which completes the induction for this case. In the remaining case where
the point \( Q \) is on the curve \( k_0 \), the only difference is that an arc, \( l_1 \), of a curve
of intersection in \( \alpha_1 \), and not necessarily an entire curve, approaches the
curve \( k_0 \) as \( \alpha_1 \) approaches \( \alpha_0 \). The necessary deformation of \( \alpha_1 \) is one such
that the arc (or curve) \( l_1 \) shrinks to the point \( Q \) as \( \alpha_1 \) approaches \( \alpha_0 \). We
perform a similarly modified deformation on \( \sigma_2 \) and complete the argument
just as before, thereby proving the theorem.

A similar reduction may be applied for the case \( p = 1 \), but at some stage
of the process the curve \( k_0 \) will be non-bounding. The side of \( \sigma \) containing
the plane surface \( C \) bounded by \( k_0 \) will thus have to be tubular, that is to
say, homeomorphic with the interior of an anchor ring. This is the the-
orem predicted by Tietze. For a general value of \( p \), it is easy to show that
the linear connectivity of either region bounded by \( \sigma \) is \((P_1 - 1) = P\), but
the group of the region may be very complicated.

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AN EXAMPLE OF A SIMPLY CONNECTED SURFACE BOUND-
ING A REGION WHICH IS NOT SIMPLY CONNECTED

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The following construction leads to a simplified example of a surface \( \Sigma \)
of genus zero situated in spherical 3-space and such that its exterior is
not a simply connected region. The surface \( \Sigma \) is obtained directly without
the help of Antoine's inner limiting set.

The surface \( \Sigma \) will be the combination, modulo 2, of a denumerable
infinity of simply connected surfaces \( S_i \) (\( i = 1, 2, \ldots \)), all precisely similar
in shape, though their dimensions diminish to zero as \( i \) increases without
bound. The shape of the surface \( S_i \) may perhaps be described most readily
by referring to the accompanying figure in which the surfaces \( S_2 \) and \( S_3 \)
are represented. By comparison with \( S_2 \) to which, by hypothesis, all the
other surfaces \( S_i \) are similar, we see that the general surface \( S_i \) is roughly
like a tube twisted into the shape of the letter \( C \) and terminating in a pair
of circular 2-cells, \( \beta_i \) and \( \gamma_i \). There is, however, a slight protuberance in
the side of the tube terminating in a 2-cell \( \alpha_i \).

The position of the surfaces \( S_1 \), \( S_2 \), and \( S_3 \) with respect to one another
is indicated in the figure, though only the two ends of \( S_1 \) terminating in
\( \alpha_1 \) and \( \beta_1 \) are shown. It will be noticed that the faces \( \alpha_2 \) of \( S_3 \) and \( \alpha_3 \) of \( S_3 \)
are subfaces of the faces \( \beta_1 \) and \( \gamma_1 \) of \( S_1 \), respectively, and that the surfaces
$S_2$ and $S_3$ are hooked around one another, so to speak. When $S_2$ and $S_3$ are added modulo 2 to $S_1$ (which means that the points of $\alpha_2$ and $\alpha_3$ must be deleted from the combined surfaces), a simple closed surface, $\Sigma_1$, is obtained. The surface $\Sigma_1$ will be regarded as the first approximation of the desired surface $\Sigma$. The next approximating surface, $\Sigma_2$, is obtained by adjoining to $\Sigma_1$, modulo 2, the next four surfaces, $S_4$, $S_5$, $S_6$, $S_7$. The first two of these will be related to $S_2$ and the last two to $S_3$ in exactly the same way that the surfaces $S_2$ and $S_3$ are related to $S_1$; that is to say, a similarity transformation of the 3-space carrying $S_1$ into $S_3$ would carry $S_2$ and $S_3$ into $S_4$ and $S_5$, respectively, while one carrying $S_1$ into $S_3$ would carry $S_2$ and $S_3$ into $S_6$ and $S_7$, respectively. The third approximation $\Sigma_3$ is obtained by adjoining the next eight surfaces $S_8$, ..., $S_{15}$ in a similar manner, so that the pair $S_8$ and $S_9$ are attached to $S_4$ just as the pair $S_2$ and $S_3$ are attached to $S_1$, and so on. The surface $\Sigma$ is the limiting surface approached by the sequence $\Sigma_1$, $\Sigma_2$, $\Sigma_3$, ... It will be seen without difficulty that the interior of the limiting surface $\Sigma$ is simply connected, and that the surface itself is of genus zero and without singularities, though a hasty glance at the surface might lead one to doubt this last statement. The exterior $R$ of $\Sigma$ is not simply connected, however, for a simple closed curve in $R$ differing but little from the boundary of one of the cells $\gamma_i$ cannot be deformed to a point within $R$. It is easily shown, in fact, that the group of $R$ requires an infinite number of generators.

The points $K$ of $\Sigma$ which are not points of the approximating surfaces $\Sigma_i$ form an inner limiting set of a much simpler type than the inner limiting set of Antoine, as was pointed out to me by Professor Veblen. For we
may close down upon the points $K$ by a system of spheres rather than by a complicated system of linking anchor rings.

This example shows that a proof of the generalized Schönflieess theorem announced by me two years ago, but never published, is erroneous.

**REMARKS ON A POINT SET CONSTRUCTED BY ANTOINE**

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From the consideration of a remarkable point set discovered by Antoine, the following two theorems may be derived:

*Theorem 1.* There exists a simple closed surface of genus 0 in 3-space such that the interior of the surface is not simply connected but has, on the contrary, an infinite group.

*Theorem 2.* There exists a simple closed curve in 3-space which is not knotted, inasmuch as it bounds a 2-cell without singularities, and yet such that its group (as defined by Dehn) is not the same as the group of a circle in 3-space.

It follows without difficulty from the second theorem that if an isotopic deformation be defined in the customary manner (cf. for example, Veblen's Cambridge Colloquium Lectures), the group of a curve in 3-space is not an isotopic invariant. This suggests that a modified definition of isotopy might be advisable.

Antoine's point set is obtainable as follows. Within an anchor ring $\pi$ in 3-space, we first construct a chain $C$ or anchor rings $\pi_i$ ($i = 1, 2, \ldots, s$) such that each ring $\pi_i$ is linked with its immediate predecessor and immediate successor, after the manner of links in an ordinary chain, and such, also, that the last ring $\pi_s$ is linked with the first $\pi_1$, thereby making the chain closed. We further suppose that the chain $C$ is constructed in such a way that it winds once around the axis of the ring $\pi$. Secondly, we make a similar construction within each of the anchor rings $\pi_i$, thereby obtaining chains $C_i$ made up of rings $\pi_{ij}$ ($j = 1, 2, \ldots, s$), and repeat the process indefinitely, obtaining chains $C_{ij}$ within $\pi_{ij}$, $C_{ijk}$ within $\pi_{ijk}$, and so on. If we think of the system of rings within one of the rings $\pi_i$ as the image of the system of rings within the ring $\pi$ under a similarity transformation carrying the interior and boundary of $\pi$ into the interior and boundary of $\pi_i$, it is clear that the diameters of the rings $\pi_{ijk} \ldots$ decrease towards zero as the number of subscripts to their symbols increases. The inner limiting set $\Sigma$ determined by the infinite sequences of rings $\pi^i, \pi_{ii}, \pi_{ijk}, \ldots$ is the