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On the calculation of UNil

Frank Connolly[a], Andrew Ranicki[b,*]

[a]*Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-5683, USA*
[b]*School of Mathematics, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, Scotland, UK*

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Abstract

Cappell's codimension 1 splitting obstruction surgery group UNil_n is a direct summand of the Wall surgery obstruction group of an amalgamated free product. For any ring with involution R we use the quadratic Poincaré cobordism formulation of the L-groups to prove that

$$L_n(R[x]) = L_n(R) \oplus \mathrm{UNil}_n(R; R, R).$$

We combine this with Weiss' universal chain bundle theory to produce almost complete calculations of $\mathrm{UNil}_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ and the Wall surgery obstruction groups $L_*(\mathbb{Z}[D_\infty])$ of the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$.

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0. Introduction

The nilpotent K- and L-groups of rings are a rich source of algebraic invariants for geometric topology, giving results of two types: if the groups are zero it is possible to solve the associated splitting and classification problems, while if they are nonzero

* Corresponding author.
E-mail addresses: connolly.1@nd.edu (F. Connolly), a.ranicki@ed.ac.uk (A. Ranicki).

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the groups are infinitely generated and the solutions to the problems are definitely obstructed, see [2,4,6,8,9,10,11,16].

The unitary nilpotent L-groups UNil_* arise as follows. Suppose given a closed n-dimensional manifold X which is expressed as a union of codimension 0 submanifolds $X_1, X_{-1} \subseteq X$

$$X \;=\; X_1 \cup X_{-1}$$

with

$$X_0 \;=\; X_1 \cap X_{-1} \;=\; \partial X_{-1} \;=\; \partial X_1 \subseteq X$$

a codimension 1 submanifold. Assume X, X_{-1}, X_0, X_1 are connected, and that the maps $\pi_1(X_0) \to \pi_1(X_{\pm 1})$ are injective, so that by the van Kampen theorem the fundamental group of X is an amalgamated free product

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(X_0)} \pi_1(X_{-1})$$

with $\pi_1(X_i) \to \pi_1(X)$ $(i = -1, 0, 1)$ injective. Given another closed n-dimensional manifold M and a simple homotopy equivalence $f : M \to X$ there is a single obstruction

$$s(f) \in \mathrm{UNil}_{n+1}(R; \mathcal{B}_1, \mathcal{B}_{-1})$$

to deforming f by an h-cobordism of domains to a homotopy equivalence of the form

$$f_1 \cup f_{-1} \;:\; M_1 \cup M_{-1} \;\to\; X_1 \cup X_{-1}$$

with $f_{\pm 1} : (M_{\pm 1}, \partial M_{\pm 1}) \to (X_{\pm 1}, \partial X_{\pm 1})$ homotopy equivalences of manifolds with boundary such that

$$f_1| \;=\; f_{-1}| \;:\; \partial M_1 \;=\; \partial M_{-1} \to \partial X_1 \;=\; \partial X_{-1}$$

and

$$R \;=\; \mathbb{Z}[\pi_1(X_0)],\;\; \mathcal{B}_{\pm 1} \;=\; \mathbb{Z}[\pi_1(X_{\pm 1}) \backslash \pi_1(X_0)].$$

Cappell [5,6] proved geometrically that the free Wall [21] surgery obstruction groups $L_* = L^h_*$ of the fundamental group ring

$$\Lambda \;=\; \mathbb{Z}[\pi_1(X)] \;=\; \mathbb{Z}[\pi_1(X_1)] *_{\mathbb{Z}[\pi_1(X_0)]} \mathbb{Z}[\pi_1(X_{-1})]$$

have direct sum decompositions

$$L_*(\Lambda) \;=\; \mathrm{UNil}_*(R;\mathcal{B}_1,\mathcal{B}_{-1}) \oplus L'_*(\mathbb{Z}[\pi_1(X_0)] \to \mathbb{Z}[\pi_1(X_1)] \times \mathbb{Z}[\pi_1(X_{-1})])$$

with L'_* appropriately decorated intermediate relative L-groups. The split monomorphism

$$\mathrm{UNil}_{n+1}(R;\mathcal{B}_1,\mathcal{B}_{-1}) \;\to\; L_{n+1}(\Lambda);\;\; s(f) \mapsto \sigma(g)$$

sends the splitting obstruction $s(f)$ to the surgery obstruction $\sigma(g)$ of the 'unitary nilpotent cobordism' of [6], an $(n+1)$-dimensional normal map cobordism between f and a split homotopy equivalence. The 4-periodicity $L_*(\Lambda) = L_{*+4}(\Lambda)$ extends to a 4-periodicity

$$\mathrm{UNil}_*(R;\mathcal{B}_1,\mathcal{B}_{-1}) \;=\; \mathrm{UNil}_{*+4}(R;\mathcal{B}_1,\mathcal{B}_{-1}).$$

Farrell [10] obtained a remarkable factorization

$$\mathrm{UNil}_{n+1}(R;\mathcal{B}_1,\mathcal{B}_{-1}) \;\to\; \mathrm{UNil}_{n+1}(\Lambda;\Lambda,\Lambda) \;\to\; L_{n+1}(\Lambda).$$

For this reason (and some others too) the groups $\mathrm{UNil}_*(R;R,R)$ for any ring with involution R are of especial significance to us, and we introduce the abbreviation:

$$\mathrm{UNil}_n(R) \;=\; \mathrm{UNil}_n(R;R,R).$$

But even the groups $\mathrm{UNil}_*(\mathbb{Z})$ have remained opaque for the last 30 years. Cappell [3–5] proved that $\mathrm{UNil}_{4k}(\mathbb{Z}) = 0$ and that $\mathrm{UNil}_{4k+2}(\mathbb{Z})$ is infinitely generated. The UNil-groups $\mathrm{UNil}_*(R;\mathcal{B}_1,\mathcal{B}_{-1})$ are 2-primary torsion groups. Farrell [10] proved that $4\mathrm{UNil}_*(R) = 0$, for any ring R. Connolly and Koźniewski [8] obtained an isomorphism

$$\mathrm{UNil}_{4k+2}(\mathbb{Z}) \;\cong\; \bigoplus_1^\infty \mathbb{F}_2,$$

together with information on $\mathrm{UNil}_{4k+2}(R)$ for various Dedekind domains and division rings. But that is nearly all that is known.

The infinite dihedral group is a free product of two copies of the cyclic group $\mathbb{Z}_2$ of order 2

$$D_\infty \;=\; \mathbb{Z}_2 * \mathbb{Z}_2.$$

Since the surgery obstruction groups $L_*(R[D_\infty])$ are hard to compute directly, the split monomorphisms $\mathrm{UNil}_*(R) \to L_*(R[D_\infty])$ are more useful in computing $L_*(R[D_\infty])$

from $\mathrm{UNil}_*(R)$ than the other way round. Connolly and Koźniewski [8] expressed $\mathrm{UNil}_*(R;\mathcal{B}_1,\mathcal{B}_{-1})$ as the L-groups $L_*(\mathbb{A}_\alpha[x])$ of an *additive category* with involution $\mathbb{A}_\alpha[x]$. Although this expression did give new computations of $\mathrm{UNil}_*(R)$, the L-theory of additive categories with involution [17] is not in general very computable.

The first goal of this paper therefore, is to provide a new description for $\mathrm{UNil}_n(R)$ in terms of L-groups, which can be used to computational advantage. Cappell and Farrell observed that the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ can also be viewed as an extension of $\mathbb{Z}$ by $\mathbb{Z}_2$

$$\{1\} \to \mathbb{Z} \to D_\infty \to \mathbb{Z}_2 \to \{1\},$$

so that the classifying space can be viewed both as a one-point union

$$K(D_\infty,1) \ = \ K(\mathbb{Z}_2,1) \vee K(\mathbb{Z}_2,1) \ = \ \mathbb{RP}^\infty \vee \mathbb{RP}^\infty$$

and as the total space of a fibration

$$K(\mathbb{Z},1) \ = \ S^1 \to K(D_\infty,1) \to K(\mathbb{Z}_2,1) \ = \ \mathbb{RP}^\infty,$$

and that this should have implications for codimension 1 surgery obstruction theory with $\pi_1 = D_\infty$. This observation was used in [16, pp. 737–745] to prove geometrically that for the group ring $R = \mathbb{Z}[\pi]$ of a finitely presented group π

$$\mathrm{UNil}_*(R) \ = \ NL_*(R) \ = \ \ker(L_*(R[x]) \to L_*(R))$$

with the involution on R extended to $R[x]$ by $\overline{x} = x$, and $R[x] \to R; x \mapsto 0$ the augmentation map. The NL-groups are L-theoretic analogues of the nilpotent K-group

$$NK_1(R) \ = \ \ker(K_1(R[x]) \to K_1(R)) \ = \ \widetilde{\mathrm{Nil}}_0(R)$$

of Chapter XII of Bass [2], which is such that

$$K_1(R[x]) \ = \ K_1(R) \oplus NK_1(R).$$

Theorem A. *For any ring with involution R*

$$\mathrm{UNil}_*(R) \ = \ NL_*(R)$$

so that

$$L_n(R[x]) \ = \ L_n(R) \oplus \mathrm{UNil}_n(R).$$

We develop a new method for calculating $\mathrm{UNil}_*(R)$, adopting the following strategy. The symmetric L-groups $L^*(R)$ of a ring R with an involution $R \to R; x \mapsto \overline{x}$ were defined by Mishchenko [12] and Ranicki [14,15] to be the cobordism groups of symmetric Poincaré complexes over R. The quadratic L-groups $L_*(R)$ were expressed in [14,15] as the cobordism groups of quadratic Poincaré complexes over R, and the two types of L-groups were related by an exact sequence

$$\cdots \to L_n(R) \to L^n(R) \to \widehat{L}^n(R) \to L_{n-1}(R) \to \cdots$$

with the hyperquadratic L-groups $\widehat{L}^*(R)$ the cobordism groups of (symmetric, quadratic) Poincaré pairs. The symmetric and hyperquadratic L-groups are not 4-periodic in general, but there are defined natural maps

$$L^n(R) \to L^{n+4}(R),\ \ \widehat{L}^n(R) \to \widehat{L}^{n+4}(R)$$

(which are isomorphisms for certain R, e.g. a Dedekind ring or the polynomial extension of a Dedekind ring). The 4-periodic versions of the symmetric and hyperquadratic L-groups

$$L^{n+4*}(R) \ = \ \lim_{k\to\infty} L^{n+4k}(R),\ \ \widehat{L}^{n+4*}(R) \ = \ \lim_{k\to\infty} \widehat{L}^{n+4k}(R)$$

are related by an exact sequence

$$\cdots \to L_n(R) \to L^{n+4*}(R) \to \widehat{L}^{n+4*}(R) \to L_{n-1}(R) \to \cdots \ .$$

The theory of Weiss [22,23] identified $\widehat{L}^{n+4*}(R)$ with the 'twisted Q-group' $Q_n(B^R, \beta^R)$ of the 'universal chain bundle' (B^R, β^R) over R, which can be computed (more or less effectively) from the Tate $\mathbb{Z}_2$-cohomology groups of the involution on R

$$\begin{aligned} H_n(B^R) \ &= \ \widehat{H}^n(\mathbb{Z}_2; R) \\ &= \ \{a \in R \,|\, \overline{a} = (-1)^n a\}/\{b + (-1)^n \overline{b} \,|\, b \in R\}. \end{aligned}$$

In Proposition 11 we show that for a Dedekind ring with involution R

$$L^n(R[x]) \ = \ L^n(R), \quad NL^n(R) \ = \ 0$$

making the UNil-groups

$$\mathrm{UNil}_n(R) \ = \ \ker(Q_n(B^{R[x]}, \beta^{R[x]}) \to Q_n(B^R, \beta^R))$$

accessible to computation.

Theorem B. *For the ring* $\mathbb{Z}$, *we have*

$$\mathrm{UNil}_0(\mathbb{Z}) \ = \ 0, \ \mathrm{UNil}_1(\mathbb{Z}) \ = \ 0$$

and there is an exact sequence

$$0 \to \mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{\psi^2-1} \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathrm{UNil}_2(\mathbb{Z}) \to 0$$

with

$$\psi^2 \ : \ \mathbb{F}_2[x] \to \mathbb{F}_2[x]; \ a \mapsto a^2$$

the Frobenius map. $\mathrm{UNil}_3(\mathbb{Z})$ *is not finitely generated, with* $4\mathrm{UNil}_3(\mathbb{Z}) = 0$.

We now give an outline of the rest of this paper.

In Section 1 we define the groups $\mathrm{UNil}_n(R)$ and the map $c : \mathrm{UNil}_n(R) \to L_n(R[x])$, as well as the various other morphisms and groups with which we will be working. Theorem A is proved in Section 1.

In Section 2 we relate $\mathrm{UNil}_n(R)$ for Dedekind R to the group of symmetric structures on the universal chain bundle of Weiss. We then make the calculations necessary to prove Theorem B.

1. Fundamental concepts. The proof of Theorem A

1.1. Algebraic L-groups

Throughout this paper R denotes a ring with an involution

$$R \to R; \ r \mapsto \bar{r}.$$

An R-module is understood to be a left R-module, unless a right R-module action is specified. Given an R-module P let P^t be the right R-module with the same additive group and

$$P^t \times R \to P^t; \ (x, r) \mapsto \bar{r}x.$$

The dual of an R-module P is the R-module

$$\begin{aligned} P^* \ &= \ \mathrm{Hom}_R(P, R), \\ R \times P^* \to P^*; \ &(r, f) \mapsto (x \mapsto f(x)\bar{r}). \end{aligned}$$

Write the evaluation pairing as

$$\langle\ ,\ \rangle\ :\ P^*\times P\to R;\ \ (f,x)\mapsto\langle f,x\rangle\ =\ f(x).$$

An element $\phi\in \mathrm{Hom}_R(P,P^*)$ determines a sesquilinear form on P

$$\langle\ ,\ \rangle_\phi\ :\ P\times P\to R;\ \ (x,y)\mapsto\langle\phi(x),y\rangle,$$

and we identify $\mathrm{Hom}_R(P,P^*)$ with the additive group of such forms. The dual of a finitely generated (f.g.) projective R-module P is a f.g. projective R-module P^*, and the morphism

$$P\to P^{**};\ \ x\mapsto(f\mapsto\overline{f(x)})$$

is an isomorphism, which we shall use to identify

$$P^{**}\ =\ P,$$

and to define the ε-duality involution

$$T_\varepsilon\ :\ \mathrm{Hom}_R(P,P^*)\to\mathrm{Hom}_R(P,P^*);\ \ \phi\mapsto\varepsilon\phi^*,\quad \langle x,y\rangle_{\phi^*}\ =\ \overline{\langle y,x\rangle}_\phi.$$

For $\varepsilon=\pm1$, any R-module chain complex C and any $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex X define the $\mathbb{Z}$-module chain complexes

$$\begin{aligned} X^{\%}(C,\varepsilon)\ &=\ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(X,C^t\otimes_R C),\\ X_{\%}(C,\varepsilon)\ &=\ X\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_R C)\end{aligned}$$

with $T\in\mathbb{Z}_2$ acting on $C^t\otimes_R C$ by the signed transposition isomorphisms

$$T_\varepsilon\ :\ C^t_p\otimes_R C_q\to C^t_q\otimes_R C_p;\ \ x\otimes y\mapsto(-1)^{pq}\varepsilon y\otimes x.$$

We shall be mainly concerned with finite chain complexes C of f.g. projective R-modules, in which case we identify

$$C^t\otimes_R C\ =\ \mathrm{Hom}_R(C^*,C)$$

using the natural $\mathbb{Z}$-module isomorphisms

$$C^t_p\otimes_R C_q\to\mathrm{Hom}_R(C^p,C_q);\ \ x\otimes y\mapsto(f\mapsto\overline{f(x)}.y)$$

with $C^p = (C_p)^*$. The signed transposition isomorphisms correspond to the signed duality isomorphisms

$$T_\varepsilon \ : \ \mathrm{Hom}_R(C^p, C_q) \to \mathrm{Hom}_R(C^q, C_p); \ \ \phi \mapsto (-1)^{pq}\varepsilon\phi^*.$$

As in [14,15] the group of n-dimensional ε-symmetric (resp. ε-hyperquadratic, resp. ε-quadratic) structures on C is defined by:

$$\begin{aligned} Q^n(C,\varepsilon) &= H_n(W^{\%}C), \quad \widehat{Q}^n(C,\varepsilon) = H_n(\widehat{W}^{\%}C), \\ Q_n(C,\varepsilon) &= H_n(W_{\%}C) = H_n((W^{-*})^{\%}C) \end{aligned}$$

where W (resp. $\widehat{W}$) denotes the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ (resp. complete resolution) and

$$W^{-*} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathbb{Z}[\mathbb{Z}_2]).$$

If $S^{-1}W^{-*}$ denotes the desuspension of W^{-*}, the short exact sequence

$$0 \to S^{-1}W^{-*} \to \widehat{W} \to W \to 0$$

induces the exact sequence:

$$\cdots \to Q_n(C,\varepsilon) \to Q^n(C,\varepsilon) \xrightarrow{J} \widehat{Q}^n(C,\varepsilon) \to Q_{n-1}(C,\varepsilon) \to \cdots \ . \tag{1}$$

Given a f.g. projective R-module P define the 0-dimensional f.g. projective R-module chain complex

$$C \ : \ \cdots \to 0 \to C_0 = P^* \to 0 \to \cdots \ .$$

At the risk of notational confusion, the 0-dimensional ε-symmetric and ε-quadratic Q-groups of C are written

$$\begin{aligned} Q^0(C,\varepsilon) &= Q^\varepsilon(P) = \ker(1 - T_\varepsilon : \mathrm{Hom}_R(P, P^*) \to \mathrm{Hom}_R(P, P^*)), \\ Q_0(C,\varepsilon) &= Q_\varepsilon(P) = \mathrm{coker}(1 - T_\varepsilon : \mathrm{Hom}_R(P, P^*) \to \mathrm{Hom}_R(P, P^*)). \end{aligned}$$

Definition 1. An *ε-symmetric form* (P, ϕ) (resp. an *ε-quadratic form* (P, ψ)) over R is a f.g. projective R-module P together with an element $\phi \in Q^\varepsilon(P)$ (resp. $\psi \in Q_\varepsilon(P)$). The form is *nonsingular* if the R-module morphism

$$\phi : P \to P^* \ \text{(resp. } N_\varepsilon(\psi) = (1 + T_\varepsilon)\psi : P \to P^*)$$

is an isomorphism.

We refer to Ranicki [14–16,19] for various accounts of the construction of the free ε-symmetric (resp. quadratic) L-groups $L^n(R,\varepsilon)$ (resp. $L_n(R,\varepsilon)$) as the cobordism groups of n-dimensional ε-symmetric (resp. ε-quadratic) Poincaré complexes over R $(C,\phi\in Q^n(C,\varepsilon))$ (resp. $(C,\psi\in Q_n(C,\varepsilon))$) with

$$C\ :\ \cdots\to 0\to C_n\to C_{n-1}\to\cdots\to C_0\to 0\to\cdots$$

an n-dimensional f.g. free R-module chain complex. The projective L-groups $L_p^*(R,\varepsilon)$ (resp. $L_*^p(R,\varepsilon)$) are constructed in the same way, using f.g. projective C.

The suspension of an R-module chain complex C is the R-module chain complex SC with

$$d_{SC}\ =\ d_C\ :\ (SC)_{r+1}\ =\ C_r\to(SC)_r\ =\ C_{r-1}.$$

As in [14,15, p. 105] use the natural $\mathbb{Z}$-module isomorphisms

$$S^2(W^{\%}(C,\varepsilon))\ \cong\ W^{\%}(SC,-\varepsilon),\quad S^2(W_{\%}(C,\varepsilon))\ \cong\ W_{\%}(SC,-\varepsilon)$$

to identify

$$Q^n(C,\varepsilon)\ =\ Q^{n+2}(SC,-\varepsilon),\quad Q_n(C,\varepsilon)\ =\ Q_{n+2}(SC,-\varepsilon)$$

and to define the skew-suspension maps

$$\begin{aligned}\overline{S}^n\ &:\ L^n(R,\varepsilon)\to L^{n+2}(R,-\varepsilon);\ \ (C,\phi)\mapsto(SC,\phi),\\ \overline{S}_n\ &:\ L_n(R,\varepsilon)\to L_{n+2}(R,-\varepsilon);\ \ (C,\psi)\mapsto(SC,\psi).\end{aligned}$$

Definition 2. A ring R is 1-*dimensional* if it is hereditary and noetherian, or equivalently if every submodule of a f.g. projective R-module is f.g. projective.

In particular, Dedekind rings are 1-dimensional.

Proposition 3 (*Ranicki [14,15]*). (i) *For every ring with involution R the $\pm\varepsilon$-quadratic skew-suspension maps $\overline{S}_n$ are isomorphisms, so that*

$$L_n(R,\varepsilon)\ =\ L_{n+2}(R,-\varepsilon)\ =\ L_{n+4}(R,\varepsilon),$$

with $L_{2n}(R,\varepsilon)=L_0(R,(-1)^n\varepsilon)$ the Witt group of stable isometry classes of nonsingular $(-1)^n\varepsilon$-quadratic forms over R.

(ii) *If R is 1-dimensional then the $\pm\varepsilon$-symmetric skew-suspension maps $\overline{S}^n$ are isomorphisms, so that*

$$L^n(R,\varepsilon)\ =\ L^{n+2}(R,-\varepsilon)\ =\ L^{n+4}(R,\varepsilon)$$

with $L^{2n}(R,\varepsilon) = L^0(R,(-1)^n\varepsilon)$ *the Witt group of stable isometry classes of nonsingular* $(-1)^n\varepsilon$*-symmetric forms over* R.

Proof. By algebraic surgery below the middle dimension, given by Proposition I.4.3 of [14,15] for (i), and Proposition I.4.5 of [14,15] for (ii). □

For $\varepsilon = 1$ we write

$$\begin{aligned} X^{\%}(C,1) &= X^{\%}C, \quad X_{\%}(C,1) = X_{\%}C, \\ Q^*(C,1) &= Q^*(C), \quad \widehat{Q}^*(C,1) = \widehat{Q}^*(C), \quad Q_*(C,1) = Q_*(C), \\ L^*(R,1) &= L^*(R), \quad L_*(R,1) = L_*(R). \end{aligned}$$

The hyperquadratic Q-groups $\widehat{Q}^*(C)$ are used in Section 2 to define chain bundles.

1.2. *The nilpotent L-groups* LNil, $L\widetilde{\text{Nil}}$

Theorem A identifies the unitary nilpotent L-groups $\text{UNil}_*(R)$ with the nilpotent L-groups $L\widetilde{\text{Nil}}_*(R)$, whose definition we now recall.

We start with nilpotent K-theory.

Definition 4. (i) An *R-nilmodule* (P,ν) is a f.g. projective R-module P together with a nilpotent endomorphism $\nu : P \to P$, so that

$$\nu^N = 0 : P \to P$$

for some $N \geqslant 1$.

(ii) A *morphism* of R-nilmodules $f : (P,\nu) \to (P',\nu')$ is an R-module morphism $f : P \to P'$ such that $\nu' f = f\nu : P \to P'$.

(iii) The *nilpotent K-groups* of R are defined to be the K-groups

$$\text{Nil}_*(R) = K_*(\text{Nil}(R))$$

of the exact category $\text{Nil}(R)$ be of R-nilmodules. The *reduced nilpotent K-groups*

$$\widetilde{\text{Nil}}_*(R) = \ker(\text{Nil}_*(R) \to K_*(R))$$

are such that

$$\text{Nil}_*(R) = K_*(R) \oplus \widetilde{\text{Nil}}_*(R).$$

(iv) The *NK-groups* of R are defined by

$$NK_*(R) = \ker(K_*(R[x]) \to K_*(R)),$$

so that

$$K_*(R[x]) \;=\; K_*(R) \oplus NK_*(R).$$

Proposition 5 (*Bass [2]*). (i) *There is a natural identification*

$$NK_1(R) \;=\; \widetilde{\text{Nil}}_0(R)$$

using the split injection

$$\widetilde{\text{Nil}}_0(R) \to K_1(R[x]);\;\; (P, v) \mapsto \tau(1 + xv : P[x] \to P[x]).$$

(ii) *If R is* 1*-dimensional then*

$$\widetilde{\text{Nil}}_0(R) \;=\; 0.$$

Proof. (i) See Chapter XII of [2].

(ii) Given a nilmodule (P, v) with $v^N = 0 : P \to P$ for some $N \geqslant 1$ define the nilmodules

$$(P', v') \;=\; (\ker(v), 0), \quad (P'', v'') \;=\; (\text{im}(v), v|),$$

using the 1-dimensionality of R to ensure that the R-modules $\ker(v), \text{im}(v) \subseteq P$ are f.g. projective. It follows from the exact sequence

$$0 \to (P', v') \to (P, v) \to (P'', v'') \to 0$$

that

$$[P, v] \;=\; [P', v'] + [P'', v''] \in \text{Nil}_0(R).$$

Now $v' = 0$, $(v'')^{N-1} = 0$, so proceeding inductively we obtain

$$[P, v] \;=\; \sum_{i=1}^{N} [\ker(v^i)/\ker(v^{i-1}), 0] \in K_0(R) \subseteq \text{Nil}_0(R)$$

and hence that $\widetilde{\text{Nil}}_0(R) = 0$. □

Definition 6. An n-dimensional *R-nilcomplex* (C, v) is a n-dimensional f.g. projective R-module chain complex

$$C \;:\; \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0$$

together with a chain map $v : C \to C$ which is chain homotopy nilpotent, i.e. such that $v^N \simeq 0 : C \to C$ for some integer $N \geqslant 1$.

Proposition 7. *The chain equivalence classes of the following types of chain complexes are in one–one correspondence*:

(i) *n-dimensional chain complexes of R-nilmodules*

$$(C, v) \ : \ \cdots \to 0 \to (C_n, v) \to (C_{n-1}, v) \to \cdots \to (C_1, v) \to (C_0, v),$$

(ii) *n-dimensional R-nilcomplexes* (C, v),

(iii) $(n+1)$*-dimensional f.g. projective* $R[x]$*-module chain complexes*

$$D \ : \ \cdots \to 0 \to D_{n+1} \to D_n \to \cdots \to D_1 \to D_0$$

such that

$$H_*(R[x, x^{-1}] \otimes_{R[x]} D) \ = \ 0.$$

Proof. (i) $\Longrightarrow$ (ii): An n-dimensional chain complex of R-nilmodules is an n-dimensional R-nilcomplex.

(ii) $\Longrightarrow$ (iii): Given an n-dimensional R-nilcomplex (C, v) define the $(n+1)$-dimensional f.g. projective $R[x]$-module chain complexes

$$D \ = \ \mathcal{C}(x - v : C[x] \to C[x])$$

such that

$$H_*(R[x, x^{-1}] \otimes_{R[x]} D) \ = \ 0,$$
$$x \ = \ v \ : \ H_*(D) \ = \ H_*(C) \to H_*(D) \ = \ H_*(C).$$

(i) $\Longleftrightarrow$ (iii): See Proposition 3.1.2 of Ranicki [16]. $\square$

In particular, it follows from Proposition 7 that every n-dimensional R-nilcomplex is chain equivalent to an n-dimensional R-nilcomplex (C, v) with $v^N = 0 : C \to C$ for some $N \geqslant 1$ (rather than just $v^N \simeq 0$).

Now for nilpotent L-theory.

Definition 8 (*Ranicki [16, p. 440; 18, p. 470]*). (i) The *ε-symmetric* $Q\,\mathrm{Nil}$*-groups* $Q\,\mathrm{Nil}^*(C, v, \varepsilon)$ of an R-nilcomplex (C, v) are the relative Q-groups in the exact sequence

$$\cdots \to Q^{n+1}(C, -\varepsilon) \to Q\,\mathrm{Nil}^n(C, v, \varepsilon) \to Q^n(C, \varepsilon) \xrightarrow{\Gamma_v} Q^n(C, -\varepsilon) \to \cdots$$

with

$$\Gamma_v \ : \ W^{\%}(C,\varepsilon) \to W^{\%}(C,-\varepsilon);\ \ \phi \mapsto (1\otimes v)\phi - \phi(v\otimes 1).$$

Similarly for the *ε-quadratic* Q Nil-*groups* $Q\,\mathrm{Nil}_*(C,v,\varepsilon)$, with an exact sequence

$$\cdots \to Q_{n+1}(C,-\varepsilon) \to Q\,\mathrm{Nil}_n(C,v,\varepsilon) \to Q_n(C,\varepsilon) \xrightarrow{\Gamma_v} Q_n(C,-\varepsilon) \to \cdots \ .$$

(ii) An *n-dimensional ε-symmetric Poincaré nilcomplex over R* $(C,v,\delta\phi,\phi)$ is an n-dimensional R-nilcomplex (C,v) together with an element

$$(\delta\phi,\phi) \in Q\,\mathrm{Nil}^n(C,v,\varepsilon)$$

such that $(C,\phi\in Q^n(C,\varepsilon))$ is an n-dimensional ε-symmetric Poincaré complex over R. The *ε-symmetric* LNil-*group* $\mathrm{LNil}^n(R,\varepsilon)$ is the cobordism group of n-dimensional ε-symmetric Poincaré nilcomplexes over R. Similarly in the ε-quadratic case, with $\mathrm{LNil}_n(R,\varepsilon)$.

(iii) The *reduced ε-symmetric* LNil-*groups* are defined by

$$L\widetilde{\mathrm{Nil}}^*(R,\varepsilon) \ = \ \ker(\mathrm{LNil}^*(R,\varepsilon) \to L^*_p(R,\varepsilon))$$

with

$$\mathrm{LNil}^*(R,\varepsilon) \ = \ L^*_p(R,\varepsilon) \oplus L\widetilde{\mathrm{Nil}}^*(R,\varepsilon).$$

Similarly in the ε-quadratic case, with $L\widetilde{\mathrm{Nil}}_*(R,\varepsilon)$.

(iv) Extend the involution to $R[x]$ by $\overline{x} = x$. Use the augmentation map

$$R[x] \to R;\ \ x \mapsto 0$$

to define the *nilpotent ε-symmetric L-groups of R*

$$NL^*(R,\varepsilon) \ = \ \ker(L^*(R[x],\varepsilon) \to L^*(R,\varepsilon))$$

with

$$L^*(R[x],\varepsilon) \ = \ L^*(R,\varepsilon) \oplus NL^*(R,\varepsilon).$$

Similarly for the *nilpotent ε-quadratic L-groups* $NL_*(R,\varepsilon)$.

Proposition 9. (i) *The* $Q\,\mathrm{Nil}$-*groups of an R-nilcomplex* (C, v) *are the Q-groups of the* $R[x, x^{-1}]$-*contractible f.g. projective* $R[x]$-*module chain complex*

$$D \;=\; \mathcal{C}(x - v : C[x] \to C[x])$$

with

$$x \;=\; v \;:\; H_*(D) \;=\; H_*(C) \to H_*(D) \;=\; H_*(C),$$
$$[1.ex]Q\,\mathrm{Nil}^n(C, v, \varepsilon) \;=\; Q^{n+1}(D, -\varepsilon),$$
$$[1ex]Q\,\mathrm{Nil}_n(C, v, \varepsilon) \;=\; Q_{n+1}(D, -\varepsilon).$$

An element $(\delta\phi, \phi) \in Q\,\mathrm{Nil}^n(C, v, \varepsilon)$ *corresponds to an element*

$$\Phi \in Q^{n+1}(D, -\varepsilon) \;=\; Q^{n-1}(S^{-1}D, \varepsilon)$$

with

$$\phi_0 \;=\; \Phi_0 \;:\; H^{n+1-*}(D) \;=\; H^{n-*}(C) \to H_*(D) \;=\; H_*(C),$$

so that $(C, v, \delta\phi, \phi)$ *is an* ε-*symmetric Poincaré nilcomplex if and only if* $(S^{-1}D, \Phi)$ *is an* ε-*symmetric Poincaré complex. Similarly in the* ε-*quadratic case.*

(ii) *The nilpotent* ε-*symmetric L-group of a ring with involution R fits into a split exact sequence*:

$$0 \to L^n_{\widetilde{K}_0(R)}(R[x], \varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon) \to \mathrm{LNil}^n(R, \varepsilon) \to 0$$

with the surjection split by the injection

$$\mathrm{LNil}^n(R, \varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon),$$
$$[1ex](C, v, \delta\phi, \phi) \mapsto (C[x, x^{-1}], [v, \delta\phi, \phi]) \oplus (C[x, x^{-1}], -\phi),$$
$$[1ex]([v, \delta\phi, \phi]_s = (x - v)\phi_s + T_\varepsilon\delta\phi_{s-1}\,,\; s \geqslant 0\,,\; \delta\phi_{-1} = 0).$$

Similarly in the ε-*quadratic case, with a split exact sequence*:

$$0 \to L_n^{\widetilde{K}_0(R)}(R[x], \varepsilon) \to L_n^{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon) \to \mathrm{LNil}_n(R, \varepsilon) \to 0,$$

where the surjection split by the injection

$$\mathrm{LNil}_n(R, \varepsilon) \to L_n^{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon),$$
$$(C, v, \delta\psi, \psi) \mapsto (C[x, x^{-1}], [v, \delta\psi, \psi]) \oplus (C[x, x^{-1}], -\psi),$$
$$([v, \delta\psi, \psi]_s = (x - v)\psi_s + T_\varepsilon\delta\psi_{s+1}\,,\; s \geqslant 0).$$

(iii) *The morphism*

$$\mathrm{LNil}^n(R,\varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x],\varepsilon);\ \ (C,\nu,\delta\phi,\phi) \mapsto (C[x],\widetilde{\Phi})$$
$$(\widetilde{\Phi}_s = (1-x\nu)\phi_s + xT_\varepsilon\delta\phi_{s-1},\ \ s\geqslant 0,\ \ \delta\phi_{-1}=0)$$

is an isomorphism, and

$$L^n(R[x],\varepsilon) \ = \ L^n(R,\varepsilon)\oplus L\widetilde{\mathrm{Nil}}^n(R,\varepsilon),\ \ NL^n(R,\varepsilon) \ = \ L\widetilde{\mathrm{Nil}}^n(R,\varepsilon).$$

Similarly in the ε*-quadratic case, with the morphism*[1]

$$\mathrm{LNil}_n(R,\varepsilon) \to L_n^{\widetilde{K}_0(R)}(R[x],\varepsilon),\ \ (C,\nu,\delta\psi,\psi) \mapsto (C[x],\widetilde{\Psi}),$$
$$(\widetilde{\Psi}_s = (1-x\nu)\psi_s + xT_\varepsilon\delta\psi_{s+1},\ \ s\geqslant 0)$$

an isomorphism, and

$$L_n(R[x],\varepsilon) \ = \ L_n(R,\varepsilon)\oplus L\widetilde{\mathrm{Nil}}_n(R,\varepsilon),\ \ NL_n(R,\varepsilon) \ = \ L\widetilde{\mathrm{Nil}}_n(R,\varepsilon).$$

Proof. (i) Ranicki [18, Propositions 34.5].

(ii) The ε-symmetric L-theory localization exact sequence of Proposition 3.7.2 of Ranicki [16]

$$\cdots \to L^n_I(A,\varepsilon) \to L^n_{S^{-1}I}(S^{-1}A,\varepsilon) \xrightarrow{\partial} L^n_I(A,S,\varepsilon) \to L^{n-1}_I(A,\varepsilon) \to \cdots$$

is defined for any ring with involution A, a central multiplicative subset $S\subseteq A$ of non-zero divisors, and any $*$-invariant subgroup $I\subseteq \widetilde{K}_0(A)$, with $L^n_I(A,S,\varepsilon)$ the cobordism group of $(n-1)$-dimensional ε-symmetric Poincaré complexes (C,ϕ) over A such that

$$S^{-1}A\otimes_A C \ \simeq \ 0,\ \ [C]\in I.$$

The boundary map is defined by

$$\partial \ : \ L^n_{S^{-1}I}(S^{-1}A,\varepsilon) \to L^n_I(A,S,\varepsilon);\ \ S^{-1}(C,\phi)\mapsto \partial(C,\phi)$$

with (C,ϕ) an n-dimensional $S^{-1}A$-Poincaré ε-symmetric complex over A such that $[C]\in I$, and $\partial(C,\phi)=(\partial C,\partial\phi)$ the $(n-1)$-dimensional $S^{-1}A$-contractible ε-symmetric

[1] As noted by the referee the cycles $\widetilde{\Phi}, (1+T)\widetilde{\Psi}\in (W^{\%}C[x])_n$ differ by a boundary involving $\delta\psi_0$.

Poincaré complex over A given by the boundary construction of p. 48 of [16], with $\partial C = \mathcal{C}(\phi_0 : C^{n-*} \to C)_{*+1}$. For

$$(A, S) = (R[x], \{x^k | k \geqslant 0\}), \quad S^{-1}A = R[x, x^{-1}], \ I = \widetilde{K}_0(R) \subseteq \widetilde{K}_0(R[x])$$

the localization exact sequence breaks up into split exact sequences

$$0 \to L^n_I(A, \varepsilon) \to L^n_{S^{-1}I}(S^{-1}A, \varepsilon) \xrightarrow{\partial} L^n_I(A, S, \varepsilon) \to 0$$

with

$$\begin{aligned} L^n_I(A, \varepsilon) &= L^n_{\widetilde{K}_0(R)}(R[x], \varepsilon), \\ L^n_{S^{-1}I}(S^{-1}A, \varepsilon) &= L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon), \\ L^n_I(A, S, \varepsilon) &= \mathrm{LNil}^n(R, \varepsilon). \end{aligned}$$

(Propositions 5.1.3, 5.1.4 of [16]). The formulae for $[v, \delta\phi, \phi]$ and $[v, \delta\psi, \psi]$ are from p. 445 of [16]. The identification $L^n_I(A, S, \varepsilon) = \mathrm{LNil}^n(R, \varepsilon)$ can be deduced from (i), noting that by Proposition 7 a finite f.g. projective $R[x]$-module chain complex D with projective class $[D] \in I$ is such that $R[x, x^{-1}] \otimes_{R[x]} D \simeq 0$ if and only if D is chain equivalent to $\mathcal{C}(x - v : C[x] \to C[x])$ for an R-nilcomplex (C, v), with C R-module chain equivalent to D and $v \simeq x : C \simeq D \to C \simeq D$. The map

$$L\,\mathrm{Nil}^n(R, \varepsilon) \to L^n_I(R[x], S, \varepsilon), \ (C, v, \delta\phi, \phi) \mapsto (S^{-1}D, \Phi)$$

is an isomorphism, which factors as

$$L\,\mathrm{Nil}^n(R, \varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon) \xrightarrow{\partial} L^n_I(R[x], S, \varepsilon)$$

with

$$\begin{aligned} &L\,\mathrm{Nil}^n(R, \varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon), \\ &(C, v, \delta\phi, \phi) \mapsto (C[x, x^{-1}], \{(x - v)\phi_s + T_\varepsilon \delta\phi_{s-1} | s \geqslant 0\}) \quad (\delta\phi_{-1} = 0). \end{aligned}$$

(iii) The inclusion $R[x^{-1}] \to R[x, x^{-1}]$ induces a split injection

$$L^n_{\widetilde{K}_0(R)}(R[x^{-1}], \varepsilon) \to L^n_{\widetilde{K}_0(R)}(R[x, x^{-1}], \varepsilon) = L^n_{\widetilde{K}_0(R)}(R[x], \varepsilon) \oplus \mathrm{LNil}^n(R, \varepsilon)$$

with image

$$L^n_p(R, \varepsilon) \oplus L\widetilde{\mathrm{Nil}}^n(R, \varepsilon) = L\,\mathrm{Nil}^n(R, \varepsilon).$$

Replacing $R[x^{-1}]$ by $R[x]$, it follows that the morphism

$$\begin{aligned}&L\,\mathrm{Nil}^n(R,\varepsilon)\to L^n_{\widetilde{K}_0(R)}(R[x^{-1}],\varepsilon),\\ &(C,\nu,\delta\phi,\phi)\mapsto (C[x^{-1}],\{(1-x^{-1}\nu)\phi_s+x^{-1}T_\varepsilon\delta\phi_{s-1}|s\geqslant 0\})\quad(\delta\phi_{-1}=0)\end{aligned}$$

is an isomorphism. The inclusion $R[x^{-1}]\to R[x,x^{-1}]$ induces a split injection

$$L^n_{\widetilde{K}_0(R)}(R[x^{-1}],\varepsilon)\to L^n_{\widetilde{K}_0(R)}(R[x,x^{-1}],\varepsilon)\;=\;L^n_{\widetilde{K}_0(R)}(R[x],\varepsilon)\oplus \mathrm{LNil}^n(R,\varepsilon)$$

with image

$$L^n_p(R,\varepsilon)\oplus L\widetilde{\mathrm{Nil}}^n(R,\varepsilon),$$

and an isomorphism

$$L^n_p(R,\varepsilon)\oplus L\widetilde{\mathrm{Nil}}^n(R,\varepsilon)\to L^n_{\widetilde{K}_0(R)}(R[x^{-1}],\varepsilon).\qquad\square$$

In the applications of the nilpotent L-groups to the unitary nilpotent L-groups we shall be particularly concerned with the Witt groups of 'nilforms' over R.

Define the $Q\,$Nil-*groups* of an R-nilmodule (P,ν) to be the $Q\,$Nil-groups of the 0-dimensional R-nilcomplex (C,ν^*) with

$$C\;:\;\cdots\to 0\to C_0=P^*\to 0\to\cdots\;,$$

as given in the ε-symmetric case by

$$\begin{aligned}Q\,\mathrm{Nil}^\varepsilon(P,\nu)\;&=\;Q\,\mathrm{Nil}^0(C,\nu^*,\varepsilon)\\ &=\;\{\phi\in\mathrm{Hom}_R(P,P^*)\,|\,\varepsilon\phi^*=\phi,\,\nu^*\phi=\phi\nu:P\to P^*\}\end{aligned}$$

and in the ε-quadratic case by

$$\begin{aligned}&Q\,\mathrm{Nil}_\varepsilon(P,\nu)\\ &=Q\,\mathrm{Nil}_0(C,\nu^*,\varepsilon)\\ &=\frac{\{(\delta\psi,\psi)\in\mathrm{Hom}_R(P,P^*)\oplus\mathrm{Hom}_R(P,P^*)\,|\,\nu^*\psi-\psi\nu=\delta\psi+\varepsilon\delta\psi^*\,:\,P\to P^*\}}{\{(\delta\chi-\varepsilon\delta\chi^*+\nu^*\chi-\chi\nu,\chi-\varepsilon\chi^*)\,|\,(\delta\chi,\chi)\in\mathrm{Hom}_R(P,P^*)\oplus\mathrm{Hom}_R(P,P^*)\}}.\end{aligned}$$

There is an evident ε-symmetrization map

$$N_\varepsilon\;:\;Q\,\mathrm{Nil}_\varepsilon(P,\nu)\to Q\,\mathrm{Nil}^\varepsilon(P,\nu);\;\;(\delta\psi,\psi)\mapsto N_\varepsilon(\psi).$$

Definition 10 (*Ranicki [16, p. 452]*). (i) A *non-singular ε-symmetric nilform over R* (P, ν, ϕ) consists of

(a) an R-nilmodule (P, ν),
(b) an element $\phi \in Q\,\mathrm{Nil}^{\varepsilon}(P, \nu)$ such that $\phi : P \to P^*$ is an isomorphism.

Thus (P, ϕ) is a nonsingular ε-symmetric form over R, and there is defined an isomorphism of R-nilmodules

$$\phi \ : \ (P, \nu) \to (P^*, \nu^*).$$

A *lagrangian* for (P, ν, ϕ) is a direct summand $L \subseteq P$ such that

(c) $\nu(L) \subseteq L$,
(d) the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^*\phi} L^* \to 0$$

is exact, with $i : L \to P$ the inclusion.

In particular, L is a lagrangian for the nonsingular ε-symmetric form (P, ϕ).

(ii) A *nonsingular ε-quadratic nilform over R* $(P, \nu, \delta\psi, \psi)$ consists of

(a) an R-nilmodule (P, ν)
(b) an element $(\delta\psi, \psi) \in Q\,\mathrm{Nil}_{\varepsilon}(P, \nu)$ such that $N_{\varepsilon}(\psi) : P \to P^*$ is an isomorphism.

Thus (P, ψ) is a nonsingular ε-quadratic form over R, and there is defined an isomorphism of R-nilmodules

$$N_{\varepsilon}(\psi) \ : \ (P, \nu) \to (P^*, \nu^*).$$

A *lagrangian* for $(P, \nu, \delta\psi, \psi)$ is a direct summand $L \subseteq P$ such that

(c) $\nu(L) \subseteq L$,
(d) the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* N_{\varepsilon}(\psi)} L^* \to 0$$

is exact, with $i : L \to P$ the inclusion,
(e) $(i^*\delta\psi i, i^*\psi i) = (0, 0) \in Q\,\mathrm{Nil}_{\varepsilon}(L, \nu|)$.

In particular, L is a lagrangian for the nonsingular ε-quadratic form (P, ψ).

The notion of stable isometry of nilforms is now defined in the usual way using lagrangians and orthogonal direct sums, and $\mathrm{LNil}^0(R, \varepsilon)$ (resp. $\mathrm{LNil}_0(R, \varepsilon)$) is the Witt group of nonsingular ε-symmetric (resp. ε-quadratic) nilforms over R. See [16, pp. 456–457] for the identification of $\mathrm{LNil}^1(R,\varepsilon)$ (resp. $\mathrm{LNil}_1(R,\varepsilon)$) with the Witt group of nonsingular ε-symmetric (resp. ε-quadratic) nilformations over R.

Proposition 11 (*Ranicki [18, Proposition 41.3]*). (i) *For any ring with involution R the skew-suspension maps in the nilpotent $\pm\varepsilon$-quadratic L-groups are isomorphisms, so that*

$$\mathrm{LNil}_n(R,\varepsilon) \;=\; \mathrm{LNil}_{n+2}(R,-\varepsilon) \;=\; \mathrm{LNil}_{n+4}(R,\varepsilon),$$

with $\mathrm{LNil}_{2n}(R,\varepsilon)=\mathrm{LNil}_0(R,(-1)^n\varepsilon)$ *the Witt group of nonsingular* $(-1)^n\varepsilon$*-quadratic nilforms over R. Similarly for* $L\widetilde{\mathrm{Nil}}_*(R,\varepsilon)$.

(ii) *If R is a Dedekind ring with involution then*

$$\begin{aligned}\mathrm{LNil}^n(R,\varepsilon) \;&=\; \mathrm{LNil}^{n+2}(R,-\varepsilon) \;=\; \mathrm{LNil}^{n+4}(R,\varepsilon),\\ \mathrm{LNil}^n(R,\varepsilon) \;&=\; L^n_p(R,\varepsilon),\quad L\widetilde{\mathrm{Nil}}^n(R,\varepsilon) \;=\; 0 \quad (n\geqslant 0).\end{aligned}$$

Proof. (i) In order to establish the 4-periodicity use algebraic surgery below the middle dimension, as for the ordinary ε-quadratic L-groups $L_n(R,\varepsilon)$ in Proposition I.4.3 of [14,15] (cf. Proposition 3 above).

(ii) The explicit proof in the case $n=0$ [18, p. 588] extends to the general case as follows. Let $(C,\nu,\delta\phi,\phi)$ be an n-dimensional ε-symmetric Poincaré nilcomplex over R, representing an element of $\mathrm{LNil}^n(R,\varepsilon)$, with

$$\nu^N \;=\; 0 \;:\; C\to C$$

for some $N\geqslant 1$. We reduce to the case $N=1$ using the structure theory of f.g. modules over the Dedekind ring R : every f.g. R-module M fits into a split exact sequence

$$0\to T(M)\to M\to M/T(M)\to 0$$

with

$$T(M) \;=\; \{x\in M\,|\,ax=0\in M \text{ for some } a\neq 0\in R\}$$

the torsion R-submodule and the quotient torsion-free R-module $M/T(M)$ is f.g. projective. In particular, for any R-nilmodule (P,ν) with

$$\nu^N \;=\; 0 \;:\; P\to P$$

the R-submodule of P defined by

$$T_N(P,\nu) \;=\; \{x\in P\,|\,ax\in \nu^{N-1}(P) \text{ for some } a\neq 0\in R\}$$

is such that

$$T_N(P,v)/v^{N-1}(P) \ = \ T(P/v^{N-1}(P)).$$

The torsion-free quotient R-module

$$(P/v^{N-1}(P))/T(P/v^{N-1}(P)) \ = \ P/T_N(P,v)$$

is f.g. projective, so that $T_N(P,v)$ is a direct summand of P. The inclusion defines a morphism of R-nilmodules

$$i \ : \ (T_N(P,v),0) \to (P,v).$$

Moreover, if (P',v') is another R-nilmodule with $v'^N = 0$ and

$$\theta \ : \ (P,v) \to (P',v')^* \ = \ (P'^*, v'^*).$$

is a morphism of R-nilmodules then

$$i'^*\theta i \ = \ 0 \ : \ T_N(P,v) \to T_N(P',v')^*$$

since for any $x \in T_N(P,v)$, $x' \in T_N(P',v')$ there exist $a, a' \neq 0 \in R$, $y \in P$, $y' \in P'$ with

$$ax \ = \ v^{N-1}(y) \in P, \quad a'x' \ = \ v'^{N-1}(x') \in P'$$

and

$$\begin{aligned} a'\theta(x)(x')\overline{a} \ &= \ \theta(ax)(a'x') \\ &= \ \theta(v^{N-1}(y))(v'^{N-1}(y')) \\ &= \ \theta(v^{2N-2}(y))(y') \\ &= \ 0 \in R \quad (\text{since } 2N-2 \geqslant N) \end{aligned}$$

so that

$$\theta(x)(x') \ = \ 0 \in R.$$

Returning to the n-dimensional ε-symmetric Poincaré nilcomplex $(C, v, \delta\phi, \phi)$ with $v^N = 0 : C \to C$, let $i : (B,0) \to (C^{n-*}, v^*)$ be the inclusion of the subcomplex defined by

$$B_r \ = \ T_N(C^{n-r}, v^*).$$

The chain map of R-nilmodule chain complexes defined by

$$f \;=\; i^* \;:\; (C, \nu) \to (D, 0) \;=\; (B^{n-*}, 0)$$

is such that

$$f^*(\delta\phi, \phi) \;=\; 0 \in Q\,\mathrm{Nil}^n(D, 0, \varepsilon).$$

Algebraic surgery on $(C, \nu, \delta\phi, \phi)$ using the $(n+1)$-dimensional ε-symmetric nil-pair $(f : (C, \nu) \to (D, 0), (0, (\delta\phi, \phi)))$ over R results in a cobordant n-dimensional ε-symmetric Poincaré nilcomplex $(C', \nu', \delta\phi', \phi')$ over R with

$$\nu' \;\simeq\; 0 \;:\; C' \to C'. \qquad \square$$

1.3. The unitary nilpotent L-groups UNil

Let R be any ring. An *involution* on an R–R bimodule $\mathcal{A}$ is a homomorphism

$$\mathcal{A} \to \mathcal{A};\;\; a \mapsto \bar{a}$$

which satisfies

$$\bar{\bar{a}} \;=\; a,\;\; \overline{ras} \;=\; \bar{s}\bar{a}\bar{r} \text{ for all } a \in \mathcal{A},\;\; r, s \in R.$$

For any R-module P there is defined an R-module

$$\mathcal{A}P \;=\; \mathcal{A} \otimes_R P.$$

As in the special case $\mathcal{A} = R$ write the evaluation pairing as

$$\langle\;,\;\rangle \;:\; \mathcal{A}P^* \times P \to \mathcal{A};\;\; (a \otimes f, x) \mapsto \langle a \otimes f, x\rangle \;=\; af(x).$$

An element $\phi \in \mathrm{Hom}_R(P, \mathcal{A}P^*)$ determines a $\mathcal{A}$-valued sesquilinear form on P

$$\langle\;,\;\rangle_\phi \;:\; P \times P \to \mathcal{A};\;\; (x, y) \mapsto \langle \phi(x), y\rangle,$$

and we identify $\mathrm{Hom}_R(P, \mathcal{A}P^*)$ with the additive group of such forms. For $\varepsilon = \pm 1$ and a f.g. projective P define an involution

$$T_\varepsilon \;:\; \mathrm{Hom}_R(P, \mathcal{A}P^*) \to \mathrm{Hom}_R(P, \mathcal{A}P^*);\;\; \phi \mapsto \varepsilon\phi^*, \quad \langle x, y\rangle_{\phi^*} \;=\; \overline{\langle y, x\rangle}_\phi.$$

One then defines a map

$$N_\varepsilon \;=\; 1+T_\varepsilon \;:\; \mathrm{Hom}_R(P,\mathcal{A}P^*) \to \mathrm{Hom}_R(P,\mathcal{A}P^*);\;\; \phi \mapsto \phi+\varepsilon\phi^t \tag{2}$$

with

$$\langle x,y\rangle_{N_\varepsilon(\phi)} \;=\; \langle x,y\rangle_\phi + \varepsilon\overline{\langle y,x\rangle}_\phi .$$

An $\mathcal{A}$-valued ε-symmetric form (P,λ) (resp. ε-quadratic form (P,μ)) over R is a f.g. projective R-module P together with an element of the group

$$\begin{aligned}\lambda \in Q^\varepsilon(P,\mathcal{A}) \;&=\; \ker(1-T_\varepsilon : \mathrm{Hom}_R(P,\mathcal{A}P^*) \to \mathrm{Hom}_R(P,\mathcal{A}P^*)),\\ \mu \in Q_\varepsilon(P,\mathcal{A}) \;&=\; \mathrm{coker}(1-T_\varepsilon : \mathrm{Hom}_R(P,\mathcal{A}P^*) \to \mathrm{Hom}_R(P,\mathcal{A}P^*)).\end{aligned}$$

As usual, for $\lambda \in Q^\varepsilon(P,\mathcal{A})$ we write

$$\lambda(x,y) \;=\; \langle \lambda(x),y\rangle \in \mathcal{A}$$

and for $\mu \in Q_\varepsilon(P,\mathcal{A})$ we write

$$\mu(x) \;=\; \langle \mu(x),x\rangle \in \mathcal{A}/\{a-\varepsilon\bar{a}\,|\,a\in\mathcal{A}\}.$$

The map N_ε induces a well defined map:

$$N_\varepsilon \;:\; Q_\varepsilon(P,\mathcal{A}) \to Q^\varepsilon(P,\mathcal{A});\;\; [\mu] \mapsto \mu+\varepsilon\mu^t . \tag{3}$$

Definition 12 (*Cappell [5]*). (i) Let $\mathcal{B}_1,\mathcal{B}_{-1}$ be R-bimodules with involution. Assume $\mathcal{B}_1,\mathcal{B}_{-1}$ are free as right R-modules. A *nonsingular ε-quadratic unilform over* $(R;\mathcal{B}_1,\mathcal{B}_{-1})$ is a quadruple

$$(P_1,P_{-1},\mu_1,\mu_{-1})$$

where, for $\delta=\pm 1$, we require:

(a) (P_δ,μ_δ) is a stably f.g. free $\mathcal{B}_\delta$-valued ε-quadratic form over R,
(b) $P_\delta = P^*_{-\delta}$; we then identify $(P^*_\delta)^* = P_\delta$ in the usual way, and write the evaluation pairing as

$$\langle\;,\;\rangle \;:\; P_1\times P_{-1} \to R;\;\; (x,f)\mapsto f(x).$$

(c) If $\lambda_\delta = N_\varepsilon(\mu_\delta)$ is the associated ε-symmetric form to μ_δ, then the composite

$$P_1 \xrightarrow{\lambda_1} \mathcal{B}_1 P_{-1} \xrightarrow{\lambda_{-1}\otimes 1} \mathcal{B}_{-1}\mathcal{B}_1 P_1 \xrightarrow{\lambda_1 \otimes 1} \mathcal{B}_1\mathcal{B}_{-1}\mathcal{B}_1 P_{-1} \xrightarrow{\lambda_{-1}\otimes 1} \cdots$$

is eventually zero. (That is to say, for some k, the composite map $P_1 \to (\mathcal{B}_{-1}\mathcal{B}_1)^k P_1$ is zero.)

(ii) A *sublagrangian* for $(P_1, P_{-1}, \mu_1, \mu_{-1})$ is a pair of stably f.g. free direct summands $V_1 \subseteq P_1$, $V_{-1} \subseteq P_{-1}$ such that, for $\delta = \pm 1$

$$\langle V_1, V_{-1}\rangle \;=\; 0,\;\; \lambda_\delta(V_\delta) \subseteq \mathcal{B}_\delta V_{-\delta}, \quad \mu_\delta(x) \;=\; 0 \text{ for all } x \in V_\delta. \tag{4}$$

We call (V_1, V_{-1}) a *lagrangian* if in addition:

$$V_1 \;=\; V_{-1}^{\perp}. \tag{5}$$

One can form orthogonal direct sums of ε-quadratic unilforms over $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ in a rather obvious way. Cappell [5] defined $\mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1})$ to be the Witt group of stable isometry classes of nonsingular $(-1)^n$-quadratic unilforms over $(R; \mathcal{B}_1, \mathcal{B}_{-1})$ modulo those admitting lagrangians, and showed (geometrically) that if π_{-1}, π_0, π_1 are finitely presented groups with $\pi_0 \subseteq \pi_{-1}$, $\pi_0 \subseteq \pi_1$ and

$$\pi \;=\; \pi_{-1} *_{\pi_0} \pi_1, \quad R \;=\; \mathbb{Z}[\pi_0], \;\; \mathcal{B}_{\pm 1} \;=\; \mathbb{Z}[\pi_{\pm 1}\backslash \pi_0]$$

then the morphism defined by

$$\begin{aligned} &\mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}) \to L_{2n}(\mathbb{Z}[\pi]),\\ &(P_1, P_{-1}, \mu_1, \mu_{-1}) \mapsto \left(\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_0]} (P_1 \oplus P_{-1}), \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_{-1} \end{pmatrix}\right) \end{aligned}$$

is a split monomorphism.

If an ε-quadratic unilform $u = (P_1, P_{-1}, \mu_1, \mu_{-1})$ has a sublagrangian (V_1, V_{-1}), then one can form a new ε-quadratic unilform (see [8, 6.3(f)])

$$u' \;=\; (V_{-1}^{\perp}/V_1, V_1^{\perp}/V_{-1}, \mu_1', \mu_{-1}'),$$

so that

$$[u] \;=\; [u'] \in \mathrm{UNil}_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1}).$$

1.4. The proof of Theorem A in the even-dimensional case

We begin by defining maps:

$$L\widetilde{\mathrm{Nil}}_{2n}(R) \xrightarrow{c} \mathrm{UNil}_{2n}(R;R,R) \xrightarrow{r} NL_{2n}(R) \subseteq L_{2n}(R[x]).$$

The proof will show that the maps c, r are both isomorphisms.
Let $\varepsilon = (-1)^n$.

Definition 13. The map

$$r \ : \ \mathrm{UNil}_{2n}(R;R,R) \to NL_{2n}(R); \ \ u \mapsto r(u)$$

sends an ε-quadratic unilform $u = (P_1, P_{-1}, \mu_1, \mu_{-1})$ over $(R;R,R)$ to the ε-quadratic form $r(u)$ over $R[x]$ given by:

$$r(u) \ = \ (P_1[x] \oplus P_{-1}[x], \psi_0 + x\psi_1),$$

where

$$\psi_0 \ = \ \begin{pmatrix} 0 & 1 \\ 0 & \mu_{-1} \end{pmatrix}, \ \ \psi_1 \ = \ \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, $\psi_i : (P_1 \oplus P_{-1})[x] \to (P_{-1} \oplus P_1)[x]$ $(i = 0, 1)$ is the $R[x]$-module morphism induced, using change of coefficients, from the R-module morphism of the same name

$$\psi_i \ : \ (P_1 \oplus P_{-1}) \to (P_1 \oplus P_{-1})^* \ = \ (P_{-1} \oplus P_1).$$

In order to verify that r is well-defined, first notice that

$$N_\varepsilon(\psi_0 + x\psi_1) \ = \ \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}(1+v) \ : \ (P_1 \oplus P_{-1})[x] \to (P_1^* \oplus P_{-1}^*)[x],$$

where

$$v \ = \ \begin{pmatrix} 0 & \varepsilon\lambda_{-1} \\ x\lambda_1 & 0 \end{pmatrix} \ : \ (P_1 \oplus P_{-1})[x] \to (P_1 \oplus P_{-1})[x], \quad \lambda_{\pm 1} : \ = \ N_\varepsilon(\mu_{\pm 1}).$$

Because

$$v^2 \ = \ \begin{pmatrix} x\varepsilon\lambda_{-1}\lambda_1 & 0 \\ 0 & x\lambda_1\lambda_{-1} \end{pmatrix}.$$

Definition 12 shows that v is obviously nilpotent. Therefore $N_\varepsilon(\psi_0 + x\psi_1)$ is nonsingular.

To see that $[r(u)] \in NL_{2n}(R)$, notice that $\eta_*[r(u)] = [P_1 \oplus P_{-1}, \psi_0]$, and that $P_1 \oplus 0$ is a lagrangian for $(P_1 \oplus P_{-1}, \psi_0)$.

The rule $u \mapsto r(u)$ preserves orthogonal direct sums of forms. If (V_1, V_{-1}) is a lagrangian for u, then $V_1[x] \oplus V_{-1}[x]$ is a lagrangian for $r(u)$. We thus have a well-defined homomorphism:

$$r \ : \ \mathrm{UNil}_{2n}(R; R, R) \to NL_{2n}(R).$$

Definition 14. The map

$$c \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to \mathrm{UNil}_{2n}(R; R, R); \ z \mapsto c(z)$$

sends a nonsingular ε-quadratic nilform $z = (P, v, \delta\psi, \psi)$ over R (see Definition 10) to $c(z) = (P_1, P_{-1}, \mu_1, \mu_{-1})$, where

$$P_1 = P, \quad P_{-1} = P^*, \quad \mu_1 = \delta\psi - v^*\psi, \quad \mu_{-1} = -\phi^{-1}\psi^*\phi^{-1}$$

with $\phi = N_\varepsilon(\psi) : P \to P^*$ an isomorphism.

Using Definition 10 set

$$\lambda_1 = N_\varepsilon(\mu_1) = -\phi v = -v^*\phi,$$

noting that $N_{-\varepsilon}N_\varepsilon(\delta\psi) = 0$. Set also

$$\lambda_{-1} = N_\varepsilon(\mu_{-1}) = -\varepsilon\phi^{-1}.$$

Because $\lambda_{-1}\lambda_1 = \varepsilon v$, and v is nilpotent, it follows that $c(z)$ is an ε-quadratic unilform over $(R; R, R)$. The rule $z \mapsto c(z)$ preserves orthogonal direct sums. Moreover, if N is a lagrangian for z , then $(N, N^\perp)$ is a lagrangian for $c(z)$. Therefore Definition 14 gives a homomorphism:

$$c \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to \mathrm{UNil}_{2n}(R; R, R). \tag{6}$$

Definition 15. The morphism

$$j \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to NL_{2n}(R); \ y \mapsto j(y)$$

sends $y = [P, v, \delta\psi, \psi]$ to

$$j(y) = [P[x], \psi + x(\delta\psi - v^*\psi)].$$

It was proved in [16, p. 445] that j is in fact an isomorphism. See Remark 16 for the precise matching up of the formula in Definition 15 with the morphism defined there.

The right-hand side in Definition 15 gives a nonsingular form because

$$N_\varepsilon(\psi + x(\delta\psi - v^*\psi)) \;=\; N_\varepsilon(\psi)(1 - xv),$$

an isomorphism by Definition 10. Moreover this right-hand side is in $NL_{2n}(R)$, also by Definition 10.

Remark 16. In order to obtain the formula in Definition 15 for $j(y)$ from the formula in [16, p. 445] one must make the following translation of the terminology there to our terminology:

$$\begin{aligned} A \;&=\; R, \quad C^0 \;=\; P, \quad C^i \;=\; 0 \text{ for } i \neq 0, \\ \psi_0 \;&=\; \psi, \quad \delta\psi_1 \;=\; \delta\psi, \end{aligned}$$

noting that the x^{-1} is our x, and the v^* there is our v. In the following argument we shall use the Witt group

$$\mathrm{LNil}^h_{2n}(R) \;=\; L_{2n}(R) \oplus L\widetilde{\mathrm{Nil}}_{2n}(R)$$

of nonsingular $(-)^n$-quadratic nilforms $(P, v, \delta\psi, \psi)$ over R with P a f.g. free R-module, and the split injection

$$\Delta \;:\; \mathrm{LNil}^h_{2n}(R) \to L_{2n}(R[x, x^{-1}]);\;\; [P, v, \delta\psi, \psi] \mapsto [P[x, x^{-1}], (x^{-1} - v^*)\psi + \delta\psi]$$

defined there, along with the splitting map

$$\partial \;:\; L_{2n}(R[x, x^{-1}]) \to \mathrm{LNil}^h_{2n}(R)$$

and the natural inclusion and projection:

$$L\widetilde{\mathrm{Nil}}_{2n}(R) \xrightarrow{i} \mathrm{LNil}^h_{2n}(R) \xrightarrow{p} L\widetilde{\mathrm{Nil}}_{2n}(R).$$

Let $\tilde{E} : NL_{2n}(R) \to L_{2n}(R[x, x^{-1}])$ be the restriction of the natural monomorphism

$$E \;:\; L_{2n}(R[x]) \to L_{2n}(R[x, x^{-1}]).$$

Also, set

$$\tilde{\partial} \;=\; p\partial \;:\; L_{2n}(R[x,x^{-1}]) \to L\widetilde{\mathrm{Nil}}_{2n}(R),$$
$$\tilde{\Delta} \;=\; \Delta i \;:\; L\widetilde{\mathrm{Nil}}_{2n}(R) \to L_{2n}(R[x,x^{-1}]).$$

Because $\partial\Delta = 1$, we get $\tilde{\partial}\tilde{\Delta} = 1$. According to the braid on p. 448 of [16], $\tilde{\partial}\tilde{E}$ is an isomorphism. The map j of Definition 15 is $j = (\tilde{\partial}\tilde{E})^{-1}$. To get the formula for j in Definition 15, note that the "devissage" map $\tilde{\partial}$ satisfies

$$\tilde{\partial} \;=\; \tilde{\partial}M$$

with

$$M \;:\; L_n(R[x,x^{-1}]) \to L_n(R[x,x^{-1}]);\;\; (P,\psi) \mapsto (P,x\psi).$$

Then from [16, p. 445], we translate and find:

$$\begin{aligned}\tilde{\Delta}(y) &= \tilde{\Delta}([P,v,\delta\psi,\psi]) \\ &= [P[x,x^{-1}], x^{-1}\{\psi + x(\delta\psi - v^*\psi)\}].\end{aligned}$$

So

$$\begin{aligned} j(y) &= j(\tilde{\partial}M\tilde{\Delta}(y)) \\ &= \tilde{E}^{-1}M\tilde{\Delta}(y) \\ &= \tilde{E}^{-1}([P[x,x^{-1}], \psi + x(\delta\psi - v^*\psi)]) \\ &= [P[x], \psi + x(\delta\psi - v^*\psi)],\end{aligned}$$

as in Definition 15.

As explained above, [16] proves that j is an isomorphism.

Remark 17. The inverse of j

$$k \;=\; j^{-1} \;:\; NL_{2n}(R) \to L\widetilde{\mathrm{Nil}}_{2n}(R) \tag{7}$$

can be computed via Higman linearization (see [8, 3.6 (a)]) in the following way. By Higman linearization, each element of $NL_{2n}(R)$ can be represented in the form $[P[x], \psi_0 + x\psi_1]$. In these terms, the formula for $k = j^{-1}$ is

$$k[P[x], \psi_0 + x\psi_1] \;=\; [P, v, \delta\psi, \psi], \tag{8}$$

where

$$\psi \;=\; \psi_0, \quad v \;=\; (N_\varepsilon(\psi_0))^{-1} N_\varepsilon(\psi_1), \quad \delta\psi \;=\; v^*\psi_0 + \psi_1.$$

It is clear that $jk = 1$.

We now turn to the proof of Theorem A in even dimensions. We only have to show that:

$$\text{(i) } ckr \;=\; 1, \quad \text{(ii) } rc \;=\; j. \tag{9}$$

The proof of (9) (i) is easiest: let $(P_1, P_{-1}, \mu_1, \mu_{-1})$ be an ε-quadratic unilform over $(R; R, R)$. By Definitions 13, 14 and (8), and direct calculation, we obtain:

$$ckr[P_1, P_{-1}, \mu_1, \mu_{-1}] \;=\; [P_1 \oplus P_{-1}, P_{-1} \oplus P_1, \tilde{\mu}_1, \tilde{\mu}_{-1}], \tag{10}$$

where

$$\tilde{\mu}_1 \;=\; \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mu}_{-1} \;=\; \begin{pmatrix} \mu_{-1} & 1 \\ 0 & 0 \end{pmatrix}.$$

Perform a sublagrangian construction on the right-hand side of (10), using the sublagrangian

$$V_1 \;=\; 0 \oplus P_{-1}, \quad V_{-1} \;=\; 0.$$

This yields

$$[P_1 \oplus P_{-1}, P_{-1} \oplus P_1, \tilde{\mu}_1, \tilde{\mu}_{-1}] \;=\; [P_1, P_{-1}, \mu_1, \mu_{-1}].$$

Therefore $ckr = 1$, proving Eq. (9)(i).

Next we prove Eq. (9)(ii).

Suppose $a = [P, v, \delta\psi, \psi] \in L\widetilde{\text{Nil}}_{2n}(R)$. By direct calculation and Definitions 13 and 14, we have

$$rc(a) \;=\; [P[x] \oplus P^*[x], \Psi_0 + x\Psi_1], \tag{11}$$

where

$$\Psi_0 \;=\; \begin{pmatrix} 0 & 1 \\ 0 & -\phi^{-1}\psi^*\phi^{-1} \end{pmatrix}, \quad \Psi_1 \;=\; \begin{pmatrix} \delta\psi - v^*\psi & 0 \\ 0 & 0 \end{pmatrix}$$

with $\phi = N_\varepsilon(\psi)$. By hypothesis (see Definition 10), (P, ψ) admits a lagrangian, say $N \subseteq P$. Let

$$V = (\phi N)[x] \subseteq P^*[x] \subseteq P[x] \oplus P^*[x].$$

By (11) V is a sublagrangian for $\Psi_0 + x\Psi_1$. In fact, setting $\Phi = N_\varepsilon(\Psi_0 + x\Psi_1)$, one readily computes that the Φ-orthogonal complement of V is

$$V_\Phi^\perp = \{(u, v) \in P[x] \oplus P^*[x] \mid \phi(u) - v \in V\}.$$

Therefore one obtains an isomorphism

$$g : P[x] \to V_\Phi^\perp / V;\; u \mapsto (u, \phi(u)).$$

Let $(V_\Phi^\perp / V, \Psi')$ be the sublagrangian construction on $cr(a)$ using V. We claim that

$$g : (P[x], \psi + x(\delta\psi - v^*\psi)) \to (V_\Phi^\perp / V, \Psi') \tag{12}$$

is an isometry. Since the right-hand side of (12) represents $rc(a)$, and the left-hand side is $j(a)$, this claim (12) will prove (9)(ii).

We prove (12) using the duality pairing

$$\begin{aligned} \{\ ,\ \} &: (P^*[x] \oplus P[x]) \times (P[x] \oplus P^*[x]) \to R[x], \\ &((\xi, \eta), (\eta', \xi')) \mapsto \{(\xi, \eta), (\eta', \xi')\} = \langle \xi, \eta' \rangle + \overline{\langle \xi', \eta \rangle}. \end{aligned}$$

Eq. (12) amounts to the identity:

$$\langle [\psi + x(\delta\psi - v^*\psi)](u), v \rangle = \{\Psi(u, \phi(u)), (v, \phi(v))\} \quad (u, v \in P[x]), \tag{13}$$

where $\Psi = \Psi_0 + x\Psi_1 : P[x] \oplus P^*[x] \to P^*[x] \oplus P[x]$. The right-hand side of (13) is computed from (11) as

$$\begin{aligned} &\langle [\phi + x(\delta\psi - v^*\psi)](u), v \rangle + \overline{\langle \phi(v), -\phi'\psi^*(u) \rangle} \\ &\quad = \langle [\phi + x(\delta\psi - v^*\psi](u), v \rangle + \langle -\phi^*\phi^{-1}\psi^*(u), v \rangle \\ &\quad = \langle [(\phi - \varepsilon\psi^*) + x(\delta\psi - v^*\psi)](u), v \rangle, \end{aligned}$$

which is the left-hand side of (13). This proves (12) and therefore also (9)(ii). Therefore the proof of Theorem A, when n is even, is complete.

Remark 18. It seems appropriate to record here an explicit formula for the inverse isomorphism

$$c^{-1} \; : \; \mathrm{UNil}_{2n}(R; R, R) \to L\widetilde{\mathrm{Nil}}_{2n}(R)$$

which can be derived from (8), (9) and Definition 13, as follows.

For $[P_1, P_{-1}, \mu_1, \mu_{-1}] \in \mathrm{UNil}_{2n}(R; R, R)$ we have:

$$c^{-1}([P_1, P_{-1}, \mu_1, \mu_{-1}]) \;=\; (P_1 \oplus P_{-1}, v, \delta\psi, \psi), \tag{14}$$

where

$$\begin{aligned}
\psi \;&=\; \begin{pmatrix} 0 & 1 \\ 0 & \mu_{-1} \end{pmatrix} : P_1 \oplus P_{-1} \to P_{-1} \oplus P_1, \quad \psi_1 \;=\; \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \end{pmatrix},\\
\delta\psi \;&=\; v^*\psi + \psi_1 \;:\; P_1 \oplus P_{-1} \to P_{-1} \oplus P_1,\\
v \;&=\; -N_\varepsilon(\psi)^{-1} N_\varepsilon(\psi_1) \;=\; \begin{pmatrix} \varepsilon\lambda_{-1}\lambda_1 & 0 \\ -\lambda_1 & 0 \end{pmatrix}
\end{aligned}$$

with $\lambda_{\pm 1} = N_\varepsilon(\mu_{\pm 1})$.

1.5. The proof of Theorem A in the odd-dimensional case

We begin by commenting that the "simple L-theory" version of Theorem A, in even dimensions, proceeds uneventfully, along the same lines as above. We explain this in some detail now.

$\mathrm{UNil}^s_{2n}(R; \mathcal{B}_1, \mathcal{B}_{-1})$ is defined in [5, p. 1118]. Also,

$$\mathrm{LNil}^s_n(R) \;=\; L^s_n(R) \oplus L\widetilde{\mathrm{Nil}}^s_n(R)$$

is defined in [16, pp. 466–468], where there are also constructed exact sequences:

$$0 \to L^{I_+}_n(R[x]) \xrightarrow{E^s} L^J_n(R[x, x^{-1}]) \longrightarrow L^p_n(R) \oplus L\widetilde{\mathrm{Nil}}^s_n(R) \to 0,$$

$$0 \to L^{I_-}_n(R[x^{-1}]) \longrightarrow L^J_n(R[x, x^{-1}]) \xrightarrow{\partial^s} L^p_n(R) \oplus L\widetilde{\mathrm{Nil}}^s_n(R) \to 0,$$

where

$$\begin{aligned}
I_\pm \;&=\; \widetilde{K}_1(R) \subseteq \widetilde{K}_1(R[x^{\pm 1}]) \;=\; \widetilde{K}_1(R) \oplus \widetilde{\mathrm{Nil}}_0(R),\\
J \;&=\; \widetilde{K}_1(R) \oplus K_0(R) \subseteq \widetilde{K}_1(R[x, x^{-1}]) \;=\; \widetilde{K}_1(R) \oplus K_0(R) \oplus \widetilde{\mathrm{Nil}}_0(R) \oplus \widetilde{\mathrm{Nil}}_0(R).
\end{aligned}$$

Define

$$NL^s_n(R) \;=\; \ker(\eta_* : L^{K_1(R)}_n(R[x]) \to L_n(R))$$

and let $\widetilde{\Delta}^s, \tilde{E}^s, \tilde{\partial}^s$ be as in Remark 16, concluding that $\tilde{\partial}^s \tilde{E}^s$ is an isomorphism. As before, define

$$j^s = (\tilde{\partial}^s \tilde{E}^s)^{-1} \ : \ L\widetilde{\mathrm{Nil}}_{2n}(R) \to NL^s_{2n}(R)$$

using the formula of Definition 15. The maps

$$L\widetilde{\mathrm{Nil}}^s_{2n}(R) \xrightarrow{c^s} \mathrm{UNil}^s_{2n}(R; R, R) \xrightarrow{r^s} NL^s_{2n}(R), \text{ and } k^s \ = \ (j^s)^{-1}$$

are now defined exactly as in Definitions 13, 8, and the proof that these are isomorphisms can now be repeated without change. In summary, we have:

Proposition 19. *The maps $L\widetilde{\mathrm{Nil}}^s_{2n}(R) \xrightarrow{c^s} \mathrm{UNil}^s_{2n}(R; R, R) \xrightarrow{r^s} NL^s_{2n}(R)$ described in the paragraph above are isomorphisms. Moreover, $j^s = r^s c^s$.*

We now complete the proof of Theorem A in odd dimensions.
Let $S = R[z, z^{-1}]$, extending the involution on R to S by

$$\bar{z} \ = \ z^{-1}.$$

Let $i : R \to S$ be the inclusion. The split exact sequence of Shaneson [20] and Ranicki [13]

$$0 \to L^s_n(R) \stackrel{i_*}{\to} L^s_n(S) \to L_{n-1}(R) \to 0$$

yields the split exact sequence

$$0 \to NL^s_n(R) \stackrel{i_*}{\to} NL^s_n(S) \to NL_{n-1}(R) \to 0.$$

Cappell [5] defined $\mathrm{UNil}_{2n-1}(R; R, R)$ as the cokernel in the split exact sequence:

$$0 \to \mathrm{UNil}^s_{2n}(R; R, R) \to \mathrm{UNil}^s_{2n}(S; S, S) \to \mathrm{UNil}_{2n-1}(R; R, R) \to 0.$$

The isomorphism r^s of Proposition 19, being functorial, therefore induces an isomorphism:

$$r \ : \ \mathrm{UNil}_{2n-1}(R; R, R) \to NL_{2n-1}(R).$$

This proves Theorem A.

2. Chain bundles and the proof of Theorem B

2.1. Universal chain bundles

We begin with a resumé of the results of Ranicki [14,15,19] and Weiss [22,23] which we need. As in Section 1, R is a ring with involution.

A *chain bundle* (B, β) over R is a projective R-module chain complex B together with a 0-cycle

$$\beta \in (\widehat{W}^{\%} B^{-*})_0.$$

(We shall be mainly concerned with cases when the chain modules B_r are f.g. projective.) A map of chain bundles $f : (C, \gamma) \to (B, \beta)$ is a chain map $f : C \to B$ such that

$$[\widehat{f}^{\%}(\beta)] \;=\; [\gamma] \in \widehat{Q}^0(C^{-*}),$$

with $\widehat{f}^{\%} : \widehat{W}^{\%} B^{-*} \to W^{\%} C^{-*}$ the chain map induced by f. Each chain bundle (B, β) determines a homomorphism

$$J_\beta \;:\; Q^n(B) \to \widehat{Q}^n(B);\;\; \phi \mapsto J(\phi) - \widehat{\phi}_0^{\%}(S^n \beta), \tag{15}$$

where J is as in (1), $\phi_0^{\%}$ is the map induced by $\phi_0 : B^{n-*} \to B$, and $S^n : \widehat{W}^{\%} C \to \Sigma^{-n} \widehat{W}^{\%} \Sigma^n(C)$ is the natural isomorphism of chain complexes. The map J_β is not induced by a chain map.

The Tate $\mathbb{Z}_2$-cohomology group

$$\widehat{H}^r(\mathbb{Z}_2; R) \;=\; \{x \in R \mid \bar{x} = (-1)^r x\}/\{y + (-1)^r \bar{y} \mid y \in R\}$$

is an R-module via

$$R \times \widehat{H}^r(\mathbb{Z}_2; R) \to \widehat{H}^r(\mathbb{Z}_2; R);\;\; (a, x) \mapsto a x \bar{a}.$$

The *Wu classes* of a chain bundle (B, β) are the R-module morphisms

$$v_r(\beta) \;:\; H_r(B) \to \widehat{H}^r(\mathbb{Z}_2; R);\;\; x \mapsto \langle \beta_{-2r}, x \otimes x \rangle \;\; (r \in \mathbb{Z}). \tag{16}$$

The *universal chain bundle* (B^R, β^R) exists for each R. It is the chain bundle (unique up to equivalence) characterized by the requirement that map (16) is an isomorphism for each r. This implies the more general property that for each f.g. free chain complex C the map

$$k_C \;:\; H_n(C \otimes_R B^R) \to \widehat{Q}^n(C);\;\; f \mapsto S^{-n} f^{\%}(\beta) \tag{17}$$

is an isomorphism. A cycle $f \in (C \otimes_R B^R)_n$ is a chain map $f : (B^R)^{-*} \to S^{-n}C$, inducing a morphism

$$S^{-n} f^{\%} \ : \ \widehat{Q}^0((B^R)^{-*}) \to \widehat{Q}^0(S^{-n}C) \ = \ \widehat{Q}^n(C).$$

See [19,22,23].

2.2. *The chain bundle exact sequence and the theorem of Weiss*

For each chain bundle (B, β), the map J_β above fits into an exact sequence:

$$\cdots \to \widehat{Q}^{n+1}(B) \xrightarrow{H} Q_n(B, \beta) \xrightarrow{N_\beta} Q^n(B) \xrightarrow{J_\beta} \widehat{Q}^n(B) \to \cdots \ , \tag{18}$$

where the group $Q_n(B, \beta)$ of "*twisted quadratic structures*" and the maps N_β and H are defined as follows.

$Q_n(B, \beta)$ is defined as the abelian group of equivalence classes of pairs (ϕ, θ) (called *symmetric structures on* (B, β)) where $\phi \in (W^{\%}B)_n$, $\theta \in (\widehat{W}^{\%}B)_{n+1}$ satisfy

$$d\phi \ = \ 0, \quad d\theta \ = \ J_\beta(\phi).$$

The addition is defined by

$$(\phi, \theta) + (\phi', \theta') \ = \ (\phi + \phi', \theta + \theta' + \xi) \ \text{ where } \xi_s \ = \ \phi_0 \beta_{s-n+1} \phi'_0.$$

One says that (ϕ, θ) is equivalent to (ϕ', θ') if there exist $\zeta \in (W^{\%}B)_{n+1}$, $\eta \in (\widehat{W}^{\%}B)_{n+2}$ such that

$$d\zeta \ = \ \phi' - \phi, \quad d\eta \ = \ \theta' - \theta + J(\zeta) + (\zeta_0, \phi_0, \phi'_0)^{\%}(S^n\beta).$$

Here $(\zeta_0, \phi_0, \phi'_0)^{\%} : (\widehat{W}^{\%}B^{-*})_n \to (\widehat{W}^{\%}B)_{n+1}$ is the chain homotopy from $\phi_0^{\%}$ to $(\phi')_0^{\%}$ induced by ζ_0 (see [19, Section 3]).

The map H is defined by: $H(\theta) = [0, \theta]$.

The map N_β is defined by: $N_\beta([\phi, \theta]) = [\phi]$.

When $\beta = 0$, then $Q_n(B, 0) = Q_n(B)$ and (18) reduces to (1).

Recall now from [16, pp. 19, 39, 137], the cobordism groups $L_n(R, \varepsilon)$ (resp. $L^n(R, \varepsilon)$, $\widehat{L}^n(R, \varepsilon)$) of free n-dimensional ε-quadratic (resp. symmetric, resp. hyperquadratic) Poincaré complexes over R, where $\varepsilon = \pm 1$. These are related by a long exact sequence and a skew-suspension functor:

$$\begin{array}{ccccccc} \widehat{L}^{n+1}(R,\varepsilon) & \xrightarrow{H} & L_n(R,\varepsilon) & \longrightarrow & L^n(R,\varepsilon) & \longrightarrow & \widehat{L}^n(R,\varepsilon) \\ \big\downarrow \widehat{\overline{S}}^{n+1} & & \big\downarrow \overline{S}_n & & \big\downarrow \overline{S}^n & & \big\downarrow \widehat{\overline{S}}^n \\ \widehat{L}^{n+3}(R,-\varepsilon) & \longrightarrow & L_{n+2}(R,-\varepsilon) & \longrightarrow & L^{n+2}(R,-\varepsilon) & \longrightarrow & \widehat{L}^{n+2}(R,-\varepsilon). \end{array} \tag{19}$$

$\overline{S}_n$ is an isomorphism for all n, and $L_n(R,1)$ is the Wall surgery obstruction group, $L_n(R)$. But $\widehat{\overline{S}}^n$ and $\overline{S}^n$ are not isomorphisms in general. Instead, the main result of Weiss [22,23] (see also [19]) identifies the limit of the maps $\widehat{\overline{S}}^n$ in terms of a functorial isomorphism:

$$\lim_{k\to\infty} \widehat{L}^{n+2k}(R,(-1)^k) \overset{\cong}{\to} Q_n(B^R,\beta^R). \tag{20}$$

The skew-suspension maps $\widehat{S}^n$, S^n are isomorphisms for 1-dimensional R.

2.3. UNil *and* 1-*dimensional rings*

Recall from Definition 2 that a ring R is said to be 1-dimensional if it is hereditary and noetherian.

Proposition 20. *For any* 1*-dimensional ring R with involution, and any* $n \geqslant 0$*, there is a short exact sequence*:

$$0 \to \mathrm{UNil}_n(R;R,R) \to Q_{n+1}(B^{R[x]},\beta^{R[x]}) \to Q_{n+1}(B^R,\beta^R) \to 0.$$

Proof. Following Definition 8 set

$$NQ_n(R) = \ker\{Q_n(B^{R[x]},\beta^{R[x]}) \to Q_n(B^R,\beta^R)\}. \tag{21}$$

By Propositions 9 and 11

$$NL^n(R) = L\widetilde{\mathrm{Nil}}^n(R) = 0 \text{ for all } n \geqslant 0.$$

So by (19) we get a square of isomorphisms, for all $n \geqslant 0$:

$$\begin{array}{ccc} N\widehat{L}^{n+1}(R,\varepsilon) & \xrightarrow{\cong} & NL_n(R,\varepsilon) \\ \widehat{S}^n \big\downarrow \cong & & S_n \big\downarrow \cong \\ N\widehat{L}^{n+3}(R,-\varepsilon) & \xrightarrow{\cong} & NL_{n+2}(R,-\varepsilon). \end{array} \tag{22}$$

By Theorem A, (20)–(22), for all $n \geqslant 0$, we have:

$$\begin{aligned}\mathrm{UNil}_n(R; R, R) \ &\cong \ NL_n(R,1) \ \cong \ N\widehat{L}^{n+1}(R,1) \\ &\cong \ \lim_k \ N\widehat{L}^{n+1+2k}(R,(-1)^k) \ \cong \ NQ_{n+1}(R).\end{aligned}$$

This proves (20). □

2.4. Rules for calculating $Q_n(C,\gamma)$

Our goal, in the light of Proposition 20, is to compute $Q_n(B^A, \beta^A)$, especially when $A = \mathbb{Z}$. But first we explain three tools for computing $Q_n(C,\gamma)$ for any chain bundle (C,γ) over any ring with involution A.

(A) *Suppose (C,γ) is a chain bundle and $C \otimes_A C$ is n-connected. Then*

$$Q_i(C,\gamma) \ = \ 0 \ \textit{for} \ i \leqslant n-1 \quad \textit{and} \quad Q^{n+1}(C) \xrightarrow{J_\gamma^{n+1}} \widehat{Q}^{n+1}(C) \xrightarrow{H^{n+1}} Q_n(C,\gamma) \to 0$$

is exact. Moreover, for $i \leqslant n$, $J^i_\gamma = J^i : Q^i(C) \to \widehat{Q}^i(C)$, and J^i_γ is an isomorphism.

Proof. Use the spectral sequence

$$E^2_{p,q} \ = \ H_p(\mathbb{Z}_2; H_q(C \otimes_A C)) \Rightarrow H_{p+q}((W^{-*})^{\%}C) \ = \ Q_{p+q}(C).$$

This proves $Q_i(C) = 0$, for $i \leqslant n$. Next,

$$J^i_\gamma([\phi]) \ = \ J^i([\phi]) - \phi_0^{\%}([S^i\gamma])$$

for any $[\phi] \in Q^i(C)$. But if $i \leqslant n$, ϕ_0 is null homotopic because $[\phi_0] = 0 \in H_i(C \otimes_A C)$. Consequently, $\phi_0^{\%} = 0$ and $J^i_\gamma = J^i$ for all $i \leqslant n$. But by the exact sequence 1, it follows that J^i_γ is an isomorphism for all $i \leqslant n$, and H^{n+1} is an epimorphism. This proves (A). □

(B) *Suppose (C,γ) is a chain bundle for which the chain complex C splits as*

$$C \ = \ \sum_{i=-\infty}^{\infty} C(i).$$

Then

$$\gamma \ = \ \sum_{i=-\infty}^{\infty} \gamma(i),$$

where $\gamma(i) \in \widehat{Q}^0(C(i))$, *and the inclusions* $C(i) \to C$ *induce a long exact sequence:*

$$\cdots \to \sum_{i=-\infty}^{\infty} Q_n(C(i), \gamma(i)) \to Q_n(C, \gamma)$$
$$\to \sum_{i<j} H_n(C(i) \otimes C(j)) \to \sum_{i=-\infty}^{\infty} Q_{n-1}(C(i), \gamma(i)) \to \cdots . \quad (23)$$

Proof. On general principles

$$\widehat{Q}^n(C) \;=\; \sum_i \widehat{Q}^n(C(i))$$

and

$$Q^n(\sum_i C(i)) \;=\; \sum_i Q^n(C(i)) \oplus \sum_{i<j} H_n(C(i) \otimes C(j)).$$

Therefore, (B) is a consequence of a diagram chase applied to the following map of exact sequences obtained from (18):

$$\begin{array}{ccccccc}
\sum_i Q_n(C(i), \gamma(i)) & \longrightarrow & \sum_i Q^n(C(i)) & \xrightarrow{\Sigma J_{\beta(i)}} & \sum_i \widehat{Q}^n(C(i)) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow \cong & & \\
Q_n(C, \gamma) & \longrightarrow & Q^n(C) & \xrightarrow{J_\beta} & \widehat{Q}^n(C) & \longrightarrow & .
\end{array} \qquad \square$$

(C) *Suppose the chain complex* C *is concentrated in degrees* $\leqslant n$. *Then* $Q^k(C) = 0$ *if* $k > 2n$. *If, in addition,* $H_n(C) = 0$, *then* $Q^{2n}(C) = 0$ *as well.*

The proof of (C) is straightforward from the definition of $W^{\%}C$.

The A-modules $\widehat{H}^r(\mathbb{Z}_2; A)$ $(r = 0, 1)$ will be said to be *k-dimensional* if they admit k-dimensional f.g. free A-module resolutions. In the next two subsections we compute $Q_n(B^A, \beta^A)$ for A with $2A = 0$ and $k = 0, 1$.

2.5. $Q_n(B^A, \beta^A)$ *for* 0*-dimensional* $\widehat{H}^*(\mathbb{Z}_2; A)$

Throughout this section we suppose $2A = 0$, the involution on A is trivial (and consequently A is commutative), and that $\widehat{H}^r(\mathbb{Z}_2; A)$ is a f.g. free A-module for each r.

This occurs, for example, when $A = \mathbb{F}$ or $\mathbb{F}[x]$, where $\mathbb{F}$ is a perfect field of characteristic 2.

The Frobenius map

$$\psi^2 \ : \ A \to A; \ a \mapsto \psi^2(a) = a^2.$$

is a ring homomorphism which makes the target copy of A a module over the source copy of A. We denote the target copy A-module as A'; thus A' is the additive group of A with A acting by

$$A \times A' \to A'; \ (a, x) \mapsto a^2 x$$

and there is defined an A-module isomorphism

$$A' \to \widehat{H}^r(\mathbb{Z}_2; A); \ x \mapsto x.$$

In this case one can easily construct the universal chain bundle (B^A, β^A) for A with

$$d \ = \ 0 \ : \ (B^A)_r \ = \ A' \to (B^A)_{r-1} \ = \ A'.$$

The 0-cycle of $\widehat{W}^{\%} B^{-*}$

$$\beta \ = \ \sum_r \beta^{-2r} \in (\widehat{W}^{\%} B^{-*})_0 \ = \ \sum_r (\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, B_r^{-*} \otimes_R B_r^{-*}))_0$$

is obtained as follows. Here and below we view B_r as a chain complex concentrated in degree r. Its dual chain complex, B_r^{-*}, concentrated in degree $-r$, consists of $B^r = \mathrm{Hom}_A(B_r, A)$.

Let $x_1 \dots x_k$ be a basis of A' over A. Let $x^1 \dots x^k$ be the dual basis. Write $\underline{x}_i$ for the element x_i, viewed as a member of the ring A. Note that $B^r \otimes_A B^r$ is the A-module of bilinear forms on B_r with values in A, which is canonically identified with

$$(\widehat{W}^{\%} B_r^{-*})_0 \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{-2r}, B_r^{-*} \otimes_R B_r^{-*}).$$

Therefore the elements $x^i \otimes x^i$, $\underline{x}_i(x^i \otimes x^i)$ and

$$\beta_{-2r} \ := \ \sum_{i=1}^{k} \underline{x}_i (x^i \otimes x^i)$$

are 0-cycles in $\widehat{W}^{\%}B_r^{-*}$, and bilinear forms on B_r. The matrix of the symmetric bilinear form β_{-2r} is diagonal:

$$\begin{bmatrix} \underline{x}_1 & 0 & 0 & \dots \\ 0 & \underline{x}_2 & 0 & \dots \\ 0 & 0 & \underline{x}_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It follows that $\widehat{v}_r : H_r(B) \to A'$ is the identity map. So (B, β) is universal. Inclusion induces a map of chain bundles, $(B_r, \beta_{-2r}) \xrightarrow{\iota_r} (B, \beta)$.

Lemma 21. *Assume $2A = 0$, the involution on A is trivial, and A' is free and finitely generated over A. With notation as above, the map $\iota_r : Q_*(B_r, \beta_{-2r}) \to Q_*(B, \beta)$, and the exact sequence* (18) *for (B_r, β_{-2r}), combine to give an exact sequence for each r*:

$$0 \to Q_{2r}(B, \beta) \to Q^{2r}(B_r) \xrightarrow{J_{\beta_{-2r}}} \widehat{Q}^{2r}(B_r) \to Q_{2r-1}(B, \beta) \to 0. \tag{24}$$

Proof. By (17) we have an isomorphism $B_r \otimes B_{n-r} \overset{k_{B_r}}{\cong} \widehat{Q}^n(B_r)$. By 2.4(A), we have $Q_n(B_s, \beta_{-2s}) = 0$ for $n < 2s - 1$. Therefore (23) can be written

$$\begin{aligned} &\sum_{s \leqslant r} \widehat{Q}^{2r+1}(B_s) \to \sum_{s \leqslant r} Q_{2r}(B_s, \beta_{-2s}) \to Q_{2r}(B, \beta) \to \sum_{s<r} \widehat{Q}^{2r-1}(B_s) \\ &\quad \to \sum_{s \leqslant r} Q_{2r-1}(B_s, \beta_{-2s}) \to Q_{2r-1}(B, \beta) \to \sum_{s<r} \widehat{Q}^{2r}(B_s) \\ &\quad \to Q_{2r-2}(B_s, \beta_{-2s}). \end{aligned} \tag{25}$$

Now, for dimensional reasons, if $n > 2s$, $Q^n(B_s) = 0$, and so $\widehat{Q}^{n+1}B_s) \xrightarrow{H} Q_n(B_x, \beta_{-2s})$ is an isomorphism. So (25) reduces to two pieces:

$$\begin{aligned} &\widehat{Q}^{2r+1}(B_r) \xrightarrow{H} Q_{2r}(B_r, \beta_{-2r}) \to Q_{2r}(B, \beta) \to 0, \\ &\qquad Q_{2r-1}(B_r, \beta_{-2r}) \overset{\iota_r}{\cong} Q_{2r}(B, \beta). \end{aligned} \tag{26}$$

Now apply the exact sequence (18) and Rule 2.4 A to B_r to get

$$0 \to \operatorname{coker}(H_{\beta_{-2r}}) \longrightarrow Q^{2r}(B_r) \xrightarrow{J_{\beta_{-2r}}} \widehat{Q}^{2r}(B_r) \to Q_{2r-1}(B_r, \beta_{-2r}) \to 0,$$

which, together with (26) implies Lemma 21. $\square$

We now restrict ourselves to the case when $A = \mathbb{F}[x]$ where $\mathbb{F}$ is a perfect field of characteristic 2. Then A' is free of rank 2 over A, generated by 1 and x. Since $B_r = A'$ for all r, the abelian group $Q^{2r}(B_r)$ can be identified with the additive group, $\mathrm{Sym}_2(A)$, of 2×2 symmetric matrices over A. The A-module $\widehat{Q}^{2r}(B_r)$ can be identified with $\mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$ where $\mathrm{Quad}_2(A)$ denotes the matrices of the form $M + M^t$. The map $J_{\beta_{2r}} : \mathrm{Sym}_2(A) \to \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$ then has the form

$$J_\beta \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} - \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + a + xb^2 & * \\ * & b^2 + d + xd^2 \end{bmatrix}.$$

We intend to show that the kernel and cokernel of J_β can be identified with the kernel and cokernel of the map $\psi^2 - 1 : A \to A$.

We have two inclusion maps $A \xrightarrow{\imath} \mathrm{Sym}_2(A)$, and $A \xrightarrow{\imath'} \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$, both of the form:

$$a \to \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote the images of these two maps as X, X'. Note that $(J_\beta)\imath = \imath'(\psi^2 - 1)$.

We use the following easily proved lemma:

Lemma 22. *Suppose X, X' are subgroups of two abelian groups Y, Y'. Suppose $j : Y \to Y'$ is a homomorphism such that $j(X) \subseteq X'$, and the induced map $\tilde{j} : Y/X \to Y'/X'$ is an isomorphism. Set $k = j|X : X \to X'$. Then $\ker(k) = \ker(j)$, and the inclusion $X' \to Y'$ induces an isomorphism*

$$\imath : \mathrm{coker}(k) \cong \mathrm{coker}(j).$$

We want to apply this lemma when X, X' are as mentioned earlier and the role of $j : Y \to Y'$ is played by

$$J_\beta \ : \ \mathrm{Sym}_2(A) \to \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A).$$

This means we must first check that $\tilde{j}$ is an isomorphism. In other words, we must check that each element $p \in \mathbb{F}[x]$ can be written in one and only one way in the form $b^2 + d + xd^2$ where $b, d \in \mathbb{F}[x]$.

Write

$$p = \sum_{j=0}^{2n+1} a_j x^j, \quad b = \sum_i b_i x^i, \quad d = \sum_i d_i x^i.$$

Then

$$b^2 + xd^2 + d = \sum_i (b_i^2 + d_{2i})x^{2i} + \sum_i (d_i^2 + d_{2i+1})x^{2i+1}.$$

Therefore the equation $p = b^2 + xd^2 + d$ reduces to equations,

$$d_i^2 + d_{2i+1} = a_{2i+1}; \quad b_i^2 + d_{2i} = a_{2i}.$$

One solves these recursively for d_i and b_i, working from higher to lower indices. Note that the first equation implies that $d_i = 0$ for all $i > n$. Therefore recursively, the equations

$$d_i^2 = d_{2i+1} + a_{2i+1}$$

specify d. Then the equations

$$b_i^2 = d_{2i} + a_{2i}$$

specify b. Here we use that $\mathbb{F}$ is perfect. Therefore $\tilde{j}$ is an isomorphism.

Applying the lemma, we conclude that if $A = \mathbb{F}[x]$ then

$$\ker(\psi^2 - 1) \overset{\iota}{\cong} \ker J_\beta; \quad \operatorname{coker}(\psi^2 - 1) \overset{\tilde{\imath}}{\cong} \operatorname{coker}(J_\beta). \tag{27}$$

The map $\tilde{\imath}$ is induced by $A \xrightarrow{\iota'} \mathrm{Sym}_2(A)/\mathrm{Quad}_2(A)$.

Note that if $A = \mathbb{F}_2[x]$, then $\ker(\psi^2 - 1) = \mathbb{F}_2$ and the cokernel of $A \xrightarrow{\psi^2 - 1} A$ can be identified with the vector space $\{\sum_i a_i x^i \mid a_{2i} = 0 \text{ for } i > 0\}$.

Summarizing, we have a confirmation of the calculation of Connolly and Koźniewski [8]:

Theorem 23. *For all k, we have*

$$\begin{aligned}\mathrm{UNil}_{2k+1}(\mathbb{F}_2;\mathbb{F}_2,\mathbb{F}_2) &= 0,\\ \mathrm{UNil}_{2k}(\mathbb{F}_2;\mathbb{F}_2,\mathbb{F}_2) &\cong \mathrm{coker}(\mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{\psi^2-1} \mathbb{F}_2[x]/\mathbb{F}_2)\\ &\cong \{\sum_i a_i x^i : a_{2i}=0 \text{ for } i\geqslant 0, a_i\in\mathbb{F}_2\}.\end{aligned}$$

Proof. This is a consequence of Corollary 20, Lemma 21 and (27). □

2.6. $Q_n(B^A,\beta^A)$ for 1-dimensional $\widehat{H}^*(\mathbb{Z}_2;A)$

In this subsection we deal with a ring A whose universal chain bundle (B^A,β^A) satisfies:

$$\text{For all } i,\ B^A_{2i}\xrightarrow{d} B^A_{2i-1} \text{ is zero; } B^A_{2i+1}\xrightarrow{d} B^A_{2i} \text{ is injective} \tag{28}$$

with B^A_r f.g. free A-modules. Thus $\widehat{H}^0(\mathbb{Z}_2;A)$ has a 1-dimensional f.g. free A-module resolution

$$0\to B^A_{2i+1}\xrightarrow{d} B^A_{2i}\to \widehat{H}^0(\mathbb{Z}_2;A)\to 0$$

and $\widehat{H}^1(\mathbb{Z}_2;A)=0$. (We shall see that this holds for $A=\mathbb{Z}$ or $\mathbb{Z}[x]$. The point is that Corollary 20 reduces the calculation of $\mathrm{UNil}_*(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ to that of $Q_*(B^A,\beta^A)$ for such rings A.)

We clearly have

$$(B^A,\beta^A) = \sum_{i=-\infty}^{\infty}(B^A(i),\beta^A(i)), \text{where } B^A(i) \text{ is:} \cdots 0\to B^A_{2i+1}\xrightarrow{d} B^A_{2i}\to 0\cdots .$$

We first relate $Q_n(B^A,\beta^A)$ to $Q_n(B^A(0),\beta^A(0))$, for $n=-1,0,1,2$, by analyzing the exact sequence (23), of the above direct sum splitting. By (2.4 A), we have

$$\sum_{i=-\infty}^{\infty} Q_m(B^A(i),\beta^A(i)) = \sum_{i\leqslant\frac{m+1}{4}} Q_m(B^A(i),\beta^A(i)).$$

Next, because of (17), and dimensional reasons, we have

$$\begin{aligned}\sum_{i<j} H_m(B^A(i)\otimes B^A(j)) &= \sum_{2i<[\frac{m}{2}]} H_m(B^A(i)\otimes B^A([\frac{m}{2}]-i))\\ &= \sum_{2i<[\frac{m}{2}]} H_m(B^A(i)\otimes B^A)\\ &= \sum_{i<\frac{1}{2}[\frac{m}{2}]} \widehat{Q}^m(B^A(i)).\end{aligned}$$

But, by (18) and (2.4)(C), the map $\widehat{Q}^{m+1}(B^A(i)) \to Q_m(B^A(i),\beta^A(i))$ is an isomorphism if $i \leqslant (m-2)/4$. Therefore, after we remove isomorphic direct summands from the exact sequence (23), it reduces to the much simpler long exact sequence:

$$\begin{aligned}\cdots \to \sum_{\frac{m-2}{4}<i\leqslant\frac{m+1}{4}} Q_m(B^A(i),\beta^A(i)) \to Q_m(B^A,\beta^A)\\ \to \sum_{\frac{m-3}{4}<i<\frac{1}{2}[\frac{m}{2}]} \widehat{Q}^m(B^A(i)) \to \cdots \ .\end{aligned}$$

So, we get

$$Q_m(B^A,\beta^A) \overset{\cong}{\to} Q_m(B^A(0),\beta^A(0)) \qquad \text{for } m=-1,0 \tag{29}$$

and

$$\begin{aligned}Q_1(B^A,\beta^A) &= \ker\{Q^1(B^A(0)) \xrightarrow{J^1_{\beta^A(0)}} \widehat{Q}^1(B^A(0))\},\\ Q_2(B^A,\beta^A) &= \operatorname{im}\{Q^2(B^A(0)) \xrightarrow{J^2_{\beta^A(0)}} \widehat{Q}^2(B^A(0))\} = 0 \text{ by } (2.4)(C)\end{aligned}$$

whenever (B^A,β^A) is the universal chain bundle of A, and (B^A,β^A) satisfies (28).

Next we show that (28) holds when $A=\mathbb{Z}$ or $\mathbb{Z}[x]$.

2.6.1. The construction of (B^A,β^A) for certain rings A

Suppose A is a commutative ring with no elements of order 2, and trivial involution. Write

$$A_2 = A/2A.$$

Therefore $\widehat{H}^1(\mathbb{Z}_2;A)=0$, and $\widehat{H}^0(\mathbb{Z}_2;A)=A_2'$, by which we mean the abelian group A_2, equipped with the A-module structure:

$$A\times A_2 \to A_2;\ (a,x)\mapsto (a^2x).$$

Suppose further that there are elements $x_1, x_2, \ldots, x_r \in A$, $r > 0$, such that,

$$0 \to A^r \stackrel{\times 2}{\to} A^r \stackrel{j}{\to} A'_2 \to 0$$

is exact, where

$$j \ : \ A^r \to A'_2; \ (a_1, a_2, \ldots, a_r) \mapsto a_1^2 x_1 + a_2^2 x_2 + \cdots + a_r^2 x_r.$$

(For example, if $A = \mathbb{Z}$then $r = 1, x_1 = 1$, while if $A = \mathbb{Z}[x]$ then $r = 2, x_1 = 1$, $x_2 = x$.)

We show here how to construct the universal chain bundle (B, β) for A, so that (28) holds.

First we construct B. For all i, we define

$$\begin{aligned} B_i &= A^r, \\ B_{2i} &\stackrel{d=0}{\longrightarrow} B_{2i-1}, \\ B_{2i+1} = A^r &\stackrel{d\ =\ \times 2}{\longrightarrow} A^r = B_{2i}. \end{aligned} \tag{30}$$

Next let $X \in M_r(A)$ be the diagonal matrix,

$$X = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_r \end{pmatrix}.$$

We define $\beta = \{\beta_{-i} \in (B^{-*} \otimes B^{-*})_i\}$ by

$$\begin{aligned} \beta_{-4i} &= X \in M_r(A) = (B_{2i} \otimes B_{2i})^*, \\ \beta_{-4i-1} &= (\delta \otimes 1)\beta_{-4i}, \\ \beta_{-4i-2} &= -\tfrac{1}{2}(\delta \otimes \delta)\beta_{-4i} \text{ for all } i. \end{aligned} \tag{31}$$

Here $\delta : B_0^{-*} \to B_{-1}^{-*}$ is the coboundary homomorphism.

As in (2.5), the map $\widehat{v}_{2i} : H_{2i}(B) \to A'_2$ is an isomorphism for all i, and so (B, β) is the universal chain bundle for A.

We can now apply calculation (29) to the computation of $Q_n(B^A, \beta^A)$, when $A = \mathbb{Z}$ or $\mathbb{Z}[x]$. Specifically, (29) and Corollary 20 give us the split short exact sequence if

$n = 0$ or -1:

$$0 \to \mathrm{UNil}_{n-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to Q_n(B^{\mathbb{Z}[x]}(0), \beta^{\mathbb{Z}[x]}(0)) \xrightarrow{\eta_*} Q_n(B^{\mathbb{Z}}(0), \beta^{\mathbb{Z}}(0)) \to 0, \tag{32}$$

where $\eta : \mathbb{Z}[x] \to \mathbb{Z}$ is the augmentation map.

To simplify things further we define three families of groups, K_n, C_n, I_n, by the exactness of the following three split sequences:

$$\begin{aligned}
&0 \to K_n \to \ker(J^n_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \ker(J^n_{\beta(0)}(\mathbb{Z})) \to 0,\\
&0 \to C_n \to \mathrm{coker}(J^{n+1}_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \mathrm{coker}(J^{n+1}_{\beta(0)}(\mathbb{Z})) \to 0,\\
&0 \to I_{n+1} \to \mathrm{im}(J^{n+1}_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \mathrm{im}(J^{n+1}_{\beta(0)}(\mathbb{Z})) \to 0.
\end{aligned}$$

We next claim there is an isomorphism:

$$\mathrm{UNil}_{-2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \;\cong\; C_{-1}. \tag{33}$$

To see this, note that $Q_n(B^A(0)) = 0$ for dimensional reasons if $n \leqslant -1$. Also, by (2.4)(A),

$$J^{-1}_{\beta^A(0)} \;=\; J^{-1} : Q^{-1}(B^A(0)) \to \widehat{Q}^{-1}(B^A(0)),$$

which is a monomorphism by (1). This implies that

$$Q_{-1}(B^A(0), \beta^A(0)) \;\cong\; \mathrm{coker}(J^0_{\beta^A(0)}).$$

Therefore (32) simplifies when $n = -1$, to (33).

Now 29, Corollary 20, and the exact sequence (18) for $(B^A(0), \beta^A(0))$ (when $A = \mathbb{Z}, \mathbb{Z}[x]$) yield the following calculations:

$$\begin{aligned}
&\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \;\cong\; K_1, \qquad (34)\\
&\mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \;\cong\; I_2,\\
0 \to C_0 \to\; &\mathrm{UNil}_{-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to K_0 \to 0,\\
C_{-1} \;\cong\; &\mathrm{UNil}_{-2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}).
\end{aligned}$$

Therefore our goal is to calculate C_0, C_{-1}, K_0, and K_1. This is done in the next two subsections.

2.6.2. Calculation of $Q^n(B^A(0))$ and $\widehat{Q}^n(B^A(0))$

Recall from Ranicki [14,15] that for any ring with involution A and for any A-module chain complex C an element $\phi \in (\widehat{W}^{\%}C)_n$ is specified by the sequence of elements $(\dots, \phi_{-1}, \phi_0, \phi_1, \dots)$ of $C \otimes_A C$ defined by

$$\phi_i \;=\; \phi(e_i) \in (C \otimes_A C)_{n+i} \;\; (i \in \mathbb{Z}),$$

where $e_i \in \widehat{W}_i$ is the standard basis element. Likewise, an element $\phi \in (W^{\%}C)_n$ is specified by a sequence $(\phi_0, \phi_1, \dots)$, with $\phi_i = \phi(e_i)$.

For the rest of this section we assume A is a ring satisfying the hypotheses at the beginning of Section 2.6.1.

Let $t : M_r(A) \to M_r(A)$ be the transpose map and define

$$\begin{aligned} \mathrm{Sym}_r(A) \;&=\; \ker(1 - t : M_r(A) \to M_r(A)), \\ \mathrm{Quad}_r(A) \;&=\; \mathrm{im}(1 + t : M_r(A) \to M_r(A)). \end{aligned}$$

Note that $B^A(0)$ is the algebraic mapping cone $\mathcal{C}(f)$ of the map $f : C \to D$, where $C = D = A^r$ is concentrated in degree 0, and $f = \times 2 : A^r \to A^r$. Therefore, for all m:

$$\widehat{Q}^{2m}(C) \;=\; \widehat{Q}^{2m}(D) \;\cong\; \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) : [\phi] \mapsto [\phi_{-2m}]$$

because $\phi_{-2m} \in A^r \otimes A^r = M_r(A)$ must be in the kernel of $1 - T$, for all $2m$-cycles $\phi \in (\widehat{W}^{\%}D)_{2m}$.

Also, $Q^{2m+1}(C) = \widehat{Q}^{2m+1}(D) = 0$ for all m. Since the induced map, $f^{\%} : \widehat{Q}^m(C) \to \widehat{Q}^m(D)$ is multiplication by 4, we see $f^{\%} = 0$. So the sequence:

$$0 \to \widehat{Q}^m(D) \to \widehat{Q}^m(B^A(0)) \to \widehat{Q}^m(\Sigma C) \to 0$$

is exact for all m.

If $m = 1$ the composite isomorphism,

$$\widehat{Q}^1(B^A(0)) \;\overset{\cong}{\to}\; \widehat{Q}^1(\Sigma C) \;\overset{\cong}{\to}\; \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}$$

is written as

$$\widehat{Q}^1(B^A(0)) \overset{\beta^1}{\cong} \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} : \quad \beta^1([(\phi_{-1}, \phi_0, \phi_1)]) = [\phi_1]. \tag{35}$$

If $m = 0$ we write the inverse of the composite isomorphism

$$\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \cong \widehat{Q}^0(D) \cong \widehat{Q}^0(B^A(0))$$

as

$$\widehat{Q}^0(B^A(0)) \overset{\beta^0}{\cong} \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} : \quad \beta^0([(\phi_0, \phi_1, \phi_2)]) = [\phi_0]. \tag{36}$$

The calculation of $Q^m(B^A(0))$ requires more work.

Following Ranicki [14,15] we define $Q^m(f)$ as the mth homology group of the mapping cone of $f^{\%}$:

$$Q^m(f) = H_m(f^{\%} : W^{\%}C \to W^{\%}D),$$

for any chain map $f : C \to D$ of free A-module chain complexes. We also write $\mathcal{C}(f)$ for the mapping cone of such f, and we write $g : D \to \mathcal{C}(f)$ for the inclusion. The symmetrization map

$$H_m(C \otimes_A C) \to Q^m(C); \;\; \theta \mapsto \{\phi_s = \begin{cases} (1+T)\theta & \text{if } s = 0, \\ 0 & \text{if } s \geqslant 1 \end{cases} \}$$

fits into a natural transformation of exact sequences:

$$\begin{array}{ccccccc}
H_m(C \otimes_A C) & \xrightarrow{f} & H_m(D \otimes C) & \xrightarrow{g} & H_m(\mathcal{C}(f) \otimes C) & \longrightarrow & H_{m-1}(C \otimes_A C) \\
\downarrow{\scriptstyle (1+T)} & & \downarrow{\scriptstyle (1+T)f} & & \downarrow{\scriptstyle (1+T)f} & & \downarrow{\scriptstyle (1+T)} \\
Q^m(C) & \xrightarrow{f^{\%}} & Q^m(D) & \longrightarrow & Q^m(f) & \longrightarrow & Q^{m-1}(C)
\end{array}$$

This leads to a further exact sequence relating $Q^m(f)$ to $Q^m(\mathcal{C}(f))$:

$$\cdots \to Q^{m+1}(\mathcal{C}(f)) \to H_m(\mathcal{C}(f) \otimes C) \xrightarrow{(1+T)f} Q^m(f) \to Q^m(\mathcal{C}(f)) \to \cdots .$$

Now in the case at hand (where $C = D = A^r$, and $\mathcal{C}(f) = B^A(0)$), we have

$$Q^m(C) \;=\; Q^m(D) \;=\; \begin{cases} \mathrm{Sym}_r(A) & \text{if } m = 0, \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}, & \text{if } m \text{ is even and } m < 0, \\ 0 & \text{in all other cases.} \end{cases}$$

But $f^{\%}$ is multiplication by 4. Thus

$$\begin{aligned} Q^0(f) &= \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)}, \\ 3pt]Q^{2m}(f) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \ (m < 0), \\ Q^k(f) &= 0 \text{ for all other } k. \end{aligned}$$

So from the above exact sequence, we extract the following diagram with exact rows:

$$\begin{array}{ccccccc} H_0(\mathcal{C}(f) \otimes C) & \xrightarrow{(1+T)f} & Q^0(f) & \longrightarrow & Q^0(B^A(0)) & \longrightarrow & 0 \\ \big\downarrow \cong & & \big\downarrow \cong & & \big\downarrow = & & \\ \dfrac{M_r(A)}{2M_r(A)} & \xrightarrow{2(1+t)} & \dfrac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} & \xrightarrow{\alpha} & Q^0(B^A(0)) & \longrightarrow & 0 \end{array}$$

Therefore α induces an isomorphism:

$$\frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \overset{\alpha^0}{\cong} Q^0(B^A(0)); \quad \alpha^0([M]) \;=\; [(M, 0, 0)], \tag{37}$$

where $(M, 0, 0)$ is a 0-cycle in $W^{\%}B^A(0)$, for any

$$M \in \mathrm{Sym}_r(A) \subseteq M_r(A) \;=\; A^r \otimes A^r \;=\; (B^A(0) \otimes B^A(0))_0.$$

Now $Q^m(B^A(0)) = 0$ if $m \geqslant 2$ by 2.4(C). Also by (1), if $m \leqslant -1$, the map $Q^m(B^A(0)) \overset{J^m}{\to} \widehat{Q}^m(B^A(0))$ is an isomorphism.

Therefore, we are only left with the calculation of $Q^1(B^A(0))$. Instead of the above method (which would yield the result) we calculate this by hand both for its therapeutic value and for its greater explicitness. The bottom line will be (38).

For each $M \in M_r(A)$, define

$$\phi^M = (\phi_0^M, \phi_1^M) \in (W^{\%}B^A(0))_1$$

by

$$\phi_1^M = M \in M_r(A) = A^r \otimes A^r = B_1 \otimes B_1,$$
$$\phi_0^M = M \oplus (-M) \in (B_1 \otimes B_0) \oplus (B_0 \otimes B_1),$$

where $B_i = B^A(0)_i$.

Lemma 24. *If* $M \in \mathrm{Sym}_r(A)$, *then* ϕ^M *is a 1-cycle in* $W^{\%}B^A(0)$, *and the rule* $M \mapsto \phi^M$ *induces an isomorphism*:

$$\alpha^1 : \frac{\mathrm{Sym}_r(A)}{2\mathrm{Sym}_r(A)} \cong Q^1(B^A(0)). \tag{38}$$

Proof. For any $\phi = (\phi_0, \phi_1) \in (W^{\%}B^A(0))_1$, $\phi = (\phi_0, \phi_1)$ where $\phi_i \in (B^A(0) \otimes_A B^A(0))_{i+1}$. We can write

$$\phi_0 = \kappa_1 \oplus \kappa_2,$$

where $\kappa_1 \in M_r(A) = B_1 \otimes_A B_0$, and $\kappa_2 \in M_r(A) = B_0 \otimes_A B_1$. ϕ is a 1-cycle if and only if:

$$(1)\ \partial\phi_0 = 0, \qquad (2)\ (T-1)\phi_0 = -\partial\phi_1, \qquad (3)\ (T+1)\phi_1 = 0,$$

where $T : B^A(0) \otimes B^A(0) \to B^A(0) \otimes B^A(0)$ is the twist chain map

$$T(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

These three conditions are equivalent to:

$$\kappa_2 = -\kappa_1, \qquad (1+t)\kappa_1 = 2\phi_1, \qquad t\phi_1 = \phi_1 \text{ in } A^r \otimes_A A^r = M_r(A).$$

Here t denotes the transpose map in $M_r(A)$. Also a cycle ϕ as above is a boundary in $W^{\%}B^A(0)$ if and only if there is an element $\psi \in B_1 \otimes B_1$, such that $\kappa_1 = 2\psi$ in $A^r \otimes A^r = M_r(A)$. Therefore the map

$$Q^1(B^A(0)) \to \mathrm{Sym}_r(A)/2\mathrm{Sym}_r(A) : [\phi] \mapsto \kappa_1 \bmod (2A) \tag{39}$$

is an isomorphism.

The above discussion shows that if $M \in \mathrm{Sym}_r(A)$, then ϕ^M is a 1-cycle, and if $M \in 2\mathrm{Sym}_r(A)$, then ϕ^M is a boundary. Since the map (39) obviously sends ϕ^M to M, the proof is complete. □

We summarize the calculations of this subsection as follows:

$$\begin{aligned}
\widehat{Q}^m(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \text{ for all } m,\\
Q^0(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)}\\
Q^1(B^A(0)) &\cong \frac{\mathrm{Sym}_r(A)}{2\mathrm{Sym}_r(A)}\\
Q^n(B^A(0)) &= 0 \text{ for } n \geqslant 2\\
Q^n(B^A(0)) &\overset{J^n}{\cong} \widehat{Q}^n(B^A(0)) \text{ if } n \leqslant -1.
\end{aligned} \tag{40}$$

2.6.3. The maps $J^0_{\beta(0)}(A)$, $J^1_{\beta(0)}(A)$ and the groups C_{-1}, C_0 and K_1, K_0

We first analyze the map $J^0_{\beta(0)}(A) : Q^0(B^A(0)) \to \widehat{Q}^0(B^A(0))$, when $A = \mathbb{Z}$ or $\mathbb{Z}[x]$ using the isomorphisms of (35)–(38). By 15, $\beta^0 \circ J^0_{\beta(0)}(A) \circ \alpha^0$ sends a matrix $M \in \dfrac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)}$ to

$$\beta^0(J^0([(M,0,0)])) - M^t XM \;=\; M - MXM \in \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}.$$

In the case when $A = \mathbb{Z}$, so that $r = 1$, and $X = 1$, we have $\beta^0 J^0_{\beta(0)}(\mathbb{Z})\alpha^0$, sending $a \in \mathbb{Z}_4$ to $a - a^2 \in \mathbb{Z}_4/2\mathbb{Z}_4 = \mathbb{Z}_2$. So $J^0_{\beta(0)}(\mathbb{Z}) = 0$. Therefore

$$\ker J^0_{\beta(0)}(\mathbb{Z}) \;=\; Q^0(B^{\mathbb{Z}}(0)) \cong \mathbb{Z}_4; \quad \mathrm{coker} J^0_{\beta(0)}(\mathbb{Z}) \;=\; \widehat{Q}^0(B^{\mathbb{Z}}(0)) \cong \mathbb{Z}_2.$$

Now we let $A = \mathbb{Z}[x]$. Set

$$\begin{aligned}
\mathcal{J}^0 \;&=\; \beta^0 \circ J^0_{\beta(0)}(\mathbb{Z}[x]) \circ \alpha^0 : \mathrm{Sym}_2(\mathbb{Z}[x])/2\mathrm{Quad}_2(\mathbb{Z}[x])\\
&\to \mathrm{Sym}_2(\mathbb{Z}[x])/\mathrm{Quad}_2(\mathbb{Z}[x]).
\end{aligned}$$

For any

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z}[x])/2\mathrm{Quad}_2(\mathbb{Z}[x]),$$

we compute from the above formula:

$$\mathcal{J}^0 \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} a - a^2 - b^2x & b - ab - bdx \\ b - ab - bdx & d - b^2 - d^2x \end{pmatrix} \in \frac{\mathrm{Sym}_2(A)}{\mathrm{Quad}_2(A)}. \tag{41}$$

We want to apply Lemma 22 again. Let $j = \mathcal{J}^0$, and

$$Y = \frac{\mathrm{Sym}_r(\mathbb{Z}[x])}{2\mathrm{Quad}_r(\mathbb{Z}[x])}, \ Y' = \frac{\mathrm{Sym}_r(\mathbb{Z}[x])}{\mathrm{Quad}_r(\mathbb{Z}[x])}, \ X = (\mathbb{Z}_4[x]) \times (\mathbb{F}_2[x]),$$
$$X' = (\mathbb{F}_2[x]).$$

X and X' include into Y and Y', respectively, by the rules: $(a, d) \mapsto \begin{pmatrix} a & 0 \\ 0 & 2d \end{pmatrix}$, and $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. We first will have to show that $Y/X \to Y'/X'$ is an isomorphism. To this end, we note an isomorphism, $\mathbb{F}_2[x] \times \mathbb{F}_2[x] \cong Y/i(X)$, defined by $(b, d) \mapsto \begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$, and an isomorphism $\mathbb{F}_2[x] \cong Y'/i'(X')$, given by : $p \mapsto \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix}$. Therefore the claim that j induces an isomorphism, $Y/X \to Y'/X'$, amounts to the statement that each $p \in \mathbb{F}_2[x]$ can be written uniquely in the form, $p = b^2 + d + xd^2$, for some $b, d \in \mathbb{F}_2[x]$. But this was proved already in Section 2.5.

Define

$$k : \mathbb{Z}_4[x] \times \mathbb{F}_2[x] \to \mathbb{F}_2[x]; \ (a, d) \mapsto a - a^2 \bmod 2.$$

Clearly,

$$\ker(k) = \{(a, d) \in \mathbb{Z}_4[x] \times \mathbb{F}_2[x] | a = a_0 + 2a_1, \text{for some } a_0 \in \mathbb{Z}_4, a_1 \in \mathbb{Z}_4[x]\},$$
$$\mathrm{coker}(k) = \mathrm{coker}(\psi^2 - 1).$$

Applying Lemma 22, we see that i and i' induce isomorphisms:

$$\ker(k) \overset{\imath}{\cong} \ker J^0_{\beta(0)}(\mathbb{Z}[x]), \quad \mathrm{coker}(\psi^2 - 1) \overset{\imath'}{\to} \mathrm{coker}(J^0_{\beta(0)}(\mathbb{Z}[x])).$$

Also, $\imath(a, d) = \alpha^0 \begin{bmatrix} a & 0 \\ 0 & 2d \end{bmatrix}$.

The augmentation map induced by η

$$Q^0(B^{\mathbb{Z}[x]}(0)) \overset{\eta_*}{\to} Q^0(B^{\mathbb{Z}}(0))$$

sends $\alpha^0 \begin{bmatrix} a & 0 \\ 0 & 2d \end{bmatrix}$ to $a_0 \in \mathbb{Z}_4$, the degree zero coefficient of a. The same formula holds as well for $\eta_* : Q^0(B^{\mathbb{Z}[x]}(0)) \to Q^0(B^{\mathbb{Z}}(0))$.

Restricting η_* to $\ker J^0_{\beta(0)}(\mathbb{Z}[x])$, we get a short exact sequence:

$$0 \to \mathbb{F}_2[x] \times \mathbb{F}_2[x] \xrightarrow{k_2} \ker(J^0_{\beta(0)}(\mathbb{Z}[x])) \xrightarrow{\eta_*} \ker(J^0_{\beta(0)}(\mathbb{Z})) \to 0,$$

where k_2 is defined by

$$k_2(a,d) \;=\; \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix}.$$

This yields isomorphisms:

$$\mathbb{F}_2[x] \times \mathbb{F}_2[x] \;\overset{k_2}{\cong}\; K_0, \quad \operatorname{coker}\{(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2\} \;\overset{k_2'}{\cong}\; C_{-1}. \qquad (42)$$

Here $(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2$ is the map induced by $\psi^2 - 1 : \mathbb{F}_2[x] \to \mathbb{F}_2[x]$, and k_2' is induced by ι'.

Now we analyze $J^1_{\beta(0)}(A)$ similarly. Recall $B^A(0)$ is a chain complex concentrated in degrees 0 and 1: $B_0 = A^r$; $B_1 = A^r$, and its boundary map is $\partial = \times 2 : B_1 \to B_0$.

In order to understand the map $J^1_{\beta(0)}(A)$, we define, for any 1-cycle, $\phi \in (W^{\%}B^A(0))_1$, another 1-cycle

$$\gamma^{\phi} \;=\; \phi_0^{\%}(S^1(\beta(0))) \in (\widehat{W}^{\%}B^A(0))_1 \;.$$

We know $\gamma^{\phi} = (\gamma^{\phi}_{-1}, \gamma^{\phi}_0, \gamma^{\phi}_1)$, where

$$\gamma^{\phi}_i \;=\; \gamma_i \;=\; \tilde{\phi}_0 \otimes \tilde{\phi}_0(\beta(0)_{i-1}).$$

Here $\tilde{\phi}_0 : B^A(0)^{1-*} \to B^A(0)$ is the chain map whose matrix is $\phi_0 \in (B^A(0) \otimes B^A(0))_1$.

We conclude:

$$\begin{aligned}
\gamma_1 &= \tilde{\phi}_0 \otimes \tilde{\phi}_0(X) \in B_1 \otimes B_1,\\
\gamma_0 &= (1 \otimes \partial)\gamma_1 \in (B^A(0) \otimes B^A(0))_1,\\
\gamma_{-1} &= \tfrac{1}{2}(\partial \otimes \partial)\gamma_1 \in B_0 \otimes B_0.
\end{aligned}$$

Therefore

$$J^1_{\beta(0)}(A) : Q^1(B^A(0)) \to \widehat{Q}^1(B^A(0)) \quad \text{is} : [\phi] \mapsto J^1([\phi]) - [\gamma^{\phi}].$$

Set

$$\mathcal{J}^1 \;=\; \beta^1 \circ J^1_{\beta(0)}(A) \circ \alpha^1.$$

We get

$$\begin{aligned}\mathcal{J}^1(M) \;&=\; \beta^1(J^1[\phi^M]) - [\gamma_1^{\phi^M}] \;=\; M - M^t X M \\ &=\; M - MXM \mod \mathrm{Quad}_r(A)\end{aligned}$$

for all $M \in \mathrm{Sym}_r(A)$. (The formulae for $\mathcal{J}^1$ and $\mathcal{J}^0$ are identical!). Therefore formula (41) can also be used for $\mathcal{J}^1$. We therefore conclude at once that we have an isomorphism, induced by β^1:

$$\mathrm{coker}\{(\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2\} \;\cong\; C_0. \tag{43}$$

To compute K_1, we note from 41 that the kernel of $J^1_{\beta(0)}(\mathbb{Z}[x]) \circ \alpha^1$ is

$$\{\begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{Sym}_2(\mathbb{F}_2[x]) : a_0 \in \mathbb{F}_2\}.$$

Since $\widehat{\eta}_* \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} \;=\; a_0 \in \mathbb{Z}_2$, we conclude at once that

$$K_1 \;=\; 0. \tag{44}$$

2.7. The calculation of $\mathrm{UNil}_n(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ *for all n*

The results of the last section allow us to prove Theorem B of the Introduction:

Theorem 25. *There are isomorphisms*:

$$\begin{aligned}\mathrm{UNil}_0(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \;&=\; 0, \\ \mathrm{UNil}_1(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \;&=\; 0, \\ \mathrm{UNil}_2(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \;&\cong\; \mathrm{coker}\{\, (\psi^2 - 1) : \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathbb{F}_2[x]/\mathbb{F}_2 \,\},\end{aligned}$$

and an exact sequence:

$$0 \to \mathbb{F}_2[x]/\mathbb{F}_2 \xrightarrow{(\psi^2-1)} \mathbb{F}_2[x]/\mathbb{F}_2 \to \mathrm{UNil}_3(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \to \mathbb{F}_2[x] \times \mathbb{F}_2[x] \to 0.$$

Proof. Note $I_2 = 0$, by (29). Therefore (44) and (34) imply the first two equations at once. The third equation is immediate from (34) and (42). The final exact sequence is immediate from (34), (42), and (43). □

See [1,7] for further computations.

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