VECTOR BUNDLES AND HOMOGENEOUS SPACES

BY

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Dedicated to Professor Marston Morse

Introduction. In [1] we introduced for a space X the "ring of complex vector bundles" $K(X)$. The Bott periodicity of the infinite unitary group [8; 9; 10] implied that K satisfied the "Künneth formula"

$$K(X \times S^2) \simeq K(X) \otimes K(S^2)$$

which was fundamental for the proof of the differentiable Riemann-Roch theorems [1; 16].

Using the Bott periodicity we construct in §1 a "periodic cohomology theory": For every integer n, the abelian group $K^n(X)$ is defined, $K^0(X)$ is $K(X)$ and $K^{n+2}(X)$ is isomorphic with $K^n(X)$, the group $K^1(X)$ is the kernel of the homomorphism $K^0(X \times S^1) \to K^0(X)$ induced from the embedding $X \to X \times S^1$. This cohomology theory satisfies all the axioms of Eilenberg-Steenrod [14] except the "dimension axiom." For the space consisting of a single point, K^n is infinite cyclic for even n and vanishes for odd n. The axioms without the dimension axiom do not characterize the theory, even if the values of K^n are given for a point. There is a spectral sequence relating the ordinary cohomology theory with our periodic theory (§2).

In §§3–5 we try to get information on K^0 and K^1 for classifying spaces and certain homogeneous spaces. An important tool is the differentiable Riemann-Roch theorem which we recall in the beginning of §3. The final goal would be to answer all those questions for the K-theory on homogeneous spaces which for the ordinary cohomology theory have been treated so successfully by A. Borel (see for example [3]). We can give only partial results in this direction. The new cohomology theory can be applied to various topological questions and may give better results than the ordinary cohomology theory, even if the latter one is enriched by cohomology operations (see [2] and M. F. Atiyah and J. A. Todd, *On complex Stiefel manifolds*, to appear in Proc. Cambridge Philos. Soc.). This justifies the new theory.

In spite of its length, the present paper is by no means a final exposition. The proofs are often sketchy and the definitions and results could be generalized in certain cases. For example, using real vector bundles and the Bott periodicity of the infinite orthogonal group, we can define a periodic cohomology theory with period 8. This is not more difficult than in the unitary case. Furthermore, the definition of $K(X)$ in 1.1 can be given for any topological space. For convenience, we have restricted the theory to the special class $\mathfrak{A}$ (see 1.1). We have then the homotopy classification theorem (1.3)

$$K(X) \cong [X, Z \times B_U], \qquad (X \in \mathfrak{A}). \tag{1}$$

For this actually $\mathfrak{A}$ could be chosen much larger. But in general (1) would be wrong. The restriction to $\mathfrak{A}$ simplifies the presentation of certain consequences drawn from the spectral sequence. For any topological space, we can take the right side of (1) as a definition of a functor $k(X)$. If $Z \times B_U$ is endowed with a natural structure of a commutative ring (up to homotopy), then $k(X)$ has a natural (commutative) ring structure for any space X and the rings $K(X)$ and $k(X)$ are isomorphic if $X \in \mathfrak{A}$. Such a "ring" structure on $Z \times B_U$ has been defined by Milnor (not published). In view of Milnor's construction it would perhaps be more natural to study the functor $k(X)$, but since Milnor's result is not yet at our disposal we have studied $K(X)$ where sum and product structure is automatically given by the Whitney sum and the tensor product of vector bundles.

For the classifying spaces B_G we have defined $\mathcal{K}(B_G)$ as an inverse limit indicating by the curly letter that we mean neither $K(B_G)$ nor $k(B_G)$. We conjecture that $\mathcal{K}(B_G)$ is isomorphic to $k(B_G)$ for any compact Lie group G. But we shall deal with this question elsewhere. We prove for a compact connected Lie group G that $\mathcal{K}(B_G)$ is isomorphic with the completed representation ring $\hat{R}(G)$ (see 4.8).

1. A cohomology theory derived from the unitary groups.

1.1. Let $\mathfrak{A}$ be the class of those spaces which can carry the structure of a finite CW-complex. For $X \in \mathfrak{A}$ we have defined in [1] an abelian group $K(X)$. There we gave the definition only for a connected X, but we may define $K(X)$ in general as the direct sum of the groups $K(X_i)$ where the X_i are the connectedness components of X. For the sake of completeness we recall the definition of $K(X)$ and give it directly for a space $X \in \mathfrak{A}$ not necessarily connected.

We adopt the usual definition of a complex vector bundle over X except that we allow the bundle to have fibres of different dimensions over the various connectedness components of X. We can now verbally repeat the definition of [1]:

Let $F(X)$ be the free abelian group generated by the set of all isomorphism classes of complex vector bundles over X. To every triple $t = (\xi, \xi', \xi'')$ of vector bundles with $\xi \cong \xi' \oplus \xi''$ we assign the element $[t] = [\xi] - [\xi'] - [\xi'']$ of $F(X)$, where $[\xi]$ denotes the isomorphism class of ξ. The group $K(X)$ is defined as the quotient of $F(X)$ by the subgroup generated by all the elements of the form $[t]$.

The tensor product of vector bundles defines a commutative ring structure in $K(X)$; the unit 1 is given by the trivial bundle of dimension 1.

K is a contravariant functor: for a continuous map $f : Y \to X$ $(Y, X \in \mathfrak{A})$ we have the natural ring homomorphism $f^! : K(X) \to K(Y)$ induced by the lifting of bundles under f. We denote it by $f^!$ to distinguish it from the analogous homomorphism f^* in ordinary cohomology theory.

1.2. Let $\tilde{\mathfrak{A}}$ be the class whose objects are the pairs (X, x_0) with $X \in \mathfrak{A}$ and $x_0 \in X$. Usually we shall write an object of $\tilde{\mathfrak{A}}$ simply by indicating the space X. Very often the base point x_0 of X is naturally given by the context. For $X \in \tilde{\mathfrak{A}}$ we define the *reduced group* $\tilde{K}(X)$ as follows: the ring $K(\{x_0\})$ is canonically isomorphic with Z (the ring of integers). The imbedding $i : \{x_0\} \rightarrow X$ induces the ring homomorphism

$$i^! : K(X) \rightarrow K(\{x_0\}) = Z.$$

We define $\tilde{K}(X)$ to be the kernel of $i^!$. It is an ideal of $K(X)$. Whenever a symbol like $\tilde{K}(X)$ occurs it is to be understood that X is a space with base point, i.e., an object of $\tilde{\mathfrak{A}}$.

We now consider the class $\mathfrak{B}$ consisting of pairs (X, Y) where X can be given the structure of a finite CW-complex in such a way that Y becomes a subcomplex. For $(X, Y) \in \mathfrak{B}$ we define

$$K(X, Y) = \tilde{K}(X/Y).$$

Here X/Y is obtained from X by collapsing Y to a point which becomes then the base point of X/Y. By [19], $X/Y \in \tilde{\mathfrak{A}}$. Note that $\tilde{K}(X) = K(X, x_0)$ for $X \in \tilde{\mathfrak{A}}$. If Y is empty ($Y = \emptyset$), then $X/\emptyset = X^+$ (where X^+ is the topological sum of X with an extra point which becomes base point of X^+) and $K(X, \emptyset) = \tilde{K}(X^+) = K(X)$.

For $X, Y \in \tilde{\mathfrak{A}}$ the objects $X \vee Y$ and $X \wedge Y$ of $\tilde{\mathfrak{A}}$ are defined. (In the literature, $X \wedge Y$ is also denoted by $X \# Y$). $X \vee Y$ is obtained from the topological sum of X and Y by identifying the base point of X with the base point of Y to one point which becomes the base point of $X \vee Y$. The space $X \wedge Y$ is $X \times Y$ with the union of the axis $x_0 \times Y$ and $X \times y_0$ collapsed to a point which becomes the base point of $X \wedge Y$. We have the natural maps

$$X \vee Y \rightarrow X \times Y \rightarrow X \wedge Y$$

and may write

$$X \wedge Y = X \times Y/X \vee Y. \tag{1}$$

The operations $\vee$ and $\wedge$ are associative and commutative and $\wedge$ is distributive over $\vee$. This means, for example, that there is a *canonical homeomorphism* between $X \wedge Y$ and $Y \wedge X$.

If $S^n \in \tilde{\mathfrak{A}}$ is the standard n-sphere with base point, we write

$$S^n(X) = S^n \wedge X, \qquad (X \in \tilde{\mathfrak{A}}). \tag{2}$$

This is the nth suspension of X. Since

$$S^n = S^1 \wedge S^1 \wedge \cdots \wedge S^1 \qquad (n \text{ times})$$

it follows that $S^n(X)$ is the n times iterated suspension of X.

DEFINITION. *For any integer* $n \geq 0$ *we put* $K^{-n}(X, Y) = \tilde{K}(S^n(X/Y))$, $((X, Y) \in \mathfrak{B})$. *For* $X \in \mathfrak{A}$ *we put* $K^{-n}(X) = K^{-n}(X, \emptyset) = \tilde{K}(S^n(X^+))$. *For* $X \in \tilde{\mathfrak{A}}$ *with base point* x_0 *we put* $\tilde{K}^{-n}(X) = K^{-n}(X, x_0) = \tilde{K}(S^n(X))$.

For $n = 0$ we have the groups already defined:

$$K^0(X, Y) = K(X, Y), \quad K^0(X) = K(X), \quad \tilde{K}^0(X) = \tilde{K}(X).$$

Of course, the K^{-n} are also contravariant functors.

1.3. We write $[A, B]$ for the set of homotopy classes of maps of the space A into the space B and correspondingly $[A, U; B, V]$ for the homotopy classes of maps of the pair (A, U) into the pair (B, V). If the spaces A and B have base points, then we write $[A, B]_0$ for the set of homotopy classes of maps preserving base points.

Let B_U be the classifying space of the infinite unitary group **[10]** and $Z \times B_U$ the cartesian product of it with the group of integers (Z having the discrete topology). In $Z \times B_U$ we choose a base point lying in $0 \times B_U$. The classification theorem for unitary bundles **[18**, §19.3] gives rise to the following natural bijective maps (compare also [**16**, §1.7, 2.1]):

$$K(X) \cong [X, Z \times B_U], \qquad (X \in \mathfrak{A}).$$

$$\tilde{K}(X) \cong [X, Z \times B_U]_0, \qquad (X \in \tilde{\mathfrak{A}}).$$

$$\begin{aligned} K^{-n}(X, Y) &\cong [S^n(X/Y), Z \times B_U]_0, && ((X, Y) \in \mathfrak{B}), \\ &\cong [X/Y, \Omega^n(Z \times B_U)]_0, \\ &\cong [X, Y; \Omega^n(Z \times B_U), \text{point}], \\ &\cong [X, Y; \Omega^{n-1}U, \text{point}], && n > 0. \end{aligned}$$

We recall that $Z \times B_U$ is weakly homotopy equivalent to an H-space, namely to ΩU (Bott, see **[8]**). Thus all the above sets of homotopy classes are endowed with a natural group structure. The above bijections are in fact all group isomorphisms. Since U is weakly homotopy equivalent to $\Omega(Z \times B_U)$, the space $\Omega^2(Z \times B_U)$ is weakly homotopy equivalent to $Z \times B_U$ and we have an isomorphism

$$K^{-(n+2)}(X, Y) \cong K^{-n}(X, Y), \qquad n \geq 0. \tag{3}$$

We shall give later an explicit description of an isomorphism between these two groups.

If x_0 denotes the space consisting of a single point, then

$$K^{-n}(x_0) = \pi_n(Z \times B_U), \qquad n \geq 0,$$

and thus **[9]**

$$K^{-n}(x_0) \cong Z \quad \text{for } n \text{ even and } K^{-n}(x_0) = 0 \quad \text{for } n \text{ odd.}$$

1.4. PROPOSITION. *If* $(X, Y) \in \mathfrak{B}$ *we have exact sequences*

$$\text{(i)} \quad \cdots \to K^{-(n+1)}(Y) \xrightarrow{\delta} K^{-n}(X, Y) \to K^{-n}(X) \to K^{-n}(Y) \to \cdots \to K^0(X, Y) \to K^0(X) \to K^0(Y),$$

$$\text{(ii)} \quad \cdots \to \tilde{K}^{-(n+1)}(Y) \xrightarrow{\delta} K^{-n}(X, Y) \to \tilde{K}^{-n}(X) \to \tilde{K}^{-n}(Y) \to \cdots \to K^0(X, Y) \to \tilde{K}^0(X) \to \tilde{K}^0(Y).$$

For (ii) *we assume* $X, Y \in \bar{\mathfrak{A}}$ *with* $x_0 = y_0 \in Y$.

Proof. We use the paper of Puppe [17]. If Y and X are arbitrary spaces with base point and $f : Y \to X$ a map preserving base points, then there is a sequence of spaces and maps (with base points)

$$Y \xrightarrow{f} X \xrightarrow{Pf} C_f \xrightarrow{Qf} S^1 Y \to S^1 X \to S^1 C_f \to S^2 Y \to S^2 X \to \cdots$$

such that the following is true: if V is any space with base point, then the functor $[\ ,\ V]_0$ gives an exact sequence of sets. Here we note that exactness is a property of sets with preferred elements—the group structure is irrelevant. The preferred element is always given by the constant map onto the base point. We recall the construction of C_f. First we take the cone

$$CY = Y \times I / Y \times 1 \cup y_0 \times I.$$

Then we take the topological sum $CY + X$ in which we identify $(y, 0) \in CY$ with $f(y)$ for each $y \in Y$. The space C_f contains X as subspace. C_f/X is (canonically homeomorphic with) the first suspension of Y. This gives rise to the maps $Y \to^f X \to^{Pf} C_f \to^{Qf} S^1 Y$. All the other maps in Puppe's sequence are suspensions of these. If Y is a subspace of X and f the injection, then we have a natural homeomorphism $X/Y \cong C_f/CY$. If (X, Y) belongs to $\mathfrak{B}$ then it satisfies the homotopy extension condition and according to Puppe the map $C_f \to C_f/CY$ followed by the above mentioned homeomorphism is a homotopy equivalence h. The composition $h \circ Pf$ is the natural projection $X \to X/Y$. Taking this into account Puppe's theorem applied to $V = Z \times B_U$ gives the exact sequence (ii) and all homomorphisms in this sequence are canonically defined by Puppe's maps. $K^{-n}(X, Y) \to \tilde{K}^{-n}(X)$ is induced from $X \to X/Y$. The sequence (i) is obtained by replacing in (ii) Y and X by Y^+ and X^+ respectively.

Remark. If $Y = \{x_0\}$ then the sequence (i) breaks off in split exact sequences

$$0 \to \tilde{K}^{-n}(X) \to K^{-n}(X) \to K^{-n}(x_0) \to 0.$$

Hence

$$K^{-n}(X) \cong \tilde{K}^{-n}(X) \oplus \pi_n(Z \times B_U), \qquad \text{(see 1.3).}$$

The exact sequence (i) is obtained from (ii) by adding to $\tilde{K}^{-n}(X)$ and also to $\tilde{K}^{-n}(Y)$ the direct summand $\pi_n(Z \times B_U)$.

1.5. We have mentioned in 1.1 that $K(X) = K^0(X)$ is a commutative ring. We wish to define more generally products also involving the groups K^{-n} $(n \geq 0)$.

Suppose $X, Y \in \tilde{\mathfrak{A}}$. Then $X \vee Y = X \times y_0 \cup x_0 \times Y$ is a subspace of $X \times Y$. We apply 1.4 (ii) to the pair $(X \times Y, X \vee Y)$. The exact sequence breaks off in this case into split exact sequences.

$$(4) \qquad 0 \to \tilde{K}^{-n}(X \wedge Y) \to \tilde{K}^{-n}(X \times Y) \to \tilde{K}^{-n}(X \vee Y) \to 0, \qquad (n \geqq 0),$$

and we have a canonical decomposition

$$(5) \qquad \tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y).$$

For the proof of (4) we observe that $\tilde{K}^{-n}(X \times Y) \to \tilde{K}^{-n}(X \vee Y)$ is surjective and that this homomorphism may be regarded as the projection onto a direct summand. For this we make use of

$$\tilde{K}^{-n}(X \vee Y) = \tilde{K}(S^n(X \vee Y)) = \tilde{K}(S^nX \vee S^nY) = \tilde{K}(S^nX) \oplus \tilde{K}(S^nY)$$
$$= \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y).$$

We have the following natural group homomorphisms which are all induced by the tensor product of vector bundles

$$(6) \qquad K(X) \otimes K(Y) \to K(X \times Y), \qquad (X, Y \in \mathfrak{A}),$$

$$(7) \qquad \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y), \qquad (X, Y \in \tilde{\mathfrak{A}}),$$

$$(8) \qquad K(X, X_0) \otimes K(Y, Y_0) \to K(X \times Y, X_0 \times Y \cup X \times Y_0),$$
$$\text{for } (X, X_0) \text{ and } (Y, Y_0) \in \mathfrak{B}.$$

It is clear how (6) is defined. If $a \in \tilde{K}(X)$ and $b \in \tilde{K}(Y)$, then the product is in the kernel of $\tilde{K}(X \times Y) \to \tilde{K}(X \times y_0 \cup x_0 \times Y)$. By (4) and (5) the product is well defined as element of $\tilde{K}(X \wedge Y)$, $(n = 0)$. If we replace in (7) X by X/X_0 and Y by Y/Y_0 we get the definition of (8). More generally we have a group homomorphism

$$(9) \qquad K^{-m}(X, X_0) \otimes K^{-n}(Y, Y_0) \to K^{-(m+n)}(X \times Y, X_0 \times Y \cup X \times Y_0),$$
$$\text{for } (X, X_0), (Y, Y_0) \in \mathfrak{B} \text{ and } m \geq 0, n \geqq 0.$$

We get this from (7) and the fact that

$$S^m(X/X_0) \wedge S^n(Y/Y_0) = S^{m+n}(X/X_0 \wedge Y/Y_0)$$
$$= S^{m+n}(X \times Y/X_0 \times Y \cup X \times Y_0).$$

The equality sign means that there is a natural homeomorphism between these spaces. If one uses the natural identification of $X \times Y$ with $Y \times X$, one gets from (9) a product

$$(9') \qquad K^{-n}(Y, Y_0) \otimes K^{-m}(X, X_0) \to K^{-(m+n)}(X \times Y, X_0 \times Y \cup X \times Y_0).$$

Lemma. *If $a \in K^{-m}(X, X_0)$ and $b \in K^{-n}(Y, Y_0)$, then $ab = (-1)^{mn}ba$ where ab is the image of $a \otimes b$ under* (9) *and ba the image of $b \otimes a$ under* (9′).

PROOF. The sign comes from the use of the various "natural identifications" between different spaces. $S^m \wedge (X/X_0) \wedge S^n \wedge (Y/Y_0)$ and $S^n \wedge (Y/Y_0) \wedge S^m \wedge (X/X_0)$ are identified just by the permutation. However, for the definition of (9) we employ the identification

$$\alpha_{m,n} : S^m \wedge S^n \to S^{m+n}$$

which comes from a map $S^m \times S^n \to S^{m+n}$ of degree $+1$ (all spheres and also the cartesian product in this order have the standard orientations). If β is the permutation $S^m \times S^n \to S^n \times S^m$, then $\alpha_{n,m} \circ \beta \circ \alpha_{m,n}^{-1}$ has degree $(-1)^{mn}$. This shows that ab and ba correspond to elements of

$$G = [S^{m+n}(X \times Y/X_0 \times Y \cup X \times Y_0), Z \times B_U]_0$$

which are related with each other by a map of S^{m+n} onto itself of degree $(-1)^{mn}$. Since the group structure of G can also be defined by the suspension coordinate like a homotopy group, the lemma follows.

1.6. Using the diagonal map as in the definition of the cup product we get:

PROPOSITION. *Let $X \in \mathfrak{A}$. Then $\sum_{n \geq 0} K^{-n}(X)$ is a graded anti-commutative ring. Let $(X, Y) \in \mathfrak{B}$. Then there is a "graded homomorphism"*

$$(\sum_{n \geq 0} K^{-n}(X)) \otimes (\sum_{m \geq 0} K^{-m}(X, Y)) \to \sum_{j \geq 0} K^{-j}(X, Y),$$

making $\sum_{m \geq 0} K^{-m}(X, Y)$ a graded module over $\sum_{n \geq 0} K^{-n}(X)$.

The products have functorial properties. For example, if $(X, X_0) \to^{f} (X', X_0')$ and $(Y, Y_0) \to^{g} (Y', Y_0')$ are maps with the pairs all belonging to $\mathfrak{B}$, then we have the commutative diagram

$$\begin{array}{ccc} K^{-m}(X', X_0') \otimes K^{-n}(Y', Y_0') & \to & K^{-(m+n)}(X' \times Y', X_0' \times Y' \cup X' \times Y_0') \\ \downarrow f^! \otimes g^! & & \downarrow (f \times g)^! \\ K^{-m}(X, X_0) \otimes K^{-n}(Y, Y_0) & \to & K^{-(m+n)}(X \times Y, X_0 \times Y \cup X \times Y_0) . \end{array} \quad (10)$$

Furthermore, for $f : Y \to X$, the induced homomorphism $f^! : \sum_{m \geq 0} K^{-m}(X) \to \sum_{m \geq 0} K^{-m}(Y)$ is a ring homomorphism, etc.

1.7. *The Bott isomorphism.* The existence of the Bott isomorphism (see 1.3 (3)) is the central and deep point of the cohomology theory we are developing. We give now the explicit description of this isomorphism.

Let x_0 be the space consisting of a single point. Then (1.3) $K^{-2}(x_0)$ is infinite cyclic. By definition $K^{-2}(x_0) = \tilde{K}(S^2)$. Let η be the complex line bundle over S^2 whose first Chern class equals the canonical generator of $H^2(S^2, Z)$. Then η represents an element $[\eta] \in K(S^2)$ and $[\eta] - 1$ is a generator of $\tilde{K}(S^2) = K^{-2}(x_0)$ which we denote by g. If $a \in K^{-m}(X, X_0)$, then $ag \in K^{-(m+2)}(X, X_0)$. Here we use 1.5 (9) with $Y = x_0$ and Y_0 empty.

THEOREM. *The map $a \to ag$ is an isomorphism of $K^{-m}(X, X_0)$ onto $K^{-(m+2)}(X, X_0)$. In particular, $\sum_{n \geq 0} K^{-n}(x_0)$ is the polynomial ring $Z[g]$.*

For a proof of this central theorem we refer to [10]. For any $(X, X_0) \in \mathfrak{B}$ the graded group $\sum_{n \geq 0} K^{-n}(X, X_0)$ is a module over $Z[g]$. Multiplication with g^k gives an isomorphism of $K^{-n}(X, X_0)$ onto $K^{-(n+2k)}(X, X_0)$. This holds in particular if X_0 is empty or reduces to the base point, i.e., $\sum_{n \geq 0} K^{-n}(X)$ and $\sum_{n \geq 0} \tilde{K}^{-n}(X)$ are both modules over $Z[g]$. Let β denote the multiplication by g. The next lemma follows from 1.6 (10).

Lemma. *If (X, Y) and (X', Y') belong to $\mathfrak{B}$ and if $f : (X, Y) \to (X', Y')$ is a continuous map, then $f^!\beta = \beta f^!$ where $f^! : K^{-n}(X', Y') \to K^{-n}(X, Y)$, $(n \geqq 0)$, is the induced homomorphism, in other words: $f^!$ is a homomorphism of $Z[g]$-modules.*

Lemma. *If $(X, Y) \in \mathfrak{B}$, then β gives a homomorphism of exact sequences* (1.4 (ii)), *i.e., we have the commutative diagram* $(n \geqq 0)$

$$\begin{array}{ccccccc}
\tilde{K}^{-(n+1)}(Y) & \xrightarrow{\delta} & K^{-n}(X, Y) & \to & \tilde{K}^{-n}(X) & \to & \tilde{K}^{-n}(Y) \\
\downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta \\
\tilde{K}^{-(n+3)}(Y) & \xrightarrow{\delta} & K^{-(n+2)}(X, Y) & \to & \tilde{K}^{-(n+2)}(X) & \to & \tilde{K}^{-(n+2)}(Y).
\end{array}$$

The corresponding statement holds for the exact sequence (1.4 (i)).

Proof. This follows from the preceding lemma. We take into account that the homomorphism δ is also induced by a map, namely by $C_f \to S^1Y$.

1.8. The group $K^{-2n}(X, Y)$ can be identified with $K^0(X, Y)$ and $K^{-(2n+1)}(X, Y)$ with $K^{-1}(X)$ by the Bott isomorphisms:

$$\beta^n : K^0(X, Y) \to K^{-2n}(X, Y),$$

$$\beta^n : K^{-1}(X, Y) \to K^{-(2n+1)}(X, Y).$$

This allows us to define $K^n(X, Y)$ for any integer n by

$$K^n(X, Y) = K^0(X, Y) \quad \text{if} \quad n \quad \text{is even},$$

$$K^n(X, Y) = K^{-1}(X, Y) \quad \text{if} \quad n \quad \text{is odd}.$$

The groups $K^n(X, Y)$ satisfy the usual axioms of a cohomology theory [14] (in the category $\mathfrak{B}$ with all continuous maps of one pair into another one being admissable) excep. that $K^n(x_0)$ does not vanish for $n \neq 0$ (1.3). The existence of an exact sequence

$$(11) \quad \cdots \to K^n(Y) \xrightarrow{\delta} K^{n+1}(X, Y) \to K^{n+1}(X) \to K^{n+1}(Y) \to \cdots (-\infty < n < \infty)$$

follows from 1.4 and the second lemma of 1.7.

Let (X, Y, Z) be a triple with $X \supset Y \supset Z$ and all the pairs (X, Y), (X, Z), (Y, Z) belonging to $\mathfrak{B}$. Then we have an exact sequence

$$(11^*) \qquad \cdots \to K^n(Y, Z) \xrightarrow{\delta} K^{n+1}(X, Y) \to K^{n+1}(X, Z) \to K^{n+1}(Y, Z) \to \cdots, \quad (-\infty < n < \infty),$$

where the δ of (11^*) is the composition $K^n(Y, Z) \to K^n(Y) \to^{\delta} K^{n+1}(X, Y)$.

The exactness of (11^*) would follow from 1.4(ii) applied to the pair $(X/Z, Y/Z)$ if this belonged to $\mathfrak{B}$. But (11^*) is also a consequence of the cohomology axioms. (Excision-, homotopy-, and dimension axioms are not needed for this formal deduction of (11^*); compare [14, Chapter I, §10].)

1.9. In 1.8 we have completed the construction of a cohomology theory satisfying all axioms except the "dimension axiom." Since these "cohomology groups" are periodic ($K^n(X, Y) = K^{n+2}(X, Y)$) it is convenient to define

$$K^*(X, Y) = K^0(X, Y) \oplus K^1(X, Y), \qquad (X, Y) \in \mathfrak{B},$$

and similarly for $K^*(X)$ and $\tilde{K}^*(X)$. $K^*(X)$ is then an anti-commutative ring, graded by Z_2, i.e., $K^0(X)$ is a subring and

$$K^0(X)\cdot K^1(X) \subset K^1(X), \qquad K^1(X)\cdot K^1(X) \subset K^0(X).$$

Moreover $K^*(X, Y)$ is a Z_2-graded module over $K^*(X)$. Since δ respects the periodicity, we have the exact triangle

$$\begin{array}{ccc} K^*(Y) & \xrightarrow{\delta} & K^*(X, Y) \\ & \nwarrow \quad \swarrow & \\ & K^*(X) & \end{array} \tag{12}$$

which resolves in an exact hexagon

$$\begin{array}{ccccc} & K^1(X, Y) & \rightarrow & K^1(X) & \\ {}^{\delta}\nearrow & & & & \searrow \\ K^0(Y) & & & & K^1(Y) \\ \nwarrow & & & & \swarrow{}_{\delta} \\ & K^0(X) & \leftarrow & K^0(X, Y) & \end{array}$$

and which has, so to speak, the exact sequence (11) as "universal covering."

For a triple X, Y, Z (see 1.8) we have the exact triangle

$$\begin{array}{ccc} K^*(Y, Z) & \rightarrow & K^*(X, Y) \\ & \nwarrow \quad \swarrow & \\ & K^*(X, Z) & \end{array} \tag{12*}$$

and the corresponding hexagon.

1.10. *The Chern character.* For each complex vector bundle ξ over the space $X \in \mathfrak{A}$ the Chern character $ch(\xi)$ is defined as an element of the rational cohomology ring $H^*(X, Q)$, [5, §9.1]. If $H^{**}(X, Q)$ denotes the direct sum of the even dimensional cohomology groups (which is a commutative subring of $H^*(X, Q)$), then $ch(\xi) \in H^{**}(X, Q)$. The definition of $ch(\xi)$ uses only the total Chern class $c(\xi)$. The classes $ch(\xi)$ and $c(\xi)$, both regarded as elements of $H^{**}(X, Q)$, determine each other. The Chern character induces a ring homomorphism [15, §12.1 (5)]

$$ch: K(X) = K^0(X) \rightarrow H^{**}(X, Q) \subset H^*(X, Q) \tag{13}$$

with

$$ch(\tilde{K}(X)) \subset \tilde{H}^*(X, Q) = \text{Kernel} \quad (H^*(X, Q) \rightarrow H^*(\{x_0\}, Q)).$$

We are now going to define a group homomorphism

$$(14) \qquad ch: K^{-n}(X, Y) \to H^*(X, Y; Q), \quad ((X, Y) \in \mathfrak{B}, n \geq 0).$$

By definition, $K^{-n}(X, Y) = \tilde{K}(S^n(X/Y))$. We have the suspension isomorphism

$$\sigma^n: \tilde{H}^*(A, Q) \to \tilde{H}^*(S^n(A), Q), \qquad A \in \mathfrak{A},$$

which raises degrees by n and is defined by tensoring $a \in \tilde{H}^*(A, Q)$ (from the left) with the canonical generator of $H^n(S^n, Z)$. If $\xi \in K^{-n}(X, Y)$, let ξ' be the "corresponding element" of $\tilde{K}(S^n(X/Y))$. Then $ch(\xi') \in \tilde{H}^*(S^n(X/Y), Q)$ and $(\sigma^n)^{-1} ch(\xi') \in \tilde{H}^*(X/Y; Q)$. We have the canonical isomorphism

$$\alpha : \tilde{H}^*(X/Y, Q) \to H^*(X, Y; Q)$$

and we define

$$ch(\xi) = \alpha((\sigma^n)^{-1} ch(\xi')).$$

In 1.7 and 1.8 we described the Bott isomorphism. Since $ch([\eta] - 1)$ is the canonical generator of $H^2(S^2, Z)$ and since ch preserves products, it follows easily, that $ch(\beta(\xi)) = ch(\xi)$ for $\xi \in K^{-n}(X, Y)$. Therefore we can define $ch(\xi)$ for $\xi \in K^n(X, Y)$, n any integer. Using the notation of 1.9 we have now defined the Chern character as a homomorphism

$$ch: K^*(X, Y) \to H^*(X, Y; Q).$$

ch maps $K^0(X, Y)$ into $H^{ev}(X, Y; Q)$ and $K^1(X, Y)$ into $H^{od}(X, Y; Q)$ which denotes the direct sum of the odd-dimensional cohomology groups. The following theorem is easy to check.

THEOREM. *The Chern character is a "natural transformation" of the "cohomology theory" described in* 1.9 *into the ordinary cohomology theory with rational coefficients for which one only considers the* Z_2*-grading* $H^* = H^{ev} \oplus H^{od}$. *In particular, ch preserves products, commutes with maps,* $ch \circ f^! = f^* \circ ch$, *and one has commutative diagrams*

$$\begin{array}{ccc} K^0(Y) & \xrightarrow{\delta} & K^1(X, Y) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^{ev}(Y, Q) & \xrightarrow{\delta} & H^{od}(X, Y; Q), \end{array} \qquad \begin{array}{ccc} K^1(Y) & \xrightarrow{\delta} & K^0(X, Y) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^{od}(Y, Q) & \xrightarrow{\delta} & H^{ev}(X, Y; Q). \end{array}$$

The commutativity of these diagrams can be deduced from the fact that the δ of both theories is induced from the map $C_f \to S^1Y$ (compare 1.4). One has to be careful with the signs. We hope to have chosen the various definitions such that commutativity (not only commutativity up to sign) holds in these diagrams.

2. The spectral sequence. Let X be a finite simplicial complex. We shall establish a spectral sequence relating the integral cohomology ring of X with $K^*(X)$.

2.1. Let X^n be the n-skeleton of X. We use the K-theory defined in 1.8. We filter $K^n(X)$ by defining

$$K_p^n(X) = \text{Kernel } [K^n(X) \to K^n(X^{p-1})].$$

THEOREM. *Let X be a finite simplicial complex. Let x_0 be the space consisting of a single point, so that $K^q(x_0) \cong Z$ if q is even and $K^q(x_0) = 0$ if q is odd. There exists a spectral sequence $E_r^{p,q}$ ($r \geqq 1$, $-\infty < p, q < \infty$) with*

(1) $$E_1^{p,q} \cong C^p(X, K^q(x_0)),$$

d_1 being the ordinary coboundary operator.

(2) $$E_2^{p,q} \cong H^p(X, K^q(x_0)),$$

(3) $$E_\infty^{p,q} \cong G_p K^{p+q}(X) = K_p^{p+q}(X)/K_{p+1}^{p+q}(X).$$

The differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanishes for r even since $E_r^{p,q} = 0$ for all odd values of q.

PROOF. We use the method of [12, Chapter XV, §7] and define the graded group

$$H(p, q) = \sum_{-\infty<n<\infty} H^n(p, q) = \sum_{-\infty<n<\infty} K^n(X^{q-1}, X^{p-1}), \qquad q \geqq p.$$

These $H(p, q)$ satisfy the axiom (SP.1)–(SP.5) of [12, loc. cit.]. For axiom (SP.4) see 1.8 (11*).

$$E_1^{p,q} = K^{p+q}(X^p, X^{p-1}) = \sum_i K^{p+q}(\sigma_i^p, \dot\sigma_i^p),$$

where σ_i^p runs through all p-simplices. But $\sigma_i^p/\dot\sigma_i^p = S^p$. Therefore $K^{p+q}(\sigma_i^p, \dot\sigma_i^p) \cong \tilde K^{p+q}(S^p) \cong \tilde K^q(S^0) \cong K^q(x_0)$. This proves (1). To get (2) one has to check that d_1 is the ordinary coboundary operator.

2.2. REMARK. The preceding spectral sequence can be generalized to a fibre bundle (Y, X, F) with projection $\pi : Y \to X$. If this fibre bundle satisfies certain conditions, then there is a spectral sequence with $E_1^{p,q} \cong C^p(X, K^q(F))$ and $E_2^{p,q} \cong H^p(X, K^q(F))$ (local coefficients). Furthermore $E_\infty^{p,q} \cong G_p K^{p+q}(Y)$ with respect to a certain filtration of $K^{p+q}(Y)$. This spectral sequence specializes to the one of the theorem for $Y = X$ and π the identity.

2.3. The whole spectral sequence of 2.1 is compatible with the Bott periodicity. This makes it possible to forget about the grading and to use the notation of 1.9.

THEOREM. *Let X be a finite simplicial complex. Let $K_p^*(X)$ be the kernel of $K^*(X) \to K^*(K^{p-1})$. There exists a spectral sequence $E_r^p(X)$, $r \geqq 1$, with*

$$E_1^p(X) \cong C^p(X, Z),$$

$$E_2^p(X) \cong H^p(X, Z),$$

$$E_\infty^p(X) \cong G_p K^*(X) = K_p^*(X)/K_{p+1}^*(X).$$

The differentials d_r vanish for even r.

This spectral sequence could also be obtained directly by the method of [12, Chapter XV, §7] by putting $H(p, q) = K^*(X^{q-1}, X^{p-1})$.

Remark. It is easy to show (by the notion of 1-equivalence, [12, p. 336]) that the $E_r^p(X)$ together with the differentials d_r are homotopy type invariants of X for $r \geqq 2$. Also $K_q^*(X)$ is a homotopy type invariant. It can be invariantly defined as follows: an element ξ of $K^*(X)$ belongs to $K_q^*(X)$ if and only if for any finite simplicial complex Y of dimension $\leqq q - 1$ and any continuous map $f : Y \to X$ we have $f^!\xi = 0$. Thus the spectral sequence $\{E_r^p(X), r \geqq 2\}$ is well-defined for any space X of the homotopy type of a finite simplicial complex. By a theorem of J. H. C. Whitehead [19, p. 239, Theorem 13] any finite CW-complex is of the homotopy type of a finite simplicial complex. Hence the spectral sequence $\{E_r^p(X), r \geqq 2\}$ is well-defined for spaces of the class $\mathfrak{A}$ (see 1.1).

The differentials d_r are certain (higher order) cohomology operations. $d_3 : E_3^p \cong H^p(X, Z) \to E_3^{p+3} \cong H^{p+3}(X, Z)$ is the Steenrod operation Sq^3.

2.4. Let X be a finite simplicial complex. We propose to study the spectral sequence of 2.3 in its relation with the Chern character. Let $'E_r^p$ be the spectral sequence with

$$'E_1^p = C^p(X, Q), \qquad d_1 \text{ the ordinary coboundary operator,}$$

$$'E_r^p = H^p(X, Q) \quad \text{for} \quad r \geqq 2, \qquad 'd_r = 0 \quad \text{for} \quad r \geq 2.$$

This trivial spectral sequence is obtained by the method of [12, Chapter XV, §7] by putting $'H(r, s) = H^*(X^{s-1}, X^{r-1}; Q)$ for $s \geqq r$. The spectral sequence of 2.3 comes from $H(r, s) = K^*(X^{s-1}, X^{r-1})$. The Chern character gives a homomorphism

$$ch: H(r, s) \to 'H(r, s),$$

and since ch is a natural transformation from the K^*-theory to the rational cohomology theory, we get a homomorphism ch from the spectral sequence $\{E_r^p\}$ of 2.3 into the spectral sequence $\{'E_r^p\}$. Using ch we can prove:

Theorem. *Suppose $X \in \mathfrak{A}$ (see 1.1). The spectral sequence $\{E_r^p(X)\}$ collapses (i.e., $d_r = 0$ for $r \geqq 2$ and thus $E_2^p(X) \cong E_\infty^p(X)$) if one of the following conditions is satisfied:*

(i) *$H^*(X, Z)$ has no torsion,*

(ii) *$H^i(X, Z) = 0$ for all odd integers i.*

Proof. We may assume that X is a finite simplicial complex. $ch : E_r^p \to 'E_r^p$ is always injective for $r = 1$, since then it is just the coefficient homomorphism $C^p(X, Z) \to C^p(X, Q)$. For $r = 2$ it is the homomorphism $H^p(X, Z) \to H^p(X, Q)$ which is injective if X has no torsion. Since the $'d_r$ vanish for $r \geqq 2$ it follows by induction on r that the d_r also vanish for $r \geqq 2$ if $E_2^p \to 'E_2^p$ is injective. This proves the theorem under the assumption (i). If (ii) holds, then $d_r (r \geqq 3$, odd) vanishes since it maps $E_r^p(X)$ in $E_r^{p+r}(X)$, and one of these groups is 0. The d_r (r even) vanish anyhow.

THEOREM. *Suppose $X \in \mathfrak{A}$ (see 1.1). The spectral sequence $\{E_r^*(X) \otimes Q\}$ collapses (i.e., $d_r \otimes Q = 0$ for $r \geq 2$).*

$$ch: K^*(X) \otimes Q \to H^*(X, Q)$$

is bijective and maps $K^0(X) \otimes Q$ onto $H^{ev}(X, Q)$ and $K^1(X) \otimes Q$ onto $H^{od}(X, Q)$.

PROOF. We may assume that X is a finite simplicial complex. The spectral sequence $\{E_r^*(X) \otimes Q\}$ is obtained by putting $''H(p, q) = K^*(X^{q-1}, X^{p-1}) \otimes Q$. The Chern character gives a homomorphism of this spectral sequence into the spectral sequence $\{'E_r^*(X)\}$ which is bijective for $r = 1$. This implies the theorem (compare [**12**, Chapter XV, Theorem 3.2]).

COROLLARY. *Suppose $X \in \mathfrak{A}$ (see 1.1). If $K^*(X)$ has no torsion, then*

$$ch: K^*(X) \to H^*(X, Q)$$

is injective.

2.5. The preceding results on the spectral sequence imply:

COROLLARY. *Let X be a space belonging to $\mathfrak{A}$ (see 1.1). Then $K^*(X)$ is additively a finitely generated abelian group. The rank of $K^0(X)$ equals the sum of the even dimensional Betti numbers of X, whereas the rank of $K^1(X)$ is the sum of the odd dimensional Betti numbers of X.*

For any $\xi \in K^*(X)$ let $ch_n(\xi)$ be the n-dimensional component of $ch(\xi)$.

COROLLARY. *Suppose $X \in \mathfrak{A}$ and that $H^*(X, Z)$ has no torsion. Then*

(i) *$\xi \in K_p^*(X)$ if and only if $ch_r(\xi) = 0$ for $r < p$, in particular*

$$ch: K^*(X) \to H^*(X, Q)$$

is injective and $K^*(X)$ is without torsion, i.e., free abelian.

(ii) *If $\xi \in K_p^*(X)$, then $ch_p(\xi) \in H^p(X, Q)$ comes from an integral class which is uniquely determined and equal to the image of ξ in $K_p^*(X)/K_{p+1}^*(X) \cong H^p(X, Z)$. To every integral p-dimensional class x, there exists $\xi \in K_p^*(X)$ with $ch_p(\xi) = x$, i.e., $ch(\xi) = x +$ higher terms.*

(iii) *Let A be a subgroup of $K^*(X)$. If for every $x \in H^p(X, Z)$, $p \geq 0$, there exists $\xi \in A$ with $ch(\xi) = x +$ higher terms, then $A = K^*(X)$.*

2.6. So far we have not studied the behaviour of the spectral sequence (2.3) with respect to the product structure of $K^*(X)$. We have only been able to get a partial result which we summarize without proof in the following theorem.

THEOREM. *Suppose $X \in \mathfrak{A}$. We consider the spectral sequence $E_r^*(X)$ $(r \geq 2)$ with the operators d_r. Let Z_r^* be the kernel and B_r^* the image of d_r. There exist pairings $\prod_r : E_r^p(X) \otimes E_r^q(X) \to E_r^{p+q}(X)$ with*

$$Z_r^p(X) \otimes Z_r^q(X) \to Z_r^{p+q}(X),$$
$$(4) \qquad Z_r^p(X) \otimes B_r^q(X) \to B_r^{p+q}(X) \quad \text{and} \quad B_r^p(X) \otimes Z_r^q(X) \to B_r^{p+q}(X),$$

and such that $\prod_{r+1}$ *is induced from* $\prod_r$ *in virtue of* (4). *Moreover,* $\prod_2$ *is the cup-product and* $\prod_\infty$ *is the product in* $GK^*(X)$ *induced by the ring structure of* $K^*(X)$ *for which*

(5) $$K_p^*(X) \cdot K_{p'}^*(X) \subset K_{p+p'}^*(X).$$

We conjecture that d_r is an anti-derivation. This would imply (4). We shall only need (5) in the sequel. (5) admits a straightforward proof.

By (5) the mth power of an element of $K_1^*(X)$ belongs to $K_m^*(X)$. If m is sufficiently large then $K_m^*(X)$ is zero, hence any element of $K_1^*(X)$ is nilpotent. Clearly, $\xi \in K_1^*(X)$ if and only if $ch_0(\xi) = 0$. This special case of 2.5 (i) holds for any $X \in \mathfrak{A}$. We conclude:

Proposition. *An element* ξ *of* $K^*(X)$ *is nilpotent if and only if* $ch_0(\xi) = 0$. *An element* η *of* $K^*(X)$ *is invertible if and only if* $ch_0(\eta) = \pm 1$.

Proof. It remains to show that η is invertible if $ch_0(\eta) = \pm 1$. In this case, $\pm\eta = 1 - \xi$ with $ch_0(\xi) = 0$ and thus ξ nilpotent. Then $\eta^{-1} = \pm(1 + \xi + \xi^2 + \cdots + \xi^{m-1})$ if $\xi^m = 0$.

3. The differentiable Riemann-Roch theorem and some applications.

3.1. We recall the Riemann-Roch theorem given in [1] in a slightly more general formulation. Let X, Y be compact oriented differentiable manifolds. By the triangulation theorem of Cairns, X and Y belong to the class $\mathfrak{A}$ of 1.1. A continuous map $f : Y \to X$ will be called a c_1-map if we are given an element $c_1(f) \in H^2(Y, Z)$ such that $c_1(f) \equiv w_2(Y) - f^*w_2(X) \bmod 2$ where $w_2(Y)$ and $w_2(X)$ are the second Stiefel-Whitney classes of Y and X respectively ($w_2 \in H^2(\ , Z_2)$). As in [5; 1], if ξ is a real vector bundle with finite-dimensional base B_ξ we define

$$\hat{\mathfrak{A}}(\xi) = \prod_i (x_i/2)/(\sinh(x_i/2)) \in H^*(B_\xi, Q)$$

where the Pontrjagin classes of ξ are the elementary symmetric functions in the x_i^2. If ξ is the tangent bundle of the differentiable manifold X we write $\hat{\mathfrak{A}}(X)$ instead of $\hat{\mathfrak{A}}(\xi)$.

Theorem. *Let* Y *and* X *be as before. Let* $f : Y \to X$ *be a* c_1*-map. Then there exists a homomorphism*

$$g : K^*(Y) \to K^*(X)$$

such that

(i) $$ch(g(y)) \cdot \hat{\mathfrak{A}}(X) = f_*(ch(y)e^{c_1(f)/2} \cdot \hat{\mathfrak{A}}(Y)), \qquad y \in K^*(Y),$$

where f_* *is the Gysin homomorphism (Poincaré dual of the homology homomorphism).*

(ii) g *maps* $K^0(Y)$ *into* $K^0(X)$ *and* $K^1(Y)$ *into* $K^1(X)$ *if* $\dim Y \equiv \dim X$ (mod 2).
g *maps* $K^0(Y)$ *into* $K^1(X)$ *and* $K^1(Y)$ *into* $K^0(X)$ *if* $\dim Y \not\equiv \dim X$ (mod 2).

(iii) *g is related to the homomorphism $f^! : K^*(X) \to K^*(Y)$ by the formula*

$$g(f^!(x)\cdot y) = x\cdot g(y), \qquad x \in K^*(X), \qquad y \in K^*(Y).$$

If we define $\hat{\mathfrak{A}}(f) = \hat{\mathfrak{A}}(Y)\cdot f^*(\hat{\mathfrak{A}}(X)^{-1})$, then (i) *may be written as*

$$\text{(i}'\text{)} \qquad ch(g(y)) = f_*(ch(y)\, e^{c_1(f)/2}\cdot\hat{\mathfrak{A}}(f)).$$

This theorem is slightly more general than Theorem 1 of [1] which was formulated for $K^0(X)$. Here we have stated it for $K^*(X)$ which makes the assumption $\dim Y \equiv \dim X \pmod 2$ superfluous. The proof does not have to be changed once one has developed the cohomology theory of §1. Moreover we assert here the existence of the homomorphism g satisfying (iii). This brings no additional difficulty. One just has to follow up the proof of Theorem 1 of [1] (see also [16]). Something new would be involved if we tried to choose g in a natural way (call it then $f_!$) and prove certain functorial properties of it. We shall take up this question in a more detailed exposition. Nevertheless we permit ourselves to call the g of the theorem $f_!$. But we are not allowed then to use for $Z \to^{\bar{f}} Y \to^{f} X$ the formula $(f\circ\bar{f})_! = f_! \circ \bar{f}_!$. (The composition of two c_1-maps is a c_1-map in a natural way.) The formula (i′) shows that $ch(g(y))$ is uniquely determined for a c_1-map f. Therefore (2.4, 2.5), $g = f_!$ is given without ambiguity if $K^*(X)$ or $H^*(X, Z)$ has no torsion.

It follows easily from (i′) that

$$ch((f\circ\bar{f})_!z) = ch(f_!(\bar{f}_!z)) \quad \text{for} \quad z \in K^*(Z).$$

By (2.4, 2.5)

$$(f\circ\bar{f})_!z = f_!(\bar{f}_!z) \quad \text{if} \quad K^*(X) \quad \text{or} \quad H^*(X, Z) \quad \text{has no torsion.}$$

3.2. Let Y be a compact oriented differentiable manifold. It is called a c_1-manifold if we are given an element $c_1(Y) \in H^2(Y, Z)$ whose restriction mod 2 is $w_2(Y)$. For a c_1-manifold Y the Todd genus $T(Y)$ is defined. It is equal to the value of the top-dimensional component of $e^{c_1(Y)/2}\cdot\hat{\mathfrak{A}}(Y)$ on the fundamental cycle of Y. By definition, $T(Y)$ is a rational number. It is an integer as follows by applying Theorem 3.1 to the map of Y onto a point. Compare [1], see also [6]. If Y is almost-complex and $c_1(Y)$ the first Chern class, then $T(Y)$ is the usual Todd genus which is equal to the arithmetic genus if Y is a projective algebraic manifold [15].

3.3. Let $\xi = (E_\xi, B_\xi, F_\xi, \pi_\xi)$ be a differentiable fibre bundle in the sense of [5, §7.4]. Assume that E_ξ, B_ξ, F_ξ are compact oriented differentiable manifolds. As in [5] we let $\hat{\xi}$ be the bundle along the fibres. This is a real vector bundle over E_ξ whose second Stiefel-Whitney class $w_2(\hat{\xi})$ equals $w_2(E_\xi) - \pi^*w_2(B_\xi)$. Assume that $\pi = \pi_\xi$ is a c_1-map. Then $c_1(\pi) \equiv w_2(\hat{\xi}) \bmod 2$. If $i : F_\xi \to E_\xi$ is the injection of a fibre in the total space then

$$i^*c_1(\pi) \equiv w_2(F_\xi) \mod 2.$$

Therefore if we put $c_1(F_\xi) = i^*c_1(\pi)$, the manifold F_ξ becomes a c_1-manifold and we can speak of the Todd genus $T(F_\xi)$. Assume that ξ is endowed with a complex structure, i.e., we are given a complex vector bundle η over E_ξ which considered as real vector bundle is ξ. Then F_ξ is almost complex in a natural way. Furthermore π is a c_1-map with $c_1(\pi) = c_1(\eta)$. The Todd genus $T(F_\xi)$ is then the same whether we consider F_ξ as c_1-manifold with $c_1(F_\xi) = i^*c_1(\eta)$ or as almost complex manifold.

3.4. THEOREM. *Let ξ be a differentiable fibre bundle as in* 3.3. *Let $\pi = \pi_\xi$ be a c_1-map. If the Todd genus $T(F_\xi) = \pm 1$ then the homomorphism*

$$\pi^1 : K^*(B_\xi) \to K^*(E_\xi)$$

is injective. Moreover $\pi^!$ identifies $K^*(B_\xi)$ with a direct summand of $K^*(E_\xi)$. The endomorphism $\pi_! \circ \pi^!$ of $K^*(B_\xi)$ is the multiplication with a fixed invertible element of $K^*(B_\xi)$.

PROOF. We shall use Theorem 3.1 for π with $Y = E_\xi$ and $X = B_\xi$. First we observe that

$$\hat{\mathfrak{A}}(\xi) = \hat{\mathfrak{A}}(E_\xi)\cdot(\pi^*\hat{\mathfrak{A}}(B_\xi))^{-1} = \hat{\mathfrak{A}}(\pi).$$

Therefore with $g = \pi_!$ we have by 3.1 (i′)

$$ch(\pi_!(y)) = f_*(ch(y)\cdot e^{c_1(\pi)/2}\cdot\hat{\mathfrak{A}}(\xi)), \qquad y \in K^*(E_\xi).$$

Now put $y = 1$, the unit of $K^*(E_\xi)$. Then $ch(y) = 1$ and it follows easily that the zero-dimensional component of $ch(\pi_! 1)$ equals $T(F_\xi)$. Since $T(F_\xi) = \pm 1$, $\pi_! 1$ is an invertible element in $K^*(B_\xi)$ (see 2.6) whose inverse we denote by a. Now let h be the homomorphism $K^*(E_\xi) \to K^*(B_\xi)$ equal to $\pi_!$ followed by multiplication with a; then (iii) of 3.1 gives

$$h(\pi^1(x)) = x \quad \text{for all} \quad x \in K^*(B_\xi)$$

which proves the theorem.

The preceding theorem admits various generalisations. For example, if the Todd genus $T(F_\xi) = m \neq 0$, $(m \in Z)$, then π^1 is injective on the direct sum of those p-primary components of $K^*(B_\xi)$ with $p = 0$ or a prime not dividing m. This type of theorem is analogous to 3.2 of [4].

3.5. Let G be a compact connected Lie group and T a maximal torus of G. Let ξ be a principal G-bundle whose base space B_ξ is a compact oriented differentiable manifold. Consider the associated bundle with G/T as fibre. Its total space is E_ξ/T, its base space is B_ξ. With these assumptions we have:

PROPOSITION. *Let π be the projection $E_\xi/T \to B_\xi$. Then*

$$\pi^! : K^*(B_\xi) \to K^*(E_\xi/T)$$

is injective. $\pi^! K^*(B_\xi)$ is a direct summand of $K^*(E_\xi/T)$.

PROOF. We may assume that ξ is differentiable. The bundle along the fibres of E_ξ/T admits a complex structure such that G/T has Todd genus 1 (see

[5, §§7.4, 22.3]). The complex structure along the fibres and the orientation of B_ξ define an orientation for the compact differentiable manifold E_ξ/T. The proposition follows from 3.4.

THEOREM. *We make the preceding assumptions. Let U be a closed connected subgroup of G of maximal rank, i.e., we may assume $U \supset T$. Then E_ξ/U is the total space of the bundle associated to ξ and with G/U as fibre. Let σ be the projection $E_\xi/U \to B_\xi$. Then*

$$\sigma^! : K^*(B_\xi) \to K^*(E_\xi/U)$$

is injective. $\sigma^! K^*(B_\xi)$ is a direct summand of $K^*(E_\xi/U)$.

PROOF. We have the diagram

$$E_\xi/T \xrightarrow{\rho} E_\xi/U \xrightarrow{\sigma} B_\xi, \qquad \sigma \circ \rho = \pi, \qquad \pi^! = \rho^! \circ \sigma^!.$$

By the above proposition $\pi^!$ is injective which implies $\sigma^!$ is injective. Also the last statement of the theorem follows immediately.

REMARK. We have proved this theorem under the assumption that B_ξ is a compact oriented differentiable manifold. A small generalization of the Riemann-Roch Theorem 3.1 makes it possible to drop the assumption on orientability. It is probably also true when B_ξ is any finite CW-complex.

The preceding theorem holds in particular for bundles with an even dimensional sphere as fibre and the special orthogonal group as structure group. If $\pi. : Y \to X$ is such a bundle (X compact oriented differentiable), then $\pi^! : K^*(X) \to K^*(Y)$ is injective. The corresponding theorem for integral cohomology holds if X has no 2-torsion (more generally, π^* is injective on the direct sum of the p-primary components of $H^*(X, Z)$ with $p = 0$ or p an odd prime).

3.6. THEOREM. *Let G be a compact connected Lie group, U a closed connected subgroup of G of maximal rank. Then $K^1(G/U) = 0$ and $K^0(G/U)$ is a free abelian group with rank equal to the quotient of the order of the Weyl group of G by the order of the Weyl group of U.*

PROOF. The theorem is true if $U = T$ (maximal torus of G). In this case G/T has no torsion in integral cohomology and its odd dimensional cohomology groups vanish [7]. The theorem follows then from 2.5 if one takes into account that the order of $W(G)$ (Weyl group of G) is the Euler number of G/T which equals $\dim_Q H^{ev}(G/T, Q)$. For the general case, we assume that $U \supset T$ and consider the map $\pi : G/T \to G/U$. Then $\pi^!$ is injective by 3.5. It follows that $K^1(G/U) = 0$ and that $K^0(G/U)$ has no torsion. It is well-known [3] that the odd-dimensional Betti numbers of G/U vanish and that the Euler number of G/U equals ord $W(G)$/ord $W(U)$. Thus $\dim_Q H^{ev}(G/U, Q) =$ ord $W(G)$/ord $W(U)$ which completes the proof in virtue of 2.5.

REMARK. As in the case of G/T, Theorem 3.6 follows immediately from 2.5 if $H^*(G/U, Z)$ has no torsion.

4. The classifying space of a compact connected Lie group.

4.1. *Completions of modules.* We shall summarize here some known results of commutative algebra which we learned from J. P. Serre. These results are needed in the sequel. For references see Zariski and Samuel, *Commutative algebra*, Van Nostrand, and [**13**, Exposé 18 (Godement)].

Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A. We give every finitely generated A-module M the topology defined by the submodules $\mathfrak{a}^n M$. The completion of M for this "$\mathfrak{a}$-adic topology" is by definition

$$\hat{M} = \varprojlim M/\mathfrak{a}^n M \qquad \text{(inverse limit)}.$$

(i) *Let N be a submodule of M. Then the $\mathfrak{a}$-adic topology of N coincides with the topology induced on N by the $\mathfrak{a}$-adic topology of M.*

This is a consequence of the lemma of Artin-Rees which says that there exists a positive integer h such that $(\mathfrak{a}^n M) \cap N = \mathfrak{a}^{n-h}((\mathfrak{a}^h M) \cap N)$ for $n \geqq h$; see [**13**, Exposé 2, Théorème 2].

(ii) *Let* $0 \to N \to M \to P \to 0$ *be an exact sequence of (finitely generated) A-modules; then*

$$0 \to \hat{N} \to \hat{M} \to \hat{P} \to 0$$

is exact. Thus "completion" is an exact functor [**12**, Chapter II, §4].

PROOF. We have the exact sequence

$$0 \to N/(\mathfrak{a}^n M \cap N) \to M/\mathfrak{a}^n M \to P/\mathfrak{a}^n P \to 0.$$

By (i), $\hat{N}$ is the inverse limit of the first inverse system in this sequence. Since $N/(\mathfrak{a}^{n+k} M \cap N) \to N/(\mathfrak{a}^n M \cap N)$ is onto for all n and all $k \geqq 0$, this inverse system satisfies the "Mittag-Leffler condition." According to the forthcoming book of Dieudonné-Grothendieck (Complements to Chapter 0) the assertion (ii) follows. There is, of course, a direct proof along the lines of [**11**, §3].

(iii) *Let B be a commutative ring, G a finite group of automorphisms of B and let $A = B^G$ be the subring of those elements of B which are invariant under all automorphisms of G. Assume B is, as an A_0-algebra, finitely generated over a Noetherian subring A_0 of A. Then B and A are Noetherian and B is a finitely generated A-module.*

PROOF. Since A_0 is Noetherian, B (as a quotient ring of a polynomial ring over A_0) is also Noetherian. If $x \in B$, then $\prod_{\sigma \in G} (x - \sigma(x)) = 0$. Thus x is integral over A. Let $x_1, \cdots, x_n$ be generators of B over A_0. Then we have equations

$$x_i^q + a_{i1} x_i^{q-1} + \cdots + a_{iq} = 0, \qquad a_{ij} \in A, \qquad q = \text{order of } \; G.$$

Thus B is generated as an A-module by the monomials $x_1^{m_1} \cdots x_n^{m_n}$ ($m_i \leqq q - 1$), hence is a finitely generated A-module. Let A' be the subring of A generated over A_0 by the a_{ij}. The ring A' is Noetherian since it is a finitely generated

A_0-algebra. B is even a finitely generated A'-module. Thus also A is a finitely generated A'-module. If $c_1, \cdots, c_k$ are generators of A as module over A', then the c_r and the a_{ij} generate A as A_0-algebra. Hence A is a Noetherian ring.

(iv) *We make the assumptions of* (iii). *Let* $\mathfrak{b}$ *be an ideal of* B *which is stable under* G ($\sigma(\mathfrak{b}) = \mathfrak{b}$ *for* $\sigma \in G$). *Put* $\mathfrak{a} = \mathfrak{b} \cap A$. *Let* $\mathfrak{b}' = \mathfrak{a} \cdot B$ *be the ideal of* B *generated by* $\mathfrak{a}$. *Then there exists a positive integer* n *such that* $\mathfrak{b}^n \subset \mathfrak{b}' \subset \mathfrak{b}$. *Thus* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *define the same topology on* B.

Proof. In a noetherian ring, to prove that a power of the ideal $\mathfrak{b}$ is contained in $\mathfrak{b}'$, it is enough to show that all prime ideals $\mathfrak{p}$ containing $\mathfrak{b}'$ also contain $\mathfrak{b}$ (see for example [13, Exposé 2]). Let $\mathfrak{p}$ be a prime containing $\mathfrak{b}'$ and let $x \in \mathfrak{b}$. Then $x' = \prod_{\sigma \in G} \sigma(x) \in A \cap \mathfrak{b} = \mathfrak{a} \subset \mathfrak{b}'$. Hence $x' \in \mathfrak{p}$. Hence there is a σ with $\sigma(x) \in \mathfrak{p}$ and thus $x \in \sigma^{-1}(\mathfrak{p})$. Hence $\mathfrak{b}$ is contained in the union of the prime ideals $\sigma(\mathfrak{p})$, $\sigma \in G$. But it is an easy lemma (see Northcott, *Ideal theory*, Cambridge Tracts, pp. 12–13), true in any ring, that if an ideal $\mathfrak{b}$ is contained in the union of a finite number of prime ideals, it is contained in one of them. Thus in our case, $\mathfrak{b} \subset \sigma(\mathfrak{p})$ for some $\sigma \in G$. But $\mathfrak{b} = \sigma^{-1}(\mathfrak{b})$ by assumption. Thus $\mathfrak{b} \subset \mathfrak{p}$ as contended.

We consider A and B both as A-modules and complete them with respect to the $\mathfrak{a}$-adic topology. We have a map $\hat{A} \to \hat{B}$ which is injective by (ii). In view of (iv) $\hat{B}$ is also the completion of B with respect to the $\mathfrak{b}$-adic topology of B. The group G operates naturally on $\hat{B}$. Let $(\hat{B})^G$ be the ring of invariants.

(v) *Under the preceding assumptions the map* $\hat{A} \to \hat{B}$ *maps* $\hat{A}$ *(bijectively) onto* $(\hat{B})^G$. *Thus* $(B^G)^\wedge = (\hat{B})^G$.

Proof. Let $B(G)$ be the ring of all maps from G into B. This is a direct sum of g copies of B where g is the order of G. We consider the exact sequence

$$0 \to B^G \to B \xrightarrow{\alpha} B(G)$$

where $\alpha(b)$, $b \in B$, is the map which attaches to $\sigma \in G$ the element $b - \sigma(b) \in B$. All rings in this exact sequence have to be considered as A-modules ($A = B^G$). We complete them with respect to the $\mathfrak{a}$-adic topology. "Completion" is an exact functor, hence $(B(G))^\wedge = \hat{B}(G)$ and the resulting sequence

$$0 \to (B^G)^\wedge \to \hat{B} \xrightarrow{\hat{\alpha}} \hat{B}(G)$$

is exact which proves (v).

4.2. *The representation ring of a compact Lie group*. Let G be a compact Lie group. Let $(\rho_1, \rho_2, \cdots)$ be the (equivalence classes of) irreducible complex representations of G. Let $R(G)$ be the free abelian group generated by the ρ_i. The tensor product of representations makes $R(G)$ into a ring which we shall call the representation ring of G. The complex representations of G may be identified with the elements $\sum n_i \rho_i$ of $R(G)$ where the n_i are non-negative integers.

Let $\epsilon : R(G) \to Z$ be the "augmentation homomorphism" obtained by attaching to each representation of G its dimension. Let $I(G)$ be the kernel of ϵ; it will be called the augmentation ideal of $R(G)$. We define the completed representation ring with respect to the $I(G)$-adic topology:

$$\hat{R}(G) = \varprojlim R(G)/I(G)^n \qquad \text{(inverse limit)}.$$

Let G, H be compact Lie groups and $G \to H$ a homomorphism; then we have an induced homomorphism $R(H) \to R(G)$ which maps $I(H)$ in $I(G)$ and is therefore continuous with respect to the $I(H)$-adic topology of $R(H)$ and the $I(G)$-adic topology of $R(G)$. It induces therefore a homomorphism $\hat{R}(H) \to \hat{R}(G)$. Suppose now $G = H$. Then any automorphism of G induces automorphisms of $R(G)$ and of $\hat{R}(G)$. An inner automorphism induces the identity. If G is connected and T a maximal torus of G, then the Weyl group $W(G)$ is a group of automorphisms of T and thus operates also on $R(T)$ and $\hat{R}(T)$.

4.3. *The completed representation ring of a torus.* Let T be a torus. We write it as the group of k-tuples of reals mod 1. Every irreducible representation of T is 1-dimensional and given by a homomorphism

$$(x_1, \cdots, x_k) \to \exp(2\pi i(a_1x_1 + \cdots + a_kx_k)), \qquad a_i \in Z,$$

of T into $U(1)$. The ring $R(T)$ may be identified with that subring of the ring of formal power series $C[[x_1, \cdots, x_k]]$ which is generated over Z by

$$\exp(2\pi ix_1), \exp(-2\pi ix_1), \cdots, \exp(2\pi ix_k), \exp(-2\pi ix_k).$$

Hence $R(T)$ *is Noetherian.*

Let $z_1, \cdots, z_k$ be indeterminates. We give the polynomial ring $Z[z_1, \cdots, z_k]$ the $(z_1, \cdots, z_k)$-adic topology, and define a ring homomorphism

$$\phi : Z[z_1, \cdots, z_k] \to R(T)$$

by setting $\phi(z_i) = \exp(2\pi ix_i) - 1$. Then $\phi(z_i) \in I(T)$, thus ϕ is continuous and induces a homomorphism $\hat{\phi}$ of the completed rings.

PROPOSITION. *The homomorphism*

$$\hat{\phi} : Z[[z_1, \cdots, z_k]] \to \hat{R}(T)$$

is bijective. ($Z[[z_1, \cdots, z_k]]$ *is the ring of formal power series.*)

PROOF. Put $A = Z[z_1, \cdots, z_k]$. Under ϕ we may identify A with a subring of $R(T)$. For the latter ring we may write

$$R(T) = Z[z_1, \cdots, z_k, (1 + z_1)^{-1}, \cdots, (1 + z_k)^{-1}].$$

We have then $I(T) = (z_1, \cdots, z_k, (1 + z_1)^{-1} - 1, \cdots, (1 + z_k)^{-1} - 1)$. Thus the ideal $I(T)^n$ of $R(T)$ contains only formal power series with lowest term of degree $\geqq n$. Thus $I(T)^n \cap A$ contains only polynomials with lowest term of degree $\geqq n$. Therefore, $I(T)^n \cap A \subset (z_1, \cdots, z_k)^n$. Clearly, $(z_1, \cdots, z_k)^n \subset I(T)^n \cap A$. Thus

$$I(T)^n \cap A = (z_1, \cdots, z_k)^n = (I(T) \cap A)^n.$$

This shows that the $(z_1, \cdots, z_k)$-adic topology of A coincides with the topology induced from the embedding of A in $R(T)$. Thus $\hat{\phi}$ is injective. Since $\hat{\phi}(\hat{A})$ contains $R(T)$, the map $\hat{\phi}$ is surjective.

NOTE. We have just considered $R(T)$ as subring of $\hat{R}(T)$. This is all right, since $R(T)$ *is Hausdorff* in its $I(T)$-adic topology. In fact, $\bigcap_n I(T)^n = 0$, since an element of this intersection would be a power series whose lowest term has an arbitrarily high degree.

4.4. *The completed representation ring of a compact connected Lie group.* Let G be a compact *connected* Lie group and T a maximal torus of G. The Weyl group $W(G)$ operates on $R(T)$; see 4.2. We have a ring homomorphism $R(G) \to R(T)$ (by the restriction map) which is injective. $R(G)$ maps (bijectively) onto the ring of invariants of $R(T)$ under the action of $W(G)$. This classical result follows from the fact that the highest weight of an irreducible representation has multiplicity one (compare [5, §3.4]). We denote this ring of invariants by $R(T)^{W(G)}$ and identify $R(G)$ with it. We have the situation of 4.1 (iii). A_0 is here the ring Z of integers. Thus we know that $R(G)$ is Noetherian and that $R(T)$ is a finitely generated module over $R(G)$.

$W(G)$ operates naturally on $\hat{R}(T)$ and we have an induced map $\hat{R}(G) \to \hat{R}(T)$ (see 4.2).

THEOREM. *Let G be a compact connected Lie group, T a maximal torus of G. Then $\hat{R}(G) \to \hat{R}(T)$ maps $\hat{R}(G)$ bijectively onto $(\hat{R}(T))^{W(G)}$, the ring of invariants of $W(G)$ in $\hat{R}(T)$.*

PROOF. We are exactly in the situation of 4.1. Here $R(T)$ plays the role of B, $R(G)$ of A, $W(G)$ of G, and Z of A_0. The ideal $\mathfrak{b}$ corresponds to $I(T)$, the ideal $\mathfrak{a}$ to $I(T) \cap R(G) = I(G)$.

NOTE. $R(G)$ is Hausdorff, since $\bigcap_n I(G)^n \subset \bigcap_n I(T)^n = 0$. The homomorphism $R(G) \to \hat{R}(G)$ is injective. This is in general not true if G is not connected (Atiyah, *Characters and cohomology*, in preparation).

4.5. Let X be a space belonging to the class $\mathfrak{A}$ of 1.1. Let ξ be a principal G-bundle over X where G is a compact Lie group. ξ induces a ring homomorphism

$$\alpha_\xi : R(G) \to K^0(X) \subset K^*(X)$$

in the following way. Consider a representation of G viewed as a homomorphism $\rho : G \to U(m)$. Then $\rho(\xi)$ is a principal $U(m)$-bundle and defines an element $\alpha_\xi(\rho)$ of $K^0(X)$. Since the (equivalence classes of) irreducible representations are free generators of the additive group $R(G)$ the homomorphism α_ξ is well-defined.

If we have a map $f : Y \to X$ ($Y, X \in \mathfrak{A}$), if ξ is a principal G-bundle over X and $\eta = f^*\xi$ the principal G-bundle over Y induced from ξ by f, then we have the commutative diagram

(1)
$$\begin{array}{ccc} & \overset{\alpha_\xi}{\nearrow} & K^*(X) \\ R(G) & & \Big\downarrow f^! \\ & \underset{\alpha_\eta}{\searrow} & K^*(Y). \end{array}$$

If Y consists of a single point, then $K^*(Y) \cong Z$ and α_η is just the augmentation $\epsilon : R(G) \to Z$. This shows that the ideal $I(G)$ is mapped by α_ξ into $K_1^*(X)$ (see 2.3 Remark). By 2.6 (5) there exists an n_0 such that $\alpha_\xi(I(G)^n) = 0$ for $n \geqq n_0$. Since $\hat{R}(G)$ is the inverse limit of the $R(G)/(I(G))^n$ with $n \geqq n_0$, we have a natural ring homomorphism

$$\hat{\alpha}_\xi : \hat{R}(G) \to K^*(X).$$

Obviously, α_ξ is $R(G) \to \hat{R}(G)$ followed by $\hat{\alpha}_\xi$.

If we have as before a map $f : Y \to X$, then we have the commutative diagram

$$(2) \qquad \begin{array}{ccc} & & K^*(X) \\ & \nearrow^{\hat{\alpha}_\xi} & \\ \hat{R}(G) & & \downarrow f^! \\ & \searrow_{\hat{\alpha}_\eta} & \\ & & K^*(Y) \end{array}, \qquad \eta = f^*\xi.$$

4.6. *Classifying spaces.* Let F be a contravariant functor on the class $\mathfrak{A}$ (see 1.1), i.e., F attaches to each $X \in \mathfrak{A}$ an algebraic object of a given type, say an abelian group for convenience, and for each continuous map $f : Y \to X$ ($Y, X \in \mathfrak{A}$) there is given a homomorphism $f^* : F(X) \to F(Y)$ satisfying the functorial properties and the homotopy axiom ($f^* = g^*$ if the maps $f, g : Y \to X$ are homotopic).

Let G be a compact Lie group, B_G its (infinite) classifying space determined up to homotopy type. We shall define $\mathfrak{F}(B_G)$ to be an algebraic object of the same type as all the $F(X)$, $X \in \mathfrak{A}$. The definition will be such that an element of $\mathfrak{F}(B_G)$ is completely given by the group G. The classifying space B_G is not needed for the definition, but we write $\mathfrak{F}(B_G)$ rather than $\mathfrak{F}(G)$ to avoid the confusion with $F(G)$.

DEFINITION. *An element a of $\mathfrak{F}(B_G)$ is an operator which attaches to each X and each principal G-bundle ξ over X an element $a(\xi) \in F(X)$ depending only on the equivalence class of ξ such that the following holds: for a map $f : Y \to X$ ($Y, X \in \mathfrak{A}$), a principal G-bundle ξ over X and the principal G-bundle $f^*\xi$ over Y induced from ξ by f, we have $a(f^*\xi) = f^*(a(\xi))$. Using the notation of* [15, §3] *this means that the diagram*

$$\begin{array}{cccc} f^* : & H^1(X, G_c) & \to & H^1(Y, G_c) \\ & a \downarrow & & a \downarrow \\ f^* : & F(X) & \to & F(Y) \end{array}$$

is commutative.

If U, G are compact Lie groups and $\rho : U \to G$ a homomorphism, then we have the induced homomorphism

$$\rho^* : \mathfrak{F}(B_G) \to \mathfrak{F}(B_U).$$

For $a \in \mathfrak{F}(B_G)$, $\rho^* a : H^1(X, U_c) \to F(X)$ is the composition $H^1(X, U_c) \to^{\rho} H^1(X, G_c) \to^{a} F(X)$.

If $U = G$ and ρ is an inner automorphism of G, then ρ^* is the identity since $\rho : H^1(X, G_\epsilon) \to H^1(X, G_\epsilon)$ is the identity.

According to the classification theorem [18, §19] we can choose a principal G-bundle ξ_n which is classifying up to n, i.e., $\pi_i(E_{\xi_n}) = 0$ for $i \leqq n$, and whose base space $B_{\xi_n} = B_n$ belongs to $\mathfrak{A}$. Let $n_1 < n_2 < n_3 < \cdots$ be a sequence of positive integers such that $\dim B_{n_i} \leqq n_{i+1}$. Then ξ_{n_i} is induced from $\xi_{n_{i+1}}$ by a map $B_{n_i} \to B_{n_{i+1}}$ uniquely determined up to homotopy. Thus we have a homomorphism $F(B_{n_{i+1}}) \to F(B_{n_i})$. This enables us to write $\mathfrak{F}(B_G)$ as an inverse limit

$$(3) \qquad \mathfrak{F}(B_G) \cong \varprojlim F(B_{n_i}).$$

This isomorphism is canonical. In particular, we have:

(4) *An element a of $\mathfrak{F}(B_G)$ vanishes if and only if there exists for every n_0 an integer $n \geqq n_0$ and a principal G-bundle ξ_n classifying up to n such that $a(\xi_n) = 0$.*

If we take for F the ordinary cohomology theory with coefficients in some abelian group, then $\mathfrak{F}(B_G)$ becomes $H^{**}(B_G, A)$; see [5, §6.1]. If we take for F the K^*-theory of 1.9 then we define the ring

$$\mathcal{K}^*(B_G) = \mathcal{K}^0(B_G) \oplus \mathcal{K}^1(B_G) = \mathfrak{F}(B_G)$$

$\mathcal{K}^0(B_G)$ is the $\mathcal{K}(B_G)$ mentioned in the introduction. In this theory we write $\rho^!$ instead of ρ^*. The Chern character $ch : \mathcal{K}^*(B_G) \to H^{**}(B_G, Q)$ is clearly defined.

4.7. Because of the diagrams (1) and (2) of 4.5 we have canonical ring homomorphisms

$$\alpha : R(G) \to \mathcal{K}^*(B_G), \qquad \hat{\alpha} : \hat{R}(G) \to \mathcal{K}^*(B_G).$$

α equals $R(G) \to \hat{R}(G)$ followed by $\hat{\alpha}$. Of course, α and $\hat{\alpha}$ map into $\mathcal{K}^0(B_G)$. We sometimes write more explicitly α_G instead of α and $\hat{\alpha}_G$ instead of $\hat{\alpha}$.

Let G and H be compact Lie groups and $\rho : G \to H$ a homomorphism; then we have a commutative diagram

$$\begin{array}{ccccccc} R(G) & \to & \hat{R}(G) & \xrightarrow{\hat{\alpha}} & \mathcal{K}^*(B_G) & \xrightarrow{ch} & H^{**}(B_G, Q) \\ \uparrow & & \uparrow & & \uparrow \rho^! & & \uparrow \rho^{**} \\ R(H) & \to & \hat{R}(H) & \xrightarrow{\hat{\alpha}} & \mathcal{K}^*(B_H) & \xrightarrow{ch} & H^{**}(B_H, Q). \end{array}$$

4.8 We state now the main theorem of §4 and give a corollary. The proof of the theorem will be given in the following sections.

THEOREM. *Let G be a compact connected Lie group. Then $\hat{\alpha}$ is an isomorphism of $\hat{R}(G)$ onto $\mathcal{K}^*(B_G)$.*

COROLLARY. *Let G be a compact connected Lie group. Then $\mathcal{K}^1(B_G) = 0$. Moreover, $\mathcal{K}^*(B_G) = \mathcal{K}^0(B_G)$ has no torsion and no zero divisors.*

We have seen in 4.4 that $\hat{R}(G)$ is a subring of $\hat{R}(T)$ which is a ring of formal power series over Z. Thus the corollary follows from the theorem.

REMARK. We conjecture the theorem to hold for any compact Lie group. It holds if G is finite (Atiyah, loc. cit. in 4.4 Note).

4.9. We prove Theorem 4.8 first for the case where G is a torus T which we describe as in 4.3 as the group of k-tuples of reals mod 1. Let P_n be the complex projective space of complex dimension n. Over P_n we take the $U(1)$-bundle η_n whose first Chern class is the canonical generator g of $H^2(P_n, Z) \cong Z$; see [15, §4.2]. η_n is induced from η_{n+1} by the embedding $P_n \to P_{n+1}$. Let B_{2n} be the cartesian product of k copies of P_n. Over B_{2n} we have the T-bundle ξ_{2n} which is the Whitney sum of the $\pi_i^*(\eta_n)$, $1 \leq i \leq k$, where π_i is the projection of B_{2n} on its ith factor. ξ_{2n} is classifying up to dimension $2n$. We have the embedding $B_{2n} \to B_{2n+2}$ which induces ξ_{2n} from ξ_{2n+2} and which gives rise to the homomorphism $K^*(B_{2n+2}) \to K^*(B_{2n})$. It follows from 4.6 (3) that

$$\mathcal{K}^*(B_T) \cong \varprojlim K^*(B_{2n}),$$

the inverse limit being taken with respect to the maps $K^*(B_{2n+2}) \to K^*(B_{2n})$ just defined.

Let us denote by x_i the first Chern class of $\pi_i^*(\eta_n)$, i.e., $x_i = \pi_i^*(g)$. Then

$$H^*(B_{2n}, Z) = Z[x_1, \cdots, x_k]/I_{n+1} \tag{5}$$

where I_{n+1} is the ideal $(x_1^{n+1}, \cdots, x_k^{n+1})$.

We consider the map $ch \circ \alpha_{\xi_{2n}} \circ \phi$ of the polynomial ring $Z[z_1, \cdots, z_k]$ into $H^*(B_{2n}, Q)$, see 4.3 and 4.5. It maps z_i onto $e^{x_i} - 1$. Since $e^{x_i} - 1 = x_i +$ higher terms, it follows from 2.5 (iii) that $\alpha_{\xi_{2n}} \circ \phi$ maps $Z[z_1, \cdots, z_k]$ onto $K^*(B_{2n}) = K^0(B_{2n})$, the kernel being the ideal $J_{n+1} = (z_1^{n+1}, \cdots, z_k^{n+1})$ as follows from (5). Thus

$$K^*(B_{2n}) \cong Z[z_1, \cdots, z_k]/J_{n+1}$$

and

$$\mathcal{K}^*(B_T) \cong \varprojlim Z[z_1, \cdots, z_k]/J_{n+1}. \tag{6}$$

If we identify $\hat{R}(T)$ with $Z[[z_1, \cdots, z_k]]$ (Proposition 4.3) and $\mathcal{K}^*(B_T)$ with the above inverse limit (6), then $\hat{\alpha} : \hat{R}(T) \to \mathcal{K}^*(B_T)$ is just the natural map

$$Z[[z_1, \cdots, z_k]] \to \varprojlim Z[z_1, \cdots, z_k]/J_{n+1}.$$

To prove that this map is bijective, one has to check that the $(z_1, \cdots, z_k)$-adic topology of $Z[z_1, \cdots, z_k]$ and the topology defined by the sequence J_n of ideals coincide. But this is easy to do.

4.10. PROPOSITION. *Let G be a compact connected Lie group, T a maximal torus of G and $\rho : T \to G$ the embedding. Then the map $\rho^! : \mathcal{K}^*(B_G) \to \mathcal{K}^*(B_T)$ (see 4.6) is injective.*

Proof. We first observe that there exist principal G-bundles which are classifying up to n (n arbitrary) *and which have a compact oriented differentiable manifold as base.* This is true for $G = U(m)$, since then we have the complex Grassmannians as "universal" base spaces. An arbitrary G may be embedded in $U(m)$ for m sufficiently large. G has thus "universal" base spaces which are fibred with $U(m)/G$ as typical fibre and complex Grassmannians. The bundle along the fibres is orientable, since it is an extension of a principal G-bundle and G is connected [5, §7.5]. Hence we have constructed universal base spaces for G with the desired properties (compare [18, §19.6]).

Let a be an element of $\mathcal{K}^*(B_G)$ for which $\rho^!(a) = 0$. Then we must show that $a = 0$. By 4.6 (4) and the above observation on classifying bundles, it suffices to prove that $a(\xi) = 0$ where ξ is any principal G-bundle over an arbitrary compact oriented differentiable manifold X. Using the notation and the proposition of 3.5 with $B_\xi = X$ it suffices to prove that $\pi^! a(\xi) = 0$. But $\pi^! a(\xi) = a(\pi^* \xi)$, the lifted bundle $\pi^*\xi$ equals $\rho(\eta)$ where η is a principal T-bundle. Now $a(\rho(\eta)) = (\rho^! a)(\eta) = 0$.

4.11. **Proof of Theorem 4.8.** We have the commutative diagram

$$\begin{array}{ccc} \hat{R}(T) & \xrightarrow{\hat{\alpha}_T} & \mathcal{K}^*(B_T) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \rho^!} \\ \hat{R}(G) & \xrightarrow[\hat{\alpha}_G]{} & \mathcal{K}^*(B_G). \end{array}$$

The vertical maps are injective, the upper horizontal one is bijective (4.4, 4.10, 4.9). Thus $\hat{\alpha}_G$ is injective. The Weyl group $W(G)$ as group of automorphisms of T operates on $\mathcal{K}^*(B_T)$ (see definition of $\rho^!$ in 4.6). Since these automorphisms come from inner automorphisms of G, every element of $\rho^! \mathcal{K}^*(B_G)$ is invariant under $W(G)$. The operation of $W(G)$ on $\hat{R}(T)$ and $\mathcal{K}^*(B_T)$ is the same if one identifies the two rings under $\hat{\alpha}_T$; this follows from the diagram in 4.7. Therefore by 4.4

$$\hat{\alpha}_T^{-1}(\rho^! \mathcal{K}^*(B_G)) \subset i\hat{R}(G),$$

$$\rho^! \mathcal{K}^*(B_G) \subset \hat{\alpha}_T i(\hat{R}(G)) = \rho^! \hat{\alpha}_G \hat{R}(G).$$

Since $\rho^!$ is injective, $\mathcal{K}^*(B_G) \subset \hat{\alpha}_G \hat{R}(G)$ which completes the proof.

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