism  $\|X'\| \rightarrow \|X\|$  sending the vertex  $x \in X = (X')^{(0)}$  to the barycentre*

$$\hat{x} = x\left(\frac{1}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1}\right) \in x(\overset{\circ}{\Delta}^n) \subseteq \|X\| .$$

*Proof.* It suffices to consider the special case  $X = \Delta^n$ , so that  $X$  and  $X'$  are simplicial complexes, and to define a homeomorphism  $\|X'\| \rightarrow \|X\|$  by  $x \mapsto \hat{x}$  and extending linearly.  $\square$

**Definition 3.4.** Let  $X$  be a  $\Delta$ -set, and let  $x \in X$  be a simplex.

(i) The *open star space*

$$\|\text{st}(x)\| = \bigcup_{y \in x^\perp \setminus \partial x^\perp} \overset{\circ}{\Delta}^{|y|} \subseteq \|X'\| = \|X\|$$

is the subspace of the realization  $\|X'\|$  of the barycentric subdivision  $X'$  defined by the union of the interiors of the simplices  $y \in x^\perp \setminus \partial x^\perp$ , i.e.

$$y = (x \rightarrow x_0 \rightarrow \cdots \rightarrow x_n) \in X'$$

with  $x \rightarrow x_0 = x$  the identity.

(ii) The *homology* of the open star is

$$H_*(\text{st}(x)) = H_*(\Delta(\text{st}(x)))$$

with  $\Delta(\text{st}(x))$  the chain complex defined by

$$\Delta(\text{st}(x)) = \Delta(x^\perp, \partial x^\perp)_{*-|x|} .$$

□

**Lemma 3.5.** For any simplex  $x \in X$  of a  $\Delta$ -set  $X$  the characteristic  $\Delta$ -map

$$i : x^\perp \rightarrow X' ; (x \rightarrow x_0 \rightarrow \cdots \rightarrow x_n) \mapsto (x_0 \rightarrow \cdots \rightarrow x_n)$$

is injective on  $x^\perp \setminus \partial x^\perp$ . The images  $i(\partial x^\perp), i(x^\perp) \subseteq X'$  are sub- $\Delta$ -sets such that

$$\|i(x^\perp)\| \setminus \|i(\partial x^\perp)\| = \|\text{st}(x)\| \subseteq \|X'\|$$

and there are homology isomorphisms

$$\begin{aligned} H_*(\text{st}(x)) &= H_{*-|x|}(x^\perp, \partial x^\perp) \\ &\cong H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) \\ &\cong H_*(\|X\|, \|X\| \setminus \|\text{st}(x)\|) \\ &\cong H_*(\|X\|, \|X\| \setminus \{\widehat{x}\}) . \end{aligned}$$

*Proof.* The inclusion  $(\|X\|, \|X\| \setminus \|\text{st}(x)\|) \hookrightarrow (\|X\|, \|X\| \setminus \{\widehat{x}\})$  is a deformation retraction, and the open star subspace  $\|\text{st}(x)\| \subset \|X\|$  has an open regular neighbourhood

$$\|\text{st}(x)\| \times \overset{\circ}{\Delta}^{|x|} \subset \|X\|$$

with one-point compactification

$$(\|\text{st}(x)\| \times \overset{\circ}{\Delta}^{|x|})^\infty = \|i(x^\perp)\| / \|i(\partial x^\perp)\| \wedge \Delta^{|x|} / \partial \Delta^{|x|} ,$$

so that

$$\begin{aligned} H_*(\|X\|, \|X\| \setminus \{\widehat{x}\}) &\cong H_*(\|X\|, \|X\| \setminus \|\text{st}(x)\|) \\ &\cong \widetilde{H}_*(\|i(x^\perp)\| / \|i(\partial x^\perp)\| \wedge \Delta^{|x|} / \partial \Delta^{|x|}) \\ &\cong H_{*-|x|}(i(x^\perp), i(\partial x^\perp)) . \end{aligned}$$

□

**Example 3.6.** Let  $X$  be a simplicial complex. The barycentric subdivision of  $X$  is the ordered simplicial complex  $X'$  with one  $n$ -simplex for each sequence of proper face inclusions  $x_0 < x_1 < \cdots < x_n$ . By definition, the *dual cell* of a simplex  $x \in X$  is the subcomplex  $D(x, X) \subseteq X'$  consisting of all the simplices  $x_0 < x_1 < \cdots < x_n$  with  $x \leq x_0$ . The *boundary* of the dual cell is the subcomplex  $\partial D(x, X) \subseteq D(x, X)$  consisting of all the simplices  $x_0 < x_1 < \cdots < x_n$  with  $x < x_0$ . The  $\Delta$ -sets associated to  $X, X', D(x, X), \partial D(x, X)$  are just the  $\Delta$ -sets  $X, X', x^\perp, \partial x^\perp$  of 3.2, with the characteristic map  $i : x^\perp = D(x, X) \rightarrow X'$  injective. Moreover,  $X \setminus \text{st}(x) \subset X$  is a subcomplex such that

$$\|X \setminus \text{st}(x)\| = \|X\| \setminus \|\text{st}(x)\|$$

and

$$\Delta(\text{st}(x)) = \Delta(D(x, X), \partial D(x, X))_{* - |x|} \simeq \Delta(X, X \setminus \text{st}(x)) .$$

□

**Example 3.7.** Let  $X$  be the  $\Delta$ -set (2.8) with one 0-simplex  $x_0$  and one 1-simplex  $x_1$ , with non-identity morphisms

$$x_0 \rightrightarrows x_1$$

and realization  $\|X\| = S^1$ . The barycentric subdivision  $X'$  is the  $\Delta$ -set with 2 0-simplices and 2 1-simplices:

$$X'^{(0)} = \{x_0, x_1\} , \quad X'^{(1)} = \{x_0 \rightrightarrows x_1\} .$$

The duals and their boundaries are given by

$$x_0^\perp = \{x_0 \longrightarrow x_0 , \quad x_0 \rightrightarrows x_1\} \cup \{x_0 \longrightarrow x_0 \rightrightarrows x_1\} ,$$

$$\partial x_0^\perp = \{x_0 \rightrightarrows x_1\} = \{0, 1\} ,$$

$$x_1^\perp = \{x_1 \longrightarrow x_1\} , \quad \partial x_1^\perp = \emptyset .$$

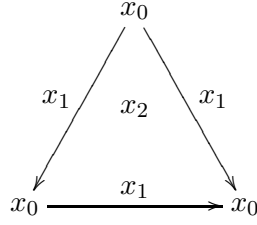
The characteristic map  $i : x_0^\perp \rightarrow X'$  is surjective but not injective, and

$$H_n(x_0^\perp, \partial x_0^\perp) = H_n(i(x_0^\perp), i(\partial x_0^\perp)) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 . \end{cases}$$

□

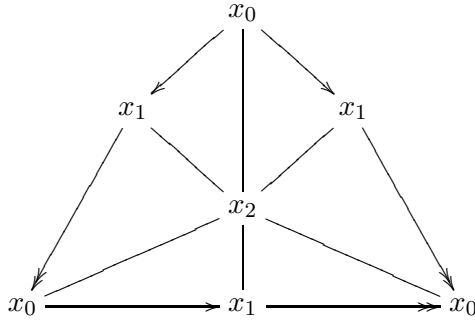
**Example 3.8.** Let  $X$  be the contractible  $\Delta$ -set with one 0-simplex  $x_0$ , one 1-simplex  $x_1$  and one 2-simplex  $x_2$ , with non-identity morphisms

$$x_0 \rightrightarrows x_1 , \quad x_1 \rightrightarrows x_2 , \quad x_0 \rightrightarrows x_2$$



The realization  $\|X\|$  is the dunce hat (Zeeman [14]). The barycentric subdivision  $X'$  is the  $\Delta$ -set with three 0-simplices, eight 1-simplices and six 2-simplices:

$$\begin{aligned} X'^{(0)} &= \{x_0, x_1, x_2\}, \\ X'^{(1)} &= \{x_0 \rightrightarrows x_1\} \cup \{x_1 \rightrightarrows x_2\} \cup \{x_0 \rightrightarrows x_2\} \\ X'^{(2)} &= \{x_0 \rightrightarrows x_1 \rightrightarrows x_2\} \end{aligned}$$



The duals and their boundaries are given by

$$\begin{aligned} x_0^\perp &= \{x_0 \longrightarrow x_0, x_0 \rightrightarrows x_1, x_0 \rightrightarrows x_2\} \\ &\cup \{x_0 \longrightarrow x_0 \rightrightarrows x_1, x_0 \longrightarrow x_0 \rightrightarrows x_2, x_0 \rightrightarrows x_1 \rightrightarrows x_2\} \\ &\cup \{x_0 \longrightarrow x_0 \rightrightarrows x_1 \rightrightarrows x_2\}, \\ \partial x_0^\perp &= \{x_0 \rightrightarrows x_1, x_0 \rightrightarrows x_2\} \cup \{x_0 \rightrightarrows x_1 \rightrightarrows x_2\}, \|\partial x_0^\perp\| \simeq S^1 \vee S^1, \\ x_1^\perp &= \{x_1 \longrightarrow x_1, x_1 \rightrightarrows x_2\} \cup \{x_1 \longrightarrow x_1 \rightrightarrows x_2\}, \\ \partial x_1^\perp &= \{x_1 \rightrightarrows x_2\}, \|\partial x_1^\perp\| \simeq \{0, 1, 2\}, \\ x_2^\perp &= \{x_2 \longrightarrow x_2\}, \partial x_2^\perp = \emptyset. \end{aligned}$$

The characteristic map  $i : x_0^\perp \rightarrow X'$  is surjective but not injective, with

$$\|i(x_0^\perp)\| \simeq \{*\}, \|\partial x_0^\perp\| \simeq S^1 \vee S^1$$

and

$$H_n(x_0^\perp, \partial x_0^\perp) = H_n(i(x_0^\perp), i(\partial x_0^\perp)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$

The characteristic map  $i : x_1^\perp \rightarrow X'$  is neither surjective nor injective, with

$$\|i(x_1^\perp)\| \simeq S^1 \vee S^1, \quad \|i(\partial x_1^\perp)\| \simeq \{*\}$$

and

$$H_n(x_1^\perp, \partial x_1^\perp) = H_n(i(x_1^\perp), i(\partial x_1^\perp)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

□

**Definition 3.9.** Given a ring  $R$  let  $\text{Mod}(R)$  be the additive category of left  $R$ -modules. For  $R = \mathbb{Z}$  write  $\text{Mod}(\mathbb{Z}) = \text{Ab}$ , as usual. □

**Definition 3.10.** (Ranicki and Weiss [8, 1.9] for simplicial complexes)

(i) The  $R$ -coefficient simplicial chain complex  $\Delta(X'; R)$  of the barycentric subdivision  $X'$  of a finite  $\Delta$ -set  $X$  is the chain complex in  $\text{Mod}(R)_*(X)$  with

$$\Delta(X'; R)(x) = \Delta(x^\perp, \partial x^\perp; R), \quad \Delta(X'; R)[x] = \Delta(x^\perp; R).$$

Compare example 2.4 case (iii).

(ii) Let  $f : Y \rightarrow X'$  be a  $\Delta$ -map from a finite  $\Delta$ -set  $Y$  to the barycentric subdivision  $X'$  of a  $\Delta$ -set  $X$ . The  $R$ -coefficient simplicial chain complex  $\Delta(Y; R)$  is the chain complex in  $\text{Mod}(R)_*(X)$  with

$$\Delta(Y; R)(x) = \Delta(x/f, \partial(x/f); R), \quad \Delta(Y; R)[x] = \Delta(x/f; R) \quad (x \in X)$$

with  $x/f, \partial(x/f)$  the  $\Delta$ -sets defined to fit into strict pullback squares of  $\Delta$ -sets

$$\begin{array}{ccccc} \partial(x/f) & \longrightarrow & x/f & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow f \\ \partial x^\perp & \longrightarrow & x^\perp & \xrightarrow{i} & X' \end{array}$$

□

#### 4. THE TOTAL COMPLEX

For a finite chain complex  $C$  in  $\mathbb{A}_*[X]$ , there is defined a chain complex in  $\mathbb{A}^*(X)$ , called the *total complex* of  $C$ .

**Definition 4.1.** The *total complex*  $\text{Tot}_* C$  of a finite chain complex  $C$  in  $\mathbb{A}_*[X]$  is the finite chain complex in  $\mathbb{A}^*(X)$  given by

$$(\text{Tot}_* C)(x)_n = C[x]_{n-|x|}$$

with differential  $d = d_{C[x]} + \sum_{i=0}^{|x|} (-1)^{i+|x|} C(\partial_i x \rightarrow x)$ . The construction is natural, defining a covariant functor

$$\mathbb{B}(\mathbb{A})_*[X] \rightarrow \mathbb{B}(\mathbb{A})^*(X); \quad C \mapsto \text{Tot}_* C.$$

□

**Remark 4.2.** There is a forgetful functor  $\mathbb{B}(\mathbb{A})_f^*(X) \rightarrow \mathbb{B}(\mathbb{A})$  taking  $C$  in  $\mathbb{B}(\mathbb{A})^*(X)$  to

$$C(X) = \bigoplus_{x \in X} C(x) .$$

Compare example 2.4. The chain complex  $(\text{Tot}_* C)(X)$  in  $\mathbb{A}$  is the ‘realization’

$$\left( \sum_{x \in X} \Delta(\Delta^{|x|}) \otimes_{\mathbb{Z}} C[x] \right) / \sim$$

with  $\sim$  the equivalence relation generated by  $a \otimes \lambda^* b \sim \lambda_* a \otimes b$  for a morphism  $\lambda : y \rightarrow z$  in  $X$ , with  $a \in \Delta(\Delta^{|y|})$ ,  $b \in C[z]$ .  $\square$

**Example 4.3.** The simplicial chain complex  $\Delta(X)$  of a finite  $\Delta$ -set  $X$  is  $(\text{Tot}_* C)(X)$  for the chain complex  $C$  in  $\text{Ab}_*[X]$  defined by  $C[x] = \mathbb{Z}$  for all  $x$  (a constant functor).  $\square$

**Remark 4.4.** There are evident forgetful functors

$$\begin{aligned} \mathbb{B}(\mathbb{A})_*(X) &\rightarrow \mathbb{B}(\mathbb{A}) ; C \mapsto C(X) , \\ \mathbb{B}(\mathbb{A})_f^*(X) &\rightarrow \mathbb{B}(\mathbb{A}) ; C \mapsto C(X) . \end{aligned}$$

The diagram

$$\begin{array}{ccc} \mathbb{B}(\mathbb{A})_*(X) & \longrightarrow & \mathbb{B}(\mathbb{A})_*[X] \xrightarrow{\text{Tot}_*} \mathbb{B}(\mathbb{A})_f^*(X) \\ & \searrow & \swarrow \\ & & \mathbb{B}(\mathbb{A}) \end{array}$$

commutes up to natural chain homotopy equivalence: for any finite chain complex  $C$  in  $\mathbb{A}_*(X)$

$$(\text{Tot}_* C)(X)_n = \sum_{x \in X} \sum_{x \rightarrow y} C(y)_{n-|x|} = \sum_{y \in X} (\Delta(X/y) \otimes_{\mathbb{Z}} C(y))_n$$

with  $X/y$  the  $\Delta$ -set defined in 2.7, which is contractible.  $\square$

**Proposition 4.5.** (i) For any objects  $M, N$  in  $\mathbb{A}_*(X)$  the abelian group  $\text{Hom}_{\mathbb{A}_*(X)}(M, N)$  is naturally an object in  $\text{Ab}_f^*(X)$ , with

$$\begin{aligned} \text{Hom}_{\mathbb{A}_*(X)}(M, N)(x) &= \text{Hom}_{\mathbb{A}}(M(x), [N][x]) \\ &= \sum_{x \rightarrow y} \text{Hom}_{\mathbb{A}}(M(x), N(y)) \quad (x \in X) . \end{aligned}$$

If  $f : M' \rightarrow M$ ,  $g : N \rightarrow N'$  are morphisms in  $\mathbb{A}_*(X)$  there is induced a morphism in  $\text{Ab}^*(X)$

$$\text{Hom}_{\mathbb{A}_*(X)}(M, N) \rightarrow \text{Hom}_{\mathbb{A}_*(X)}(M', N') ; h \mapsto ghf .$$

(ii) For any objects  $M, N$  in  $\mathbb{A}_f^*(X)$  the abelian group  $\text{Hom}_{\mathbb{A}^*(X)}(M, N)$  is naturally an object in  $\text{Ab}_*(X)$ , with

$$\begin{aligned} \text{Hom}_{\mathbb{A}^*(X)}(M, N)(x) &= \text{Hom}_{\mathbb{A}}(M(x), [N][x]) \\ &= \sum_{y \rightarrow x} \text{Hom}_{\mathbb{A}}(M(x), N(y)) \quad (x \in X) . \end{aligned}$$

Naturality as in (i).

*Proof.* Immediate from 1.9. □

**Example 4.6.** (i) For a chain complex  $C$  in  $\text{Ab}_*(X)$  the total complex in  $\text{Ab}^*(X)$  of the corresponding chain complex  $[C]$  in  $\text{Ab}_*[X]$  is given by

$$[C]_*[X] = \text{Hom}_{\text{Ab}_*(X)}(\Delta(X)^{-*}, C) .$$

(ii) For a chain complex  $D$  in  $\text{Ab}^*(X)$  the total complex in  $\text{Ab}_*(X)$  of the corresponding chain complex  $[D]$  in  $\text{Ab}^*[X]$  is given by

$$[D]^*[X] = \text{Hom}_{\text{Ab}^*(X)}(\Delta(X), D) .$$

□

## 5. CHAIN DUALITY IN $L$ -THEORY

In general, it is not possible to extend an involution  $T : \mathbb{A} \rightarrow \mathbb{A}$  on an additive category  $\mathbb{A}$  to the functor category  $\mathbb{A}_*(X)$  for an arbitrary category  $X$ . An object in  $\mathbb{A}_*(X)$  is an induced contravariant functor  $F : X \rightarrow \mathbb{A}$  and the composite of the contravariant functors

$$X \xrightarrow{F} \mathbb{A} \xrightarrow{T} \mathbb{A}$$

is a covariant functor, not a contravariant functor, let alone an induced contravariant functor. A ‘chain duality’ on  $\mathbb{A}$  is essentially an involution on the derived category of finite chain complexes and chain homotopy classes of chain maps; an involution on  $\mathbb{A}$  is an example of a chain duality. Given a chain duality on  $\mathbb{A}$  we shall now define a chain duality on the induced functor category  $\mathbb{A}_*(X)$ , for any  $\Delta$ -set  $X$ , essentially in the same way as was carried out for a simplicial complex  $X$  in [6].

**Definition 5.1.** (Ranicki [6, 1.1]) A *chain duality*  $(T, e)$  on an additive category  $\mathbb{A}$  is a contravariant additive functor

$$T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$$

together with a natural transformation

$$e : T^2 \rightarrow 1 : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$$

such that for each object  $M$  in  $\mathbb{A}$

- (i)  $e(T(M)) \circ T(e(M)) = 1 : T(M) \rightarrow T^3(M) \rightarrow T(M)$ ,
- (ii)  $e(M) : T^2(M) \rightarrow M$  is a chain equivalence.

□

A chain duality  $(T, e)$  on  $\mathbb{A}$  extends to a contravariant functor on the bounded chain complex category

$$T : \mathbb{B}(\mathbb{A}) \rightarrow \mathbb{B}(\mathbb{A}) ; C \mapsto T(C) ,$$

using the double complex construction with

$$T(C)_n = \sum_{p+q=n} T(C_{-p})_q , \quad d_{T(C)} = d_{T(C_{-p})} + (-)^q T(d : C_{-p+1} \rightarrow C_{-p}) ,$$

and  $e(C) : T^2(C) \rightarrow C$  a chain equivalence. For any objects  $M, N$  in an additive category  $\mathbb{A}$  there is defined a  $\mathbb{Z}$ -module  $\text{Hom}_{\mathbb{A}}(M, N)$ . Thus for any chain complexes  $C, D$  in  $\mathbb{A}$  there is defined a  $\mathbb{Z}$ -module chain complex  $\text{Hom}_{\mathbb{A}}(C, D)$ , with

$$\text{Hom}_{\mathbb{A}}(C, D)_n = \sum_{q-p=n} \text{Hom}_{\mathbb{A}}(C_p, D_q) , \quad d_{\text{Hom}_{\mathbb{A}}(C, D)}(f) = d_D f + (-)^q f d_C .$$

If  $(T, e)$  is a chain duality on  $\mathbb{A}$  there is defined a  $\mathbb{Z}$ -module chain map

$$\text{Hom}_{\mathbb{A}}(TC, D) \rightarrow \text{Hom}_{\mathbb{A}}(TD, C) ; f \mapsto e(C)T(f)$$

which is a chain equivalence for finite  $C$ .

**Example 5.2.** An involution  $(T, e)$  on  $\mathbb{A}$  is a contravariant functor  $T : \mathbb{A} \rightarrow \mathbb{A}$  with a natural equivalence  $e : T^2 \rightarrow 1$  such that for each object  $M$  in  $\mathbb{A}$

$$e(T(M)) = T(e(M)^{-1}) : T^3(M) \rightarrow T(M) .$$

This is essentially the same as a chain duality  $(T, e)$  such that  $T(M)$  is a 0-dimensional chain complex for each object  $M$  in  $\mathbb{A}$ . □

**Definition 5.3.** A *chain product*  $(\otimes_{\mathbb{A}}, b)$  on an additive category  $\mathbb{A}$  is a natural pairing

$\otimes_{\mathbb{A}} : \text{Ob}(\mathbb{A}) \times \text{Ob}(\mathbb{A}) \rightarrow \{\mathbb{Z}\text{-module chain complexes}\} ; (M, N) \mapsto M \otimes_{\mathbb{A}} N$   
together with a natural chain equivalence

$$b(M, N) : M \otimes_{\mathbb{A}} N \rightarrow N \otimes_{\mathbb{A}} M$$

such that up to natural isomorphism

$$\begin{aligned} (M \oplus M') \otimes_{\mathbb{A}} N &= (M \otimes_{\mathbb{A}} N) \oplus (M' \otimes_{\mathbb{A}} N) , \\ M \otimes_{\mathbb{A}} (N \oplus N') &= (M \otimes_{\mathbb{A}} N) \oplus (M \otimes_{\mathbb{A}} N') \end{aligned}$$

and

$$b(N, M) \circ b(M, N) \simeq 1 : M \otimes_{\mathbb{A}} N \rightarrow M \otimes_{\mathbb{A}} N .$$

□

**Remark 5.4.** The notion of chain product is a linear version of an ‘SW-product’ in the sense of Weiss and Williams [12], where SW = Spanier-Whitehead. □

Given an additive category  $\mathbb{A}$  with a chain product  $(\otimes_{\mathbb{A}}, b)$  and chain complexes  $C, D$  in  $\mathbb{A}$  let  $C \otimes_{\mathbb{A}} D$  be the  $\mathbb{Z}$ -module chain complex defined by

$$(C \otimes_{\mathbb{A}} D)_n = \sum_{p+q+r=n} (C_p \otimes_{\mathbb{A}} D_q)_r ,$$

$$d_{C \otimes_{\mathbb{A}} D} = d_{C_p \otimes_{\mathbb{A}} C_q} + (-)^r (1 \otimes_{\mathbb{A}} d_D + (-)^q d_C \otimes_{\mathbb{A}} 1) .$$

By the naturality of  $b$  there is defined a natural chain equivalence

$$b(C, D) : C \otimes_{\mathbb{A}} D \rightarrow D \otimes_{\mathbb{A}} C .$$

**Proposition 5.5.** *Let  $\mathbb{A}$  be an additive category.*

(i) *A chain duality  $(T, e)$  on  $\mathbb{A}$  determines a chain product  $(\otimes_{\mathbb{A}}, b)$  on  $\mathbb{A}$  by*

$$M \otimes_{\mathbb{A}} N = \text{Hom}_{\mathbb{A}}(TM, N) ,$$

$$b(M, N) : M \otimes_{\mathbb{A}} N \rightarrow N \otimes_{\mathbb{A}} M ;$$

$$(f : TM \rightarrow N) \mapsto (e(M) \circ T(f) : TN \rightarrow T^2M \rightarrow M) .$$

(ii) *If  $(\otimes_{\mathbb{A}}, b)$  is a chain product on  $\mathbb{A}$  such that*

$$M \otimes_{\mathbb{A}} N = \text{Hom}_{\mathbb{A}}(TM, N) , \quad b(M, N)(f) = e(M) \circ T(f)$$

*for some contravariant additive functor  $T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$  and natural transformation  $e : T^2 \rightarrow 1 : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$ , then  $(T, e)$  is a chain duality on  $\mathbb{A}$ .*

*Proof.* Immediate from the definitions. □

**Example 5.6.** Let  $R$  be a ring with an involution  $R \rightarrow R; r \mapsto \bar{r}$ . Regard a (left)  $R$ -module  $M$  as a right  $R$ -module by

$$M \times R \rightarrow M ; (x, r) \mapsto \bar{r}x .$$

Thus for any  $R$ -modules  $M, N$  there is defined a  $\mathbb{Z}$ -module

$$M \otimes_R N = (M \otimes_{\mathbb{Z}} N) / \{ \bar{r}x \otimes y - x \otimes ry \mid x \in M, y \in N, r \in R \}$$

with a natural isomorphism

$$b(M, N) : M \otimes_R N \rightarrow N \otimes_R M ; x \otimes y \mapsto y \otimes x$$

defining a (0-dimensional) chain product  $(\otimes_R, b)$  on the  $R$ -module category  $\text{Mod}(R)$ . As in 2.5 use the involution on  $R$  to define the contravariant duality functor

$$T : \text{Mod}(R) \rightarrow \text{Mod}(R) ; M \mapsto M^* = \text{Hom}_R(M, R)$$

with

$$R \times M^* \rightarrow M^* ; (r, f) \mapsto (x \mapsto f(x)\bar{r}) .$$

The natural  $\mathbb{Z}$ -module morphism defined for any  $R$ -modules  $M, N$  by

$$M \otimes_R N \rightarrow \text{Hom}_R(M^*, N) ; x \otimes y \mapsto (f \mapsto f(x)y)$$

is an isomorphism for f.g. projective  $M$ . The  $R$ -module morphism defined for any  $R$ -module  $M$  by

$$e'(M) : M \rightarrow M^{**} ; x \mapsto (f \mapsto f(x))$$

is an isomorphism for f.g. projective  $M$ . Let  $\text{Proj}(R) \subset \text{Mod}(R)$  be the full subcategory of f.g. projective  $R$ -modules. The natural isomorphisms

$$e(M) = e'(M)^{-1} : M^{**} \rightarrow M$$

define an involution  $(T, e)$  on  $\text{Proj}(R)$ , corresponding to the restriction to  $\text{Proj}(R)$  of the chain product  $(\otimes_R, b)$  on  $\text{Mod}(R)$ .  $\square$

**Proposition 5.7.** (Ranicki [6, 5.1,5.9,7], Weiss [11, 1.5])

A chain duality  $(T_{\mathbb{A}}, e_{\mathbb{A}})$  on an additive category  $\mathbb{A}$  extends to a chain duality  $(T_{\mathbb{A}_*(X)}, e_{\mathbb{A}_*(X)})$  on  $\mathbb{A}_*(X)$ , for any  $\Delta$ -set  $X$

$$T_{\mathbb{A}_*(X)} : \mathbb{A}_*(X) \longrightarrow \mathbb{A}_*[X] \xrightarrow{\text{Tot}_*} \mathbb{B}(\mathbb{A})_f^*(X) \xrightarrow{T_{\mathbb{A}}} \mathbb{B}(\mathbb{A})_*(X)$$

where  $T_{\mathbb{A}} : \mathbb{B}(\mathbb{A})_f^*(X) \rightarrow \mathbb{B}(\mathbb{A})_*(X)$  is the extension of the contravariant functor

$$T_{\mathbb{A}} : \mathbb{A}_f^*(X) \rightarrow \mathbb{B}(\mathbb{A})_*(X) ; M = \sum_{x \in X} M(x) \mapsto T_{\mathbb{A}}(M) = \sum_{x \in X} T_{\mathbb{A}}(M(x)) .$$

More explicitly, the chain dual of a finite chain complex  $C$  in  $\mathbb{A}_*(X)$  is given by

$$T_{\mathbb{A}_*(X)}(C) = T_{\mathbb{A}}(\text{Tot}_* C) ,$$

so that

$$\begin{aligned} T_{\mathbb{A}_*(X)}(C)(x) &= T_{\mathbb{A}}(C[x]_{*-|x|}) \\ &= \sum_{x \rightarrow y} T_{\mathbb{A}}(C(y)_{*-|x|}) \quad (x \in X) . \end{aligned}$$

**Example 5.8.** Let  $\mathbb{A} = \mathbb{A}(\mathbb{Z})$ , the additive category of f.g. free abelian groups.

(i) For any finite chain complex  $C$  in  $\mathbb{A}_*(X)$ , which we also view as a (degreewise) induced chain complex  $C$  in  $\mathbb{A}_*(X)$ , the total complex  $\text{Tot}_*(C)$  is given by 4.6 to be

$$\text{Hom}_{\mathbb{A}_*(X)}(\Delta(X)^{-*}, C) ,$$

so that the chain dual of  $C$  is given by

$$T_{\mathbb{A}_*(X)}(C) = \text{Hom}_{\mathbb{A}}(\text{Hom}_{\mathbb{A}_*(X)}(\Delta(X)^{-*}, C), \mathbb{Z}) .$$

(ii) As in 3.10 regard the simplicial chain complex  $\Delta(X')$  of the barycentric subdivision  $X'$  of a finite  $\Delta$ -set  $X$  as a chain complex in  $\mathbb{A}_*(X)$  or in  $\mathbb{A}_*[X]$  with

$$\Delta(X')(x) = \Delta(x^\perp, \partial x^\perp) , \quad \Delta(X')[x] = \Delta(x^\perp)$$

for  $x \in X$ . The chain dual  $T(\Delta(X'))$  is the chain complex in  $\mathbb{A}_*(X)$  with

$$T(\Delta(X'))(x) = \Delta(x^\perp)^{|x|-*} \quad (x \in X) .$$

□

**Remark 5.9.** See Fimmel [2] and Woolf [13] for Verdier duality for local coefficient systems on simplicial sets and simplicial complexes. In particular, [13] relates the chain duality of [6, Chapter 5] defined on  $\text{Proj}(R)_*(X)$  for a simplicial complex  $X$  to the Verdier duality for sheaves of  $R$ -module chain complexes over the polyhedron  $\|X\|$ . □

For any additive category with chain duality  $\mathbb{A}$  let  $\mathbb{L}_\bullet(\mathbb{A})$  be the quadratic  $L$ -theory  $\Omega$ -spectrum defined in Ranicki [6], with homotopy groups

$$\pi_n(\mathbb{L}_\bullet(\mathbb{A})) = L_n(\mathbb{A}) .$$

It was shown in [6, Chapter 13] that the covariant functor

$$\{\text{simplicial complexes}\} \rightarrow \{\Omega\text{-spectra}\} ; X \mapsto \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)))$$

is an unreduced homology theory, i.e. a covariant functor which is homotopy invariant, excisive and sends arbitrary disjoint unions to wedges. More generally :

**Proposition 5.10.** ([6, 13.7] for simplicial complexes)

- (i) *If  $\mathbb{A}$  is an additive category with chain duality and  $X$  is a  $\Delta$ -set then  $\mathbb{A}_*(X)$  is an additive category with chain duality.*
- (ii) *The functor*

$$\{\Delta\text{-sets}\} \rightarrow \{\Omega\text{-spectra}\} ; X \mapsto L_*(\mathbb{A}, X) = \mathbb{L}_\bullet(\mathbb{B}(\mathbb{A}_*(X)))$$

*is an unreduced homology theory, that is  $L_*(\mathbb{A}, X) = H_*(X; \mathbb{L}_\bullet(\mathbb{A}))$ .*

- (iii) *Let  $R$  be a ring with involution, so that  $\mathbb{A} = \text{Proj}(R)$  is an additive category of f.g. projective  $R$ -modules with the duality involution. If  $X$  is a  $\Delta$ -set and  $p : \tilde{X} \rightarrow X$  is a regular cover with group of covering translations  $\pi$  (e.g. the universal cover with  $\pi = \pi_1(X)$ ) the assembly functor*

$$\begin{aligned} A : \mathbb{B}(R)_*(X) &\rightarrow \mathbb{B}(R[\pi]) ; C \mapsto C(\tilde{X}) \\ (C(\tilde{X})) &= \sum_{x \in \tilde{X}} C(p(x)) \end{aligned}$$

*is a functor of additive categories with chain duality. The assembly maps  $A$  induced in the  $L$ -groups fit into an exact sequence*

$$\cdots \rightarrow H_n(X; \mathbb{L}_\bullet(R)) \xrightarrow{A} L_n(R[\pi_1(X)]) \rightarrow S_n(R, X) \rightarrow H_{n-1}(X; \mathbb{L}_\bullet(R)) \rightarrow \cdots$$

*with  $S_n(R, X)$  the cobordism group of the  $R[\pi_1(X)]$ -contractible  $(n - 1)$ -dimensional quadratic Poincaré complexes in  $\mathbb{A}_*(X)$ .*

*Proof.* Exactly as for the simplicial complex case, but using the  $\Delta$ -set duals instead of the dual cells!  $\square$

**Example 5.11.** Let  $X = S^1$  be the  $\Delta$ -set of the circle (2.8, 3.7) with one 0-simplex and one 1-simplex. Given a ring with involution  $R$  let the Laurent polynomial extension ring  $R[z, z^{-1}]$  have the involution  $\bar{z} = z^{-1}$ . An  $n$ -dimensional quadratic Poincaré complex in  $\text{Proj}(R)_*(S^1)$  is an  $n$ -dimensional fundamental quadratic Poincaré cobordism over  $R$ , with assembly the union  $n$ -dimensional quadratic Poincaré complex over  $R[z, z^{-1}]$ , and the assembly maps

$$A : H_n(S^1; \mathbb{L}_\bullet(R)) = L_n(R) \oplus L_{n-1}(R) \rightarrow L_n(R[z, z^{-1}])$$

are isomorphisms modulo the usual  $K$ -theoretic decorations (Ranicki [7, Chapter 24].  $\square$

**Remark 5.12.** Proposition 5.10 has an evident analogue for the symmetric  $L$ -groups  $L^*$ .  $\square$

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