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ON THE NON-EXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

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CHAPTER 1. INTRODUCTION

1.1. *Results.* It is the object of this paper to prove a theorem in homotopy-theory, which follows as Theorem 1.1.1. In stating it, we use one definition. A *continuous product with unit* on a space X is a continuous map $\mu : X \times X \to X$ with a point e of X such that $\mu(x, e) = \mu(e, x) = x$. An *H-space* is a space which admits a continuous product with unit. For the remaining notations, which are standard in homotopy-theory, we refer the reader to [15], [16]. In particular, $H^m(Y; G)$ is the m^{th} singular cohomology group of the space Y with coefficients in the group G.

THEOREM 1.1.1. *Unless $n = 1, 2, 4$ or 8, we have the following conclusions:*

(a) *The sphere S^{n-1} is not an H-space.*

(b) *In the homotopy group $\pi_{2n-3}(S^{n-1})$, the Whitehead product $[\iota_{n-1}, \iota_{n-1}]$ is non-zero.*

(c) *There is no element of Hopf invariant one* [17] *in $\pi_{2n-1}(S^n)$.*

(d) *Let $K = S^m \cup E^{m+n}$ be a* CW*-complex formed by attaching an $(m + n)$-cell E^{m+n} to the m-sphere S^m. Then the Steenrod square* [31]

$$\mathrm{Sq}^n : H^m(K; Z_2) \to H^{m+n}(K; Z_2)$$

is zero.

It is a classical result that the four conclusions are equivalent [36], [31]. Various results in homotopy-theory have been shown to depend on the truth or falsity of these conclusions. It is also classical that the conclusions are false for $n = 2, 4$ and 8. In fact, the systems of complex numbers, quaternions and Cayley numbers provide continuous products on the euclidean spaces R^2, R^4 and R^8; from these one obtains products on the unit spheres in these spaces, that is, on S^1, S^3 and S^7. (See [16].) The case $n = 1$ is both trivial and exceptional, and we agree to exclude it from this point on.

The remarks above, and certain other known theorems, may be summarized by the following diagram of implications.

R^n is a normed algebra over the reals $\Longleftrightarrow n = 2, 4, \text{or } 8.$

$\Downarrow$

R^n is a division algebra over the reals $\Longrightarrow n = 2^m$

$\Downarrow$

S^{n-1}, with its usual differentiable structure, is parallelisable $\Longrightarrow_{(6)} n = 2, 4, \text{or } 8$

$\Downarrow$

S^{n-1}, with some (perhaps extraordinary) differentiable structure, is parallelisable

$\Downarrow (5)$

S^{n-1} is an H-space

$\Updownarrow$

There is an element of Hopf invariant one in $\pi_{2n-1}(S^n)$

$\Downarrow (1)$ $\Downarrow (2)$ $\Downarrow (3)$ $\Downarrow (4)$

$n = 2 \text{ or } 4r$ $\quad n = 2^m$ $\quad n \neq 16$ $\quad n = 2, 4, \text{or } 8.$

The implications (1), (2), (3) represent cases of Theorem 1.1.1 which are already known. In fact, (1) is due to G. W. Whitehead [36]; (2) is due to J. Adem [4]; and (3) is due to H. Toda [34], who used an elegant lemma in homotopy-theory and extensive calculations of the homotopy groups of spheres.

The implication (4) is just Theorem 1.1.1. The implication (5) is due to A. Dold, in answer to a question of A. Borel. (We remark that Theorem 1.1.1 implies strong results on the non-parallelizability of manifolds: see Kervaire [22].) The implication (6) was proved independently by M. Kervaire [21] and by R. Bott and J. Milnor [8]. In each case, it was deduced from deep results of R. Bott [7] on the orthogonal groups.

A summary of the present work appeared as [3]. The first draft of this paper was mimeographed by Princeton; I am most grateful to all those who offered criticisms and suggestions, and especially to J. Stasheff.

1.2. *Method.* Theorem 1.1.1 will be proved by establishing conclusion (d). The method may be explained by analogy with Adem's proof [4] in the case $n \neq 2^r$. In case $n = 6$, for example, Adem relies on the relation

$$\mathrm{Sq}^6 = \mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^5\mathrm{Sq}^1 .$$

Now, in a complex $K = S^m \cup E^{m+6}$, the composite operations $\mathrm{Sq}^2\mathrm{Sq}^4$ and $\mathrm{Sq}^5\mathrm{Sq}^1$: $H^m(K; Z_2) \to H^{m+6}(K; Z_2)$ will be zero, since $H^{m+4}(K; Z_2)$ and $H^{m+1}(K; Z_2)$ are zero. Therefore Sq^6: $H^m(K; Z_2) \to H^{m+6}(K; Z_2)$ is zero.

The method fails in the case $n = 2^r$, because in this case Sq^n is not decomposable in terms of operations of the first kind.

We therefore proceed by showing that Sq^n *is decomposable in terms of operations of the second kind* (in case $n = 2^r$, $r \geqq 4$). (cf. [2, § 1]). We next explain what sort of decomposition is meant.

Suppose that $n = 2^{k+1}$, $k \geqq 3$, and that $u \in H^m(X; Z_2)$ is a cohomology class such that $\mathrm{Sq}^{2^i}(u) = 0$ for $0 \leqq i \leqq k$. Then certain cohomology operations of the second kind are defined on u; for example,

$$\beta_4(u) \in H^{m+1}(X; Z_2) \,/\, \mathrm{Sq}^1 H^m(X; Z_2)$$

$$\Phi(u) \in H^{m+3}(X; Z_2) \,/\, \mathrm{Sq}^2 H^{m+1}(X; Z_2) + \mathrm{Sq}^3 H^m(X; Z_2) \ ,$$

and so on. In fact, in § 4.2 of this paper we shall obtain a system of secondary operations $\Phi_{i,j}$, indexed by pairs (i, j) of integers such that $0 \leqq i \leqq j, j \neq i + 1$. These operations will be such that (with the data above) $\Phi_{i,j}(u)$ is defined if $j \leqq k$. The value of $\Phi_{i,j}(u)$ will be a coset in $H^q(X; Z_2)$, where $q = m + (2^i + 2^j - 1)$; let us write

$$\Phi_{i,j}(u) \in H^*(X; Z_2) \,/\, Q^*(X; i, j) \ .$$

We do not need to give the definition of $Q^*(X; i, j)$ here; however, as in the examples above, it will be a certain sum of images of Steenrod operations. (By a Steenrod operation, we mean a sum of composites of Steenrod squares.)

Suppose it granted, then, that we shall define such operations $\Phi_{i,j}$. In § 4.6 we shall also establish a formula, which is the same for all spaces X:-

$$\mathrm{Sq}^n(u) = \textstyle\sum_{i,j;\, j \leqq k} a_{i,j,k} \Phi_{i,j}(u) \qquad \text{modulo } \sum_{i,j;\, j \leqq k} a_{i,j,k} Q^*(X; i, j) \ .$$

In this formula, each $a_{i,j,k}$ is a certain Steenrod operation. We recall that $n = 2^{k+1}$, $k \geqq 3$.

Suppose it granted, then, that we shall prove such a formula. Then we may apply it to a complex $K = S^m \cup E^{m+n}$. If $u \in H^m(K; Z_2)$, then $\mathrm{Sq}^{2^i}(u) = 0$ for $0 \leqq i \leqq k$. The cosets $\Phi_{i,j}(u)$ will thus be defined for $j \leqq k$; and they will be cosets in zero groups. The formula will be applicable, and will show that $\mathrm{Sq}^n(u) = 0$, modulo zero. Theorem 1.1.1 will thus follow immediately.

1.3. *Secondary operations.* It is clear, then, that all the serious work involved in the proof will be concerned with the construction and properties of secondary cohomology operations. Two methods have so far been used to define secondary operations which are stable. The first method is that of Adem [4]. This possesses the advantage that the operations

defined are computable, at least in theory. Unfortunately, it does not give us much insight into the properties of such operations. Moreover, not all the operations we need can be defined by this method as it now stands.

The second method, which we shall use, is that of the universal example [6], [30]. This is a theoretical method; it gives us some insight, but it gives us no guarantee that the operations so defined are computable. Similarly, it sometimes shows (for example) that one operation is linearly dependent on certain others, without yielding the coefficients involved.

Both methods show that secondary operations are connected with relations between primary operations. For example, the Bockstein coboundary β_4 is connected with the relation $Sq^1Sq^1 = 0$; the Adem operation Φ is connected with the relation $Sq^2Sq^2 + Sq^3Sq^1 = 0$.

It may appear to the reader that what we say about "relations" in this section is vague and imprecise; however, it will be made precise later by the use of homological algebra [13]; this is the proper tool to use in handling relations, and in handling relations between relations.

In any event, it will be our concern in Chapter 3 to set up a general theory of stable secondary cohomology operations, and to show that *to every "relation" there is associated at least one corresponding secondary operation.* We study these operations, and the relations between them.

This theory, in fact, is not deep. However, it affords a convenient method for handling operations, by dealing with the associated relations instead. For example, we have said that Φ is "associated with" the relation $Sq^2Sq^2 + Sq^3Sq^1 = 0$. We would expect the composite operation $Sq^3\Phi$ to be "associated with" the relation

$$(Sq^3Sq^2)Sq^2 + (Sq^3Sq^3)Sq^1 = 0 . \tag{1}$$

Similarly, we have said that β_4 is "associated with" the relation $Sq^1Sq^1 = 0$. We would expect the composite operation $Sq^5\beta_4$ to be "associated with" the relation

$$(Sq^5Sq^1)Sq^1 = 0 . \tag{2}$$

But since $Sq^3Sq^2 = 0$ and $Sq^3Sq^3 = Sq^5Sq^1$, the relations (1), (2) coincide. We would therefore expect to find

$$Sq^3\Phi = Sq^5\beta_4 \qquad \text{(modulo something as yet unknown).}$$

And, in fact, the theory to be presented in Chapter 3 will justify such manipulations, and this is one of its objects.

If we can do enough algebra, then, of a sort which involves relations between the Steenrod squares, we expect to obtain relations of the form

$$\sum_{i,j} a_{i,j}\Phi_{i,j} = 0 \quad \text{(modulo something as yet unknown).}$$

(Here we use "a" as a generic symbol for a Steenrod operation, and "Φ" as a generic symbol for a secondary operation.) In particular, we shall in fact obtain (in § 4.6) a formula

$$\sum_{i,j;\, j\leq k} a_{i,j,k}\Phi_{i,j}(u) = \lambda \mathrm{Sq}^{2^{k+1}}(u)$$

such as we seek, but containing an undetermined coefficient λ.

To determine the coefficient λ, it is sufficient to apply the formula to a suitable class u in a suitable test-space X. We shall take for X the complex projective space P of infinitely-many dimensions. Our problem, then, reduces to calculating the operations $\Phi_{i,j}$ in this space P. This is performed in § 4.5.

The plan of this paper is then as follows. In Chapter 2 we do the algebraic work; in Chapter 3 we set up a general theory of stable secondary operations; in Chapter 4 we make those applications of the theory which lead to Theorem 1.1.1.

The reader may perhaps like to read § 2.1 first, and then proceed straight to Chapters 3 and 4, referring to Chapter 2 when forced by the applications.

Chapter 2. Homological Algebra

2.1. *Introduction.* In this chapter, we make those applications of homological algebra [13] which are needed for what follows. From the point of view of logic, therefore, this chapter is prior to Chapter 4; but from the point of view of motivation, Chapter 4 is prior to this one.

For an understanding of Chapter 3, only the first article of this chapter is requisite.

The plan of this chapter is as follows. In §§ 2.1, 2.2 we outline what we need from the general theory of homological algebra, proceeding from what is well known to what is less well known. In § 2.4 we state, and begin to use, Milnor's theorem on the structure of the Steenrod algebra A. In § 2.5 we perform the essential step of calculating $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ as far as we need it. This work relies on § 2.4, and also relies on a certain spectral sequence in homological algebra. This sequence is set up in § 2.3. In the last section, § 2.6, we calculate $\mathrm{Ext}_A^{s,t}(M, Z_2)$ (as far as we need it) for a certain module M that arises in Chapter 4.

We now continue by recalling some elementary algebraic notions.

The letter K will denote a field of coefficients, usually the field Z_p of residue classes modulo a fixed prime p.

A *graded algebra* A over K; *qua* vector space over K, it is the direct sum of components, $A = \sum_{q \geqq 0} A_q$; and these satisfy $1 \in A_0$, $A_q \cdot A_r \subset A_{q+r}$. The elements lying in one component A_q are called *homogeneous* (of degree q). We shall be particularly concerned with the Steenrod algebra [5], [12]. We shall give a formal, abstract definition of the Steenrod algebra in § 3.5; for present purposes the following description is sufficient. If $p = 2$, the generators are the symbols Sq^k, and the relations are those which hold between the Steenrod squares in the (mod 2) cohomology of every topological space. If $p > 2$, the generators are the symbols β_p and P_p^k, and the relations are those which hold between the Bockstein coboundary and the Steenrod cyclic reduced powers, in the (mod p) cohomology of every topological space. (Here we suppose the Bockstein coboundary defined without signs, so that it anticommutes with suspension.) The Steenrod algebra is graded; the degrees of Sq^k, β_p and P_p^k are k, 1 and $2(p-1)k$.

A *graded left module* M over the graded algebra A is a left module [9] over the algebra A; *qua* vector space over K, it is the direct sum of components, $M = \sum_q M_q$; and these satisfy $A_q \cdot M_r \subset M_{q+r}$. The elements lying in one component M_q are called *homogeneous* (of degree q). We shall write $\deg(m)$ for the degree of a homogeneous element m, and the use of this notation will imply that m is homogeneous. When we speak of *free* graded modules over the graded algebra A, we understand that they have bases consisting of homogeneous elements.

We must also discuss maps between graded modules. A K-linear function $f: M \to M'$ is said to be *of degree* r if we have $f(M_q) \subset M'_{q+r}$. We say that it is a *left* A-map if we propose to write it on the left of its argument, and if it is A-linear in the sense that

$$f(am) = (-1)^{qr} a f(m) \qquad \text{(where } a \in A_q\text{)}.$$

Similarly, we call it a *right* A-map if we propose to write it on the right of its argument, and if it is A-linear in the sense that

$$(am)f = a(mf) \ .$$

There is, of course, a (1-1) correspondence between left A-maps and right A-maps (of a fixed degree r); it is given by

$$f(m) = (-1)^{qr}(m)f^* \qquad \text{(where } m \in M_q\text{)}.$$

The two notions are thus equivalent.

Sometimes we have to deal with bigraded modules; in that case the

degree r which should appear in these signs is the total degree.

It is clear that we can avoid some signs by using right A-maps; and in Chapter 3 we will do this. In the present chapter, however, it is convenient to follow the received notation for the bar construction, so we will use left A-maps. The passage from one convention to the other will cause no trouble, as the applications are in characteristic 2.

Let

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ P & \xrightarrow{f'} & Q \end{array}$$

be a diagram of A-maps in which f and f' have degree r, while g and g' have degree s. Then we say that the diagram is *anticommutative* if

$$g'f = (-1)^{rs} f'g \ .$$

We now begin to work through the elementary notions of homological algebra, in the case when our modules are graded. We shall suppose given a graded algebra A over K which is locally finite-dimensional; that is, $\sum_{q \leq r} A_q$ is finite-dimensional for each r. We shall also assume that A is connected, that is, $A_0 = K$. Let M be a graded module over A which is locally finitely-generated; this is equivalent to saying that $\sum_{q \leq r} M_q$ is finite-dimensional. A *free resolution* of M consists of the following.

(i) A bigraded module $C = \sum_{s,t} C_{s,t}$ such that $A_q \cdot C_{s,t} \subset C_{s,t+q}$. We set $C_s = \sum_t C_{s,t}$, and require that each C_s is a (locally finitely-generated) free module over A.

(ii) An A-map $d : C \to C$ of bidegree $(-1, 0)$, so that $dC_{s,t} \subset C_{s-1,t}$. (Thus the total degree of d is -1). We write d_s: $C_s \to C_{s-1}$ for the components of d.

(iii) An A-map $\varepsilon : C_0 \to M$ of degree zero. We require that the sequence

$$0 \longleftarrow M \xleftarrow{\varepsilon} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \longleftarrow C_{s-1} \xleftarrow{d_s} C_s \longleftarrow \cdots$$

should be exact, and we regard C as an acyclic chain complex.

Next, let L and N be left and right graded A-modules. We have a group

$$\mathrm{Hom}_A^t(C_s, L)$$

whose elements are the A-maps $\mu : C_s \to L$ of total degree $-(s + t)$, so that $\mu(C_{s,u}) \subset L_{u-t}$. Since C_s is graded, we define

$$\mathrm{Hom}_A(C_s, L) = \sum_t \mathrm{Hom}_A^t(C_s, L)$$

and

$$\mathrm{Hom}_A(C, L) = \sum_s \mathrm{Hom}_A(C_s, L) .$$

We also have a group [13]

$$N \otimes_A C = \sum_s N \otimes_A C_s;$$

we bigrade it as follows; if $n \in N_u$, $c \in C_{s,t}$, we set

$$n \otimes_A c \in (N \otimes_A C)_{s,t+u} .$$

We may regard $\mathrm{Hom}_A(C, L)$ as a cochain complex and $N \otimes_A C$ as a chain complex, using the boundaries

$$\mathrm{Hom}_A(C_0, L) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(C_{s-1}, L) \xrightarrow{(d_s)^*} \mathrm{Hom}_A(C_s, L) \longrightarrow \cdots ,$$

$$N \otimes_A C_0 \longleftarrow \cdots \longleftarrow N \otimes_A C_{s-1} \xleftarrow{(d_s)_*} N \otimes_A C_s \longleftarrow \cdots$$

(Note that we have to define $(d_s)_*$, that is, $1 \otimes d_s$, by the rule

$$(1 \otimes d_s)(n \otimes c) = (-1)^u (n \otimes d_s c)$$

where $n \in N_u$). We may write $\delta = \sum_s (d_s)^*$, $\partial = \sum_s (d_s)_*$.

These complexes are determined up to chain equivalence by L, M, N. In fact, given resolutions C, C' of M, M', and given a map $f : M \to M'$ we may extend it to a chain map $g : C \to C'$; moreover, such a map is unique up to chain homotopy. Such chain maps (and homotopies) yield cochain maps (and homotopies) of $\mathrm{Hom}_A(C, L)$. We thus see that the cohomology groups of $\mathrm{Hom}_A(C, L)$ are independent of C (up to a natural isomorphism), and are natural in M. We write $\mathrm{Ext}_A^{s,t}(M, L)$ for $\mathrm{Ker}\,(d_{s+1})^*/\mathrm{Im}\,(d_s)^*$ in the sequence

$$\cdots \longrightarrow \mathrm{Hom}_A^t(C_{s-1}, L) \xrightarrow{(d_s)^*} \mathrm{Hom}_A^t(C_s, L) \xrightarrow{(d_{s+1})^*} \mathrm{Hom}_A^t(C_{s+1}, L) \longrightarrow \cdots$$

We also define

$$\mathrm{Ext}_A^s(M, L) = \sum_t \mathrm{Ext}_A^{s,t}(M, L) .$$

Similarly for the homology groups of $N \otimes_A C$, which are written $\mathrm{Tor}_{s,t}^A(N, M)$ or $\mathrm{Tor}_s^A(N, M)$.

We shall be particularly concerned with the cases $L = K$, $N = K$. We grade $L = K$ by setting $L_0 = K$, $L_q = 0$ for $q \neq 0$. The structure of $L = K$ as an A-module is thus unique (and trivial) since $a(L_0) = 0$ if $\deg(a) > 0$, while the action of A_0 is determined by that of the unit. Similarly for $N = K$.

In the case $L = K$, $N = K$ we have a formal duality between Tor and Ext. In fact, if V is a finite-dimensional vector space over K, we write V^* for its dual. If V is a locally finite-dimensional graded vector space over K, we set

$$V^* = \sum_q (V_q)^* ,$$

and regard this as the dual in the graded case. Thus $K \otimes_A C_s$ and $\mathrm{Hom}_A(C_s, K)$ are dual (graded) vector spaces over K; the pairing is given by

$$h(k \otimes_A c) = h(kc)$$

for $h \in \mathrm{Hom}_A(C_s, K)$, etc. The maps $(d_s)_*$ and $(d_s)^*$ are dual. Thus

$$\mathrm{Tor}^A_{s,t}(K, M) \quad \text{and} \quad \mathrm{Ext}^{s,t}_A(M, K)$$

are dual vector spaces over K.

We now introduce some further notions which are applicable because $A_0 = K$. We set $I(A) = \sum_{q>0} A_q$; and if N is a (graded) A-module, we set $J(N) = I(A) \cdot N$. (Thus $J(N)$ is the kernel of the usual map $N \to K \otimes_A N$.) We call a map $f : N \to N'$ *minimal* if $\mathrm{Ker}\, f \subset J(N)$. We call a resolution *minimal* if the maps d_s and ε are minimal. The word "minimal" expresses the intuitive notion that in constructing such a resolution by the usual inductive process, we introduce (at each stage) as few A-free generators as possible.

It is easy to show that each (locally finitely-generated) A-module M has a minimal resolution. Any two minimal resolutions of M are isomorphic.

We note that if C is a minimal resolution of M, then

$$\mathrm{Tor}^A_{s,t}(K, M) \cong (K \otimes_A C)_{s,t}$$
$$\mathrm{Ext}^{s,t}_A(M, K) \cong \mathrm{Hom}^t_A(C_s, K) .$$

This is immediate, since the boundary ∂ in $K \otimes_A C$ is zero, and so is the coboundary δ in $\mathrm{Hom}_A(C, K)$.

This concludes our survey of the elementary notions which are needed in Chapter 3.

2.2. *General notions.* In this section we continue to survey the general notions of homological algebra that we shall have occasion to use later.

We begin by setting up a lemma which forms a sort of converse to the last remarks of § 2.1. It arises in the following context. Let

$$M \longleftarrow C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \longleftarrow C_{s-1} \xleftarrow{d_s} C_s$$

be a partial resolution of M; and set $Z(s) = \mathrm{Ker}(d_s) \cap J(C_s)$. Then there is a homomorphism

$$\theta : Z(s) \longrightarrow \mathrm{Tor}^A_{s+1}(K, M)$$

defined as follows. Extend the partial resolution by adjoining some C_{s+1},

d_{s+1}. Given $z \in Z(s)$, take $w \in C_{s+1}$ such that $d_{s+1}w = z$, and define

$$\theta(z) = \{1 \otimes_A w\} .$$

We easily verify that θ is well-defined, epimorphic, and natural for maps of M.

Next, let $\{g_i\}$ be a K-base for $\mathrm{Tor}^A_{s+1}(K, M)$; choose $z_i \in Z(s)$ so that $\theta(z_i) = g_i$. Let C_{s+1} be an A-free module on generators c_i in (1-1) correspondence with the z_i, and of the same t-degrees; and define

$$d_{s+1} : C_{s+1} \to C_s$$

by

$$d_{s+1}(c_i) = z_i .$$

LEMMA 2.2.1. *The map d_{s+1} is minimal, and if d_s is also minimal, then*

$$C_{s-1} \xleftarrow{d_s} C_s \xleftarrow{d_{s+1}} C_{s+1}$$

is exact. In this case $\{1 \otimes_A c_i\} = g_i$.

We shall use this lemma to construct minimal resolutions in a convenient fashion. In this application, since d_s will be minimal, θ will be defined on $\mathrm{Ker}(d_s)$.

PROOF. By the construction, we have $d_s d_{s+1} = 0$. Moreover, if $C_{s-1} \xleftarrow{d_s} C_s \xleftarrow{d_{s+1}} C_{s+1}$ is not exact, we may add further generators to C_{s+1} (obtaining C'_{s+1}, d'_{s+1}, say) so that the sequence becomes exact. By using the definition of θ, the condition $\theta(z_i) = g_i$ now yields

$$\{1 \otimes_A c_i\} = g_i .$$

We will now prove that d_{s+1} is minimal. In fact, take an element

$$z = \textstyle\sum_i (\lambda_i + a_i)c_i$$

of $\mathrm{Ker}(d_{s+1})$, with $\lambda_i \in K$, $a_i \in I(A)$. Then, on extending our resolution to

$$C_s \xleftarrow{d'_{s+1}} C'_{s+1} \xleftarrow{d'_{s+2}} C'_{s+2} ,$$

we can find w such that $d'_{s+2}w = z$. Hence

$$\partial(1 \otimes_A w) = \textstyle\sum_i (1 \otimes_A \lambda_i c_i) .$$

That is, $\sum \lambda_i g_i = 0$. Thus $\lambda_i = 0$ for each i, and z lies in $J(C_{s+1})$. We have shown that d_{s+1} is minimal.

We now suppose that d_s is minimal, so that $Z(s) = \mathrm{Ker}(d_s)$. We wish to prove the exactness. Suppose, as an inductive hypothesis, that

$$C_{s-1,t} \xleftarrow{d_s} C_{s,t} \xleftarrow{d_{s+1}} C_{s+1,t}$$

is exact for $t < n$; this hypothesis is vacuous for sufficiently small n. One may verify that the kernel of

$$\theta\colon Z(s) \to \mathrm{Tor}^{A}_{s+1}(K, M)$$

is $J(\ker(d_s))$. Hence any x in $Z(s) \cap C_{s,n}$ can be written in the form

$$x = \sum_i \lambda_i z_i + \sum_j (-1)^{\varepsilon(j)} a_j x_j$$

with $\lambda_i \in K$, $a_j \in I(A)$, $x_j \in \mathrm{Ker}(d_s)$, $\varepsilon(j) = \deg(a_j)$. By the inductive hypothesis, $x_j = d_{s+1} w_j$, say; hence

$$x = d_{s+1}\left(\sum_i \lambda_i c_i + \sum_j a_j w_j\right).$$

This completes the induction, and the proof of Lemma 2.2.1.

We next introduce products into the cohomology groups Ext. One method of doing this is due to Yoneda [37]. Let M, M', M'' be three A-modules, with resolutions C, C', C''. Let

$$f : C_s \to M', \qquad g : C'_{s'} \to M''$$

be A-maps of total degrees $-(s+t)$, $-(s'+t')$, such that $f d_{s+1} = 0$ and similarly for g, so that f, g represent elements of

$$\mathrm{Ext}^{s,t}_{A}(M, M'), \qquad \mathrm{Ext}^{s',t'}_{A}(M', M'').$$

Then we may form an anticommutative diagram, as follows.

$$\begin{array}{ccccccccccccc}
M & \xleftarrow{\varepsilon} & C_0 & \longleftarrow \cdots \longleftarrow & C_s & \xleftarrow{d_{s+1}} & C_{s+1} & \longleftarrow \cdots & \cdots \longleftarrow & C_{s+s'} & \longleftarrow \cdots \\
 & & & \swarrow^{f} & \downarrow f_0 & & \downarrow f_1 & & & \downarrow f_{s'} & \\
 & & M' & \xleftarrow{\varepsilon'} & C'_0 & \xleftarrow{d'_1} & C'_1 & \longleftarrow \cdots & \cdots \longleftarrow & C'_{s'} & \longleftarrow \cdots \\
 & & & & & & & & & \downarrow g & \\
 & & & & & & & & & M'' &
\end{array}$$

The composite map $(-1)^{(s+t)(s'+t')} g f_{s'}$ represents an element of

$$\mathrm{Ext}^{s+s',t+t'}_{A}(M, M'').$$

(The sign is introduced for convenience later.) By performing the obvious verifications, we see that this "composition" product gives us an invariantly-defined pairing from $\mathrm{Ext}^{s',t'}_{A}(M', M'')$ and $\mathrm{Ext}^{s,t}_{A}(M, M')$ to $\mathrm{Ext}^{s+s',t+t'}_{A}(M, M'')$. This product is bilinear and associative.

Our next lemma states an elementary relation between this product and the homomorphism

$$\theta : Z(s) \to \mathrm{Tor}^{A}_{s+1}(K, M)$$

introduced above. We first set up some data.

Suppose given a K-linear function $\alpha : A \to K$, of degree $-t'(t' > 0)$, and such that $\alpha(ab)=0$ if $a \in I(A)$, $b \in I(A)$. That is, α is a primitive element of A^*. It follows that $\alpha \mid I(A)$ is A-linear. Let us take a resolution C' of K, such that $C'_0 = A$ and $\varepsilon' : C'_0 \to K$ is the projection of A on A_0. Then the composite

$$C'_1 \xrightarrow{d'_1} A \xrightarrow{\alpha} K$$

is A-linear, and defines an element

$$h_\alpha \in \mathrm{Ext}^{1,t'}_{A}(K, K)$$

depending only on α.

Suppose given an element h of $\mathrm{Ext}^{s,t}_{A}(M, K)$; let C be a partial resolution of M over A, and define $Z(s) = \mathrm{Ker}(d_s) \cap J(C_s)$, as above. Let x be an element of $Z(s) \cap C_{s,t+t'}$, and suppose that x can be written in the form $x = \sum_i a_i c_i$, where $a_i \in I(A)$ and $d_s c_i \in J(C_{s-1})$. For example, if d_{s-1} is minimal, then we can always write x in this form, by taking the elements c_i from an A-base of C_s. In any case, we have

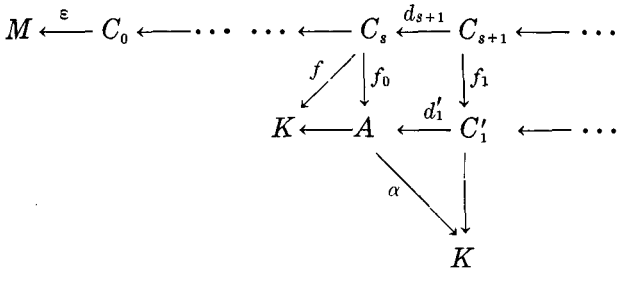
$$\{1 \otimes_A c_i\} \in \mathrm{Tor}^{A}_{s}(K, M) .$$

LEMMA 2.2.2.

$$(h_\alpha h)(\theta x) = \sum_i (\alpha a_i)(h\{1 \otimes_A c_i\}) .$$

Here, of course, the product $(h_\alpha h)(\theta x)$ can be formed because $\mathrm{Ext}^{s+1,t+t'}_{A}(M, K)$ and $\mathrm{Tor}^{A}_{s+1,t+t'}(K, M)$ are dual vector spaces; similarly for the product $h\{1 \otimes_A c_i\}$.

PROOF. In order to obtain the product $h_\alpha h$, it is proper to extend the partial resolution C and set up the following diagram, in which f is a representative cocycle for h.



We see that

$$(-1)^{(s+1)(1+t')}(h_\alpha h)(\theta x)$$
$$= \alpha d_1' f_1(d_{s+1}^{-1}x) = (-1)^{s+t}\alpha f_0 x = (-1)^{s+t}\alpha f_0 \textstyle\sum_i a_i c_i = \alpha \sum_i (-1)^{\varepsilon(i)} a_i (f_0 c_i)$$
$$= \textstyle\sum_i (-1)^{\varepsilon(i)}(\alpha a_i)(fc_i) \qquad \text{where } \varepsilon(i) = (s+t)(1+\deg(a_i)) .$$

But $h\{1 \otimes_A C_i\} = (fc_i)$, and the only terms which contribute to the sum have $\deg(a_i) = t'$. This proves the lemma.

We next introduce the bar construction, which gives us a standard resolution of K over A. Let us write $M \otimes M'$ for $M \otimes_K M'$. Then we set

$$\bar{A} = A/A_0 , \quad (\bar{A})^0 = K , \quad (\bar{A})^s = \bar{A} \otimes (\bar{A})^{s-1} \qquad (s > 0),$$
$$\bar{B}(A) = \textstyle\sum_{s \geq 0} (\bar{A})^s , \qquad B(A) = A \otimes \bar{B}(A) .$$

Thus, $B(A)$ is a free A-module, and $\bar{B}(A) \cong K \otimes_A B(A)$. We write the elements of $(\bar{A})^s$ and $A \otimes (\bar{A})^s$ in the forms

$$[a_1 \mid a_2 \mid \cdots \mid a_s], \qquad a[a_1 \mid a_2 \mid \cdots \mid a_s] .$$

We also write a for $a[\ ]$. We define an augmentation $\varepsilon : B(A) \to K$ by $\varepsilon(1) = 1$, $\varepsilon(I(A)) = 0$, $\varepsilon(A \otimes (\bar{A})^s) = 0$ if $s > 0$. We define a contracting homotopy S in $B(A)$ by $Sa_0[a_1 \mid a_2 \mid \cdots \mid a_s] = 1\,[a_0 \mid a_1 \mid a_2 \mid \cdots \mid a_s]$. We define a boundary d in $B(A)$ by the inductive formulae $d(1) = 0$, $dS + Sd = 1 - \varepsilon$, $d(am) = (-1)^q a(dm)$ $(a \in A_q)$. $B(A)$ thus becomes a free, acyclic resolution of K over A. We take the induced boundary $\bar{d}$ in $\bar{B}(A)$; its homology is therefore $\mathrm{Tor}^A(K, K)$.

Explicit forms for these boundaries are as follows (where $a_i \in I(A)$ for $i \geqq 1$.)

$$da_0[a_1 \mid a_2 \mid \cdots \mid a_s] = (-1)^{\varepsilon(0)} a_0 a_1 [a_2 \mid \cdots \mid a_s]$$
$$+ \textstyle\sum_{1 \leq r \leq s} (-1)^{\varepsilon(r)} a_0 [a_1 \mid \cdots \mid a_r a_{r+1} \mid \cdots \mid a_s] ,$$
$$\bar{d}[a_1 \mid a_2 \mid \cdots \mid a_s] = \textstyle\sum_{1 \leq r \leq s} (-1)^{\eta(r)} [a_1 \mid \cdots \mid a_r a_{r+1} \mid \cdots \mid a_s] ,$$

where

$$\varepsilon(r) = r + \textstyle\sum_{0 \leq i \leq r} \deg(a_i) , \qquad \eta(r) = r + \textstyle\sum_{1 \leq i \leq r} \deg(a_i) .$$

It would therefore be equivalent to set

$$I(A) = \textstyle\sum_{q>0} A_q, \quad I(A)^0 = K , \quad I(A)^s = I(A) \otimes I(A)^{s-1} , \quad \bar{B}(A) = \textstyle\sum_{s \geq 0} I(A)^s$$

and define the boundary $d_s : I(A)^s \to I(A)^{s-1}$ in $\bar{B}(A)$ by the formula given above for $\bar{d}$.

It is now easy to obtain the vector-space dual $(\bar{B}(A))^*$ of $\bar{B}(A)$. Let A^* be the dual of A. We define $\overline{A^*} = A^*/A_0^*$; $\overline{A^*}$ is dual to $I(A)$; $(\overline{A^*})^s$ is dual to $(I(A))^s$. We may define

$$F(A^*) = \textstyle\sum_{s \geq 0} (\overline{A^*})^s;$$

$F(A^*)$ is dual to $\bar{B}(A)$. We write elements of $F(A^*)$ in the form

$$[\alpha_1 | \alpha_2 | \cdots | \alpha_s]$$

where $\alpha_i \in A^*$.

From the product map $\varphi : A \otimes A \to A$ we obtain a diagonal map $\psi = \varphi^*$: $A^* \to A^* \otimes A^*$. (Here we define the pairing of $A^* \otimes A^*$ and $A \otimes A$ by $(\alpha \otimes \beta)(a \otimes b) = (\alpha a)(\beta b)$; we thus omit the sign introduced by Milnor [25]. Since the applications are in characteristic 2, this is immaterial). Let us write $\psi(\alpha) = \sum_u \alpha'_u \otimes \alpha''_u$; we may now define the coboundary

$$\bar{d}^s : (\overline{A^*})^{s-1} \to (\overline{A^*})^s$$

by

$$\bar{d}^s[\alpha_1 | \alpha_2 | \cdots | \alpha_{s-1}] = \sum_{1 \leq r \leq s; u} (-1)^{\varepsilon(r,u)} [\alpha_1 | \cdots | \alpha'_{r,u} | \alpha''_{r,u} | \cdots | \alpha_{s-1}] ,$$

where

$$\varepsilon(r, u) = r + \deg(\alpha'_{r,u}) + \sum_{1 \leq i < r} \deg(\alpha_i) .$$

The coboundary $\bar{d}^s$ is dual to $\bar{d}_s$; the cohomology of $F(A^*)$ is therefore $\mathrm{Ext}_A(K, K)$. Of course, $F(A^*)$ is nothing but the cobar construction [1] on the coalgebra A^*.

There is a second method of introducing products, which uses the bar construction. In fact, we may define a cup-product of cochains in $F(A^*)$ by

$$[\alpha_1 | \alpha_2 | \cdots | \alpha_s][\alpha_{s+1} | \alpha_{s+2} | \cdots | \alpha_{s+s'}] = [\alpha_1 | \alpha_2 | \cdots | \alpha_s | \alpha_{s+1} | \cdots | \alpha_{s+s'}] .$$

It is clear that it is associative, bilinear and satisfies

$$\delta(xy) = (\delta x)y + (-1)^{s+t} x(\delta y)$$

(where $x = [\alpha_1 | \alpha_2 | \cdots | \alpha_s]$, $t = \sum_{1 \leq i \leq s} \deg(\alpha_i)$.) Therefore it induces an associative, bilinear product in the cohomology of $F(A^*)$, that is, in $\mathrm{Ext}_A(K, K)$. Indeed, this cup-product of cochains even allows us to define Massey products [23][24][35], etc.

We should show that this product coincides with the previous one (in case $M = M' = M'' = K$.) Let

$$f : A \otimes (\bar{A})^s \to K$$

be an A-linear map of degree $-(s+t)$ such that $f d_{s+1} = 0$. The previous method requires us to construct certain functions

$$f_{s'} : A \otimes (\bar{A})^{s+s'} \to A \otimes (\bar{A})^{s'} .$$

We may do this by setting

$$f_{s'}(a_0[a_1 | a_2 | \cdots | a_{s+s'}]) = (-1)^{\varepsilon} a_0 [a_1 | a_2 | \cdots | a_{s'}] f[a_{s'+1} | \cdots | a_{s+s'}] ,$$

where $\varepsilon = (s + t)(s' + t')$ and $t' = \sum_{0 \leq i \leq s'} \deg(a_i)$. From this it is immediate that the two products coincide.

A third method of defining products is useful under a different set of conditions. Before proceeding to state them, we give $A \otimes A$ the structure of a graded algebra by setting

$$\deg(a_1 \otimes a_2) = \deg(a_1) + \deg(a_2)$$

and

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{\varepsilon}(a_1 a_3 \otimes a_2 a_4)$$

where

$$\varepsilon = \deg(a_2)\deg(a_3) .$$

Similarly, if M_1, M_2 are (graded) A-modules, we give $M_1 \otimes M_2$ the structure of a graded $A \otimes A$-module by setting

$$\deg(m_1 \otimes m_2) = \deg(m_1) + \deg(m_2)$$

and

$$(a_1 \otimes a_2)(m_1 \otimes m_2) = (-1)^{\varepsilon}(a_1 m_1 \otimes a_2 m_2)$$

where

$$\varepsilon = \deg(a_2)\deg(m_1) .$$

We now suppose that A is a Hopf algebra [25][26]. That is, there is given a "diagonal" or "co-product" map

$$\psi : A \to A \otimes A$$

which is a homomorphism of algebras. It is required that ψ should be "co-associative", in the sense that the following diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ \psi \downarrow & & \downarrow 1 \otimes \psi \\ A \otimes A & \xrightarrow{\psi \otimes 1} & A \otimes A \otimes A \end{array}$$

(This diagram is obtained from that which expresses the associativity of a product map, by reversing the arrows.) Lastly, it is required that ψ should have a " co-unit", in a similar sense.

Let C be a resolution of K over A. We may form $C \otimes C$, and give it a first grading by setting

$$(C \otimes C)_{s''} = \sum_{s+s'=s''} C_s \otimes C_{s'} .$$

Here, each summand is a (graded) module over $A \otimes A$. We give $C \otimes C$ a boundary by the rule

$$d(x \otimes y) = dx \otimes y + (-1)^{s+t} x \otimes dy \qquad (x \in C_{s,t});$$

the complex $C \otimes C$ is acyclic. We may thus construct a map

$$\Delta : C \to C \otimes C$$

compatible with the map ψ of operations, and with the canonical isomorphism $K \to K \otimes K$. The map Δ induces a product

$$\mu : \mathrm{Hom}_A(C, K) \otimes \mathrm{Hom}_A(C, K) \to \mathrm{Hom}_A(C, K) .$$

By performing the obvious verifications, one sees that this product yields an invariantly-defined pairing from $\mathrm{Ext}_A^{s,t}(K, K)$ and $\mathrm{Ext}_A^{s',t'}(K, K)$ to $\mathrm{Ext}_A^{s+s',t+t'}(K, K)$. This product is bilinear and associative. Moreover, if the diagonal map ψ of A is anticommutative, then one easily sees (as in [13, Chapter XI]) that this product is anticommutative, the sign being $(-1)^{(s+t)(s'+t')}$.

We should next show that this product coincides with the previous one, defined using the bar construction. In fact, if we take $C = B(A)$, we may construct a map Δ by using the contracting homotopy

$$T = S \otimes 1 + \varepsilon \otimes S$$

in $B(A) \otimes B(A)$; we use the inductive formulae

$$\Delta(1) = 1 \otimes 1 , \qquad \Delta S = T\Delta ,$$
$$\Delta(am) = (\psi a)(\Delta m) .$$

Let us write $\psi(a_r) = \sum a_r' \otimes a_r''$, leaving the parameter in this summation to be understood; and suppose $\deg(a_r) > 0$ for $1 \leqq r \leqq q$. Then we find

$$\Delta[a_1 \mid a_2 \mid \cdots \mid a_q] = \sum_{0 \leqq r \leqq q} (-1)^{\varepsilon} [a_1' \mid a_2' \mid \cdots \mid a_r'] \otimes a_1'' a_2'' \cdots a_r'' [a_{r+1} \mid \cdots \mid a_q]$$

where $\varepsilon = \sum_{1 \leqq i < j \leqq r} \deg(a_i'')(1 + \deg(a_j'))$. The resulting product μ in $\mathrm{Hom}_A(B(A), K)$ coincides with the cup-product in $F(A^*)$ given by

$$[\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r][\alpha_{r+1} \mid \alpha_{r+2} \mid \cdots \mid \alpha_q] = [\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r \mid \alpha_{r+1} \mid \cdots \mid \alpha_q] .$$

These two products in $\mathrm{Ext}_A(K, K)$ thus coincide. From this we can make two deductions. First, the product defined using the diagonal map ψ in A is independent of ψ. Secondly, if the algebra A should happen to admit an anticommutative diagonal ψ, then the cup-products defined using the bar construction are anticommutative. This would not be true for a general algebra A.

Let us now assume that the diagonal ψ in A is anticommutative; let us set $C = B(A)$, and let $\rho : C \otimes C \to C \otimes C$ be the map which permutes the two factors and introduces the appropriate sign. We will define an explicit chain homotopy χ between the maps

$$\Delta, \rho\Delta : C \to C \otimes C .$$

We do this by the following inductive formulae:

$$\chi(1) = 0$$
$$\chi S = T(\rho\Delta - \Delta)S - T\chi$$
$$\chi(am) = (-1)^{\deg a}\psi(a)\chi(m) .$$

We will now assume $K = Z_2$, since this is satisfied in the applications; we may thus omit the signs. We have the following explicit formula for χ:

$$\chi[a_1 \mid a_2 \mid \cdots \mid a_q] = \sum_{0 \le t < r \le q} [a_1' \mid a_2' \mid \cdots \mid a_t' \mid a_{t+1}'a_{t+2}' \cdots a_r' \mid a_{r+1} \mid a_{r+2} \mid \cdots \mid a_q] \otimes a_1''a_2'' \cdots a_t''[a_{t+1}'' \mid a_{t+2}'' \mid \cdots \mid a_r''] .$$

(Here, of course, we have again used the convention that $\psi(a_r) = \sum a_r' \otimes a_r''$).

Passing to the complex $\bar{B}(A)$, and then to its dual $F(A^*)$, we obtain a product $\smile_1$ in $F(A^*)$, satisfying the usual formula

$$\delta(x \smile_1 y) = \delta x \smile_1 y + x \smile_1 \delta y + x \smile y + y \smile x .$$

To give an explicit form for this product, we recall that if A is a Hopf algebra, then its dual A^* is also a Hopf algebra, with ψ^* for product and φ^* for diagonal. This gives sense to the following explicit formula.

$$[\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_p] \smile_1 [\beta_1 \mid \beta_2 \mid \cdots \mid \beta_q] = \sum_{1 \le r \le p} [\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_{r-1} \mid \alpha_r^{(1)}\beta_1 \mid \alpha_r^{(2)}\beta_2 \mid \cdots \mid \alpha_r^{(q)}\beta_q \mid \alpha_{r+1} \mid \cdots \mid \alpha_p] .$$

Here we have written the iterated diagonal $\Psi : A^* \to (A^*)^q$ in the form

$$\Psi(\alpha_r) = \sum \alpha_r^{(1)} \otimes \alpha_r^{(2)} \otimes \cdots \otimes \alpha_r^{(q)} ,$$

the parameter in the summation being left to be understood.

In particular, we have

$$[\alpha_1] \smile_1 [\alpha_2] = [\alpha_1\alpha_2]$$
$$[\alpha_1 \mid \alpha_2] \smile_1 [\alpha_3] = [\alpha_1\alpha_3 \mid \alpha_2] + [\alpha_1 \mid \alpha_2\alpha_3]$$
$$[\alpha_1 \mid \alpha_2] \smile_1 [\alpha_3 \mid \alpha_4] = \sum [\alpha_1'\alpha_3 \mid \alpha_1''\alpha_4 \mid \alpha_2] + \sum [\alpha_1 \mid \alpha_2'\alpha_3 \mid \alpha_2''\alpha_4] .$$

This concludes the present survey of general notions.

2.3. *A spectral sequence.* In this section we establish a spectral sequence which is needed in our calculations. It arises in the following situation.

Let Γ be a (connected) Hopf algebra [26] over a field K, and let Λ be a Hopf subalgebra of Γ. We will suppose that Λ is *central* in Γ, in the sense that

$$ab = (-1)^{tu}ba \qquad \text{if } a \in \Lambda_t, b \in \Gamma_u .$$

This already implies that Λ is *normal* in Γ, in the sense that

$$I(\Lambda)\cdot\Gamma = \Gamma\cdot I(\Lambda)\ ,$$

where $I(\Lambda) = \sum_{i>0}\Lambda_i$ (cf. [13, Chapter XVI, §6]). We define the quotient $\Omega = \Gamma//\Lambda$ by

$$\Gamma//\Lambda = \Gamma/(I(\Lambda)\cdot\Gamma) \qquad \text{(cf. [13], loc. cit.)}$$

To simplify the notation, we define $H^{s,t}(A) = \mathrm{Ext}_A^{s,t}(K, K)$, $H^s(A) = \mathrm{Ext}_A^s(K, K)$, where A is a connected, graded algebra over K.

We now take the (bigraded) cochain complex $F(\Gamma^*)$, with the cup-product defined above. We filter it by setting

$$[\alpha_1 | \alpha_2 | \cdots | \alpha_s] \in F(\Gamma^*)^{(p)}$$

if α_i annihilates $I(\Lambda)$ for p values of i.

THEOREM 2.3.1. *This filtration of $F(\Gamma^*)$ defines a spectral sequence with cup-products, such that:*

(i) *E_∞ gives a composition-series for $H^*(\Gamma)$.*

(ii) *$E_2 \cong H^*(\Lambda)\otimes H^*(\Omega)$.*

Here the ring-structure of the right-hand side is defined by

$$(x\otimes y)(z\otimes w) = (-1)^{(p+t)(q+u)}(xz\otimes yw)\ ,$$

where $y \in H^{p,t}(\Omega)$, $z \in H^{q,u}(\Lambda)$.

(iii) *The isomorphism*

$$E_2^{0,q} \cong H^q(\Lambda)\otimes K \cong H^q(\Lambda)$$

is induced by the natural map $F(\Gamma^*) \to F(\Lambda^*)$. The isomorphism

$$E_2^{p,0} \cong K \otimes H^p(\Omega) \cong H^p(\Omega)$$

is induced by the natural map $F(\Omega^*) \to F(\Gamma^*)$.

This spectral sequence was used in [2]; the author supposes that it coincides with that given in [13, Chapter XVI], but this is not relevant to the applications.

In [13] it is assumed that Γ is free (or at least projective) as a (left or right) module over Λ. In our case, this follows from the assumption that Γ is a Hopf algebra and Λ a Hopf subalgebra. Thus, if $\{\omega_i\}$ is a K-base for Ω, with $\omega_1 = 1$, and if γ_i is a representative for ω_i in Γ, with $\gamma_1 = 1$, then $\{\gamma_i\}$ is a (left or right) Λ-base for Γ (see [26]). This is the only use we make of the diagonal structure of Γ and Λ.

We begin the proof of the theorem in homology, by considering the (bigraded) chain complex $\bar{B}(\Gamma)$. We filter it by setting

$$[a_1 | a_2 | \cdots | a_s] \in \bar{B}(\Gamma)^{(p)}$$

if $a_i \in I(\Lambda)$ for $(s - p)$ values of i. Each $\bar{B}(\Gamma)^{(p)}$ is closed for $\bar{d}$; we thus

obtain a spectral sequence, whose term E^∞ gives a composition series for $H_*(\Gamma) = \mathrm{Tor}^\Gamma(K, K)$. We have to calculate E^2. To this end, we begin by calculating the homology of certain subcomplexes of $B(\Gamma)$.

Let us consider

$$\Lambda \otimes \bar{B}(\Gamma)^{(p)} + \Gamma \otimes \bar{B}(\Gamma)^{(p-1)}, \qquad \Gamma \otimes \bar{B}(\Gamma)^{(p-1)}.$$

Both are closed for d, and the first is also closed for S, hence acyclic. Consider

$$C^{(p)} = \Lambda \otimes \bar{B}(\Gamma)^{(p)} + \Gamma \otimes \bar{B}(\Gamma)^{(p-1)} / \Gamma \otimes \bar{B}(\Gamma)^{(p-1)}.$$

LEMMA 2.3.2.

$$H_s(C^{(p)}) \cong \begin{cases} (\bar{\Omega})^p & (s = p) \\ 0 & (s \neq p). \end{cases}$$

The isomorphism for $s = p$ is obtained by projecting Λ to K and $(\bar{\Gamma})^p$ to $(\bar{\Omega})^p$.

(Here, of course, the suffix s refers to the first grading.)

PROOF. This is certainly true for $p = 0$, since $\Lambda \otimes \bar{B}(\Gamma)^{(0)} = B(\Lambda)$. As an inductive hypothesis, suppose it true for p. Consider the following chain complexes.

$$C' = (\Gamma \otimes \bar{B}(\Gamma)^{(p)}) / (\Lambda \otimes \bar{B}(\Gamma)^{(p)} + \Gamma \otimes \bar{B}(\Gamma)^{(p-1)})$$

$$C'' = (\Lambda \otimes \bar{B}(\Gamma)^{(p+1)} + \Gamma \otimes \bar{B}(\Gamma)^{(p)}) / (\Lambda \otimes \bar{B}(\Gamma)^{(p)} + \Gamma \otimes \bar{B}(\Gamma)^{(p-1)})$$

$$C^{(p+1)} = (\Lambda \otimes \bar{B}(\Gamma)^{(p+1)} + \Gamma \otimes \bar{B}(\Gamma)^{(p)}) / (\Gamma \otimes \bar{B}(\Gamma)^{(p)})$$

We have an exact sequence

$$0 \to C' \to C'' \to C^{(p+1)} \to 0$$

and hence an exact homology sequence. We know that $H_*(C'') = 0$; we may find $H_*(C')$ by expressing C' as a direct sum. Let $\{\gamma_i\}$ be a right Λ-base for Γ, with $\gamma_1 = 1$, as above; we may define an antichain map $f_i : C^{(p)} \to C'$ by

$$f_i(x) = \gamma_i x;$$

f_i is monomorphic (if $i > 1$) and

$$C' = \textstyle\sum_{i>1} f_i(C^{(p)}).$$

We deduce that

$$H_s(C') \cong \begin{cases} (\bar{\Omega})^{p+1} & (s = p) \\ 0 & (s \neq p). \end{cases}$$

The isomorphism for $s = p$ is obtained by projecting Γ to $\bar{\Omega}$ and $(\bar{\Gamma})^p$ to $(\bar{\Omega})^p$. Now, in the exact homology sequence, we have

$$d\colon H^{s+1}(C^{(p+1)}) \cong H^s(C') ;$$

from this we see that the lemma is true for $(p+1)$. This proves the lemma, by induction over p.

We now note that $C^{(p)}$ is a free (left) Λ-module, and

$$\bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)} \cong K \otimes_{\Lambda} C^{(p)};$$

this isomorphism is a chain map. Therefore the term E^1 of the spectral sequence is given by

$$\begin{aligned} E^1_{p,q} &= H_{p+q}(\bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)}) \\ &\cong H_{p+q}(K \otimes_{\Lambda} C^{(p)}) \\ &\cong \mathrm{Tor}^{\Lambda}_q(K, (\bar{\Omega})^p) \\ &\cong \mathrm{Tor}^{\Lambda}_q(K, K) \otimes (\bar{\Omega})^p . \end{aligned}$$

The last step uses the fact that $(\bar{\Omega})^p$, qua $H_p(C^{(p)})$, has trivial operations from Λ (see Lemma 2.3.2.)

In order to calculate $E^2_{p,q}$, we need explicit chain equivalences between

$$\bar{B}(\Lambda) \otimes \bar{\Omega}^p \quad \text{and} \quad \bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)} .$$

In one direction the map is easy. Let $\pi : \Gamma \to \Omega$ be the projection. Define a map

$$\nu\colon C^{(p)} \to B(\Lambda) \otimes (\bar{\Omega})^p$$

by

$$\nu a[a_1 | a_2 | \cdots | a_{p+q}] = a[a_1 | \cdots | a_q] \otimes [\pi a_{q+1} | \cdots | \pi a_{p+q}] .$$

Then ν is a Λ-map, a chain map (if the boundary in $B(\Lambda) \otimes (\bar{\Omega})^p$ is $d \otimes 1$) and induces the isomorphism of homology established above. By the uniqueness theorem in homological algebra, the induced map

$$\nu_*\colon \bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)} \to \bar{B}(\Lambda) \otimes (\bar{\Omega})^p$$

is a chain equivalence.

We have not yet made any essential use of the fact that Λ is central in Γ. However, this fact is required; it ensures that $H_*(\Lambda)$ (which is analogous to the homology of the fibre) has simple operations from Ω. We use it in constructing the equivalence in the other direction.

We define a product

$$\mu\colon B(\Lambda) \otimes B(\Gamma) \to B(\Gamma)$$

as follows, using shuffles [14]. Take elements

$$a[a_1 \mid a_2 \mid \cdots \mid a_q] \in B(\Lambda)$$
$$b[b_1 \mid b_2 \mid \cdots \mid b_p] \in B(\Gamma) .$$

Let $c_1, c_2, \cdots, c_{p+q}$ consist of the a_i and b_j in some order; we require that the a_i occur in their correct relative order and that the b_j occur in their correct relative order. We call such a set $c = \{c_k\}$ a *shuffle*. Let us write $\sum n \mid P$ instead of $\sum_P n$ if the proposition P has a complicated form. Then we define the *signature* $(-1)^{\varepsilon(c)}$ of a shuffle by

$$\varepsilon(c) = \sum(1 + \deg(a_i))(1 + \deg(b_j)) \mid a_i = c_k,\ b_j = c_l,\ k > l .$$

We also set

$$\eta = \sum_i (1 + \deg(a_i))(\deg(b)) .$$

We now define

$$\mu(a[a_1 \mid a_2 \mid \cdots \mid a_q] \otimes b[b_1 \mid b_2 \mid \cdots \mid b_p]) = \sum_c (-1)^{\varepsilon(c)+\eta} ab[c_1 \mid c_2 \mid \cdots \mid c_{p+q}] .$$

LEMMA 2.3.3. $d\mu(x \otimes y) = \mu(dx \otimes y) + (-1)^{q+t}\mu(x \otimes dy)$ *(where* $x \in B(\Lambda)_{q,t}$*).*

This lemma, of course, is the usual one for shuffle-products (see [14]); it depends on the fact that Λ is central in Γ.

If we restrict y to lie in $(\bar{\Gamma})^p$, and pass to the tensor-product with K, we obtain an induced map

$$\mu_* : \bar{B}(\Lambda) \otimes (\bar{\Gamma})^p \to \bar{B}(\Gamma)^{(p)} .$$

Its explicit form is

$$\mu_*([a_1 \mid a_2 \mid \cdots \mid a_q] \otimes [b_1 \mid b_2 \mid \cdots \mid b_p]) = \sum_c (-1)^{\varepsilon(c)} [c_1 \mid c_2 \mid \cdots \mid c_{p+q}] .$$

It satisfies

$$\bar{d}\mu_*(x \otimes y) = \mu_*(\bar{d}x \otimes y) + (-1)^{q+t}\mu_*(x \otimes \bar{d}y) .$$

We define a K-map $l : \bar{\Omega} \to \bar{\Gamma}$ by $l(\omega_i) = \gamma_i$, where $\{\omega_i\}$, $\{\gamma_i\}$ are bases for Ω, Γ over K, Λ, as above. We may now define

$$\mu' : B(\Lambda) \otimes (\bar{\Omega})^p \to C^{(p)}$$

by

$$\mu'(x \otimes y) = \{\mu(x \otimes l^p y)\} .$$

Then μ' is a Λ-map, a chain map, and induces the correct isomorphism of homology. On passing to tensor products with K, we obtain an induced map

$$\mu'_* : \bar{B}(\Lambda)\otimes(\bar{\Omega})^p \to \bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)}$$

which, as before, is a chain equivalence. We even have $\nu_*\mu'_* = 1$.

The equivalence μ'_*, then, is just the composite

$$\bar{B}(\Lambda)\otimes(\bar{\Omega})^p \xrightarrow{1\otimes l^p} \bar{B}(\Lambda)\otimes(\bar{\Gamma})^p \xrightarrow{\mu_*} \bar{B}(\Gamma)^{(p)} \longrightarrow \bar{B}(\Gamma)^{(p)}/\bar{B}(\Gamma)^{(p-1)} .$$

LEMMA 2.3.4. *The following diagram is commutative.*

$$\begin{array}{ccc} E^1_{p,q} & \xrightarrow{\quad d^1 \quad} & E^1_{p-1,q} \\ {\scriptstyle \mu'_{**}}\uparrow & & \downarrow{\scriptstyle \nu_{**}} \\ H_q(\bar{B}(\Lambda)\otimes(\bar{\Omega})^p) & \xrightarrow{(-1)^{q+t}(1\otimes\bar{d})} & H_q(\bar{B}(\Lambda)\otimes(\bar{\Omega})^{p-1}) \end{array}$$

(It is, of course, implied that this ν_{**} is the one defined for dimension $(p-1)$.)

PROOF. Take $x \in \bar{B}(\Lambda)_{q,t}$ such that $\bar{d}x = 0$, and $y \in (\bar{\Omega})^p$, so that $x\otimes y$ is a representative for an element of $H_q(B(\Lambda)\otimes(\bar{\Omega})^p)$. Then the following elements represent $\nu_{**}d^1\mu'_{**}\{x\otimes y\}$:

$$\begin{aligned} \nu_*\bar{d}\mu_*(1\otimes l^p)(x\otimes y) &= \nu_*\mu_*((-1)^{q+t}x\otimes\bar{d}l^p y) \\ &= (-1)^{q+t}(1\otimes\pi^{p-1})(x\otimes\bar{d}l^p y) \\ &= (-1)^{q+t}(x\otimes\bar{d}\pi^p l^p y) \\ &= (-1)^{q+t}(x\otimes\bar{d}y) . \end{aligned}$$

This proves the lemma.

We conclude that

$$E^2_{p,q} \cong H_q(\bar{B}(\Lambda))\otimes H_p(\bar{B}(\Omega)) .$$

Inverse isomorphisms are induced by μ'_{**} and ν_{**}. We note that the isomorphisms

$$\begin{aligned} H_q(\bar{B}(\Lambda)) &\cong H_q(\bar{B}(\Lambda)\otimes K) \to E^2_{0,q} \\ E^2_{p,0} &\to K\otimes H_p(B(\bar{\Omega})) \cong H_p(B(\bar{\Omega})) \end{aligned}$$

are induced by the natural maps $\bar{B}(\Lambda) \to \bar{B}(\Gamma)$, $\bar{B}(\Gamma) \to \bar{B}(\Omega)$.

Let us now pass to the vector-space dual of this spectral sequence. It is obtained by giving $F(\Gamma^*)$ a filtration in which $F(\Gamma^*)^{(p)}$ is the annihilator of $\bar{B}(\Gamma)^{(p-1)}$. This is the filtration originally described. The cup-products satisfy

$$F(\Gamma^*)^{(p)}\cdot F(\Gamma^*)^{(p')} \subset F(\Gamma^*)^{(p+p')} .$$

We thus have a spectral sequence with products. We have obtained the whole of Theorem 2.3.1, except that part which relates to the ring-structure of E_2.

To obtain this ring-structure, we consider the isomorphism

$$E_1^{p,q} \cong H^q(F(\Lambda^*)) \otimes (\bar{\Omega}^*)^p$$

dual to μ'_{**} and ν_{**}. It may be described as follows. Let x be a cocycle of dimension q in $F(\Lambda^*)$, and let y be a cochain in $(\bar{\Omega}^*)^p$. Let x' be a cochain in $F(\Gamma^*)$ such that $i^*x' = x$, where $i : \Lambda \to \Gamma$ is the inclusion. Then $\{x' \cdot \pi y\}$ is an element of $E_1^{p,q}$, independent of the choice of x'; and ν^{**}, the dual of ν_{**}, maps $\{x\} \otimes y$ to $\{x' \cdot \pi^* y\}$.

Next, let x, z be cocycles in $F(\Lambda^*)$, representing elements of $H^{q',u'}(\Lambda)$, $H^{q,u}(\Lambda)$. Let y, w be cochains in $F(\Omega^*)$, of bidegrees (p, t), (p', t'). Let x', z' be cochains in $F(\Gamma^*)$ such that $i^*x' = x$, $i^*z' = z$. Then

$$\nu^{**}(\{x\} \otimes y) \cdot \nu^{**}(\{z\} \otimes w) = \{x' \cdot \pi^* y \cdot z' \cdot \pi^* w\} \ .$$

Now let X be a cycle in $\bar{B}(\Lambda)$, of dimension $q+q'$, and let Y be a chain in $\bar{B}(\Omega)$, of dimension $p+p'$. Inspecting the definition of the shuffle-product, we see

$$\{x' \cdot \pi^* y \cdot z' \cdot \pi^* w\} \cdot \mu'_{**}(\{X\} \otimes Y) = (-1)^{(p+t)(q+u)}(\{xz\}\ \{X\})((yw)Y) \ .$$

That is,

$$\mu'_{**}(\nu^{**}(\{x\} \otimes y) \cdot \nu^{**}(\{z\} \otimes w)) = (-1)^{(p+t)(q+u)}\ \{xz\} \otimes yw \ .$$

We have shown that the isomorphism

$$E_1^{p,q} \cong H^q(F(\Lambda^*)) \otimes (\bar{\Omega}^*)^p$$

preserves the ring-structure. Therefore the induced isomorphism of $E_2^{p,q}$ does so. This completes the proof of Theorem 2.3.1.

2.4. *Milnor's description of* A. In this section we recall J. Milnor's elegant description of the Steenrod algebra A [25], and begin to deduce from it the results we shall need later.

We recall that the mod p Steenrod algebra A is a Hopf algebra; that is, besides having a product map $\varphi : A \otimes A \to A$, it has a diagonal map $\psi : A \to A \otimes A$, and these satisfy certain axioms. The diagonal ψ may be described as follows. We have an isomorphism

$$\nu : H^*(X \times Y; Z_p) \to H^*(X; Z_p) \otimes H^*(Y; Z_p)$$

given by the external cup-product. The left-hand side admits operations from A; the right-hand side admits operations from $A \otimes A$, defined as in § 2.3. There is one and only one function

$$\psi : A \to A \otimes A$$

such that

$$\nu(ah) = \psi(a)\nu(h)$$

for all X, Y, a and h. (This may be shown, for example, by the method of § 3.9.)

We make the identification $(A \otimes A)^* = A^* \otimes A^*$; A^* thus becomes a Hopf algebra, whose product and diagonal maps are the duals of ψ and φ. We now quote Milnor's theorem [25] on the structure of A^*, in the case $p = 2$.

THEOREM 2.4.1.

(i) *A^* is a polynomial algebra on generators ξ_i, $i=1, 2, \cdots$, of grading $2^i - 1$.*

(ii) $\varphi^* \xi_k = \sum_{i+j=k} \xi_j^{2^i} \otimes \xi_i$ *(where $\xi_0 = 1$).*

(iii) $\xi_1^k(\mathrm{Sq}^k) = 1$ *and* $m(\mathrm{Sq}^k) = 0$ *for any other monomial m in the ξ_i.*

It is possible to describe the elements ξ_i very simply. In fact, consider $H^*(\pi, 1; Z_2)$ for $\pi = Z_2$; this is a polynomial algebra on one generator x of dimension 1. If $a \in A$, then ax is primitive, so that

$$ax = \sum_{i \geqq 0} \lambda_i x^{2^i} \qquad \text{with } \lambda_i \in Z_2 .$$

We define ξ_i by $\xi_i(a) = \lambda_i$. The elements ξ_i are thus closely connected with the Thom-Serre-Cartan representation [30], [12] of A. In fact, consider $H^*(\pi, 1; Z_2)$, where π is a finite vector space over Z_2; this is a polynomial algebra on generators $x_1, x_2, \cdots, x_n$ of dimension 1. If $a \in A$, we have

$$a(x_1 x_2 \cdots x_n) = \sum_{i,j,\cdots,l} ((\xi_i \xi_j \cdots \xi_l)a) x_1^{2^i} x_2^{2^j} \cdots x_n^{2^l} .$$

We now pass on to the study of certain quotient algebras of A. It is immediate that the generators ξ_i of A^* which satisfy $1 \leqq i \leqq n$ generate a Hopf subalgebra of A^*; call it A_n^*. The dual of A_n^* is a Hopf algebra Q_n, which is a quotient of A. We have

$$H^{s,t}(Q_n) \cong H^{s,t}(A) \qquad \text{if } t \leqq 2^{n+1} - 2 .$$

For each n, Q_n is a quotient of Q_{n+1}.

Now, suppose we are given an epimorphism $\pi : \Gamma \to \Omega$ of connected Hopf algebras, and suppose that the dual monomorphism $\pi^*: \Omega^* \to \Gamma^*$ embeds Ω^* as a normal subalgebra of Γ^* (see § 2.3). Then we may form the quotient $\Gamma^*//\Omega^*$, which is again a Hopf algebra; let us call its dual Λ, so that $\Lambda^* = \Gamma^*//\Omega^*$; then Λ is embedded monomorphically in Γ.

LEMMA 2.4.2. *If Λ, Γ, Ω are as above, then Λ is normal in Γ, and $\Gamma//\Lambda \cong \Omega$.*

This lemma is an exercise in handling Hopf algebras [26]; we only sketch the proof. It is trivial that

$$I(\Lambda)\cdot\Gamma \subset \operatorname{Ker}\pi\,, \qquad \Gamma\cdot I(\Lambda) \subset \operatorname{Ker}\pi;$$

we have to prove the opposite inclusions. We see that Λ is the kernel of the composite map

$$\Gamma \xrightarrow{\psi} \Gamma\otimes\Gamma \longrightarrow \Gamma\otimes\bar{\Omega};$$

call this composite χ. We see that if

$$\chi(x) \in \textstyle\sum_{r\geqq m}\Gamma_r\otimes\bar{\Omega}\,,$$

then

$$\chi(x) \in \Lambda_m\otimes\bar{\Omega} + \textstyle\sum_{r>m}\Gamma_r\otimes\bar{\Omega}\,.$$

Now suppose $\pi x = 0$. By an inductive process, using $\chi(x)$, we may subtract from x products yz with $y \in \Lambda_m$, $m>0$; we finally obtain a new x' with $\chi(x') = 0$. This shows that $\operatorname{Ker}\pi \subset I(\Lambda)\cdot\Gamma$; similarly for $\Gamma\cdot I(\Lambda)$.

We may apply this lemma to the epimorphism $Q_n \to Q_{n-1}$ introduced above. We see that in this application, Γ^* and Ω^* become A_n^* and A_{n-1}^*. Thus $\Lambda^* = \Gamma^*//\Omega^*$ becomes $A_n^*//A_{n-1}^*$, that is, a polynomial algebra on one generator ξ_n, whose diagonal is given by

$$\psi\xi_n = \xi_n\otimes 1 + 1\otimes\xi_n\,.$$

We write K_n for the corresponding algebra A; it is a divided polynomial algebra.

Since we propose to apply the spectral sequence of § 2.3 to the case $\Lambda = K_n$, $\Gamma = Q_n$, we should show that K_n is central in Q_n. We may proceed in the duals, by showing that the following diagram is commutative.

$$\begin{array}{ccc} A_n^* & \xrightarrow{\psi} & A_n^*\otimes A_n^*//A_{n-1}^* \\ & \searrow^{\psi} & \downarrow\rho \\ & & A_n^*//A_{n-1}^*\otimes A_n^* \end{array}$$

(Here, of course, ρ is the map which permutes the two factors.) Since each map is multiplicative, we need only check the commutativity on the generators, for which it is immediate.

The reader may care to compare the work of this section with that of [2, § 5].

2.5. *Calculations.* In this section we prove what we need about the cohomology of the (mod 2) Steenrod algebra A. The result is:-

THEOREM 2.5.1.

(0) $H^0(A)$ *has as a base the unit element* 1.

(1) $H^1(A)$ *has as a base the elements* $h_i = \{[\xi_1^{2^i}]\}$ *for* $i = 0, 1, 2, \cdots$.

(2) *In* $H^2(A)$ *we have* $h_{i+1}h_i = 0$. $H^2(A)$ *admits as a base the products* h_jh_i *for which* $j \geqq i \geqq 0$ *and* $j \neq i+1$.

(3) *In* $H^3(A)$ *we have the relations*

$$h_{i+2}h_i^2 = h_{i+1}^3, \qquad h_{i+2}^2 h_i = 0 .$$

If we take the products $h_kh_jh_i$ *for which* $k \geqq j \geqq i \geqq 0$ *and remove the products*

$$h_{j+1}h_jh_i , \qquad h_kh_{i+1}h_i , \qquad h_{i+2}h_i^2 , h_{i+2}^2h_i ,$$

then the remaining products are linearly independent in $H^3(A)$.

We propose to prove this theorem by considering a family of spectral sequences. The n^{th} spectral sequence, say ${}_nE_r^{p,q}$, will be obtained by applying § 2.3 to the algebras

$$\Lambda = K_n , \qquad \Gamma = Q_n , \qquad \Omega = Q_{n-1} .$$

In these spectral sequences, we know $H^*(K_n)$; we have $Q_1 = K_1$, so that $H^*(Q_1)$ is known; we propose to obtain information about $H^*(Q_n)$ by induction over n.

We will now give names to the cohomology classes which will appear in our calculations. Let $h_{n,i}$ be the generator $\{[\xi_n^{2^i}]\}$ in $H^1(K_n)$; $H^*(K_n)$ is thus a polynomial algebra on the generators $h_{n,i}$, where $i = 0, 1, 2, \cdots$. We shall write h_i (instead of $h_{1,i}$) for the generator $\{[\xi_1^{2^i}]\}$ in $H^1(Q_1)$, or for its image in $H^1(Q_n)$.

We define the class $g_{n-1,i}$ in $H^2(Q_{n-1})$ by

$$g_{n-1,i} = {}_n\tau h_{n,i} ,$$

where ${}_n\tau : H^1(K_n) \to H^2(Q_{n-1})$ is the transgression. The class $g_{n-1,i}$ can be represented by the explicit cocycle

$$\delta[\xi_n^{2^i}] = \sum [\xi_j^{2^{k+i}} \mid \xi_k^{2^i}]$$

where the sum extends over $j + k = n$, $j > 0$, $k > 0$.

Similarly, we can define a class $f_{n-1,i}$ in $H^3(Q_{n-1})$ by

$$f_{n-1,i} = {}_n\tau h_{n,i}^2 ,$$

since $h_{n,i}^2$ is clearly transgressive. The class $f_{n-1,i}$ can be represented by the explicit cocycle

$$\delta(x \smile x + \delta x \smile_1 x) = \delta x \smile_1 \delta x ,$$

where $x = [\xi_n^{2^i}]$, so that δx is the cocycle obtained above. The $\smile_1$ product $\delta x \smile_1 \delta x$ can be expanded by the formula given at the end of § 2.2.

LEMMA 2.5.2.

(i) *We have* $g_{1,i} = h_{i+1}h_i$, $f_{1,i} = h_{i+2}h_i^2 + h_{i+1}^3$.

(ii) *If $n > 1$, then $g_{n,i}$ is of filtration 1 in $H^*(Q_n)$, $h_{n,i+1}h_i + h_{n,i}h_{n+i}$ is a cycle in ${}_nE_2^{1,1}$, and in ${}_nE_\infty^{1,1}$ we have*

$$\{g_{n,i}\} = \{h_{n,i+1}h_i + h_{n,i}h_{n+i}\} .$$

(iii) *If $n>1$, then $f_{n,i}$ is of filtration 1 in $H^*(Q_n)$, $h_{n,i+1}^2h_{i+1}+h_{n,i}^2h_{n+i+1}$ is a cycle in ${}_nE_2^{1,2}$ and in ${}_nE_3^{1,2}$, and in ${}_nE_\infty^{1,2}$ we have*

$$\{f_{n,i}\} = \{h_{n,i+1}^2h_{i+1} + h_{n,i}^2h_{n+i+1}\} .$$

These conclusions follow from the explicit cocycles given above. For example, the explicit cocycle for $g_{1,i}$ is

$$[\xi_1^{2^{i+1}} | \xi_1^{2^i}] ;$$

the explicit cocycle for $f_{1,i}$ is

$$[\xi_1^{2^{i+2}} | \xi_1^{2^i} | \xi_1^{2^i}] + [\xi_1^{2^{i+1}} | \xi_1^{2^{i+1}} | \xi_1^{2^{i+1}}] ;$$

the explicit cocycle for $g_{n,i}$ differs from

$$[\xi_n^{2^{i+1}} | \xi_1^{2^i}] + [\xi_1^{2^{n+i}} | \xi_n^{2^i}]$$

by a cochain of filtration 2; and so on.

We will now begin the calculations.

LEMMA 2.5.3.

(i) *The elements h_i in $H^1(Q_n)$ are linearly independent.*

(ii) *If $n > 1$, the elements $\{g_{n,i}\}$ in ${}_nE_\infty^{1,1}$ are linearly independent.*

(iii) *In $H^2(Q_n)$, the elements $g_{n,k}$ and h_jh_i (where $j \geqq i \geqq 0, j \neq i+1$) are linearly independent.*

(iv) *$H^1(Q_n)$ is spanned by the elements h_i.*

PROOF. Part (i) is immediate, since no differential maps into ${}_nE_r^{1,0}$.

Part (ii) follows from Lemma 2.5.2 (ii), since no differential maps into ${}_nE_r^{1,1}$.

Part (iii) is true for $n = 1$, since $g_{1,k} = h_{k+1}h_k$. We proceed by induction over n; let us assume that part (iii) holds for Q_{n-1}. We must examine the differential

$${}_nd_2 \colon {}_nE_2^{0,1} \to {}_nE_2^{2,0} .$$

It is described by

$${}_nd_2(h_{n,i}) = g_{n-1,i} .$$

We conclude that:-

(a) ${}_nE_\infty^{0,1} = 0$.

(b) The classes $\{h_jh_i\}$ (for $j \geqq i \geqq 0, j \neq i+1$) are linearly independent in ${}_nE_\infty^{2,0}$.

Using part (ii), we see that part (iii) holds for Q_n.

Part (iv) follows immediately from the fact that ${}_nE_\infty^{0,1} = 0$. This completes the proof.

LEMMA 2.5.4. *In $H^2(Q_n)$ for $n \geqq 2$ we have*

(i) $h_{i+1}h_i = 0$

(ii) $\langle h_i, h_{i+1}, h_i\rangle = h_{i+1}^2$

(iii) $\langle h_{i+1}, h_i, h_{i+1}\rangle = h_{i+2}h_i$.

In $H^2(Q_2)$ we have

(iv) $\langle h_{i+2}, h_{i+1}, h_i\rangle = g_{2,i}$

(v) $h_{i+2}^2 h_i = g_{2,i}h_{i+1}$.

PROOF. The following formulae show that $h_{i+1}h_i = 0$, $h_ih_{i+1} = 0$ in $H^2(Q_n)$ for $n \geqq 2$:

$$\delta[\xi_2^{2^i}] = [\xi_1^{2^{i+1}} \mid \xi_1^{2^i}]$$

$$\delta[\xi_2^{2^i} + \xi_1^{3\cdot 2^i}] = [\xi_1^{2^i} \mid \xi_1^{2^{i+1}}] .$$

These formulae will also help us to write down explicit cocycles representing the Massey products mentioned in the lemma. For example, $\langle h_{i+2}, h_{i+1}, h_i\rangle$ is represented by the explicit cocycle

$$[\xi_2^{2^{i+1}} \mid \xi_1^{2^i}] + [\xi_1^{2^{i+2}} \mid \xi_2^{2^i}] ,$$

which coincides with that given above for $g_{2,i}$. This proves (iv).

To prove (ii) and (iii), we may quote the formula

$$\langle x, y, x\rangle = (x \smile_1 x)y ;$$

or by substituting appropriate values in the proof of this formula, we obtain the following:-

$$\delta[\xi_1^{2^i}\xi_2^{2^i}] = [\xi_2^{2^i} + \xi_1^{3\cdot 2^i} \mid \xi_1^{2^i}] + [\xi_1^{2^i} \mid \xi_2^{2^i}] + [\xi_1^{2^{i+1}} \mid \xi_1^{2^{i+1}}]$$

$$\delta[\xi_1^{2^{i+1}}\xi_2^{2^i}] = [\xi_2^{2^i} \mid \xi_1^{2^{i+1}}] + [\xi_1^{2^{i+1}} \mid \xi_2^{2^i} + \xi_1^{3\cdot 2^i}] + [\xi_1^{2^{i+2}} \mid \xi_1^{2^i}] .$$

These formulae prove (ii) and (iii).

Since we now know $H^1(Q_n)$, it it easy to check that the Massey products considered above are defined modulo zero.

To prove (v), it is sufficient to make the following manipulation:

$$\begin{aligned} h_{i+2}^2 h_i &= h_{i+2}\langle h_{i+1}, h_i, h_{i+1}\rangle \\ &= \langle h_{i+2}, h_{i+1}, h_1\rangle h_{i+1} = g_{2,i}h_{i+1} . \end{aligned}$$

Alternatively, by substituting appropriate values in the proof of the relation

$$a\langle b, c, d\rangle = \langle a, b, c\rangle d ,$$

we obtain the following:-

$$\delta[\xi_2^{2^{i+1}} \mid \xi_2^{2^i} + \xi_1^{3\cdot 2^i}] + \delta[\xi_1^{2^{i+2}} \mid \xi_1^{2^{i+1}} \xi_2^{2^i}]$$
$$= [\xi_1^{2^{i+2}} \mid \xi_1^{2^{i+2}} \mid \xi_1^{2^i}] + [\xi_2^{2^{i+1}} \mid \xi_1^{2^i} \mid \xi_1^{2^{i+1}}] + [\xi_1^{2^{i+2}} \mid \xi_2^{2^i} \mid \xi_1^{2^{i+1}}] .$$

This formula proves (v). This completes the proof.

LEMMA 2.5.5.

(i) *If $n > 1$, the elements $h_{n,i}^2$ form a base for ${}_nE_3^{0,2}$.*

(ii) *If $n > 1$, the elements $\{f_{n,i}\}$ in ${}_nE_\infty^{1,2}$ are linearly independent.*

(iii) *If $n > 2$, the elements $\{g_{n,j}h_i\}$ for which $j \neq i + 1$ are linearly independent in ${}_nE_\infty^{2,1}$. The same is true for $n = 2$, provided we exclude also the elements $\{g_{2,i}h_{i+1}\}$.*

PROOF. To prove part (i), it is sufficient to note that the differential

$${}_nd_2 : {}_nE_2^{0,2} \to {}_nE_2^{2,1}$$

is described by

$${}_nd_2(h_{n,j}h_{n,i}) = h_{n,j}g_{n-1,i} + h_{n,i}g_{n-1,j} .$$

Part (ii) follows from Lemma 2.5.2 (iii), since no differential maps into ${}_nE_n^{1,2}$.

To prove part (iii), we note that the following formula holds in ${}_nE_\infty^{2,1}$:

$$\{g_{n,j}h_i\} = \{h_{n,j+1}h_jh_i + h_{n,j}h_{n+j}h_i\} .$$

Moreover, the only differential mapping into ${}_nE_r^{2,1}$ is

$${}_nd_2 : {}_nE_2^{0,2} \to {}_nE_2^{2,1} ,$$

which has just been described. It is now easy to obtain part (iii). This completes the proof.

LEMMA 2.5.1.

(i) *If $n > 1$, the following elements are linearly independent in $H^3(Q_n)$: the elements $f_{n,i}$; the elements $g_{n,j}h_i$ for which $j \neq i+1$; the products $h_kh_jh_i$ for which $k \geqq j \geqq i \geqq 0$, with the following exceptions:*

$$h_{j+1}h_jh_i , \quad h_kh_{i+1}h_i , \quad h_{i+2}h_i^2 , \quad h_{i+2}^2h_i .$$

The same conclusion remains true for $n=1$ if we include the products $h_{i+2}^2h_i$.

(ii) *$H^2(Q_n)$ is spanned by the elements $g_{n,k}$ and h_jh_i (where $j \geqq i \geqq 0$, $j \neq i + 1$).*

PROOF. It is elementary to check part (i) for $n = 1$. We proceed by induction over n; let us assume that part (i) holds for Q_{n-1}. We must examine the differentials

$${}_nd_2 : {}_nE_2^{1,1} \to {}_nE_2^{3,0} , \qquad {}_nd_3 : {}_nE_3^{0,2} \to {}_nE_3^{3,0} .$$

These are described by

$$ {}_nd_2(h_{n,j}h_i) = g_{n-1,j}h_i \ , \qquad {}_nd_3(h_{n,t}^2) = f_{n-1,t} \ . $$

We obtain the following conclusions.

(a) ${}_nE_3^{1,1}$ has as a base the elements

$$ \{g_{n,i}\} = \{h_{n,i+1}h_i + h_{n,t}h_{n+i}\} \ . $$

(b) ${}_nE_4^{0,2} = 0$.

(c) In ${}_nE_4^{3,0}$ the products $h_kh_jh_i$ named above are linearly independent. In the case $n = 2$, this conclusion remains true when we include also the products $h_{i+2}^2h_i$.

Using Lemma 2.5.5 parts (ii) and (iii), we see that part (i) holds for Q_n. (If $n = 2$, we need to know that $g_{2,i}h_{i+1} = h_{i+2}^2h_i$; this was proved in Lemma 2.5.4.) This completes the proof of part (i).

Part (ii) follows immediately from the facts (a) and (b) established during the proof of part (i).

Since $H^{s,t}(Q_n) \to H^{s,t}(A)$ as $n \to \infty$ (for fixed s and t), the work which we have done completes the proof of Theorem 2.5.1.

2.6. *More calculations.* In this section we shall calculate $\mathrm{Ext}_A^{s,t}(M, Z_2)$ for a certain module M which arises in the applications (and for a limited range of s and t). The results are stated in Theorem 2.6.2.

We first obtain a lemma which is true for a general algebra A over Z_2. Suppose given a primitive element α in A_n^* $(n > 0)$, that is, an element α such that

$$ \varphi^*\alpha = \alpha \otimes 1 + 1 \otimes \alpha \ . $$

According to § 2.2, it defines an element h_α in $\mathrm{Ext}_A^{1,n}(Z_2, Z_2)$, which for example may be written $\{[\alpha]\}$, using the cobar construction.

In terms of α, we define a module $M = M(\alpha)$ as follows. *Qua* vector space over Z_2, it has a base containing two elements m_1, m_2 of degrees 0, n. The operations A are defined by

$$ am_1 = (\alpha a)m_2 \qquad (a \in A_n) \ . $$

These operations do give M the structure of an A-module, since α annihilates all decomposable elements of A.

The element m_2 generates a submodule M_2 of M isomorphic with Z_2; we define $M_1 = M/M_2$, so that $M_1 \cong Z_2$. We agree to write $x^{(i)}$ $(i = 1, 2)$ for the element of $\mathrm{Ext}_A^s(M_i, Z_2)$ corresponding to the element x of $\mathrm{Ext}_A^s(Z_2, Z_2)$.

From the exact sequence

$$0 \to M_2 \to M \to M_1 \to 0$$

we obtain an exact sequence

$$\cdots \leftarrow \mathrm{Ext}_A^{s+1,t}(M_1, Z_2) \xleftarrow{\delta} \mathrm{Ext}_A^{s,t}(M_2, Z_2) \leftarrow \mathrm{Ext}_A^{s,t}(M, Z_2) \leftarrow \cdots$$

which one might use to calculate $\mathrm{Ext}_A^{s,t}(M, Z_2)$.

LEMMA 2.6.1. *The coboundary δ is given by*

$$\delta(x^{(2)}) = (xh_\alpha)^{(1)}$$

(where $x \in \mathrm{Ext}_A^s(Z_2, Z_2)$).

PROOF. Let us take two resolutions C, C' of Z_2 over A, as follows:

$$\begin{array}{lllll} Z_2 \xleftarrow{\varepsilon} A \xleftarrow{d} C_1 \longleftarrow \cdots & & \cdots \longleftarrow C_q \longleftarrow \cdots \\ Z_2 \xleftarrow{\varepsilon'} A \xleftarrow{d'} C_1' \longleftarrow \cdots & & \cdots \longleftarrow C_q' \longleftarrow \cdots . \end{array}$$

To calculate the cup-product with h_α, we must construct a diagram, as considered in § 2.2.

$$\begin{array}{lllllll} Z_2 \xleftarrow{\varepsilon} A \xleftarrow{d} C_1 \xleftarrow{d} C_2 \longleftarrow \cdots & & \cdots \longleftarrow C_{q+1} \longleftarrow \cdots \\ \quad\alpha\downarrow \quad f\swarrow & \downarrow f_0 & \downarrow f_1 & & \downarrow f_q \\ Z_2 \xleftarrow[\varepsilon']{} A \xleftarrow[d']{} C_1' \longleftarrow \cdots & & \cdots \longleftarrow C_q' \longleftarrow \cdots \end{array}$$

Let us now define a boundary d on $C + C'$ and an augmentation ε: $C + C' \to M$ by setting

$$\begin{aligned} d(x, y) &= (dx, d'y + f_{q-1}(x)) && (x \in C_q, q > 0) \\ \varepsilon(1, 0) &= m_1 , \qquad \varepsilon(0, 1) = m_2 . \end{aligned}$$

$C + C'$ is a chain complex in which C' is embedded, with quotient C; the exact cohomology sequence shows that $C + C'$ is acyclic. $C + C'$ is thus a resolution of M; it contains C', which is a resolution of M_2, and the quotient is C, which is a resolution of M_1. We obtain an exact sequence of cochain complexes

$$0 \leftarrow \mathrm{Hom}_A(C', Z_2) \leftarrow \mathrm{Hom}_A(C + C', Z_2) \leftarrow \mathrm{Hom}_A(C, Z_2) \leftarrow 0 .$$

The corresponding exact cohomology sequence is the one required. The coboundary in

$$\mathrm{Hom}_A(C + C', Z_2) \cong \mathrm{Hom}_A(C, Z_2) + \mathrm{Hom}_A(C', Z_2)$$

is given by

$$\delta(x, y) = (\delta x + yf_{q-1}, \delta y)$$

(if $y \in \mathrm{Hom}_A(C'_{q-1}, Z_2)$). It follows immediately that in the exact cohomology sequence we have

$$\delta(\{y\}^{(2)}) = \{yf_{q-1}\}^{(1)} = (\{y\}h_\alpha)^{(1)} .$$

This proves Lemma 2.6.1.

We now suppose that A is the (mod 2) Steenrod algebra. Take an integer $k \geqq 2$; we define a module $M = M(k)$ as follows. *Qua* vector space over Z_2, it has a base containing three elements m_1, m_2, m_3 of degrees 0, 2^k, 2^{k+1}. The operations from A are defined by

$$\left.\begin{aligned} am_1 &= (\xi_1^{2^k}a)m_2 \\ am_2 &= 0 \end{aligned}\right\} \quad (\text{if } \deg(a) = 2^k)$$

$$am_1 = (\xi_1^{2^{k+1}}a)m_3 \quad (\text{if } \deg(a) = 2^{k+1}) .$$

The elements m_2, m_3 generate submodules M_2, M_3 isomorphic with Z_2; we write i_2, i_3 for their injections. We define $M_1 = M/(M_2 + M_3)$, so that $M_1 \cong Z_2$; we have an exact sequence

$$0 \longrightarrow M_1 + M_2 \xrightarrow{i_2 + i_3} M \xrightarrow{j} M_1 \longrightarrow 0 .$$

We continue the previous convention about $x^{(i)}$.

THEOREM 2.6.2. *In dimensions $t < 3\cdot 2^k$, $\mathrm{Ext}_A^1(M, Z_2)$ has as a base the elements*

$$\begin{aligned} &j^*h_i^{(1)} && (0 \leqq i \leqq k-1) \\ &(i_2^*)^{-1}h_{k-1}^{(2)} \end{aligned}$$

and $\mathrm{Ext}_A^2(M, Z_2)$ has as a base the elements

$$\begin{aligned} &j^*(h_ih_l)^{(1)} && (0 \leqq i \leqq l \leqq k-1,\ l \neq i+1) \\ &(i_2^*)^{-1}(h_ih_{k-1})^{(2)} && (0 \leqq i \leqq k-1,\ i \neq k-2) \\ &(i_2^*)^{-1}(h_{k-2}h_k)^{(2)} . \end{aligned}$$

PROOF. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 + M_3 & \longrightarrow & M & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & M/M_3 & \longrightarrow & M_1 & \longrightarrow & 0 . \end{array}$$

Now, Lemma 2.6.1 applies to M/M_3, with $\alpha = \xi_1^{2^k}$; thus $\delta(x^{(2)}) = (xh_k)^{(1)}$. By naturality, the same formula holds in the exact sequence

$$\cdots \xleftarrow{\delta} \mathrm{Ext}_A^{s,t}(M_2 + M_3, Z_2) \longleftarrow \mathrm{Ext}_A^{s,t}(M, Z_2) \xleftarrow{j^*} \mathrm{Ext}_A^{s,t}(M_1, Z_2) \xleftarrow{\delta} \cdots .$$

Similarly,

$$\delta(x^{(3)}) = (xh_{k+1})^{(1)} .$$

It follows (using Theorem 2.5.1) that in this exact sequence, Coker δ and Ker δ have the following bases, at least in degrees $t < 3 \cdot 2^k$.

$$\begin{array}{lll} s = 1, \text{ Coker } \delta: & h_i^{(1)} & (0 \leqq i \leqq k-1) . \\ s = 1, \text{ Ker } \delta: & h_{k-1}^{(2)} . & \\ s = 2, \text{ Coker } \delta: & (h_i h_l)^{(1)} & (0 \leqq i \leqq l \leqq k-1,\ l \neq i+1). \\ s = 2, \text{ Ker } \delta: & (h_i h_{k-1})^{(2)} & (0 \leqq i \leqq k-1,\ i \neq k-2) , \\ & (h_{k-2} h_k)^{(2)} . & \end{array}$$

This proves Theorem 2.6.2.

We have now completed all the homological algebra which is necessary for our applications.

Chapter 3. Secondary Cohomology Operations

3.1. *Introduction.* In this chapter we shall develop the general theory of stable secondary operations. The results at issue are not deep; the author hopes that this fact will not be obscured by the language necessary to express them in the required generality.

The plan of this chapter is as follows. In § 3.6 we give axioms for the sort of operations we shall consider; in Theorems 3.6.1, 3.6.2 we prove the existence and uniqueness of operations satisfying these axioms. In § 3.7 (and in Theorems 3.7.1, 3.7.2. in particular) we consider relations between composite operations. In § 3.9 (and in Theorem 3.9.4 in particular) we consider Cartan formulae for such operations. All the theorems mentioned above are essential for the applications.

In an attempt to arrange the proofs of these theorems lucidly, we begin by giving a formal status to some of the ideas involved. These ideas concern cohomology operations (of kinds higher than the first), universal examples for such operations, the suspension of such operations, and so on. Of course, these notions are common property; but by giving a connected account, we can build up the lemmas which we need later. This preliminary work occupies §§ 3.2 to 3.5.

In § 3.8 we give a short account of the application of homological algebra to the study of stable secondary operations.

In this chapter, the following conventions will be understood. We shall assume that all cohomology groups have coefficients in a fixed finitely-generated group G; in § 3.8 we assume that G is a field, and in § 3.9 that G is the field Z_p. We shall omit the symbol for the fixed coefficient group G, except where special emphasis is needed.

We shall assume that all the spaces considered are arcwise-connected. There is nothing essential in this assumption, but it serves to simplify some statements. Symbols such as x_0, y_0 will denote base-points in the corresponding spaces X, Y. It is understood that the base-points in CW-complexes are chosen to be vertices.

The study of stable operations forces us to work with suspension. We therefore agree that in §§ 3.2 to 3.8 the symbols $H^*(X)$, $H^n(X)$ denote augmented cohomology (with coefficients in G).

In § 3.9, however, we have to work with products. In this section, therefore, the symbol $H^n(X)$ denotes ordinary cohomology. We write $H^+(X)$ for $\sum_{n>0} H^n(X)$; since our spaces are arcwise-connected, we may use H^+ instead of augmented cohomology.

We have to take a little care with signs. It is usual to write cohomology operations on the left of their arguments; we shall follow Milnor [25] in taking the signs which arise naturally from this convention. We shall also try to keep our theoretical work as free from signs as possible. For this purpose it seems best to write our homomorphisms on the right of their arguments, accepting the signs which arise naturally from this convention.

In particular, we introduce a "right" coboundary

$$\delta^*: H^n(Y) \to H^{n+1}(X, Y)$$

whose definition in terms of the usual coboundary δ is

$$(h)\delta^* = (-1)^n \delta(h) \qquad \text{(where } h \in H^n(Y)\text{)}.$$

We shall use this signed coboundary in discussing suspension. For example, let Y be a space with base-point y_0; let $\Omega Y \longrightarrow LY \xrightarrow{\pi} Y$ be the path-space fibering introduced by Serre [29]. Then we define the "suspension" homomorphism

$$\sigma : H^{n+1}(Y) \to H^n(\Omega Y)$$

to be the composite map

$$H^{n+1}(Y) \xleftarrow{\cong} H^{n+1}(Y, y_0) \xrightarrow{\pi^*} H^{n+1}(LY, \Omega Y) \xleftarrow{\delta^*} H^n(\Omega Y) .$$

Similarly for the transgression τ.

3.2. *Theory of universal examples.* It is the object of this section to set up a general theory of universal examples for cohomology operations of higher kinds. This is done by making the obvious changes in the corresponding theory for primary operations, which is due to Serre [30]. The considerations which guide our definitions are the following. We ex-

pect our operations to be defined on a subset of all cohomology classes; we expect their values to lie in a quotient set of cohomology classes; we expect them to be natural.

It would be convenient, in some ways, if we set up our theory for the category of CW-complexes. However, it would be inconvenient in other ways, since we need to use fiberings (in the sense of Serre). The difficulty could be avoided by working in the category of CSS-complexes; but it seems preferable to work, at first, with concepts as geometrical as possible. We therefore work in the category of all spaces. This forces us to use the device of replacing a space by a weakly equivalent CW-complex. We recall that a map $f: X \to Y$ (between arcwise-connected spaces) is said to be a weak homotopy equivalence if it induces isomorphisms of all homotopy groups. Two spaces X, Y are said to be weakly equivalent if they can be connected by a finite chain of weak homotopy equivalences. These notions have the following properties. For each space Y, there is a CW-complex X and a map $f: X \to Y$ which is a weak homotopy equivalence; it is sufficient, for example, to take X to be a geometrical realisation of the singular complex of Y, or of a minimal complex. Let $\mathrm{Map}(X, Y)$ denote the set of homotopy classes of maps from X to Y; if $f: X \to Y$ is a weak homotopy equivalence and W is a CW-complex, then the induced function $f_*: \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y)$ is a (1-1) correspondence.

Our first definition is phrased so as to cover the case of cohomology operations in several variables; the reader may prefer to consider first the case of one variable. Let J be a set of indices j. We call S a *natural subset of cohomology* (in J variables, of degrees n_j) if S associates with each space X a set $S(X)$ of J-tuples $\{x_j\}$ (where $x_j \in H^{n_j}(x)$) and satisfies the following axioms.

Axiom 1. If $f: X \to Y$ is a map and $\{y_j\} \in S(Y)$, then $\{y_j f^*\} \in S(X)$.

Axiom 2. If $f: X \to Y$ is a weak homotopy equivalence and $\{y_j f^*\} \in S(x)$, then $\{y_j\} \in S(Y)$.

Although we shall not assume that the indexing set J is finite, we shall assume that for each integer N we have $n_j < N$ for only a finite number of j.

For the next definition, we suppose given such a natural subset S. We call Φ a *cohomology operation* (defined on S, and with values of degree m) if Φ associates with each space X and each J-tuple $\{x_j\}$ in $S(X)$ a non-empty subset $\Phi\{x_j\} \subset H^m(X)$ which satisfies the following axioms.

Axiom 3. If $f: X \to Y$ is a map and $\{y_j\} \in S(Y)$, then

$$(\Phi\{y_j\})f^* \subset \Phi\{y_j f^*\} .$$

Axiom 4. If $f: X \to Y$ is a weak homotopy equivalence and $\{y_j\} \in S(Y)$, then

$$(\Phi\{y_j\})f^* = \Phi\{y_j f^*\} .$$

There is nothing in our definition to limit the size of the subsets $\Phi\{x_j\}$. We therefore make the definitions which follow. They refer to cohomology operations defined on a fixed natural subset S. We write $\Phi \subset \Psi$ if we have $\Phi\{x_j\} \subset \Psi\{x_j\}$ for each X and each $\{x_j\}$ in $S(X)$. We call Ψ *minimal* if there is no operation Φ such that $\Phi \subset \Psi$ and $\Phi \neq \Psi$. We are mainly interested in operations which are minimal.

We now introduce the notion of a universal example. Let S be a natural subset of cohomology (in J variables, of degrees n_j). Let U be a space and $\{u_j\}$ a J-tuple in which $u_j \in H^{n_j}(U)$. We say that $(U, \{u_j\})$ is a *universal example* for S if $\{u_j\} \in S(U)$ and the following axiom is satisfied.

Axiom 5. For each CW-complex X and each J-tuple $\{x_j\}$ in $S(X)$ there is a map $g: X \to U$ such that $\{u_j g^*\} = \{x_j\}$.

LEMMA 3.2.1. *For each space U and each J-tuple $\{u_j\}$ there is one and only one natural subset S admitting $(U, \{u_j\})$ as a universal example.*

PROOF. The uniqueness of S is immediate; for if X is a CW-complex, $S(x)$ is the set of J-tuples $\{u_j g^*\}$, where g runs over all maps from X to U; while if X is a general space, we may take a weak homotopy equivalence $f: W \to X$ such that W is a CW-complex; then $S(X)$ is the set of J-tuples $\{x_j\}$ such that $\{x_j f^*\} \in S(W)$.

This procedure also shows how to construct S from $(U, \{u_j\})$; it is not hard to verify that the S so constructed is well-defined, is such that $\{u_j\} \in S(U)$ and satisfies Axioms 1 and 2.

For our next definition, let S be a natural subset admitting a universal example $(U, \{u_j\})$. Let Φ be a cohomology operation defined on S, and with values of degree m; let v be a class in $H^m(U)$. We say that $(U, \{u_i\}, v)$ is a *universal example* for Φ if $v \in \Phi\{u_j\}$ and the following axiom is satisfied.

Axiom 6. For each CW-complex X, each J-tuple $\{x_j\}$ in $S(X)$ and each class y in $\Phi\{x_j\}$ there is a map $g: X \to U$ such that $\{u_j g^*\} = \{x_j\}$, $vg^* = y$.

LEMMA 3.2.2. *For each space U, each J-tuple $\{u_j\}$ and each class v there is one and only one cohomology operation Φ admitting $(U, \{u_j\}, v)$ as a universal example.*

The proof is closely similar to that of Lemma 3.2.1.

If the space U is understood, we may write "$\{u_j\}$ is a universal ex-

ample for S''; similarly, if U and $\{u_j\}$ are understood, we may write "v is a universal example for Φ''.

For our last lemma, suppose that S is a natural subset admitting a universal example $\{u_j\}$, and that Ψ is an operation defined on S.

LEMMA 3.2.3. *If $v \in \Psi\{u_j\}$ and Φ is the operation given by the universal example v, then $\Phi \subset \Psi$.*

The proof is obvious.

This lemma shows that, if we wish to study the operations defined on such a natural subset S, it is sufficient to consider the ones given by universal examples.

3.3. *Construction of universal examples.* It is the object of this section to construct the universal examples on which our theory of secondary operations will be based. In fact, our secondary operations will be defined on natural subsets which can be described using primary operations. We shall show that these natural subsets admit universal examples, in the sense of § 3.2.

Our universal examples will be fiberings in which both base and fibre are weakly equivalent to Cartesian products of Eilenberg-MacLane spaces. It will be clear that the method of this section is only the beginning of an obvious induction; we might equally well construct an example-space of the $(n+1)^{\text{th}}$ kind as a fibering with the same sort of fibre, but with an example-space of the n^{th} kind as a base. However, we shall not do this.

We say that Φ is a *primary* operation (acting on J variables of degrees n_j, and with values of degree m) if it has the following properties.

(1) $\Phi\{x_j\}$ is defined for every J-tuple $\{x_j\}$ such that $x_j \in H^{n_j}(X)$.

(2) The values $\Phi\{x_j\}$ of Φ are single elements of $H^m(X)$.

Let K be a set of indices k. When we speak of a *K-tuple $\{a_k\}$ of primary operations*, we shall understand that each a_k acts on J variables whose degrees n_j do not depend on k. We shall also suppose that for each integer M we have $m_k < M$ for only a finite number of k in K.

We define *the natural subset T determined by* $\{a_k\}$ as follows: $T(X)$ is the set of J-tuples $\{x_j\}$ such that $x_j \in H^{n_j}(X)$ and $a_k\{x_j\} = 0$ for each k in K. It is clear that T is indeed a natural subset.

We shall call a cohomology operation Φ *secondary* if it is defined on a natural subset T of this kind. We shall prove that every natural subset T of this kind admits a universal example; this is formally stated as Theorem 3.3.7.

We must begin by considering universal examples for primary operations. Suppose given integers $n_j > 0$, for $j \in J$. Then we can form a Cartesian product

$$X = \times_{j \in J} X_j$$

in which X_j is an Eilenberg-MacLane space of type (G, n_j). Let x_j be the fundamental class in $H^{n_j}(X_j; G)$; let $\pi_j: X \to X_j$ be the projection map onto the j^{th} factor; then we have classes $x_j\pi_j^*$ in $H^*(X)$. Suppose given a space Y and classes y_j in $H^{n_j}(Y)$. We will say that Y is a *generalised Eilenberg-MacLane space* (with fundamental classes y_j) if Y is weakly equivalent to a product such as X in such a way that the classes y_j correspond to the classes $x_j\pi_j^*$.

LEMMA 3.3.1. *If Y is a generalised Eilenberg-MacLane space, W is a* CW*-complex and $w_j \in H^{n_j}(W)$ for each j, then there is one and only one homotopy class of maps $f: W \to Y$ such that $y_j f^* = w_j$ for each j.*

This follows immediatly from the corresponding fact for Eilenberg-MacLane spaces.

Next, let Y and W be generalised Eilenberg-MacLane spaces, for the same integers n_j, and with fundamental classes y_j, w_j. We call a map $f: Y \to W$ a *canonical equivalence* if $w_j f^* = y_j$ for each j. Such a map is necessarily a weak homotopy equivalence.

Let Y be a generalised Eilenberg-MacLane space, for integers $n_j > 1$; Y is thus weakly equivalent to a product $X = \times_{j \in J} X_j$ of Eilenberg-MacLane spaces. Then, if the base points are chosen consistently, the loop-space ΩY is weakly equivalent to ΩX, which is homeomorphic to $\times_{j \in J} \Omega X_j$; thus ΩY is a generalised Eilenberg-MacLane space. In each loop-space ΩX_j we take the fundamental class x_j^Ω given by

$$x_j^\Omega = (x_j)\sigma$$

where σ denotes "suspension". In ΩX we have fundamental classes $x_j^\Omega \pi_j^*$, and in ΩY we have the corresponding fundamental classses y_i^Ω.

We have said that we propose to construct our universal examples as fiberings. All our fiberings will be fiberings in the sense of Serre [29]. We must recall that the notion of an induced fibering is valid in this context.

Let X, B be spaces with base-points x_0, b_0. Let $f: X, x_0 \to B, b_0$ be a map, and let $\pi: E \to B$ be a fibering (in the sense of Serre). Let f^-E be the subspace of $X \times E$ consisting of pairs (x, e) such that $fx = \pi e$. We define maps $\varpi: f^-E \to X$ and $f_-: f^-E \to E$ by $\varpi(x, e) = x$, $f_-(x, e) = e$;

we thus obtain the following commutative diagram.

$$\begin{array}{ccc} f^-E & \xrightarrow{f_-} & E \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f} & B \end{array}$$

LEMMA 3.3.2. *The map ϖ is a fibering (in the sense of Serre). The map f_- maps the fibre $\varpi^{-1}x_0$ of ϖ homeomorphically onto the fibre $\pi^{-1}b_0$ of π.*

The verification is trivial. The original reference for this lemma, so far as the author knows, is [10].

The next lemma states that this construction is natural, in an obvious sense. Suppose given the following diagram.

$$\begin{array}{ccc} E & \xrightarrow{\varepsilon} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B,\ b_0 & \xrightarrow{\beta} & B',\ b_0' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ X,\ x_0 & \xrightarrow{\xi} & X',\ x_0' \end{array}$$

LEMMA 3.3.3. *We can define a map*

$$\xi \otimes \varepsilon : f^-E \to (f')^-E'$$

by the rule $(\xi \otimes \varepsilon)(x, e) = (\xi x, \varepsilon e)$. This map makes the following diagram commutative.

$$\begin{array}{ccc} E & \xrightarrow{\varepsilon} & E' \\ {\scriptstyle f_-}\uparrow & & \uparrow{\scriptstyle f'_-} \\ f^-E & \xrightarrow{\xi\otimes\varepsilon} & (f')^-E' \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\varpi'} \\ X & \xrightarrow{\xi} & X' \end{array}$$

The verification is trivial. The diagram shows in particular, that the effect of $\xi \otimes \varepsilon$ on the fibres is the same as that of ε, up to the homeomorphisms of Lemma 3.3.2.

We can now describe the fibre-spaces which we shall use as universal examples. Suppose given, as above, a fixed K-tuple $\{a_k\}$ of primary operations, such that each a_k acts on J variables of degrees n_j and has values of degree m_k. We suppose $n_j > 0$, $m_k > 1$. We call $\varpi: E \to B$ a

canonical fibering associated with $\{a_k\}$ if it satisfies the following conditions.

(1) The fibering $\varpi : E \to B$ is induced by a map μ: $B, b_0 \to Y, y_0$ from the fibering $\pi : LY \to Y$.

(2) The spaces B and Y are generalised Eilenberg-MacLane spaces with fundamental classes b_j and y_k of degrees n_j and m_k.

(3) We have $y_k\mu^* = a_k\{b_j\}$.

We remark that if these conditions are fulfilled, then the fibre $F = \varpi^{-1}b_0$ of ϖ is a generalised Eilenberg-MacLane space. In fact, the map μ_- maps F homeomorphically onto ΩY, by Lemma 3.3.2; and by our remarks above, ΩY is a generalised Eilenberg-MacLane space, with fundamental classes y_k^Ω of degree $m_k - 1$. Therefore F is also a generalised Eilenberg-MacLane space, with fundamental classes f_k corresponding to the y_k^Ω under the map μ_-.

LEMMA 3.3.4. *The class* $f_k\delta^*$ *in* $H^*(E, F)$ *is the image of* $a_k\{b_j\}$ *under the composite homomorphism*

$$H^*(B) \xleftarrow{\cong} H^*(B, b_0) \xrightarrow{\varpi^*} H^*(E, F) .$$

This is immediate, by naturality.

LEMMA 3.3.5. *For each* $\{a_k\}$ *and each corresponding space* Y *there exists a canonical fibering associated with* $\{a_k\}$ *in which the space* B *is a* CW*-complex.*

This is clear; for we can construct B to be a CW-complex, and there is then a map $\mu : B \to Y$ of the sort required.

Our next theorem will assert that all the canonical fiberings associated with $\{a_k\}$ are "equivalent", in a suitable sense. In fact, let $\varpi: E \to B$ and $\varpi': E' \to B'$ be two such fiberings; then we define an equivalence between them to be a diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\varepsilon} & E' \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\varpi'} \\ B & \xrightarrow{\beta} & B' \end{array}$$

in which $\varphi : F \to F'$ and $\beta : B, b_0 \to B', b_0'$ are canonical equivalences, in the sense explained above. Such a diagram, of course, induces an isomorphism of the exact homotopy sequences, so that ε will be a weak

homotopy equivalence.

THEOREM 3.3.6. *Any two canonical fiberings associated with the same* $\{a_k\}$ *may be connected by a finite chain of equivalences.*

PROOF. Let E', E'' be two such canonical fiberings. Let $\varpi: E \to B$ be another canonical fibering, constructed using spaces B and Y which are CW-complexes. It is sufficient to show how to connect E to E' by a chain of equivalences, since we may connect E to E'' similarly. We may construct canonical equivalences

$$\beta : B, b_0 \to B', b'_0 . , \qquad \eta : Y, y_0 \to Y', y'_0 ,$$

so that $b'_j\beta^* = b_j$, $y'_k\eta^* = y_k$. It follows that $\mu'\beta \sim \eta\mu$. Let $h : I \times B/I \times b_0 \to Y'$ be a homotopy between them; we may write $hi' = \mu'\beta$, $hi = \eta\mu$, where i and i' are the embeddings of B in $I \times B/I \times b_0$. Using Lemma 3.3.3 we obtain the following chain of equivalences.

$$E = \mu^{-}LY \xrightarrow{i \otimes L(\eta)} h^{-}LY' \xleftarrow{i' \otimes 1} (\mu'\beta)^{-}LY' \xrightarrow{\beta \otimes 1} (\mu')^{-}LY' = E'$$

Thus E, E' can be connected as required. This completes the proof of Theorem 3.3.6.

For our next theorem, let $\varpi: E \to B$ be a canonical fibering associated with $\{a_k\}$; we define a J-tuple $\{e_j\}$ of classes in $H^*(E)$ by $e_j = b_j\varpi^*$. Let T be the natural subset determined by $\{a_k\}$, as defined at the beginning of this section.

THEOREM 3.3.7. *The natural subset* T *admits the universal example* $\{e_j\}$.

PROOF. We must begin by proving that $\{e_j\} \in T(E)$. In fact, we have

$$a_k\{e_j\} = a_k\{b_j\varpi^*\} = (a_k\{b_j\})\varpi^* = y_k\mu^*\varpi^* = y_k\pi^*(\mu_-)^* .$$

But since LY is acyclic, we have $y_k\pi^* = 0$ and hence $a_k\{e_j\} = 0$. Thus $\{e_j\} \in T(E)$.

It remains to show that if X is a CW-complex and $\{x_j\} \in T(x)$, then there is a map $g : X \to E$ such that $\{e_jg^*\} = \{x_j\}$. For this purpose we introduce the following lemma.

LEMMA 3.3.8. *If* X *is a* CW*-complex and* $f : X \to B$ *is a map such that* $a_k\{b_jf^*\} = 0$, *then there is a map* $g : X \to E$ *such that* $\varpi g = f$.

From this lemma, the theorem follows immediately. In fact, suppose we have a CW-complex X and a J-tuple $\{x_j\}$ such that $a_k\{x_j\} = 0$; then we can construct $f: X \to B$ such that $b_jf^* = x_j$ and $g : X \to E$ such that $\varpi g = f$; it follows that

$$e_j g^* = b_j \varpi^* g^* = b_j f^* = x_j \ .$$

PROOF OF LEMMA 3.3.8. By Theorem 3.3.6, it is sufficient to consider the case in which Y is a Cartesian product of Eilenberg-MacLane spaces; say $Y = \times_{k \in K} Y_k$, where Y_k is of type (G, m_k). We can now consider the canonical fibering E in a different way. We have a map $\mu: B \to Y$; its k^{th} component is $\pi_k \mu: B \to Y_k$. We may define $E_k = (\pi_k \mu)^- L Y_k$; the product $E' = \times_{k \in K} E_k$ is fibred over the base $B' = \times_{k \in K} B$. Let $\Delta : B \to B'$ be the diagonal map; then $\Delta^- E'$ coincides with E, up to a homeomorphism. It is therefore sufficient to lift the map f in each factor E_k separately. But to this there is only one obstruction; and up to a sign, the obstruction is

$$y_k \mu^* f^* = (a_k \{b_j\}) f^* = a_k \{b_j f^*\} = 0 \ .$$

The lifting is therefore possible. This completes the proof of Lemma 3.3.8. and of Theorem 3.3.7.

3.4. *Suspension.* In this section we discuss the suspension of cohomology operations, and show that if Φ admits a universal example, so does its suspension. This is formally stated as Theorem 3.4.6. We also show that the application of this principle does not enlarge the class of universal examples considered in § 3.3.

In this section the symbol sX will denote the suspension of X, so that $sX = I \times X / (0 \times X, 1 \times X)$. We shall also use s to denote the canonical isomorphism

$$s : H^{n+1}(sX) \to H^n(X)$$

which we define to be the following composite map.

$$H^{n+1}(sX) \xleftarrow{\cong} H^{n+1}(sX, s_0) \longrightarrow H^{n+1}(tX, X) \xleftarrow{\delta^*} H^n(X) \ .$$

Here, tX is the cone on X, so that $tX = I \times X / 0 \times X$; the embedding of X in tX, the base-point s_0 in sX and the map from tX to sX are the obvious ones.

Let S be a natural subset of cohomology (in J variables, of degrees n_j). Then we can define a natural subset S^s (in J variables, of degrees $n_j - 1$) by taking $S^s(X)$ to be the set of J-tuples $\{x_j s\}$, where $\{x_j\} \in S(sX)$. We call S^s the *suspension* of S.

Similarly, let Φ be a cohomology operation defined on S, and with values of degree m. Then we can construct an operation Φ^s, defined on S^s and with values of degree $(m - 1)$, by setting

$$\Phi^s \{x_j s\} = (\Phi \{x_j\}) s \ .$$

We call Φ^s the *suspension* of Φ.

For our first lemma, let $\{a_k\}$ be a K-tuple of primary operations, and let T be the natural subset determined by $\{a_k\}$.

LEMMA 3.4.1. *The natural subset determined by* $\{a_k^s\}$ *is* T^s.

The verification is trivial.

LEMMA 3.4.2. *If* S *is a natural subset and* $\{x_j\} \in S^s(X)$, *then* $\{-x_j\} \in S^s(X)$.

If Φ *is a cohomology operation defined on* S, *while* $\{x_j\} \in S^s(X)$ *and* $y \in \Phi^s\{x_j\}$, *then* $-y \in \Phi^s\{-x_j\}$.

To prove this lemma, one merely considers the map ν: $sX \to sX$ defined by

$$\nu(t, x) = (1 - t, x) .$$

For the sake of similar arguments later on, we set down the following lemma, which is well known.

LEMMA 3.4.3. *The set of homotopy classes*

$$\text{Map } [sX, sx_0; Y, y_0]$$

is a group. Let φ *be the function which assigns to each homotopy class of maps* $g : sX, sx_0 \to Y, y_0$ *the induced map* g^*: $H^*(Y) \to H^*(sX)$; *then* φ *is a homomorphism.*

Our next results relate the notions of "suspension" and "universal example".

LEMMA 3.4.4. *There is a* (1-1) *correspondence between maps* $f : X, x_0 \to \Omega Y, \omega_0$ *and maps* $g : sX, sx_0 \to Y, y_0$. *For corresponding maps we have* $g^*s = \sigma f^*$: $H^*(Y) \to H^*(X)$.

This lemma is well-known. The (1-1) correspondence is set up by the equation

$$(f(x))(u) = g(u, x) \qquad \text{(where } u \in I.)$$

The equation $g^*s = \sigma f^*$ is proved by passing to cohomology from the following diagram.

$$\begin{array}{ccccc} X & \longrightarrow & tX & \longrightarrow & sX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ \Omega Y & \longrightarrow & LY & \longrightarrow & Y \end{array}$$

Here, the map h is defined by

$$(h(u, x))(v) = g(uv, x) \qquad \text{(where } u, v \in I.)$$

For our next lemma, we suppose that the space X is 1-connected, so that ΩX is 0-connected. Similarly, in Theorem 3.4.6 we shall assume that U is 1-connected.

LEMMA 3.4.5. *If* $\{x_j\} \in S(x)$, then $\{x_j\sigma\} \in S^s(\Omega X)$. *If (further)* Φ *is defined on* S, *then*

$$(\Phi\{x_j\})\sigma \subset \Phi^s\{x_j\sigma\} \ .$$

This is immediate, by considering the map $g : s\Omega X \to X$ corresponding to $f = 1 : \Omega X \to \Omega X$.

THEOREM 3.4.6. *If the natural subset* S *admits the universal example* $(U, \{u_j\})$, *then* S^s *admits the universal example* $(\Omega U, \{u_j\sigma\})$. *In this case* $S^s(X)$ *is a subgroup of* $\prod_{j\in J} H^{n_j-1}(X)$.

If the cohomology operation Φ *admits the universal example* $(U, \{u_j\}, v)$, *then* Φ^s *admits the universal example* $(\Omega U, \{u_j\sigma\}, v\sigma)$. *In this case* $\Phi^s\{0\}$ *is a subgroup of* $H^{m-1}(X)$, $\Phi^s\{x_j\}$ *is a coset of* $\Phi^s\{0\}$ *and* Φ^s *is a homomorphism.*

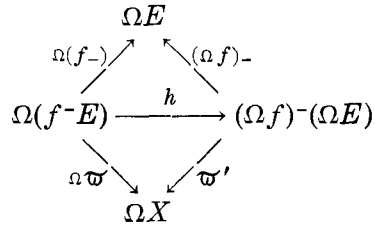
This theorem follows easily from Lemmas 3.4.3 to 3.4.5.

Our next theorem will show that if U lies in the class of universal examples considered in § 3.3, then the universal example ΩU lies in the same class. For this purpose we need two lemmas.

LEMMA 3.4.7. *If* $\pi : E, e_0 \to B, b_0$ *is a fibering (in the sense of Serre) then* $\Omega\pi : \Omega E \to \Omega B$ *is a fibering (in the sense of Serre) with fibre* ΩF.

The verification is trivial. The next lemma concerns induced fiberings, so we adopt the notation of Lemma 3.3.2.

LEMMA 3.4.8. *There is a canonical homeomorphism* $h : \Omega(f^-E) \to (\Omega f)^-(\Omega E)$ *which makes the following diagram commutative.*



The verification is trivial.

THEOREM 3.4.9. *If* $\varpi : E, e_0 \to B, b_0$ *is a canonical fibering associated with* $\{a_k\}$, *and if we take in* ΩB, ΩF *the fundamental classes* $b_j\sigma$, $-f_k\sigma$, *then* $\Omega\varpi : \Omega E \to \Omega B$ *is (up to a canonical homeomorphism) a canonical fibering associated with* $\{a_k^s\}$.

PROOF. Suppose that E is induced by a map $\mu : B, b_0 \to Y, y_0$. There is an obvious homeomorphism between the fiberings $\Omega(LY) \xrightarrow{\Omega\pi} \Omega Y$ and $L(\Omega Y) \xrightarrow{\pi'} \Omega Y$; both fiberings have fibre $\Omega^2 Y$, but the homeomorphism induces a non-trivial automorphism α of $\Omega^2 Y$, defined by

$$[(\alpha\omega)(u)](v) = [\omega(v)](u) \qquad \text{(where } u, v \in I\text{)}.$$

Under this automorphism, the class $-y_k\sigma^2$ in the fibre of $\Omega\pi$ corresponds to the class $y_k\sigma^2$ in the fibre of π'.

We now remark that the fiberings induced by $\Omega\mu$ from $\Omega\pi$ and from π' must still be homeomorphic. The first is homeomorphic to $\Omega\varpi : \Omega E \to \Omega B$, by Lemma 3.4.8. The second is a canonical fibering, induced by $\Omega\mu$, and satisfying

$$(y_k\sigma)(\Omega\mu)^* = a_k^s\{b_j\sigma\}$$

(as we see using Lemma 3.4.5.) It is thus a canonical fibering associated with $\{a_k^s\}$. It is now easy to check that the k^{th} fundamental class in its fibre corresponds to $-f_k\sigma$ in ΩF. This completes the proof.

3.5. *Stable operations.* In this section we shall study stable operations, and prove two lemmas needed in § 3.6.

In § 3.4 we defined the suspension of natural subsets and of cohomology operations. We use this notion to make the following definitions.

A *stable natural subset* S associates to each (positive or negative) integer l a natural subset S^l in such a way that $S^l = (S^{l+1})^s$. We may take our notations for degrees so that the variables in S^l are of degrees $n_j + l$. We admit, of course, that the natural subsets S^l may be trivial if l is large and negative. A similar remark applies to the next definition.

A *stable cohomology operation* (defined on such an S) associates to each integer l a cohomology operation Φ^l defined on S^l in such a way that $\Phi^l = (\Phi^{l+1})^s$. We may take our notations for degrees so that Φ^l has values of degree $m + l$.

We also allow ourselves to write $S(X) = \bigcup_l S^l(X)$, and to regard Φ as a function defined on $S(X)$ by the rule $\Phi \mid S^l = \Phi^l$. This is done in order to preserve the analogy between Φ and symbols such as Sq^i, which denote operations applicable in each dimension. We may call S^l, Φ^l the *components* of S, Φ.

As a particular case of the above, we have the notion of a stable primary operation a in one variable. Its l^{th} component is a primary operation

$$a^l : H^{n+l}(X) \to H^{m+l}(X);$$

we assign to a the degree $(m - n)$. By Theorem 3.4.6, a^l is a homomorphism. Since natural homomorphisms can be composed and added, we

easily see that the set of stable primary operations in one variable is a graded ring. We write A for this ring, or A_G if we wish to emphasise its dependence on the coefficient group; we call A the *Steenrod ring*. If X is any space, then $H^*(X)$ is a graded module over the graded ring A. If $f: X \to Y$ is a map, then $f^*: H^*(Y) \to H^*(X)$ is an A-map (of degree zero).

Let X be an Eilenberg-MacLane space of type (G, n), with fundamental class x; and let C be a free A-module, on one generator c of degree n. We can define an A-map $\theta : C \to H^*(X)$ by $c\theta = x$. It is both clear and well-known that $\theta \mid C_m: C_m \to H^m(X)$ is an isomorphism if $m < 2n$ (and a monomorphism if $m = 2n$).

Similarly, let Y be a generalised Eilenberg-MacLane space, with fundamental classes y_j of degrees n_j. Set $\nu = \mathrm{Min}_{j\in J} n_j$. Let C be a free A-module on generators c_j of dimension n_j. We can define an A-map $\theta : C \to H^*(Y)$ by $c_j\theta = y_j$. Then, as before, $\theta \mid C_m : C_m \to H^m(Y)$ is an isomorphism if $m < 2\nu$ (and a monomorphism if $m = 2\nu$).

It follows, incidentally, that every stable primary cohomology operation in J variables is of the form $a\{x_j\} = \sum_{j\in J} a_j x_j$, where the sum is finite and the coefficients a_j lie in the Steenrod ring.

We next take a K-tuple $\{a_k\}$ of stable primary operations. Each stable operation a_k has components a_k^l; we shall suppose that a_k^l acts on J variables of degrees $n_j + l$ and has values of degree $m_k + l$. (We suppose, as always, that for each integer N we have $n_j < N$ for only a finite number of j and $m_k < N$ for only a finite number of k.)

Such a K-tuple evidently determines a stable natural subset T; we define T^l to be the natural subset determined by $\{a_k^l\}$. We shall call a stable cohomology operation Φ *secondary* if it is defined on a stable subset T of this kind.

In considering such stable secondary operations, it is natural to introduce a sequence of canonical fiberings E_l in which E_l is associated (in the sense of § 3.3) with the K-tuple $\{a_k^l\}$. For such fiberings to exist we require $l + n_j > 0$, $l + m_k > 1$; so we should assume that $l > -\nu$, where $\nu = \mathrm{Min}_{j\in J, k\in K}(n_j, m_k - 1)$.

The next lemma will show that if a relation (between stable secondary operations) holds in the canonical fibering E_l (where l is sufficiently large), then it holds universally.

We will assume that T^l is the natural subset determined by $\{a_k^l\}$, as above, and that χ is a stable operation such that χ^l is defined on T^l and has values of degree $q + l$. We will also assume that χ satisfies the following axiom.

Axiom 1. If $g : X \to Y$ is a map such that $g^*: H^r(Y) \to H^r(X)$ is an isomorphism for $r \leqq q + l$, and if $\{y_j\} \in T^l(Y)$, then

$$(\chi^l\{y_j\})g^* = \chi^l\{y_j g^*\} .$$

Of course, this axiom is slightly stronger than Axiom 4, § 3.2; however, it is satisfied in the applications.

Let E_l be a canonical fibering associated with $\{a_k^l\}$, and let e_j^l be the fundamental classes in E_l.

LEMMA 3.5.1. *If* $0 \in \chi^\lambda\{e_j^\lambda\}$ *for one value* λ *of* l *such that* $\lambda > \mathrm{Max}(-\nu, q - 2\nu)$, *then* $0 \in \chi^l\{x_j\}$ *for all* l, *all* X *and all* $\{x_j\}$ *in* $T^l(X)$.

PROOF. We first note that if $0 \in \chi^l\{e_j^l\}$, then $0 \in \chi^l\{x_j\}$ for all X and all $\{x_j\}$ in $T^l(X)$; this is immediate, by naturality. We will show that if this holds for some l (where $l \geqq \lambda$) then it holds also for $l + 1$. We may find a space X and a map $g : sX \to E_{l+1}$ such that

$$g^*: H^r(E_{l+1}) \to H^r(sX)$$

is an isomorphism for $r \leqq q + l + 1$; for it is sufficient to take X to be ΩE_{l+1}, and g to be the map which corresponds to $f = 1$ in Lemma 3.4.3. We now have

$$\begin{aligned} 0 \in \chi^l\{e_j^{l+1}g^*s\} &= (\chi^{l+1}\{e_j^{l+1}g^*\})s \\ &= (\chi^{l+1}\{e_j^{l+1}\})g^*s \qquad \text{(by Axiom 1.)} \end{aligned}$$

Hence $0 \in \chi^{l+1}\{e_j^{l+1}\}$.

This proves that $0 \in \chi^l\{x_j\}$ if $l \geqq \lambda$. The corresponding result for $l < \lambda$ follows immediately, by suspension.

Our next lemma gives information about the group $H^{l+q}(E_l)$, at least if l is sufficiently large. We suppose given a K-tuple $\{a_k\}$ of stable primary operations, as described above; we may express each a_k in the form

$$a_k\{x_j\} = \sum_{j \in J} \beta_{k,j} x_j$$

where the sum is finite and the coefficients $\beta_{k,j}$ lie in the Steenrod ring A. Let C_0, C_1 be free A-modules on generators $c_{0,j}$, $c_{1,k}$ of degrees n_j, m_k. We can define an A-map $d : C_1 \to C_0$ by setting

$$c_{1,k}d = \sum_{j \in J} \beta_{k,j} c_{0,j} .$$

Let us take a value of l such that $l > -\nu$, and let $\varpi: E \to B$ be a canonical fibering associated with the K-tuple $\{a_k^l\}$. We can define A-maps

$$\theta_0: C_0 \to H^*(B) , \qquad \theta_1: C_1 \to H^*(F)$$

by setting

$$c_{0,j}\theta_0 = b_j\ , \quad c_{1,k}\theta_1 = (-1)^l f_k\ .$$

We take the "total degree" of $c_{s,t}$ to be $t - s (s = 0, 1)$, so that both θ_0 and θ_1 have degree l. The sign in the definition of θ_1 is essential in order that θ_1 should be compatible with σ; see Theorem 3.4.9.

Finally, we recall the following convention. Let

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ P & \xrightarrow{f'} & Q \end{array}$$

be a diagram of A-maps in which f and f' have total degree r, while g and g' have total degree s. Then we say that the diagram is *anticommutative* if

$$fg' = (-1)^{rs} gf'\ .$$

LEMMA 3.5.2. *With the above data, we have the following anticommutative diagram.*

$$\begin{array}{ccccccc} C_{1,q} & & \xrightarrow{\qquad\qquad\theta_1\qquad\qquad} & & & & H^{l+q-1}(F) \\ {\scriptstyle d}\downarrow & & & & & & \downarrow{\scriptstyle \delta^*} \\ C_{0,q} & \xrightarrow{\theta_0} & H^{l+q}(B) & \xleftarrow{\cong} & H^{l+q}(B, b_0) & \xrightarrow{\varpi^*} & H^{l+q}(E, F) \end{array}$$

If $q < l + 2\nu$, we have also the following anticommutative diagram.

$$\begin{array}{ccccccccc} \cdots\longrightarrow & H^{l+q}(B) & \xrightarrow{\varpi^*} & H^{l+q}(E) & \xrightarrow{i^*} & H^{l+q}(F) & \xrightarrow{\tau} & H^{l+q+1}(B) \\ & {\scriptstyle\theta_0}\uparrow\cong & & & & {\scriptstyle\theta_1}\uparrow\cong & & \uparrow{\scriptstyle\theta_0} \\ \cdots\longrightarrow & C_{0,q} & & & & C_{1,q+1} & \xrightarrow{d} & C_{0,q+1} \end{array}$$

The horizontal sequence in this diagram is exact; the maps θ_0 and θ_1 marked as isomorphisms are such; the remaining map θ_0 is a monomorphism.

PROOF. We take first the anticommutativity of the first diagram. By using Lemma 3.3.4 and the definitions of the various homomorphisms, it is easy to check that the two ways of chasing $c_{1,k}$ round the diagram agree, up to the sign $(-1)^l$. The corresponding result for a general element of C_1 now follows by linearity over A.

In the second diagram, the squares are provided by the first diagram. We have already noted the behaviour of maps such as θ_0 and θ_1. The exact sequence is due to Serre [29]; it is valid up to $H^{2l+2\nu-1}(F)$ because

B and F are $(l+\nu-1)$-connected. We can add the last τ, since it is defined on the whole of $H^{2l+2\nu-1}(F)$. This completes the proof.

3.6. *Axiom for stable secondary operations.* In §§ 3.2, 3.4 we were concerned with operations in general. It is the object of this section to give a system of axioms for stable secondary operations. This work is essential for the applications. After giving the axioms, we state Theorems 3.6.1 and 3.6.2, which assert the existence and (essential) uniqueness of operations satisfying the axioms. We then give some explanation of the axioms. Finally, we prove the two theorems.

It is generally understood that a secondary operation corresponds to a relation between primary operations. For example, the Massey product [23] [24] [35] corresponds to the relation $(uv)w = u(vw)$; the Adem operation [4] corresponds to the relation $\mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^3\mathrm{Sq}^1 = 0$; and so on. We aim to get a hold on stable secondary operations by dealing with their associated relations. The essential feature of our axioms is that they axiomatise the connection between the secondary operation and its associated relation.

The notion of a "relation" between primary operations will be formalised in a suitable way. In fact, we shall replace the notion of a "relation" by the notion of a pair (d, z), of the following algebraic nature. The first entry d is to be a map $d: C_1 \to C_0$. Here, the objects C_0 and C_1 are to be graded modules over the Steenrod ring A (see §§ 2.1, 3.5); they are to be locally finitely-generated and free, and d is to be a right A-map such that $(C_{0,t})d \subset C_{1,t}$. Following § 3.5, we ascribe to $c_{s,t}$ the "total degree" $t-s$. The second entry z is to be a homogeneous element of $\mathrm{Ker}\, d$.

We must next explain the connection between pairs (d, z) and relations in the intuitive sense. The equations

$$\mathrm{Sq}^1\mathrm{Sq}^1 = 0\ , \quad \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^3\mathrm{Sq}^1 = 0\ ,$$
$$\mathrm{Sq}^1\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^4\mathrm{Sq}^1 = 0$$

are relations in the intuitive sense. More generally, suppose given an integer q and a finite number of elements α_k, β_k in A such that

$$\sum_{k\in K}\alpha_k\beta_k = 0\ , \quad \deg(\alpha_k) + \deg(\beta_k) = q+1\ .$$

Then the equation $\sum_{k\in K}\alpha_k\beta_k = 0$ is a (homogeneous) relation in the intuitive sense; we shall associate it with the pair (d, z) constructed in the following way. We take C_0 to be free on one generator c_0 in $C_{0,0}$; we take C_1 to be free on generators $c_{1,k}$ in $C_{1,t(k)}$, where $t(k) = \deg(\beta_k)$. We define $d: C_1 \to C_0$ by $c_{1,k}d = \beta_k c_0$; we define z by $z = \sum_{k\in K}\alpha_k c_{1,k}$. We thus have $zd = 0$, $z \in C_{1,q+1}$.

Our axioms will ensure that if an operation Φ is associated with such a pair (d, z), then it is defined on classes x in $H^l(X)$ such that $\beta_k(x) = 0$ for each k, and has values in

$$H^{l+q}(X)/\sum_{k\in K}\alpha_k H^{l+q-r_k}(X)$$

where $r_k = \deg(\alpha_k)$. (Note that a relation of degree $(q+1)$ corresponds to an operation of degree q.)

The reader may like to keep in mind some explicit examples of pairs (d, z), to illustrate the considerations of this section.

We next remark that, according to our axioms, stable secondary operations are defined, not on J-tuples $\{x_j\}$ of cohomology classes, but on right A-maps $\varepsilon : C_0 \to H^*(X)$. There is no essential difference hence; if we take a base of elements $c_{0,j}$ in the free module C_0, then an A-map $\varepsilon : C_0 \to H^*(X)$ is determined uniquely by giving the images $(c_{0,j})\varepsilon$ of the base elements; and these classes $(c_{0,j})\varepsilon$ in $H^*(X)$ may be chosen at will, provided that they have the correct degrees. We set up a (1-1) correspondence between J-tuples $\{x_j\}$ and maps ε by writing

$$x_j = (c_{0,j})\varepsilon \ .$$

It is always to be understood that operations $\Phi\{x_j\}$ are to be identified with operations $\Phi(\varepsilon)$ in this way.

We now give the axioms. We will say that Φ is a *stable secondary operation associated with the pair* (d, z) if it satisfies the following axioms.

Axiom 1. $\Phi(\varepsilon)$ is defined if and only if $\varepsilon : C_0 \to H^*(X)$ is a right A-map such that $d\varepsilon = 0$.

For the next axiom, suppose that the total degrees of ε, z are l, q. Let $f : C_1 \to H^*(X)$ run over the right A-maps of total degree l, and let $Q^{l+q}(z, X)$ be the set of elements zf in $H^{l+q}(X)$.

Axiom 2. $\Phi(\varepsilon) \in H^{l+q}(X)/Q^{l+q}(z, X)$.

For the next axiom, let $g : X \to Y$ be a map, and let $\varepsilon : C_0 \to H^*(Y)$ be a right A-map such that $d\varepsilon = 0$.

Axiom 3. $(\Phi(\varepsilon))g^* = \Phi(\varepsilon g^*)$.

It is understood that the g^* on the left-hand side of this equation denotes a homomorphism of quotient groups, induced by the homomorphism g^* of cohomology groups.

For the next axiom, let sX be the suspension of X, and let s: $H^*(sX) \to$

$H^*(X)$ be the suspension isomorphism, as in § 3.4. Let $\varepsilon: C_0 \to H^*(sX)$ be a right A-map such that $d\varepsilon = 0$.

Axiom 4. $(\Phi(\varepsilon))s = \Phi(\varepsilon s)$.

The s on the left-hand side of this equation is to be interpreted like the g^* in Axiom 3.

For the next axiom, let (X, Y) be a pair of spaces, and let $\varepsilon : C_0 \to H^*(X)$ be a right A-map of degree l such that $d\varepsilon = 0$ and $\varepsilon i^* = 0$. We can now find right A-maps $\eta : C_0 \to H^*(X, Y)$ and $\zeta : C_1 \to H^*(Y)$(of total degree l) to complete the following anticommutative diagram.

$$\begin{array}{ccccccccc} H^*(Y) & \xleftarrow{i^*} & H^*(X) & \xleftarrow{j^*} & H^*(X, Y) & \xleftarrow{\delta^*} & H^*(Y) & \xleftarrow{i^*} & H^*(X) \\ & & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \zeta} & & \\ & & C_0 & \xleftarrow{\quad d \quad} & & & C_1 & & \end{array}$$

Axiom 5. $(\Phi(\varepsilon))i^* = [z\zeta] \mod (Q^{l+q}(z, X))i^*$.

It is understood that $[z\zeta]$ denotes the coset containing $z\zeta$. We easily check that this coset is independent of the choice of η and ζ.

The following theorems may help to justify this set of axioms.

THEOREM 3.6.1. *For each pair (d, z), there is at least one associated operation Φ.*

THEOREM 3.6.2. *If Φ, Ψ are two operations associated with the same pair (d, z), then they differ by a primary operation, in the sense that there is an element c in* (Coker $d)_q$ *such that* $\Phi(\varepsilon) - \Psi(\varepsilon) = [c\varepsilon]$.

We note that these theorems do not depend on any choice of bases in C_0 and C_1; however, we may of course use bases in the proofs.

We will now comment on the effect of these axioms. Let us take bases $c_{0,j}$, $c_{1,k}$ in C_0, C_1; suppose that the total degrees of $c_{0,j}, c_{1,k}$ are n_j, $m_k - 1$. We may write

$$(c_{1,k})d = \sum_{j \in J} \beta_{k,j} c_{0,j} , \quad z = \sum_{k \in K} \alpha_k c_{1,k}$$

where α_k, $\beta_{k,j}$ lie in A. We may define a stable primary operation a_k (in J variables) by

$$a_k\{x_j\} = \sum_{j \in J} \beta_{k,j} x_j .$$

Then (as we easily check) Axiom 1 is equivalent to saying that Φ is defined on J-tuples $\{x_j\}$ such that $x \in H^{l+n_j}(X)$ for each $j \in J$ and $a_k\{x_j\} = 0$ for each $k \in K$.

Axiom 2 states that the "indeterminacy" of Φ is $Q^{l+q}(z, X)$; with the above notation, we have

$$Q^{l+q}(z, X) = \sum_{k \in K} \alpha_k H^{l+m_k-1}(X) .$$

It is easy to see that any operation whose indeterminacy is given in this way satisfies Axiom 1, § 3.5.

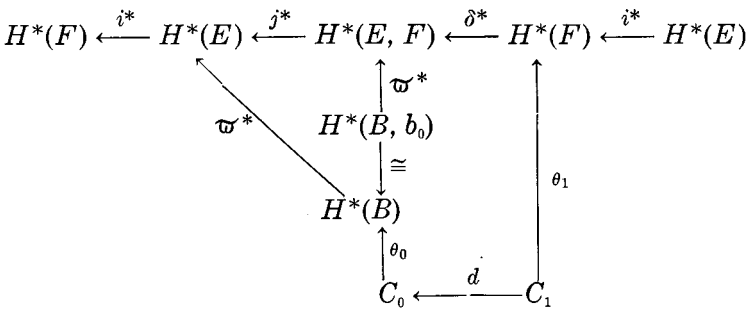
Axiom 3 states that Φ is natural. Axiom 4 states that Φ is stable, in the sense of § 3.5. It is now clear that every Φ satisfying our axioms is a stable secondary operation in the sense of § 3.5.

Axiom 5 may be regarded in two ways. On the one hand, it is a version of one of the Peterson-Stein relations [28], and is of some use in applications. On the other hand, it serves to prescribe the universal example for Φ, without making explicit mention of any such thing. This is made precise by Lemma 3.6.3; we shall need the following notation. Let Φ be an operation satisfying Axioms 1–4, and let a_k be as above. Let Φ^l, a_k^l be the l^{th} components of the stable operations Φ, a_k (as in § 3.5); and let E be a canonical fibering associated with the K-tuple $\{a_k^l\}$, as in §§ 3.3, 3.5. (For this purpose we assume that $l > -\nu$, where ν is given by $\nu = \text{Min}_{j \in J,\, k \in K}(n_j, m_k - 1)$.) We may refer to E as a "canonical fibering associated with d". Let the maps θ_0, θ_1 be as in Lemma 3.5.2; and let ε_E: $C_0 \to H^*(E)$ be the A-map corresponding to the J-tuple $\{e_j\}$, so that $\varepsilon_E = \theta_0 \varpi^*$. We may regard ε_E as analogous to the "fundamental class" in an Eilenberg-MacLane space. We have $a_k\{e_j\} = 0$, or equivalently $d\varepsilon_E = 0$, so that $\Phi^l(\varepsilon_E)$ is defined.

LEMMA 3.6.3. *Φ^l satisfies Axiom 5 if and only if*

$$z\theta_1 \in (\Phi^l(\varepsilon_E))i^* .$$

PROOF. Suppse Φ^l satisfies Axiom 5; then we may apply Axiom 5 to the following diagram (in which the square is provided by Lemma 3.5.2):



The conclusion which we obtain is

$$z\theta_1 \in (\Phi^l(\varepsilon_E))i^* .$$

Conversely, suppose that Φ^l satisfies this condition; we have to verify Axiom 5. It is sufficient to do so when the pair (X, Y) is a pair of CW-complexes. In this case, suppose given ε and $\eta : C_0 \to H^*(X, Y)$, as in Axiom 5. Since B is a generalised Eilenberg-MacLane space, we can construct a map $f : X, Y \to B, b_0$ so that the composite map

$$C_0 \xrightarrow{\theta_0} H^*(B) \xleftarrow{\cong} H^*(B, b_0) \xrightarrow{f^*} H^*(X, Y)$$

coincides with η. By Lemma 3.3.8, we can lift f to $g : X, Y \to E, F$. We can now take ζ to be the composite map

$$C_1 \xrightarrow{\theta_1} H^*(F) \xrightarrow{g^*} H^*(Y) .$$

But with this choice of ζ, we have

$$z\zeta \in (\Phi^l(\varepsilon))i^* ,$$

by applying g^* to the original condition. This completes the proof.

PROOF OF THEOREM 3.6.1. Suppose given a pair (d, z). Let us take bases $c_{0,j}$, $c_{1,k}$ in C_0, C_1; and let us keep the other notations introduced in the comments on the axioms, so that the K-tuple $\{a_k^l\}$ is as above. Let E_l be a canonical fibering associated with the K-tuple $\{a_k^l\}$. Let us fix on a value λ of l such that $\lambda > \mathrm{Max}(-\nu, q - 2\nu)$. By Lemma 3.5.2, we may choose a class v in $H^{\lambda+q}(E_\lambda)$ so that

$$vi^* = z\theta_1 .$$

Since we shall later wish to quote the part of the argument which starts at this point, we give it the status of a lemma.

LEMMA 3.6.4. *With the data above, there is at least one operation Φ associated with (d, z) (in the sense of Axioms* 1-5*) and such that $(E_\lambda, \{e_j^\lambda\}, v)$ is a universal example for Φ^λ.*

It is clear that this lemma implies the theorem.

PROOF OF LEMMA 3.6.4. We have canonical fiberings $\varpi_l : E_l \to B_l$ associated with K-tuples $\{a_k^l\}$ (at least for $l > -\nu$). By Theorem 3.4.9, $\Omega\varpi_{l+1} : \Omega E_{l+1} \to \Omega B_{l+1}$ is a canonical fibering associated with $\{a_k^l\}$; by Theorem 3.3.6, it is equivalent to $\varpi_l : E_l \to B_l$. We choose a finite chain of equivalences connecting them. Our next step is to choose a sequence of classes $v^l \in H^{l+q}(E_l)$ so that $v^\lambda = v$ and so that the class $v^{l+1}\sigma$ in $H^{l+q}(\Omega E_{l+1})$ corresponds to v^l in $H^{l+q}(E_l)$ under the finite chain of equivalences. This choice is clearly possible and unique, because the suspension

$$\sigma : H^{r+1}(E_{l+1}) \to H^r(\Omega E_{l+1})$$

is an isomorphism for $r < 2(l + \nu)$. Let Φ^l(for $l > -\nu$) be the operation given by the universal example $(E_l, \{e_j^l\}, v^l)$. Then it is clear that the operation given by v^l(namely Φ^l) coincides with that given by $v^{l+1}\sigma$ (namely $(\Phi^{l+1})^s$). To obtain a stable operation Φ (in the sense of § 3.5) we need only define its components Φ^l for $l \leqq -\nu$ by the inductive formula

$$\Phi^l = (\Phi^{l+1})^s \qquad \text{(for each } l\text{)}.$$

We have now defined the operation Φ required; it remains to verify that it has the desired properties. We have ensured that v is a universal example for Φ^λ. Further, we have satisfied Axioms 3 and 4 (since Φ is natural and stable) and also Axiom 1 (since Φ^l is defined on the natural subset determined by $\{a_k^l\}$).

We next consider the equation

$$v^l i^* = z\theta_1 .$$

It holds for $l = \lambda$, by hypothesis; we may deduce that it holds for all l (such that $l > -\nu$), since the suspension

$$\sigma: H^{r+1}(F_{l+1}) \to H^r(\Omega F_{l+1})$$

is an isomorphism for $r < 2(l + \nu)$.

We can now verify Axiom 2. By suspension, it is sufficient to do this for $l > -\nu$. In this case, Theorem 3.4.6 shows that the values of Φ^l are cosets of $\Phi^l(0)$; we will show that $\Phi^l(0) = Q^{l+q}(z, X)$. It is sufficient to do this in the case when X is a CW-complex. In this case, any element of $\Phi^l(0)$ can be written in the form $v^l g^*$, where $g : X \to E_l$ is a map such that $\{e_j^l g^*\} = 0$. Since $e_j^l = b_j^l \varpi^*$, we have $\{b_j^l \varpi^* g^*\} = 0$, and therefore $\varpi g : X \to B_l$ is homotopic to the constant map at b_0. Covering this homotopy, we find a map $h : X \to F_l$ such that $g \sim ih$: $X \to E_l$. We now have

$$v^l g^* = v^l i^* h^* = z\theta_1 h^* .$$

Since $\theta_1 h^*$: $C_1 \to H^*(X)$ is an A-map of degree l, we have shown that $\Phi^l(0) \subset Q^{l+q}(z, X)$. Since any A-map $f : C_1 \to H^*(X)$ of degree l may be written in the form $\theta_1 h^*$ by a suitable choice of h, we easily see that $\Phi^l(0) \supset Q^{l+q}(z, X)$. This completes the proof of Axiom 2.

It remains only to verify Axiom 5. But this follows immediately from Lemma 3.6.3, at least if $l > -\nu$; and the case $l \leqq -\nu$ may be deduced by suspension. This completes the proof of Lemma 3.6.4 and of Theorem 3.6.1.

PROOF OF THEOREM 3.6.2. Suppose that Φ, Ψ are two operations as-

sociated with the same pair (d, z). Let us keep the general notation used above; let λ be a value of l such that $\lambda > \mathrm{Max}(-\nu, q - 2\nu)$, and let E be a canonical fibering associated with $\{a_k^\lambda\}$. By Axiom 5 and Lemma 3.6.3, we have

$$z\theta_1 \in (\Phi(\varepsilon_E))i^* .$$

Let v be a class in $\Phi(\varepsilon_E)$ such that $vi^* = z\theta_1$. Similarly, let w be a class in $\Psi(\varepsilon_E)$ such that $wi^* = z\theta_1$; then $(v-w)i^* = 0$. By Lemma 3.5.2, there is an element c in $(C_0)_q$ such that $v - w = c\theta_0\varpi^*$. This shows that

$$\Phi(\varepsilon_E) - \Psi(\varepsilon_E) - [c\varepsilon_E] = 0 .$$

Let us define a stable operation χ by

$$\chi(\varepsilon) = \Phi(\varepsilon) - \Psi(\varepsilon) - [c\varepsilon] ;$$

then χ satisfies the conditions of Lemma 3.5.1; therefore $\chi(\varepsilon) = 0$. This completes the proof of Theorem 3.6.2.

It is clear that operations satisfying Axioms 1–5 are linear, in the sense that

$$\Phi(\varepsilon + \varepsilon') = \Phi(\varepsilon) + \Phi(\varepsilon') .$$

In fact, Theorem 3.4.6 shows that this is true for operations constructed by the method of Lemma 3.6.4; and Theorem 3.6.2 enables us to deduce the corresponding result for any operation Φ.

3.7. *Properties of the operations.* In this section we shall prove certain properties of the stable secondary operations described in the last section. In content these properties are relations which hold between certain sums of composite operations, such as $\sum_{r\in R} a_r\Phi_r$ and $\sum_{r\in R}\Phi_r a_r$; here, the a_r are primary operations, and the Φ_r are secondary operations. We shall give these properties a form in keeping with our algebraic machinery. They are stated as Theorems 3.7.1 and 3.7.2; these results are essential for the applications.

We shall not discuss operations of the form $\Phi\Psi$, since such operations are tertiary, not secondary; $\Phi\Psi\{x_j\}$ is only defined if $a_k\Psi\{x_j\} = 0$ for various a_k.

We first suppose given an A-map $d : C_1 \to C_0$ (of the usual kind) and finitely many elements z_r in $\mathrm{Ker}\ d$ and of degrees q_r. Suppose that

$$z = \sum_{r\in R} a_r z_r$$

where $a_r \in A$ and $\deg(a_r) + q_r = q$. Suppose that operations Φ_r correspond to the pairs (d, z_r); then we have the following result.

THEOREM 3.7.1. *There is an operation Φ associated with (d, z) and such that*

$$\sum_{r\in R}[a_r\Phi_r(\varepsilon)] = [\Phi(\varepsilon)] \mod \sum_r a_r Q^{l+q_r}(z_r, X)$$

for each X and each A-map $\varepsilon : C_0 \to H^*(X)$ (of degree l) such that $d\varepsilon = 0$.

We remark that $Q^{l+q}(z, X) \subset \sum_r a_r Q^{l+q_r}(z_r, X)$; this is easily verified. We may call the group $\sum_r a_r Q^{l+q_r}(z_r, X)$ the *total indeterminacy* of the operations $a_r\Phi_r$ and Φ; the expressions in square brackets denote cosets of this group. It is worth noting, for the applications, that if the set R contains only one member r, then

$$Q^{l+q}(z, X) = a_r Q^{l+q_r}(z_r, X).$$

PROOF OF THEOREM 3.7.1. Let $E = E_\lambda$ be a canonical fibering associated with d, for some $\lambda > \mathrm{Max}(-\nu, q-2\nu)$. Let v_r be an element in $\Phi_r(\varepsilon_E)$ such that $v_r i^* = z_r\theta_1$. Define v by $v = \sum_{r\in R} a_r v_r$; then $v \in H^{\lambda+q}(E)$ and $vi^* = z\theta_1$. By Lemma 3.6.4 there is an operation Φ associated with (d, z) such that $v \in \Phi(\varepsilon_E)$. We thus have

$$[\Phi(\varepsilon_E)] - \sum_{r\in R}[a_r\Phi_r(\varepsilon_E)] = 0 .$$

By Lemma 3.5.1 we have

$$[\Phi(\varepsilon)] - \sum_{r\in R}[a_r\Phi_r(\varepsilon)] = 0 .$$

This completes the proof.

For the next theorem, we suppose given the following anticommutative diagram, in which d and d' are A-maps of the usual kind, while ρ_0 and ρ_1 are A-maps of degree r.

$$\begin{array}{ccc} C_1 & \xrightarrow{\rho_1} & C_1' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ C_0 & \xrightarrow{\rho_0} & C_0' \end{array}$$

Let z be an element of Ker d, of total degree q, and let Φ be an operation associated with (d, z). Then we have the following result.

THEOREM 3.7.2. *There is an operation Φ' associated with $(d', z\rho_1)$ such that*

$$\Phi(\rho_0\varepsilon') = [\Phi'(\varepsilon')] \mod Q^{l+q+r}(z, X)$$

for each X and each A-map $\varepsilon' : C_0' \to H^*(X)$ (of degree l) such that $d'\varepsilon' = 0$.

We remark that $\Phi(\rho_0\varepsilon')$ is defined and that $Q^{l+q+r}(z', X) \subset Q^{l+q+r}(z, X)$; this is easily verified.

EXAMPLE. Consider the case in which C_0, C'_0 are free on single generators c_0, c'_0 of degree zero, and ρ_0 is defined by $c_0\rho_0 = ac'_0$, where $a \in A_r$. We may replace the map ε' by a class x' in $H^*(X)$, and our conclusion becomes

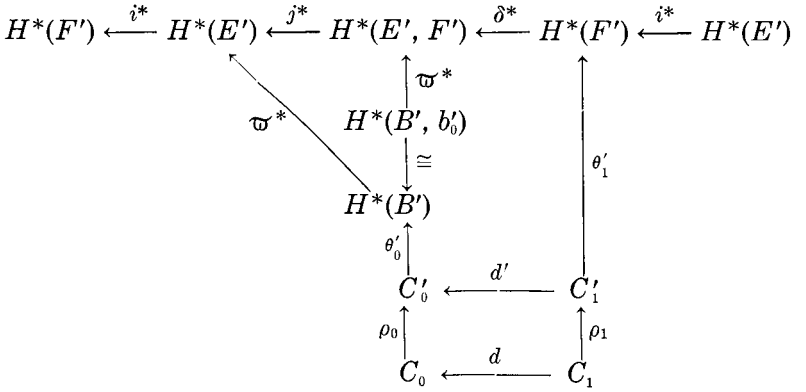
$$\Phi(ax') = [\Phi'(x')] \ .$$

For later use, we give the first step in the proof the status of a lemma. Let λ be a value of l such that $\lambda > \mathrm{Max}(-\nu', q + r - 2\nu')$; let $E' = E'_\lambda$ be a canonical fibering associated with d'.

LEMMA 3.7.3. *There is a class v' in $H^{\lambda+q+r}(E')$ such that*

$$v' \in \Phi(\rho_0\varepsilon_{E'}) \ , \quad v'i^* = z\rho_1\theta'_1 \ .$$

This is immediate, by applying Axiom 5 (for Φ) to the following anticommutative diagram.



PROOF OF THEOREM 3.7.2. Let v' be as in the lemma. By Lemma 3.6.4, there is an operation Φ' associated with $(d', z\rho_1)$ and such that $v' \in \Phi'(\varepsilon_{E'})$. It is now easy to check that the operation

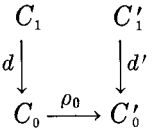
$$\chi(\varepsilon') = \Phi(\rho_0\varepsilon') - [\Phi'(\varepsilon')]$$

satisfies the conditions of Lemma 3.5.1. Therefore

$$\Phi(\rho_0\varepsilon') - [\Phi'(\varepsilon')] = 0 \ .$$

This completes the proof.

Theorem 3.7.2 has a converse, which we state as Lemma 3.7.4. Suppose given the following diagram, in which d and d' are A-maps of the usual kind, while ρ_0 is an A-map of degree r.



Let z, z' be elements of Ker d, Ker d' of total degrees $q, q+r$, and let Φ, Φ' be operations associated with the pairs $(d, z), (d', z')$. Assume the following conditions:-

(i) Wherever $\Phi'(\varepsilon')$ is defined, $\Phi(\rho_0\varepsilon')$ is defined.

(ii) For one value λ of l such that $\lambda > \mathrm{Max}(-\nu', q+r-2\nu')$ we have

$$\Phi(\rho_0\varepsilon') \supset \Phi'(\varepsilon')$$

for each A-map $\varepsilon'\colon C_0 \to H^*(X)$ of degree λ such that $\Phi'(\varepsilon')$ is defined.

Then we have the following conclusions.

LEMMA 3.7.4. *With the above data, there is an A-map $\rho_1\colon C_1 \to C_1'$ (of degree r) such that $\rho_1 d' = (-1)^r d\rho_0$ and $z' = z\rho_1$. Moreover, we have*

$$\Phi(\rho_0\varepsilon') \supset \Phi'(\varepsilon')$$

for A-maps ε' of any degree.

PROOF. Our first step is to deduce from (i) that $\mathrm{Im}(d\rho_0) \subset \mathrm{Im}\, d'$. Let $E' = E_l'$ be a canonical fibering associated with d'; then $d'\varepsilon_{E'} = 0$, and hence $d\rho_0\varepsilon_{E'} = 0$; using Lemma 3.5.2, we see that

$$(\mathrm{Im}\, d\rho_0)_t \subset (\mathrm{Im}\, d')_t$$

if $t < l + 2\nu'$. Since l is arbitrary, we have $\mathrm{Im}(d\rho_0) \subset \mathrm{Im}\, d'$.

We can now construct some map $\rho_1'\colon C_1 \to C_1'$ (of degree r) such that $\rho_1 d' = (-1)^r d\rho_0$. Let $E' = E_\lambda'$ be a canonical fibering associated with d'. By Lemma 3.7.3, there is a class v' in $H^{\lambda+q+r}(E')$ such that

$$v' \in \Phi(\rho_0\varepsilon_{E'}), \quad v'i^* = z\rho_1'\theta_1' .$$

On the other hand, there is a class w' in $H^{\lambda+q+r}(E')$ such that

$$w' \in \Phi'(\varepsilon_{E'}) \subset \Phi(\rho_0\varepsilon_{E'}), \quad w'i^* = z'\theta_1' .$$

By Axiom 2 for Φ, we have an A-map $f\colon C_1 \to H^*(E')$ such that $w' - v' = zf$; thus $(z' - z\rho_1')\theta_1' = zfi^*$. Using Lemma 3.5.2, we can define an A-map $g\colon C_1 \to \mathrm{Ker}\, d'$ such that

$$g\theta_1' | C_{1,t} = fi^* | C_{1,t} \qquad \text{for} \quad t \leqq q+1 .$$

We now have $z' - z\rho_1' = zg$. We can take $\rho_1 = \rho_1' + g$.

It is now clear that $Q^{l+q+r}(z', X) \subset Q^{l+q+r}(z, X)$ for each l, so we may apply Lemma 3.5.1 to the operation

$$\chi(\varepsilon') = \Phi(\rho_0\varepsilon') - [\Phi'(\varepsilon')]$$

and show that it is zero. This completes the proof.

We now give a subsidiary result, which may however serve to justify some of our concepts. We shall suppose that the coefficient group G is a

field, that Φ is an operation associated with a pair (d, z), and that $l > \mathrm{Max}(-\nu, q - 2\nu)$ (with the notations used above); in other words, we shall only prove this result in a stable range of dimensions.

LEMMA 3.7.5. *If the map $d : C_1 \to C_0$ is minimal (in the sense of* § 2.1) *then the l^{th} component Φ^l of Φ is minimal (in the sense of* § 3.2).

PROOF. Let $E = E_l$ be a canonical fibering associated with d. If there is an operation Ψ such that $\Psi \subset \Phi^l$, choose a class v in $\Psi(\varepsilon_E)$; let χ^l be the operation determined by the universal example v; then $\chi^l \subset \Psi \subset \Phi^l$, by Lemma 3.2.3. By Lemma 3.6.4, χ^l is one component of a stable operation χ associated with some pair (d, z'). By Lemma 3.7.4, there is an A-map $\rho_1 : C_1 \to C_1$ such that $\rho_1 d = d$ and $z' = z\rho_1$. Since d is minimal, ρ_1 is an isomorphism. Therefore $Q^{l+q}(z, X) = Q^{l+q}(z', X)$, and $\chi^l = \Psi^l = \Phi^l$. This completes the proof.

3.8. *Outline of applications.* Throughout this section we shall assume that the coefficient group G is a field. Under this condition, we shall give a general scheme for applying the results of §§ 3.6, 3.7. We wish to show, in particular, how homological algebra helps us to find secondary operations to serve given purposes, and to find relations between such operations. For example, if a class x in $H^n(X)$ generates a sub-A-module M of $H^*(X)$, we shall be led to consider operations Φ_r defined on x and in (1-1) correspondence with a base of $\mathrm{Tor}_2^A(G, M)$. The particular formulae used later were found by applying the principles outlined in this section.

Suppose given a sub-A-module M of $H^*(X)$ which is locally finite-dimensional. Suppose that we wish to study the stable secondary operations Φ which are defined on J-tuples $\{x_j\}$ of classes in M. It is equivalent to say that such operations Φ are defined on A-maps $\varepsilon : C_0 \to M$. Each such Φ will be associated with a pair (d, z) such that $d\varepsilon = 0$.

It is sufficient to consider one particular pair of A-maps $d : C_1 \to C_0$ and $\varepsilon : C_0 \to M$ such that

$$C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact. For suppose that $d' : C_1' \to C_0'$ and $\varepsilon' : C_0' \to M$ are some other A-maps such that $d'\varepsilon' = 0$, and let Φ' be an operation corresponding to a pair (d', z'). Then we can form the following diagram.

$$\begin{array}{ccccc} C_1' & \xrightarrow{d'} & C_0' & \xrightarrow{\varepsilon'} & M \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle 1} \\ C_1 & \xrightarrow{d} & C_0 & \xrightarrow{\varepsilon} & M \end{array}$$

By Theorem 3.7.2, we have

$$\Phi'(\varepsilon') = [\Phi(\varepsilon)]$$

for some Φ associated with $(d, z'\rho_1)$.

We may therefore suppose that the A-maps $d : C_1 \to C_0$ and $\varepsilon : C_0 \to M$ considered are the beginning of a minimal resolution, in the sense of § 2.1; let its first few terms be

$$C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M .$$

We may now consider a subset of the operations Φ. Take an A-base of elements $c_{2,r}$ in C_2; set $z_r = c_{2,r}d_2$; let Φ_r be an operation corresponding to (d_1, z_r). It is a property of the operations Φ_r that the other operations Φ are linearly dependent on them, in a suitable sense. To be precise, let Φ be an operation associated with a pair (d_1, z). Since $zd_1 = 0$ and Ker $d_1 = \text{Im } d_2$, we have $z = \sum_r a_r z_r$ for some a_r in A.
By Theorems 3.7.1, 3.6.2 we have

$$[\Phi(\varepsilon)] = [c\varepsilon] + \textstyle\sum_r [a_r \Phi_r(\varepsilon)]$$

(modulo the total indeterminacy involved). It is therefore sufficient to consider the operations $\Phi_r(\varepsilon)$, provided that their indeterminacies are small enough for our purposes.

By § 2.1, the basic operations Φ_r are in (1-1) correspondence with a G-base of $\text{Tor}_2^A(G, M)$. It may happen that we can calculate $\text{Tor}_2^A(G, M)$ without using resolutions; if so, we can count how many basic operations Φ_r are needed.

We have now shown how we may consider a set of basic operations; we proceed to show how we may consider relations between them.

First take an element c_3 of C_3. We may write

$$c_3 d_3 = \textstyle\sum_r a_r c_{2,r} \qquad \text{(where } a_r \in A) .$$

Applying d_2, we have $\sum_r a_r z_r = 0$. By Theorems 3.7.1, 3.6.2 we have $\sum_r [a_r \Phi_r(\varepsilon)] = [c\varepsilon]$ (modulo the total indeterminacy involved). Now let c_3 run over an A-base of C_3; we obtain basic relations between the Φ_r, in (1-1) correspondence with a base of $\text{Tor}_3^A(G, M)$. As before, it may happen that we can calculate $\text{Tor}_3^A(G, M)$ without using resolutions. In this case we can count how many basic relations are available.

We will now consider a slightly different application, which concerns composite operations of the form Φa. Suppose that M, M' are (locally finite-dimensional) submodules of $H^*(X)$ such that $M' \subset M$. We can take minimal resolutions of M, M' and form the following diagram.

$$\begin{array}{ccccccc} C_2' & \xrightarrow{d_2'} & C_1' & \xrightarrow{d_1'} & C_0' & \xrightarrow{\varepsilon'} & M' \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle i} \\ C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\varepsilon} & M \end{array}$$

Let Ψ' be a basic operation corresponding to a pair $(d_1', c_2'd_2')$; we seek to evaluate $\Psi'(\varepsilon')$ in terms of basic operations $\Phi_r(\varepsilon)$. For example, if M is generated (*qua* A-module) by one generator x, and if M' is generated by ax (where $a \in A$), then the problem is equivalent to evaluating $\Psi'(ax)$ in terms of operations $\Phi_r(x)$.

Let us write

$$c_2'\rho_2 = \sum_r b_r c_{2,r} \qquad \text{(where } b_r \in A\text{)} .$$

Then we have

$$c_2'd_2'\rho_1 = \sum_r b_r z_r .$$

By Theorems 3.6.2, 3.7.1, 3.7.2 we have

$$[\Psi'(\varepsilon')] = [c\varepsilon] + \sum_r [b_r \Phi_r(\varepsilon)]$$

(modulo the total indeterminacy involved).

It may happen that this formula is useful to us only if the coefficients b_r are of positive degree. To locate the Ψ' which admit a formula of this sort, one should find the kernel of

$$i_*: \operatorname{Tor}_2^A(G, M') \to \operatorname{Tor}_2^A(G, M);$$

for the coefficients b_r of degree zero in $c_2'\rho_2 = \sum_r b_r c_{2,r}$ are determined by i_*.

This concludes our outline of the use of homological algebra in searching for operations and relations to serve given purposes.

3.9. *The Cartan formula.* Throughout this section, we shall assume that the coefficient domain G is the field Z_p of integers modulo p, where p is a prime. Under this condition, we shall prove the existence of a Cartan formula [11] for expanding $\Phi(xy)$, where xy is a cup-product and Φ is an operation of the sort considered in § 3.6. The expansion which we obtain is of the form

$$\sum_r (-1)^{\eta(r)} \Phi_r'(x) \Phi_r''(y) .$$

where the signs are given by

$$\eta(r) = \deg(x) \deg(\Phi_r'') .$$

We can give one elementary example of the sort of Cartan formula at issue; for the Bockstein coboundary β_{p^f} is a stable operation, and is a secondary operation if $f = 2$; it satisfies the formula

$$\beta_{p^r}(xy) \supset (\beta_{p^r}x)y + (-1)^{\eta}x(\beta_{p^r}y)$$

where $\eta = \deg(x)$.

The precise result we require is stated as Theorem 3.9.4; the remainder of this section is devoted to proving it. The proof uses the method of the universal example. The obvious universal example for this purpose is a Cartesian product, as considered by Serre [30]. However, we are particularly concerned with stable operations; we must therefore show how our Cartan formulae behave under suspension. For this purpose we use a "product" more conveniently related to the suspension. Let Y', Y'' be enumerable CW-complexes with base-points y_0', y_0''; we may form the "reduced product" [18]

$$Y' \times Y'' = Y' \times Y''/(Y' \times y_0'' \cup y_0' \times Y'') .$$

This is again an enumerable CW-complex. The quotient map $q\colon Y' \times Y'' \to Y' \times Y''$ induces a homomorphism

$$q^+\colon H^+(Y' \times Y'') \to H^+(Y' \times Y'') ,$$

and q^+ embeds $H^+(Y' \times Y'')$ as a direct summand of $H^+(Y' \times Y'')$, complementary to $H^+(Y')$ and $H^+(Y'')$. If $y' \in H^+(Y')$ and $y'' \in H^+(Y'')$, we have the "external" cup-product $y' \times y''$ in $H^+(Y' \times Y'')$ and a "reduced" cup-product $y' \times y''$ in $H^+(Y' \times Y'')$ defined by

$$(y' \times y'')q^+ = y' \times y'' .$$

We now set up some more notation. If K, L are subsets of $H^q(X)$, $H^r(X)$ then we define KL to be the set of cup-products kl, where $k \in K$, $l \in L$; thus $KL \subset H^{q+r}(X)$. If K, L are subsets of $H^q(X)$ and λ, μ lie in Z_p, then we define $\lambda K + \mu L$ to be the set of linear combinations $\lambda k + \mu l$, where $k \in K$, $l \in L$. These definitions give a precise sense to formulae such as

$$\sum_r (-1)^{\eta(r)}\Phi_r'(x)\Phi_r''(y);$$

an expression of this sort denotes some set of cohomology classes.

If K, L are subsets of $H^q(Y')$, $H^q(Y'')$ then we define the subset $K \times L$ of $H^{q+r}(Y' \times Y'')$ in a similar way.

We will now use the reduced product to study Cartan formulae valid in a fixed pair of dimensions. We will suppose that each of S, S', S'' is a natural subset of cohomology, in one variable, whose dimension is l, l', l'' in the three cases. Let R be a finite set of indices r; let Φ, Φ_r', Φ_r'' be operations defined on S, S', S'', and of degrees (say) q, q_r', q_r''. We will suppose that

$$l' + l'' = l , \quad q_r' + q_r'' = q \qquad \text{(for each } r\text{)}$$

and that our operations have arguments and values of positive degree.

We will say that Φ *can be expanded on* S', S'' *in terms of* Φ'_r, Φ''_r if the following two conditions hold. First, for each space X and each $x' \in S'(X)$, $x'' \in S''(X)$ we have

(i) $x'x'' \in S(X)$, and $\Phi(x'x'')$ has a non-empty intersection with

$$\sum_r (-1)^{\eta(r)} \Phi'_r(x') \Phi''_r(x'')$$

where $\eta(r) = l'q''_r$.

Secondly, whenever Y', Y'' are enumerable CW-complexes and $y' \in S'(Y')$, $y'' \in S''(Y'')$ we have

(ii) $y' \times y'' \in S(Y' \times Y'')$, and $\Phi(y' \times y'')$ has a non-empty intersection with

$$\sum_r (-1)^{\eta(r)} \Phi'_r(y') \times \Phi''_r(y'') .$$

For our first lemma, we suppose that we can choose enumerable CW-complexes Y', Y'' and classes $y' \in H^+(Y')$, $y'' \in H^+(Y'')$ so that (Y', y'), (Y'', y'') are universal examples for S', S''. (We can certainly do this if the subsets S', S'' are determined by K-tuples of primary operations, since we can replace the canonical fiberings of § 3.3 by weakly equivalent enumerable CW-complexes.)

LEMMA 3.9.1. *If condition* (ii) *above holds for one such pair of universal examples* (Y', y') *and* (Y'', y''), *then conditions* (i) *and* (ii) *hold in general, so that* Φ *can be expanded on* S', S'' *in terms of* Φ'_r, Φ''_r.

This lemma is immediate, by naturality.

For our next lemma, we suppose that not only the subsets S', S'' but also the subsets $(S')^s$, $(S'')^s$ admit universal examples which are enumerable CW-complexes.

LEMMA 3.9.2. *If* Φ *can be expanded on* S', S'' *in terms of* Φ'_r, Φ''_r *then* Φ^s *can be expanded on* $(S')^s$, S'' *in terms of* $(\Phi'_r)^s$, Φ''_r *and on* S', $(S'')^s$ *in terms of* Φ'_r, $(\Phi''_r)^s$.

PROOF. It is sufficient to prove that half of the lemma which relates to $(S')^s$, S'', since the other half may be proved similarly,

If Y is a CW-complex, we may interpret the suspension sY to be the reduced product $S^1 \times Y$, since this is homotopy-equivalent to the ordinary suspension in this case. Suppose that $y \in H^n(Y)$ $(n > 0)$; let s^1 be the generator of $H^1(S^1)$; and let s be the suspension isomorphism. Then we have the equation

$$ys^{-1} = (-1)^n s^1 \times y .$$

If Y', Y'' are enumerable CW-complexes, we have the "associativity" formula

$$(S^1 \times Y') \times Y'' = S^1 \times (Y' \times Y'') .$$

In $H^+(S^1 \times Y' \times Y'')$ we have

$$(y' \times y'')s^{-1} = (-1)^{n''}(y's^{-1}) \times y''$$

(where $n'' = \deg(y'')$); this follows easily from the equation above and the associativity of the cup-product.

Now let $(Y', y'), (Y'', y'')$ be universal examples for $(S')^s$, S''. Since $(-1)^{t''}y' \in (S')^s(Y')$, we have $(-1)^{t''}y's^{-1} \in S'(S^1 \times Y')$; also $y'' \in S''(Y'')$. If Φ can be expanded on S', S'', we have

$$(-1)^{t''}(y's^{-1}) \times y'' \in S(S^1 \times Y' \times Y'') ,$$

whence $(y' \times y'')s^{-1} \in S(S^1 \times Y' \times Y'')$ and $y' \times y'' \in S^s(Y' \times Y'')$.

Again, if Φ can be expanded on S', S'', we have

$$\Phi\big((-1)^{t''}(y's^{-1}) \times y''\big) \cap \textstyle\sum_r (-1)^{\eta(r)}\Phi'_r\big((-1)^{t''}y's^{-1}\big) \times \Phi''_r(y'') \neq 0 ,$$

where $\eta(r) = l'q''_r$. In this expression we have

$$\Phi'_r\big((-1)^{t''}y's^{-1}\big) = \big((\Phi'_r)^s\big((-1)^{t''}y'\big)\big)s^{-1} = (-1)^{t''}\big((\Phi'_r)^s(y')\big)s^{-1} ,$$

by Lemma 3.4.2. A little manipulation now shows that

$$\Phi^s(y' \times y'') \cap \textstyle\sum_r (-1)^{\zeta(r)}(\Phi'_r)^s(y') \times \Phi''_r(y'') \neq 0 ,$$

where $\zeta(r) = (l' - 1)q''_r$. By Lemma 3.9.1, Φ^s can be expanded on $(S')^s$, S'' in terms of $(\Phi'_r)^s$, Φ''_r. This completes the proof.

It will be convenient if we now set up the data for Theorem 3.9.4. Let C_0 be a free A-module on one generator c_0 of degree zero. We suppose given three A-maps $d\colon C_1 \to C_0$, $d'\colon C'_1 \to C'_0$ and $d''\colon C''_1 \to C''_0$, as in § 3.6. We shall suppose that $C'_{1,q+1} = 0$ for $q < 0$ and that $d' \mid C'_{1,1}\colon C'_{1,1} \to C'_{0,1}$ is monomorphic; this assumption is automatically satisfied if d' is minimal. The result of this assumption about d' is that if E'_l is a canonical fibering associated with d', as in § 3.6, then E'_l is $(l-1)$-connected and $H^l(E'_l) = Z_p$, generated by the "fundamental class" e^l. We make the corresponding assumption about d''.

Corresponding to d, d', d'' we have stable natural subsets T, T', T'' in one variable. We now make an essential assumption restricting d, d', d''; we suppose that for each space X and for each $x' \in T'(X)$, $x'' \in T''(X)$ the cup-product $x'x''$ lies in $T(X)$.

LEMMA 3.9.3.

(a) *If Y', Y'' are enumerable* CW*-complexes and $y' \in T'(Y')$, $y'' \in T''(Y'')$ then $y' \times y'' \in T(Y' \times Y'')$.*

(b) *If X is any space, then*

$$T'(X) \subset T(X) , \quad T''(X) \subset T(X) .$$

PROOF OF PART (a). By our assumptions, the external cup-product $y' \times y''$ lies in $T(Y' \times Y'')$; and if a_k is a primary operation such that

$a_k(y' \times y'') = 0$, then $a_k(y' \times y'') = 0$, since q^+ is a monomorphism.

PROOF OF PART (b). It is sufficient to prove one inclusion, say the second. If $l > 0$, let (Y'', y'') be a universal example for $(T'')^l$ such that Y'' is an enumerable CW-complex; then $y'' \in (T'')^l(Y'')$. The element $(-1)^l s^1$ in $H^1(S^1)$ certainly lies in $(T')^1(S^1)$, since $a(-1)^l s^1 = 0$ for each primary operation a of positive degree. Therefore $(-1)^l s^1 \times y'' \in T'^{+1}(S^1 \times Y'')$; since $y''s^{-1} = (-1)^l s^1 \times y''$ and T is stable, we have $y'' \in T'^l(Y'')$. The inclusion $(T'')^l(X) \subset T'^l(X)$ for a general space X now follows by naturality.

We will now state Theorem 3.9.4. This theorem will allow us to expand $\Phi(x'x'')$ in the form

$$\Phi'_\alpha(x')x'' + \sum_{r \neq \alpha, \omega} (-1)^{\eta(r)} \Phi'_r(x') \Phi''_r(x'') + (-1)^{\eta(\omega)} x' \Phi''_\omega(x'')$$

whenever $x' \in T'(X)$ and $x'' \in T''(X)$. It will also give us some information about the operations Φ'_α, Φ''_ω which occur in the end terms of this expansion; they are essentially the same as Φ.

THEOREM 3.9.4. *If d, d', d'', T, T' and T'' are as above, and if Φ is an operation of degree $q > 0$ associated with a pair (d, z), then there are operations Φ'_r, Φ''_r which are associated with pairs (d', z'_r), (d'', z''_r) and satisfy the following conditions.*

(i) *For each pair of dimensions (l, m), the component Φ^{l+m} can be expanded on $(T')^l$, $(T'')^m$ in terms of $(\Phi'_r)^l$, $(\Phi''_r)^m$.*

(ii) *There are two values α, ω of r such that Φ''_α, Φ'_ω are identity operations. For $r \neq \alpha, \omega$ we have*

$$0 < \deg(\Phi'_r) < q \ , \quad 0 < \deg(\Phi''_r) < q \ .$$

(iii) *There is an A-map $\rho'_1 \colon C_1 \to C'_1$ such that $\rho'_1 d' = d$ and $z\rho'_1 = z'_\alpha$. For each space X and each $x' \in T'(X)$ we have*

$$\Phi'_\alpha(x') \subset \Phi(x') \ .$$

(iv) *There is an A-map $\rho''_1 \colon C_1 \to C''_1$ such that $\rho''_1 d'' = d$ and $z\rho''_1 = z''_\omega$. For each space X and each $x'' \in T''(X)$ we have*

$$\Phi''_\omega(x'') \subset \Phi(x'') \ .$$

We require a further lemma, which is a converse of Lemma 3.9.2. We shall suppose that Φ, Φ'_r, Φ''_r are stable operations associated with pairs (d, z), (d', z'_r), (d'', z''_r); we set $q = \deg(\Phi)$.

LEMMA 3.9.5. *If Φ^{l+m} can be expanded on $(T')^l$, $(T'')^m$ in terms of $(\Phi'_r)^l$, $(\Phi''_r)^m$ and if $l > q$, $m > q$ then Φ^{l+m+1} can be expanded on $(T')^{l+1}$, $(T'')^m$ in terms of $(\Phi'_r)^{l+1}$, $(\Phi''_r)^m$ and on $(T')^l$, $(T'')^{m+1}$ in terms of $(\Phi'_r)^l$, $(\Phi''_r)^{m+1}$.*

Roughly speaking, the effect of Lemmas 3.9.2 and 3.9.5 is that in order to prove Theorem 3.9.4, it is sufficient to consider a single pair of dimensions (l, m).

PROOF OF LEMMA 3.9.5. As for Lemma 3.9.2, it is sufficient to prove that half of the lemma which passes from dimensions (l, m) to dimensions $(l+1, m)$. Let (Y', y'), (Y'', y'') be universal examples for $(T')^{l+1}$, $(T'')^m$ such that Y', Y'' are enumerable CW-complexes. As in Lemma 3.5.1, we can take a space X and a map $g : sX \to Y'$ such that

$$g^*: H^r(Y') \to H^r(sX)$$

is an isomorphism for $r \leqq l + q + 1$; we may suppose that X is an enumerable CW-complex, and that sX is the reduced product $S^1 \times X$. We can now form the map

$$g \times 1: S^1 \times X \times Y'' \to Y' \times Y'';$$

this induces isomorphisms of cohomology up to dimension $l + m + q + 1$ at least. It is now easy to check that

$$\Phi^{l+m+1}(y' \times y'') \cap \textstyle\sum_r (-1)^{\eta(r)} (\Phi'_r)^{l+1}(y') \times (\Phi''_r)^m(y'') \neq 0$$

by applying $(g \times 1)^+$ and using the data. The conclusion now follows by Lemma 3.9.1. This completes the proof.

PROOF OF THEOREM 3.9.4. We begin by fixing attention on a pair of dimensions (l, m) such that $l > q$, $m > q$. Let E'_l, E''_m be canonical fiberings associated with d', d'', as in § 3.6. We may take weakly equivalent enumerable CW-complexes Y', Y''; we write y', y'' for the classes corresponding to the fundamental classes e', e''. Since $H^*(Y')$, $H^*(Y'')$ are locally finite-dimensional, the reduced cup-product

$$\mu : H^+(Y') \otimes H^+(Y'') \to H^+(Y' \times Y'')$$

is an isomorphism. The coset $\Phi(y' \times y'')$ is defined; we may choose a class v in $\Phi(y' \times y'')$, and expand v in the form

$$v = \textstyle\sum_r (-1)^{\eta(r)} v'_r \times v''_r$$

where $v'_r \in H^+(Y')$, $v''_r \in H^+(Y'')$ and $\eta(r) = l(\deg(v''_r) - m)$. By our original data, Y' is $(l-1)$-connected and $H^l(Y') = Z_p$, generated by y'; similarly for Y''. We may therefore take our expansion so that $\deg(v'_r) > l$ and $\deg(v''_r) > m$, except that $v'_\omega = y'$ and $v''_\alpha = y''$. By Lemma 3.6.4, there are operations Φ'_r (for $r \neq \omega$), Φ''_r (for $r \neq \alpha$) which are associated with pairs (d', z'_r), (d'', z''_r) and are such that v'_r, v''_r are universal examples for $(\Phi'_r)^l$, $(\Phi''_r)^m$. We define Φ'_ω and Φ''_α directly, defining them to be identity operations. By Lemma 3.9.1, Φ^{l+m} can be expanded on $(T')^l$, $(T'')^m$

in terms of $(\Phi'_r)^l$, $(\Phi''_r)^m$. By Lemmas 3.9.2 and 3.9.5, a similar conclusion follows in every pair of dimensions. This establishes parts (i) and (ii) of the theorem.

We will now examine the class v''_ω which occurs in the above expansion. Let us take $W = S^l$, so that $H^l(W) = Z_p$, generated by w. We may take a map $f\colon W \to Y$ such that $y'f^+ = (-1)^{lm}w$; thus $f^+\colon H^r(Y') \to H^r(W)$ is zero for $r > l$. We have

$$\sum_r (-1)^{\eta(r)} v'_r \times v''_r \in \Phi^{l+m}(y' \times y'');$$

applying $(f \times 1)^+$, we find

$$(-1)^{l(q+m)} w \times v''_\omega \in \Phi^{l+m}\big((-1)^{lm} w \times y''\big)$$

in $W \times Y''$. But $W \times Y''$ is the l-fold suspension of Y'', and we have the equation

$$ys^{-l} = (-1)^{lt} w \times y \qquad (\text{for } y \in H^t(Y'')).$$

Since Φ is stable, we deduce $v''_\omega \in \Phi^m(y'')$. Lemma 3.2.3 now shows that

$$(\Phi''_\omega)^m(x'') \subset \Phi^m(x'') \qquad (\text{for } x'' \in (T'')^m(X)).$$

Lemma 3.7.4 now shows the existence of an A-map $\rho''_1\colon C_1 \to C''_1$ such that $\rho''_1 d'' = d$ and $z\rho''_1 = z''_\omega$; it also guarantees that

$$\Phi''_\omega(x'') \subset \Phi(x'')$$

for classes x'' of any degree. This establishes part (iv) of the theorem; we may establish part (iii) similarly. The proof of Theorem 3.9.4 is complete.

Chapter 4. Particular Operations

4.1. *Introduction.* In this chapter we shall use the theory of Chapter 3 to define and study a particular set of secondary cohomology operations $\Phi_{i,j}$. These operations act on cohomology with mod 2 coefficients; they will be defined in § 4.2. The object of our work is to prove the formula

$$\sum_{i,j} a_{i,j,k} \Phi_{i,j}(u) = [\mathrm{Sq}^{2^{k+1}}(u)]$$

mentioned in Chapter 1; this formula is proved in § 4.6. The line of proof is as follows. We first apply the theory of Chapter 3 to prove a formula

$$\sum_{i,j} a_{i,j,k} \Phi_{i,j}(u) = [\lambda \mathrm{Sq}^{2^{k+1}}(u)]$$

containing an undetermined coefficient λ. We then determine the coefficient λ by applying the formula to a suitable class u in the cohomology of a suitable space. For this purpose we use complex projective space of infinitely-many dimensions, which we shall call P. We therefore need to

know the values of the operations $\Phi_{i,j}$ in P; these are found in § 4.5. It turns out that an inductive calculation is possible; there are many relations between the different operations in P, and these enable us to deduce, from the value of one selected operation, the values of all the others. In § 4.4 we find the value of this one operation. In § 4.3 we apply the theory of Chapter 3 to prove those relations between the operations which are needed for the calculation in § 4.5. This work, therefore, will complete the proof of Theorem 1.1.1.

In this chapter we have to use the Steenrod squares Sq^k for values of k which may have a complicated form. We therefore make a convention, by which we write $\mathrm{Sq}(k)$ instead of Sq^k in such cases. Similarly, we may write $\xi_i(k)$ instead of ξ_i^k in dealing with A^* (see § 2.4). Again, we write $H^m(X)$ instead of $H^m(X; Z_2)$, and $H^*(X)$ instead of $H^*(X; Z_2)$, since we shall not have to deal with any coefficients except Z_2.

4.2. *The operations* $\Psi(u)$ *and* $\Phi_{i,j}(u)$. In this section we apply the theory of Chapter 3 to define certain particular secondary operations, acting on cohomology with mod 2 coefficients. These will be operations on one variable.

To define our first operation, we have to give a pair (d, z) (see § 3.1). We take C_0 to be A-free on one generator c of degree zero; we take C_1 to be A-free on three generators c_1, c_3, c_4 of degrees 1, 3, 4. We define d by

$$c_i d = \mathrm{Sq}^i c \qquad (i = 1, 3, 4).$$

We define z by

$$z = \mathrm{Sq}^4 c_1 + \mathrm{Sq}^2 c_3 + \mathrm{Sq}^1 c_4 .$$

This pair (d, z) corresponds, of course, to the relation

$$\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^1\mathrm{Sq}^4 = 0 .$$

We note that $(C_0/dC_1)_4 = 0$; so by Theorems 3.6.1, 3.6.2, there is a unique operation Ψ (of degree 4) associated with this pair (d, z). This is the first operation we require.

To define further operations, we introduce further pairs (d, z). We begin by constructing the first terms

$$C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} Z_2$$

of a minimal resolution (see § 2.1) of Z_2 over A. We may do this as follows. We take C_0 to be A-free on one generator c of degree zero, and define $c\varepsilon = 1$. We take C_1 to be A-free on generators c_i of degrees 2^i, for $i = 0, 1, 2, \cdots$. We define d by setting

$$c_i d = \mathrm{Sq}(2^i)c \ .$$

It is clear that $\mathrm{Im}(d) = \mathrm{Ker}(\varepsilon)$, since the elements $\mathrm{Sq}(2^i)$ are multiplicative generators for A (see [4]). It is also easy to show that d is minimal. In fact, to do this, we should take an element $x = \lambda c_i + \sum_{j<i} a_j c_j$ of degree 2^i in C_1, assume $xd=0$, and deduce that $\lambda=0$. This is immediate from the equation

$$\xi_1(2^i)((\lambda c_i + \textstyle\sum_{j<i} a_j c_j)d) = \lambda \ .$$

Next, we make use of the epimorphism θ: $\mathrm{Ker}(d) \to \mathrm{Tor}_2^{A}(Z_2, Z_2)$ introduced in § 2.2. It was shown in § 2.5 that the elements $h_i h_j$ (with $0 \leqq i \leqq j$, $j \neq i+1$) in $\mathrm{Ext}_A^{2}(Z_2, Z_2)$ form a base for it. For $0 \leqq i \leqq j$, $j \neq i+1$, then, we may take cycles $z_{i,j}$ (of degree $2^i + 2^j$) in $\mathrm{Ker}(d)$ such that

$$(h_i h_j)(\theta z_{i,j}) = 1 \ .$$

Let $C_1(j)$ be the submodule of C_1 generated by $c_0, c_1, \cdots, c_j$. Then the cycle $z_{i,j}$ lies in $C_1(j)$. This is clear from the degrees if $i < j$; if $i = j$, it follows by using also the fact that d is minimal. We set $d(j) = d \mid C_1(j)$.

By Theorem 3.6.1, once $z_{i,j}$ is chosen, there is an operation $\Phi_{i,j}$ (of degree $2^i + 2^j - 1$) associated with the pair $(d(j), z_{i,j})$ (where $0 \leqq i \leqq j$, $j \neq i+1$). Such an operation is unique, by Theorem 3.6.2, since $(C_0/dC_1(j))_n = 0$ if $n < 2^{j+1}$. These operations $\Phi_{i,j}$ are the ones which we require. They are defined on classes u such that

$$\mathrm{Sq}(2^r)(u) = 0 \qquad \text{for } 0 \leqq r \leqq j \ .$$

The indeterminacy of $\Phi_{i,j}$ may depend on the choice of the cycle $z_{i,j}$, and *a fortiori* the operation $\Phi_{i,j}$ may do so. However, all the propositions that we shall state about the operations $\Phi_{i,j}$ remain equally true, whatever choice of the $z_{i,j}$ is made. We shall therefore only need to suppose that the $z_{i,j}$ are chosen in some fixed fashion.

For completeness, we should perhaps consider the operation $\Phi'_{i,j}$ associated with the pair $(d(k), z_{i,j})$ for some $k > j$. It has the same indeterminacy as $\Phi_{i,j}$, but is defined on fewer classes u. Moreover, by Theorem 3.7.2, we have

$$\Phi'_{i,j}(u) = \Phi_{i,j}(u)$$

whenever $\Phi'_{i,j}(u)$ is defined. Thus, in what follows, we shall not need to distinguish $\Phi'_{i,j}$ from $\Phi_{i,j}$ by a separate symbol.

It may be of interest to display a particular relation, holding in A, which corresponds to a cycle which one might choose for $z_{i,j}$. We consider first the case $i < j$, so that $i \leqq j-2$. Then the Adem relations [4] for $\mathrm{Sq}(2^i)\,\mathrm{Sq}(2^j)$ and $\mathrm{Sq}(2^{i+1})\,\mathrm{Sq}(2^j - 2^i)$ both contain the term $\mathrm{Sq}(2^i + 2^j)$. Their sum is therefore an equation of the form

$$\mathrm{Sq}(2^i)\mathrm{Sq}(2^j) = \sum_{0<k<2^j} \lambda_k \mathrm{Sq}(2^i + 2^j - k)\mathrm{Sq}(k)$$

with certain coefficients λ_k. We may use Adem's method [4] to express $\mathrm{Sq}(k)$ in the form

$$\mathrm{Sq}(k) = \sum_{0 \leqq l < j} a_{k,l} \mathrm{Sq}(2^l)$$

with certain coefficients $a_{k,l}$ in A. Substituting, we obtain an equation of the form

$$\mathrm{Sq}(2^i)\mathrm{Sq}(2^j) = \sum_{0 \leqq l < j} b_l \mathrm{Sq}(2^l)$$

with certain coefficients b_l in A. Hence the expression

$$z_{i,j} = \mathrm{Sq}(2^i)c_j + \sum_{0 \leqq l < j} b_l c_l$$

is a cycle in $C_1(j)$. The fact that it satisfies the equation

$$(h_i h_j)(\theta z_{i,j}) = 1$$

follows from Lemma 2.2.2.

The case $i = j$ may be treated similarly, but even more simply, using the Adem relation for $\mathrm{Sq}(2^j)\mathrm{Sq}(2^j)$ (cf. [4]).

It may be remarked that the above process allows us to choose the cycles $z_{i,j}$ in a way which is quite definite, if this should be required. In fact, we have only to remark that Adem's method for reducing $\mathrm{Sq}(k)$ to a sum of products of the generators $\mathrm{Sq}(2^r)$ leads to a well-determined answer. And this is clear, since it proceeds by a well-determined reductive process, using at each step a well-determined substitution.

We conclude by remarking that the operations $\Phi_{0,0}$, $\Phi_{1,1}$ and $\Phi_{0,2}$ do not depend on the choice of the cycles $z_{0,0}$, $z_{1,1}$ and $z_{0,2}$. In fact, it is easy to see that there is only one choice for the cycle $z_{0,0}$, namely $\mathrm{Sq}^1 c_0$. There are only two choices for the cycle $z_{1,1}$, namely

$$(\mathrm{Sq}^2 c_1 + \mathrm{Sq}^3 c_0) + \lambda \mathrm{Sq}^2(\mathrm{Sq}^1 c_0) \qquad (\lambda = 0, 1)\,.$$

These two cycles can be mapped into one another by an automorphism $\rho_1 : C_1(1) \to C_1(1)$ defined as follows:

$$c_0\rho_1 = c_0$$
$$c_1\rho_1 = c_1 + \mathrm{Sq}^1 c_0\,.$$

By Theorem 3.7.2, the operations corresponding to the two cycles coincide.

Similarly, there are only four choices for the cycle $z_{0,2}$, namely

$$(\mathrm{Sq}^1 c_2 + \mathrm{Sq}^2\mathrm{Sq}^1 c_1 + \mathrm{Sq}^4 c_0) + \lambda \mathrm{Sq}^1(\mathrm{Sq}^2 c_1 + \mathrm{Sq}^3 c_0) + \mu \mathrm{Sq}^3(\mathrm{Sq}^1 c_0)$$
$$(\lambda, \mu \in Z_2).$$

As above, the choice does not affect $\Phi_{0,2}$.

4.3. *Relations between the operations* Ψ *and* $\Phi_{i,j}$. In this section we shall obtain those relations between the operations Ψ and $\Phi_{i,j}$ which we need in § 4.5.

For our first lemma, let $u \in H^m(X)\,(m>0)$ be a class such that $\mathrm{Sq}^1(u)=0$, $\mathrm{Sq}^3(u)=0$, $\mathrm{Sq}^4(u)=0$.

LEMMA 4.3.1. *There is a formula*

$$\Phi_{0,2}\mathrm{Sq}^4\mathrm{Sq}^2(u) = [\mathrm{Sq}^6\Psi(u) + \lambda\mathrm{Sq}^{10}(u)]$$

valid in

$$H^{m+10}(X)/\,(\mathrm{Sq}^1H^{m+9}(X) + \mathrm{Sq}^2\mathrm{Sq}^1H^{m+7}(X) + \mathrm{Sq}^4H^{m+6}(X))$$

for a fixed $\lambda \in Z_2$.

We require this formula in order to apply it to the fundamental class in complex projective space. The actual value of λ is not relevant, although at a later stage in our calculations it would be possible to show that $\lambda = 0$.

PROOF. Informally, the proof consists in showing that the relations.

$$(\mathrm{Sq}^1\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^1)\mathrm{Sq}^4\mathrm{Sq}^2 = 0\ ,$$

$$\mathrm{Sq}^6(\mathrm{Sq}^1\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^4\mathrm{Sq}^1) = 0$$

are the "same". Formally, we shall obtain this lemma as an application of Theorem 3.7.2, and we use the notation of that Theorem. In particular, we take $d\colon C_1 \to C_0$ to be the map $d(2)$, as used to define $\Phi_{0,2}$ in § 4.2; thus, we have

$$c_i d = \mathrm{Sq}(2^i)c \qquad (i = 0, 1, 2)\ .$$

Again, we take $d'\colon C_1' \to C_0'$ to be the d used in defining Ψ. That is, we take

$$c_i' d' = \mathrm{Sq}^i c' \qquad (i = 1, 3, 4)\ .$$

We define an A-linear map $\rho_0\colon C_0 \to C_0'$(of degree 6) by $c\rho_0 = \mathrm{Sq}^4\mathrm{Sq}^2c'$. To apply Theorem 3.7.2, we have to construct a map $\rho_1\colon C_1 \to C_1'$ such that $\rho_1 d' = d\rho_0$. It is sufficient to take

$$\begin{aligned} c_0\rho_1 &= \mathrm{Sq}^4c_3' \\ c_1\rho_1 &= \mathrm{Sq}^7c_1' + \mathrm{Sq}^4c_4' \\ c_2\rho_1 &= \mathrm{Sq}^6\mathrm{Sq}^3c_1' + (\mathrm{Sq}^7 + \mathrm{Sq}^6\mathrm{Sq}^1)c_3'\ . \end{aligned}$$

Taking

$$z = \mathrm{Sq}^4c_0 + \mathrm{Sq}^2\mathrm{Sq}^1c_1 + \mathrm{Sq}^1c_2$$

we find

$$z\rho_1 = \mathrm{Sq}^6(\mathrm{Sq}^4 c_1' + \mathrm{Sq}^2 c_3' + \mathrm{Sq}^1 c_4') \ .$$

Theorems 3.7.2, 3.6.2 now yield the conclusion, since $(\mathrm{Coker}\ d')_{10} = Z_2$, generated by the image of $\mathrm{Sq}^{10}c'$.

For our next lemma, let

$$C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} Z_2$$

be the first part of a minimal resolution of z_2 over A, as constructed in § 4.2. Let $z \in C_1$ be such that $zd = 0$ and $\deg(z) \leqq 2^k$; thus $z \in C_1(k-1)$. Let χ be an operation associated with the pair $(d(k-1), z)$; thus $\deg(\chi) < 2^k$. Let u be a class such that $\mathrm{Sq}(2^r)u = 0$ for $0 \leqq r < k$. Then we have the following conclusion.

LEMMA 4.3.2. *If $\Phi_{i,j}(u)$ has zero indeterminacy, and is zero, for each pair (i, j) with $0 \leqq i \leqq j < k$, $j \neq i+1$, then $\chi(u)$ has zero indeterminacy and is zero.*

PROOF. We first define C_2 to be A-free on generators $c_{i,j}$ of degree $2^i + 2^j$ with $c_{i,j}d = z_{i,j}$; then by Lemma 2.2.1, the terms

$$C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} Z_2$$

form part of a minimal resolution of Z_2 over A. Since $z \in C_1$ and $zd = 0$, we have

$$z = (\textstyle\sum_{i,j} a_{i,j} c_{i,j})d \ .$$

By considering degrees, all the terms in this sum have $j < k$. Thus we have

$$z = \textstyle\sum_{i,j;\, j<k} a_{i,j} z_{i,j} \ .$$

We may apply Theorems 3.7.1, 3.6.2; since $(\mathrm{Coker}\ d(k-1))_n = 0$ if $n < 2^k$, we find

$$\begin{aligned} [\chi(u)] &= \textstyle\sum_{i,j;\, j<k} a_{i,j} \Phi_{i,j}(u) \\ &= 0 \pmod{\text{zero}} \ . \end{aligned}$$

This proves the lemma.

For the next lemma, we take an integer $k \geqq 2$ and suppose that $u \in H^m(X)(m > 0)$ is a class such that $\mathrm{Sq}(2^r)u = 0$ for $0 \leqq r \leqq k$ and $\mathrm{Sq}(2^k)\mathrm{Sq}(2^{k+1})(u) = 0$.

LEMMA 4.3.3. *There is a formula*

$$\begin{aligned} \Phi_{0,k+1}\mathrm{Sq}(2^{k+1})(u) = &\textstyle\sum_{\substack{0 \leqq i \leqq j \leqq k \\ j \neq i+1}} a_{i,j,k} \Phi_{i,j}(u) \\ &+ \textstyle\sum_{\substack{0 \leqq i \leqq k \\ i \neq k-1}} a_{i,k} \Phi_{i,k} \mathrm{Sq}(2^{k+1})(u) + \lambda_k \mathrm{Sq}(2^{k+2})(u) \end{aligned}$$

in which $a_{i,j,k} \in A$, $a_{i,k} \in A$ *and* $\lambda_k \in Z_2$. *It holds modulo the total indeterminacy of both sides. The coefficient* $a_{0,k,k}$ *of* $\Phi_{0,k}$ *satisfies*

$$(\xi_1(3\cdot 2^k) + \xi_2(2^k))a_{0,k,k} = 1 .$$

We require this formula in order to apply it to a power y^{2^k} of the fundamental class y in complex projective space. The actual value of λ_k is not relevant, although at a later stage in our calculations it would be possible to show that $\lambda_k = 0$.

This lemma should be considered as strictly analogous to Lemma 4.3.1; it is another application of the theorems of § 3.7. The only difference is that we do not propose to carry out the calculations explicitly.

We begin by constructing a partial minimal resolution

$$C_2'' \xrightarrow{d_2''} C_1'' \xrightarrow{d_1''} C_0'' \xrightarrow{\varepsilon''} M .$$

Here M is the module of Theorem 2.6.2, except that the integer k of that theorem is replaced by $(k+1)$. Thus, M is a module whose Z_2-base consists of three elements m, $\mathrm{Sq}(2^{k+1})m$, $\mathrm{Sq}(2^{k+2})m$.

We take C_0'' to be free on one generator c'', and define ε'' by $c''\varepsilon'' = m$. We know, by Theorem 2.6.2, that C_1'' will require generators $c_i''(0 \leqq i \leqq k)$, c_*'' of degrees 2^i, $3\cdot 2^k$, plus other generators of degrees $t \geqq 3\cdot 2^{k+1}$. By Lemma 2.2.1, it is sufficient to specify $c_i''d_1''$, $c_*''d_1''$, etc., in a suitable fashion. We may take

$$c_i''d_1'' = \mathrm{Sq}(2^i)c'', \quad c_*''d_1'' = \mathrm{Sq}(2^k)\mathrm{Sq}(2^{k+1})c'' .$$

The choice of d_1'' on the other generators does not concern us.

We postpone the construction of d_2'', in order to indicate how we propose to apply Theorem 3.7.2. Using the notations of that theorem, we shall take $d\colon C_1 \to C_0$ to be $d(k+1)$, as used in § 4.2 to define $\Phi_{0,k+1}$. We shall take C_1' to be the submodule of C_1'' generated by the $c_i''(0 \leqq i \leqq k)$ and c_*''; we take $C_0' = C_0''$ and $d' = d_1'' \mid C_1'$.

The map $\rho_0\colon C_0 \to C_0'$ will be defined by

$$c\rho_0 = \mathrm{Sq}(2^{k+1})c'' .$$

Since this induces a map from C_0/dC_1 to M_1 it is possible to construct a map $\rho_1\colon C_1 \to C_1''$ so that $\rho_1 d_1'' = d\rho_0$. By considering dimensions we see that ρ_1 will map into C_1'.

The map ρ_1 may be taken in any way; we only require the following property.

LEMMA 4.3.4. *If*

$$c_0\rho_1 = \textstyle\sum_{0 \leqq i \leqq k} a_i c_i'' \qquad (a_i \in A)$$

then

$$\xi_1(2^{k+1})a_0 = 1 .$$

PROOF. Set $h = 2^{k+1}$. If $a, b \in I(A)$, we have

$$\xi_1^{h+1}(ab) = (\xi_1^h a)(\xi_1 b) + (\xi_1 a)(\xi_1^h b) .$$

We define a function $\xi\colon C_0' \to Z_2$ by setting

$$\xi(ac'') = \xi_1^{h+1}a .$$

Thus

$$\begin{aligned}\xi(c_0\rho_1 d') &= \xi\left(\textstyle\sum_{0\le i\le k} a_i \mathrm{Sq}(2^i)c''\right)\\ &= \xi_1^h a_0 .\end{aligned}$$

But

$$\begin{aligned}\xi(c_0 d\rho_0) &= \xi(\mathrm{Sq}^1\mathrm{Sq}^h c'')\\ &= 1 .\end{aligned}$$

This proves the lemma.

We now revert to the construction of C_2''. We know by Theorem 2.6.2, that C_2'' will require generators

$c''_{i,j}$ $(0 \le i \le j \le k, j \ne i+1)$	of degrees $2^i + 2^j$
$c''_{i,*}$ $(0 \le i \le k, i \ne k-1)$	of degrees $2^i + 3\cdot 2^k$
$c''_{*,*}$	of degree $2^{k-1} + 2^{k+2}$

plus other generators of degrees $t \ge 3\cdot 2^{k+1}$. Using Lemma 2.2.1, we may construct d_2'' as follows. Define an embedding $e\colon C_1(k) \to C_1''$ by $c_i e = c_i''$. Then we may take

$$c''_{i,j}d_2'' = z_{i,j}e , \quad c''_{i,*}d_2'' = z_{i,k}\rho_1 ,$$
$$c''_{*,*}d_2'' = z_{k-1,k+1}\rho_1 .$$

The choice of d_2'' on the other generators does not concern us.

We now observe that $z_{0,k+1}\rho_1$ is a cycle in C_1''. Thus it lies in Im d_2'', and must have the following form:

$$z_{0,k+1}\rho_1 = \sum_{\substack{0\le i\le j\le k\\ j\ne i+1}} a_{i,j,k}(z_{i,j}e) + \sum_{\substack{0\le i\le k\\ i\ne k-1}} a_{i,k}(z_{i,k}\rho_1) . \tag{4.3.5}$$

Since $k \ge 2$, there is no term in $z_{k-1,k+1}\rho_1$, by considering degrees. Applying Theorems 3.7.2, 3.7.1, 3.6.2, we obtain the formula which was to be proved.

It remains only to obtain the required information about the coefficient $a_{0,k,k}$. To this end, we define a function $\xi\colon C_1' \to Z_2$ by setting

$$\xi(a'_* c''_* + \textstyle\sum_{0 \le i \le k} a'_i c''_i) = (\xi_1(2^k)\xi_2(2^k))a'_0 .$$

We shall apply ξ to both sides of the equation (4.3.5). We first note that

$$\psi(\xi_1(2^k)\xi_2(2^k)) = \xi_1(2^k)\xi_2(2^k) \otimes 1 + (\xi_1(3 \cdot 2^k) + \xi_2(2^k)) \otimes \xi_1(2^k) + \xi_1(2^{k+1}) \otimes \xi_1(2^{k+1}) + \xi_1(2^k) \otimes \xi_2(2^k) + 1 \otimes \xi_1(2^k)\xi_2(2^k) .$$

In particular, $\xi_1(2^k)\xi_2(2^k)(ab) = 0$ unless $\deg(a)$ and $\deg(b)$ are both divisible by 2^k.

Let us expand $z_{0,k+1}$ in the form

$$z_{0,k+1} = \textstyle\sum_{0 \le i \le k+1} b_i c_i \qquad (b_i \in A) .$$

Then we have

$$\xi(z_{0,k+1}\rho_1) = \xi\left(\textstyle\sum_i b_i (c_i \rho_1)\right) .$$

But $\deg(b_i) = 2^0 + 2^{k+1} - 2^i$, which is odd unless $i = 0$. Let us write $c_0\rho_1 = \sum_i a_i c''_i$. Then

$$\begin{aligned} \xi(z_{0,k+1}\rho_1) &= \xi(b_0(c_0\rho_1)) \\ &= (\xi_1(2^{k+1})b_0)(\xi_1(2^{k+1})a_0) \\ &= 1 , \end{aligned}$$

using Lemma 2.2.2. (for b_0) and Lemma 4.3.4 (for a_0).

Let us apply ξ to the right-hand side of the equation (4.3.5). We have

$$\begin{aligned} \deg(a_{i,j,k}) &= 2^0 + 2^{k+2} - 2^i - 2^j \\ \deg(a_{i,k}) &= 2^0 + 2^k - 2^i \end{aligned}$$

and these are odd, except in the cases with $i = 0$. Moreover,

$$\deg(a_{0,j,k}) = 2^{k+2} - 2^j ,$$

and this is not divisible by 2^k, except in the case $j = k$. Thus it remains only to evaluate

$$\xi(a_{0,k,k}(z_{0,k}e)) , \quad \xi(a_{0,k}(z_{0,k}\rho_1)) .$$

We deal with the latter first. In C_1, let us write $z_{0,k} = \sum_{0 \le i \le k} b_i c_i$, with a new set of coefficients b_i in A. Let us write

$$c_i\rho_1 = b_{i,*}c''_* + \textstyle\sum_{0 \le j \le k} b_{i,j}c''_j .$$

Then

$$\xi(a_{0,k}(z_{0,k}\rho_1)) = (\xi_1(2^k)\xi_2(2^k))(a_{0,k}\textstyle\sum_{0 \le i \le k} b_i b_{i,0}) .$$

Here we have

$$\begin{aligned} \deg(a_{0,k}) = 2^k, \quad \deg(b_i) &= 2^0 + 2^k - 2^i , \\ \deg(b_{i,0}) &= 2^i + 2^{k+1} - 2^0 . \end{aligned}$$

Thus

$$\xi(a_{0,k}(z_{0,k}\rho_1)) = (\xi_1(2^k)a_{0,k})\textstyle\sum_{0\leq i\leq k}(\xi_2(2^k)b_ib_{i,0}) = 0\ .$$

Lastly, we consider $\xi(a_{0,k,k}(z_{0,k}e))$. Let us write $z_{0,k}e = \sum_{0\leq i\leq k}b_ic_i''$. Then

$$\begin{aligned}\xi(a_{0,k,k}(z_{0,k}e)) &= ([\xi_1(3\cdot 2^k) + \xi_2(2^k)]a_{0,k,k})(\xi_1(2^k)b_0)\\ &= (\xi_1(3\cdot 2^k) + \xi_2(2^k))a_{0,k,k}\ .\end{aligned}$$

We conclude that

$$(\xi_1(3\cdot 2^k) + \xi_2(2^k))a_{0,k,k} = 1\ .$$

This completes the proof of Lemma 4.3.3.

4.4. *The operation Ψ in P^6.* In this section we find the value of that operation which is needed to start the induction in § 4.5.

Let P be complex projective space of infinitely-many dimensions; let y be a generator of $H^2(P)$, so that $H^*(P)$ is a polynomial algebra (over Z_2) generated by y. Let Ψ be the operation defined in § 4.2.

THEOREM 4.4.1. $\Psi(y) = y^3$.

The operation Ψ is defined on y because the elements Sq^1y, Sq^3y and Sq^4y are zero. It is defined modulo zero because the elements Sq^2y^2 and Sq^4y are zero.

Before proving Theorem 4.4.1, we insert some remarks on its proof. It is easy to show (by considering the universal example) that if $u \in H^2(X)$ is any class such that $\mathrm{Sq}^1(u) = 0$, we have

$$\Psi(u) = [\lambda u^3]$$

for some fixed coefficient λ. Theorem 4.4.1 is therefore equivalent to the proposition that if u is any class such that $\mathrm{Sq}^1u = 0$, we have

$$\Psi(u) = [u^3]\ .$$

It would be desirable, in some ways, to prove this latter proposition by arguments lying wholly inside homology-theory. This is indeed possible, by using the methods of Steenrod and Adem (see [32], [5]) to give a construction for Ψ and to discuss its properties. However, to employ such methods here would lengthen the present paper by a chapter; for brevity, therefore, we make an *ad hoc* application of the methods of homotopy theory.

In fact, the space P may be decomposed as a CW-complex $S^2 \cup E^4 \cup E^6 \cup \cdots \cup E^{2n} \cup \cdots$, where the subcomplex $S^2 \cup E^4 \cup E^6 \cup \cdots \cup E^{2n}$ is just P^{2n}, the complex projective space of n complex dimensions. The stable cohomology operations in $H^*(P)$ depend on the attaching maps of these

cells, or rather, on their stable or S-homotopy classes.

The attaching map for E^4 is just the Hopf map $\eta : S^3 \to S^2$. Similarly, the attaching map $\varpi : S^5 \to P^4$ for E^6 is just the usual fibering, with fibre S^1. Let $S\varpi : S^6 \to SP^4$ be the suspension of ϖ, and let $Si : S^3 \to SP^4$ be the suspension of the embedding $i : S^2 \to P^4$.

LEMMA 4.4.2. *In $\pi_6(SP^4)$ we have*

$$\{S\varpi\} = (Si)_*\omega ,$$

where ω is a generator of $\pi_6(S^3)$.

This lemma is essentially due to H. Toda [33, Chapter 7]; but unfortunately, he does not state it explicitly. A variety of proofs are available; the neatest I have seen is the following, for which I am indebted to Dr. I. M. James. It depends on the following lemma.

LEMMA 4.4.3. *Let $B = S^q \cup E^n \cup E^{n+q}$ be a q-sphere bundle over S^n, decomposed into cells in the obvious way. Let $\alpha \in \pi_{n-1}(R_{q+1})$ be the characteristic element for B, and let $\beta \in \pi_{n+q-1}(S^q \cup E^n)$ be the attaching element for E^{n+q}. Then*

$$S\beta = \pm (Si)_*(\omega) ,$$

where

$$(Si)_* : \pi_{n+q}(S^{q+1}) \to \pi_{n+q}(S(S^q \cup E^n))$$

is the injection and ω is obtained from α by the Hopf construction.

This lemma is cognate with the work done in [20] (see § 7 in particular).

For the application, we take $q = 2$, $n = 4$, and take B to be the standard fibering

$$S^2 \to P^6 \to S^4 .$$

The element α is a generator of $\pi_3(R_3)$, and hence ω is a generator of $\pi_6(S^3)$.

We now proceed to deduce Theorem 4.4.1 from Lemma 4.4.2. We need one more lemma. Let $K = S^n \cup E^{n+4}$ be a complex (with $n \geqq 5$) in which the class of the attaching map is $2r\nu$, where ν is a generator of $\pi_{n+3}(S^n)$ and r is an integer.

LEMMA 4.4.4. *$\Psi : H^n(K) \to H^{n+4}(K)$ is zero if r is even, non-zero if r is odd.*

PROOF. We first observe that Ψ is defined, and is defined modulo zero, because

$$\mathrm{Sq}^4 : H^n(K) \to H^{n+4}(K)$$

is zero. We now construct a space X, equivalent to K, as follows; take the mapping-cylinder of a map $f: S^{n+3} \to S^n$ representing ν; then attach a cell E^{n+4} to S^{n+3} by a map of degree $2r$. Let Y be the subspace $E^{n+4} \cup S^{n+3}$. We shall apply Axiom 5, § 3.6, to the pair X, Y. Inspecting the exact cohomology sequence of this pair, we see that we may take generators as follows.

$$\begin{aligned} u &\in H^n(X, Y), & j^*u &\in H^n(X) \\ v &\in H^{n+3}(Y), & \delta v &\in H^{n+4}(X, Y) \\ w &\in H^{n+4}(X), & i^*w &\in H^{n+4}(Y). \end{aligned}$$

We have $\mathrm{Sq}^1 u = 0$, $\mathrm{Sq}^3 u = 0$, $\mathrm{Sq}^4 u = \delta v$, and $\mathrm{Sq}^1 v = r(i^*w)$. By Axiom 5, $i^*\Psi(j^*u) = r(i^*w)$ and hence $\Psi(j^*u) = rw$. This proves Lemma 4.4.4.

We now prove Theorem 4.4.1. Consider S^3P^6, the threefold suspension of P^6. By Lemma 4.4.2, the attaching map of S^3E^6 lies in the class $(S^3i)_*(S^2\omega)$, where ω is some generator of $\pi_6(S^3)$. But $S^3\omega = 2r\nu$ with r odd. Let K be a complex, as considered in Lemma 4.4.4, for $n = 5$ and this value of r; then there is a map $f: K^5 \to S^3P^6$ inducing isomorphisms of H^5, H^9. Since Ψ is non-zero in K by Lemma 4.4.4, it is non-zero in S^3P^6. Since Ψ commutes with suspension, it is non-zero in P^6, and hence in P. This completes the proof of Theorem 4.4.1.

We remark that the operation Ψ is by no means the only secondary operation in P which we can evaluate directly. In particular, one can evaluate $\Phi_{0,2}(y^{4t})$ using James's results on the attaching maps in quaternionic projective spaces—see (2.10a) of [19].

4.5. *The operations $\Phi_{i,j}$ in P.* In this section we shall obtain the values of the operations $\Phi_{i,j}$ when they act in complex projective space of infinitely-many dimensions. We write P for this projective space, and write y for the generator of $H^2(P)$, so that $H^*(P)$ is a polynomial algebra (over Z_2) generated by y. The Steenrod squares in $H^*(P)$ are easily calculated; we have

$$\begin{aligned} \mathrm{Sq}^{2k+1}(y^t) &= 0 \\ \mathrm{Sq}^{2k}(y^t) &= (t-k, k)y^{t+k}. \end{aligned}$$

(Here (h, k) stands for the (mod 2) binomial coefficient $(h+k)!/h!\,k!$).

Let χ be an operation associated with a pair $(d(j), z)$ (using the notation of § 4.2). Then χ is defined on y^t if and only if $\mathrm{Sq}(2^r)(y^t) = 0$ for $0 \leqq r \leqq j$. For this, it is necessary and sufficient that $t \equiv 0 \bmod 2^j$. Now set $\deg(\chi) = n$; and suppose that $n < 2^{j+1}$. If n is odd, then $\chi(y^t)$ is a coset in a zero group. If n is even, we should examine the Steenrod operations a_r which enter into the indeterminacy of χ (and are defined by $z = \sum_r a_r c_r$). We see

$$\deg(a_r) = n + 1 - 2^r.$$

If $r>0$, this degree is odd, so that a_r contributes nothing to the indeterminacy. On the other hand, if $r = 0$, then this degree is n; and if $t \equiv 0 \mod 2^j$, then

$$a_0 H^{2t}(P) = 0$$

for any a_0 in A such that $\deg(a_0) = n < 2^{j+1}$. We conclude that $\chi(y^t)$, if defined at all, has zero indeterminacy.

In particular, we conclude that $\Phi_{i,j}(y^t)$ is defined if and only if $t \equiv 0 \mod 2^j$; that its indeterminacy is then zero; and that it is zero (since of odd degree) unless $i = 0, j \geqq 2$.

For our next result, which gives the values of the $\Phi_{0,j}$, we set $h = 2^j$.

THEOREM 4.5.1. $\Phi_{0,j}(y^{ht}) = ty^{h(t+1/2)}$.

PROOF. We first obtain the case $j = 2, t = 1$. In fact, by applying Lemma 4.3.1 to the case $u = y$ and using Theorem 4.4.1, we see

$$\Phi_{0,2}(y^4) = \Phi_{0,2}(\mathrm{Sq}^4\mathrm{Sq}^2 y) = \mathrm{Sq}^6 \Psi y + \lambda \mathrm{Sq}^{10} y = \mathrm{Sq}^6 y^3 = y^6 .$$

This case serves to start an induction. Suppose, as an inductive hypothesis, that we have established the result for all $j<k$ (where $k\geqq 2$) and for the case $j = k, t = 1$. We now note that if χ is any secondary operation associated with $d(k)$ such that $\deg(\chi) < h = 2^k$, then $\chi(y^{ht}) = 0$ (modulo zero). This is immediate by Lemma 4.3.2, using the inductive hypothesis. We also note that it is possible to apply the Cartan formula (Theorem 3.9.4) in case $d = d' = d'' = d(k)$ (with the notations of §§ 3.9, 4.2). We will verify the main condition on d, d' and d'' imposed in § 3.9. In fact, if u, v are such that $\mathrm{Sq}(2^r)u = 0$, $\mathrm{Sq}(2^r)v = 0$ for $0 \leqq r \leqq k$, then $\mathrm{Sq}^i(u) = 0$, $\mathrm{Sq}^i(v) = 0$ for $1 \leqq i \leqq 2^k$; by the ordinary Cartan formula [11], we deduce $\mathrm{Sq}^i(uv) = 0$ for $1 \leqq i \leqq 2^k$; *a fortiori* $\mathrm{Sq}(2^r)(uv)=0$ for $0 \leqq r \leqq k$.

We can thus obtain the result for $j = k$ and any t, by induction over t, using Theorem 3.9.4. In fact, suppose

$$\Phi_{0,k}(y^{ht}) = ty^{h(t+1/2)} \qquad \text{(where } h = 2^k\text{)}.$$

Then

$$\begin{aligned}\Phi_{0,k}(y^{h(t+1)}) &= \Phi_{0,k}(y^{ht} \cdot y^h) \\ &= \Phi_{0,k}(y^{ht}) \cdot y^t + y^{ht} \cdot \Phi_{0,k}(y^h)\end{aligned}$$

(by Theorem 3.9.4, since the intermediate terms yield zero). That is

$$\Phi_{0,k}(y^{h(t+1)}) = (t + 1)y^{h(t+3/2)} .$$

This completes the induction over t; we have obtained the result for $j \leqq k$ and all t.

We now apply Lemma 4.3.3 to the class $u = y^h$, where $h = 2^k$. The left-hand side of the formula yields $\Phi_{0,k+1}(y^{2h})$, modulo zero. On the right, the term $\lambda_k \mathrm{Sq}(2^{k+2})u$ yields zero. The term $a_{i,k}\Phi_{i,k}\mathrm{Sq}(2^{k+1})u$ yields zero, modulo zero, by what we have already proved. The term $a_{i,j,k}\Phi_{i,j}u$ yields zero, modulo zero, except in the case $i = 0, j = k$. In this case, $\Phi_{0,k}(u)$ becomes $y^{3h/2}$, modulo zero. Now, we easily see that if $a \in A_{3h}$, then

$$a \cdot y^{3h/2} = ([\xi_1(3 \cdot 2^k) + \xi_2(2^k)]a)y^{3h} .$$

Using the last part of Lemma 4.3.3, we conclude that the term $a_{0,k,k}\Phi_{0,k}(u)$ yields y^{3h}, modulo zero, and

$$\Phi_{0,k+1}(y^{2h}) = y^{3h} .$$

We have proved the result for the case $j = k + 1$, $t = 1$. This completes the induction over k; Theorem 4.5.1 is proved.

4.6. *The final relation.* In this section we obtain the relation required to carry out the argument indicated in Chapter 1.

Take an integer $k \geqq 3$. Let $u \in H^m(X)(m > 0)$ be a class such that $\mathrm{Sq}(2^r)u = 0$ for $0 \leqq r \leqq k$.

THEOREM 4.6.1. *There is a relation*

$$\sum_{\substack{0 \leqq i \leqq j \leqq k \\ j \neq i+1}} a_{i,j,k}\Phi_{i,j}(u) = [\mathrm{Sq}(2^{k+1})u]$$

(independent of X) which holds modulo the total indeterminacy of the left-hand side.

PROOF. Let us consider the first few terms

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} Z_2$$

of a minimal resolution of Z_2, as constructed in §§ 4.2, 4.3. Let us choose a cycle z in C_2 such that $(h_0 h_k^2)(\theta z) = 1$; this is possible by Theorem 2.5.1, since $k \geqq 3$. Let us write $z = \sum_{i,j} a_{i,j,k}c_{i,j}$. By considering degrees and using the fact that d_1 is minimal, we see that this sum consists of terms with $j \leqq k$. By Lemma 2.2.2, we have

$$\xi_1(2^k)a_{0,k,k} = 1 .$$

Since z is a cycle, we have

$$0 = zd = (\textstyle\sum_{i,j} a_{i,j,k}c_{i,j}d) = \sum_{i,j} a_{i,j,k}z_{i,j} .$$

Now, this relation holds in the submodule $C_1(k)$ of C_1. Appealing to the theorems of §§ 3.6, 3.7, we find the required relation

$$\sum_{\substack{0\leqq i\leqq j\leqq k\\ j\neq i+1}} a_{i,j,k}\Phi_{i,j}(u) = [\lambda \mathrm{Sq}(2^{k+1})u] \ .$$

It remains only to determine the coefficient λ.

To do this, we apply both sides to a power y^h (where $h = 2^k$) of the fundamental class y in $H^2(P)$. The total indeterminacy of the left-hand side is then zero. The right-hand side yields λy^{2h}. By § 4.5 (and Theorem 4.5.1 in particular), each term on the left-hand side yields zero, modulo zero, except the term

$$a_{0,k,k}\Phi_{0,k}(y^h) = a_{0,k,k}y^{3h/2} \qquad \text{(mod zero)} \ .$$

But we see that for any $a \in A_h$, we have

$$ay^{3h/2} = (\xi_1^h a)y^{2h} \ .$$

Since $\xi_1^h a_{0,k,k} = 1$, we have $\lambda = 1$, and the proof is complete.

This establishes the decomposability of Sq^i for $i = 2^r$, $r \geqq 4$, which implies that the corresponding groups $\pi_{2i-1}(S^i)$ contain no elements of Hopf invariant one.

ADDENDUM

1. The paper to which this is an addendum makes use of the following as a key lemma:

LEMMA 1. *Let P denote complex projective space of infinitely-many dimensions; let u be the generator of $H^2(P; Z_2)$; let Ψ be the secondary operation associated with the relation*

$$\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^1\mathrm{Sq}^4 = 0 \ .$$

Then we have

$$\Psi(u) = u^3 \ .$$

(See Theorem 4.4.1).

It is the object of this addendum to give a simple proof of this lemma. We will actually prove:

LEMMA 2. *Let $u \in H^2(X; Z_2)$ be a cohomology class such that $\mathrm{Sq}^1 u = 0$. Then the coset Ψu contains the element u^3.*

The proof to be given employs a method which I owe to A. Liulevicius; he uses it in his treatment of the problem of "elements of Hopf invariant one mod p." I am most grateful to him for interesting letters on this subject. Liulevicius, in turn, ascribes the basic idea of his method to W. Browder.

2. The basic idea of the method is as follows. One considers a universal example consisting of a space E and a class $v \in H^*(E; Z_p)$, as in §3.3.

Then the loop-space ΩE and the suspension $\sigma v \in H^*(\Omega E; Z_p)$ constitute another universal example, in which the dimensions have been decreased by one. It is possible that the space ΩE may be equivalent to a Cartesian product $X \times Y$, although the space E does not split in the same way. If this happens, then the Pontrjagin product in $H_*(\Omega E; Z_p)$ gives us a ring-structure on

$$H_*(X \times Y; Z_p) = H_*(X; Z_p) \otimes H_*(Y; Z_p) ;$$

in general, this ring-structure does not split as the tensor-product of ring-structures on $H_*(X; Z_p)$ and $H_*(Y; Z_p)$. Since the element $\sigma v \in H^*(\Omega E; Z_p)$ is primitive, it is possible to make deductions about its value.

3. We will now apply this method to our case. In what follows, all cohomology groups have coefficients in the group Z_2. In order to prove Lemma 2, it is sufficient to prove it when u is the fundamental class in a suitable universal example. The universal examples we must consider are those used to define Ψ; they can be constructed as follows.

For any positive integer n, let $K(Z_2, n)$, $K(Z_2, n+1)$, $K(Z_2, n+3)$ and $K(Z_2, n+4)$ be Eilenberg-MacLane spaces of the types indicated; we suppose that the first is a CW-complex, and write b^n for its fundamental class. Then there is a map

$$m : K(Z_2, n) \longrightarrow K(Z_2, n+1) \times K(Z_2, n+3) \times K(Z_2, n+4)$$

which maps the fundamental classes on the right-hand side into $\mathrm{Sq}^1 b^n$, $\mathrm{Sq}^3 b^n$, $\mathrm{Sq}^4 b^n$. The map m induces fibre-space over $K(Z_2, n)$ with fibre $K(Z_2, n) \times K(Z_2, n+2) \times K(Z_2, n+3)$; this fibre-space we call E_n. If we write $f^{n,0}, f^{n,2}, f^{n,3}$ for the fundamental classes in the fibre of E_n, then we have

$$\begin{aligned} \tau f^{n,0} &= \mathrm{Sq}^1 b^n \\ \tau f^{n,2} &= \mathrm{Sq}^3 b^n \\ \tau f^{n,3} &= \mathrm{Sq}^4 b^n . \end{aligned}$$

The class $v^{n+4} \in H^{n+4}(E_n)$ which serves as a universal example for Ψ satisfies

$$i^* v^{n+4} = \mathrm{Sq}^4 f^{n,0} + \mathrm{Sq}^2 f^{n,2} + \mathrm{Sq}^1 f^{n,3} .$$

This condition defines v^{n+4} uniquely, so long as n is sufficiently large.

It would be equivalent, however, to induce our fibering in two stages; first induce a fibering E'_n with fibre $K(Z_2, n)$ over $K(Z_2, n)$; then induce a fibering E''_n with fibre $K(Z_2, n+2) \times K(Z_2, n+3)$ over E'_n, If we adopt this procedure, the first stage evidently gives $E'_n = K(Z_4, n)$; therefore we may regard E_n as a fibering with base $K(Z_4, n)$ and fibre

$$K(Z_2, n+2) \times K(Z_2, n+3) .$$

Let us re-appropriate the symbols b^n, $f^{n,2}$, $f^{n,3}$ for the fundamental classes in these spaces; we have

$$\tau f^{n,2} = \mathrm{Sq}^3 b^n$$
$$\tau f^{n,3} = \mathrm{Sq}^4 b^n$$
$$i^* v^{n+4} = \mathrm{Sq}^2 f^{n,2} + \mathrm{Sq}^1 f^{n,3} .$$

We will now examine what happens to E_n when n is small. Let us take $n = 3$; we find that E_3 is equivalent to a product $X \times K(Z_2, 6)$, owing to the fact that Sq^4 vanishes on classes of dimension 3. We can therefore choose a class $g^6 \in H^6(E_3)$ whose image in the fibre $K(Z_2, 5) \times K(Z_2, 6)$ is the fundamental class in the second factor. Similarly, let us take $n=2$; we find that E_2 is equivalent to a product $K(Z_4, 2) \times K(Z_2, 4) \times K(Z_2, 5)$, owing to the fact that Sq^3 and Sq^4 vanish on classes of dimension 2. We can therefore choose classes $g^4 \in H^4(E_2)$, $g^5 \in H^5(E_2)$ whose images in the fibre $K(Z_2, 4) \times K(Z_2, 5)$ are the fundamental classes. Let $g^2 \in H^2(E_2)$ be the fundamental class. We can now write down the following base for $H^6(E_2)$:

$$(g^2)^3 , \quad (\beta_4 g^2) , \quad g^2 g^4 , \quad \mathrm{Sq}^2 g^4 , \quad \mathrm{Sq}^1 g^5 .$$

We wish to know how the element v^6 can be expressed in terms of this base.

Since E_2 is equivalent to ΩE_3, we are precisely in the situation envisaged in § 2. In fact, v^6 is primitive, since by construction it corresponds to σv^7 in the equivalence $E_2 \sim \Omega E_3$. Let μ denote the product: then the homomorphism μ^* of $H^6(E_2)$ is determined by the following equations.

(1) $\mu^* g^2 = g^2 \otimes 1 + 1 \otimes g^2$

(2) $\mu^* g^4 = g^4 \otimes 1 + g^2 \otimes g^2 + 1 \otimes g^4$

(3) $\mu^* g^5 = g^5 \otimes 1 + 1 \otimes g^5$.

Here the equation (1) holds for dimensional reasons; while (3) holds provided we choose g^5 to correspond to σg^6, as we evidently may. As for (2), the only alternative is to suppose that g^4 is primitive; and if it were primitive, then (for dimensional reasons) it would be the suspension of some element $\gamma^5 \in H^5(E_3)$; this would satisfy $i^* \gamma^5 = f^{3,2}$, contradicting the fact that $\tau f^{3,2} = \mathrm{Sq}^3 b^3 \neq 0$.

It is now easy to calculate such values as

$$\mu^*(g^2)^3 = (g^2)^3 \otimes 1 + (g^2)^2 \otimes g^2 + g^2 \otimes (g^2)^2 + 1 \otimes (g^2)^3$$
$$\mu^*(\mathrm{Sq}^2 g^4) = \mathrm{Sq}^2 g^4 \otimes 1 + (g^2)^2 \otimes g^2 + g^2 \otimes (g^2)^2 + 1 \otimes \mathrm{Sq}^2 g^4 .$$

We find the following base of primitive elements in $H^6(E_2)$:

$$(\beta_4 g^2)^2 , \qquad (g^2)^3 + \mathrm{Sq}^2 g^4 , \qquad \mathrm{Sq}^1 g^5 .$$

Since

$$i^* v^6 = \mathrm{Sq}^2 f^{2,2} + \mathrm{Sq}^1 f^{2,3} ,$$

we have

$$v^6 = \lambda(\beta_4 g^2)^2 + (g^2)^3 + \mathrm{Sq}^2 g^4 + \mathrm{Sq}^1 g^5$$

for some $\lambda \in Z_2$.

Now, the indeterminacy of $\Psi(g^2)$ is a subgroup Q of $H^6(E_2)$ which has the following base:

$$\mathrm{Sq}^2 g^4 , \qquad (\beta_4 g^2)^2 , \qquad \mathrm{Sq}^1 g^5 .$$

Moreover, $\Psi(g^2)$ is by definition that coset of Q which contains v^6. Therefore $\Psi(g^2)$ contains $(g^2)^3$. This completes the proof.

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