Splitting homotopy equivalences along codimension 1 submanifolds

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Abstract

The thesis addresses the problem of deciding whether a homotopy equivalence of manifolds is splittable along a codimension 1 submanifold. If $g : W \to Y$ is a homotopy equivalence of manifolds, and $X \subset Y$ is a codimension 1 submanifold, g is *split* if it is transverse to X and letting $M = g^{-1}(X)$, $f = g|_M : M \to X$ is a homotopy equivalence. g is *splittable* if it is h-cobordant to a split homotopy equivalence $g' : W' \to Y$. We restrict our attention here to the case where $Y = Y_1 \cup_X Y_2$ and $H = \pi_1(X) \to \pi_1(Y_i) = G_i$ are injections.

Such problems were first studied in detail by Cappell in the 1970s using high dimensional surgery theory, initially by considering the effect of handle exchanges — which vary g by a homotopy and perform surgery on the map $f : M \to X$ inside $g : W \to Y$. Cappell showed that not every homotopy equivalence of the above form is splittable. In particular, in the case when X is even-dimensional ($\dim X = 2k \geq 6$) there are 2 obstructions to g being splittable: the first is a K-theory obstruction $\overline{\phi}(\tau(g))$; the second is an L-theory type obstruction $\chi(g)$ lying in the so-called *unitary nilpotent group* $\mathrm{UNil}_{2k+2}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$ which consists of UNil forms: pairs of quadratic forms taking values in $\mathbb{Z}[G_i]$.

The thesis begins with a slightly modified presentation of Cappell's results, which is more closely linked to the language of the quadratic forms which define the L-groups. In the same way that Ranicki defined a correspondence between quadratic formations and short odd complexes and defined the L-groups as highly connected cobordism classes of short odd complexes, a correspondence between UNil formations and *short odd nilcomplexes* is established, and $\mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$ is defined as cobordism classes of short odd nilcomplexes. This definition is related to the chain complex formulation of the splitting problem due to Ranicki and a map is defined $\mathrm{UNil}_{2k+3} \to L_{2k+3}$.

It is shown that given a splitting problem of the form above, with $\dim X = 2k + 1 \geq 5$ and $\overline{\phi}(\tau(g)) = 0$, there is an obstruction $\chi(g) \in \mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$ such that $\chi(g) = 0$ if and only if g is splittable.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Jeremy Brookman*)

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Chapter 1

Introduction

Let $Y^{n+1} = Y_1 \cup_X Y_2$ where X^n is a codimension 1 submanifold of Y, and suppose that $g : W \to Y$ is a homotopy equivalence of manifolds. Assume that g is transverse regular to X, so that $M = g^{-1}(X)$ is a codimension 1 submanifold of W, with $W = W_1 \cup_M W_2$. g is *split*, if $f = g|_M : M \to X$ is a homotopy equivalence, and g is *splittable* if there exists an h-cobordism V of $g : W \to Y$ with $g' : W' \to Y$ where g' is split. This thesis addresses the question:

Question 1.1 Is every homotopy equivalence $g : W \to Y$ splittable?

Note that the fundamental group of $Y$is given by the Seifert-van Kampen theorem as the pushout of the diagram:

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d] & \pi_1(Y_1) \arrow[d, dotted] \\ \pi_1(Y_2) \arrow[r, dotted] & \pi_1(Y) \end{tikzcd}$$

Henceforth, assume that the maps $\pi_1(X) \to \pi_1(Y_i)$ are injective (so that the maps $\pi_1(Y_i) \to \pi_1(Y)$ are also injective), and write $H = \pi_1(X)$, $G_i = \pi_1(Y_i)$ and $G = \pi_1(Y)$. Then G is an amalgamated free product $G = G_1 *_H G_2$.

We shall call the question of deciding whether a homotopy equivalence of the above form is splittable a *splitting problem*. The techniques to be used are essentially the techniques of surgery theory.

For many Y, the answer to this question is yes, but it is not always so; there is a counter-example due to Cappell (see Cappell [2]). In later work Cappell constructs two obstructions to g being split when n is even; the first is a K-theory obstruction $\overline{\phi}(\tau(g)) \in H^n(\mathbb{Z}_2; I = \ker(\tilde{K}_0(\mathbb{Z}[H] \to \tilde{K}_0(\mathbb{Z}[G_1]) \oplus \tilde{K}_0(\mathbb{Z}[G_2])))$; the second is an L-theoretic obstruction $\chi(g)$ lying in an obstruction group (the unitary nilpotent group UNil) which depends only upon the fundamental groups H, G_1, G_2 (see Cappell[3]). He was then able to show the vanishing of the obstruction group for a wide class of fundamental groups; in particular for the square-root closed condition where $g^2 \in H \Rightarrow g \in H$. The thesis addresses the remaining case when n is odd.

It has been shown that the UNil groups fit into a Mayer-Vietoris-like sequence of surgery

groups (Cappell[4], Ranicki[9]):

$$\ldots \to L_n^I(\mathbb{Z}[H]) \oplus \mathrm{UNil}_{n+1} \to L_n(\mathbb{Z}[G_1]) \oplus L_n(\mathbb{Z}[G_2]) \to L_n(\mathbb{Z}[G]) \to L_{n-1}^I(\mathbb{Z}[H]) \oplus \mathrm{UNil}_n \to \ldots$$

(where the undecorated L-groups are the free L-groups $L_n(R) = L_n^h(R)$). Here UNil_n is described in terms of chain complexes as described in chapter 10. The map $L_n(\mathbb{Z}[G]) \to \mathrm{UNil}_n$ is then a split surjection.

The splitting question has many similarities to the question of surgery theory:

Question 1.2 Suppose $f : M \to X$ is a degree 1 normal map of n-dimensional manifolds. Is f normal bordant to a homotopy equivalence?

The solutions to these 2 problems are very heavily related, although the theory relating to the surgery question is more developed. In this thesis, we try to recap the relevant surgery theory in parallel with developing the solution of the splitting obstruction, to make the similarities clear. For this reason, in prose, when we refer the a splitting problem as being even-dimensional, we mean that the codimension 1 submanifold X is even-dimensional, and that Y is odd-dimensional, in contrast with Cappell's use. Throughout the thesis, n will be used to refer to the dimension of X when considering surgery or splitting problems.

The methods in use are as usual restricted to high-dimensional manifolds and it is assumed that $n \geq 5$.

In order to modify a map to become closer to a homotopy equivalence, the simplest operation that can be used is a handle exchange: given a homotopy equivalence $g : W \to Y$ cut along $f : M \to X$, a handle exchange on g has the effect of a surgery on f. In this chapter we recall the effects of surgeries and handle exchanges on maps.

1.1 Surgeries

Here, let M and X be compact n-dimensional manifolds, and $f : M \to X$ a degree 1 map such that $f|\partial M : \partial M \to \partial X$ is a homotopy equivalence. (In fact X need not be a manifold — it could also be any CW-complex with Poincaré duality; however for most of our applications it will be a manifold.) Then the map $H_*(M) \to H_*(X)$ is a split surjection; the kernel homology groups are to be denoted $K_*(M)$, so that $H_*(M) = H_*(X) \oplus K_*(M)$.

The goal of the surgery program is to make f increasingly connected, by first making $\pi_1(M) \to \pi_1(X)$ an isomorphism, and then 'killing off' the kernel homology groups of the lowest dimension, in which the Hurewicz map $K_k(M) \to \pi_{k+1}(f)$ is an isomorphism; by Whitehead's theorem, f is a homotopy equivalence if and only if $\pi_1(M) \to \pi_1(X)$ is an isomorphism and $K_k(M) = 0$ for all k.

The primary tool for this is that of surgery:

Definition 1.3 The result of a *k-surgery* on a commutative square

$$\begin{tikzcd} S^k \times D^{n-k} \arrow[r, "\partial\theta"] \arrow[d] & M \arrow[d, "f"] \\ D^{k+1} \times D^{n-k} \arrow[r, "\theta"] & X \end{tikzcd}$$

in a degree 1 normal map $f : M \to X^n$, and where $\partial\theta$ is an embedding, is the map $f' : M' \to X$ where

$$M' = \overline{M \setminus (S^k \times D^{n-k})} \cup_{S^m \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}$$

with $f' = f$ on $M \setminus (S^k \times D^{n-k})$ and $f' = g$ on $D^{k+1} \times S^{n-k-1}$.

Implicitly, we are noting that $\partial(D^{k+1} \times D^{n-k}) = S^k \times D^{n-k} \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}$, so we can cut out the embedding of $S^k \times D^{n-k}$ and replace it by an embedding $D^{k+1} \times S^{n-k-1}$.

Examples 1.4 (i) Suppose that M is disconnected, with 2 components M_1 and M_2. Then there is an obvious embedding $S^0 \times D^n \to M$ with $0 \times D^n \subset M_1$ and $1 \times D^n \subset M_2$. Then a surgery on this embedding corresponds to forming the connected sum of M_1 and M_2.

(ii) Suppose that M is $n = 2k$-dimensional. Then a surgery on a trivial $(k-1)$-sphere has the effect of taking the connected sum with $S^k \times S^k$.

Definition 1.5 The *trace* of the above surgery is the cobordism $(W; M, M')$ where

$$W = M \times I \cup_{S^k \times D^{n-k} \times \{1\}} D^{k+1} \times D^{n-k},$$

together with a (normal) map to $X \times I$.

Hence manifolds which are related by surgery are cobordant. The converse is also true: given a cobordism of two manifolds, a Morse function can be constructed on the cobordism; then by considering the critical points of the Morse function a handle decomposition of the cobordism can be defined. But every handle addition arises as the trace of a surgery, and hence two manifolds are cobordant if and only if there is a finite sequence of surgeries from one to the other.

1.2 Handle exchanges

For this section, (and this notation is to be fixed whenever a splitting problem is referred to), assume that $g : W \to Y$ is a homotopy equivalence of $(n+1)$-dimensional manifolds which is transverse to X, so that $M = g^{-1}(Y)$ is a codimension 1 submanifold of W and $f = g|_M$ is then a map $f : M \to X$.

In order to make f increasingly connected, we would like to kill off the homotopy groups by surgery — but in this case we need to perform ambient surgery inside W. This is made precise in the following proposition:

Proposition 1.6 (Handle exchange , Cappell[5]) Suppose that

$$\alpha : (D^i, S^{i-1}) \times D^{n+1-i} \to (W_1, M)$$

is an embedding and let T be a neighbourhood of $M \cup \operatorname{Im} \alpha$. Then f is homotopic to a map f' by a homotopy fixed outside of T with ${f'}^{-1}(Y_2) = W_2 \cup \operatorname{Im} \alpha$ and ${f'}^{-1}(X) = M'$ where M' is obtained from M by a surgery on the restriction of α to $\partial\alpha : S^{i-1} \times D^{n+1-i} \to M$.

Examples 1.7 (i) Suppose again that M is disconnected, so is a disjoint union $M = M_1 \sqcup M_2$ as in example 1.4. Then there is a homotopy $G : W \times I \to Y$ to a map $g' : W \to Y$ such that the effect on f' is the effect of a 0-surgery. Furthermore, $G^{-1}(X)$ is the trace of the surgery.

(ii) Performing a handle exchange on an embedding of a trivial sphere in M has the effect of taking the connected sum with a product of spheres.

The following proposition (Cappell [5]) gives sufficient conditions to be able to represent a homotopy class by an embedding.

Proposition 1.8 Let $\alpha \in \pi_i(W_j, M)$ so that $f_*(\alpha) = 0 \in \pi_i(Y_j, X)$. Then if $2i < n + 1$, α can be represented by an embedding $\alpha : (D^i, S^{i-1}) \times D^{n+1-i} \to (W_j, M)$

Lemma 1.9 (Cappell [5]) Suppose that $n \geq 5$. Then g is homotopic to a map g' where the restriction $f' : M' \to X$ is 2-connected.

f is then made highly connected inductively by starting at the bottom dimension and killing off the lowest dimensional homology groups. Surgery and handle exchanges are unobstructed below the middle dimension. The details of how to make f highly connected are given in chapter 6.

1.3 Obstructions

The first splitting obstruction is a K-theory obstruction due to Cappell and Waldhausen, which is well understood. We recall the details in chapter 5 using the treatment given in Ranicki[7].

In answering both the surgery and splitting problems, surgeries or handle exchanges can be performed in order to make $f : M \to X$ k-connected, if $\dim X = n = 2k$ or $2k + 1$.

The cohomology of M also splits as $H^k(M) = K^k(M) \oplus K^k(X)$ and the Poincaré duality map $- \cap [M] : H^k(M) \to H_k(M)$ splits as $(- \cap [M]) \oplus (- \cap [X]) : K^k(M) \oplus K^k(X) \to H_k(M) \oplus H_k(X)$.

Hence if $\dim X = 2k$, $K_j(M) = 0$ unless $j = k$. In this case the surgery obstruction is given by an equivalence class of $\mathbb{Z}[H]$-valued quadratic forms on $K_k(M)$; those forms which are zero are those which admit a Lagrangian, a submodule of maximal rank on which the form vanishes. In the case of the splitting problem, $K_k(M) = P \oplus Q$, and the splitting obstruction is given by two quadratic forms, over P and Q, over $\mathbb{Z}[G_1]$ and $\mathbb{Z}[G_2]$ respectively. In chapter 7, we give a

new treatment of this splitting obstruction, in terms of the $\mathfrak{Nil}$ category defined by Waldhausen in [15]. In chapter 8 the definitions of the even-dimensional surgery and splitting obstructions are recalled.

The key step of Cappell in showing that vanishing of his obstruction is sufficient for g being splittable was the construction of the 'nilpotent normal cobordism'. This is a cobordism of g with a split problem $g : W' \to Y$; the surgery obstruction of this was computed. Details of this are given in chapter 10 in some detail, as they will be relied upon later.

Ranicki also gave a homotopy invariant definition of the surgery obstruction in terms of quadratic structures on chain complexes (a homotopy invariant generalization of quadratic forms on modules); this obviates the need to make all maps highly connected, and makes it much easier to follow through the results of surgeries. This theory is reviewed in chapters 11 and 12. In Ranicki[7] a description of the UNil groups in terms of chain complexes was given. In chapter 12 this is revisited (with a slight extra assumption). The algebraic version of the nilpotent normal cobordism given there is considered in chapter 13 and its surgery obstruction is computed explicitly.

If $\dim X = 2k+1$, $K_j(M) = 0$ unless $j = k, k+1$. In this case, Ranicki defines the surgery obstruction groups in terms of formations or in terms of short odd complexes — these are highly connected chain complexes with a quadratic structure, so much of the theory follows as a restriction of the previous theory. Chapter 14 recalls this theory, and constructs the equivalent structures, UNil formations and short odd nilcomplexes; these bear the same relation to formations and nilcomplexes as our redefined UNil forms bore to quadratic forms. Thus in chapter 15 we are able to define the odd-dimensional splitting obstruction and show that it is well-defined.

In chapter 16, we construct an odd-dimensional nilpotent normal cobordism, and compute its obstruction. This completes the proof of our main theorem:

Theorem 1.10 If $k \geq 2$ and $g : W \to Y^{2k+2}$ is a splitting problem such that $\overline{\phi}(\tau(g)) = 0$, then there is an obstruction $\chi(g) \in \mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$ such that g is splittable if and only if $\chi(g) = 0$. Furthermore there is a map $\alpha : \mathrm{UNil}_{2k+3} \to L_{2k+3}$.

Chapter 2

Algebraic Preliminaries

2.1 Rings with involution

In most of our applications, we shall be considering left modules over R where $R = \mathbb{Z}[\pi]$ is the group ring of a fundamental group of a manifold. When the group is not abelian the group ring is not commutative; however the orientation character determines an involution on the ring, which is used to convert right R-modules into left R-modules as described in this section, and obtain a homology theory with coefficients in the group ring with involution for both oriented and non-orientable manifolds.

Definition 2.1 An *involution* on a ring R is a map $\bar{\ } : R \to R$ such that for all $r, s \in R$:

- $\overline{r+s} = \bar{r} + \bar{s}$;
- $\overline{r.s} = \bar{s}.\bar{r}$;
- $\bar{1} = 1$;
- $\bar{\bar{r}} = r$.

Example 2.2 Let π be a group, and $w : \pi \to \mathbb{Z}_2 = \{\pm 1\}$ a group homomorphism. Then defining $\bar{\ } : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ by $\overline{\Sigma a_g g} = \Sigma a_g w(g) g^{-1}$ makes $\mathbb{Z}[\pi]$ into a ring with involution. In particular, taking $w(g) = 1$ for all $g \in \pi$ gives an involution on $\mathbb{Z}[\pi]$. The integral group ring with involution w is denoted by $\mathbb{Z}^w[\pi]$, or simply by $\mathbb{Z}[\pi]$ if $w = 1$.

Definition 2.3
Let K and K' be modules over a ring with involution $(R, \bar{\ })$, $f : K \to K'$ an R-module homomorphism.

- $K^* = \mathrm{Hom}_R(K, R)$ is the abelian group of left R-module homomorphisms $\theta : K \to R$, made into a left R-module via $(r.\theta)(k) = \theta(k).\bar{r}$;
- $f^* : K'^* \to K^*$ is the R-module homomorphism defined by $f^*(\theta)(k) = \theta(f(k))$;
- $e_K : K \to K^{**}$ is the R-module homomorphism defined by $e_K(k) = (\theta \to \overline{\theta(k)})$.

Proposition 2.4 Let K be a f.g. projective R-module. Then e_K is an isomorphism.

Definition 2.5 Let K be an R-module. Then let K^t be K considered as a right R-module, with the right action given by $k.r = \overline{r}.k$.

2.2 Chain complexes

Definition 2.6 Let C be a chain complex. C is:

- *n-dimensional* if $C_r = 0$ for $r \notin \{0, \ldots, n\}$;
- *finite dimensional* if it is n-dimensional for some $n \in \mathbb{Z}$;
- *finite* if C_r is a f.g. free R-module for all r.

Definition 2.7 (Ranicki[13]) Let C be a left R-module chain complex. Define C^t to be the right A-module chain complex formed by using the involution as above.

Definition 2.8 (Ranicki[13]) Let C and D be R-module chain complexes such that for some R, $C_r = D_r = 0$ when $r \geq R$. Define $\mathbb{Z}$-module chain complexes:

- $(C^t \otimes D)_n = \bigoplus_{r+s=n} C^t_r \otimes_R D_s$, $d(x \otimes y) = x \otimes d(y) + (-1)^s d(x) \otimes y$;
- $\mathrm{Hom}(C^t, D)_n = \bigoplus_{s-r=n} \mathrm{Hom}_R(C^t_r, D_s)$, $d(\theta)(x) = d(f(x)) + (-1)^s f(d(x))$;
- $C^{-*}_r = C^{-r} := C^*_{-r}$, $d = d^* : C^{-*}_r \to C^{-*}_{r+1}$;

Definition 2.9 Let C be a finite-dimensional projective R-module chain complex.

- The *transposition involution* $T : C^t \otimes C \to C^t \otimes C$ is defined by

$$T(x \otimes y) = (-1)^{rs} y \otimes x (x \in C_r, y \in C_q)$$

- The *slant isomorphism* is the map $\backslash : C^t \otimes C \to \mathrm{Hom}(C^{-*}, C)$, given by $x \backslash y = (f \to \overline{f(x)}.y)$;
- The transposition involution on the chain complex $C^t \otimes C$ is given by:

$$T(f) = (-1)^{rs} f^* \qquad (f : C^{-r} \to C_s)$$

Definition 2.10

Let $f : C \to D$ be a map of R-module chain complexes. Define the algebraic mapping cone $C(f)$ by

$$\begin{aligned} C(f)_r &= D_r \oplus C_{r-1} \\ d_r &= \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix} : C(f)_{r+1} \to C(f)_r \end{aligned}$$

Then define the homology of the map $H_n(f) = H_n(C(f))$.

Proposition 2.11 Let $f : C \to D$ be a map of finite-dimensional projective R-module complexes. f is a homotopy equivalence iff $C(f)$ is contractible.

2.3 Triads

Definition 2.12 A *triad* of projective R-module chain complexes is a diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "k"] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

such that $dk + kd = hf - f'g$.

A triad determines a map $\begin{pmatrix} h & (-1)^r k \\ 0 & g \end{pmatrix} : C(f) \to C(f')$ so that we can define a mapping cone:

Definition 2.13 Given a triad Γ as above, define $C(\Gamma)$ by:

$$C(\Gamma)_r = D'_r \oplus C'_{r-1} \oplus D_{r-1} \oplus C_{r-2}$$
$$d^r = \begin{pmatrix} d_{D'} & (-1)^r f' & (-1)^{r-1} h & k \\ 0 & d_{C'} & 0 & (-1)^{r-1} g \\ 0 & d_D & 0 & (-1)^{r-1} f \\ 0 & 0 & 0 & d_C \end{pmatrix}$$

Then define the homology groups $H_n(\Gamma) = H_n(C(\Gamma))$.

Then these fit into a commutative diagram:

$$\begin{tikzcd} & \arrow[d] & \arrow[d] & \arrow[d] & \\ \arrow[r] & H_n(C) \arrow[r, "g"] \arrow[d, "f"] & H_n(C') \arrow[r] \arrow[d, "f'"] & H_n(g) \arrow[r] \arrow[d] & \\ \arrow[r] & H_n(D) \arrow[r, "h"] \arrow[d] & H_n(D') \arrow[r] \arrow[d] & H_n(h) \arrow[r] \arrow[d] & \\ \arrow[r] & H_n(f) \arrow[r] \arrow[d] & H_n(f') \arrow[r] \arrow[d] & H_n(\Gamma) \arrow[r] \arrow[d] & \\ & & & & \end{tikzcd}$$

Chapter 3

Geometric Preliminaries

In this chapter, for the sake of consistency and completion, we recall some standard definitions and results upon which we shall later rely.

3.1 Homology and homotopy groups

Recall that the homotopy groups $\pi_n(A)$ are defined to be the set of (based) homotopy classes of maps $S^n \to A$.

Now for $n \geq 2$, $\pi_n(A) \cong \pi_n(\tilde{A})$. Furthermore, the action of $\pi_1(A)$ on $\tilde{A}$ induces an action on $\pi_n(\tilde{A}) = \pi_n(A)$. Therefore the following is an equivalent definition of the homotopy groups, which makes it easier to describe the action of $\mathbb{Z}[\pi_1(A)]$; we shall take it as our definition:

Definition 3.1 Given a space A, together with basepoint a_0, the homotopy group $\pi_n(A)$ is the set of homotopy classes of pairs (g, γ) where $g : S^n \to A$ and $\gamma : [0, 1] \to A$ is a path such that $\gamma(0) = a_0$ and $\gamma(1) = g(1, 0, \ldots, 0)$ (homotopy keeping $\gamma(0)$ fixed). Given a loop $\alpha \in \pi_1(A)$ and $(g, \gamma) \in \pi_n(A)$, $\alpha.(g, \gamma) = (\alpha \cdot \gamma, g)$ (where $\cdot$ means 'take the join of the two paths').

In future we shall give all further definitions of homotopy groups in this way, to facilitate the description of the group action.

Definition 3.2 Given a map of spaces with basepoints $f : (A, a_0) \to (B, b_0)$, let $\pi_{k+1}(f)$ be the set of homotopy classes of commutative squares:

$$\begin{tikzcd} S^k \arrow[r, "g"] \arrow[d] & A \arrow[d, "f"] \\ D^{k+1} \arrow[r, "h"] & B \end{tikzcd}$$

together with paths $\gamma : [0, 1] \to B$ such that $\gamma(0) = b_0$ and $\gamma(1) = f(g(1, 0, \ldots, 0))$.

Definition 3.3 Suppose that $f : (B, A) \to (Y, X)$ is a map of pairs (with basepoints). Then:

- $\pi_{k+1}(B, A) = \pi_{k+1}(i : A \to B)$

- $\pi_{k+2}(f)$ = homotopy classes of diagrams

$$\begin{tikzcd}
S^k \arrow[rr, "g"] \arrow[dr] \arrow[dd] & & A \arrow[dr] \arrow[dd] & \\
& D_L^{k+1} \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
D_U^{k+1} \arrow[rr] \arrow[dr] & & B \arrow[dr, "f"] & \\
& D^{k+2} \arrow[rr] & & Y
\end{tikzcd}$$

together with paths $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_0$ and $\gamma(1) = f(i(g(1,0,\ldots,0)))$; where D_U^{k+1} and D_L^{k+1} are the upper and lower hemispheres of $S^{k+1} = \partial D^{k+2}$, with intersection in S^k.

Definition 3.4 Let ϕ be a space, pair, map, or map of pairs (where all spaces are connected). Then ϕ is k-connected if $\pi_i(\phi) = 0$ for $i \leq k$.

In addition to acting on homotopy groups, the fundamental group also acts upon the singular chain complex of the universal cover $C_*^{\text{sing}}(\tilde{M})$, so that there is also an action on homology. Also, if M is a CW complex then there exists a cellular chain complex for $\tilde{M}$ with an action of $\mathbb{Z}[\pi_1(M)]$, so that $C_*^{CW}(\tilde{M})$ can be considered as a $\mathbb{Z}[\pi]$-complex. In this thesis, unless otherwise specified, $C_*(\tilde{M})$ will be used to refer to a $\mathbb{Z}[\pi]$-module complex, either singular or cellular, with cochain complex $C^*(\tilde{M}) = \text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}), \mathbb{Z}[\pi])$.

Definition 3.5 Given a connected manifold M with fundamental group π and universal cover $\tilde{M}$, define the homology with local coefficients:

$$H_k(M; \mathbb{Z}[\pi]) = H_k(C_*(\tilde{M}))$$

and the cohomology with local coefficients:

$$H^k(M; \mathbb{Z}[\pi]) = H_k(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}, \mathbb{Z}[\pi])))$$

both considered as $\mathbb{Z}[\pi]$-modules.

Note that $H_k(M; \mathbb{Z}[\pi]) = H_k(\tilde{M}; \mathbb{Z})$ considered as $\mathbb{Z}$-modules. The same is not true for cohomology.

Convention 3.6 From now on, unless otherwise specified, all homology will be taken with local coefficients; specifically $H_*(M)$ shall mean $H_*(M; \pi_1(M))$.

Proposition 3.7 (Prop. 10.21, Ranicki[11]) Suppose that $f : M \to X$ is a degree 1 map of connected manifolds, and suppose that $\pi_1(M) = \pi_1(X)$. Then the homology and cohomology of M decompose as:

$$H_k(M) = K_k(M) \oplus H_k(X), \qquad H^k(M) = K^k(M) \oplus H^k(X)$$

and the Poincaré duality isomorphisms split as $[M] \cap - = ([M] \cap -) \oplus ([X] \cap -)$.

These groups $K_k(M)$, the homology kernel groups, are clearly such that f induces an isomorphism on homology if and only if they are all zero. They are also the homology groups of the mapping cone of the map $f : M \to X$ (sometimes denoted by $H_{k+1}(f)$), and as such there is a Hurewicz homomorphism from the above homotopy group which permits representations of elements of the groups:

Theorem 3.8 (Hurewicz) If f is $(k-1)$-connected then

$$\pi_k(f) \cong H_k(f) \cong K_{k-1}(M).$$

Furthermore, the kernel homology groups behave as a homology theory, respecting the relative long exact sequence, as well as excision, (and hence the Mayer-Vietoris sequence.) We shall also need the following variant of Mayer-Vietoris (the proof is a trivial modification of that of Mayer-Vietoris.)

Theorem 3.9 There is an exact sequence:

$$\cdots \longrightarrow H_i(A \cup B) \longrightarrow H_i(A \cup B, B) \oplus H_i(A \cup B, A) \longrightarrow H_{i-1}(A \cap B) \longrightarrow H_{i-1}(A \cup B) \longrightarrow \cdots$$

Proof. See, for example, Bredon [1] problem IV.18.4. □

Theorem 3.10 (Whitehead)

- There is a Hurewicz map $\pi_{k+1}(f) \to H_{k+1}(f)$.
- For $k \geq 2$, f is k-connected if and only if f induces an isomorphism of fundamental groups, and $K_i(M) = 0$ for $i < k$.
- For $k \geq 2$, if f is k-connected then the Hurewicz map is an isomorphism $\pi_{k+1}(f) \cong K_k(M)$ (or $K_k(N, M)$ when f is a map of pairs).

Finally we shall need a technical lemma of Wall ([17], Lemma 2.3).

Lemma 3.11 Let $f : (N, M) \to (Y, X)$ be a map of pairs with Y connected (M and X may be empty). Suppose that $H_i(f) = 0$ for $i < k$, as a module with $\Lambda = \mathbb{Z}[\pi_1(Y)]$ coefficients. Then:

(a) If $H^{k+1}(f; B) = 0$ for every Λ-module B, then $H_k(f)$ is a projective Λ-module.

(b) If N and Y are finite, $H_k(f)$ is finitely generated.

(c) If, in addition to (a) and (b), $H_i(f) = 0$ for $i \neq k$, then $H_k(f)$ is stably free.

3.2 Covering spaces

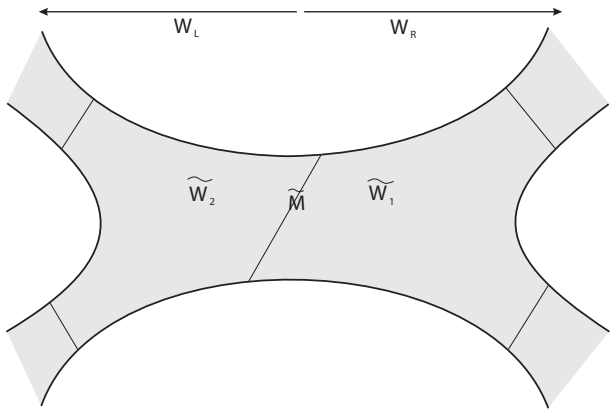
Suppose that $g : W \to Y$ is a splitting problem. In order to understand the maps on homology induced by the inclusions $M \to W_{\{1,2\}}$, it is necessary first to consider the structure of the covering spaces of W and Y.

Let $\tilde{Y}$ denote the universal covering space of Y, with covering map $\pi_Y : \tilde{Y} \to Y$. Choose a fixed point $x \in X$, and choose a lift $z \in \tilde{Y}$.

Now $\pi_Y^{-1}(X)$ will have many connected components, with each component giving a simply connected covering space for X. The component containing z will be denoted $\tilde{X}$. Similarly we denote the covering spaces containing Y_1 and Y_2 containing z by $\tilde{Y_1}$ and $\tilde{Y_2}$ respectively.

Now denote by $\hat{Y}$ the quotient space of $\tilde{Y}$ by the action of H on $\tilde{Y}$. Similarly $\hat{Y_1}$ and $\hat{Y_2}$

Figure 3.1: The covering space $\tilde{W}$



Lemma 3.12 The covering space of Y has the following properties:

(i) $\pi_Y^{-1}(X) = \bigcup_{\alpha \in [G;H]} \tilde{X}\alpha$

(ii) $\pi_Y^{-1}(Y_1) = \bigcup_{\alpha \in [G;G_1]} \tilde{Y_1}\alpha$

(iii) $\pi_Y^{-1}(Y_2) = \bigcup_{\alpha \in [G;G_2]} \tilde{Y_2}\alpha$

Note now that $\tilde{Y} \setminus \tilde{X}$ has 2 components, one of whose closure contains $\tilde{Y_1}$. Following Cappell, denote this component by Y_R and the other by Y_L.

The quotients Y_R/H and Y_L/H are to be denoted by Y_r and Y_l respectively.

Since g is 2-connected, the covering space of W satisfies the same properties. (See figure 3.1 for a picture.)

Chapter 4

Nilpotency

We stated in the introduction that Cappell's obstruction to splitting a homotopy equivalence lies in a group called UNil, the unitary nilpotent group. In our attempts to kill off all kernel homology of f by embedding representatives of homology classes, we find that we can not extend an arbitrary embedding of a sphere to an embedding of a disk in W_1. However, it is the nilpotency of a certain map which provides a filtration of the homology modules, and provides the means below the middle dimension to always kill off the homology.

In this chapter, we describe an additive category, $\mathfrak{Nil} = \mathfrak{Nil}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$, first defined by Waldhausen ([15], pg. 148), and put an involution on it. We shall describe how a splitting problem determines objects in this category, in the same way that a map determines objects in the category of modules via the homology with local coefficients. These objects share many desirable properties with the usual homology.

This will enable us in following chapters to show how a splitting problem can be made increasingly highly connected, and how in the middle dimensions certain maps determine the splitting obstruction. When the obstructions are formulated in terms of $\mathfrak{Nil}$ ojects, the similarities with the surgery obstructions are particularly noticeable.

Definition 4.1 Given a subgroup $H \leq G_i$, define $\widetilde{\mathbb{Z}[G_i]}$ to be the additive subgroup of $\mathbb{Z}[G_i]$ generated by $G_i \setminus H$. This is then also a $\mathbb{Z}[H]$-bimodule, and as a $\mathbb{Z}[H]$-bimodule, $\mathbb{Z}[G_i] \cong \mathbb{Z}[H] \oplus \widetilde{\mathbb{Z}[G_i]}$. Furthermore, $\widetilde{\mathbb{Z}[G_i]}$ is free as a right $\mathbb{Z}[H]$-module, with basis representatives of the right cosets of $\mathbb{Z}[H]$ in $\mathbb{Z}[G_i]$.

Convention 4.2 In this thesis, $\otimes\mathbb{Z}[G_i]$ is always to be interpreted as $\otimes_{\mathbb{Z}[H]}\mathbb{Z}[G_i]$ unless otherwise specified.

4.1 Nilpotent category

The next 2 definitions define the nilpotent category of Waldhausen.

Definition 4.3

- Let P and Q be $\mathbb{Z}[H]$-modules. A *nilpotent structure* on (P, Q) is a pair of maps (ρ_1, ρ_2) where $\rho_1 : P \to \widetilde{\mathbb{Z}[G_1]} \otimes Q$, $\rho_2 : Q \to \widetilde{\mathbb{Z}[G_2]} \otimes P$ such that there exists a

filtration associated to $(P, Q; \rho_1, \rho_2)$ of $\mathbb{Z}[H]$-modules:

$$\begin{array}{ccccccc} P & = & P_0 & \supseteq P_1 & \supseteq & \dots & \supseteq & P_r = 0 \\ Q & = & Q_0 & \supseteq Q_1 & \supseteq & \dots & \supseteq & Q_s = 0 \end{array}$$

such that $\rho_1(P_j) \subseteq \widetilde{\mathbb{Z}[G_1]} \otimes Q_{j+1}$ and similarly for ρ_2.

- Define $\mathrm{Obj}(\mathfrak{Nil}) = \{(P, Q; \rho_1, \rho_2) : P, Q$ are projective and (ρ_1, ρ_2) is a nilpotent structure on $(P, Q)\}$.

Definition 4.4 • If (P, Q) and (P', Q') have nilpotent structures (ρ_1, ρ_2) and (ρ_1', ρ_2') respectively, then a map $(f, g) : (P, Q) \to (P', Q')$ is compatible with the nilpotent structures if the following diagram (and its obvious counterpart in ρ_2, ρ_2') commutes

$$\begin{tikzcd} P \arrow[r, "\rho_1"] \arrow[d, "f_P"'] & \mathbb{Z}[G_1] \otimes Q \arrow[d, "1 \otimes f_Q"] \\ P' \arrow[r, "\rho_1'"] & \mathbb{Z}[G_1] \otimes Q' \end{tikzcd}$$

- $\mathrm{Hom}_{\mathfrak{Nil}}((P, Q; \rho_1, \rho_2), (P', Q'; \rho_1', \rho_2')) = \{(f_P, f_Q) | f_P : P \to P', f_Q : Q \to Q'$ compatible with the nilpotent structures$\}$

We now define an involution on the category (for definition of category with involution, see Ranicki([8]).

Definition 4.5 • Given $(P, Q; \rho_1, \rho_2) \in \mathfrak{Nil}$, define $\rho_1^* : Q^* \to \widetilde{\mathbb{Z}[G_1]} \otimes P^*$ that $\rho_1^*(g)$ is the unique element of $\widetilde{\mathbb{Z}[G_1]} \otimes P^* \cong \mathrm{Hom}(P, \widetilde{\mathbb{Z}[G_1]})$ satisfying $\rho_1^*(g)(p) = (1 \otimes g)(\rho_1(p))$ for all $p \in P$.

- We define an involution functor $\mathfrak{Nil} \to \mathfrak{Nil}$ by:
  - $(P, Q; \rho_1, \rho_2)^* = (Q^*, P^*; -\rho_1^*, -\rho_2^*)$
  - $(f, g)^* = (g^*, f^*)$

Lemma 4.6 The above involution is a well-defined involution on the category.

Proof. We must show that $(Q^*, P^*; \rho_1^*, \rho_2^*) \in \mathrm{Obj}(\mathfrak{Nil})$, i.e. construct filtrations of modules for P^* and Q^*. Let P and Q have filtrations:

$$\begin{array}{ccccccc} P & = & P_0 & \supseteq P_1 & \supseteq & \dots & \supseteq & P_r = 0 \\ Q & = & Q_0 & \supseteq Q_1 & \supseteq & \dots & \supseteq & Q_s = 0 \end{array}$$

such that $\rho_1(P_i) \subseteq \widetilde{\mathbb{Z}[G_1]} \otimes Q_i$ etc. Assume, by adding zero terms to the end of one sequence if necessary, that $r = s$. We claim that the following is a filtration associated to $(Q^*, P^*; \rho_1^*, \rho_2^*)$:

$$\begin{array}{ccccccc} P^* & = & P_r^\circ & \supseteq P_{r-1}^\circ & \supseteq & \dots & \supseteq & P_0^\circ = 0 \\ Q^* & = & Q_r^\circ & \supseteq Q_{r-1}^\circ & \supseteq & \dots & \supseteq & Q_0^\circ = 0 \end{array}$$

Suppose that $f \in P_{i+1}^\circ$. Then $f(p) = 0$ for all $p \in P_{i+1}$. Let $q \in Q_i$. Then $\rho_2^*(f)(q) = f(\rho_2(q)) = 0$ since $\rho_2(q) \in \widetilde{\mathbb{Z}[G_2]} \otimes P_{i+1}$. Hence $\rho_2^*(f) \in \widetilde{\mathbb{Z}[G_2]} \otimes Q_i^\circ$ and the above is a filtration as claimed. Similarly for ρ_1^*. □

Definition 4.7 If A_i are objects in $\mathfrak{Nil}$, and $f_i : A_i \to A_{i-1}$ are morphisms, then the sequence:

$$\cdots \longrightarrow A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \cdots$$

is exact if for all i, $\ker f_{i-1} = \operatorname{Im} f_i$.

Remark 4.8 Not every short exact sequence splits — for example, we shall see later that not every Lagrangian in a UNil form has a complementary Lagrangian, which gives an example of a short exact sequence which does not split.

4.2 Objects in the category

In the same way that with suitable connectivity assumptions, a manifold determines homology and cohomology objects in the category of projective modules, a splitting problem determines objects in the nilpotent category. In this section, we shall demonstrate the construction of these objects, and show that (in some cases) they satisfy the usual reasonable properties, such as relative exact sequences, Poincare duality and universal coefficient theorem.

For clarity of presentation, we shall first describe the case where W and M are closed. This is precisely the case described by Cappell in [5], so we omit details of many of the proofs where they are not relevant to further work.

4.2.1 Homology splitting

In this section, we assume that $g : W^n \to Y^n$ is a splitting problem, and consider those homology groups of the restriction $f : M \to X$ which are well behaved (finitely generated projective). An example is the group $K_k(M)$ where $n = 2k+1$ and f is k-connected. The nilpotent object that we construct will be a splitting of $K_k(M)$. In the next section we shall consider the more general case of a pair, however for ease of exposition we shall consider the simpler case separately first.

Proposition 4.9 (Cappell([5])) Suppose that $g : W \to Y$ is a splitting problem and that $f : M \to X$ is k-connected. Then:

(i) $K_k(M) = P \oplus Q$ where $P = K_{k+1}(W_r, M) \cong K_k(W_l)$ and $Q = K_{k+1}(W_l, M) \cong K_k(W_r)$;

(ii) $K_k(W_1) \cong \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} Q$ and $K_k(W_2) \cong \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P$;

(iii) The maps $P \to K_k(M) \to K_k(W_1) \cong \mathbb{Z}[G_1] \otimes Q$ and $Q \to K_k(M) \to K_k(W_2) \cong \mathbb{Z}[G_2] \otimes P$ factor through maps $\rho_1 : P \to \widetilde{\mathbb{Z}[G_1]} \otimes Q$ and $Q \to \widetilde{\mathbb{Z}[G_2]} \otimes P$ respectively;

(iv) The map $\begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix} : \mathbb{Z}[G] \otimes (P \oplus Q) \to \mathbb{Z}[G] \otimes (P \oplus Q)$ is an isomorphism;

(v) $(P, Q; \rho_1, \rho_2)$ is an object in $\mathfrak{Nil}$.

Definition 4.10 With the terms as defined above:

$$\mathrm{Spl}_k(M) := (P, Q; \rho_1, \rho_2)$$

Before we proceed further, we shall say a little about the meaning of the terms defined so far. ρ_1 represents the obstruction to being able to represent $\alpha \in P$ by an embedding $(D^{k+1}, S^k) \to (W_1, M)$. In particular $\alpha \in \ker(K_k(M) \to K_k(W_1))$ iff $\alpha \in P$ and $\rho_1(\alpha) = 0$ (and similarly for W_2). The map ρ is nilpotent, and we use this nilpotency below the middle dimension to show that f can be made highly connected. In the middle dimension it will form part of our obstruction.

The proof of this will be deferred until the next section, when it will be given in more generality. We shall, however, note that the nilpotent structure follows from (4) by the following lemma from [5], Lemma I.9:

Lemma 4.11 Let P, Q be finitely generated $\mathbb{Z}[H]$-modules and $\rho : \mathbb{Z}[G] \otimes (P \oplus Q) \to \mathbb{Z}[G] \otimes (P \oplus Q)$ a $\mathbb{Z}[G]$-linear map, satisfying:

(i) $I + \rho$ is an isomorphism, I the identity map of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (P \oplus Q)$

(ii) $\rho(P) \subset \widetilde{\mathbb{Z}[G_1]} \otimes Q$, $\rho(Q) \subset \widetilde{\mathbb{Z}[G_2]} \otimes_{\mathbb{Z}[H]} P$

Then ρ is nilpotent, and (P, Q) has an upper-triangular filtration.

Before we leave this section, we make one further observation which will be needed later on:

Lemma 4.12 $K_{k+1}(W_1, M; \mathbb{Z}[G_1]) \cong \mathbb{Z}[G_1] \otimes P$. The map

$$K_{k+1}(W_1, M; \mathbb{Z}[G_1]) \to K_{k+1}(M; \mathbb{Z}[G_1]) \cong \mathbb{Z}[G_1] \otimes (P \oplus Q)$$

is given by $\begin{pmatrix} 1 \\ -\rho_1 \end{pmatrix}$.

Proof. There is a Mayer-Vietoris exact sequence with coefficients in $\mathbb{Z}[G_1]$.

$$\begin{tikzcd} K_{k+1}(M, M) = 0 \arrow[r] & K_{k+1}(W_1, M) \oplus K_{k+1}(\bigcup W_l, M) \arrow[r, "\cong"] & K_{k+1}(W, M) \arrow[r] \arrow[d, "\cong"] & 0 \\ & & \mathbb{Z}[G_1] \otimes (P \oplus Q) \cong K_k(M) & \end{tikzcd}$$

So consider the isomorphism, call it $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. By the definition of the isomorphism $K_k(M) = P \oplus Q$, and the long exact sequence of the pair (W_l, M) it follows that $\beta = 0$ and $\delta = 1$. Hence α is an isomorphism. Henceforth, we use α to identify $K_{k+1}(W_1, M)$ with $\mathbb{Z}[G_1] \otimes P$.

The composite

$$\mathbb{Z}[G_1] \otimes P \cong K_{k+1}(W_1, M; \mathbb{Z}[G_1]) \to K_k(M) \cong \mathbb{Z}[G_1] \otimes (P \oplus Q) \to K_k(W_1; \mathbb{Z}[G_1]) \cong \mathbb{Z}[G_1] \otimes Q$$

is zero, by the long exact sequence of the pair (W_1, M). The second map is $\begin{pmatrix} \rho_1 & 1 \end{pmatrix}$ by definition. The first component of the first map is 1 since it is simply the identification made above. Hence the map $\mathbb{Z}[G_1] \otimes P \to \mathbb{Z}[G_1] \otimes (P \oplus Q)$ is $\begin{pmatrix} 1 \\ -\rho_1 \end{pmatrix}$ as claimed. $\square$

4.2.2 Relative homology splitting

Proposition 4.13 We now suppose that $g : T \to Y^n$ is a splitting problem with boundary $\partial g : W \to \partial Y$, and that $K_k(f, \partial f) : (N, M) \to (X, \partial X)$ is f.g. projective. Then:

(i) $K_k(N, M) = P \oplus Q$ where $P = K_{k+1}(T_r, W_r \cup N) = K_k(T_l, W_l)$ and $Q = K_{k+1}(T_l, W_l \cup N) = K_k(T_r, W_r)$;

(ii) $K_k(T_1, W_1) \cong \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} Q$ and $K_k(T_2, W_2) \cong \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P$;

(iii) The maps

$$P \to K_k(N, M) \to K_k(T_1, W_1) \cong \mathbb{Z}[G_1] \otimes Q$$

and

$$Q \to K_k(N, M) \to K_k(T_2, W_2) \cong \mathbb{Z}[G_2] \otimes P$$

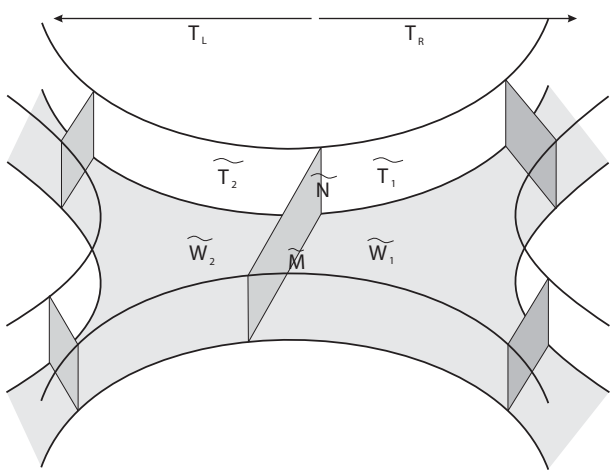
factor through maps $\rho_1 : P \to \widetilde{\mathbb{Z}[G_1]} \otimes Q$ and $Q \to \widetilde{\mathbb{Z}[G_2]} \otimes P$ respectively;

(iv) The map $\begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix} : \mathbb{Z}[G] \otimes (P \oplus Q) \to \mathbb{Z}[G] \otimes (P \oplus Q)$ is an isomorphism;

(v) $(P, Q; \rho_1, \rho_2)$ is an object in $\mathfrak{Nil}$.

See figure 4.1 for picture.

Figure 4.1: The covering space $\tilde{T}$



Definition 4.14 With the terms as defined above:

$$\mathrm{Spl}_k(N, M) := (P, Q; \rho_1, \rho_2)$$

Proof of 4.13. The proof is simply the relative version of [5].

(i) Follows from the braid of $\mathbb{Z}[H]$-modules:

$$\begin{tikzcd}
 & K_{k+1}(T_r, W_r \cup_M N) \arrow[rr, bend left] \arrow[dr] & & K_k(T_l, W_l) \arrow[dr] & \\
K_{k+1}(\tilde{T}, \tilde{W}) = 0 \arrow[ur] \arrow[dr] & & K_k(N, M) \arrow[ur] \arrow[dr] & & K_k(\tilde{T}, \tilde{W}) = 0 \\
 & K_{k+1}(T_l, W_l \cup_M N) \arrow[ur] \arrow[rr, bend right] & & K_k(T_r, W_r) \arrow[ur] &
\end{tikzcd}$$

(ii) Consider the decomposition $\tilde{T} = \tilde{T}_1 \cup_{[G_1;H]\tilde{N}} [G_1;H]T_l$ and similarly for W. Since $K_i(T,W) = 0$ for all i, the Mayer-Vietoris sequence gives isomorphisms:

$$\begin{aligned}
K_k(\tilde{N}, \tilde{M}; \mathbb{Z}[G_1]) &\cong K_k(T_l, W_l; \mathbb{Z}[G_1]) \oplus K_k(\tilde{T}_1, \tilde{W}_1; \mathbb{Z}[G_1]) \\
\mathbb{Z}[G_1] \otimes (P \oplus Q) &\cong \mathbb{Z}[G_1] \otimes P \oplus K_k(T_1, W_1)
\end{aligned}$$

By construction, the map $\mathbb{Z}[G_1] \otimes (P \oplus Q) \to \mathbb{Z}[G_1] \otimes P$ is the map $\begin{pmatrix} 1 & 0 \end{pmatrix}$, so we have an isomorphism $Q \otimes \mathbb{Z}[G_1] \cong K_k(T_1, W_1)$. Similarly for $K_k(T_2, W_2)$.

(iii) Follows from commutativity of the diagram:

$$\begin{tikzcd}
P \arrow[r] & \tilde{\tilde{K}}_{k+1}(T_r.W_r \cup_M N) \arrow[r] \arrow[d] & K_{k+1}(T_r, W_r \cup T_1) \arrow[d] \\
 & K_k(N, M) \arrow[r] & K_k(T_1, W_1)
\end{tikzcd}$$

(iv) From the Mayer-Vietoris sequence for W, we have an isomorphism of $\mathbb{Z}[G]$-modules:

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} K_k(N, M) \xrightarrow{\cong} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} K_k(T_1, W_1) \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} K_k(T_2, W_2)$$

Note that in (2) above, the map $\mathbb{Z}[G_1] \otimes (P \oplus Q) \to \mathbb{Z}[G_1] \otimes P \oplus \mathbb{Z}[G_1] \otimes Q$ is given by $\begin{pmatrix} 1 & 0 \\ \rho_1 & 0 \end{pmatrix}$. In particular, therefore, the map $K_k(N, M; \mathbb{Z}[G_1]) = \mathbb{Z}[G_1] \otimes (P \oplus Q) \to K_k(T_1, W_1; \mathbb{Z}[G_1]) = \mathbb{Z}[G_1] \otimes Q$ is given by $\begin{pmatrix} \rho_1 & 1 \end{pmatrix}$. Similarly, the map $K_k(N, M; \mathbb{Z}[G_1]) \to K_k(T_2, W_2) = \mathbb{Z}[G_2] \otimes P$ is given by $\begin{pmatrix} 1 & \rho_2 \end{pmatrix}$. Hence the Mayer-Vietoris map is $\begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix}$ as claimed.

□

Lemma 4.15 Wherever the objects referred to are defined:

(i) If $\phi : W \to W'$ is a map of splitting problems over Y, there is a map $\phi_* : \mathrm{Spl}_k(M) \to \mathrm{Spl}_k(M')$;

(ii) Given a splitting problem with boundary (T, W), there is a connecting homomorphism

$$\partial : \mathrm{Spl}_{k+1}(N, M) \to \mathrm{Spl}_k(M);$$

(iii) The sequence $\ldots \to \mathrm{Spl}_{k+1}(N, M) \to \mathrm{Spl}_k(M) \to \mathrm{Spl}_k(N) \to \mathrm{Spl}_k(N, M) \to \ldots$ is exact.

Proof. (i) Clear from the construction.

(ii) Consider the diagram in figure (4.1). The maps are all natural, and all squares are commutative.

□

4.2.3 Cohomology splitting

We shall now proceed further and show that cohomology as well as homology defines an object in UNil. Later on we shall show analogues of Poincaré duality and the universal coefficient theorem. There are two choices of cohomology splitting: one construction is essentially the Poincaré dual of the homology splitting; the other is the dual of the homology splitting. These are related by a factor of -1. It is for this reason that the $-$ signs appeared in the definition of the involution on UNil given earlier.

Proposition 4.16 Suppose again that $g : W \to Y$ is a splitting problem and that $f : M \to X$ is k-connected. Then:

(i) $K^k(M) = P \oplus Q$ where $P = K^k(W_r) \cong K^{k+1}(W_l, M)$ and $Q = K^k(W_r) \cong K^{k+1}(W_l, M)$;

(ii) $K^k(W_1, M; \mathbb{Z}[G_1]) \cong \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} Q$ and $K^k(W_2, M; \mathbb{Z}[G_2]) \cong \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P$;

(iii) The maps $P \to \mathbb{Z}[G_1] \otimes K^k(M) \to K_k(W_1, M) \cong \mathbb{Z}[G_1] \otimes Q$ and $Q \to K^k(M) \to K_k(W_2, M) \cong \mathbb{Z}[G_2] \otimes P$ factor through maps $\rho_1 : P \to \widetilde{\mathbb{Z}[G_1]} \otimes Q$ and $Q \to \widetilde{\mathbb{Z}[G_2]} \otimes P$ respectively;

(iv) The map $\begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix} : \mathbb{Z}[G] \otimes (P \oplus Q) \to \mathbb{Z}[G] \otimes (P \oplus Q)$ is an isomorphism;

(v) $(P, Q; \rho_1, \rho_2)$ is an object in $\mathfrak{Nil}$.

Definition 4.17 With P, Q, ρ_1, ρ_2 as above:

$$\mathrm{Spl}^k(M) := (P, Q; \rho_1, \rho_2)$$

Proof of 4.16. (i) Follows from the braid of $\mathbb{Z}[H]$-modules:

$$\begin{tikzcd}
 & K^k(W_r) \arrow[rr, bend left] \arrow[dr] & & K^{k+1}(W_l, M) \arrow[dr] & \\
K^k(W) = 0 \arrow[ur] \arrow[dr] & & K^k(M) \arrow[ur] \arrow[dr] & & K^{k+1}(W) = 0 \\
 & K^k(W_l) \arrow[rr, bend right] \arrow[ur] & & K^{k+1}(W_r, M) \arrow[ur] &
\end{tikzcd}$$

(ii) Consider the decomposition of $\tilde{W} = \tilde{W}_1 \cup_{[G_1;H]\tilde{M}} [G_1; H]W_l$. All components are equipped with a $\mathbb{Z}[G_1]$-action giving us a Mayer-Vietoris sequence (see theorem 3.9):

$$K^k(\tilde{M}; \mathbb{Z}[G_1]) \cong K^{k+1}(\tilde{W}_1, \partial\tilde{W}_1; \mathbb{Z}[G_1]) \oplus K^{k+1}(W_l, M; \mathbb{Z}[G_1]).$$

Table 4.1: Commutative diagram used in the construction of splittings

$$
\begin{tikzcd}
P_1 \oplus Q_1 \arrow[r] & K_{k+2}(W_r, M) \oplus K_{k+2}(W_l, M) \arrow[r] & K_{k+2}(W, M) \arrow[r] & K_{k+1}(M) \arrow[r] & K_{k+1}(W_1) \arrow[r] & \cong & P_1 \otimes \mathbb{Z}[G_1] \\
P_2 \oplus Q_2 \arrow[r] & K_{k+2}(T_r, N) \oplus K_{k+2}(T_l, N) \arrow[r] & K_{k+2}(T, N) \arrow[r] & K_{k+1}(N) \arrow[r] & K_{k+1}(T_1) \arrow[r] & \cong & P_2 \otimes \mathbb{Z}[G_1] \\
P_3 \oplus Q_3 \arrow[r] & K_{k+2}(T_r, W_r \cup N) \oplus K_{k+2}(T_l, W_l \cup N) \arrow[r] & K_{k+2}(T, W \cup N) \arrow[r] & K_{k+1}(N, M) \arrow[r] & K_{k+1}(T_1, N_1) \arrow[r] & \cong & P_3 \otimes \mathbb{Z}[G_1] \\
P_4 \oplus Q_4 \arrow[r] & K_{k+1}(W_r, M) \oplus K_{k+1}(W_l, M) \arrow[r] & K_{k+1}(W, M) \arrow[r] & K_k(M) \arrow[r] & K_k(W_1) \arrow[r] & \cong & P_4 \otimes \mathbb{Z}[G_1]
\end{tikzcd}
$$

If $C_*(\tilde{M})$ is a free $\mathbb{Z}[H]$-module chain complex, then:

$$\begin{aligned} H^k(C_*(M);\mathbb{Z}[G_1]) &= H_k(\mathrm{Hom}_{\mathbb{Z}[G_1]}(\mathbb{Z}[G_1]\otimes_{\mathbb{Z}[H]} C_*(\tilde{M}),\mathbb{Z}[G_1])) \\ &\cong H_k(\mathbb{Z}[G_1]\otimes_{\mathbb{Z}[H]} \mathrm{Hom}_{\mathbb{Z}[H]}(C_*(\tilde{M}),\mathbb{Z}[H]) \\ &\cong \mathbb{Z}[G_1]\otimes_{\mathbb{Z}[H]} H_k(\mathrm{Hom}_{\mathbb{Z}[H]}(C_*(\tilde{M}),\mathbb{Z}[H])) \\ &= \mathbb{Z}[G_1]\otimes_{\mathbb{Z}[H]} H^k(M;\mathbb{Z}[H]) \end{aligned}$$

where the first isomorphism is from the fact that $C_*(\tilde{M})$ is free and the second is from the fact that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module.

The rest of the result follows precisely as before, with ρ_1 being the map $P \to \mathbb{Z}[G_1]\otimes(P\oplus Q) \to K^{k+1}(W_1, M;\mathbb{Z}[G_1]) \cong \mathbb{Z}[G_1]\otimes Q$.

□

A straight-forward compilation of the results in the previous 2 sections then allows us to make the following definition

Definition 4.18 Suppose that $g : T \to Y^n$ is a splitting problem with boundary $\partial g : W \to Y$, and that $K^k(N, M)$ is f.g. projective. Then:

- Let $P := K^k(T_l, W_l)$ and $Q := K^k(T_r, W_r)$;
- Let $\rho_1 := P \to K^k(N, M) \to K^k(N, M) \to \widetilde{\mathbb{Z}[G_1]}\otimes Q \subset K^{k+1}(T_1, W_1 \cup_M N)$
- Let $\rho_2 := Q \to K^k(N, M) \to K^k(N, M) \to \widetilde{\mathbb{Z}[G_2]}\otimes P \subset K^{k+1}(T_2, W_2 \cup_M N)$

Proposition 4.19 Wherever the objects referred to are defined:

(i) If $\phi : W \to W'$ is a map of splitting problems over Y, there is a map $\phi^* : \mathrm{Spl}_k(M') \to \mathrm{Spl}_k(M)$;

(ii) Given a splitting problem with boundary (T, W), there is a connecting homomorphism $\partial : \mathrm{Spl}^k(M) \to \mathrm{Spl}^{k+1}(N, M)$;

Proposition 4.20 (Poincaré duality) If $(T^n; W, W')$ is a cobordism of splitting problems, then there is a Poincaré duality isomorphism

$$\mathrm{Spl}^{n-k}(N, M) \cong \mathrm{Spl}_k(N, M').$$

Proof. Consider the following commutative diagram, where the horizontal arrows are all the

Poincaré duality isomorphisms.

$$\begin{tikzcd}
H_{n-k}(\mathrm{Hom}_{\mathbb{Z}[H]}(C(T_l,W_l),\mathbb{Z}[H])) \arrow[r] \arrow[d] \arrow[dr, phantom, "(-1)^{n-k}"] & H_{k+1}(C(T_l,W_l'\cup_M N)) \arrow[d] \\
H_{n-k}(\mathrm{Hom}_{\mathbb{Z}[H]}(C(N,M),\mathbb{Z}[H])) \arrow[r] \arrow[d] & H_k(C(N,M')) \arrow[d] \\
\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} H_{n-k}(\mathrm{Hom}_{\mathbb{Z}[H]}(C(N,M),\mathbb{Z}[H])) \arrow[r] \arrow[d] & \mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} H_k(C(N,M')) \arrow[d] \\
H_{n-k}(\mathrm{Hom}_{\mathbb{Z}[G_i]}(\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} C(N,M),\mathbb{Z}[G_i])) \arrow[r] \arrow[d] \arrow[dr, phantom, "(-1)^{n-k+1}"] & H_k(\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} C(N,M')) \arrow[d] \\
H_{n-k+1}(\mathrm{Hom}_{\mathbb{Z}[G_i]}(C(T_1,W_1\cup_M N),\mathbb{Z}[G_i])) \arrow[r] & H_k(C(T_1,W_1'))
\end{tikzcd}$$

The composite on the left hand side is precisely the map ρ_1 in the cohomology splitting. The composite on the right hand side is the map ρ_1 in the homology splitting. □

Proposition 4.21 (Universal coefficient theorem) Suppose that $H_{k-1}(M) = 0$. Then there is an isomorphism

$$\mathrm{Spl}^k(M) \cong \mathrm{Spl}_k(M)^*.$$

Proof. From the commutative diagram, with the horizontal maps all isomorphisms from the universal coefficient theorem.

$$\begin{tikzcd}
H^k(\mathrm{Hom}_{\mathbb{Z}[H]}(C(W_l),\mathbb{Z}[H])) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[H]}(H_k(W_l),\mathbb{Z}[H]) \arrow[d] \\
H^k(\mathrm{Hom}_{\mathbb{Z}[H]}(C(M),\mathbb{Z}[H])) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[H]}(H_k(C(M)),\mathbb{Z}[H]) \arrow[d] \\
\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} H^k(\mathrm{Hom}_{\mathbb{Z}[H]}(C(M),\mathbb{Z}[H])) \arrow[r] \arrow[d] & \mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} \mathrm{Hom}_{\mathbb{Z}[H]}(H_k(C(M),\mathbb{Z}[H]) \arrow[d] \\
H^k(\mathrm{Hom}_{\mathbb{Z}[G_i]}(\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} C(M),\mathbb{Z}[G_i])) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[G_i]}(H_k(\mathbb{Z}[G_i]\otimes_{\mathbb{Z}[H]} C(M)),\mathbb{Z}[G_i]) \arrow[d] \\
H^k(\mathrm{Hom}_{\mathbb{Z}[G_i]}(C(W_1,M),\mathbb{Z}[G_i])) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[G_i]}(H_k(C(W_1,M)),\mathbb{Z}[G_i])
\end{tikzcd}$$

The composite map on the left hand side is ρ_1' in the definition of the cohomology splitting. The composite map on the right hand side is $-\rho_1^*$, (simply the dual of the map in lemma 4.12). □

Chapter 5

K-theory

In this chapter the K-theoretic part of the obstruction is discussed. As in the introduction, fix $I = \ker(\tilde{K}_0(\mathbb{Z}[H]) \to \tilde{K}_0(\mathbb{Z}[G_1]) \oplus \tilde{K}_0(\mathbb{Z}[G_2]))$. Let $\mathbb{Z}_2$ act on $\tilde{K}_0(\mathbb{Z}[H])$ in the usual way $T[M] = [M^*]$ It was shown by Cappell that there is an obstruction lying in $H^n(\mathbb{Z}_2; I)$ which is the obstruction to g being normal bordant to a split homotopy equivalence. In the following, the relevant theory is revised using the treatment in Ranicki [7] pp. 672–678.

Definition 5.1 Let E be a $\mathbb{Z}[G]$-module chain complex. A *Mayer-Vietoris presentation* of E consists of:

- A free $\mathbb{Z}[H]$-module chain complex C;
- A free $\mathbb{Z}[G_1]$-module chain complex D_1;
- A free $\mathbb{Z}[G_2]$-module chain complex D_2;
- Maps $f_i : \mathbb{Z}[G_i] \otimes_{\mathbb{Z}[H]} C \to D_i$;
- Maps $g_i : \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_i]} D_i \to E$

such that

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} D_1 \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} D_2 \to E$$

is exact and has zero torsion (i.e. the mapping cone $C(C(C \to D_1 \oplus D_2) \to E)$ has zero torsion).

Note that in particular if $f_1 \oplus f_2$ is a $\mathbb{Z}[G]$-isomorphism then E is contractible and $\tau(E) = \tau(f_1 \oplus f_2)$.

Every f.g. free $\mathbb{Z}[G]$-module chain complex admits a presentation. Denote E_r when considered as a $\mathbb{Z}[H]$-module rather than a $\mathbb{Z}[G]$-module by $E_r|_{\mathbb{Z}[H]}$. Then:

Proposition 5.2 (Ranicki [9], Remark 8.7) Let E be a f.g. free $\mathbb{Z}[G]$-module chain complex. There exist f.g. free subcomplexes $D_1 \subset E|_{\mathbb{Z}[G_1]}$, $D_2 \subset E|_{\mathbb{Z}[G_2]}$, and $C \subset E|_{\mathbb{Z}[H]}$ such that

$$0 \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} D_1 \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} D_2 \to E \to 0$$

is a Mayer-Vietoris presentation.

To define the map $\mathrm{Wh}(G) \to \tilde{K}_0(\mathbb{Z}[H])$, it is necessary to decompose $\mathbb{Z}[G]$ as a $\mathbb{Z}[H]$-bimodule. We use the notation of Cappell:

Lemma 5.3 (Cappell [5], pg. 84) Define $A_i, B_i, \Gamma_i, \Delta_i$ inductively:

$$A_1 = \widetilde{\mathbb{Z}[G_1]}, \quad B_1 = 0, \quad \Gamma_1 = \widetilde{\mathbb{Z}[G_2]}, \quad \Delta_1 = 0$$
$$A_{i+1} = \Delta_i \otimes_{\mathbb{Z}[H]} \widetilde{\mathbb{Z}[G_1]}, \quad B_{i+1} = \Gamma_i \otimes_{\mathbb{Z}[H]} \widetilde{\mathbb{Z}[G_1]}$$
$$\Gamma_{i+1} = B_i \otimes_{\mathbb{Z}[H]} \widetilde{\mathbb{Z}[G_2]}, \quad \Delta_{i+1} = A_i \otimes_{\mathbb{Z}[H]} \widetilde{\mathbb{Z}[G_2]}$$

Then as a $\mathbb{Z}[H]$-bimodule,

$$\mathbb{Z}[G] \cong \mathbb{Z}[H] \oplus \sum_{i=1}^{\infty} A_i \oplus \sum_{i=1}^{\infty} B_i \oplus \sum_{i=1}^{\infty} C_i \oplus \sum_{i=1}^{\infty} D_i$$

The following algebraic result corresponds to the decomposition of the covering space shown in the previous chapter.

Definition 5.4 ([15] pg. 146, [7] pg. 673) Suppose that (C, D_i, f_i, g_i) is a Mayer-Vietoris presentation such that $f_1 \oplus f_2$ is a $\mathbb{Z}[G]$-isomorphism. Then consider the following diagram:

$$\begin{tikzcd}
\ldots \oplus & \Gamma_1 \otimes C \arrow[dl] \arrow[dr] & \oplus & C \arrow[dl] \arrow[dr] & \oplus & A_1 \otimes C \arrow[dl] \arrow[dr] & \oplus & \Delta_2 \otimes C \arrow[dl] \arrow[dr] & \oplus & A_3 \otimes C \arrow[dl] \arrow[dr] & \oplus & \cdots \arrow[dl] \\
\ldots & \oplus & D_2 & \oplus & D_1 & \oplus & A_1 \otimes D_2 & \oplus & \Delta_2 \otimes D_1 & \oplus & A_3 \otimes D_2 \oplus \ldots &
\end{tikzcd}$$

The top row is a decomposition of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C$, the bottom is $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} D_1 \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} D_2$, and the arrows are the only component maps which can be non-zero.

Define D_+ to be the union of everything on the right hand side of C, i.e. the mapping cone of the map:

$$(A_1 \oplus \Delta_2 \oplus A_3 \oplus \Delta_4 \oplus \ldots) \otimes C \to D_1 \oplus (A_1 \otimes D_2) \oplus (\Delta_2 \otimes D_1) \oplus (A_3 \otimes D_2) \oplus \ldots.$$

Define D_- similarly.

Definition 5.5 ([7], pg. 674) Let $\tau(E) \in \mathrm{Wh}(G)$ for some contractible $\mathbb{Z}[G]$-module chain complex E, and let

$$0 \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} D_1 \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} D_2 \to E \to 0$$

be a Mayer-Vietoris presentation of E. Define $\phi(\tau(E)) = [C(C \to D_+)] \in \tilde{K}_0(\mathbb{Z}[H])$. ϕ is a well-defined map $\mathrm{Wh}(G) \to \tilde{K}_0\mathbb{Z}[H]$.

Lemma 5.6 ([5], Lemma II.2) Suppose that P and Q are f.g. projective $\mathbb{Z}[H]$-module chain complexes such that $[P] = -[Q] \in \tilde{K}_0(\mathbb{Z}[H])$, and such that $\mathbb{Z}[G_i] \otimes_{\mathbb{Z}[H]} P, Q$ are free $\mathbb{Z}[G_i]$ module chain complexes, with maps

$$\begin{pmatrix} \rho_1 & 1 \end{pmatrix} : \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} (P \oplus Q) \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} Q$$

and

$$\begin{pmatrix} 1 & \rho_2 \end{pmatrix} : \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} (P \oplus Q) \to \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P$$

so that $\rho = \begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix}$ is a chain equivalence. Then

$$0 \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (P \oplus Q) \xrightarrow[\rho]{} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} (P \oplus Q) \longrightarrow C(\rho) \longrightarrow 0$$

is a Mayer-Vietoris presentation of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P \oplus Q$, and $\tau(C(\rho)) = [P]$.

The K-theory splitting obstruction is a $\mathbb{Z}_2$ cohomology class:

Lemma 5.7 (Cappell[5], Lemma II.4) Let $g : W \to Y^{n+1}$ be a splitting problem. Then $\phi(\tau(g)) = (-1)^{n+1}\phi(\tau(f))^*$, and so determines an element

$$\overline{\phi}(\tau(g)) \in H^{n+2}(\mathbb{Z}_2; \ker(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus K_0(G_2))).$$

$\overline{\phi}(\tau(g)) = 0$ if and only if g is bordant to $g' : W' \to Y$ such that $\phi(\tau(g')) = 0$.

This will always be the first splitting obstruction. For the remainder of this thesis, we assume that $\phi(\tau(g)) = 0$.

Chapter 6

Below the Middle Dimensions

It was seen in the introduction that a framed embedding $S^k \times D^{n-k} \to M$ extending to a map $D^{k+1} \times D^{n-k} \to X$ is the data required to perform surgery. Given a homotopy class $\alpha \in \pi_{k+1}(f)$, there are two ways of deciding whether it can be represented by a framed embedding: either make it an embedding first, and then try to find a framing, else find a framing first and then try to change it to an embedding by a regular homotopy. We follow Wall, and fix a framing first, and then try to represent it by an embedding.

It was stated in the introduction that it is always possible to perform surgeries / handle exchanges to make surgery/splitting problems highly connected. In this chapter we assume that M is connected and that $f : M \to X$ induces an isomorphism of fundamental groups. We then proceed to state how surgery problems can be made highly connected, and the extent to which splitting problems can be made homotopy equivalences is determined in proposition 6.11.

6.1 Surgery

In this section we review how framed embeddings of spheres can be constructed inside a normal map and the result of surgery on such embeddings.

First recall the definition of a normal map (following Wall):

Definition 6.1 Let X be covered by a bundle η:

- A *normal map* is a map $f : M \to X$ together with a stable trivialisation F of $\tau_M \oplus f^*\eta$.
- A *normal bordism* is a cobordism $(W; M, M')$ together with maps $(g; f, f') : (W; M, M') \to X$ together with a stable trivialisation G of $\tau_W \oplus g^*\eta$

The following theorem of Wall provides the regular homotopy classes of framed immersions which must be used to perform surgery in order to produce a normal bordant map.

Theorem 6.2 (Wall[17], Theorem 1.1) Let M^n be a smooth or PL manifold (with boundary), $f : M \to X$ a continuous map, ν a vector bundle or PL bundle over X, and F stable trivialisation of $\tau_N \oplus f^*\nu$. Then any $\alpha \in \pi_{r+1}(f), f \leq n-2$, determines a regular homotopy

class of immersions $S^r \times D^{n-r} \to M$. The embedding $\phi : S^r \times D^{n-r} \to M$ can be used for a surgery killing α if and only if f is in this class.

We shall be using this result frequently, often without direct reference. A general position argument then gives the corollary:

Corollary 6.3 With notation as above, if $n > 2r$ then we can do surgery on α.

The following result says that it is always possible to perform surgery up to the middle dimension. Moreover, there exist normal maps $f : M \to X$ which are not normal bordant to homotopy equivalences, so that this result is the best general result.

Proposition 6.4 (Wall [17], chapter 1) Suppose that $f : M \to X^n$ is a degree 1 normal map. Then f is normal bordant to a $[n/2]$-connected map.

The proof is by induction, by showing that, given a k-connected map of n-dimensional manifolds, with $2 \leq k < [n/2]$, there is a bordant $(k+1)$-connected map: (The arguments which show that it is always possible to make a map 2-connected are similar to the following.)

Assume that f is k-connected. Since there is a Hurewicz isomorphism $K_k(M) \cong \pi_{k+1}(f)$, every non-zero kernel homology class α is represented by a map $(D^{k+1}, S^k) \to (X, M)$. By the above, there is a framed embedding $(D^{k+1}, S^k) \times D^{n-k} \to (X, M)$ on which surgery to produces a normal bordant map with the class α 'killed' in the following sense.

Proposition 6.5 Suppose that M is the result of a k-surgery on a class $\alpha \in K_k(M)$ where $k < [n/2]$. Then the resulting manifold M' has the following homology:

$$K_i(M') = \begin{cases} K_i(M) & \text{if } i < k \\ K_k(M)/\langle\alpha\rangle_{\mathbb{Z}[\pi]} & \text{if } i = k \end{cases}$$

6.1.1 Manifolds with boundary

Suppose that $f : (N^{n+1}, M^n) \to (Y^{n+1}, X^n)$ is a degree 1 normal map of manifolds with boundary, where $\partial N = M, \partial Y = X$. There are 2 ways of performing surgery to obtain a bordant map: firstly by performing surgeries on the interior as above, and secondly by performing surgeries on the boundary:

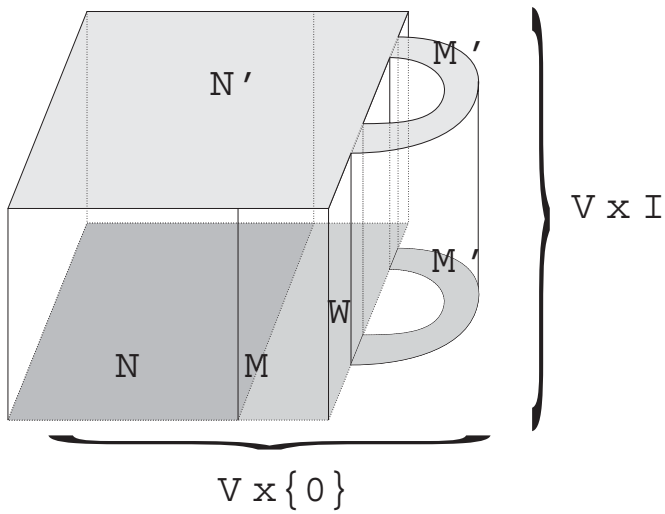
Proposition 6.6 Suppose that N is a manifold with boundary M (together with a degree 1 normal map $f : (N, M) \to (Y, X)$). Let M' be the result of a surgery on M with trace W. Then there is a cobordism $g : (V, \partial V) \to (Y, X) \times I$ with $f : (N, M) \to (Y, X)$ of $f : (N', M') \to (Y, X)$.

Proof. The cobordism is constructed in the following way:

Let $V = (N \cup_M W) \times I$. Then $\partial V = (N \cup W) \times \{0\} \cup M' \times I \cup (N \cup W) \times \{1\}$. This can be rebracketed as $\partial V = N \cup_M (W \cup M' \times I) \cup_{M'} (N \cup W)$, hence V is a cobordism of manifolds

with boundary, of (N, M) with $(N \cup W, M')$. See figure 6.1.1. Note that we need also to smooth over the corner at $M' \times \{0\}$.

Figure 6.1: A cobordism of manifolds related by surgery on the boundary



□

Hence surgeries on the boundary of a manifold can be used to make the restriction map highly connected, and then surgeries on the interior of the manifold make the map highly connected.

If $N = N^{2k+1}$ then the above results imply that there is a bordant manifold with boundary (which we continue to denote by N and M) such that $K_{k-1}(N) = K_{k-1}(M) = 0$. However we can do better than this:

Proposition 6.7 (Wall[17], Theorem 1.4) Suppose that N^{2k+1} is a manifold with boundary M and $f : (N, M) \to (Y, X)$ a degree 1 normal map, with $K_i(N) = K_i(M) = 0$ for $i < k$. Then there is a bordant manifold with boundary (M', N') such that in addition $K_k(N, M) = 0$.

Proof. We outline the proof of this theorem, since we wish to generalize it later on.

From the above, surgery below the middle dimension on M can be used to make $M \to X$ k-connected, followed by surgery on N relative to M to make $N \to Y$ k-connected.

Hence there is an exact sequence $\cdots \to K_k(N) \to K_k(N, M) \to 0$.

$K_k(N, M)$ is finitely generated by 3.11. Take a finite set of generators. Then each is represented by a sphere $\alpha_i \in \pi_k(N)$. Remove a disc $D_i^k \subset \alpha_i$ and join a tube $S_i^{k-1} \times I$ with $S_i^{k-1} \times \{0\} \subset \alpha_i$ and $S_i^{k-1} \times \{1\} \subset M$. By theorem 6.2 this procedure can be framed, to give embeddings $(D^k, S^{k-1}) \times D^{k+1} \to (N, M)$. We denote the union of these handles by H, and let $N_0 = \overline{N \setminus H}$, $M_0 = \partial N_0$.

By excision $H_{k+1}(N, H \cup \partial N) \cong H_{k+1}(N_0, \partial N_0)$

Then there is a commutative braid diagram:

$$\begin{tikzcd}
H_k(H \cup \partial N_0, \partial N_0) \arrow[rr, bend left] \arrow[dr] & & H_k(N, \partial N) \arrow[rr, bend left] \arrow[dr] & & H_k(Y, X) \\
& H_{k+1}(f) \arrow[ur] \arrow[dr] & & H_k(N_0, \partial N_0) \arrow[ur] \arrow[dr] & \\
H_{k+1}(Y, X) \arrow[ur] \arrow[rr, bend right] & & H_{k+1}(f_0) \arrow[ur] \arrow[rr, bend right] & & H_{k-1}(N \cup \partial N_0, \partial N_0)
\end{tikzcd}$$

Since three of the sequences are exact, it follows from Wall [16] that the fourth must also be exact. Combining this with the excision isomorphism above, gives the following exact sequence:

$$\longrightarrow H_i(H \cup \partial N_0, \partial N) \longrightarrow K_i(N, M) \longrightarrow K_i(N_0.M_0) \longrightarrow H_{i-1}(H \cup \partial N_o, \partial N) \longrightarrow$$

But by excision, $H_i(H \cup \partial N_0, \partial N) \cong H_i(H, H \cap \partial N) = \begin{cases} \langle \alpha_i \rangle & \text{if } i - k \\ 0 & \text{else} \end{cases}$.

Hence $H_{k-1}(H \cup \partial N_0, \partial N) = 0$ and $H_k(H \cup \partial N_0, \partial N) \to K_k(N, M)$ is surjective. It follows from exactness that $K_k(N_0, M_0) = 0$.

The effect on M is to perform trivial $(k-1)$-surgeries, each of which has the effect of forming the connected sum with $S^k \times S^k$, and therefore leaves untouched homology below the kth dimension, □

6.1.2 Connectivity results

For convenience, we now put the previous sections together in the following key results:

Corollary 6.8 Suppose that $f : M^n \to X^n$ is a degree 1 normal map. Then there exists a bordant map $f' : M' \to X$ which is $[n/2]$-connected.

Corollary 6.9 Suppose that $f : (N, M) \to (Y^{n+1}, X^n)$ is a degree 1 map of pairs. Then there is a bordant map $f_0 : (N_0, M_0) \to (Y, X)$ such that

- f_0 is $[(n+1)/2]$-connected
- $f_0|M$ is $[n/2]$-connected
- $K_k(N, M) = 0$ if $n = 2k$.

6.2 Handle exchanges

The following procedure of Cappell [5] performs an equivalent procedure to make a splitting problem highly connected. We are not able, as we want, to represent every element by an embedding $(D^{k+1}, S^k) \to (W_i, M)$ — although we have shown that we can find a basis with respect to which every element is represented by an embedding $(D^{k+1}, S^k) \to (W_{\{r,l\}}, M)$. However, there are associated to $\mathrm{Spl}_k(M)$ filtrations of P and Q. We perform handle exchanges

to make f highly connected by reducing the length of these filtrations. This works in the following manner:

- Suppose that $\alpha \in P^r$ is such that $\rho(\alpha) = 0$. Then $\alpha \in \ker(K_j(M) \to K_j(W_1))$. Then α can be represented by a null-homotopic embedding $(D^{j+1}, S^j) \to (W_1, M)$ and hence can be killed by a handle-exchange.
- The resulting map is such that the homology groups are the same up to the jth dimension, where the result is that P is replaced by $P/\langle\alpha\rangle$, so we can kill off the submodule P^r.
- Inductively we then kill off all other submodules until $r + s = 0$.

6.2.1 Manifolds with Boundary

We want to be able to apply the methods of section 6.1.1 to the split problem; in particular if we are to mirror the results in the splitting case, we wish to prove the following theorem:

Proposition 6.10 Suppose that $g : (V, W) \to (Y^{2k+2}, \partial Y)$ is a homotopy equivalence of pairs, X is a codimension 1 submanifold of Y and $g^{-1}(X, \partial X) = (N, M)$, $f = g|_N$. Suppose that f and $f|_M$ are k-connected. Then g is h-cobordant to $g' : V' \to Y$ such that $K_k(N', M') = 0$.

This will be instrumental in proving the necessity of the vanishing of the surgery obstruction. The proof will run along the same lines as 6.7, once we show how to perform the geometric moves in the context of the codimension 1 splitting problem.

Proof. Let $\mathrm{Spl}_k(N, M) = (P, Q; \rho_1, \rho_2)$. As with the previous handle exchanges, proceed by induction on the length of the filtration. Let $\hat{P}$ be such that $\rho_1(\hat{P}) = 0$ and such that $\rho_2(\hat{Q}) \subset \hat{P}$ i.e. let $\hat{P}$ and $\hat{Q}$ be the top of the filtrations of P and Q. Take a set of generators for $\hat{P}$, $\{x_i\}$. Then as before, for each x_i there is an immersion $\theta_i : (D^{k+1}, S^k) \to (V_1, N)$ which is an embedding on the boundary since $\dim N = 2k + 1$. As in the surgery case, join S^k to M by a tube $S^{k-1} \times I$ bounding $D^k \times I$ which joins $\theta_i(D^{k+1})$ to the boundary W_1. This gives an immersed disk $\phi_i : D^{k+1} \to V_1$ with embedded boundary $D^k \cup_{S^{k-1}} D^k \to N \cup_M V_1$. Then the methods of the $\pi - \pi$ theorem of Wall (Wall[17], pg 40) apply. Namely the only intersections and self-intersections of the ϕ_i are isolated points in the interior of V_1. Therefore at each intersection point either 2 branches of the same disc intersect or else 2 different discs intersect. Take a path along each branch to W_1 (NB not to $W_1 \cup_M N$). Then since $\pi_1(V_1) = \pi_1(W_1)$ there exists a triangle in V_1 bounding these 2 paths and a path in W_1 joining the 2 endpoints. Then the ϕ_i can be changed by regular homotopy to new embeddings removing the intersection points and leaving fixed all but a neighbourhood of these triangles. Then in particular the representatives $x_i \in K_k(N, M)$ have been left fixed. These embeddings can be framed in the usual way. Displace the embedding ϕ_i slightly into W_2 so that the boundary of ϕ_i is transverse to N. Now remove the embeddings $D^{k+1} \times D^{k+1}$. Let V^0 be the space formed by removing the

ϕ_i. Then $V = V^0 \cup_{D^k \times D^{k+1} \cup_{D^k \times S^k} D^{k+1} \times S^k} D^{k+1} \simeq V$ so $V \simeq V^0$, and in particular $g^0 : V^0 \to Y$ is a homotopy equivalence. Also $V_2 = V_2^0 \cup_{D^k \times D^{k+1}} D^{k+1} \times D^{k+1} \simeq V_2^0$.

Then the exact sequence of 6.7 extends to an exact sequence of $\mathfrak{Nil}$ objects:

$$(\mathbb{Z}[H]^n, 0; 0, 0) \to (P, Q; \rho_1, \rho_2) \to (P', Q'; \rho_1', \rho_2') \to 0.$$

Clearly then $Q = Q'$. We claim that if $q \in Q$ is such that $\rho_2(q) \in \hat{P}$, then $\rho_2'(q) = 0$ and therefore the length of the filtration has been reduced. For the following diagram commutes and the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Q \arrow[r, "j_Q"] \arrow[d, "\rho_2"] & Q' \arrow[r] \arrow[d, "\rho_2'"] & 0 \\
\mathbb{Z}[H]^n \arrow[r, "x_i"'] & P \arrow[r, "j_P"'] & P' \arrow[r] & 0
\end{tikzcd}$$

Suppose that $\rho_2(q) \in \hat{P}$. Then $\rho_2(q) = \sum(a_i x_i)$, so $\rho_2'(j_Q(q)) = j_P(\rho_2(q)) = 0$ by commutativity and exactness.

Continue the argument by induction in the following way:

Take generators for a filtration to construct a sequence $f^r : P^r \to P$, $g^r : Q^r \to Q$ such that

- $P^0 = Q^0 = 0$.
- For each r either $P^{r+1} = P^r \oplus \mathbb{Z}[H]^{n_r}$ and $Q^{r+1} = Q_r$ or vice versa;
- For each r, $\rho_1(f^r) \subset \mathbb{Z}[G_1] \otimes g^{r-1}(Q^{r-1})$ and vice versa.
- For some R, $P_R = P$, $Q_R = Q$.

The inductive claim is then that there is a sequence of handle subtractions of the above form so that

$$P^r \oplus Q^r \to K_k(N, M) \to K_k(N^r, M^r) \to 0$$

is exact. Clearly when $r = R$ this implies that $K_k(N^R, M^R) = 0$ as required. To prove it, suppose that the claim is true for r (clearly true for 0) and assume wlog that $Q^{r+1} = Q^r$ and that $P^{r+1} = P_r \oplus \langle x_1, \ldots, x_n \rangle$. Then the images $f^{r+1}(x_i)$ are represented by discs in (N, M) which can be taken as disjoint from any previous embeddings. Furthermore, since $\rho_1(f^{r+1}) \subset \mathbb{Z}[G_1] \otimes Q^r$ and the map $Q^r \to Q'$ is zero by exactness. Hence these embeddings bound in V_1' as before and handle subtractions can be performed as above. The result of the handle subtractions is a pair (N^{r+1}, M^{r+1}) such that

$$\mathbb{Z}[H]^n \oplus 0 \to P' \oplus Q' \to P'' \oplus Q'' \to 0$$

is exact. We claim that the sequence

$$P^{r+1} \oplus Q^{r+1} \to P \oplus Q \to P'' \oplus Q'' \to 0$$

is therefore exact.

To see this consider the commutative braid diagram:

$$\begin{tikzcd}
P^r \arrow[rr, bend left] \arrow[dr] & & P \arrow[rr, bend left] \arrow[dr] & & P'' \arrow[dr] & \\
& P^{r+1} \arrow[ur] \arrow[dr] & & P' \arrow[ur] \arrow[dr] & & 0 \\
& & \mathbb{Z}[H]^n \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] &
\end{tikzcd}$$

Three of the sequences are commutative hence the fourth is also and the result is shown. □

6.2.2 Connectivity results

Once again, we conclude the section with a summary of the results regarding handle exchanges below the middle dimension.

Corollary 6.11 Suppose that $g : W^{n+1} \to Y^{n+1}$ is a splitting problem restricting to $f : M^n \to X^n$. Then there exists a bordant splitting problem $g' : W' \to Y$ such that $f' : M' \to X$ is $[n/2]$-connected.

Corollary 6.12 Suppose that $g : (V, W) \to (Y, \partial Y)$ is a pair of splitting problems, restricting to $(f, \partial f) : (N, M) \to (X, \partial X)$ a degree 1 map of pairs. Then there is a bordant splitting problem g' restricting to $(f', \partial f') : (N', M') \to (X, \partial X)$ such that

- f' is $[(n+1)/2]$-connected
- $f'|M$ is $[n/2]$-connected
- $K_k(N, M) = 0$ if $n = 2k$.

Chapter 7

Forms

The even-dimensional L-groups and UNil groups are both defined as equivalence classes of forms. In this chapter we describe the groups algebraically, and give some relevant results, before describing in the next chapter how a geometric problem determines an obstruction in these groups. We shall first recall the theory of quadratic forms which define the surgery obstruction groups, together with the definition of Ranicki of split quadratic forms. We shall then recall the definition of UNil forms of Cappell, give a slight reformulation in terms of the category defined in chapter 4 to make them resemble quadratic forms more closely, and define split quadratic UNil forms.

7.1 Basic properties of forms

In this section we establish the basic definitions of quadratic forms over a ring with involution, and some results which we shall use later on.

Throughout this section, R is a ring with involution, M is an R-bimodule with involution, K and L are projective R-modules.

Definition 7.1 A sesquilinear pairing is a map $\lambda : K \times L \to M$, additive in each component, such that $\lambda(rk, sl) = s.\lambda(k, l).\overline{r}$. The additive group of sesquilinear pairings is denoted $S(K, L; M)$. In the case that $M = R$, we shall write simply $S(K, L)$.

Lemma 7.2 $S(K, L; M) \cong \mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M)) \cong M \otimes_R \mathrm{Hom}_R(K, L^*)$. If $\lambda \in S(K, L; M)$ then $\Lambda : K \to \mathrm{Hom}(L, M)$ is given by $\Lambda(k) = (l \to \lambda(k, l))$.

Definition 7.3

- $S(K; M) := S(K, K; M)$;
- $T_\epsilon : S(K; M) \to S(K; M)$, is the ϵ-transposition morphism given by $T_\epsilon(\lambda)(x, y) = \epsilon\overline{\lambda(y, x)}$
- The *ϵ-symmetric group over M* is $Q^\epsilon(K; M) = \{\lambda \in S(K; M) : \lambda(x, y) = \epsilon\overline{\lambda(y, x)}\} = \ker(1 - T_\epsilon)$;
- An *ϵ-symmetric form over M* is a pair $(K, \lambda)$where$\lambda \in Q^\epsilon(K; M)$;
- The *ϵ-quadratic group over M* is $Q_\epsilon(M) := M/\{x - \epsilon\overline{x} : x \in M\}$;

- An *ϵ-quadratic form over M* is a triple (K, λ, μ) such that (K, λ) is an ϵ-symmetric form and $\mu : K \to Q_\epsilon(M; M)$ such that

$$\begin{aligned} \mu(x+y) - \mu(x) - \mu(y) &= \lambda(x,y) \in Q_\epsilon(M) \\ \mu(x) + \epsilon\overline{\mu(x)} &= \lambda(x,x) \in M) \\ \mu(ax) &= a\mu(x)\overline{a} \in Q_\epsilon(M) \end{aligned}$$

Lemma 7.4 The ϵ-transposition T_ϵ on $M \otimes_R \mathrm{Hom}_R(K, K^*)$ defined by $\Lambda \to \epsilon\Lambda^*$ corresponds to the ϵ-transposition defined above under the isomorphism $S(K, K; M) \cong M \otimes_R \mathrm{Hom}_R(K, K^*)$.

7.2 The surgery obstruction group, $L_{2k}(R)$

7.2.1 Quadratic forms

We shall explain in detail in the next chapter how a surgery problem is represented by a quadratic intersection form on its middle dimensional homology $(K_k(M), \lambda, \mu)$. If M bounds some (highly connected) manifold N then there is a boundary map $(\partial : K_{k+1}(N, M) \to K_k(M))$ such that $\lambda(\partial x, \partial y) = 0$. Hence the even-dimensional surgery obstruction group is defined to be a Witt group of quadratic forms over a ring with involution R, where a form represents 0 in the group if and only if it has a *Lagrangian*.

Definition 7.5 (i) A *sublagrangian* L in a symmetric form (K, λ) is a direct summand $L \subseteq K$, such that $\lambda(L, L) = 0$.

(ii) Given a sublagrangian L in a symmetric form (K, λ), define $L^\perp(x) = \{x \in K : \lambda(x, L) = 0\}$.

(iii) A *sublagrangian* L in a quadratic form (K, λ, μ) is a sublagrangian L of (K, λ) such that $\mu(L) = 0$.

(iv) A *lagrangian* L (of a symmetric or quadratic form) is a sublagrangian such that $L = L^\perp$.

Note that L is a Lagrangian of a form (K, λ) if and only if the sequence:

$$0 \longrightarrow L \xrightarrow{i} K \xrightarrow{i^*\lambda} L^* \longrightarrow 0$$

is exact.

In fact any $(-1)^k$-symmetric quadratic form over a ring with involution R takes a particular form; namely the quadratic intersection form of the map $S^k \times S^k \to S^{2k}$. In other words, the surgery obstruction remains unchanged by taking the connected sum with a torus. The next

Definition 7.6 (i) The *hyperbolic ϵ-symmetric form* of a f.g. projective R-module K is $(K \oplus K^*, \lambda)$ where $\lambda((x, f), (g, y)) = f(y) + \epsilon\overline{g(x)}$

(ii) The *hyperbolic ϵ-quadratic form* of a f.g. projective R-module K is $(K \oplus K^*, \lambda, \mu)$ where λ is as above and $\mu(x, f) := f(x)$

Lemma 7.7 A quadratic form (K, λ, μ) over R is hyperbolic if and only if it has a Lagrangian.

Another fact we shall need later is the following:

Lemma 7.8 Suppose that (K, λ, μ) is a representative of $x \in L_n(\mathbb{Z}[G])$. Then there is another representative given by $(\tilde{K}, \tilde{\lambda}, \tilde{\mu})$ with

$$\begin{aligned} \tilde{K} &= K^* \\ \tilde{\lambda} &= \lambda^{-1} \\ \tilde{\mu}(x) &= \mu(\lambda^{-1}(x)) \end{aligned}$$

We complete this section by noting the group structure on the L groups.

Lemma 7.9 For any form (K, λ, μ), there exists an isomorphism

$$\phi : (K, \lambda, \mu) \oplus (K, -\lambda, -\mu) \to H_\epsilon(K)$$

Definition 7.10 The $2k$-dimensional L-group $L_{2k}(A)$ is the set of non-degenerate ϵ-quadratic forms (K, λ, μ), modulo the equivalence relation $(K, \lambda, \mu) \sim (K', \lambda', \mu')$ if there exist r, s such that

$$(K, \lambda, \mu) \oplus H_{(-1)^k}(A^r) \cong (K', \lambda', \mu') \oplus H_{(-1)^k}(A^s)$$

It is given an additive group structure with addition given by

$$(K, \lambda, \mu) \oplus (K' \lambda', \mu') := (K \oplus K', \lambda \oplus \lambda', \mu \oplus \mu')$$

and inverse by

$$-(K, \lambda, \mu) := (K, -\lambda, -\mu)$$

7.2.2 Split quadratic forms

It is readily seen that a symmetric form in which all diagonal entries are even can be expressed as the sum of an upper triangular matrix and its transpose. This observation motivates the consideration of ϵ-quadratic forms by representing each as the sum of a map plus its ϵ-transpose. Such maps will be called split ϵ-quadratic forms.

Definition 7.11

- A split ϵ-quadratic form (K, ψ) is a stably f.g. free A-module K together with an element $\psi \in \mathrm{Hom}_A(K, K^*)$.
- An equivalence of split ϵ-quadratic forms $(K, \psi), (K, \psi')$ is an element $\chi \in Q_{-\epsilon}(K)$ such that $\psi' - \psi = (1 - T_\epsilon)\chi$.
- A morphism of split ϵ-quadratic forms $(f, \chi) : (K, \psi) \to (K', \psi')$ is an A-module morphism $f \in \mathrm{Hom}_A(K, K')$ together with an element $\chi \in Q_{-\epsilon}(K)$ such that $f^*\psi' f - \psi = \chi - \epsilon\chi' : K \to K^*$.

Proposition 7.12 The ϵ-quadratic structures (λ, μ) on a stably f.g. free A-module K are in one-one correspondence with the equivalence classes $\psi \in Q_\epsilon(K)$ of split ϵ-quadratic forms (K, ψ).

Proof. We shall simply state the maps.

A split ϵ-quadratic form (K, ψ) determines an ϵ-quadratic form (K, λ, μ) where $\lambda = (1+T_\epsilon)\psi$ and $\mu(x) = \psi(x)(x)$.

For the reverse, choose a basis $x_1, \ldots, x_n$ for K, with dual basis $f_1, \ldots, f_k$. Define

$$\psi(x_j) = \sum_{i<j} \lambda(x_i, x_j) f_i + \tilde{\mu}(x_j) f_j$$

where $\tilde{\mu}(x_j)$ is some lift of $\mu(x_j)$ to R. □

Proposition 7.13 Let (K, ψ) be a representative for the split form corresponding to the ϵ-quadratic form (K, λ, μ). Then $i : L \subset K$ is a Lagrangian iff the inclusion $i : L \to K$ is such that $\chi - \epsilon\chi^* = i^*\psi i$.

Proof. Again choose a basis $x_1, \ldots, x_k$ for K and a dual basis $f_1, \ldots, f_k$. Then define

$$\chi(x_j) = \sum_{i<j} i^*\psi i(x_i)(x_j) f_i + i^*\psi i(x_j)(x_j)$$

which is such that $\chi - \epsilon\chi^* = 0$ since $i^*(1 + T_\epsilon)\psi i = 0$ and $\mu(ix) = 0$ for all $x \in L$. □

There is also a (trivial) correspondence between split forms and quadratic complexes (to be defined later). This is an important use, and split UNil forms are defined to bridge the gap between Cappell's theory and the algebraic theory of codimension 1 splitting problems (see Ranicki [7]).

7.3 The even-dimensional splitting obstruction group UNil

The UNil obstruction group was defined by Cappell to consist of pairs of forms, with values in $\widetilde{\mathbb{Z}[G_1]}$ and $\widetilde{\mathbb{Z}[G_2]}$, with a nilpotency condition on their adjoints, as in the $\mathfrak{Nil}$ category defined in section 3. We shall show that these can also be considered as forms on objects in the $\mathfrak{Nil}$ category. We shall extend the concept of split quadratic forms to split UNil forms. Whilst being of apparently little benefit at this stage, this reformulation will play a large part of our more general theory.

Let M_1 and M_2 be A-bimodules with involution which are free over R.

Definition 7.14 (i) An $\epsilon-$UNil *form over* (M_1, M_2) is a pair $C = ((K_1, \lambda_1, \mu_1), (K_2, \lambda_2, \mu_2))$ where:

- (K_i, λ_i, μ_i) is an ϵ-quadratic form over M_i.

- There exist finite filtrations of A-modules such that

$$\begin{array}{rcccccc} K_1 & = & K_1^0 & \supseteq K_1^1 & \supseteq & \ldots & \supseteq & K_1^r = 0 \\ K_2 & = & K_2^0 & \supseteq K_2^1 & \supseteq & \ldots & \supseteq & K_2^s = 0 \end{array}$$

- Letting $\rho_1 : K_1 \to M_1 \otimes_R K_2$ denote the adjoint of λ_1 and $\rho_2 : M_2 \otimes_R K_2 \to K_1$ denote the adjoint of λ_2,

$$\begin{array}{l} \rho_1(K_1^i)(M_1) \subseteq M_1 \otimes_R K_2^{i+1} \\ \rho_2(K_2^i)(M_2) \subseteq M_2 \otimes_r K_1^{i+1} \end{array}$$

(ii) Set $-C = ((K_1, -\lambda_1, -\mu_1), (K_2, -\lambda_2, -\mu_2))$.

(iii) A UNil Lagrangian of a form C is a pair of free direct summands $V_i \subseteq K_i$ such that V_2 is the annihilator of V_1 ($V_2 \subseteq K_2 = K_1^*$) with $\lambda_i|_{V_i \times V_i} = 0$ and $\mu_i|_{V_i} = 0$.

(iv) A UNil form C is a kernel if it has a UNil Lagrangian.

(v) Define $\mathrm{UNil}_{2k}(R; M_1, M_2)$ to be the set of equivalence classes of $(-1)^k$-UNil forms under the equivalence relation

$$C_1 \sim C_2 \quad \text{if} \quad C_1 \oplus (-C_2) \quad \text{is a kernel.}$$

Note that although a quadratic form has a Lagrangian if and only if it is hyperbolic, there is no corresponding result for UNil forms, as the following example shows:

Example 7.15 There exists a $(\mathbb{Z}; \langle t_1 \rangle_{\mathbb{Z}}, \langle t_2 \rangle_{\mathbb{Z}})$-UNil form which possesses a Lagrangian which has no complementary Lagrangian.

Proof. Let $P = \mathbb{Z}^{\oplus 3} = \langle x_1, x_2, x_3 \rangle, Q = P^* = \langle y_1, y_2, y_3 \rangle$.

Let

$$\begin{aligned} \rho_1 &= \begin{pmatrix} 0 & t_1 & 0 \\ t_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \rho_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2t_2 \end{pmatrix} \\ \mu_1(ax_1 + bx_2 + cx_3) &= abt_1 \\ \mu_2(ay_1 + by_2 + cy_3) &= c^2 t_2 \end{aligned}$$

Let L be the Lagrangian $(\langle x_3 \rangle, \langle y_1, y_2 \rangle)$.

Suppose that there exists a complementary Lagrangian $(\langle x_1', x_2' \rangle, \langle y_3' \rangle)$.

Then $y_3' = \alpha y_1 + \beta y_2 + \gamma y_3$ where $\gamma \neq 0$.

Then $\mu_2(y_3') = \mu_2(\alpha y_1 + \beta y_2) + \mu_2(\gamma y_3) + \lambda_2(\alpha y_1 + \beta y_2, \gamma y_3) = \gamma^2 t_2 \neq 0$.

But this is a contradiction of the assumption that $(\langle x_1', x_2' \rangle, \langle y_3' \rangle)$ is a Lagrangian. □

7.3.1 UNil again

We can reformulate symmetric UNil forms in terms of the $\mathfrak{Nil}$ defined earlier. This will draw an even closer parallel with the surgery obstruction group, and will be more useful for the formulation of the odd-dimensional obstruction group in terms of formations.

Definition 7.16

An ϵ-symmetric $\mathfrak{Nil}$-form (KS, λ) is an object $KS \in \mathfrak{Nil}$ together with a morphism $\lambda : KS \to KS^*$ such that $\lambda = \epsilon\lambda^*$.

An ϵ-quadratic $\mathfrak{Nil}$-form $(KS, \lambda, \mu_P, \mu_Q)$ is an ϵ-symmetric $\mathfrak{Nil}$-form (KS, λ) together with maps $\mu_P : P \to Q_\epsilon(\widetilde{\mathbb{Z}[G_1]}$, $\mu_Q : Q \to Q_\epsilon(\widetilde{\mathbb{Z}[G_2]}$, such that $\mu_P(x) + \epsilon\overline{\mu_P(x)} = (-\rho_1^*\lambda)(x)(x)$, and $\mu_Q(x) + \epsilon\overline{\mu_Q(x)} = (-\rho_2^*\lambda)(x)(x)$.

Definition 7.17 A Lagrangian of a symmetric $\mathfrak{Nil}$-form (KS, λ) is an object LS in $\mathfrak{Nil}$ with an injection $i : LS \to KS$ such that the sequence

$$0 \longrightarrow LS \xrightarrow{i} KS \xrightarrow{i^*\lambda} LS^* \longrightarrow 0$$

is exact.

Similarly a Lagrangian $LS = (L_P, L_Q)$ of a $\mathfrak{Nil}$-form $(KS, \lambda, \mu_P, \mu_Q)$ is a Lagrangian of the symmetric $\mathfrak{Nil}$-form (KS, λ) such that $\mu_P(L_P) = 0$ and $\mu_q(L_Q) = 0$.

Lemma 7.18 Symmetric/quadratic UNil forms are in 1-1 correspondence with symmetric/quadratic $\mathfrak{Nil}$ forms, and UNil lagrangians are in 1-1 correspondence with $\mathfrak{Nil}$ lagrangians.

Proof. Suppose that $((P, Q; \rho_1, \rho_2), (\lambda_1, \lambda_2))$ is a symmetric $\mathfrak{Nil}$ form. Define a UNil form by $(P, Q, \lambda_1\rho_1^*, \lambda_2\rho_2^*)$, using $\lambda_1 = \lambda_2^*$ to identify $P \cong Q^*$. This correspondence is easily seen to be reversible.

Suppose that $(i_1, i_2) : (L_1, L_2; \sigma_1, \sigma_2) \to (P, Q; \rho_1, \rho_2)$ is the inclusion of a $\mathfrak{Nil}$ lagrangian. Then in particular

$$0 \longrightarrow L_1 \xrightarrow{i_1} P \xrightarrow{i_2^*\lambda_1} L_2^* \longrightarrow 0$$

is exact.

Hence $L_1 = \ker(P \to L_2^*)$ i.e. L_1 is the annihilator of L_2 with respect to the isomorphism $P \cong Q^*$ given by λ_1. L_1 is a direct summand since the short exact sequence is a short exact sequence of projective modules and therefore splits. The form $L_1 \to L_1^* = i_1^*\lambda_2\rho_1 i_1 = \sigma_1^* i_2 * \lambda_1 i_1 = 0$ (up to sign). Similarly with L_2, and hence (L_1, L_2) is a symmetric UNil lagrangian.

Conversely, suppose that (L_1, L_2) is a UNil lagrangian. Then $P = L_1 \oplus L_2^*$, $Q = L_2 \oplus L_1^*$. Define σ_1 to be the composite $L_1 \to P \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} Q \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} L_2$. □

It would be nice to have an equivalent of the result that every quadratic form with a Lagrangian is hyperbolic. Unfortunately, as shown above, this is not true. However, we can still define hyperbolic forms which are quadratic forms with Lagrangians

Definition 7.19 Given a $\mathfrak{Nil}$-module $KS = (P, Q; \rho_1, \rho_2)$, define the hyperbolic form

$$H_\epsilon(KS) = (P \oplus Q^*, Q \oplus P^*; \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_1^* \end{pmatrix}, \begin{pmatrix} \rho_2 & 0 \\ 0 & \rho_2^* \end{pmatrix}).$$

7.3.2 Split UNil forms

Definition 7.20 A split ϵ-quadratic UNil form $(KS, \theta, \psi_P, \psi_Q)$ consists of:

- $KS = (P, Q; \rho_1, \rho_2) \in \mathfrak{Nil}$;
- An isomorphism $\theta : P \to Q^*$;
- $\psi_P : P \to P^* \otimes \widetilde{\mathbb{Z}[G_1]}, \psi_Q : Q \to Q^* \otimes \mathbb{Z}[G_2]$, such that $\rho_1^*\theta = \psi_P + \epsilon\psi_P^*$.

Definition 7.21 An equivalence of split ϵ-quadratic UNil forms

$$(KS, \theta, \psi_P, \psi_Q) \sim (KS', \theta', \psi'_P, \psi'_Q)$$

is a pair $(\chi_P, \chi_Q) \in Q_{-\epsilon}(P; \widetilde{\mathbb{Z}[G_1]} \oplus Q_{-\epsilon}(Q; \widetilde{\mathbb{Z}[G_2]})$ such that $\psi_P - \psi'_P = (1 - T + \epsilon)\chi_P$ and $\psi_Q - \psi'_Q = (1 - T_\epsilon)\chi_Q$.

Lemma 7.22 The ϵ-quadratic forms $(KS, \lambda, \mu_P, \mu_Q)$ on $KS \in \mathfrak{Nil}$ are in 1-1 correspondence with the equivalence classes of split ϵ-quadratic forms over KS.

Chapter 8

The Even-dimensional UNil Obstruction

In this chapter, assume that $f : M^{2k} \to X$ is a highly connected degree 1 normal map (sitting inside a homotopy equivalence $g : W \to Y$ for the splitting case). We recall the definition of the surgery and splitting obstructions defined in the even-dimensional case by Wall and Cappell respectively.

The even-dimensional surgery obstruction $\sigma(f)$ was defined by Wall to be the quadratic intersection on the $\mathbb{Z}[H]$-valued kernel homology of M, and the splitting obstruction $\chi(g)$ was defined by Cappell similarly, so long as a K-theoretic obstruction vanishes.

These definitions are here recalled, for use later in the thesis.

8.1 The Surgery Obstruction

Once again, before defining the splitting obstruction, we recall the theory of the surgery obstruction. In this chapter let $f : M^n \to X^n$ where $n = 2k$, X is a Poincaré complex and M is a manifold. By previous results, we can further assume that f is highly connected. Then by Poincaré duality and the Universal Coefficient Theorem, $K_j(M) = 0$ for $j \neq k$.

Given such a map f, the surgery obstruction $\sigma(f) \in L_{2k}(\mathbb{Z}[\pi_1(X)])$ is the quadratic form (K, λ, μ) with the following components:

- $K = K_k(M)$
- $\lambda =$ the homology intersection form on K.
- μ, the 'self-intersection', measures the obstruction to being able to represent an element by a framed embedding $S^k \times D^k \to M$.

We follow Wall in not defining μ on the whole homotopy class $\pi_{k+1}(f)$, but instead defining it only on the regular homotopy class determined by the normal bundle data.

Construction 8.1 With $f : M \to X$ as above, suppose that $x \in K_k(M) \cong \pi_{k+1}(f)$. Let $\theta : S^k \to M$ be in the unique regular homotopy class of immersions determined by the normal

data, and assume that θ is in general position. Then θ has only a finite set of self-intersections. Suppose that $p = (y_1, y_2)$ is a self-intersection, i.e. $\theta(y_1) = \theta(y_2)$ with $y_1 \neq y_2$. Let γ be a path in S^k from y_1 to y_2 via the base point, and avoiding all other self-intersection points. Fix a local orientation at $\theta(y_1)$, and let $\epsilon(p) = 1$ if transporting the orientation around $\theta_*(\gamma)$ gives the same orientation and -1 else. Let $g(p) = \theta_*(\gamma) \in \pi_1(X)$. Then let $\mu(x)$ be the sum over all self-intersections p, $\sum g(p)\epsilon(p)$.

$\mu(x)$ is then a regular homotopy invariant and is the obstruction to representing x by an embedded sphere framed in a manner compatible with the normal map.

Proposition 8.2 (Chap. 5, Wall[17]) For $f : M \to X^{2k}$, where $k \geq 3$, $\sigma(f) = 0$ if and only if f is normal bordant to a homotopy equivalence.

Outline of proof. Suppose that $F : N \to X \times I$ is a normal bordism with $f' : M \to X$. Then after making F highly connected $K_{k+1}(N, M)$ is a Lagrangian of M.

Suppose that $\sigma(f) = (K, \lambda, \mu)$ is stably hyperbolic. Then surgeries on $0 \in K_k(M)$ have the effect of adding a hyperbolic form (the effect of surgery is to take the connected sum with a torus). Hence it can be assumed that the obstruction is hyperbolic.

Let L be a Lagrangian generated by $x_1, \ldots, x_r$. Since the self-intersection of these elements is zero, they can be represented by framed embeddings. Surgery on these has the result of killing the middle-dimensional homology, resulting in a homotopy equivalence. □

Remark 8.3 Suppose that $f : M \to X^{2k}$ is a highly connected normal map of manifolds, not necessarily open, but such that the homology and cohomology do satisfy Poincaré duality. Then $\sigma(f)$ is defined in the same way as above and is the surgery obstruction to there existing a relative Poincaré cobordism to a homotopy equivalence. All the steps outlined above follow through without modification.

8.2 Splitting obstruction

Again assume that $f : M \to X$ is k-connected, then from the previous chapter, $K_k(M) = P \oplus Q$. Assume that $\overline{\phi}(\tau(g)) = 0 \in H^0(\mathbb{Z}_2; \ker(\tilde{K}_0(\mathbb{Z}[H]) \to \tilde{K}_0(\mathbb{Z}[G_1]) \oplus \tilde{K}_0(\mathbb{Z}[G_2])))$. Then by the results of chapter 5, it can be assumed that $\phi(\tau(g)) = 0 = [P]$, so that P is stably f.g. free.

Then trivial $(k-1)$-handle exchanges can be performed to assume that P is f.g. free, and then $\mathrm{Spl}_k(M)$ determines a free $\mathfrak{Nil}$ object. Then the Poincaré duality map defines a symmetric UNil form $\lambda : \mathrm{Spl}_k(M) \to \mathrm{Spl}_k(M)^*$.

The splitting obstruction is to be a quadratic UNil form, so a quadratic refinement must now be defined. Its definition is slightly hidden in the literature — the 'nilpotent normal cobordism' is constructed, a cobordism with homology kernel $\mathbb{Z}[G] \otimes (P \oplus Q)$, and the splitting intersection forms μ_P and μ_Q must be the self-intersection in this cobordism of $x - \rho_1(x)$ and $x - \rho_2(x)$ respectively.

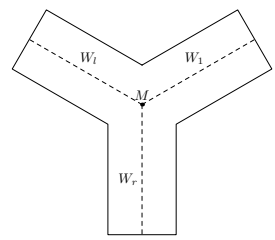
Construction 8.4 Let Δ be a planar triangle, with three edges e_1, e_2, e_3. Define:

$$W_1' = M \times \Delta \cup_{M \times e_1} W_l \times I \cup_{M \times e_2} W_l \times I \cup_{M \times e_3} W_1 \times I$$

(and smoothing corners). Let $x \in P = K_{k+1}(W_r, M)$. Then x can be represented by an immersed disc $\phi : D^{k+1} \to W_l \cup_M W_1$ with boundary $\overline{\phi} : S^k \to M$. Since P is a Lagrangian of the kernel form of $f : M \to X$, the boundary can be taken to be an embedding.

In addition, $\overline{\phi}$ bounds an immersed disc ϕ' in W_r. Thus $\phi \cup_{\overline{\phi}} \phi'$ defines an immersion of a $k+1$-dimensional sphere into $(W_1 \cup_M W_l) \cup_M W_r$ which embeds into W_1' as shown by the dotted lines in the figure. Let $\mu_1(x) \in Q_(-1)^{k+1}(\mathbb{Z}[G_1])$ be the self-intersection of the immersion.

Figure 8.1: W_1' used in defining the self-intersection $\mu_1(x)$



$\mu(x)$ is the self-intersection of an immersed sphere in the $2k+2$-dimensional manifold with boundary W_1' in figure 8.1. Namely, let Δ be a planar triangle. Then W_1' is formed by joining $W_1 \times I$, $W_l \times I$ and $W_r \times I$ onto the three edges $M \times I$ of $\Delta \times I$ (and smoothing corners). Then $\pi\phi : D^{k+1} \to W_1 \cup_M W_l \subset W_1'$ is an immersed disc, with boundary $\pi\overline{\phi} = \overline{\phi} : S^k \to M \subset W_1'$ an embedding. Let $\theta : S^{k+1} \to W_1'$ be given by:

$$\theta(S^{k+1}) = \theta(D^{k+1} \cup_{S^k} D^{k+1}) = \pi\phi(D^{k+1}) \cup_{S^k} \phi(D^{k+1}) \subset (W_1 \cup_M W_l) \cup_M W_r \subset W_1'.$$

Remark 8.5 $\mu_1(x) \in Q_{(-1)^{k+1}}(\mathbb{Z}[G_1])$ is the self-intersection of the above class θ; the map $\mu : P \to Q_{(-1)^{k+1}}(\mathbb{Z}[G_1])$ factors through the inclusion $Q_{(-1)^{k+1}}(\widetilde{\mathbb{Z}[G_1]}) \subset Q_{(-1)^{k+1}}(\mathbb{Z}[G_1])$.

This will follow once we have seen that it is the self-intersection of a certain sphere in the nilpotent normal cobordism constructed by Cappell and which is described in the next chapter.

Definition 8.6 We define the UNil obstruction of the map g above to be

$$\chi(g) \in \mathrm{UNil}_{n+2}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$$

represented by the UNil form:

$$((P, \lambda_1, \mu_1), (Q, \lambda_2, \mu_2))$$

Then Cappell proved:

Theorem 8.7 (Cappell [3]) g is h-cobordant to a split homotopy equivalence if and only if $\chi(g) = 0$ and $\overline{\phi}(\tau(g)) = 0$. Furthermore, given any $\alpha \in \mathrm{UNil}_{2k+2}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$, there exist manifolds W, M and a map $f : M \to W$ such that $\chi(f) = \alpha$.

The ‘if’ part follows from the nilpotent normal cobordism construction of the next chapter. The ‘only if’ part follows also from proposition 6.12 in which case N determines a UNil lagrangian.

Chapter 9

Even-dimensional Nilpotent Normal Cobordism

In this chapter we describe the nilpotent normal cobordism. This is a cobordism between a given splitting problem and a split homotopy equivalence (i.e. a cobordism which is a homotopy equivalence on the boundary) whose surgery obstruction can therefore be computed. If the surgery obstruction is zero, then the splitting problem is solved. We have defined the splitting obstruction $\chi(g)$.

The realization for elements of the UNil group will be achieved via the realization of the nilpotent normal cobordism. The monomorphism UNil $\to L$ is given by the nilpotent normal cobordism. Hence we can think of UNil $\subset L$ as consisting of those obstructions which arise as the nilpotent normal cobordism of a splitting obstruction.

We describe the nilpotent normal cobordism construction here since we shall soon give a generalization of this construction in terms of algebraic surgery, and it will be useful to have some geometric intuition behind it.

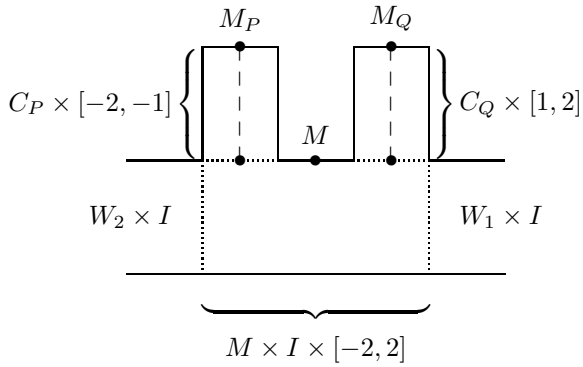
Theorem 9.1 [5] Suppose that $g : W \to Y^{2k+1}$ is an even-dimensional splitting problem with $\phi(\tau(g)) = 0$ and splitting obstruction $\chi(g) \in \mathrm{UNil}_{2k+2}(H; G_1, G_2)$. Then there is a split monomorphism $\alpha : \mathrm{UNil}_{2k+2}(H; G_1, G_2) \to L_{2k+2}(G_1 *_H G_2)$ and a cobordism $h : V^{2k+2} \to Y$ from $g : W \to Y$ to $g' : W' \to Y$, where g' is split, with $\sigma(h) = \alpha\chi(g)$. Then in particular if $\chi(g) = 0$ then $\sigma(h) = 0$ so g is splittable.

The K-theoretic obstruction has already been described as the relative finiteness obstruction of the kernel of the maps of $\mathbb{Z}[H]$-covers $W_l \to Y_l$ and $W_r \to Y_r$. When these kernels are finitely dominated, (i.e. finite in this case), P and Q are stably free and so determine Lagrangians of $K_k(M)$. Hence it is possible to perform surgeries on $f : M \to X$ on spheres representing generators of P or Q. This gives two cobordisms $(C_P; M, M_P) \to X$ and $(C_Q; M, M_Q) \to X$, where $f_P : M_P \to X$ and $f_Q : M_Q \to X$ are homotopy equivalences.

Construction 9.2 (Nilpotent normal cobordism) Take an embedding $M \times [-2, 2] \subset W$; glue $C_P \times [-2, -1]$ and $C_Q \times [1, 2]$ onto $W \times I$ by joining $M \times [-2, -1] \subset C_P \times [-2, -1]$ to

$M \times [-2,-1] \times \{1\} \subset W \times \{1\}$ and $M \times [1,2] \subset C_Q \times [1,2]$ to $M \times [1,2] \times \{1\} \subset W \times \{1\}$. Call the resulting cobordism $h : T \to Y$, Denote $W \times I$ by $\partial_- T$ and the remainder of the boundary of T $\partial_+(T)$. See figure 9.1

Figure 9.1: Construction of the nilpotent normal cobordism



Proposition 9.3 (Cappell [5]) The nilpotent normal cobordism has the following properties:

- $\partial_-(T) = W$;
- $h_+ : \partial_+(T) \to Y$ is a split homotopy equivalence;
- The kernel homology $K_{k+1}(T) = \mathbb{Z}[G_1 *_H G_2] \otimes (P \oplus Q)$
- The intersection form λ_T is such that $\lambda_T((1-\rho)x, (1-\rho)y) = \begin{pmatrix} -\lambda_1 & (-1)^{k+1} \\ 1 & -\lambda_2 \end{pmatrix} : K_{k+1}(T) \to K_{k+1}(T)^*$, and $\mu_T((1-\rho)x) = \mu_P(x)$ for $x \in P$, $\mu_T((1-\rho)x) = \mu_Q(x)$ for $x \in Q$.

Proof. That it is split is clear, since the maps $M_P \to X$ and $M_Q \to X$ are both homotopy equivalences. Note that there is an inclusion of $C_P \cup_M C_Q \subset W'$ and the restriction $C_P \cup_M C_Q \to X$ is a homotopy equivalence (in fact $C_P \cup_M C_Q$ is a compact manifold homotopy equivalent to $\hat{W}$. Then

$$W' = (W_2 \cup_M C_P) \cup_{M_P} (C_P \cup_M C_Q) \cup_{M_Q} (C_Q \cup_M W_1).$$

Since $K_k(W_2) = P \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G_2]$ and C_P is formed by attaching cells to a basis of P, $K_k(W_2 \cup C_P) = 0$. Similarly with $K_k(W_1 \cup C_Q)$ and then the Mayer-Vietoris sequence implies that $K_k(W')$ vanishes as claimed.

Cappell proves that the nilpotent normal cobordism has surgery obstruction λ_T, which satisfies $\lambda_T((1-\rho)x, y) = L(x,y)$, where $x, y \in P \text{or} Q$ and $L = \left(\begin{smallmatrix} 0 & 1 \\ (-1)^{k+1} & 0 \end{smallmatrix}\right)$. This is proved by constructing explicit embeddings representing x and $(1-\rho)x$ for $x \in P, Q$, and showing that their intersection in T is given by the form L. On the other hand, if $\epsilon = (-1)^k$, then $L = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$, $1 - \rho = \begin{pmatrix} 1 & -\rho_2 \\ -\rho_1 & 1 \end{pmatrix}$, $\epsilon\rho_1 = \lambda_1$ and $\epsilon\rho_2 = \lambda_2$ so

$$L(1-\rho) = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\rho_2 \\ -\rho_1 & 1 \end{pmatrix} = \begin{pmatrix} -\epsilon\rho_1 & \epsilon \\ 1 & -\rho_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \epsilon \\ 1 & -\rho_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \epsilon \\ 1 & -\lambda_2 \end{pmatrix}$$

.

In particular the construction of $(1-\rho)x$ is the following:

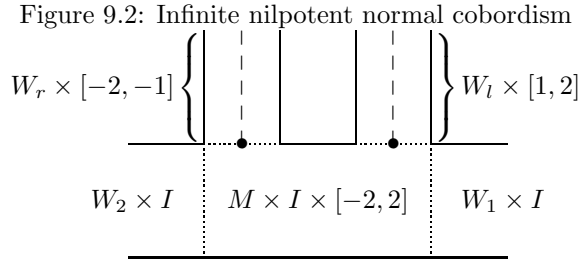
Construction 9.4 Let $x \in P$. Then x is represented by a sphere S^k which bounds discs in C_Q and in $W_1 \cup_M C_P$. Thus the union of these discs is a sphere $S^{k+1} \to T_M$; as always there is a unique regular homotopy class in this homotopy class determined by the normal data.

□

At this point we shall depart slightly from the proof of Cappell and define an infinite version of the preceding: this will (a) justify the definition of the self-intersection form given earlier, and (b) not depend upon the dimension of X being odd.

Lemma 9.5 Let $g : W \to Y^{n+1}$ be a homotopy equivalence, and define T_M^∞ to be the open surgery (Maumary [6]) problem given by glueing copies of W_r and W_l where C_P and C_Q were glued before. Then there is defined a surgery obstruction, $\sigma(T_M^\infty) = \alpha(\chi(g)) \in L_{n+2}^h(\mathbb{Z}[G])$.

Figure 9.2: Infinite nilpotent normal cobordism



Lemma 9.6 Suppose that (W, M) is a split homotopy equivalence. Then $T_M^\infty \to T_X^\infty$ is a homotopy equivalence.

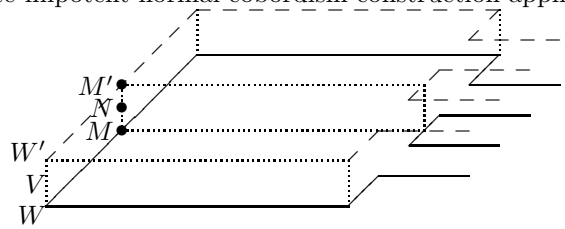
Proof. $T_M^\infty = W \times I \cup_{M\times I} \hat{W} \times I$, so the map is a homotopy equivalence if and only if $M \to X$ is, since $W \to Y$ and $\hat{W} \to \hat{Y}$ are homotopy equivalences. □

Proposition 9.7 Let $g : W \to Y^{n+1}$ be a homotopy equivalence, and let $\sigma(T_M^\infty)$ be the surgery obstruction of the infinite nilpotent normal cobordism construction. Suppose that $(V; W, W')$ is a cobordism of splitting problems with W' split and with $\phi(\tau(W)) = 0$. Then $\sigma(T_M^\infty) = \sigma(V) \in L_{n+2}(V)$.

The benefit of this result is that it does not rely upon the polarity of the dimension, or upon highly-connectedness, and we will be able to apply this result directly in the odd-dimensional case.

Proof. As before, define $T_N^\infty = N \times I \cup V_r \times I \cup V_l \times I$. Let $V_0 = V_1 \cup_M V_r \cup V_r \cup_M V_l \cup V_l \cup_M V_2$, which is homotopy equivalent to $Y_1 \cup_X Y_r \cup Y_r \cup_X Y_l \cup Y_l \cup_X Y_2$. Then the boundary of T_N^∞ is the union of T_M, $T_{M'}^\infty$, V and V'. Regard T_N^∞ as a cobordism rel ∂ of T_M with $V \cup'_W T_{M'} \cup_{M'_0} N_0$,

Figure 9.3: Infinite nilpotent normal cobordism construction applied to a cobordism



so that these have the same surgery obstruction. Then we claim that the surgery obstruction of the latter is the surgery obstruction of V. This is because the following are all homotopy equivalences:

- $W' \to Y'$;
- $T^{\infty}_{M'} \to T^{\infty}_{X}$ since W' is split;
- $V_0 \to Y_0$.

□

Remark 9.8 In the case when $f : M \to X^{2k}$ is highly connected, the computation of the surgery obstruction of the infinite nilpotent normal cobordism is identical to the surgery obstruction of the nilpotent normal cobordism, since $K_i(W_r, M) \cong K_i(C_P, M)$. In particular, the definition we gave of $\mu_P(x)$ is such that $\mu_P(x) = \mu^{\infty}_{T}((1-\rho_1)(x))$.

For convenience, we restate the results of this chapter in terms of split UNil forms.

Corollary 9.9 Let $\Psi = (KS, \theta, \delta\psi_P, \delta\psi_Q)$ be a split UNil-form. $\alpha(\Psi)$ is the split quadratic form $(\mathbb{Z}[G] \otimes (P \oplus Q), \begin{pmatrix} (-1)^k \delta\psi_P & 0 \\ (-1)^{k+1}\theta & (-1^k \delta\psi_Q \end{pmatrix})$

Chapter 10

Principles of the Algebraic Theory of Surgery

We have now covered in detail the theory which related to highly connected even-dimensional surgery problems and splitting problems.

In the next few chapters we consider the work of Ranicki, which constructs the surgery obstruction without first performing surgery below the middle dimension. The purpose of this chapter is to lay out the foundations: to define quadratic structures on chain complexes, pairs and triads, to relate them to the geometry, and give some basic results which we shall need later.

For the purposes of surgery, a degree 1 normal map of CW complexes can be represented by an algebraic 'quadratic complex', a chain complex with a quadratic structure. If the CW complexes satisfy Poincaré duality, there is a corresponding notion for quadratic complexes. Given a Poincaré n-ad of CW complexes (e.g. a CW complex satisfying Poincaré duality, or a pair satisfying Poincaré-Lefschetz duality), there is a Poincaré pair of quadratic complexes.

10.1 Quadratic structures

Example 10.1 An ϵ-quadratic form ($\epsilon = \pm 1$) over a module M is an equivalence class of split quadratic forms $\psi \in \mathrm{Hom}(M, M^*)$, with $\psi \sim \psi'$ if $\psi - \psi' = \chi - \epsilon\chi^*$ for some χ. Let $\mathbb{Z}_2 = \{1, T\}$, and let $\mathbb{Z}_2$ act on $\mathrm{Hom}(M, M^*)$ by $T\psi = \epsilon\psi^*$. Then ϵ-quadratic forms over M are $\mathbb{Z}_2$-hyperhomology classes in $H_0(\mathbb{Z}_2; \mathrm{Hom}(M, M^*))$.

Definition 10.2 Let W be the $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of Z:

$$W = \ldots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] = W_1 \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] = W_0$$

Definition 10.3 Let C be a finite R-module chain complex. Define the chain complex:

$$(W_{\%}C) = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes C).$$

Then:

$$(W_{\%}C)_n = \bigoplus_{s\geq 0}\bigoplus_{r} \mathrm{Hom}_R(C^{n-r-s}, C_r)$$

$$d(\psi) = \sum_{s\geq 0}\sum_{r} d\psi_s^{r+1} + (-1)^r\psi_s^r d^* + (-1)^{n-s-1}(\psi_{s+1}^r + (-1)^{s+1+r(n-s)}(\psi_{s+1}^{n-r-s-1})^*)$$

$$\text{where } \psi \in (W_{\%}C)_n, \text{ so that } \psi_s^r \in \mathrm{Hom}_R(C^{n-r-s}, C_r)$$

Then define:

- The group of n-dimensional quadratic structures on C, $Q_n(C) := H_n(W_{\%}(C))$;
- An *n-dimensional quadratic complex* is a pair $(C, [\psi])$ where C is a chain complex, and $\psi \in (W_{\%}C)_n$ is a representative for $[\psi] \in Q_n(C)$;
- An n-dimensional quadratic complex (C, ψ) is *Poincaré* if $(1+T)\psi_0^r : C^{n-r} \to C_r$ is a chain equivalence.

Convention 10.4 When the meaning is clear from the context we shall drop the [ ] notation.

Remark 10.5 It is sometimes helpful to represent quadratic complexes diagrammatically. So given a chain complex C, we shall represent $\psi \in (W_{\%}C)_n$ by a diagram of maps:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-k-1} \arrow[r] \arrow[d] \arrow[dr, "\psi_1^k"] \arrow[drr, dotted] & C^{n-k} \arrow[r] \arrow[d] \arrow[dr] & C^{n-k+1} \arrow[r] \arrow[d, "\psi_0^{k-1}"] & \cdots \\
\cdots \arrow[r] & C_{k+1} \arrow[r] & C_k \arrow[r] & C_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the complexes C_* and C^{n-*} are aligned so that $\psi_0^k : C^{n-k} \to C_k$ is represented by a vertical arrow, and then ψ_s^k is represented by an arrow with 'gradient' s.

Now, any chain map $f : C \to D$ gives rise to a chain map $f_{\%} : W_{\%}C \to W_{\%}D$ given by $f_{\%}(\psi_s^r) = f\psi_s^r f^*$. So $(W_{\%}C(f))_j = (W_{\%}C)_{j-1} \oplus (W_{\%}D)_j$, and there are defined relative quadratic structure groups:

Definition 10.6
- $Q_n(f : C \to D) = H_n(C(f_{\%}))$
- An $(n+1)$-dimensional quadratic pair is a triple $(f : C \to D, [(\delta\psi, \psi)])$ where $(\delta\psi, \psi)$ is a representative for $[(\delta\psi, \psi)] \in Q_n(f : C \to D)$.
- An $(n+1)$-dimensional quadratic pair $(f : C \to D, (\delta\psi, \psi))$ is Poincaré if $\begin{pmatrix} (1+T)\delta\psi_0^r \\ (1+T)\psi_0^{r-1}f^* \end{pmatrix} : D^{n+1-r} \to C(f)_r$ is a chain equivalence.

Similarly with triads, with sign modifications:

Lemma 10.7 Given a triad Γ of chain complexes:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, "k"] & D \arrow[d, "h"] \\
C' \arrow[r, "f'"] & D'
\end{tikzcd}$$

so that k is a null-homotopy of $f'g-hf$, with $f'g-hf = dk+kd$, define the map $\Gamma_n : (W_\% C)_n \to (W_\% D')_{n+1}$ by

$$\begin{aligned}\Gamma(\psi)^r_s &= (-1)^{n+1} k\psi^{r-1}_s f^* h^* + (-1)^{1+r+n} f' g \psi^r_s k^* + (-1)^{1+r+(r+1)(n+s)} k(\psi^{n-r-s}_{s+1})^* k^*. \\ &= (-1)^{n+1} k\psi^{r-1}_s f^* h^* + (-1)^{1+r+n} f' g \psi^r_s k^* + (-1)^{r+1} kT(\psi^{n-r-s}_{s+1}) k^*.\end{aligned}$$

Then $\Gamma_{n-1} d_n - d_{n+1}\Gamma_n = (-1)^n((f'g)_\% - (hf)_\%) : (W_\% C)_n \to (W_\% D')_n$.

Corollary 10.8 The map $(g,h;k)_\% = \begin{pmatrix} h_\% & \Gamma_n \\ 0 & g_\% \end{pmatrix} : C(f_\%)_{n+1} \to C(f'_\%)_{n+1}$ is a chain map.

Proof. Recall that $C(f_\%)_{n+1} = (W_\% D)_{n+1} \oplus (W_\% C)_n$, $d = \left(\begin{smallmatrix} d & (-1)^n f_\% \\ 0 & d \end{smallmatrix}\right) : C(f_\%)_{n+1} \to C(f_\%)_n$.

The condition that the map be a chain map is that

$$\begin{pmatrix} d & (-1)^n f'_\% \\ 0 & d \end{pmatrix} \begin{pmatrix} h_\% & \Gamma_n \\ 0 & g_\% \end{pmatrix} = \begin{pmatrix} h_\% & \Gamma_{n-1} \\ 0 & g_\% \end{pmatrix} \begin{pmatrix} d & (-1)^n f_\% \\ 0 & d \end{pmatrix}$$

i.e. that $d\Gamma_n + (-1)^n f'_\% g_\% = (-1)^n h_\% f_\% + \Gamma_{n-1} d$, or that $\Gamma_{n-1} d - d\Gamma_n = (-1)^n (f'_\% g_\% - h_\% f_\%)$. □

Corollary 10.9 Suppose that $g : C \to C'$ is a homotopy equivalence with homotopy inverse $h : C' \to C$ and null-homotopy such that $hg - 1 = kd + dk$. Suppose that $(f : C \to D, (\delta\psi, \psi))$ is an $n+1$-dimensional quadratic pair. Then the triad

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "kf"] & D \arrow[d, "1"] \\ C' \arrow[r, "fh"] & D \end{tikzcd}$$

induces a homotopy equivalence of pairs, with the quadratic structure in $Q_{n+1}(fh)$ given by

$$(\delta\psi'^r_s, \psi') = (\delta\psi^r_s + (-1)^{n+1} kf\psi^{r-1}_s f^* + (-1)^{n+1-r} fhg\psi^r_s f^* k^* + (-1)^{r+1} kfT(\psi^{n-r-s}_{s+1}) f^* k^*, g_\% \psi).$$

Definition 10.10

The triad quadratic Q-groups $Q_{n+2}(\Gamma)$ are defined by:

$$Q_{n+2}(\Gamma) = H_{n+2}(-(g,h;k)_\%).$$

An $(n+2)$-dimensional quadratic triad $(\Gamma, (\delta\chi, \chi, \delta\psi, \psi))$ is *Poincaré* if

- (C, ψ) is an n-dimensional Poincaré complex;
- $(g : C \to C', (\chi, \psi))$ is an $(n+1)$-dimensional Poincaré pair;
- $(f : C \to D, (\delta\psi, \psi))$ is an $(n+1)$-dimensional Poincaré pair;
- $$\begin{pmatrix} (1+T_\epsilon)\delta\chi_0 \\ (-1)^{n-r}(1+T_\epsilon)\delta\psi_0 h^* \\ h(1+T_\epsilon)\chi_0 k^* + (-1)^{n-r}(1+T_\epsilon) f'^* \\ (1+T_\epsilon)\psi_0 f^* h^* \end{pmatrix} : D'^{n+2-*} \to C(\Gamma)$$

  is a chain equivalence

Thus the triad quadratic Q-groups fit into the commutative diagram:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
\arrow[r] & Q_n(C) \arrow[r, "g"] \arrow[d, "f"] & Q_n(C') \arrow[r] \arrow[d, "f'"] & Q_n(g) \arrow[r] \arrow[d] & \\
\arrow[r] & Q_n(D) \arrow[r, "h"] \arrow[d] & Q_n(D') \arrow[r] \arrow[d] & Q_n(h) \arrow[r] \arrow[d] & \\
\arrow[r] & Q_n(f) \arrow[r] \arrow[d] & Q_n(f') \arrow[r] \arrow[d] & Q_n(\Gamma) \arrow[r] \arrow[d] & \\
 & & & &
\end{tikzcd}$$

Lemma 10.11

$$W_{\%}(\Gamma)_{n+2} = W_{\%} f'_{n+2} \oplus W_{\%} f_{n+2} = W_{\%} D'_{n+2} \oplus W_{\%}(C')_{n+1} \oplus W_{\%}(D)_{n+1} \oplus W_{\%}(C)_n$$

with differential given by:

$$\begin{pmatrix} d & (-1)^{n+1} f'_{\%} & (-1)^n h_{\%} & (-1)^n \Gamma_n \\ 0 & d & 0 & (-1)^n g_{\%} \\ 0 & 0 & d & (-1)^n f_{\%} \\ 0 & 0 & 0 & d \end{pmatrix}$$

Proposition 10.12 (Ranicki) Any degree 1 normal map $f : M \to X^n$ determines an n-dimensional Poincaré complex (C, ψ), with $H_k(C) \cong K_k(M)$. A map of $(n+1)$-dimensional Poincaré pairs $f : (N, M) \to (Y, X)$ determines an $n+1$-dimensional Poincaré pair $(f : C \to D, (\delta\psi, \psi))$. An $n+2$-dimensional triad of manifolds likewise determines an $n+2$-dimensional Poincaré triad.

Example 10.13 (Quadratic forms revisited) Let $f : M \to X$ be a highly connected $2k$-dimensional normal map. Let $C_k = K_k(M)^*$. Let ψ_0^k be a split quadratic form representing the surgery obstruction. ψ is Poincaré since $(1+T)\psi_0 = \lambda$, the intersection form.

10.2 Cobordism of quadratic complexes

The surgery obstruction group of a ring R is defined to be cobordism classes of f.g. free Poincaré complexes over R: so we now define cobordism.

Definition 10.14

- An $n+1$-dimensional cobordism of Poincaré complexes (C, ψ) and (C', ψ') is an $n+1$-dimensional quadratic Poincaré pair $(\begin{pmatrix} j_1 & j_2 \end{pmatrix} : C \oplus C' \to D, (\delta\psi, \psi \oplus \psi'))$.
- An $n+2$-dimensional cobordism of Poincaré pairs, $(f : C \to D, (\delta\psi, \psi))$ and $(f' : C' \to$

$D', (\delta\psi', \psi'))$ is an $n+2$-dimensional quadratic Poincaré triad:

$$\left(\begin{array}{ccc} C \oplus C' & \xrightarrow{(\,i_1\ i_2\,)} & \delta C \\ {\scriptstyle (\,f\oplus f'\,)}\downarrow & & {\scriptstyle g}\downarrow \\ D \oplus D' & \xrightarrow{(\,j_1\ j_2\,)} & \delta D \end{array}, (\psi \oplus \psi', \delta\psi \oplus \delta\psi', \chi, \delta\chi)\right)$$

Example 10.15 (Forms again) As before, a highly connected degree 1 normal map $f : M \to X^{2k}$ determines a Poincaré complex with $C_k = K^k(M)$. If $f : M \to X$ bounds $g : N \to Y$ with $\pi_1(N) \cong \pi_1(Y)$ then it was seen before that g can be made highly connected so that $K_j(N) = 0 (j \neq k+1)$, when the map $K_{k+1}(N, M) \to K_k(M)$ is the inclusion of a Lagrangian. Then letting $D_k = K^{k+1}(N, M)$, and setting $j : C_k \to D_k$ y $j = \partial^* : K^k(M) \to K^{k+1}(N, M)$, $(j : C \to D, (0, \psi))$ is a null-cobordism of (C, ψ).

The identification is reversible so that given a highly connected map $f : M \to X^{2k}$ and a cobordism $j : C \to D$, D^k is a stably free $\mathbb{Z}[\pi_1(X)]$-module; then the inclusion $j^* : D^k = \mathbb{Z}[\pi]^n \to C_k$ is such that surgeries can be performed on the images of the generators of D^k giving a homotopy equivalence $f' : M' \to X$.

As geometric cobordisms can be glued together, so algebraic cobordisms can be glued together:

Proposition 10.16 (Glueing, Ranicki([7], p.77)) Let $(\begin{pmatrix} j_1 & j_2 \end{pmatrix} : C \oplus C' \to D, (\delta\psi, \psi \oplus \psi'))$ and $(\begin{pmatrix} j'_1 & j'_2 \end{pmatrix} : C' \oplus C'' \to D', (\delta\psi', \psi' \oplus \psi''))$ be $n+1$-dimensional Poincaré cobordisms. Let D'' be the chain complex defined by $D''_r = D_r \oplus C'_{r-1} \oplus D'_r$, with differential given by

$$d_{D''} = \begin{pmatrix} d_D & (-1)^{r-1} j_2 & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-1)^{r-1} j'_1 & d'_D \end{pmatrix}$$

and define $\delta\psi \in (W_{\%} D'')_{n+1}$ by

$$\delta\psi''^r_s = \begin{pmatrix} \delta\psi^r_s & 0 & 0 \\ (-1)^{n-r} \psi'^r_s j_2^* & (-1)^{n-r-s-1} (\psi^{n-r-s}_{s+1})^* & 0 \\ 0 & (-1)^s j'_1 \psi'^r_s & \delta\psi'^r_s \end{pmatrix}.$$

Then the result of glueing the cobordisms along the common boundary component (C', ψ') is the Poincaré cobordism:

$$\left(\begin{pmatrix} j_1 & 0 \\ 0 & 0 \\ 0 & j'_2 \end{pmatrix} : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus \psi'') \right).$$

and there is also a relative version:

Proposition 10.17 (Glueing, Ranicki([7], p.117)) Let Γ and Γ' be cobordisms of pairs:

$$\begin{array}{ccc} C \oplus C' & \xrightarrow{f \oplus f'} & D \oplus D' \\ {\scriptstyle (\,g\ g'\,)}\downarrow & \searrow^{(\,k\ k'\,)} & \downarrow{\scriptstyle (\,h\ h'\,)} \\ \delta C & \xrightarrow[\delta f]{} & \delta D \end{array} \quad \text{and} \quad \begin{array}{ccc} C' \oplus C'' & \xrightarrow{f' \oplus f''} & D \oplus D' \\ {\scriptstyle (\,\tilde{g}'\ g''\,)}\downarrow & \searrow^{(\,k'\ k''\,)} & \downarrow{\scriptstyle (\,\tilde{h}'\ h''\,)} \\ \delta C' & \xrightarrow[\delta f']{} & \delta D' \end{array}$$

and let $(\delta\chi, \chi, \delta\psi \oplus -\delta\psi', \psi \oplus -\psi')$ and $(\delta\chi', \chi', \delta\psi' \oplus -\delta\psi'', \psi' \oplus -\psi'')$ be cycles in $W_{\%}(\Gamma)$ and $W_{\%}(\Gamma')$ respectively, so that there are determined Poincaré cobordisms of pairs with a common boundary component. Then the union is the cobordism, with

$$\Gamma'' = \begin{tikzcd} C \oplus C'' \arrow[r, "f \oplus f''"] \arrow[d, "(\tilde{g}\ \tilde{g}')"'] \arrow[dr, "(\tilde{k}\ \tilde{k}')"] & D \oplus D'' \arrow[d, "(\tilde{h}\ \tilde{h}')"] \\ \delta C'' \arrow[r, "\delta f''"'] & \delta D'' \end{tikzcd}$$

with:

$$\begin{aligned}
\delta C'' &= \delta C_r \oplus C'_{r-1} \oplus \delta C'_r \\
d_{\delta C''} &= \begin{pmatrix} d_{\delta C} & g' & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-1)^{r-1}\tilde{g}' & d_{\delta C'} \end{pmatrix} \\
\delta D'' &= \delta D_r \oplus D'_{r-1} \oplus \delta D'_r \\
d_{\delta D''} &= \begin{pmatrix} d_{\delta D} & h' & 0 \\ 0 & d_{D'} & 0 \\ 0 & (-1)^{r-1}\tilde{H}' & d_{\delta D'} \end{pmatrix} \\
\begin{pmatrix} \tilde{g} & \tilde{g''} \end{pmatrix} &= \begin{pmatrix} g & 0 \\ 0 & 0 \\ 0 & g'' \end{pmatrix} \\
\begin{pmatrix} \tilde{h} & \tilde{h''} \end{pmatrix} &= \begin{pmatrix} h & 0 \\ 0 & 0 \\ 0 & h'' \end{pmatrix} \\
\begin{pmatrix} \tilde{k} & \tilde{k''} \end{pmatrix} &= \begin{pmatrix} k & 0 \\ 0 & 0 \\ 0 & k'' \end{pmatrix} \\
\chi''_s &= \begin{pmatrix} \chi_s & 0 & 0 \\ (-1)^{n-r+1}\psi'_s g'^* & (-1)^{n-r-s}T\psi'_{s+1} & 0 \\ 0 & (-1)^s g'\psi'_s & \chi'_s \end{pmatrix} \\
\delta\chi''_s &= \begin{pmatrix} \delta\chi_s & 0 & 0 \\ (-1)^{n-r+1}\delta\psi'_s h'^* & (-1)^{n-r-s}T\delta\psi'_{s+1} & 0 \\ 0 & (-1)^s \tilde{h}'\psi'_s & \delta\chi'_s \end{pmatrix}
\end{aligned}$$

10.3 Algebraic surgery

In the same way that two normal maps are bordant if and only if they are related by a sequence of surgeries, there is a corresponding notion of algebraic surgery so that Poincaré complexes (C, ψ) and (C', ψ') are cobordant if and only if (C', ψ') is homotopy equivalent to the result of surgery on (C, ψ). In 'Topology of high-dimensional manifolds' ([14]), Ranicki described how algebraic surgery can be used to calculate the result of geometric surgery. We review this material now, as we shall need to use it later on.

Definition 10.18 (Ranicki([12]), pg. 145) Let (C, ψ) be an n-dimensional quadratic complex.

- *Surgery data* is an $(n+1)$-dimensional quadratic pair (not necessarily Poincaré)

$$(j : C \to D, (\delta\psi, \psi)).$$

- The *result of surgery* on the data is the quadratic Poincaré complex (C', ψ') where

$$\begin{aligned}
C_r &= C_r \oplus D_{r+1} \oplus D^{n-r+1} \\
d_{C'} &= \begin{pmatrix} d_C & 0 & (-1)^{n+1}(1+T_\epsilon)\psi_0 f^* \\ (-1)^r f & d_D & (-1)^r(1+T_\epsilon)\delta\psi_0 \\ 0 & 0 & (-1)^r d_D^* \end{pmatrix} \\
{\psi'}_0^r &= \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
{\psi'}_s^r &= \begin{pmatrix} \psi_s & (-1)^{r+s}T_\epsilon\psi_{s-1}f^* & 0 \\ 0 & (-1)^{n-r-s}T_\epsilon\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

- The *trace* of the surgery is the $(n+1)$-dimensional quadratic Poincaré cobordism

$$((g \quad g') : C \oplus C' \to D', (p, \psi \oplus -\psi'))$$

  where

$$\begin{aligned}
D'_r &= C_r \oplus D^{n+1-r} \\
d^r_{D'} &= \begin{pmatrix} d_C & (-1)^{n+1}(1+T_\epsilon)\psi_0 f^* \\ 0 & (-1)^{r+1}d_D^* \end{pmatrix} \\
g &= \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : C_r \to D'_r \\
g' &= \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : C'_r \to D'_r
\end{aligned}$$

This gives the effect of geometric surgeries in the following way:

Lemma 10.19 (Ranicki ([14])) Suppose that (C, ψ) is the quadratic kernel of an n-dimensional degree 1 normal map $f : M \to X$. Let $D = \ldots \to 0 \to D_{n-m} \to 0 \to \ldots$ for some $n \geq m \geq n/2$, with $D_k = \mathbb{Z}[\pi]^l = \langle e_1, \ldots, e_l \rangle$ and $(j : C \to D, (\delta\psi, \psi))$ be surgery data as above. Then C' is the result of surgeries on the homology classes $(1+T)\psi_0 j^* e_i \in K_k(M)$.

Moreover, if the surgeries succeed in giving a homotopy equivalence, then the surgery data is just the trace:

Lemma 10.20 The following are equivalent:

- The surgery data $(j : C \to D, (\delta\psi, \psi))$ is a Poincaré pair;
- The result of surgery (C', ψ') is contractible and the trace of surgery is homotopy equivalent as a Poincaré pair to $(j : C \to D, (\delta\psi, \psi))$.

We shall also need to compute the effect of surgery on the interior of manifolds with boundary (algebraically on Poincaré pairs). This is accomplished by means of the following correspondence between Poincaré pairs and (non-Poincaré) quadratic complexes.

Definition 10.21 (Ranicki [12] pp. 141–144) Let $(f : C \to D, (\delta\psi, \psi))$ be an $(n+1)$-dimensional Poincaré pair and let (E, χ) be an $(n+1)$-dimensional quadratic complex.

- The *boundary* of (E,χ) is the Poincaré pair $\partial(E,\chi) = (\partial E, \partial\chi)$ where

$$\begin{aligned}
\partial E_r &= E_{r+1} \oplus E^{n+1-r} \\
d^r_{\partial E} &= \begin{pmatrix} d_E & (-1)^{r+1}(1+T)\chi_0 \\ 0 & (-1)^{r+1} d^*_E \end{pmatrix} \\
\partial\psi^r_s &= \begin{pmatrix} (-1)^{n-r-s-1} T_\epsilon \chi^{r+1}_{s-1} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

- The *Poincaré thickening* of (E,χ) is the Poincaré pair

$$(i_E : \partial E \to E^{n+1-*}, (0, \partial\chi))$$

  where $i_E = \begin{pmatrix} 0 & 1 \end{pmatrix} : \partial E_r = E_{r+1} \oplus N^{n+1-r} \to D^{n+1-r}$.

- The *algebraic Thom complex* of $(f : C \to D, (\delta\psi, \psi))$ is the quadratic complex (C', ψ') where

$$\begin{aligned}
C' &= C(f) \\
{\psi'}^r_s &= \begin{pmatrix} \delta\psi_s & 0 \\ (-1)^{n-r}\psi^{r-1}_s f^* & (-1)^{n-r-s} T_\epsilon \psi^{r-1}_{s+1} \end{pmatrix}
\end{aligned}$$

Proposition 10.22 (Ranicki [7], Prop. 1.3.3) The algebraic Thom complex construction and algebraic Poincaré thickening operations are inverse to each other up to homotopy equivalence, defining a natural 1-1 correspondence between homotopy equivalence classes of $(n+1)$-dimensional Poincaré pairs and homotopy equivalence classes of $(n+1)$-dimensional quadratic complexes. The correspondence preserves boundaries.

Moreover, algebraic surgery does not change the homotopy type of the boundary, so the effect of surgery on the interior of a Poincaré pair can be computed by as the Poincaré thickening on the result of surgery on the algebraic Thom complex.

Remark 10.23 We can also perform handle additions on the boundary. Suppose that

$$(f : C \to D, (\delta\psi, \psi))$$

is a Poincaré pair, and

$$(j : C \to E, \delta\chi, \psi)$$

is surgery data on the boundary. Then the letting the result of the surgery be (C', ψ') and the trace of the surgery be the cobordism $(\begin{pmatrix} g & g' \end{pmatrix} : C \oplus C' \to E', (0, \psi \oplus \psi'))$, the result of handle additions is the union

$$(f : C \to D, (\delta\psi, \psi)) \cup (\begin{pmatrix} g & g' \end{pmatrix} : C \oplus C' \to E', (0, \psi \oplus \psi')),$$

which is a Poincaré pair of the form $(f' : C' \to D', (\delta\psi', \psi'))$.

Chapter 11

Surgery and Splitting Obstruction Groups

In the final chapter of 'Surgery on Compact Manifolds' ([17]), after defining the L-groups separately in the odd and even-dimensional cases, Wall suggested that it should be possible to replace these definitions of the L-groups with one in terms of some kind of generalized quadratic form on chain complexes, independent of the polarity of the dimension.

This was completed by Ranicki in 'The Algebraic Theory of Surgery' (Ranicki [12]). The techniques developed in these papers were then applied in 'Exact Sequences in the Algebraic Theory of Surgery' ([7]), section 7, to give a definition of the UNil groups independent of the polarity of the dimension of the splitting problem. In Ranicki([10]), the odd-dimensional L groups were defined in terms of 'short odd complexes', a slight refinement of 1-dimensional complexes in the theory above and it was shown that every presentation determines a short odd complex.

This then will be our outline for the next few chapters:

In this chapter, we shall recall the chain complex version of the definition of the L-groups, and give (a slight reworking of) Ranicki's definition of the UNil groups in terms of algebraic splitting problems. Furthermore, we shall give a map from this group into the corresponding surgery obstruction group, analogous to the even-dimensional nilpotent normal cobordism.

In the next section, we shall again restrict to the odd-dimensional splitting problems, and define UNil_{2k+3} as a group of 'short odd nilcomplexes', which will be highly connected algebraic splitting problems. We shall need to use the notion of a highly connected cobordism of splitting problems, which we shall show is an equivalence relation. We shall also see that the construction of this section gives a well-defined map $\mathrm{UNil}_{2k+3} \to L_{2k+3}(\mathbb{Z}[G])$.

In the final chapter, we shall show that every highly connected odd-dimensional splitting problem determines a well-defined element of this UNil group, such that the obstruction vanishes if and only if the splitting problem is soluble.

11.1 Surgery obstruction groups

Lemma 11.1 Cobordism is an equivalence relation on n-dimensional Poincaré complexes.

Proof.

- Symmetry is clear;
- Transitivity follows from the glueing formula;
- Reflexivity follows from the fact that $(\begin{pmatrix}1 & 1\end{pmatrix} : C \oplus C \to C, (0, \delta\psi \oplus -\delta\psi))$ is a Poincaré pair.

□

Definition 11.2 $L_n(\mathbb{Z}[\pi])$ is the group of cobordism classes of quadratic Poincaré complexes of f.g. free $\mathbb{Z}[\pi]$-modules.

Example 11.3 (Construction of $L_{2k}(\mathbb{Z}[\pi])$) Let K be a f.g. free $\mathbb{Z}[\pi]$-module, and let C be the chain complex with $C_k = K^*$, $C_j = 0$ else. Then $(W_{\%}C)_{2k} = (W_{\%}C)_{2k+1} = \mathrm{Hom}(K, K^*)$, and the differential is given by $d(\chi) = \chi + \chi^*$.

So $Q_{2k}(C)$ is precisely the equivalence classes of split quadratic forms over K, which are given by the surgery obstruction.

Now let D be the chain complex with $D_k = L^*$, and let $j^* : C_k \to D_k$. $(W_{\%}D)_{2k+1} = \mathrm{Hom}(L, L^*)$. Suppose that $(j : C \to D, (\delta\psi, \psi))$ is a quadratic pair. Then $\psi = \psi_0 : K \to K^*$, $\delta\psi = \delta\psi_1 : L \to L^*$, and $d(\psi, \delta\psi) = 0 \implies \delta\psi_1 + \delta\psi_1^* = j\psi j^*$, so that j is the inclusion of a Lagrangian in a split form $(j, \delta\psi); (L, 0) \to (K, \psi)$.

11.2 Splitting obstruction groups

In ([7]), Ranicki defined the LS and UNil groups in terms of Poincaré complexes, and provided a dimension-invariant definition of the UNil groups. In this section, we recall (a slight specialization of) this definition, and show how it gives rise to a map from the UNil groups defined in this way, to the L groups defined in terms of chain complexes. We show that this agrees with the previous definitions in the case of highly connected even-dimensional codimension 1 splitting problems. Note — in this section the letters P and Q will, unless otherwise specified, be free $\mathbb{Z}[H]$-module chain complexes.

11.2.1 Nilcomplexes

Definition 11.4 A nilcomplex $CS = (P, Q; \rho_1, \rho_2)$ consists of:

- Free $\mathbb{Z}[H]$-module chain complexes P and Q;
- Chain maps $\rho_1 : P \to Q \otimes \mathbb{Z}[G_1]$, $\rho_2 : Q \to P \otimes \mathbb{Z}[G_2]$

such that $\left(\begin{smallmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{smallmatrix}\right)$ is a homotopy equivalence (of $\mathbb{Z}[G]$-modules).

Definition 11.5 A map of nilcomplexes

$$F = (f_P, f_Q; k_P, k_Q) : CS = (P, Q; \rho_1, \rho_2) \to CS' = (P', Q'; \rho'_1, \rho'_2)$$

is a pair of chain maps $f_P : P \to P'$ and $f_Q : Q \to Q'$, with homotopies $k_1 : f_Q\rho_1 \simeq \rho_1 f_P$, and $k_2 : f_P\rho_2 \simeq \rho_2 f_Q$.

Lemma 11.6 Suppose that $(P, Q; \rho_1, \rho_2)$ is a nilcomplex and that $f_P : P \to P'$ and $f_Q : Q \to Q'$ are homotopy equivalences. Then there exists a nilcomplex $(P', Q'; \rho'_1, \rho'_2)$ and homotopies k_P, k_Q such that $(f_P, f_Q; k_P, k_Q)$ is a map of nilcomplexes.

Proof. Choose homotopy inverses f_P and f_Q for g_P and g_Q respectively. Let $\rho'_1 = f_Q\rho_1 g_P$ and $\rho'_2 = f_P\rho_2 g_Q$. Let h_P be such that $g_P f_P - 1 = dh_P + h_P d$, h_Q be such that $g_Q f_Q - 1 = dh_Q + h_Q d$. Then $f_P\rho_2 - \rho'_2 f_Q = f_P\rho_2(1 - g_Q f_Q) = f_P\rho_2(dk_Q + h_Q d) = d(f_P\rho_2 h_Q) + (f_P\rho_2 h_Q)d$ and similarly for ρ_1.

Then $(f_P, f_Q; f_Q\rho_1 h_P, f_P\rho_2 h_Q)$ is a map of nilcomplexes. □

Lemma 11.7 Suppose that $(f_P, f_Q; k_P, k_Q) : CS \to CS'$ is a map of nilcomplexes, such that each map is a homotopy equivalence. Then there exists a map of nilcomplexes $(g_P, g_Q; k'_P, k'_Q) : CS' \to CS$ such that g_P and g_Q are homotopy inverses for f_P and f_Q respectively.

Proof. Let g_P and g_Q be homotopy inverses for f_P and f_Q. Then $\rho_1 g_P f_P \simeq g_Q f_Q \rho_1 \simeq g_Q \rho'_1 f_P$. So $\rho_1 g_P f_P g_P \simeq g_Q \rho'_1 f_P g_P$, and therefore $\rho_1 g_P \simeq g_Q \rho'_1$. □

11.2.2 Quadratic structures on nilcomplexes

Lemma 11.8 Let $(g : W \to Y^{n+1}, f : M \to X^n)$ be a splitting problem, so that:

- $C(f) \simeq P \oplus Q$;
- $C(g_l) \simeq P$;
- $C(g_r) \simeq Q$;
- $C(g_2) \simeq \mathbb{Z}[G_2] \otimes P$;
- $C(g_1) \simeq \mathbb{Z}[G_1] \otimes Q$;
- $\begin{pmatrix} 1 & 0 \end{pmatrix} : C(f) \to C(g_l)$;
- $\begin{pmatrix} 0 & 1 \end{pmatrix} : C(f) \to C(g_r)$;
- $\begin{pmatrix} 1 & \rho_2 \end{pmatrix} : C(f) \to C(g_2)$;
- $\begin{pmatrix} \rho_1 & 1 \end{pmatrix} : C(f) \to C(g_1)$;

Then $(P, Q; \rho_1, \rho_2)$ is a nilcomplex, and there exists $\theta : P^{n-*} \to Q$ a homotopy equivalence, $\delta\psi^P \in (W_{\%}(\mathbb{Z}[G_2] \otimes P)_{n+1}), \delta\psi^Q \in (W_{\%}(\mathbb{Z}[G_1] \otimes Q)_{n+1})$ such that letting

$$\psi_s = \begin{cases} \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix} & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

the quadratic signatures are:

- $\sigma(f) = (P \oplus Q, \psi)$;
- $\sigma(g_l) = (\begin{pmatrix} 1 & 0 \end{pmatrix} : P \oplus Q \to P, (0, \psi))$;
- $\sigma(g_r) = (\begin{pmatrix} 0 & 1 \end{pmatrix} : P \oplus Q \to Q, (0, \psi))$;
- $\sigma(g_2) = (\begin{pmatrix} 1 & \rho_2 \end{pmatrix} : P \oplus Q \to \mathbb{Z}[G_2] \otimes P, (\delta\psi^P, \psi))$;
- $\sigma(g_1) = (\begin{pmatrix} \rho_1 & 1 \end{pmatrix} : P \oplus Q \to \mathbb{Z}[G_1] \otimes Q, (\delta\psi^Q, \psi))$;

Proof. By Prop. 1.4 (Ranicki [12]), $Q_n(P \oplus Q) \cong Q_n(P) \oplus Q_n(Q) \oplus \mathrm{Hom}(P^{n-*}, Q)$. Let $\psi = \sigma(f) = \begin{pmatrix} \alpha & 0 \\ \theta & \beta \end{pmatrix}$, where $\alpha \in (W_{\%}P)_n, \beta \in (W_{\%}Q)_n, \theta \in \mathrm{Hom}(P^{n-*}, Q)$ and let $\sigma(g_l) = (\begin{pmatrix} 1 & 0 \end{pmatrix} : P \oplus Q \to P, (\chi, \psi))$ (so $\theta_s = 0$ for $s \geq 1$). The content of the lemma is that α and β can be taken to be 0, and that $\sigma(g_l)$ and $\sigma(g_r)$ are of the form claimed.

Since $\sigma(g_l)$ is a quadratic pair:

$$d(\chi) = \begin{pmatrix} 1 & 0 \end{pmatrix}_{\%}(\psi) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \theta & \beta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha.$$

Then $(0, \left(\begin{smallmatrix} \chi & 0 \\ 0 & 0 \end{smallmatrix}\right)) \in (W_{\%}\begin{pmatrix} 1 & 0 \end{pmatrix})_{n+2}$, and $d(0, \left(\begin{smallmatrix} \chi & 0 \\ 0 & 0 \end{smallmatrix}\right)) = (\chi, \left(\begin{smallmatrix} d(\chi)=\alpha & 0 \\ 0 & 0 \end{smallmatrix}\right))$. Hence

$$(\chi, \psi) = (\chi, \psi) - d(0, \begin{pmatrix} \chi & 0 \\ 0 & 0 \end{pmatrix}) = (0, \begin{pmatrix} 0 & 0 \\ \theta & \beta \end{pmatrix}) \in Q_{n+1}(g_l).$$

Similarly, we can arrange that $\beta = 0$ so that $\psi = \left(\begin{smallmatrix} 0 & 0 \\ \theta & 0 \end{smallmatrix}\right) \in Q_n(f)$, and then $\sigma(g_l)$ and $\sigma(g_r)$ are in the stated form. □

Lemma 11.9 Suppose that θ is a homotopy equivalence. Then $(P \oplus Q, \psi)$ is a Poincaré complex, and $(\sigma(g_i))$ is a Poincaré pair for $i = l, r, 1, 2$.

Proof. Up to sign,

$$(1 + T)\psi_0 = \begin{pmatrix} 0 & T\theta \\ \theta & 0 \end{pmatrix} : P^{n-*} \oplus Q^{n-*} \to P \oplus Q$$

is a chain homotopy equivalence since both θ and $T\theta$ are.

$$C(\begin{pmatrix} 1 & \rho_2 \end{pmatrix})_r = P_r \oplus P_{r-1} \oplus Q_{r-1}, d = \begin{pmatrix} d_P & (-1)^r & (-1)^r\rho_2 \\ 0 & d_P & 0 \\ 0 & 0 & d_Q \end{pmatrix} : P_{r+1} \oplus P_r \oplus Q_r \to P_r \oplus P_{r-1} \oplus Q_{r-1}$$

The Poincaré duality map $P^{n+1-*} \to C(\begin{pmatrix} 1 & \rho_2 \end{pmatrix})$ is given by $\alpha = \begin{pmatrix} (1+T)\delta\psi_0^P \\ T\theta \\ \theta \end{pmatrix}$, which is a homotopy equivalence if and only if the mapping cone is contractible.

The mapping cone is $E_r = P_r \oplus P_{r-1} \oplus Q_{r-1} \oplus P^{n+2-r}$, and the differential is given by:

$$d = \begin{pmatrix} d_P & 1 & \rho_2 & (1+T)\delta\psi_0^P \\ 0 & d_P & 0 & T\theta \\ 0 & 0 & d_Q & \theta \\ 0 & 0 & 0 & d_P^* \end{pmatrix}$$

Let E' be the complex with $E'_r = E_r$ and

$$d' = \begin{pmatrix} d_P & 1 & 0 & 0 \\ 0 & d_P & 0 & 0 \\ 0 & 0 & d_Q & \theta \\ 0 & 0 & 0 & d_P^* \end{pmatrix}$$

The map

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho_2 & (1+T)\delta\psi^P \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a chain isomorphism $E \to E'$, and E' is contractible (since it is the direct sum of the mapping cone of $1 : P \to P$ and $\theta : P^{n-*} \to P$). Hence the Poincaré duality map is a homotopy equivalence, and so $(\begin{pmatrix}1 & \rho_2\end{pmatrix}, (\delta\psi^P, \psi))$ is a Poincaré pair. The case $\begin{pmatrix}\rho_1 & 1\end{pmatrix}$ is similar, and the remaining 2 claims follow from these by applying the above with $\rho_i = 0$ and $\delta\psi^i = 0$. □

Definition 11.10 Let $(P, Q; \rho_1, \rho_2)$ be a nilcomplex. Then define a chain complex $W_{\%}(P, Q; \rho_1, \rho_2)$ by:

$$\begin{aligned} (W_{\%}(P, Q; \rho_1, \rho_2)))_n &= \mathrm{Hom}(P^{n-*}, Q) \oplus (W_{\%}(P \otimes \mathbb{Z}[G_2]))_{n+1} \oplus (W_{\%}(Q \otimes \mathbb{Z}[G_1]))_{n+1} \\ d_{\%}(\theta, \delta^P\psi, \delta^Q\psi) &= (d\theta + (-1)^r\theta d^*, d_{\%}(\delta^P\psi) + (-1)^n\rho_2\theta, d_{\%}(\delta^Q\psi) + (-1)^n\theta\rho_1^*) \end{aligned}$$

Then as before, define $Q_n(P, Q; \rho_1, \rho_2) = H_n(W_{\%}(P, Q; \rho_1, \rho_2)))$, and call a triple $(\theta, \delta^P\psi, \delta^Q\psi)$ a quadratic structure on $(P, Q; \rho_1, \rho_2)$.

Definition 11.11 An *n-dimensional quadratic nilcomplex* is a pair $(CS, \underline{\psi} = (\theta, \delta^P\psi, \delta^Q\psi))$, where CS is a nilcomplex and $(\theta, \delta^P\psi, \delta^Q\psi)$ is an n-dimensional quadratic structure on it. It is Poincaré if in addition θ is a homotopy equivalence.

Lemma 11.12 A map of nilcomplexes $F : CS \to CS'$ induces a map $F_{\%} : W_{\%}(CS) \to W_{\%}(CS')$.

Proof. Define

$$F_{\%}(\theta, \delta^P\psi, \delta^Q\psi) = ((-1)^n f_Q\theta f_P^*, (-1)^n f_P\delta^P\psi f_P^* + (-1)^{n+1}k_2\theta f_P^*, (-1)^n f_Q\delta^Q\psi f_Q^* + (-1)^{n+r+1}f_Q\theta k_1^*)$$

□

Definition 11.13 Given a map of nilcomplexes F as above, define $Q_n(F) = H_n(C(F_{\%}))$.

Definition 11.14 An $(n+1)$-dimension quadratic nilpair $(F : CS \to DS, (\underline{\chi}, \underline{\psi}))$ is:

- An n-dimensional quadratic nilcomplex $(CS, \underline{\psi})$;

- A map of nilcomplexes $F = (f_P, f_Q; k_P, k_Q) : CS \to DS$;
- $(\underline{\chi}, \underline{\psi}) \in Q_{n+1}(F))$ where $\underline{\chi} = (\phi, \delta^P\chi, \delta^Q\chi)$. It is Poincaré if $\begin{pmatrix} \phi \\ \theta f_P^* \end{pmatrix}$ is a chain equivalence

Lemma 11.15 Let $(F : CS \to DS, (\underline{\chi}, \underline{\psi}))$ be a quadratic Poincaré nilpair as above, and $CS = (P, Q; \rho_1, \rho_2)$, $DS = (\hat{P}, \hat{Q}; \hat{\rho}_1, \hat{\rho}_2)$. Then

$$\left(\begin{pmatrix} f^P & 0 \\ 0 & f_Q \end{pmatrix} : P \oplus Q \to \hat{P} \oplus \hat{Q}, \left(\begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix}\right)\right)$$

is a Poincaré pair; $(\Gamma_P, \omega_P \in Q_{n+2}(\Gamma_P))$ is a Poincaré triad, where Γ_P is the triad:

$$\begin{tikzcd}
P \oplus Q \arrow[r, "(1\ \rho_2)"] \arrow[d, "f_P \oplus f_Q"'] \arrow[dr, "(0\ k_P)"] & P \arrow[d, "f_P"] \\
\hat{P} \oplus \hat{Q} \arrow[r, "(1\ \hat{\rho}_2)"'] & P
\end{tikzcd}$$

and $\Gamma_P = \left(\delta^P\chi, \left(\begin{smallmatrix} 0 & 0 \\ \theta & 0 \end{smallmatrix}\right), \delta^P\psi, \left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)\right)$.

Proof. The first statement is immediate from the definition of the Poincaré property of pairs (cf. 11.9). That the structures claimed are quadratic structures, quadratic pair structures, quadratic triad structures etc. is immediate from the definitions. It remains to check that the triad is Poincaré.

Note first that the mapping cone of the triad $C\Gamma$ is homotopy equivalent to $SC(f_Q) : Q \to \hat{Q}$, where S is the suspension of the chain complex. This is because $C(\Gamma)_r = \hat{P}_r \oplus \hat{P}_{r-1}\hat{Q}_{r-1} \oplus P_{r-1} \oplus P_{r-2} \oplus Q_{r-2}$ and the map $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is a chain equivalence. The triad is Poincaré if and only if

$$\begin{pmatrix} (1+T)\delta^P\chi_0 \\ \phi^*\hat{\rho}_2^* \\ \phi \\ \rho_2\theta^* k_P^* + (1+T)\delta^P\psi_0 f_P^* \\ \theta^* f_Q^* \hat{\rho}_s^* \\ \theta f_P^* \end{pmatrix} : P^{n+2-r} \to C(\Gamma)_r$$

is a chain equivalence, which is (by the above)true iff $\left(\begin{smallmatrix} \phi \\ \theta f_P^* \end{smallmatrix}\right)\hat{P}^{n+2-r} \to \hat{Q}_{r-1} \oplus Q_{r-2} = SC(f_Q)$ is a chain equivalence. □

Definition 11.16 A cobordism of n-dimensional quadratic Poincaré nilcomplexes $\alpha = (CS, \chi)$ and $\hat{\alpha} = (\widehat{CS}, \hat{\chi})$, $(\beta; \alpha, \hat{\alpha})$ is an $n+1$-dimensional Poincaré quadratic nilpair $(F : CS \oplus \widehat{CS} \to DS, (\delta\chi, \chi \oplus -\hat{\chi})$.

Definition 11.17
$\mathrm{UNil}_{n+2}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$ is the group of cobordism classes of quadratic n-dimensional Poincaré nilcomplexes.

Example 11.18 (Even-dimensional UNil groups) Suppose that P and Q are just chain complexes of the form $0 \to P_k \to 0$ and $0 \to Q_k \to 0$, so ρ_1 and ρ_2 are just module homomorphisms.

Then $(W_{\%}(P,Q;\rho_1,\rho_2))_{2k} = \mathrm{Hom}(P^*,Q) \oplus \mathrm{Hom}(P^*, P\otimes\mathbb{Z}[G_2]) \oplus \mathrm{Hom}(Q^*, Q\otimes\mathbb{Z}[G_1])$, and a cycle is just $(\theta, \delta\psi^P, \delta\psi^Q)$ such that $\rho_2\theta = \delta\psi^P + \delta\psi^{P*}$, and $\rho_1\theta^* = \delta\psi^Q + \delta\psi^{Q*}$, which again is precisely a split UNil form.

Chapter 12

Nilpotent normal cobordism

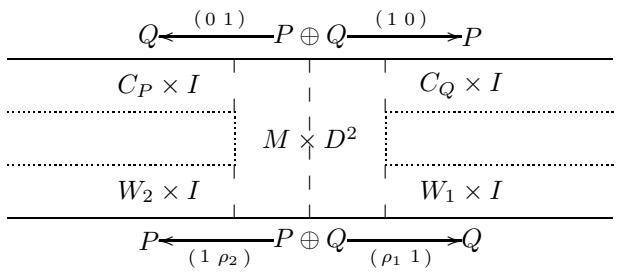
Before we move on to consider odd-dimensional obstruction, we consider the nilpotent normal cobordism in the generality of the preceding chapter. The nilpotent normal cobordism should be interpreted as a cobordism of the original splitting problem

$$P \xleftarrow{(1\ \rho_2)} P \oplus Q \xrightarrow{(\rho_1\ 1)} Q$$

with the (splittable) splitting problem

$$Q \xleftarrow{(0\ 1)} P \oplus Q \xrightarrow{(1\ 0)} P.$$

When $f : M \to X$ is a highly connected map of even-dimensional manifolds, the nilpotent normal cobordism of 9.1 is seen to have this effect algebraically by considering the following diagram. (Note that since C_Q is defined so that $K_{k+1}(C_P, M) = P$, $K_k(C_P) = Q$ and similarly $K_k(C_Q) = P$.



Geometrically, we obtain a map from splitting problems to surgery problems by computing the nilpotent normal cobordism. Algebraically, the following proposition does the same thing:

Proposition 12.1 Given an algebraic splitting problem $\chi \in \mathrm{UNil}_{n+2}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$, the algebraic nilpotent normal cobordism is $\alpha(\chi) \in L_{n+2}(\mathbb{Z}[G])$, given by $(P \oplus Q, \hat{\psi})$, with:

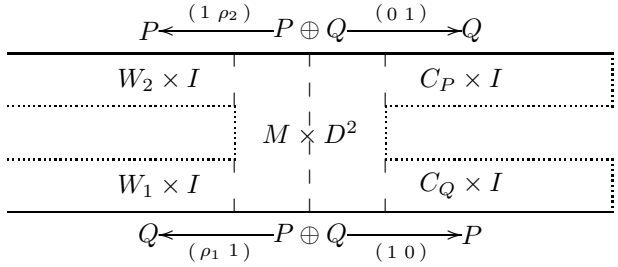
$$\hat{\psi}^r_s = \begin{pmatrix} (-1)^{n+r+s}\delta^P\psi^{r-1}_{s+1} & 0 \\ (-1)^r\theta^{r-1} & (-1)^{n+r+s}\delta^Q\psi^{r-1}_{s+1} \end{pmatrix}. \tag{12.1}$$

We use the definition of $L_k(\mathbb{Z}[G])$ as the group of cobordism classes of free Poincaré $\mathbb{Z}[G]$-module complexes; this map is then well-defined.

Lemma 12.2 The surgery obstruction of the nilpotent normal cobordism is given by $(\hat{C}, \hat{\psi})$ with

$$\hat{\psi}^r_s = \begin{pmatrix} (-1)^{n+r+s}\delta^P\psi^{r-1}_{s+1} & 0 \\ (-1)^r\theta^{r-1} & (-1)^{n+r+s}\delta^Q\psi^{r-1}_{s+1} \end{pmatrix}.$$

Proof. We construct the nilpotent normal cobordism algebraically by mimicking Cappell's nilpotent normal cobordism construction. The nilpotent normal cobordism is to be subdivided as $(C_P \cup_M C_Q) \times I \cup_{M\times I} W \times I$ as in the following diagram, where again the dotted lines represent homotopy equivalences.



Hence, for the purposes of algebraic glueing, it can be considered as the union of two null-cobordisms of pairs corresponding to

$$((W \times I, W_1 \sqcup W_2); (M \times I, M \sqcup M), (W, 0))$$

and

$$(((C_P \cup_M C_Q) \times I, C_P \sqcup C_Q); (M \times I, M \sqcup M), (C_P \cup_M C_Q, M_P \sqcup M_Q)).$$

Thus on the chain level, the surgery obstruction of the nilpotent normal cobordism is the union of glueing two triads:

$$\begin{tikzcd}
D^0 = P\oplus Q \arrow[d] & C = P\oplus Q\oplus P\oplus Q \arrow[l, "f_0=\left(\begin{smallmatrix}1 & \rho_2 & 0 & 0\\ 0 & 0 & \rho_1 & 1\end{smallmatrix}\right)"'] \arrow[r, "f_1=\left(\begin{smallmatrix}0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{smallmatrix}\right)"] \arrow[d, "g=(1\ 1)"] & D^1 = P\oplus Q \arrow[d] \\
0 & D^2 = P\oplus Q \arrow[l] \arrow[r] & 0
\end{tikzcd}$$

with quadratic structures

$$\psi = \begin{pmatrix} 0&0&0&0\\ 0&\psi&0&0\\ 0&0&0&0\\ 0&0&\psi&0 \end{pmatrix} \in Q_n(C) \tag{12.2}$$

$$\delta\psi^0 = \begin{pmatrix} \delta^P\psi & 0 \\ 0 & \delta^Q\psi \end{pmatrix} \in Q_{n+1}(f_0) \tag{12.3}$$

$$\delta\psi^1 = 0 \in Q_{n+1}(f_1) \tag{12.4}$$

$$\delta\psi^2 = 0 \in Q_{n+1}(g) \tag{12.5}$$

By Ranicki([7]), the result of the glueing is the Poincaré pair

$$\begin{pmatrix} 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1 \end{pmatrix} : C'_r \to D'_r$$

with

$$C'_r = P_{r-1} \oplus Q_{r-1} \oplus P_{r-1} \oplus Q_{r-1} \oplus P_r \oplus Q_r \oplus P_r \oplus Q_r \tag{12.6}$$

$$D'_r = P_{r-1} \oplus Q_{r-1} \tag{12.7}$$

$$d'_C = \begin{pmatrix} d_P & 0 & 0 & 0 & (-1)^r & (-1)^r\rho_2 & 0 & 0 \\ 0 & d_Q & 0 & 0 & 0 & 0 & (-1)^r\rho_1 & (-1)^r \\ 0 & 0 & d_P & 0 & 0 & 0 & (-1)^r & 0 \\ 0 & 0 & 0 & d_Q & 0 & (-1)^r & 0 & 0 \\ 0 & 0 & 0 & 0 & d_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_P & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_Q \end{pmatrix} \tag{12.8}$$

$$d'_D = \begin{pmatrix} d_P & 0 \\ 0 & d_Q \end{pmatrix} \tag{12.9}$$

$$\psi' = \begin{pmatrix} \delta^P\psi^r_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^Q\psi^r_s & 0 & 0 & \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (-1)^s\theta^r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (-1)^{n-r}\theta^{r-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (-1)^{n-r}\theta^{r-1}\rho_1^* & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{12.10}$$

$$\delta\psi' = 0 \tag{12.11}$$

Now d'_r is of the form $\begin{pmatrix} d & (-1)^r A \\ 0 & d \end{pmatrix}$, with

$$A = \begin{pmatrix} 1 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\rho_2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -\rho_1 & 0 \end{pmatrix}$$

.

Then there is a null-homotopy of the above pair with $(0 : 0 \to D, (\hat{\psi}, 0))$ which is the image of the above structure under the map of pairs given by the triad:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] \arrow[dr, "\delta f"] & D \arrow[d, "1"] \\ 0 \arrow[r] & D \end{tikzcd}$$

By corollary 10.8,

$$\hat{\psi}^r_s = (-1)^{n+2} f\Delta\psi^{r-1}_s f^* + (-1)^{r1+1} kT(\psi^{n+1-r-s}_{s+1})k^* : D^{n+2-r-s} \to D_r$$

where $\Delta = \begin{pmatrix} 0 & (-1)^{r-1}A^{-1} \\ 0 & 0 \end{pmatrix}$, which satisfies $d'_C\Delta + \Delta d'_C = 1$.
This gives that

$$\hat{\psi}_s = \begin{pmatrix} (-1)^{n+s+r}T(\delta^P\psi_{s+1}) + (-1)^{r+s}\rho_2\theta^{r-1} & 0 \\ (-1)^r\theta^{r-1} & (-1)^{n+s+r}T\delta^q\psi^Q_{s+1} \end{pmatrix} \tag{12.12}$$

$$= \begin{pmatrix} (-1)^{n+s+r}T(\delta^P\psi_{s+1}) & 0 \\ (-1)^r\theta^{r-1} & (-1)^{n+s+r}T\delta^q\psi^Q_{s+1} \end{pmatrix}, \tag{12.13}$$

since

$$\rho_2\theta = \begin{pmatrix}1 & \rho_2\end{pmatrix}\begin{pmatrix}0 & 0\\ \theta & 0\end{pmatrix}\begin{pmatrix}1\\ \rho_2^*\end{pmatrix} = 0 \in Q_{n+1}(P\otimes\mathbb{Z}[G_1]).$$

This structure is equal to that claimed since $T(\delta^P\psi) = \delta^P\psi \in Q_{n+1}(P)$. □

The proposition is then proved once the following lemma is proved:

Lemma 12.3 Let $F = (f_P, f_Q; k_P, k_Q) : CS \to DS$ where $DS = (\hat{P}, \hat{Q}; \hat{\rho}_1, \hat{\rho}_2)$ be a pair of nilcomplexes, and let $(F : CS \to DS, (\underline{\chi}, \underline{\psi}))$ be a Poincaré nilpair. Let $(\mathbb{Z}[G]\otimes(P\oplus Q)_{*-1}, \psi^{NNC})$ be the result of the nilpotent normal cobordism on $(CS, \underline{\psi})$. Then there is a Poincaré pair

$$(f_P\oplus f_Q : \mathbb{Z}[G]\otimes(P\oplus Q)_{*-1} \to \mathbb{Z}[G]\otimes(\hat{P}\oplus\hat{Q})_{*-1}, (\delta\psi^{NNC}, \psi^{NNC})),$$

where

$$(\psi^{NNC})^r_s = \begin{pmatrix}(-1)^{n+r+s}\delta^P\psi^{r-1}_{s+1} & 0\\ (-1)^r\theta & (-1)^{n+r+s}\delta^Q\psi\end{pmatrix}$$

and

$$(\delta\psi^{NNC})^r_s = \begin{pmatrix}(\delta\chi^P)^{r-1}_{s+1} & 0\\ \theta^{r-1}_s & (\delta\chi^Q)^{r-1}_{s+1}\end{pmatrix}$$

Note that this apparently is a different structure on the nilpotent normal cobordism, but this $\psi^{NNC} = \alpha(CS, \underline{\psi}) \in Q_{n+2}((P\oplus Q)_{*-1})$ above.

Proof. First check that the structures are indeed quadratic structures. For convenience from now on we omit the NNC subscript in the notation. Let $\underline{\psi} = (\theta, \delta^P\psi, \delta^Q\psi)$ and $\underline{\chi} = (\phi, \delta^P\chi, \delta^Q\chi)$. We must check first that

$$d\psi^{r+1}_s + (-1)^r\psi^r_s d^* + (-1)^{n+2-s-1}(\psi^r_{s+1} + (-1)^{s-1}(T\psi)^r_{s+1}) = 0.$$

It can be checked that $(T\psi)^r_{s+1} = (-1)^{s+r}\begin{pmatrix}T(\delta^P\psi)^{r-1}_{s+2} & 0\\ 0 & T(\delta^Q\psi)^{r-1}_{s+2}\end{pmatrix}$. Hence, checking first the top left entry in the matrix (which is identical to the bottom right):

$$(-1)^{r+s}\Big(d\delta^p\psi^r_{s+1} + (-1)^{r-1}\delta^P\psi^{r-1}_{s+1}d^* + (-1)^{n+1-s}\big(\delta^P\psi^{r-1}_{s+2} + (-1)^sT(\delta^P\psi)^{r-1}_{s+2}\big)\Big) = 0$$

since the bracket is $d(\delta\psi^P)^{r-1}_{s+1}$. The other brackets follow similarly. The verification that the pair structure is a quadratic pair structure is almost identical.

Finally check that the pair is Poincaré: Since

$$\begin{pmatrix}0 & T(\phi)\\ \phi & 0\\ 0 & T(\theta)\\ \theta & 0\end{pmatrix}$$

is a homotopy equivalence, and

$$\begin{pmatrix}1 & -\hat{\rho}_1 & 0 & k_1\\ -\hat{\rho}_2 & 1 & k_2 & 0\\ 0 & 0 & 1 & -\rho_1\\ 0 & 0 & -\rho_2 & 0\end{pmatrix}$$

is a homotopy equivalence,

$$\begin{pmatrix} -\hat{\rho}_1\phi + k_1\theta f^* & T(\phi) \\ \phi & -\hat{\rho}_2 T(\phi) + k_2 T(\theta) f^* \\ -\rho_1\theta f^* & T(\theta) f^* \\ \theta f^* & -\rho_2 T(\theta f^*) \end{pmatrix}$$

is also a homotopy equivalence. Hence

$$\begin{aligned} \begin{pmatrix} (1+T)\delta\psi_0^r \\ (1+T)\psi_0^{r-1} f^* \end{pmatrix} &= \begin{pmatrix} (-1)^{n+1-r}(1-T)\delta^P\psi_1^{r-1} & (-1)^{r+1}T(\phi)^{r-1} \\ (-1)^r\phi^{r-1} & (-1)^{n+1+r}(1-T)\delta^Q\chi_1^{r-1} \\ (-1)^{n+1+r}(1-T)\delta^P\psi_1^{r-1}j^* & (-1)^{r+1}T(\theta)^{r-1} \\ (-1)^r\theta^{r-1} & (-1)^{n+1+r}(1-T)\delta^Q\psi_1^{r-1} \end{pmatrix} \\ &\simeq \begin{pmatrix} (-1)^{r+1}\hat{\rho}_1\phi + k_1\theta f^* & (-1)^{r+1}T(\phi) \\ (-1)^r\phi & (-1)^r T(\phi\hat{\rho}_2^* + f\theta k_2^*) \\ (-1)^r\rho_s\phi & (-1)^r T(\theta) \\ (-1)^r\theta & (-1)^r T(\theta\rho_1^*) \end{pmatrix} \end{aligned}$$

is a homotopy equivalence, hence the pair is Poincaré. □

Note that the above formula for $(1+T)\psi_0$ agrees with the formula given by Cappell for the intersection form on the even-dimensional nilpotent normal cobordism.

Chapter 13

Formations and Short Odd Complexes

The odd-dimensional surgery obstruction groups have been described in several equivalent ways. The first, by Wall in [17], was in terms of automorphisms of forms. Later, they were described by Ranicki as formations, forms with pairs of Lagrangians with hyperbolic forms. As forms had a reformulation in terms of split forms, so formations carried an equivalent notion of split formations.

Once again, we begin by reviewing the surgery obstruction— it is necessary to compute the surgery obstruction of the odd-dimensional nilpotent normal cobordism; and then we extend the ideas to the splitting obstruction.

13.1 Surgery obstruction group

The surgery obstruction group will be described in two ways in this section. The first, in terms of formations, is more closely related to the original definition due to Wall, who described the odd-dimensional L-groups as groups of automorphisms of forms. The second, in terms of short odd complexes, is the highly connected version of the chain complex description of the odd-dimensional L-groups, and was explicitly described by Ranicki in [10].

The definitions of the odd-dimensional UNil groups will be given analogously with the short odd complexes, and when the surgery obstruction of the nilpotent normal cobordism is computed it will be given as a short odd complex. However, where possible connections will be made in the text between short odd complexes and formations and between their UNil equivalents. The reason for this is a trade-off between the merits of short odd nilcomplexes and UNil formations : the odd-dimensional L-groups are described as equivalence classes : representatives of equivalence classes are best described by UNil formations, but the equivalence relation is most easily expressed in terms of short odd nilcomplexes.

13.1.1 Formations

A formation is a quadratic form with a pair of Lagrangians. For surgery problems, a formation describes a Heegaard-type decomposition, although no generalization of this is known for splitting problems. Therefore this will not be described in this thesis; instead an alternative means of obtaining formations will in fact be used.

Definition 13.1

- A quadratic formation over a ring with involution R, $(K, \lambda, \mu; F, G)$ is a nonsingular quadratic form (K, λ, μ) together with an ordered pair of lagrangians (F, G).
- An isomorphism of quadratic formations

$$f : (K, \lambda, \mu) \to (K', \lambda', \mu')$$

is an isomorphism of forms $f; (K, \lambda, \mu) \to (K', \lambda', \mu')$ such that $f(F) = F'$, $f(G) = G'$.

Lemma 13.2 Every quadratic formation is isomorphic to one of the type $(H_\epsilon(F); F, \alpha(F))$ for some automorphism $\alpha : H_\epsilon(F) \to H_\epsilon(F)$.

Definition 13.3

- A formation $T = (K, \lambda, \mu; F, G)$ is *trivial* if it is isomorphic to $(H_\epsilon(F); F, F^*)$.
- A *stable isomorphism* of formations

$$f : (K, \lambda, \mu; F, G) \to (K', \lambda', \mu';' F, G')$$

is an isomorphism of quadratic formations of the type

$$f : (K, \lambda, \mu; F, G) \oplus T \to (K', \lambda', \mu'; F', G') \oplus T'$$

with T and T' trivial.

- Given a $(-\epsilon)$-quadratic form (K, λ, μ), the graph lagrangian is the lagrangian

$$\Gamma_{(K,\lambda)} = \{(x, \lambda(x)) \in K \oplus K^* | x \in K\}$$

in the hyperbolic ϵ-quadratic form $H_\epsilon(K)$.

- The boundary of (K, λ, μ) is the graph formation

$$\partial(K, \lambda, \mu) = (H_\epsilon(K); K; \Gamma_{(K,\lambda)}).$$

- Quadratic formations $(K, \lambda, \mu; F, G)$ and $(K', \lambda', \mu'; F', G')$ are cobordant if there exists a stable isomorphism

$$f : (K, \lambda, \mu; F, G) \oplus B \to (K', \lambda', \mu; F', G') \oplus B'$$

Definition 13.4 The $(2k+1)$-dimensional L-group $L_{2k+1}(A)$ of a ring with involution A is the group of cobordism classes of $(-1)^k$-quadratic formations $(K, \lambda, \mu; F, G)$ over R, with addition and inverses given by:

$$(K_1, \lambda_1, \mu_1; F_1, G_1) + (K_2, \lambda_2, \mu_2; F_2, G_2)$$
$$= (K_1 \oplus K_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2; F_1 \oplus F_2, G_1 \oplus G_2)$$

$$(K, \lambda, \mu; F, G) = (K, -\lambda, -\mu; F, G)$$

13.1.2 Short odd complexes

Definition 13.5 A $2k+1$-dimensional short odd complex (denoted (C, ψ)) over a ring R consists of:

- f.g. free R-modules C_{k+1}, C_k;
- $d : C_{k+1} \to C_k$;
- $\psi_0 : C^k \to C_{k+1}$ where $C^k := C_k^*$;
- $\psi_1 : C^k \to C_k$

such that:

- $d\psi_0 + (\psi_1 + (-1)^{k+1}\psi_1^*) = 0$;
- The chain complex:

$$C^k \xrightarrow{\binom{d^*}{\psi_0}} C^{k+1} \oplus C_{k+1} \xrightarrow{(\psi_0^* \ d)} C_k$$

  is contractible.

In other words, a short odd complex, is just a highly connected $2k+1$-dimensional Poincaré complex (C, ψ) (in the notation of 10.3) with $\psi_0^{k+1} = \psi_0, \psi_1^k = \psi_1, \psi_0^k = 0$.

The equivalence relation is given by (highly connected) Poincaré cobordism:

Definition 13.6 A null-cobordism of (C, ψ), denoted $(j : C \to D, (\delta\psi, \psi))$ is

- A f.g. free R-module D_{k+1};
- $j : C_{k+1} \to D_{k+1}$;
- $\delta\psi : D^{k+1} \to D_{k+1}$;

such that $\begin{pmatrix} (\delta\psi_0 + (-1)^{k+1}\delta\psi_0^* & j \\ \psi_0^* j^* & d \end{pmatrix}$ is an isomorphism.

Definition 13.7 A cobordism of short odd complexes (C, ψ) and (C', ψ'),$(j : C \to D, (\delta\psi, \psi \oplus -\psi'))$, is a null-cobordism of $(C \oplus C', \psi \oplus -\psi')$.

Definition 13.8 A map of short odd complexes $(f,\chi) : (C,\psi) \to (C',\psi')$ is a chain map $f : C \to C'$, $\chi = \{\chi_1 : C'^k \to C'_{k+1}, \chi_2 : C'^k \to C'_k\}$ such that

$$f\psi_0 f^* - \psi'_0 = \chi_1 \qquad \text{and} \qquad f\psi_1 f^* - \psi'_1 = d\chi_1 + \chi_2 + (-1)^k \chi_2^*.$$

A homotopy equivalence of short odd complexes is a map of short odd complexes which is a homotopy equivalences of chain complexes.

Then the odd-dimensional surgery obstruction groups are defined as follows:

Definition 13.9 $L_{2k+1}(R)$ is the group of cobordism classes of $(2k+1)$-dimensional short odd complexes, with addition given by direct sum, and $-(C,\psi) = (C,-\psi)$.

Remark 13.10 The correspondence between formations and short odd complexes is as follows:

Suppose that $(H_{(-1)^{k+1}}(F); F, G)$ is a formation. Let $i = \begin{pmatrix} \gamma \\ \mu \end{pmatrix}$ be the inclusion $G \to F \oplus F^*$. Define a short odd complex (C,ψ) by letting $C_{k+1} = F$, $C_k = G^*$, $d = \mu^*$, $\psi_0 = \gamma$, ψ_1 is any map such that $\gamma\mu^* = \psi_1 + (-1)^{k+1}\psi_1^*$ (since i is the inclusion of a Lagrangian, using 7.13). For details, see Ranicki([10]).

Under this correspondence, two short odd complexes are homotopy equivalent iff the corresponding formations are stably isomorphic, and are cobordant iff the corresponding formations are cobordant. This result has not been extended to short odd nilcomplexes.

13.2 Splitting obstruction group

13.2.1 Short odd nilcomplexes

In 'An Introduction to the Algebraic Theory of Surgery' (Ranicki, [10]), the odd-dimensional L-groups were expressed in terms of short odd complexes - which are essentially highly connected quadratic Poincaré complexes, with equivalence relation given by highly connected cobordisms of quadratic Poincaré complexes. However instead of working with homology classes $[\psi] \in Q_n(C)$, short odd complexes were defined as highly connected complexes together with cycles $\psi \in (W\%C)_n$ of a particular form, and the cobordism relation was such that if $[\psi] = [\psi'] \in Q_n(C)$, then the short odd complexes (C,ψ) and $(C'.\psi')$ are cobordant.

In this section, we do the same thing with nilcomplexes - short odd nilcomplexes are defined in terms of highly connected nilcomplexes CS together with a cycle in $(W_{\%}CS)_n$.

For the sake of clarity, note that in this section, if the symbols P and Q are used, they will refer to $\mathbb{Z}[H]$-modules, not chain complexes.

Definition 13.11 A $(2k+1)$-*dimensional short odd nilcomplex* $(CS_{k+1}, CS_k, d, \theta, \delta\psi^P, \delta\psi^Q)$ consists of:

- $CS_{k+1} = (P_{k+1}, Q_{k+1}; \rho_1, \rho_2) \in \mathfrak{Nil}$;

- $CS_k = (P_k, Q_k; \rho_1, \rho_2) \in \mathfrak{Nil}$; together with morphisms in $\mathfrak{Nil}$
- $d = (d_P, d_Q) : CS_{k_1} \to CS_k$;
- $\theta = (\theta_P, \theta_Q) : CS^k \to CS_{k+1}$ such that $d\theta = (-1)^{k+1}\theta^* d^*$ is a $(-1)^{k+1}$-symmetric UNil form with quadratic refinement $\delta\psi^P, \delta\psi^Q$ such that
- $\delta\psi^P + (-1)^{k+2}\delta\psi^{P*} = \rho_2 d_P \theta_Q$;
- $\delta\psi^Q + (-1)^{k+2}\delta\psi^{Q*} = \rho_1 d_Q \theta_P$,

such that the mapping cone:

$$0 \longrightarrow CS^k \longrightarrow CS^{k+1} \oplus CS_{k+1} \longrightarrow CS_k \longrightarrow 0$$

is contractible.

In the next chapter, we shall show that any highly connected odd-dimensional splitting problem gives rise to a short odd nilcomplex. The connection between this definition and the preceding chapter is given by the following lemma:

Lemma 13.12 Let $(CS_{k+1}, CS_k, \theta, d, \psi_P, \psi_Q)$ be a short odd nilcomplex. Then there is a Poincaré nilcomplex $(\phi, \delta^P\omega, \delta^Q\omega)$ given by:

$$\begin{aligned} \phi_{k+1} &= \theta_Q \\ \phi_k &= \theta_P^* \\ \delta^P\omega &: \end{aligned}$$

$$\begin{tikzcd} P^k \arrow[r] \arrow[dr, "-\rho_2'\theta_Q"'] \arrow[drr, bend right, "\delta\psi_P"'] & P^{k+1} \arrow[d, "0"] \arrow[dr, "0"] & \\ & P_{k+1} \arrow[r, "d_P"'] & P_k \end{tikzcd}$$

Proof. We have to prove that $d\left(\left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)\right) = 0$, and that $d(\delta^P\omega) + (-1)^{2k+1}\left(\begin{smallmatrix} 1 & \rho_2 \end{smallmatrix}\right)_{\%}\left(\left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)\right) = 0$.

Firstly,

$$\begin{aligned} & \begin{pmatrix} d_P & 0 \\ 0 & d_P \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \theta_Q & 0 \end{pmatrix} + (-1)^k \begin{pmatrix} 0 & 0 \\ \theta_P^* & 0 \end{pmatrix}\begin{pmatrix} d_P^* & 0 \\ 0 & d_Q^* \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0 \\ d_Q\theta_Q + (-1)^k\theta_P^* d_P^* & \end{pmatrix} = 0 \end{aligned}$$

since it was assumed, that $d_\theta = (-1)^{k+1}\theta^* d^*$.

Secondly, to check that $\left(\begin{smallmatrix} 1 & \rho_2 \end{smallmatrix}\right)_{\%}\left(\left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)\right) = d_{\%}(\delta^P\omega)$, there are 3 things to check:

$$\begin{aligned} d(\delta^P\omega)_0^{k+1} &= \rho_2'\theta_Q \\ d(\delta^P\omega)_0^k &= \rho_2\theta_P^* \\ d(\delta^P\omega)_1^k &= 0 \end{aligned}$$

Firstly, $d(\delta^P\omega)_0^{k+1} = -\delta^P\omega_1^{k+1} = -(-\rho_2'\theta_Q)$. Secondly, $d(\delta^P\omega)_0^k = (-\rho_2'\theta_Q)^* = -\theta_Q^*\rho_2'^* = \rho_2\theta_P^*$. Finally, $d(\delta^P\omega)_1^k = -d_P\rho_2'\theta_Q + \delta\psi_P + (-1)^k\delta\psi_P^* = 0$.

□

13.2.2 Cobordism of short odd nilcomplexes

We now define highly connected cobordisms:

Definition 13.13 A cobordism of short odd nilcomplexes $\alpha = (CS_{k+1}, CS_k, d, \theta, \delta\psi_2^P, \delta\psi_2^Q)$ and $\alpha' = (CS'_{k+1}, CS'_k, d', \theta', \delta\psi_2'^P, \delta\psi_2'^Q)$: $(\beta = ((\hat{P}, \hat{Q}, \hat{\rho}_1, \hat{\rho}_2), (f_P, f_Q), (\hat{\theta}, \delta\hat{\psi}_1^P, \delta\hat{\psi}_1^Q)); \alpha, -\alpha')$, is:

- $(\hat{P}, \hat{Q}; , \hat{\rho}_1, \hat{\rho}_2) \in \mathfrak{Nil}$
- Maps $\delta\hat{\psi}_P : \hat{P}^* \to \hat{P}$, $\delta\hat{\psi}_Q : \hat{Q}^* \to \hat{Q}$;
- Maps $f_P : P_{k+1} \oplus P'_{k+1} \to \hat{P}$ and $f_Q : Q_{k+1} \oplus Q'_{k+1} \to \hat{Q}$,
- $k_P : Q_{k+1} \oplus Q_{k+1}; \to \hat{P} \otimes \mathbb{Z}[G_1], k_Q : P_{k+1} \oplus P_{k+1}; \to \hat{Q} \otimes \mathbb{Z}[G_1]$

such that

- $f_P(\rho_2 \oplus \rho_2') - \hat{\rho}_2 f_Q = k_P(d_Q \oplus d'_Q)$;
- $f_Q(\rho_1 \oplus \rho_1') - \hat{\rho}_1 f_P = k_Q(d_P \oplus d'_P)$;
- $\delta\psi_P + \delta\psi_P^* + \hat{\rho}_2\hat{\theta} + k_P\hat{\theta}f_P^* = 0$;
- $\delta\psi_Q + \delta\psi_Q^* + \hat{\rho}_1\hat{\theta} + k_Q\hat{\theta}^* f_Q^* = 0$.

Lemma 13.14 Cobordisms according to the above definition are precisely those cobordisms in the previous section which arise from maps of UNil triads from $(P, Q; \rho_1, \rho_2) \oplus (P', Q'; \rho_1', \rho_2') \to (\hat{P}, \hat{Q}; , \hat{\rho}_1, \hat{\rho}_2)$ considering modules as 0-dimensional chain complexes in the obvious way.

It is not obvious that this notion of cobordism is in fact an equivalence relation on the set of short odd nilcomplexes. Symmetry is obvious. To show transitivity, we need to be able to glue 2 cobordisms to give another highly connected cobordism. To show reflexivity, we show that given any short odd nilcomplex, there exists a cobordism with another short odd nilcomplex. Reflexivity then follows by applying symmetry and transitivity.

Lemma 13.15 The union of 2 highly connected cobordisms is again a highly connected cobordism.

Proof. Suppose that $\alpha, \alpha', \alpha''$ are short odd nilcomplexes, and let $(\beta; \alpha, -\alpha')$ and $(\beta'; \alpha', -\alpha'')$ be cobordisms. Then by lemma 13.14, the cobordisms determine Poincaré nilpairs. Apply lemma 10.17 giving another Poincaré nilpair, $\hat{\beta} = (\hat{P}, \hat{Q}, \ldots)$.

Claim: $H_j(P) = H_j(Q) = 0$ unless $j = k + 1$.

Proof: Given a nilcomplex, $\alpha = (P, Q, \ldots)$ define $C_\alpha = P \oplus Q$. Given a cobordism $\beta = (P, Q, \ldots)$, define $D_\beta = P \oplus Q$.

Then $D_{\hat\beta} = D_\beta \cup_{C_{\alpha'}} D_{\beta'}$, and there is a Mayer-Vietoris sequence:

$$\cdots \longrightarrow H_j(C_{\alpha'}) \longrightarrow H_j(D_\beta) \oplus H_j(D_{\beta'}) \longrightarrow H_j(D_{\hat\beta}) \longrightarrow H_{j-1}(C_{\alpha'}) \longrightarrow \cdots$$

For $j \leq k$ and $j \geq k+3$, since D_β and $D_{\beta'}$ are highly connected cobordisms, and $C_{\alpha'}$ is a short odd nilcomplex, there is an exact sequence

$$H_j(D_\beta) \oplus H_j(D_{\beta'}) \longrightarrow H_j(D_{\hat\beta}) \longrightarrow H_{j-1}(C_{\alpha'})$$

so $H_j(D_{\hat\beta}) = 0$ since all the other terms are.

It remains to consider $H_{k+2}(D_{\hat\beta})$. By Poincaré-Lefschetz duality, this is isomorphic to $H^k(C_\alpha \oplus C_{\alpha''} \to D_{\hat\beta}) \cong H_k(C_\alpha \oplus C_{\alpha''} \to D_{\hat\beta})^*$. From the long exact sequences of the pairs, $H_k(C_\alpha \to D_\beta) = 0 = H_k(C_{\alpha''} \to D_{\beta'})$. Then there is a Mayer-Vietoris sequence

$$\cdots \longrightarrow H_k(C_\alpha \to D_\beta) \oplus H_k(C_{\alpha''} \to D_{\beta'}) \longrightarrow H_k(C_\alpha \oplus C_{\alpha''} \to D_{\hat\beta}) \longrightarrow H_{k-1}(C_{\alpha'}) \longrightarrow \cdots$$

Hence $0 = H_k(C_\alpha \oplus C_{\alpha''} \to D_{\hat\beta})$, so $H_{k+2}(D_{\hat\beta}) = 0$ □

The following purely algebraic result will be seen later to have a geometric background — it also guarantees that given a short odd nilcomplex α, there exists a short odd nilcomplex α' and a highly connected cobordism $(\beta; \alpha, \alpha')$

Lemma 13.16 Given a short odd nilcomplex $(CS_{k+1}, CS_k, d = (d_P, d_Q), \theta = (\theta, \theta'), \delta\psi_2^P, \delta\psi_2^Q)$, there exists a highly connected cobordism between

$$\alpha = (CS_{k+1}, CS_k, d, \theta, \delta\psi_2^P, \delta\psi_2^Q)$$

and

$$\alpha' = (CS^{k+1}, CS_k, (-1)^k\theta^*, d^*, (-1)^{k+1}\delta\psi_2^{P*}, (-1)^{k+1}\delta\psi_2^{Q*})$$

Proof. First we must check that α' is a short odd nilcomplex. All properties are clear apart from the quadratic refinement, where $(-1)^{k+1}\delta\psi^{P*} + (-1)^{k+2}((-1)^{k+1}\delta\psi^{P*})^* = -(\delta\psi^P + (-1)^{k+2}\delta\psi^{P*}) = -(\rho_2 d_P \theta_Q) = (-1)^{k+1}\theta_Q^*\rho_2^* d_P^* = \rho_2(-1)^k\theta_Q^* d_P^*$.

Now we construct a (not highly connected) cobordism. However it will be homotopy equivalent to a highly connected cobordism. In particular:

Let

- $\hat P_{k+1} = P_{k+1} \oplus Q^{k+1}, \hat Q_{k+1} = Q_{k+1} \oplus P^{k+1}$;
- $\hat P_k = P_k, \hat Q_k = Q_k$;
- $\hat d_P = \begin{pmatrix} d_P & (-1)^k\theta_Q^* \end{pmatrix}, \hat d_Q = \begin{pmatrix} d_Q & (-1)^k\theta_P^* \end{pmatrix}$
- ρ_2 is the chain map $\begin{pmatrix} \rho_2 & 0 \\ 0 & -\rho_2^* \end{pmatrix}, \begin{pmatrix} \rho_2 \end{pmatrix}$, similarly ρ_1;

- $\hat{\theta}$ is the chain map $\left(\begin{smallmatrix} 0 & -1 \\ (-1)^k & 0 \end{smallmatrix}\right) : \hat{P}^{k+1} \to \hat{Q}_{k+1}$;

- $\delta\chi^P \in (W_{\%}\hat{P})_{2k+3}$ is the structure:

$$\begin{tikzcd}
\hat{P}^k = P^k \arrow[r, "\left(\begin{smallmatrix} d_P^* \\ \theta \end{smallmatrix}\right)"] \arrow[drr, "\left(\begin{smallmatrix} \rho_2\theta \\ 0 \end{smallmatrix}\right)"'] & \hat{P}^{k+1} = P^{k+1} \oplus Q_{k+1} \arrow[drr, "\left(\begin{smallmatrix} 0 & 0 \\ (-1)^k\rho_2^* & 0 \end{smallmatrix}\right)"] \arrow[drr, "0"] & & \\
& & \hat{P}_{k+1} = P_{k+1} \oplus Q^{k+1} \arrow[r, "(d\ \theta^*)"'] & \hat{P}_k = P_k
\end{tikzcd}$$

  and $\delta\chi^Q$ similarly.

- f_P is the chain map

$$\begin{tikzcd}
P_{k+1} \oplus Q^{k+1} \arrow[r, "\left(\begin{smallmatrix} d_P & 0 \\ 0 & (-1)^k\theta \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & P_k \oplus P_k \arrow[d, "(1\ 1)"] \\
P_{k+1} \oplus Q_{k+1} \arrow[r, "(d_P\ (-1)^k\theta)"'] & P_k
\end{tikzcd}$$

  and f_Q is similar.

Now we claim that $(\hat{P}, \hat{Q}, \hat{\rho}_1, \hat{\rho}_2, \hat{\theta}, \delta\chi^P, \delta\chi^Q)$ is a cobordism in the sense of the preceding chapter.

First we have to show that $f(\theta \oplus -d^*)f^* = \hat{d}\hat{\theta} + (-1)^r\hat{\theta}\hat{d}^*$. There are only two non-zero terms:

$$\begin{aligned}
\hat{d}\hat{\theta} &= \begin{pmatrix} d_Q & (-1)^k\theta_P^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ (-1)^k & 0 \end{pmatrix} = \begin{pmatrix} \theta_P^* & -d_Q \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \theta_P^* & 0 \\ 0 & -d_Q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \hat{P}^{k+1} \to \hat{Q}_k \\
(-1)^r\hat{\theta}\hat{d}^* &= (-1)^{k+1}\begin{pmatrix} 0 & -1 \\ (-1)^k & 0 \end{pmatrix}\begin{pmatrix} d_P^* \\ (-1)^k\theta_Q \end{pmatrix} = \begin{pmatrix} \theta_Q \\ -d_P^* \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \theta_P^* & 0 \\ 0 & -d_Q \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} : \hat{P}^k \to \hat{Q}_{k+1}
\end{aligned}$$

Let $\chi = \left(\begin{smallmatrix} 0 & 0 \\ \theta & 0 \end{smallmatrix}\right)$. Then we have to show that

$$\begin{tikzcd}
\Phi = \quad P \oplus P' \oplus Q \oplus Q' \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & \rho_2 & 0 \\ 0 & 1 & 0 & \rho_2' \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} f_P & 0 \\ 0 & f_Q \end{smallmatrix}\right)"'] & P \oplus P' \arrow[d, "f_P"] \\
\hat{P} \oplus \hat{Q} \arrow[r, "(1\ \hat{\rho}_2)"] & P
\end{tikzcd}$$

is a triad, and that $((\delta\chi, \chi), (\delta\psi \oplus -\delta\psi', \psi \oplus -\psi')) \in (W_{\%}(\Phi))_{2k+3}$ is a cycle.

First note that the maps $\rho_i, \hat{\rho}_i$ actually commute with the maps f_Q, f_Q, not just up to homotopy. Hence Φ is a triad. It just remains to show that

$$d(\delta\chi) = f_{P\%}(\delta\psi^P \oplus -\delta\psi'^P) - \begin{pmatrix} 1 & \hat{\rho}_2 \end{pmatrix}_{\%}\begin{pmatrix} 0 & 0 \\ \hat{\theta} & 0 \end{pmatrix} \tag{13.1}$$

All of the terms in the above expression are terms in $(W_{\%}\hat{P} \oplus \hat{Q})_{2k+2}$, and so consist of 4 maps as shown below:

$$\begin{tikzcd}
P^k \arrow[r] \arrow[dr, "\omega_1^1"'] \arrow[drr, "\omega_2"] & P^{k+1} \oplus Q_{k+1} \arrow[d, "\omega_0"] \arrow[dr, "\omega_1^0"] & \\
& P_{k+1} \oplus Q_{k+1} \arrow[r] & P_k
\end{tikzcd}$$

Now compare terms in each structure:

<u>$d(\delta\chi)$:</u>

$$\begin{aligned}
\omega_0 &= \begin{pmatrix} 0 & 0 \\ (-1)^k\rho_2^* & 0 \end{pmatrix} - (-1)^{k+1}\begin{pmatrix} 0 & (-1)^k\rho_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \rho_2 \\ (-1)^k\rho_2^* & \end{pmatrix} \\
\omega_1^1 &= (-1)^{k+1}\begin{pmatrix} 0 & 0 \\ (-1)^k\rho_2^* & 0 \end{pmatrix}\begin{pmatrix} d_P^* \\ (-1)^k\theta_Q \end{pmatrix} - \begin{pmatrix} \rho_2\theta_Q \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho_2\theta_Q \\ -\rho_2^* d_P^* \end{pmatrix} \\
\omega_1^0 &= \begin{pmatrix} d_P & (-1)^k\theta_Q^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ (-1)^k\rho_2^* & 0 \end{pmatrix} - \begin{pmatrix} (\rho_2\theta_Q)^* & 0 \end{pmatrix} = 0 \\
\omega_2 &= d_P\rho_2\theta_Q
\end{aligned}$$

<u>$\begin{pmatrix} 1 & \hat{\rho}_2 \end{pmatrix}_{\%}\begin{pmatrix} 0 & 0 \\ \hat{\theta} & 0 \end{pmatrix}$:</u>

$$\begin{aligned}
\omega_0 &= \hat{\rho}_2\hat{\theta} = \begin{pmatrix} \rho_2 & 0 \\ 0 & -\rho_2^* \end{pmatrix}\begin{pmatrix} 0 & -1 \\ (-1)^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\rho_2 \\ (-1)^{k+1}\rho_2^* & 0 \end{pmatrix} \\
\omega_1^1 &= 0; \omega_1^0 = 0; \omega_2 = 0
\end{aligned}$$

<u>$f_{P\%}(\delta\psi^P \oplus -\delta\psi'^P)$:</u>

$$\begin{aligned}
\omega_0 &= 0 \\
\omega_1^1 &= \begin{pmatrix} -\rho_2\theta_Q \\ -\rho_2^* d^* \end{pmatrix} \\
\omega_1^0 &= 0 \\
\omega_2 &= \delta\psi_P + (-1)^k\delta\psi_P^*
\end{aligned}$$

Comparing term by term, we see that equation (13.1) is satisfied. The proof for ρ_1 runs similarly, so that we have indeed defined a quadratic cobordism.

The last stage is to prove that the nilpair is Poincaré, i.e. that

$$\begin{tikzcd}
CS_k \arrow[r] \arrow[d] & CS^{k+1} \oplus CS_{k+1} \arrow[d] & \\
CS_{k+1} \oplus CS^{k+1} \arrow[r] & CS_{k+1} \oplus CS^{k+1} \oplus CS_k \oplus CS_k \arrow[r] & CS_k
\end{tikzcd}$$

is a chain equivalence. The projection of the bottom row onto $0 \to CS_k \to 0$ is a chain equivalence, and therefore, the mapping cone is contractible if and only if the mapping cone

$$\begin{tikzcd}
CS^k \arrow[r] & CS^{k+1} \oplus CS_{k+1} \arrow[d] \\
& CS_k
\end{tikzcd}$$

is contractible. But this is just the condition that $CS_{k+1} \to CS_k$ be a Poincaré complex. $\square$

Lemma 13.17 The above notion of cobordism is an equivalence relation on the set of short odd nilcomplexes defined above.

Proof. As noted before, symmetry is obvious. Transitivity follows directly from (13.16). For reflexivity, let

$$\alpha = (CS_{k+1}, CS_k, d, \theta, \delta\psi_2^P, \delta\psi_2^Q)$$

be a short odd nilcomplex. Then by lemma 13.16, there exists a cobordism $(\beta; \alpha, \alpha')$, where

$$\alpha' = (CS^{k+1}, CS_k, \theta^*, d^*, \delta\psi_2^P, \delta\psi_2^Q).$$

Then there exists a cobordism $(\beta'; \alpha', \alpha)$, so the union $(\beta \cup \beta'; \alpha, \alpha)$ is a highly connected cobordism of α with itself. □

Then finally the UNil groups can be defined:

Definition 13.18 $\mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \mathbb{Z}[G_1], \mathbb{Z}[G_2])$ is the group of highly connected cobordism classes of $(2k+1)$-dimensional short odd nilcomplexes.

Proof that this is a group. (We use additive notation.)

The sum $(C, \psi) + (C', \psi') = (C \oplus C', \psi \oplus \psi')$. The 0 is given by the complex $0 \to 0$. $-(C, \psi) = (C, -\psi)$. Then associativity is clear, 0 is a zero, and $(C, \psi) + (C, -\psi) = 0$ by the definition of the equivalence relation. □

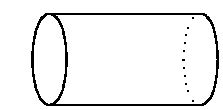
Chapter 14

The Odd-dimensional UNil Obstruction

14.1 Surgery obstruction

As mentioned before, the surgery obstruction will be defined here not as by Wall in terms of a Heegaard-type splitting, but rather as y Ranicki, in terms of a presentation:

Definition 14.1 Let $f : M \to X^{2k+1}$ be a degree 1 normal map. Then a *presentation* of f is a normal bordism $F : N \to X$ with $f' : M' \to X$ where F and f' are highly connected.



M^{2k+1} N M'

Presentations of surgery obstructions can always be constructed in a straightforward manner:

Construction 14.2 Let $K_k(M)$ be generated by $e_1, \ldots, e_r$ as a $\mathbb{Z}[H]$-module. Let these be represented by framed disjoint embeddings $\theta_i : S^k \times D^{k+1} \to M$. Let N be the trace of surgeries on these θ_i, with map $F : N \to X$. Then $F : N \to X$ is a presentation of $f : M \to X$.

A presentation determines a short odd complex (and thence a formation) in the following way:

Definition 14.3 Let $C_{k+1} = K_{k+1}(N, M)^* \cong K_{k+1}(N, M')$, $C_k = K_{k+1}(N)^* \cong K_{k+1}(N, \partial N)$, $d = p_1^*$ where $p_1 : K_{k+1}(N) \to K_{k+1}(N, M)$ is the usual projection map, ψ_0 is the composite $K_{k+1}(N) \to K_{k+1}(N, M') \xrightarrow{\cong} K_{k+1}(N, M)^*$, and $\psi_1 : K_{k+1}(N) \to K_{k+1}(N)^*$ is a splitting of the (usually singular) quadratic intersection form. It is Poincaré since the sequence

$$0 \longrightarrow K_{k+1}(N) \longrightarrow K_{k+1}(N, M) \oplus K_{k+1}(N, M') \longrightarrow K_{k+1}(N, \partial N) \longrightarrow 0$$

is exact.

The surgery obstruction is then defined to be $\sigma(f) = (C, \psi) \in L_{2k+1}(\mathbb{Z}[H])$.

Remark 14.4 Given two different sets of generators e_i and f_i, disjoint embeddings $\theta_i, \phi_i : S^K \times D^{k+1}$ can be found representing e_i and f_i respectively. Then the presentations N_1 and N_2 from these generators can both be embedded inside the presentation N which is obtained from surgery on the generators $e_i \cup f_i$ and this induces a homotopy equivalence of short odd complexes (see example 6.9, Ranicki([10]). This proof will not generalize when we consider splitting problems.

From the short odd complex, the result of surgery can be computed (which is otherwise quite hard):

Lemma 14.5 Let $f : M \to X$ be a degree 1 normal map with $\sigma(f) = (C, \psi)$. Algebraic surgery data $(j, D_{k+1}.\delta\psi)$ consists of an f.g. free module D_{k+1} together with map $j : C_{k+1} \to D_{k+1}$ and $\delta\psi_0 : D^*_{k+1} \to D_{k+1}$.

The result of surgery on the data, is the short odd complex $(C'.\psi')$ where $C'_{k+1} = C_{k+1} \oplus D^{k+1}$, $C'_k = C_k \oplus D_{k+1}$, $d' = \begin{pmatrix} d & \psi_0^* j^* \\ j & \delta\psi_0 + (-1)^k \delta\psi_0^* \end{pmatrix}$, $\psi'_0 = \begin{pmatrix} \psi_0 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi'_1 = \begin{pmatrix} \psi_1 & \psi_0^* j^* \\ 0 & \delta\psi_0 \end{pmatrix}$.

If $(j : C \to D, (\delta\psi_0, \psi))$ is a Poincaré cobordism, then the result of surgery is a contractible chain complex, and D is the trace of a cobordism of f with a homotopy equivalence.

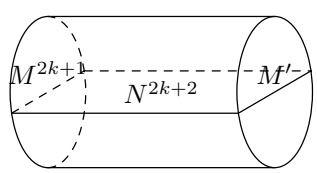
Moreover, all algebraic surgeries on (C, ψ) are realized by geometric surgeries. (See Prop. 12.36 (Ranicki [11]).)

14.2 Splitting problem

Assume that $Y = Y^{2k+2}$ and that f is k-connected. In order to define the splitting obstruction, we shall first construct a presentation of our splitting problem. We shall then use the associated UNil objects to define a formation.

Definition 14.6 Given a splitting problem $g : W \to Y^{2k+2}$ cut along $f : M \to X^{2k+1}$ such that $\phi(\tau(f) = 0$, a presentation is a cobordism T, with $\partial T = W \cup W'$, a map $h : T \simeq Y \times I$ transverse to X so that $N = h^{-1}(X \times I)$ is a $(k+1)$-connected cobordism of M with a manifold M', and such that $\phi(\tau(h)) = 0$. (See figure (14.1).)

Figure 14.1: A presentation



Convention 14.7 The above notation shall be fixed for the remainder of this section - furthermore, we shall neglect to mention the maps g and h, taking their existence as read.

Proposition 14.8 Any odd-dimensional splitting problem $g : W \to Y$ has a presentation.

Proof. Consider the map $h : W \times I \to Y \times I$. This gives a $(2k+2)$-dimensional splitting problem (with boundary), so handle exchanges can be performed on the interior of $M \times I$ giving a $(k+1)$-connected map to $X \times I$ as required.

In fact this is a presentation with the same splitting problem $g : W \to Y$ on both ends. □

Proposition 14.9 Let $g : W \to Y^{2k+2}$ be an odd-dimensional splitting problem, and $h : W \times I \to Y \times I$ be a presentation of it. Then let:

- $CS_{k+1} = \mathrm{Spl}_{k+1}(N, M')$;
- $CS_k = \mathrm{Spl}_{k_1}(N, \partial N)$;
- $d = \pi : \mathrm{Spl}_{k+1}(N, M') \to \mathrm{Spl}_{k_1}(N, \partial N)$;
- $\theta = \pi \circ i : Spl^{k+1}(N, \partial N) \to \mathrm{Spl}_{k+1}(N) \to \mathrm{Spl}_{k+1}(N, M')$

Then $\rho_1 d\theta = \delta\psi_Q + \delta\psi_Q^*$, along with a similar expression in ψ_P is given by the split quadratic UNil form on W. Then $\chi(g;h) = (CS_{k+1}, CS_k, d, \theta, \delta\psi_P, \delta\psi_Q) \in \mathrm{UNil}_{2k+3}$. The map

$$\begin{array}{ccc} CS^k = Spl_{k+1}(N) & \xrightarrow{d^*} & CS^{k+1} = Spl_{k+1}(N, M) \\ \theta \big\downarrow & & \big\downarrow \theta^* \\ CS_{k+1} = Spl_{k+1}(N, M') & \xrightarrow[d]{} & CS_k = Spl_{k+1}(N, \partial M) \end{array}$$

is a chain equivalence, since the mapping cone is just the Mayer Vietoris sequence.

This apparently depends upon the choice of presentation. However, it follows from the following 2 results that it is independent of choice of presentation:

Lemma 14.10 Let $(h; g, g') : W \times I \to Y \times I$ be a presentation of the splitting problem $g : W \to Y$. Then this can also be regarded as a presentation of $g' : W \to Y$. Then $\chi(g;h) = \chi(g';h) \in \mathrm{UNil}_{2k+3}$.

Proof. The proof of this is the promised geometrical foundation of lemma 13.16. Consider again the diagram:

$$\begin{array}{ccc} CS^k = Spl_{k+1}(N) & \xrightarrow{d^*} & CS^{k+1} = Spl_{k+1}(N, M) \\ \theta \big\downarrow & & \big\downarrow \theta^* \\ CS_{k+1} = Spl_{k+1}(N, M') & \xrightarrow[d]{} & CS_k = Spl_{k+1}(N, \partial M) \end{array}$$

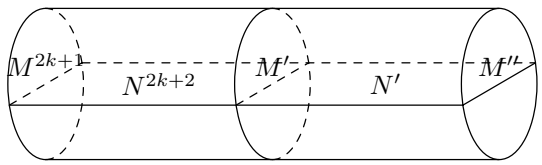
As seen previously, the presentation determines a short odd nilcomplex for g, by taking the short odd nilcomplex to be the bottom row. But it also determines a short odd complex for g', which is no more than the right hand column. So if $\chi(g;h) = (CS_{k+1}, CS_k, d, \theta, \delta\psi_P, \delta\psi_Q)$ then $\chi(g';h) = (CS^{k+1}, CS_k, \theta^*, d, \delta\psi_P, \delta\psi_Q)$. The cobordism of these two complexes is given in lemma 13.16.

In fact we can say more, the cobordism constructed is the cobordism $C(M) \to C(N) \leftarrow C(M')$, where in this case $C(M)$ is constructed as the mapping cone $(C(N, M) \oplus C(N, M') \to$

$C(N, \partial N))$, which is homotopy equivalent to $C(N)$ precisely because the mapping cone of the above map is contractible. $\square$

Lemma 14.11 Suppose that $(h; g, g')$ is a presentation of g, and $(h'; g', g'')$ is a presentation of g'. Then $(h \cup h'; g, g'')$ is a presentation of g' and is such that $\chi(g; h) = \chi(g; h')$.

Figure 14.2: A union of presentations



Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spl}_{k+1}(N'', M'') \arrow[r] \arrow[d] & \mathrm{Spl}_{k+1}(N'', M \cup M') \arrow[d] \\
\mathrm{Spl}_{k+1}(N'', N') \arrow[r] \arrow[d, "\cong"] & \mathrm{Spl}_{k+1}(N'', M \cup N') \arrow[d, "\cong"] \\
\mathrm{Spl}_{k+1}(N, M') \arrow[r] & \mathrm{Spl}_{k+1}(N'', M \cup M'')
\end{tikzcd}$$

This shows 2 maps of nilcomplexes — the second is an isomorphism, since both maps are isomorphisms by excision. The first is a homotopy equivalence, since the mapping cone

$$\mathrm{Spl}_{k+1}(N'', M'') \longrightarrow \mathrm{Spl}_{k+1}(N'', M \cup M') \oplus \mathrm{Spl}_{k+1}(N'', N') \longrightarrow \mathrm{Spl}_{k+1}(N'', M \cup N')$$

is just a Mayer-Vietoris sequence, and is therefore contractible. $\square$

It follows immediately from these results that the UNil obstruction is independent of the choice of presentation:

Lemma 14.12 Let $(h; g, g')$ and $(h'; g, g'')$ both be presentations of g. Then $\chi(g; h) = \chi(g; h')$. Hence an obstruction $\chi(g) \in \mathrm{UNil}_{2k+3}$ is defined.

Proof. Form the union $h'' = h \cup_g h'$. Then by the previous 2 results, $\chi(g; h) = \chi(g'; h) = \chi(g'; h'') = \chi(g''; h'') = \chi(g''; h') = \chi(g; h')$. $\square$

Theorem 14.13 Let $k \geq 2$ and $g : W \to Y^{2k+2}$ be a homotopy equivalence. There are 2 obstructions to g being splittable: $\overline{\phi}(\tau(g))$ and $\chi(g) \in \mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$. If $\overline{\phi}(\tau(g)) = 0$ and g is splittable then $\chi(g) = 0$.

Proof. The K-theory has already been covered in chapter 5. From above, $\chi(g)$ is well-defined, and independent of choice of presentation.

An h-cobordism with a split homotopy equivalence can be made highly connected and then gives a presentation, which then gives a null-cobordism, so if g is h-cobordant to a split homotopy equivalence then $\chi(g) = 0$. $\square$

Chapter 15

Odd-dimensional Nilpotent Normal Cobordism

Given a splitting problem, an obstruction $\chi(g) \in \mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$ has been defined. Furthermore, short odd nilcomplexes have been identified with Poincaré nilcomplexes, and the surgery obstruction associated to the algebraic nilpotent normal cobordism has been calculated. In this chapter, it is shown that this defines a map from $\mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]}) \to L_{2k+3}(\mathbb{Z}[G])$ such that if $g : W \to Y$ is a highly connected $2k+1$-dimensional splitting problem such that $\overline{\phi}(\tau(g)) = 0$, there is a cobordism with a split homotopy equivalence with surgery obstruction $\alpha(\chi(g))$.

The nilpotent normal cobordism has already been described algebraically when the dimension of X is odd; the purpose of this chapter is to show how to realize this geometrically. It has been shown that the quadratic kernel of f is the chain complex $P \oplus Q$ where P and Q are projective chain complexes, and if we assume that $\phi(\tau(g)) = 0$, then $[P] = 0$, so that $[P_{k+1}] = [P_k]$.

15.1 Bordisms of $f : M \to X$

A critical part of the construction of the even-dimensional nilpotent normal cobordism was the construction of the spaces C_P and C_Q which were cobordisms with homotopy equivalences $f_P : M_P \to X$ and $f_Q : M_Q \to X$ with the same homology kernels as W_r and W_l.

Before constructing the odd-dimensional nilpotent normal cobordism, it is useful to understand how the pairs (W_r, M) and (W_l, M) determine, when $[P] = 0$, bordisms of $f : M \to X$ with a homotopy equivalence. The crucial result is the following:

Proposition 15.1 Suppose that $[P] = 0$. Then $f_r : W_r \to Y_r$ is bordant (by finitely many surgeries on the interior) to a map $f'_r : V_r \to Y_r$, where $V_r = C_Q \cup_{M_Q} U_r$, $f'_r|_{C_Q} : C_Q \to X$ is the trace of surgeries on M, with $f'_r|_{M_P} : M_P \to X$ a homotopy equivalence and $f'_r : U_r \to Y_r$ a homotopy equivalence.

The proof will proceed by making W_r highly connected, and then using the proof of the

$\pi - \pi$ theorem. The following lemma is therefore useful, computing the result algebraically of making W_r highly connected by surgeries on the interior.

Lemma 15.2 Suppose that $(P \oplus Q \to P, (0, \theta))$ is a Poincaré pair of free $\mathbb{Z}[H]$-modules, where $P_r = 0$ if $r \notin \{k, k+1\}$. Then surgeries can be performed on the interior to give a highly connected Poincaré pair $(P \oplus Q \to P', (\delta\psi, \theta))$ where $P'_r = 0$ for $r \neq k+1$.

Proof. The Thom complex of the pair gives the quadratic complex

$$\begin{tikzcd}
P^k \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 \\ \theta_Q \end{smallmatrix}\right)"'] & P^{k+1} \oplus P^k \oplus Q^k \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \theta_P^* & 0 & 0 \end{smallmatrix}\right)"] & P^{k+1} \oplus Q^{k+1} \arrow[d] \\
P_{k+1} \oplus Q_{k+1} \arrow[r, "\left(\begin{smallmatrix} (-1)^{k+1} & 0 \\ d_P & 0 \\ 0 & d_Q \end{smallmatrix}\right)"'] & P_{k+1} \oplus P_k \oplus Q_k \arrow[r, "\left(\begin{smallmatrix} d_P & (-1)^k & 0 \end{smallmatrix}\right)"'] & P_k
\end{tikzcd}$$

and projection onto SQ is a homotopy equivalence with the quadratic complex $(SQ, 0)$,where SQ is the suspended chain complex $SQ_{r+1} = Q_r$.

Since the algebraic mapping cone

$$\begin{tikzcd}
P^k \arrow[r, "\left(\begin{smallmatrix} d^* \\ \theta_Q \end{smallmatrix}\right)"] & P^{k+1} \oplus Q_{k+1} \arrow[r, "\left(\begin{smallmatrix} \theta_P^* & d_Q \end{smallmatrix}\right)"] & Q_k \arrow[r] & 0
\end{tikzcd}$$

is a short exact sequence of projective modules, it splits, and so we can find maps $\left(\begin{smallmatrix} \alpha & \beta \end{smallmatrix}\right) : P^{k+1} \oplus Q_{k+1} \to Q_k \oplus P^k$ and $\left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right) : Q_k \to P^{k+1} \oplus Q_{k+1}$ such that $\left(\begin{smallmatrix} \gamma & d_P^* \\ \delta & \theta_Q \end{smallmatrix}\right)\left(\begin{smallmatrix} \theta_P^* & d_Q \\ \alpha & \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Hence perform surgery on SQ, corresponding to surgery on the interior of the pair, to make SQ highly connected by taking the following surgery data:

$$\begin{tikzcd}
SQ_{k+2} = Q_{k+1} \arrow[r] \arrow[d, "\beta"] & SQ_{k+1} = Q_k \arrow[r] & 0 \\
P^k & &
\end{tikzcd}$$

The result of the surgery is the quadratic complex:

$$\begin{tikzcd}
 & Q^k \oplus P_k \oplus P^k \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)"] & Q^{k+1} \\
Q_{k+1} \arrow[r, "\left(\begin{smallmatrix} d_Q \\ \beta \\ 0 \end{smallmatrix}\right)"'] & Q_k \oplus P^k \oplus P_k & &
\end{tikzcd}$$

The following is a chain equivalence:

$$\begin{tikzcd}
Q_{k+1} \arrow[r, "\left(\begin{smallmatrix} d_Q \\ \beta \\ 0 \end{smallmatrix}\right)"] \arrow[d] & Q_k \oplus P^k \oplus P_k \arrow[d, "\left(\begin{smallmatrix} \gamma & d_P^* & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \\
0 \arrow[r] & P^{k+1} \oplus P_k
\end{tikzcd}$$

inducing the quadratic structure on the target given by $\psi_0 = \left(\begin{smallmatrix} 0 & 0 \\ d_P & 0 \end{smallmatrix}\right) : P_{k+1} \oplus P^k \to P^{k+1} \oplus P_k$.

Then the resulting pair, given by the Poincaré thickening is:

$$\begin{tikzcd} P_{k+1} \oplus P^k \arrow[r, "\begin{pmatrix} 0 & d_P^* \\ d_P & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] & P^{k+1} \oplus P_k \\ P_{k+1} \oplus P^k & \end{tikzcd}$$

The quadratic structure on the pair is $(0, \psi') \in Q_{k+2}(j)$ where j is the map in the pair, and where ψ' is the structure:

$$\begin{tikzcd} P_{k+1} \oplus P^k \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} 0 & d_P^* \\ 0 & 0 \end{pmatrix}"] & P^{k+1} \oplus P_k \arrow[d, "0"] \\ P_{k+1} \oplus P^k \arrow[r, "\begin{pmatrix} 0 & d_P^* \\ d_P & 0 \end{pmatrix}"'] & P^{k+1} \oplus P_k \end{tikzcd}$$

(As a check, this is easily seen to be equivalent to $(P \oplus Q, \psi)$ where ψ is the usual quadratic structure). □

Proof of 15.1. Suppose that $P = C(W_r)$ given by the presentation is such that $[P] = 0$. Then $[P_{k+1}] = [P_k]$, so letting M be any module so that $[M] = -[P_k]$, P is homotopy equivalent to the free $\mathbb{Z}[H]$-module chain complex $P_{k+1} \oplus M \to P_k \oplus M$ with differential given by $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Similarly Q can be stabilized, and the above analysis then gives surgery data for performing surgery on W_r to give V_r which is highly connected.

Now apply the $\pi - \pi$ theorem to the preceding example. Since V_r is $(2k+2)$-dimensional, M is $(2k+1)$-dimensional and everything is highly connected, $K_{k+1}(V_r, M)$ is the only relative homology kernel and is free; therefore choose a basis e_i. The proof of the $\pi - \pi$ theorem then implies that these e_i can be represented by disjoint framed embeddings $\theta_i : (D^{k+1} \times D^{k+1}, S^k \times D^{k+1}) \to (V_r, M)$ which are null-homotopic in (Y_r, X); moreover, the result of surgery on these embeddings on M gives a homotopy equivalence $M_P \to X$, and the result of removing the embeddings from V_r is a homotopy equivalence $f_r' : U_r \to Y_r$. In other words, letting C_P be the trace of the surgeries, $V_r = C_P \cup_{M_P} U_r$ where $U_r \simeq Y_r$, $M_P \simeq X$. □

15.2 Construction of the nilpotent normal cobordism

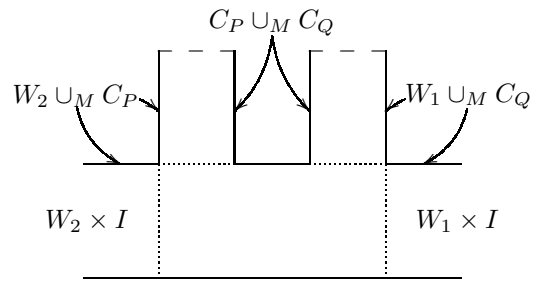
The construction of the nilpotent normal cobordism when M is odd-dimensional is somewhat less direct than when M is even-dimensional.

Proposition 15.3 Let g be a splitting problem such that $\phi(\tau(g)) = 0$. Then there exists a cobordism with a split homotopy equivalence, called the 'nilpotent normal cobordism', with surgery obstruction $\alpha\chi(g)$.

Proof. Since $\phi(\tau(g)) = 0$, $[P] = [Q] = 0$. Then by proposition 15.1, there exist cobordisms C_P and C_Q of M with M_P and M_Q homotopy equivalences, where C_P and C_Q sit inside V_l and V_r respectively.

Then perform the same construction as Cappell with these C_P and C_Q. The result is shown in figure 15.1. In that figure, the dashed lines signal that the restriction of g to those subspaces is a homotopy equivalence.

Figure 15.1: The first stage of the nilpotent normal construction



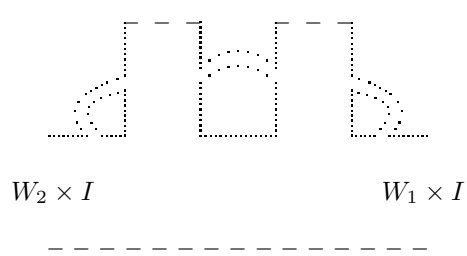
Since it is not true that the kernel chain complex $C(C_Q \cup_M W_1)$ is contractible, the boundary of the above is not a homotopy equivalence. However, the boundary can be decomposed into 3 rel boundary ∂ surgery problems: $h_1 : (W_2 \cup_M C_P, M_P) \to (Y_2, X)$, $h_2 : (C_P \cup_M C_Q) \to (X \times I, X \times \{0,1\})$, $h_3 : (W_1 \cup_M C_Q, M_Q) \to (Y_1, X)$ where the boundaries of all 3 problems are homotopy equivalences.

All of these surgery problems are soluble: Consider h_1:

By the above, h_1 : can be extended to $h_1' : (W_2 \cup_M V_l) \to (Y_2 \cup_X Y_l, X)$. Since h_1' is formed by joining a homotopy equivalence $U_l \to Y_l$ along a homotopy equivalence $M_P \to X$, $\sigma(h_1) = \sigma(h_1')$. But by the construction of V_l, h_1' is just formed by joining $W_2 \to Y_2$ to $V_l \to Y_l$, and so is formed by joining $W_2 \to Y_2$ to $W_l \to Y_l$ and performing surgeries on the interior. Since $W_2 \cup W_l \to Y_2 \cup Y_l$ is a homotopy equivalence, it follows that $\sigma(h_1') = 0$ and hence $\sigma(h_1) = 0$. Similarly for h_2 and h_3.

Hence surgery can be performed on each of these maps to homotopy equivalences. Join the trace of the handle exchanges onto the boundary, and hence obtain a cobordism with a split homotopy equivalence as claimed. (See figure 15.2; here both dashed and dotted lines are homotopy equivalences.)

Figure 15.2: The final stage of the nilpotent normal construction



□

15.3 Computation of the obstruction of the nilpotent normal cobordism

It remains to compute the surgery obstruction of the nilpotent normal cobordism. The algebraic effect has already been computed on the chain complex level, so this computation effectively verifies that the quadratic Poincaré pairs defined from the splitting obstruction are correct.

Proposition 15.4 Suppose that $\chi(g) = (\mathrm{Spl}_{k+1}, \mathrm{Spl}_k, d, \theta, \delta\psi^P, \delta\psi^Q)$ is the short odd nilcomplex coming from the presentation $((V,N);(W,M),(W'M'))$. The surgery obstruction of the nilpotent normal cobordism $\alpha(\chi)$ is the short odd complex:

$$\begin{tikzcd}
\mathbb{Z}[G] \otimes (P^k \oplus Q^k) \arrow[r] \arrow[d, "\begin{pmatrix} \rho_2\theta_Q & \theta_P \\ \theta_Q & \rho_1\theta_P \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} \delta\psi_P & 0 \\ d_P\theta_P & \delta\psi^Q \end{pmatrix}"] & \mathbb{Z}[G] \otimes (P^{k+1} \oplus Q^{k+1}) \\
\mathbb{Z}[G] \otimes (P_{k+1} \oplus Q_{k+1}) \arrow[r, "\begin{pmatrix} d_P & 0 \\ 0 & d_Q \end{pmatrix}"'] & \mathbb{Z}[G] \otimes (P_k \oplus Q_k) \arrow[u, phantom, "\searrow"]
\end{tikzcd}$$

Proof. The obstruction of the nilpotent normal cobordism will not be computed directly, since its construction was slightly involved. The proposition is immediate from proposition 9.7 after the next lemma which computes the surgery obstruction of the infinite nilpotent normal cobordism. □

Remark 15.5 The nilpotent normal cobordism construction applied to the Poincaré nilcomplex determined by the short odd nilcomplex gives the short odd complex:

$$\begin{tikzcd}
\mathbb{Z}[G] \otimes (P^k \oplus Q^k) \arrow[r] \arrow[d, "\begin{pmatrix} \rho_2\theta_Q & 0 \\ \theta_Q & \rho_1\theta_P \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} \delta\psi_P & 0 \\ 0 & \delta\psi^Q \end{pmatrix}"] & \mathbb{Z}[G] \otimes (P^{k+1} \oplus Q^{k+1}) \arrow[d, "\begin{pmatrix} 0 & 0 \\ \theta_P^* & 0 \end{pmatrix}"] \\
\mathbb{Z}[G] \otimes (P_{k+1} \oplus Q_{k+1}) \arrow[r, "\begin{pmatrix} d_P & 0 \\ 0 & d_Q \end{pmatrix}"'] & \mathbb{Z}[G] \otimes (P_k \oplus Q_k) \arrow[u, phantom, "\searrow"]
\end{tikzcd}$$

which is an equivalent quadratic structure on the same complex.

Lemma 15.6 With hypotheses as above, define T_M^∞ to be the non-compact Poincaré surgery problem given by glueing copies of W_r and W_l where C_P and C_Q as in proposition 9.7. Then the surgery obstruction, $\sigma(T_M^\infty) = \alpha(\chi(g)) \in L^h_{2k+3}(\mathbb{Z}[G])$.

Proof. Let $((V,N);(W,M),(W'M'))$ be the presentation used to give the surgery obstruction. Construct a presentation of T_M^∞ by taking $V \times I$ and glueing on copies of $V_l \times I$ and $V_r \times I$. Denote the resulting space T_N^∞ (with an implicit map to $Y \cup \hat{Y}$) which has two boundary components: one is T_M; denote the other by $T_{M'}$.

Let the short odd complex of T_M given by this presentation be the short odd complex of

$\mathbb{Z}[G]$-modules:

$$\begin{tikzcd} D^k \arrow[r] \arrow[d, "\chi_0"'] \arrow[dr, "\chi_1"] & D^{k+1} \\ D_{k+1} \arrow[r, "\delta"'] & D_k \end{tikzcd}$$

Now recall the definition of the surgery obstruction. $D_k = K_{k+2}(T_N)^* = \mathbb{Z}[G] \otimes (P_k \oplus Q_k)$ (as before by the Mayer-Vietoris sequence). Similarly $D_{k+1} = K_{k+2}(T_N, T_M)^* = \mathbb{Z}[G] \otimes (P_{k+1} \oplus Q_{k+1})$.

δ is the map induced by the natural map $K_{k+2}(T_N, T_M)^* \to K_{k+2}(T_N)^*$ which is therefore $\left(\begin{smallmatrix} d_P & 0 \\ 0 & d_Q \end{smallmatrix}\right)$. $\chi_0 : D^k \to D_{k+1}$ is the composition of the maps: $K_{k+2}(T_N) \to K_{k+2}(T_N, T_{M'}) \to K_{k+2}(T_N, T_M)$. If the map $\mathrm{Spl}_{k+1}(N) \to \mathrm{Spl}_{k+1}(N, M')$ is (f_P, f_Q), then the first of these maps is $f_P \oplus f_Q$. Let $\lambda^1 : K_{k+1}(N, M) \times K_{k+1}(N, M')$ be the bilinear form inducing the isomorphism $K_{k+1}(N, M') \cong K_{k+1}(N, M)^*$. Then the same arguments as in the straightforward even-dimensional case imply that the form $\lambda^1_T : K_{k+2}(T_N, T_M) \times K_{k+2}(T_N, T_{M'})$ is given by $\lambda^1_T((1-\rho)x, y)) = \lambda^1(x, y)$, and hence that $\lambda_T((1-\rho)x, (1-\rho)y) = L((1-\rho)x, y)$. Therefore $(1-\rho)^* \chi_0 (1-\rho)^* = \left(\begin{smallmatrix} \rho_1\theta_Q & \theta_P \\ \theta_Q & \rho_2\theta_P \end{smallmatrix}\right)$. Similarly, by 9.9 $(1-\rho)\chi_1(1-\rho) = \left(\begin{smallmatrix} \delta\psi^P & 0 \\ d_P\theta_P & \delta\psi^Q \end{smallmatrix}\right)$. Hence the chain map $(1-\rho)$ induces an isomorphism between the short odd nilcomplex associated to the presentation and that claimed. $\square$

Chapter 16

Concluding Remarks

We have now established the theorem which was the stated goal of this thesis:

Theorem 16.1 Let $k \geq 2$ and $g : W \to Y^{2k+2}$ be a splitting problem. Then g is splittable if and only if $\overline{\phi}(\tau(g)) = 0$ and $\chi(g) \in \mathrm{UNil}_{2k+3}(\mathbb{Z}[H]; \widetilde{\mathbb{Z}[G_1]}, \widetilde{\mathbb{Z}[G_2]})$.

Proof. It was seen in chapter 14 that $\chi(g)$ is an h-cobordism invariant. Therefore if g is splittable, $\alpha(\chi(g)) = 0$. Suppose that $\chi(g) = 0$. It was seen in chapter 12 that α is a group homomorphism, so $\alpha(\chi(g)) = 0$; hence the nilpotent normal cobordism constructed above has 0 surgery obstruction, and therefore is bordant rel ∂ to a homotopy equivalence, which is then an h-cobordism of W with $g' : W' \to Y$ where g' is split. □

We would like to show two things more:

- That every element of UNil_{2k+3} is realized as the splitting obstruction of some splitting problem;
- That α is a split monomorphism.

There is an obvious candidate for a splitting of α; namely let $\beta : L_{2k+3}(\mathbb{Z}[G]) \to \mathrm{UNil}_{2k+3}$ be defined in the following way: Realize $y \in L_{2k+3}(\mathbb{Z}[G])$ as the surgery obstruction of a cobordism $(h; 1, g) : (V; Y, W) \to (Y \times I; Y, Y)$ with $\phi(\tau(h)) = 0$, and where g is a homotopy equivalence, as is always possible by the realization theorem of Wall. Define $\beta(y) = \chi(g)$. It remains to prove that $\beta(\alpha(x)) = x$.

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