XXIII.—Amphicheiral Knots. By Mary Gertrude Haseman, Ph.D. *Communicated by* Dr C. G. Knott, *General Secretary*. (With One Plate.)

(MS. received May 20, 1918. Read November 4, 1918. Issued separately December 22, 1919.)

§ 1. The Intrinsic Symbol of an Amphicheiral.

The intrinsic symbol * of an amphicheiral knot is based on the idea of the sequence of the crossings; it replaces each letter of the alphabetical symbol by a number equal to one-half of the number of crossings intervening before the next occurrence of that letter as the knot is traversed in a definite direction. Hence it is seen that two knots, which have the same intrinsic symbol, are identical. Since the same number of pairs of crossings must elapse before the next occurrence of corresponding crossings of two identical knots when the knots are traversed in a given direction, it is seen that the converse is true also. It may be necessary to consider the complementary intrinsic symbol in order to detect identical knots. For example, the two knots

(1) 10 5 5 9 9 9 9 3 4 4 4 4 8 8 10 5 5 9 9 9 9 3 4 4 4 4 8 8
a g b l c m d h i j f k g b h n e i j f k a l c m d n e,

(2) 3 5 5 9 9 9 9 10 4 4 4 4 8 8 3 5 5 9 9 9 9 10 4 4 4 4 8 8
a g b l c m d a e j f k g b h n i e j f k h l c m d n i,

are found to be identical since the intrinsic symbol of (1) coincides with the complementary symbol of (2). That they are identical may be verified by the fact that their compartment symbol is

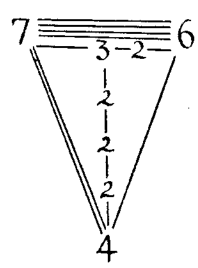


Fig. 1.

By an interchange of the two crossings, *a* and *h* in (2), it is seen that the two symbols may be made to coincide. So two amphicheiral knots are identical when their intrinsic symbols agree except for an interchange of complementary numbers.

The intrinsic symbol of an amphicheiral knot of the first order offers certain points of interest. An amphicheiral centre of an amphicheiral knot of order 1 is

* M. G. Haseman, *Trans. Roy. Soc. Edin.*, vol. lii, p. 235.

defined as a mid-point of a lap of the thread so located that corresponding crossings occur at equal arcual distances when the knot is transversed along this thread in opposite directions from the point. Let the amphicheiral centre ϕ_1 be the mid-point of the lap of thread between the two corresponding crossings p and q of a knot with n crossings, and denote by a_1, β_1 the number of pairs of crossings which elapse before the next occurrence of p, q respectively as the knot is traversed from p to ϕ_1 through q. Hence $n-1-a_1$, $n-1-\beta_1$ will be the number of pairs of crossings which elapse before the next occurrence of the crossings p, q respectively as the knot is traversed from q to ϕ_1 through p. By the definition of an amphicheiral centre, $a_1=n-1-\beta_1$ or $a_1+\beta_1=n-1$. Similarly $a_j+\beta_j=n-1$, where a_j and β_j are two crossings of the intrinsic symbol at equal arcual distances from ϕ_1. Thus an amphicheiral knot of the first order, as well as its pairs of amphicheiral centres, may be detected very easily from its intrinsic symbol. For example, the intrinsic symbol

$$\begin{matrix} 4 & 4 & 4 & 8 & 4 & 8 & \cdot & 5 & 9 & 5 & 9 & 9 & 9 & 4 & \cdot & 9 & 4 & 4 & 4 & 8 & 4 & 8 & \cdot & 5 & 9 & 5 & 9 & 9 & 9 & 4 & \cdot & 9 \\ a & f & b & k & c & l & & d & n & e & a & f & b & g & & c & h & m & i & d & j & e & & k & g & l & h & m & i & n & & j \end{matrix}$$

exhibits one pair of amphicheiral centres between the crossings l, d and e, k, as well as a second pair between g, c and n, j.

On the other hand, from TAIT'S* definition of an amphicheiral of the second order, it is seen that every crossing is separated from its correspondent by $n-1$ pairs of crossings, and hence the first n numbers of the intrinsic symbol are identically equal to the remaining numbers, the sequence of numbers being the same. Accordingly the knot whose intrinsic symbol is shown in (1) on page 1 is an amphicheiral of the second order.

§ 2. A NEW CONSTRUCTION FOR THE AMPHICHEIRALS OF ORDER 1.

The intrinsic symbol for all of the amphicheiral knots of the first class of orders 1 and 2, as constructed by TAIT † is arranged in one of the two sequences stated above, whereas this is not the case for all of the amphicheirals of the second class of orders 1 and 2. However, in a census (M. G. HASEMAN, *Trans. Roy. Soc. Edin.*, vol. lii, p. 253) of the amphicheiral knots with twelve crossings, it is found that the form shown in fig. 3, which is obtained by the unsymmetrical distortion $D_1{}^a\ D^i$ of the form shown in fig. 2, and belongs therefore to the second class, possesses the intrinsic symbol,

$$1\ 7\ 7\ 10\ 4\ 4\ 1\ 7\ 7\ 10\ 4\ 4\ 1\ 7\ 7\ 10\ 4\ 4\ 1\ 7\ 7\ 10\ 4\ 4\ ;$$

this classes it among the amphicheirals of the second order. But it is impossible to put this knot on the sphere so that corresponding compartments are opposite; that is to say, it cannot be constructed in the plane by means of a great circle and a pair of twin circuits. Hence it cannot belong to order 2 as defined by TAIT.‡ It is

* TAIT, *Trans. Roy. Soc. Edin.*, vol. xxxii, p. 497. † *Ibid.*, Plate LXXIX. ‡ *Ibid.*, p. 497.

possible, however, to construct on the sphere one of the amphicheiral forms of the knot given in fig. 2 by means of a curve C_1 in contact with a great circle at two diametrically opposite points, P_1, P_2, and of a curve C_2 which is obtained as a reflection of C_1 in the plane π_1 of the great circle, followed by a second reflection in a plane π_2 passed through the points P_1, P_2 perpendicularly to the plane π_1. To secure this construction in the plane suppose c_1, which is the projection of the curve C_1 in the plane π_1, to be the broken curve in fig. 4, with contacts at the diametral points p_1, p_2. The curve c_2, represented by the dotted curve, is the same curve as c_1, but drawn on the outside of the circle and reflected in the line p_1p_2. Now, imagine the curve c_2 to be rotated through an angle of $\frac{\pi}{4}$ to the right or left; the resulting knot, where the contacts are regarded as crossings, is found to be the knot shown in fig. 3.

The foregoing construction led me to seek for a similar construction in the plane of the amphicheirals of the first order with any number of crossings. Let the curve

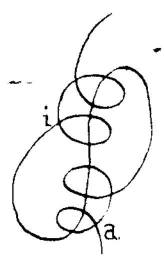


FIG. 2.

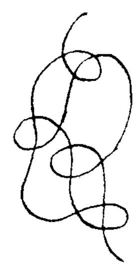


FIG. 3.

c_1 have κ contacts, 2ι intersections with the circle and σ self sections; and denote by c_2 the same curve on the outside of the circle but reflected in a diameter passing through one of the contacts. Now, imagine c_2 rotated through angles $\frac{\pi}{\kappa}$, $\frac{2\pi}{\kappa}$, etc., and the resulting curve is found to be an amphicheiral of order 1 with $2(\kappa+2\iota+\sigma)$ crossings. The curve c_2', obtained by the reflection in the plane π_1 of the great circle, ensures the desired correspondence of compartments; the reflection of the curve c_2' in the plane π_2 does not alter the number of compartments, nor the number of laps of thread bounding each compartment; instead it interchanges corresponding adjacent compartments. For instance, suppose the two regions ρ_1, ρ_2 by the first reflection to go into the two adjacent regions ρ_1', ρ_2' respectively. If, now, by the second reflection ρ_1 becomes adjacent to ρ_2', then ρ_2 must become adjacent to ρ_1' by the same reflection. Since rotation through an angle merely adds one crossing to each of the regions which are in the relation of ρ_1, ρ_2', then the desired correspondence of compartments remains unaltered. Further corresponding crossings occur at equal arcual distances from the mid-points of the lap of thread common to two corresponding compartments. Hence the knot is an amphicheiral of the first order.

It is possible to have the two curves C_1 and C_2 in their various positions intersect in σ' pairs of points, although it is not always possible to make corresponding laps of the thread intersect without introducing extra crossings. By this process I have succeeded in constructing all of the amphicheirals of order 1 with four, six, eight, and ten crossings (Nos. 1–21 in the Plate *), and find the tenfold knot No. 21 in the Plate, omitted by TAIT in his census (*Trans. Roy. Soc. Edin.*, vol. xxxii, Plate LXXIX).

Likewise the knot No. 22 in the Plate has been omitted from my census (see *Trans. Roy. Soc. Edin.*, vol. lii, pp. 253–4) of the amphicheirals with twelve crossings. It is possible that this method will reveal other omissions. It is to be noted that the maximum number of contacts were used in the constructions of these amphicheirals,

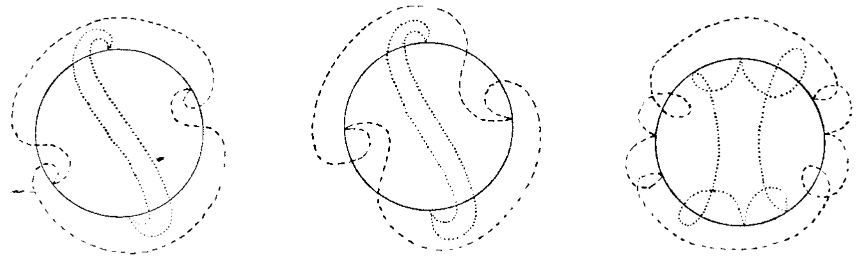


FIG. 4. FIG. 5. FIG. 6.

but I cannot say whether this is necessary in the construction of the knots with a greater number of crossings.

§ 3. SKEW AMPHICHEIRALS.

If, however, the curve c_2', obtained by a single reflection in the plane π_1, is used instead of the curve C_2, there results upon rotation a knot which can be distorted into an amphicheiral of the first order—that is to say, it belongs to TAIT's second class. When the curve C_1 is symmetrical about the line p_1p_2, the resulting knot is an amphicheiral of the first class of order 1, since then the curve c_2' is identical with the curve C_2.

In this construction there arise certain knots, called by me skew amphicheirals of the second order, which exhibit the amphicheiral symmetry in spite of the fact that they belong, by the above statement, to the second class of order 1. The intrinsic symbol of all such knots, as I have found, classes them with the amphicheirals of the first class of order 2, although corresponding regions are not opposite on the sphere. An example of a skew amphicheiral is shown in fig. 5; it is found to be identical with the knot in fig. 3. It is to be noted that in the knot shown in fig. 5 the curve c_2' possesses symmetry of such a nature that its relation to curve c_1 is the same whether c_2' be rotated to the right or left.

* The numbers at the lower right-hand corners are TAIT's numbers.

Another very good example (see fig. 6) of a skew amphicheiral is given by the symbol

1	3	3	22	1	8	8	22	20	20	15	15	1	3	3	22	1	8	8	22	20	20	15	15
a	*e*	*b*	*a*	*c*	*l*	*d*	*c*	*e*	*b*	*f*	*v*	*g*	*k*	*h*	*g*	*i*	*r*	*j*	*i*	*k*	*h*	*l*	*d*
1	3	3	22	1	8	8	22	20	20	15	15	1	3	3	22	1	8	8	22	20	20	15	15
m	*q*	*n*	*m*	*o*	*x*	*p*	*o*	*q*	*n*	*r*	*j*	*s*	*w*	*t*	*s*	*u*	*f*	*v*	*u*	*w*	*t*	*x*	*p*.

Because of the relation of the curve c_2' to the curve c_1 the crossing a may correspond to either g or s; likewise the crossing m may correspond to either g or s. Therefore we may expect not only the numbers in the last half of the sequence to be a repetition of those in the first half, but also the first half to consist of a repetition of a certain group of numbers; the number of repetitions will probably depend on the number of contacts.

The only skew amphicheirals, which I have found, may be obtained as the unsymmetrical distortions of an amphicheiral of the first class of order 1, and in view of the fact that the curve for knots with 4, 6, 8, 10, 12, 14 crossings seems to lead always to a knot which can be distorted into an amphicheiral of the first class of order 1, I am of the opinion that they do not constitute a distinct class, although it may be that they will form another class in the case of the knots with a greater number of crossings.

If, therefore, an amphicheiral knot is defined as one whose primary and secondary symbols are identical—that is to say, one whose intrinsic symbol belongs to one of the two arrangements mentioned on p. 598—it is seen that TAIT's classification given in *Trans. Roy. Soc. Edin.*, vol. xxxii, p. 499, is sufficient provided that it be admitted that an amphicheiral knot can belong to the first class of one order and to the second class of the other order.

§ 4. AMPHICHEIRAL KNOTS OF ORDER 2 WITH FOURTEEN CROSSINGS.

As has been shown by TAIT, *Trans. Roy. Soc. Edin.*, vol. xxxii, there are no amphicheiral knots of order 2 with 4, 6, 8, or 10 crossings. There are two knots* of the second order with twelve crossings, both of which may be constructed on models involving one pair of contacts, although they appear among the knots which required a greater number of contacts. In a consideration of the maximum number of contacts necessary to construct the amphicheirals with n crossings I was led to construct the amphicheirals of the second order with fourteen crossings, of which there are ten in number, as shown in Nos. 23–32 in the Plate. Nine of these were constructed on models with one pair of contacts, whereas the tenth one, No. 32, required two or more pairs of contacts. Hence it will be necessary to pass to the amphicheirals with a greater number of crossings in order to determine the maximum number of contacts required.

An interesting amphicheiral is the form which is obtained by the single distor-

* M. G. HASEMAN, *Trans. Roy. Soc. Edin.*, vol. lii, p. 254.

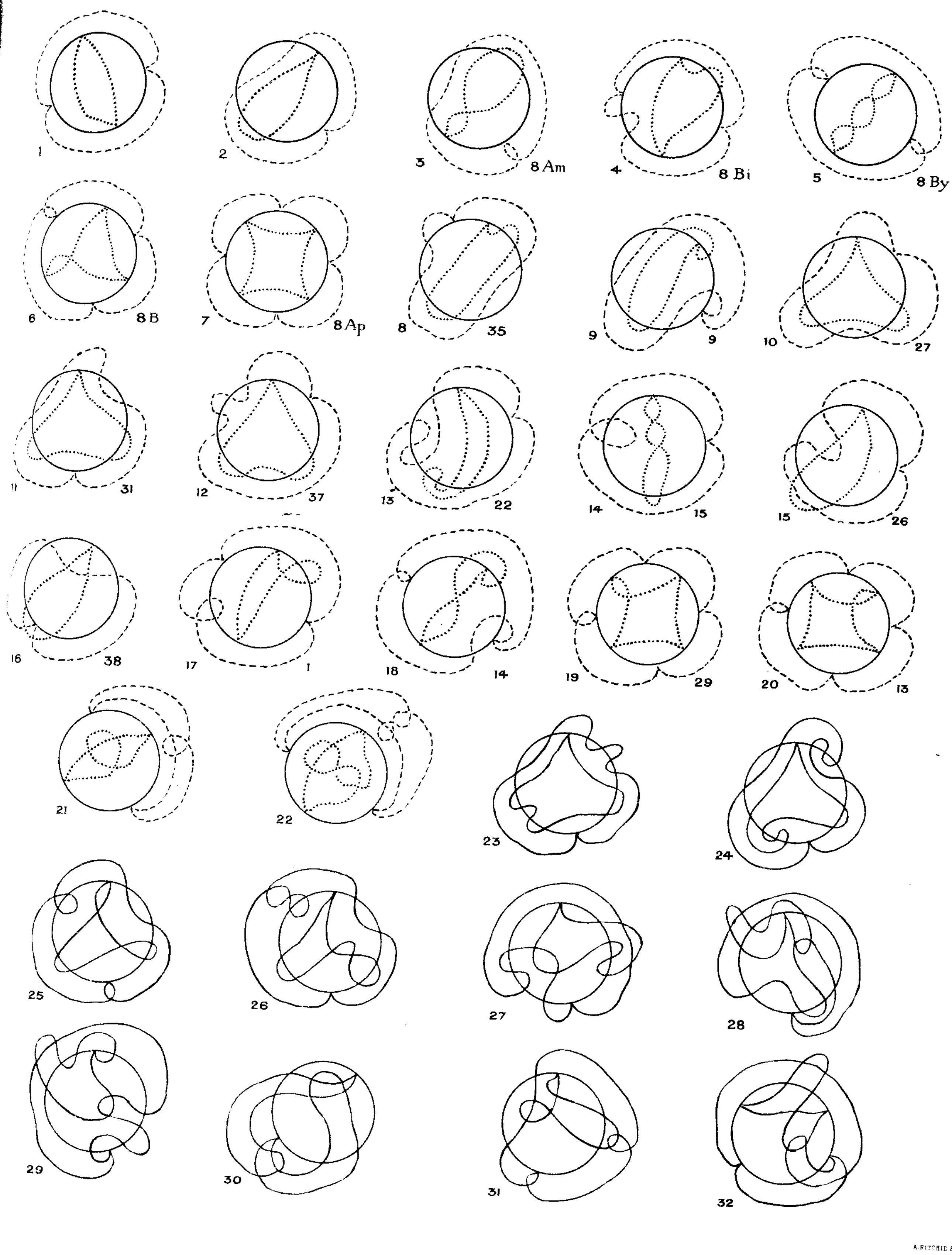
tion D_1 of the amphicheiral of the second order with fourteen crossings, No. 25 in the Plate. Its intrinsic symbol

11 10 3 3 11 10 4 3 1 10 10 12 4 3 10 9 1 3 3 12 10 9 3 2 10 10 3 2

shows that it is an amphicheiral of the first class of order 1 with one pair of amphicheiral centres. This is an example of TAIT'S supposed third class (*Trans. Roy. Soc. Edin.*, vol. xxxii, p. 499), which has the property of being changed into its own perversion by a single distortion, but, contrary to his idea, it must belong to the first class of order 1 and to the second class of order 2.

It is of interest to note that all of those amphicheirals which belong simultaneously to the first class of orders 1 and 2 exhibit two pairs of amphicheiral centres with the exception of one, which has fourteen pairs.

Mary G. Haseman: Amphicheiral Knots.



A. Ritchie & Son, Edin^r