where for nonabelian π one should interpret $\mathrm{Ext}(Z_{p^\infty}, \pi)$ and $\mathrm{Hom}(Z_{p^\infty}, \pi)$ as

$$\mathrm{Ext}(Z_{p^\infty}, \pi) = \pi_1 R_\infty K(\pi, 1), \qquad \mathrm{Hom}(Z_{p^\infty}, \pi) = \pi_2 R_\infty K(\pi, 1).$$

Note that *for π nilpotent and finitely generated one has that* $\mathrm{Hom}(Z_{p^\infty}, \pi) = 0$ *while* $\mathrm{Ext}(Z_{p^\infty}, \pi)$ *is the p-completion of* π.

We end with the observation that, although the associated spectral sequence converges to $\pi_* R_\infty X$ whenever X is connected with $\tilde{H}_n(X; Z_p)$ finite for each n (4.1), this need not be the case without this assumption, even for X a $K(\pi, n)$. Still, for a nilpotent X the tower $\{R_n X\}$ and the completion $R_\infty X$ determine each other up to homotopy (6.1) and hence it should be possible to find out what homotopy information about $R_\infty X$ is contained in $E_\infty(X; Z_p)$.

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THE KERVAIRE INVARIANT OF A MANIFOLD

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1. **Introduction.** Surgery is one of the main tools of differential topology. Typically, one wishes to construct a manifold with certain properties. By some means, usually a transverse regularity argument, one constructs a manifold enjoying some of the desired properties. Then, by surgery, one attempts to modify this manifold to meet all the required conditions. Usually, when the dimension of the manifold $\equiv 2 \bmod 4$ one meets Kervaire or Arf invariant obstruction. As an example of this we describe surgery on a map.

Suppose X is a finite CW-complex.

PROBLEM. When does X have the same homotopy type as a smooth, closed, compact orientable manifold?

Necessary conditions are that the homology and cohomology of X satisfy Poincaré duality (see [1] for details) and that there is a vector bundle η over X such that the top homology class of the Thom space $T(\eta)$ is spherical. Assuming these conditions hold, one can construct an m-manifold M and maps

$$\begin{array}{ccc} \nu_M & \xrightarrow{g} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

where ν_M is the normal bundle of M in R^{m+k} (k large), such that $f_*: H_m(M) \approx H_m(X)$. Surgery in this situation proceeds as follows: Let $\alpha \in \ker f_* \pi_l(M) \to \pi_l(X)$. Represent α by an embedding $i: S^l \subset M$ (if possible). Extend i to an embedding $j: S^l \times D^{m-l} \subset M$ (if possible). Let $N = M \times I \cup D^{l+1} \times D^{m-l}$ with $(j(x, y), 1)$ and $(x, y) \in S^l \times D^{m-l}$ identified (smooth corners). Choose j so that g can be extended to $G: \nu_N \to \eta$ (again if possible). $\partial N = M \times \{0\} \cup M'$. M' and $g' = G \mid \nu_{M'}$ are said to be obtained from (M, g) by surgery on α. One can apply

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this technique, by induction on i to produce an (M, g) such that f_* is a monomorphism on $\pi_i(M)$ for $i < [m/2]$. Furthermore, if one can carry through this construction to produce a monomorphism for $i = [m/2]$, f will be a homotopy equivalence. Wall has defined a group $L_m(\pi_1(X))$, depending only on the group $\pi_1(X)$, and an element $\sigma(M, g)$ in this group such that the surgery can be performed if and only if $\sigma(M, g) = 0$.

We consider the case $\pi_1(X) = 0$ and $m = 2n > 4$. Suppose $f_*: \pi_i(M) \to \pi_i(X)$ is a monomorphism for $i < n$ (it will be an isomorphism by Poincaré duality). Let $K = \ker(H_n(M) \to H_n(X))$. If we can kill K by surgery, f will be a homotopy equivalence. Each element of K can be represented by an embedding $i: S^n \subset M$ but in general, the normal bundle ν_i of $i(S^n)$ in M will not be trivial. ν_i is stably trivial because $i(S^n)$ represents an element of K. If n is even, ν_i is characterized by its Euler number, and when n is odd ν_i is either trivial or isomorphic to the tangent bundle of S^n. Let

$$\varphi: K \to \begin{cases} Z, & n \text{ even}, \\ Z_2, & n \text{ odd}, \end{cases}$$

be defined as follows: Let $u \in K$ and let $i: S^n \subset M$ represent u. Let $\varphi(u)$ be the Euler number of ν_i if n is even and 0 or 1 as ν_i is trivial or not when n is odd, $n \neq 1, 3, 7$. When n is even, $\varphi(u) = u \cap u$, where "$\cap$" denotes the intersection pairing, but for n odd φ cannot be expressed in terms of the intersection pairing. One does have the relation:

$$\varphi(u + v) = \varphi(u) + \varphi(v) + u \cap v \bmod 2 \tag{1.1}$$

for n odd. To perform the surgeries making $f: M \to X$ into a homotopy equivalence it is necessary and sufficient that there is a symplectic basis $\lambda_i, \mu_i, i = 1, 2, \ldots, l$ $(\lambda_i \cap \mu_j = \delta_{ij}, \lambda_i \cap \lambda_j = \mu_i \cap \mu_j = 0)$ such that $\varphi(\lambda_i) = \varphi(\mu_i) = 0$. By Poincaré duality, $\cap$ is nonsingular on K. Suppose n is odd, $n \neq 1, 3, 7$. In this case $\cap$ is skew so there is a symplectic basis for K. The Arf invariant of φ is an algebraic invariant given by $A(\varphi) = \sum \varphi(\lambda_i)\varphi(\mu_i) \in Z_2$ for any symplectic basis λ_i, μ_i. An algebraic result about A is that $A(\varphi) = 0$ if and only if λ_i, μ_i can be chosen so that $\varphi(\lambda_i) = \varphi(\mu_i) = 0$. In this case the Wall group $L_{4k+2}(0)$ is Z_2 and $\sigma(M, g) = A(\varphi)$.

Suppose n is even. $\cap$ on K is nonsingular and $u \cap u =$ Euler number ν_i is even. It follows from results on quadratic forms over Z, that there is a symplectic basis if and only if the signature of φ is zero. Thus $L_{4k}(0) = Z$ and $\sigma(M, g) =$ signature φ. Furthermore, we can give a formula for $\sigma(M, g)$ in terms of X and η. $H_n(M)$ splits, with respect to the intersection pairing, as $H_n(M) \approx K \oplus H_n(X)$, where the pairing on $H_n(X)$ comes from the fact that it satisfies Poincaré duality. Thus

Theorem 1.2. *If n is even,*

$$\sigma(M, g) = I(M) - I(X),$$

where I denotes the index.

Using the Hirzebruch index theorem we have

$$I(M) = \bar{L}_{2n}(p(\nu))(M) = \bar{L}_{2n}(p(\eta))(X),$$

where $\bar{L}_{2n}$ is the L-polynomial and p is the Pontrjagin class. Hence

Theorem 1.3. $\sigma(M,g) = \bar{L}_{2n}(p(\eta))(X) - I(X)$.

Theorems 1.2 and 1.3 provide a model for what we would like to do when n is odd. In subsequent lectures we describe a version of Theorem 1.2 for n odd.

An important special case of the above is $X = S^{2n}$, η trivial and n odd. Then g is a framing of the normal bundle of M and $\sigma(M, g)$ is the Kervaire invariant [2]. This defines a homomorphism

$$K: \Omega_{2n}(\text{framed}) \to Z_2.$$

Kervaire and Milnor conjectured that $K = 0$, $n \neq 1, 3, 7$. (K is defined for $n = 1, 3, 7$ as $\sigma(M, g)$ but in these cases $\sigma(M, g)$ is the obstruction to finding G; see page 65. $K \neq 0$, $n = 1, 3, 7$.) The results on this conjecture are:

Kervaire: $K = 0$ for $n = 5, 7$ [2],

Brown-Peterson: $K = 0$ for $n \equiv 1 \bmod 4$ [3],

Browder: $K = 0$ for $n \neq 2^i - 1$ [4], and $K \neq 0$ for $n = 2^i - 1$, if and only if, h_i^2 lives to E_∞ in the Adams spectral sequence for homotopy groups of spheres. ($K \neq 0$, if $n = 30$.)

We will indicate some of the methods used by Browder to prove his results.

2. Algebra of the Arf invariant. Let V be a finite-dimensional vector space over Z_2. The Arf invariant is defined on quadratic functions $\varphi: V \to Z_2$. Both for algebraic and geometric reasons it is useful to consider functions into Z_4.

Definition 2.1. $\varphi: V \to Z_4$ is (nonsingular) quadratic if

$$\varphi(u + v) = \varphi(u) + \varphi(v) + j\mu(u \otimes v),$$

where $\mu: V \otimes V \to Z_2$ is a nonsingular pairing and $j: Z_2 \to Z_4$ is the nontrivial homomorphism.

Remark. $2\varphi(u) = j\mu(u \otimes u)$. Thus considering φ with values in Z_4 instead of Z_2 allows us to deal with the case in which $\mu(u \otimes u) \neq 0$. This allows us to deal with manifolds in which cup product to the top dimension is nonzero.

If $\varphi_1: V_1 \to Z_4$ and $\varphi_2: V_2 \to Z_4$ are quadratic, we define $\varphi_1 \approx \varphi_2$ if there is a linear isomorphism $\lambda: V_1 \to V_2$ such that $\varphi_2\lambda = \varphi_1$. We define

$$\varphi_1 + \varphi_2: V_1 \oplus V_2 \to Z_4$$

by

$$(\varphi_1 + \varphi_2)(u, v) = \varphi_1(u) + \varphi_2(v)$$

and we define $\varphi_1\varphi_2: V_1 \otimes V_2 \to Z_4$ by

$$\varphi_1\varphi_2(u \otimes v) = \varphi_1(u)\varphi_2(v).$$

(Use the quadratic property to extend this to all of $V_1 \otimes V_2$.) Let $(-\varphi)(u) = -\varphi(u)$.

We wish to show that the Grothendieck group of these functions is Z_8. We state this in the following form:

THEOREM 2.2. *There is a unique function σ from quadratic functions to Z_8 such that*

(i) *If* $\varphi_1 \approx \varphi_2, \sigma(\varphi_1) = \sigma(\varphi_2)$.
(ii) $\sigma(\varphi_1 + \varphi_2) = \sigma(\varphi_1) + \sigma(\varphi_2)$.
(iii) $\sigma(-\varphi) = -\sigma(\varphi)$.
(iv) $\sigma(\gamma) = 1$, *where* $\gamma: Z_2 \to Z_4$ *by* $\gamma(0) = 0, \gamma(1) = 1$.

Furthermore

(v) $\sigma(\varphi_1\varphi_2) = \sigma(\varphi_1)\sigma(\varphi_2)$.
(vi) *If* $\psi: V \to Z_2$ *is nonsingular quadratic,* $\sigma(j\psi) = k(\mathrm{Arf}\,\psi)$ *where* $k: Z_2 \to Z_8$ *by* $k(1) = 4$ *and* $k(0) = 0$.
(vii) *If* $\psi: U \to Z$ *is a unimodular quadratic form over* Z, $\tilde{\psi}: U/2U \to Z_4$ *is well defined and quadratic and* $\sigma(\tilde{\psi}) =$ *signature* ψ mod 8.
(viii) *For any* φ *there is a* $\bar{\varphi}$ *such that* $\varphi + (\bar{\varphi} + (-\bar{\varphi})) \approx n\gamma + m(-\gamma)$ *and* $\sigma(\varphi) = n - m$.
(ix) $\sigma(\varphi) = \dim V \bmod 2$.

PROOF. We describe a trick due to Paul Monsky for defining σ. Let $i = (-1)^{1/2}$ and consider

$$\alpha(\varphi) = \sum_{u \in V} i^{\varphi(u)} \in \mathbf{C}.$$

It is trivial to check that $\alpha(\varphi_1 + \varphi_2) = \alpha(\varphi_1)\alpha(\varphi_2)$, $\alpha(-\varphi) = \overline{\alpha(\varphi)}$, if $L: V \to Z_4$ is linear, $\alpha(L) = 0$ if $L \neq 0$ and $\alpha(L) = 2^{\dim V}$ if $L = 0$. From the quadratic property one then sees that

$$\alpha(\varphi)\overline{\alpha(\varphi)} = 2^{\dim V} \quad \text{and} \quad \alpha(2\varphi) = \pm i 2^{\dim V}.$$

Hence $\alpha(8\varphi)$ is real and

$$\alpha(\varphi) = \sqrt{2}^{\dim V} \cdot \text{8th root of } 1 = \sqrt{2}^{\dim V}\left(\frac{1+i}{\sqrt{2}}\right)^{\sigma(\varphi)}.$$

Continuing in this vein one can prove (i) – (ix).

Suppose $\mu: V \otimes V \to Z_2$ is a nonsingular symmetric pairing. Let $Q(V, \mu) = V \times Z_2$ with the abelian group structure given by

$$(u, n) + (v, m) = (u + v, \mu(u \otimes v) + n + m).$$

It is trivial to check that quadratic functions $\varphi: V \to Z_4$ whose associated bilinear form is μ are in one-to-one correspondence with homomorphisms $\psi: Q(V, \mu) \to Z_4$ such that $\psi(0, 1) = 2$, under the correspondence $\varphi(u) = \psi(u, 0)$.

3. **The Kervaire invariant of a manifold.** Suppose M is a closed $2n$-manifold (or a Poincaré space). Let $K_n = K(Z_2, n)$, $H^n(M) = [M^+, K_n]$, and

$$\{M^+, K_n\} = \lim_{k \to \infty} [S^k M^+, S^k K_n].$$

Let $\theta: [M^+, K_n] \to \{M^+, K_n\}$ by $\theta[f] = \{f\}$. Let $d: M \to S^{2n}$ be a map of degree 1. $\{S^{2n}, K_n\} \approx Z_2$.

PROPOSITION 3.1 $\theta \times d^*: Q(H^n(M), \cup) \approx \{M^+, K_n\}$, *where* $\cup$ *denotes cup product.*

PROOF. One shows that $\theta(u + v) = \theta(u) + \theta(v) + (u \cup v)(M)\alpha$, where $\alpha = d^*1$, by using $S(K_n \times K_n) = S(K_n) \vee SK_n \vee S(K_n \wedge K_n)$. The methods for proving Proposition 3.3 then yield Proposition 3.1.

Let ν_M be the normal bundle of M in R^{2n+k}. Recall M^+ is the S-dual of $T(\nu_M)$. Hence $\{M^+, K_n\} \approx \{S^{2n+k}, T(\nu_M) \wedge K_n\}$. α corresponds to $\bar{\alpha} =$ image of the generator of $\{S^{2n+k}, S^k \wedge K_n\} \approx Z_2$ under the inclusion S^k in $T(\nu_M)$ as a fibre. Combining the results of §2 and Proposition 3.1 we have:

PROPOSITION 3.2. *The quadratic functions on* $H^n(M)$ *associated to cup product are in one-to-one correspondence with homomorphisms*

$$\{S^{2n+k}, T(\nu_M) \wedge K_n\} \to Z_4$$

taking $\bar{\alpha}$ *into* 2.

Let Y be a 0-connected spectrum such that $H^0(Y; Z_2) \approx Z_2$ and let $U: Y \to K(Z_2)$ represent the generator. A *Y orientation* for M is a map $V: T(\nu_M) \to Y_k$ such that UV is the Thom class of $T(\nu_M)$. Hence, a Y orientation of M gives a map

$$\{S^{2n+k}, T(\nu_M) \wedge K_n\} \to \{S^{2n+k}, Y_k \wedge K_n\}$$

and $\bar{\alpha}$ maps into an obvious canonical element $\bar{\bar{\alpha}}$.

PROPOSITION 3.3. *$\bar{\bar{\alpha}}$ is at most divisible by 2 and $\bar{\bar{\alpha}} \neq 0$ if and only if*

$$\chi(Sq^{n+1})U = 0.$$

PROOF.

$$\{S^{2n+k}, Y_k \wedge K_n\} \approx \{S^{2n+k+1}, Y_k \wedge SK_n\}.$$

For the dimensions under consideration, the two stage Postnikov system of SK_n, namely (K_{n+1}, Sq^{n+1}), suffices to compute this group. This gives an exact sequence

$$H_{k+n+1}(Y_k) \xrightarrow{Sq^{n+1}} H_k(Y_k) \longrightarrow \{S^{2n+k}, Y_k \wedge SK_n\} \longrightarrow H_{n+k}(Y_k) \longrightarrow 0.$$

Q.E.D.

Suppose $\chi(Sq^{n+1})U = 0$. Choose a homomorphism

$$\lambda: \{S^{2n+k}, Y_k \wedge K_n\} \to Z_4$$

such that $\lambda(\bar{\bar{\alpha}}) = 2$. Suppose V is a Y orientation of M. We then have a Kervaire invariant $K(M, V) \in Z_8$ given by $\sigma(\varphi)$, where $\varphi: H^n(M) \to Z_4$ assigns to u, λ on:

$$S^{2n+k} \xrightarrow{t} T(\nu_M) \xrightarrow{\Delta} T(\nu_M) \wedge M^+ \xrightarrow{V \wedge U} Y_k \wedge K_n,$$

where t is the Thom construction and Δ is the diagonal map.

4. **Kervaire invariant and cobordism.** Suppose $\{MG_k\}$ are the Thom spaces for some cobordism theory and $\chi(Sq^{n+1})U = 0$, where U is the Thom class of MG_k. Taking $Y = \{MG_k\}$ (and choosing λ as above) we obtain a Kervaire invariant for each G manifold of dimension $2n$.

THEOREM 4.1. *K defines a homomorphism*

$$K:\Omega_{2n}(G) \to Z_8.$$

PROOF. The proof of this is somewhat tedious but straightforward.

EXAMPLE 1. $MG_k = S^k$. λ is unique.

THEOREM 4.2. *$K:\Omega_{2n}(framed) \to Z_8$ has its image in $\{0, 4\}$ and is the Kervaire invariant.*

EXAMPLE 2. $MG_k = M\,\mathrm{Spin}_k$, $n \equiv 1 \bmod 4$. For certain choices of λ, K is the Kervaire invariant defined by Brown-Peterson.

EXAMPLE 3. $MG_k = MSU_k$, $n \equiv 1 \bmod 4$. λ is unique.

EXAMPLE 4. $MG_k = MSO_k$, n even.

CONJECTURE. For a certain choice of λ, $\varphi:H^n(M) \to Z_4$ is the Pontrjagin square and K is the index mod 8.

EXAMPLE 5. $MG_k = S^{k-1}RP_\infty$. λ is unique. $\Omega_2(G)$ is the cobordism group of surfaces immersed in R^3 and K is an isomorphism. $\varphi(u)$ may be obtained as follows: Suppose $i:S \to R^3$ is an immersion of a surface S. Represent the Poincaré dual of u by an embedded circle (or disjoint circles). Let $\varphi(u)$ = number of half twists (in R^3) of a tubular neighborhood of this circle (Mobius band has one half twist).

EXAMPLE 6. Let $v_{n+1} \in H^{n+1}(BO_k)$ be the Wu class given by $v_{n+1}U = \chi(Sq^{n+1})U$ in $H^*(MO_k)$. Let $B_k^{(n)} \xrightarrow{p} BO_k$ be the fibration with k-invariant v_{n+1}. Let $MB_k^n = T(p^*\zeta_k)$, where ζ_k is the canonical k-plane bundle. This is the cobordism theory utilized by Browder to deal with the Kervaire-Milnor conjecture. One has a commutative diagram

$$\begin{array}{ccc} \Omega_{2n}(\text{framed}) & \xrightarrow{t} & \Omega_{2n}(B^{(n)}) \\ & {\scriptstyle K}\searrow \quad \swarrow{\scriptstyle K} & \\ & Z_8 & \end{array}$$

Browder shows that $t = 0$ if $n \neq 2^i - 1$ by constructing a Postnikov system for MB_k^n up to dimension $2n$.

Suppose Y is a spectrum as in §3, $\chi(Sq^{n+1})U = 0$ and suppose λ has been chosen. Let X be a 1-connected Poincaré space of dimension $2n$, n odd, ξ its Spivak normal bundle, V a Y orientation of ξ and $\alpha \in \pi_{2n+k}(T(\xi))$ an element representing the top homology class of $T(\xi)$. The methods of §3 give an invariant

$K(X, \xi, \alpha, V) \in Z_8$. Suppose

$$\begin{array}{ccc} \nu_M & \xrightarrow{g} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

is as in §1. Let $\beta \in \pi_{2n+k}(T(\nu_M))$ be the element obtained from the Thom construction.

THEOREM 4.3. $\sigma(M, g) = K(M, \nu_M, \beta, g^*V) - K(X, \xi, T(g)_*(\beta), V)$.

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