Brought forward, 21
aris; Elias Fries,
Berlin'; August
Kirchhoff, Heidel-
Albert Kölliker,
r, Berlin ; Johann
Lepsius, Berlin ;
udolph Leuckart,
Leverrier, Paris;
Ludwig, Leipzig;
eodore Mommsen,
United States ;
in Peirce, United
Regnault, Paris ;
odor von Siebold,
ne; Otto Torell,
erlin ; Wilhelm
rich Wohler, Got-
. 34

rs deceased during

ngniart, Christian

ld Laws:—
. 1

nt Fellows at 4th
. 56

5, ... 358
cis Edward Belcombe;
. John Gibson Cazenove,
glas H. Dickson, M.A.;
J. Ballantine Hannay;
n Macleod, D.D.; John
S.C.; William Skinner,
; Rev. Francis Le Grix
. 15

Carry forward, 373

Brought forward, 373

Deduct Deceased—Dr James Warburton Begbie; David Bryce; G. Stirling Home Drummond; Lewis D. B. Gordon, C.E.; Sir George Harvey; Dr Laycock; Thomas Login, C.E.; Alexander Russel; The Marquis of Tweeddale, ... 9

Resigned—Hon. Charles Baillie (Lord Jerviswoode) . 1

10

Total number of Ordinary Fellows at November 1876, . 363
Add Honorary and Non-Resident Fellows, . . 56

Total Honorary and Ordinary Fellows at commencement of Session 1876–77, ... 419

Monday, 18th December 1876.

Sir WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. On the Roots of the Equation V $\rho\phi\rho = 0$. By Gustav Plarr. Communicated by Professor Tait.

2. Applications of the Theorem that Two Closed Plane Curves intersect an even number of times. By Prof. Tait.

(*Abstract.*)

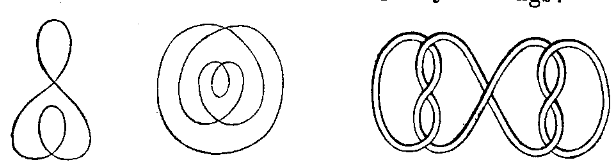
The theorem itself may be considered obvious, and is easily applied, as I showed at the late meeting of the *British Association*, to prove that in passing from any one double point of a plane closed curve continuously along the curve to the same point again, an *even* number of intersections must be passed through. Hence, if we suppose the curve to be constructed of cord or wire, and restrict the crossings to *double points*, we may arrange them throughout so that, in following the wire continuously, it goes alternately over and under each branch it meets. When this is done it is obviously as completely knotted as its scheme (defined below) will admit of, and except in a special class of cases cannot have the number of crossings reduced by any possible deformation. The excepted class is that in which the intersections are either wholly or partially nugatory, *i.e.*, not in reality contributing to the knot, whether on account of the order of their arrangement or their signs. All nugatory intersections can be detected at once by the scheme itself, and may be struck out. As will be understood from what follows, the schemes

A A B B C C and A C B B C A

are wholly nugatory, while in

A C B D C B D A E G F E G F

only the intersection A is necessarily nugatory. In fact a group like C B D C B D, when not itself nugatory by reason of its signs, is self-contained, and forms a special knot which may be drawn tight so as to present only a roughness in the string. The following sketches illustrate these essentially nugatory crossings:—



I. Given the number of its double points, to find all the essentially different forms which a closed curve can assume.

(*a*.) Going round the curve continuously, call the first, third, &c., intersections A, B, C, &c. In this category we evidently exhaust all the intersections. The complete scheme is then to be formed by properly interpolating the same letters in the even places; and the form of the curve depends solely upon the way in which this is done.

(*b*) It cannot, however, be done at random. For instance, the scheme

A *D* B *E* C *A* D *B* E *C* | A

is lawful, but

A *D* B *A* C *E* D *C* E *B* | A

is not.

The former, in fact, may be treated as the result of superposing two closed (and not self-intersecting) curves, both denoted by the letters A D B E C A, so as to make them cross one another at the points marked B, C, D, E, then cutting them open at A, and joining the free ends so as to make a continuous circuit with a crossing at A.

But in the latter scheme above, we have to deal with the curves A D B A and C E C E, and in the last of these we cannot have the junctions alternately + and − as required by our fundamental principle. In fact, the scheme would require the point C to lie simultaneously inside and outside the closed circuit A D B A.

Or we may treat A D B A and C E D C as closed curves intersecting one another and yet having only one point, D, in common.

(*c*) Thus, to test any arrangement, we may strike out from the whole scheme all the letters of any one closed part as A——*A*, and the remaining letters must satisfy the fundamental principle.

Or we may strike out all the letters of any two sets which begin and end similarly, *e.g.*, A . . . X, X . . . A, the two together being treated as one closed curve, and the test must still apply.

More generally, we may take the sides of any closed polygon as A − X, X − Y, Y − Z, Z − A, and apply them in the same way. But in this, as in the simpler case just given, the sides must all be taken the same way round in the scheme itself.

(*d.*) Such schemes as the latter of the two in (*b*) above may be made algebraically possible by slightly changing our assumptions. Thus, for instance, we might admit of a triple point, and agree not to reckon it as an intersection on a continuous oval provided one of the remaining branches goes *into* the oval, and the other comes *out* of it, these two not necessarily intersecting one another. In the case specified the triple point would be E supposed to lie *on* the oval A D B A, and not to be counted as an intersection while we pass round that oval. But this is a mere algebraic escape from a geometrical difficulty, and will not necessarily help us when we deal with knots on actual cords or wires.

II. A possible scheme being thus made, with the requisite number of intersections, let it be constructed in cord, with the intersections as above alternately + and −. Then [since all schemes involving nugatory points, like those above mentioned, must be got rid of, as they do not really possess the requisite number of intersections] no deformation which the cord can suffer will reduce, though it may increase, the number of double points. If it *do* increase the number, the added terms will be of the nugatory character presently to be explained. If it do not increase that number, the scheme will in general still represent the altered figure. Hence the scheme is a complete and definite statement of the nature of the knot.

(*a.*) One illustration depends upon the fact that all deformations of such a cord or wire may be considered as being effected by bending at a time only a limited portion of the wire, the rest being held fixed. This corresponds to changing the point of view *finitely* with regard to the part altered, and yet *infinitesimally* with regard to all the rest. This, it is clear, can always be done, as the *relative* dimensions of the various coils may be altered to any extent without altering the character of the knot. All such deformations may be obtained by altering the position of a luminous *point*, and the plane on which it casts a shadow of the knot. Any addition to the normal number of intersections which may be produced by this process is essentially nugatory.

Another mode, really depending on the same principles, consists in fixing temporarily one or more of the crossings, and considering the impossibility of unlocking in any way what is now virtually two or more *separate* interlacing closed curves, or a single closed curve with full knottings, but with fewer intersections than the original one.

Another depends upon the study of cases of knots in which one or more crossings can be got rid of. Here, it is proved that *continuations* of sign are in general lost when an intersection is lost; so that, as our system has no continuations of sign, it can lose no intersections.

(*b.*) Practical processes for producing all such deformations graphically are given at once by various simple mechanisms. Thus, taking O any fixed point whatever, let p, a point in the deformed curve, be found from its corresponding point, P, by joining PO and producing it so that

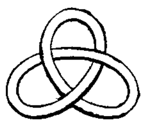
$$\mathrm{PO} \cdot \mathrm{O}p = a^2,$$

or so that

$$\mathrm{PO} + \mathrm{O}p = a, \text{ \&c., \&c.}$$

The essential thing is that points near O should have images distant from O, and *vice versâ*. And p must be taken in OP *produced*, else the distorted knot is altered from a right-handed to a left-handed one, and *vice versâ*. This distinction is shown in the cuts

1 and 3 below, where it will be seen that one turn of the coil may be regarded as wound round the other—the screw being right-handed in 3 and left-handed in 1.



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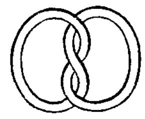

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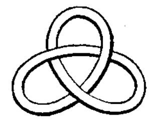
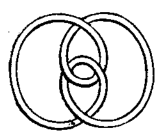


3.

It is easy to show that these methods give merely different views of the same knot. The simplest way of doing this is to suppose the knot projected on a sphere, the eye being at the centre. Arrange so that one closed branch, *e.g.*, A——A, forms nearly a great circle. Shifting the eye to opposite sides of the plane of this great circle the coil presents exactly the two appearances related to one another by the deformation processes given above. What was inside the closed branch from the one point of view is outside it from the other, and *vice versâ*.

Thus 1 and 2 above are the *only* forms with three non-nugatory intersections. 2 may be formed from 1 by putting O in either of the three border areas, each of which has two sides only. If O be placed in external space, or in the inner three-sided area, 1 is reproduced. Similar remarks apply to the deformation of 2.



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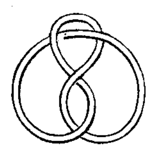
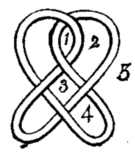


5.

Figures 4 and 5 are the only forms with four valid intersections. Like 1 and 2, the first of them is a *clear* coil (see below), the second not clear. And, of course, any deformation of either produces the other, or reproduces itself. 6, 7, 8, 9, are forms of an essentially not-clear arrangement, with five intersections. The numbers inserted in 6 show which form is produced by placing O in the corresponding area. The only other forms having five intersections, are the clear coil of two turns, whose scheme is the first given in I (*b*) above, and its solitary deformation.



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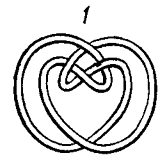

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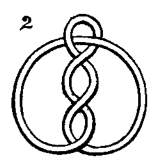

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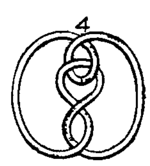


9.

(*c*.) Hence to draw a scheme, select in it any closed circuit, *e.g.*, A . . . . A—the more extensive the better, provided it do not include any less extensive one. Draw this, and build upon it the rest of the scheme; commencing always with the common point A, and passing each way from this to the *next occurring* of the junctions named in the closed circuit. [It is better to construct both parts of the rest of the scheme *inside*, and then invert one of them, as we thus avoid some puzzling ambiguities.] Inversions with respect to various origins will now give all possible forms of the scheme.

III. Thus the scheme is perfectly definite as to the general shape of the curve, if we take the possible deformations into account. And the spherical projection, already mentioned, will in general allow us to regard and exhibit the knot as a more or less perfect *plait*. It does so always when the coil is *clear*, *i.e.*, when all the turns of the cord may be regarded as passing in the same direction round a common axis thrust through the knot. When the coil is not clear some of the cords of the plait are doubled back on themselves. Thus by drawing the plait from a given scheme we can tell at once whether one of its forms is a clear coil or not.

From this point of view another notation for clear coils is given in the form

$$\begin{matrix} \alpha\,\gamma\,\beta\,\alpha \\ \beta\,\alpha\,\gamma\,\beta \end{matrix} \;\cdots\cdots$$

Here α, β, γ . . . are, in order, the several strings plaited—so that in the coil β is the prolongation of α, γ that of β, &c., and α that of the last of the series. The expression $\genfrac{}{}{0pt}{}{\alpha}{\beta}$ means that α crosses *over* β. It is sometimes useful to indicate whether a crossing takes place to the right or left. This is done by putting + or − over the symbol. Thus the four crossings above may be more fully written as

$$\begin{matrix} + & - & + & - \\ \alpha & \gamma & \beta & \alpha \\ \beta & \alpha & \gamma & \beta \end{matrix} \; \cdots\cdots$$

The properties of this notation are examined in detail. It is shown, that the combination just written cannot be simplified in itself; but that

$$\begin{matrix} + & - & - & - \\ \alpha & \gamma & \gamma & \alpha \\ \beta & \alpha & \beta & \beta \end{matrix} = \begin{matrix} - & - \\ \gamma & \gamma \\ \beta & \alpha \end{matrix}, \text{ \&c.}$$

This notation requires care. For instance, the terms

$$\begin{matrix} \alpha & \alpha \\ \beta & \beta \end{matrix}$$

are simply nugatory, and may be written off. But, on the other hand, the terms

$$\begin{matrix} \alpha & \beta \\ \beta & \alpha \end{matrix}$$

usually add to the beknottedness of the whole scheme.

When the scheme is not compatible with a clear coil there occur terms of the form

$$\begin{matrix} \alpha \\ \alpha \end{matrix},$$

and the application of this method becomes very troublesome.

(*a.*) When the scheme has no merely nugatory intersections, the most complete knotting is secured by alternate crossings above and below; or, as we may write,

$$\begin{matrix} \mathrm{A} & \mathrm{X} & \mathrm{B} & \mathrm{Y} & \mathrm{C} & \cdots & \text{\&c.} \\ + & - & + & - & + & & \end{matrix}$$

and here there is *no continuation of sign.*

(*b.*) Cases in which there is no knot at all may be obtained for any scheme by making each letter positive on its first appearance. The various coils are then, as it were, paid out over one another. This process will give rise, in general, to but few changes of sign:

but the number of such will usually depend upon the particular intersection with which we commence the scheme.

Additional changes of sign, still without any knotting, may be introduced by various processes, of which the following is the simplest:—When two letters appear together twice, not necessarily in the same order, but with like signs, these signs may be changed. Thus, the following parts of a scheme

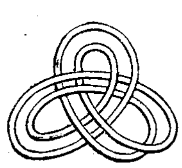
$$\begin{matrix} P & Q & \ldots & Q & P \\ - & - & & + & + \end{matrix}$$

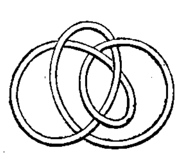
may be changed to

$$\begin{matrix} P & Q & \ldots & Q & P \\ + & + & & - & - \end{matrix}$$

and the statements already made about nugatory intersections can be applied to these and other combinations even when they occur separately once only in each of two separate knots on the same cord. This, and a great number of similar theorems, allow of a great special extension of the nugatory test already given—but an extension which cannot be made in any case until the *signs* of the intersections are given as well as the order of their occurrence.

Again, though, as has been said above, continuations of sign disappear when an intersection is lost, it does not follow that if a scheme have continuations of sign it must necessarily be reducible. The annexed diagram is an excellent instance. Its scheme contains fourteen continuations, and only twelve changes, of sign, and yet the knot is irreducible. But if we suppose it cut across twice at the single unsymmetrically placed crossing, and the ends joined so as still to preserve continuity in the string, the scheme has still fourteen continuations, but only ten changes, of sign; and it does not involve any real beknottedness.





The remaining figure illustrates a fully knotted scheme, where there are no continuations of sign, but in which the mere change of sign of one of the intersections produces four continuations of sign, and the whole beknottedness disappears. Similar remarks apply to most of the preceding figures.

IV. A great many other deductions from the fundamental proposition are given—for instance,

A closed plane curve, intersecting itself, divides the plane into separate areas whose number is greater by 2 than the number of intersections.

Regarding the curve as a wall, dividing the plane into a number of fields, if we walk along the wall and drop a coin into each field as we *reach* it, each field will get as many coins as it has corners, but those fields only will have a coin in each corner whose sides are all described in the same direction round. The number of coins is four times the number of intersections—and two coins are in each corner bounded by sides by each of which you enter—none in these bounded by sides by each of which you leave.

Cut off at any intersection and remove a portion of the curve forming a closed (not self-cutting) circuit. You thus abolish an odd number of intersections. Hence if there is an even number of coils, whether the whole be clear or not, there is an odd number of intersections, and *vice versâ*.

To form the symmetrical clear coil of two turns and of any (odd) number of intersections, make the wire into a helix, and bring one end through the axis in the same direction as the helix (not in the opposite direction, as in Ampère's *Solenoids*), then join the ends. [The solenoidal arrangement, regarded from any point of view, has only nugatory intersections.] An excellent mode of forming this coil is to twist a long strip of paper through an odd number of half-turns, and then paste its ends together,—the two longer edges become parts of one continuous curve which is the clear coil in question. This result is applied to the study of the form of soap-films obtained by Plateau's process on clear coils of wire.

V. Another question treated is the numbers of possible arrangements of given numbers of intersections in which the *cyclical* order of the letters in the 2nd, 4th, 6th, &c. places of the scheme shall be the same as that in the 1st, 3rd, 5th, &c., *i.e*, the alphabetical. Instances of such have already been given above. In the first of I (*b*), for example, the letters in the even places are

D E A B C.

Here the cyclical order of the alphabet is maintained, but A is postponed by two places.

Whatever be the number of intersections a postponement of *no* places leads to nugatory results.

A postponement of one place is possible for three and for four intersections only.

Postponement of two places is possible only for (*four*), five, and eight—three for seven and ten—four for nine and fourteen—five for (*eight*), eleven and sixteen,—six for (*ten*), thirteen, and twenty, &c. Generally there are in all cases n postponements for $2n+1$ intersections:—and for $3n+2$, or $3n+1$ intersections, according as n is even or odd. The numbers which are italicised and put in brackets above, arise from the fact that a postponement of r places, when there are n intersections, gives the same result as a postponement of $n-r-1$ places. [It will be observed that this cyclical order of the letters in the even places is possible for *any* number of intersections which is not 6 or a multiple of 6.]

When there are n postponements with $2n+1$ intersections the curve is the symmetrical double coil—*i.e.*, the plait is a simple *twist*.

The case with $3n+2$ or $3n+1$ intersections is a clear coil of three turns, corresponding to a regular plait of three strands.

VI. Numerous examples are given of the application of various methods of reduction. For instance, the scheme

$$\begin{array}{cccccccccccccccccccc|c}
\mathrm{A} & \mathit{E} & \mathrm{B} & \mathit{F} & \mathrm{C} & \mathit{G} & \mathrm{D} & \mathit{A} & \mathrm{E} & \mathit{K} & \mathrm{F} & \mathit{L} & \mathrm{G} & \mathit{D} & \mathrm{H} & \mathit{B} & \mathrm{K} & \mathit{C} & \mathrm{L} & \mathit{H} & \mathrm{A} \\
- & + & + & + & - & + & - & + & - & + & - & + & - & + & - & - & - & + & - & + &
\end{array}$$

which is rendered irreducible by changing the sign of B, is reduced by successive stages as follows:—

$$\begin{array}{cccccccccccccccc|c}
\mathrm{A} & \mathrm{B} & \mathrm{C} & \mathit{G} & \mathrm{D} & \mathit{A} & \mathit{K} & \mathit{L} & \mathrm{G} & \mathit{D} & \mathrm{H} & \mathrm{K} & \mathit{B} & \mathit{C} & \mathrm{L} & \mathit{H} & \mathrm{A}, \\
- & + & - & + & - & + & + & + & - & + & - & - & - & + & - & + &
\end{array}$$

$$\begin{array}{cccccccccccc|c}
\mathrm{B} & \mathrm{C} & \mathrm{A} & \mathit{G} & \mathrm{D} & \mathit{A} & \mathrm{G} & \mathit{D} & \mathrm{H} & \mathit{B} & \mathit{C} & \mathit{H} & \mathrm{B}, \\
+ & - & - & + & - & + & - & + & - & - & + & + &
\end{array}$$

$$\begin{array}{cccccccc|c}
\mathrm{C} & \mathrm{A} & \mathit{G} & \mathrm{D} & \mathit{A} & \mathrm{G} & \mathit{D} & \mathit{C} & \mathrm{C}; \\
- & - & + & - & + & - & + & + &
\end{array}$$

and, finally,

$$\begin{array}{cccccc|c}
\mathrm{A} & \mathit{G} & \mathrm{D} & \mathit{A} & \mathrm{G} & \mathit{D} & \mathrm{A}, \\
- & + & - & + & - & + &
\end{array}$$

which is the simple irreducible knot of figures 1 and 3 above.

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