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Embeddings and Immersions

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Translated by
Kiki Hudson

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by Masahisa Adachi

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ABSTRACT. This book provides an introduction to the theory of embeddings and immersions of smooth manifolds and then gives applications of Gromov's theorems to foliations and complex structures on open manifolds.

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Preface to the English Edition

For the convenience of readers of this English edition I have replaced the original Japanese references with the appropriate references in English or French. I have also replaced some other references that are hard to obtain with those that are more readily available.

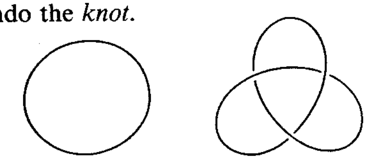
I wish to sincerely thank Professors Kobayashi and Nomizu for their advice. I am very grateful to Dr. Kiki Hudson, who has provided an excellent translation and pointed out misprints in the original edition.

Masahisa Adachi
October 28, 1992

Preface

Among closed surfaces the torus $T^2 = S^1 \times S^1$ can be thought of as sitting in three-dimensional Euclidean space $\mathbf{R}^3$, but the Klein bottle K^2 cannot be realized there. This observation naturally leads us to the question 'can a general n-dimensional manifold M^n be smoothly embedded in Euclidean space $\mathbf{R}^p$?'.

Further, it is possible to embed the circle S^1 in three-dimensional Euclidean space $\mathbf{R}^3$, but there is more than one way to do so. For example, we cannot move one of the two embeddings below to the other via an isotopy; that is, we cannot undo the *knot*.



This is generalized to the problem 'are two given embeddings $f, g : M^n \to \mathbf{R}^p$ isotopic?' Since the concept of topology was first established, these problems have been in its mainstream, and major contributions to solutions have come from H. Whitney and A. Haefliger. Still further research and development can be expected in this field.

In particular, the problem of classifying embeddings of the circle S^1 in three-dimensional Euclidean space $\mathbf{R}^3$ or the three-dimensional sphere S^3 through isotopies—a bit different from the isotopies mentioned in the previous paragraph —forms a field in topology called the *theory of knots*, which even today generates many research activities.

The problem of classifying immersions by regular homotopies is slightly easier than that of classifying embeddings by isotopies. Here is an example. In three-dimensional Euclidean space $\mathbf{R}^3$, is it possible to turn the sphere S^2 inside out smoothly allowing self-intersections? Think about it for a minute. It hardly seems likely, but a classification theorem for immersions shows that it can be done.

This classification theorem, the so-called Smale-Hirsch theorem, has been generalized step by step by A. Phillips, M. Gromov, A. Haefliger, and so on to the present stage where it now offers us a tool for finding solutions (or their candidates) to partial differential inequalities or partial differential equations of certain types. It also provides us with a method for eliminating singularites of certain C^∞ maps. There are further applications of these methods as well.

The aim of this book is to give an introduction to this theory in modern topology and its applications. In accordance with the principle of this series we have tried to make the first three chapters easy enough to understand at the level of lower-division mathematics.

In this book, unless otherwise stated, embeddings and immersions will be viewed in the C^∞ category. We first explain in detail the classification of regular closed curves in the plane by regular homotopies; this will serve as an intuitive preparation for the contents of the book.

In Chapter I, we give a summary of basic concepts about C^r manifolds and C^r maps which will be used in Chapter II and beyond.

The discussions in Chapter II evolve around Whitney's theorems. This chapter also serves as a prelude to Chapter VII. We develop Chapter III around the Smale-Hirsch theorem which is generalized to Gromov's theorem.

In Chapter IV we examine the convex integration theory due to Gromov which is another application of the Smale-Hirsch theorem.

In Chapter V we discuss an application of Gromov's theorem, namely, a classification theorem for foliations of open manifolds. In Chapter VI we study complex structures on open manifolds as an application of Gromov's theorem and Gromov's convex integration theory.

We study Haefliger's embedding theorem in Chapter VII, which is a continuation of Chapter II.

Finally, as references we give a list of books and papers we have either used, adapted, or quoted from directly, and also books and papers basic to embeddings and immersions.

The author thanks Kazuhiko Fukui, Shigeo Kawai, and Goo Ishikawa for their valuable help in writing this book.

We are deeply indebted to Professor Itiro Tamura who encouraged us to write this book and gave us valuable advice concerning the first draft.

Last but not least our deepest gratitude goes to Mr. Hideo Arai of Iwanami Shoten Publishers, without whose help this book would never have been realized.

Masahisa Adachi
May 1983

CHAPTER 0

Regular Closed Curves in the Plane

In this chapter we consider closed curves in the plane $\mathbf{R}^2$, whose tangent lines move continuously. To each closed curve we assign the "rotation number" γ, which is the angle the tangent line makes going around the curve once ($\gamma = \pm 2\pi$ for a closed curve). Our aim in this chapter is to show the following:

> Two closed curves with the same rotation number can be deformed from one to the other.

This chapter is based on Whitney [C20].

§1. Regular closed curves

We first define closed regular curves.

Let $I = [0, 1]$. Consider $f = (f_1, f_2)$, where f_1 and f_2 are C^1 functions (f is called a C^1 map). We say that f is a *parametrized regular closed curve* if it satisfies the following:

(i) $f(0) = f(1)$, $f'(0) = f'(1)$,
(ii) $f'(t) \neq 0$, for each $t \in I$.

The condition (i) shows that the curve is closed and (ii) says that f is regular in some sense with respect to the parameter t. See Figure 0.1.

To the above $f : I \to \mathbf{R}^2$ there corresponds a C^1 function $\tilde{f}$

$$\tilde{f} : (-\infty, \infty) \longrightarrow \mathbf{R}^2,$$

such that

(iii) $\tilde{f}(t) = f(t)$, $t \in I$,
(iv) $\tilde{f}(t+1) = \tilde{f}(t)$,
(v) $\tilde{f}'(t) \neq 0$.

Conversely to such an $\tilde{f}$ there corresponds an f as above. We say that $\tilde{f}$ is a *lift* of f.

DEFINITION 0.1. Let f and g be parametrized regular closed curves. We say that f and g are *equivalent* and write $f \sim g$ if there exists a C^1 function $\eta : (-\infty, \infty) \to (-\infty, \infty)$ such that

$$\eta'(t) > 0, \quad \text{for each } t \in \mathbf{R}, \qquad \eta(t+1) = \eta(t) + 1, \qquad \tilde{g}(t) = \tilde{f} \circ \eta(t).$$

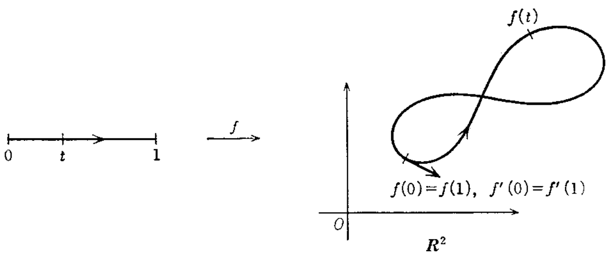


FIGURE 0.1

Clearly, $\sim$ is an equivalence relation; hence, $\sim$ divides parametrized regular closed curves into equivalence classes which are called *regular closed curves* or simply *curves*. If $f \sim g$, then we have $f(I) = g(I)$.

PROPOSITION 0.1. *Let C be a regular closed curve. Then there exists an element g in C such that $\|g'(t)\|$ is constant, where $\| \ \|$ is a norm on $\mathbf{R}^2$.*

PROOF. Let $f \in C$ and let $\tilde{f}$ be a lift of f. Set

$$L(t) = \int_0^t \|\tilde{f}'(s)\|\, ds, \qquad L = L(1).$$

Then $L = L(C)$ is the length of the curve C. Since $\tilde{f}'(t) \neq 0$, $L(t)$ is of class C^1 and monotone increasing. Hence, we can solve $s = 1/L \cdot L(t)$ for t, say $t = \eta(s)$. $\eta'(s)$ is continuous and positive. Since $\tilde{f}$ is periodic we have

$$L(t+1) - L(t) = \int_t^{t+1} \|\tilde{f}'(s)\|\, ds = \int_0^1 \|\tilde{f}'(s)\|\, ds.$$

Therefore, $\eta(s+1) = \eta(s) + 1$. Thus, if we set

$$\tilde{g}(t) = \tilde{f} \circ \eta(t),$$

we see that $\tilde{g}$ is a lift of some element g of C. Further we have

$$\tilde{g}'(t) = \tilde{f}' \circ \eta(t) \cdot \frac{L}{L'(\eta(t))}, \qquad \|\tilde{g}'(t)\| = L.$$

So $\|g'(t)\|$ is constant. $\square$

PROPOSITION 0.2. *Let C be a regular closed curve, and let g be an element of C defined as above. Suppose h is an element of C such that $\|h'(t)\|$ is a constant k. Then*

(i) $k = L$ *and*
(ii) $\tilde{h}(t) = \tilde{g}(t+a)$ *for some constant a.*

In other words, two elements of C with each $\|h'(t)\|$ constant differ by some rotation of the circle S^1.

PROOF. Since $h \sim g$ there exists $\eta : (-\infty, \infty) \to (-\infty, \infty)$ such that $\tilde{h}(t) = \tilde{g}(\eta(t))$. But

$$h'(t) = g'(\eta(t))\eta'(t),$$

and so $k = L \cdot \eta'(t)$. Hence,

$$1 = \eta(1) - \eta(0) = \int_0^1 \eta'(t)\, dt = \int_0^1 \frac{k}{L}\, dt = \frac{k}{L}.$$

Hence, we get $k = L$. It follows that $\eta'(t) = 1$ and that $\eta(t) = t + a$. $\square$

DEFINITION 0.2. Let f_0 and f_1 be parametrised regular closed curves. We say that f_0 is a *deformation* of f_1 or that f_0 and f_1 are *regularly homotopic*, and write $f_0 \underset{r}{\simeq} f_1$ if the following holds: For some continuous map $F : I \times I \to \mathbf{R}^2$

(i) $F(t, 0) = f_0$, $F(t, 1) = f_1(t)$, and
(ii) if we set $f_u(t) = F(t, u)$, then $f_u : I \to \mathbf{R}^2$ is a parametrized regular curve for each $u \in I$. Here we say that F or the $\{f_u\}$ is a *regular homotopy*.

We see that the relation $\underset{r}{\simeq}$ of being regularly homotopic is an equivalence relation. See Figure 0.2.

PROPOSITION 0.3. *Let C be a regular closed curve, and let f_0, $f_1 \in C$. Then f_0 is a deformation of f_1 in C; that is, there exists a regular homotopy $f_u \in C$, $u \in I$, connecting f_0 and f_1.*

PROOF. That f_0 and f_1 are equivalent implies that $\tilde{f}_1(t) = \tilde{f}_0 \circ \tilde{\eta}(t)$ for some function η as in Definition 0.1.

Set

$$\eta_u(t) = u\eta(t) + (1-u)t, \qquad 0 \le u \le 1,$$
$$\tilde{f}_u(t) = \tilde{f}_0 \circ \eta_u(t), \qquad \tilde{f}_0 \text{ is a lift of } f_0.$$

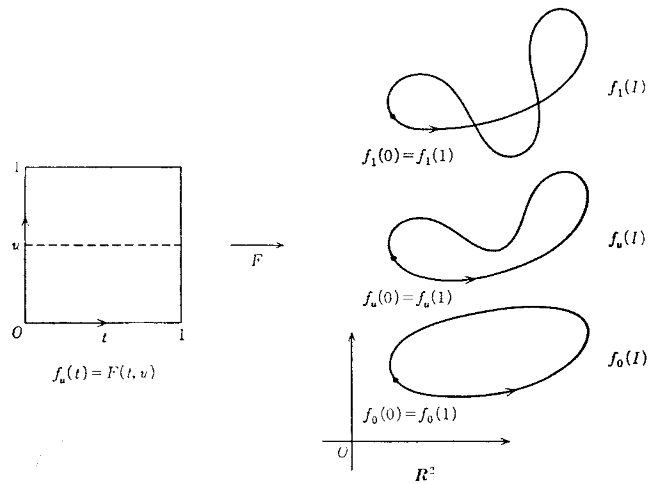


FIGURE 0.2

Here we have $\eta_0(t) = t$, $\eta_1(t) = \eta(t)$. Hence, $\tilde{f}_1$ is a lift of f_1, and we have

$$\eta_u(t+1) = u[\eta(t)+1] + (1-u)(t+1)$$
$$= \eta_u(t) + 1,$$
$$\frac{d\eta_u(t)}{dt} = u\frac{d\eta(t)}{dt} + (1-u) > 0, \qquad 0 \le u \le 1.$$

Therefore, each f_u is a parametric regular closed curve, and so we have the proposition. □

By virtue of Proposition 0.3 the expression " a regular closed curve C is a deformation of a regular closed curve C' " makes sense.

§2. Regular homotopies

We have the following basic

LEMMA 0.1. *Let $g : I \to \mathbf{R}^2$ be a continuous map and suppose $g(t) \neq 0$ for each $t \in I$. For $p \in \mathbf{R}^2$,*

$$f(t) = p + \int_0^t g(s)\,ds$$

is a parametrized regular curve if and only if

$$g(0) = g(1), \qquad \int_0^1 g(s)\,ds = 0.$$

The lemma is obvious.

DEFINITION 0.3. For a parametrized closed regular curve $f : I \to \mathbf{R}^2$ we define the rotation number $\gamma(f) \in \mathbf{R}$ of f as follows: the map

$$f^* : I \longrightarrow S^1 \subset \mathbf{R}^2, \qquad f^*(t) = \frac{f'(t)}{\|f'(t)\|}$$

defines naturally the continuous map $\hat{f}^* : S^1 \to S^1$. Now define

$$\gamma(f) = 2\pi \cdot \deg(\hat{f}^*),$$

where $\deg(\hat{f}^*)$ is the *degree* of $\hat{f}^*$ [1].

[1] In general the degree $\deg(h)$ of a continuous map $h : S^1 \to S^1$ is an integer which represents the number of the times $h(S^1)$ wraps around S^1 inclusive of the sign of $h(S^1)$. The following is a more precise definition. Notice that the fundamental group of the circle S^1, $\pi_1(S^1)$, is isomorphic to $\mathbf{Z}$. Let s be the generator of this group. On the other hand h defines the homomorphism

$$h_* : \pi_1(S^1) \longrightarrow \pi_1(S^1)$$

and the image of s by h_* is $n \cdot s$, $n \in \mathbf{Z}$. We define the *degree* or the *mapping degree* $\deg(h)$ of h to be this n.

PROPOSITION 0.4. *Let f, g be parametrized regular closed curves. If f and g are regularly homotopic then $\gamma(f) = \gamma(g)$.*

PROOF. Let $f_u : I \to \mathbf{R}^2$ be a regular homotopy connecting f and g, say $f_0 = f$ and $f_1 = g$. Then f^* and g^* are homotopic through f_u^*, and so $\hat{f}^*$ and $\hat{g}^*$ are homotopic through $\hat{f}_u^*$. Hence, $\deg(\hat{f}^*) = \deg(\hat{g}^*)$. □

DEFINITION 0.4. Let C be a regular closed curve. We define the *rotation number* $\gamma(C)$ of C by $\gamma(C) = \gamma(f)$, $f \in C$.

By Proposition 0.3 the above definition does not depend on the choice of f.

THEOREM 0.1. *Regular closed curves C_0 and C_1 are regularly homotopic if and only if $\gamma(C_0) = \gamma(C_1)$.*

This theorem is known as the *Whitney-Graustein theorem.*

COROLLARY 0.1. *The family of the regular homotopy classes of regular closed curves in the plane is in one-to-one correspondence with the set $\mathbf{Z}$ of the integers by the map $C \mapsto \gamma(C)/2\pi$.*

PROOF OF THEOREM 0.1. The 'only if' part is evident by Proposition 0.4. To prove the 'if' part set $\gamma(C_0) = \gamma(C_1) = \gamma$. Choose $g_0 \in C_0$ and $f_1 \in C_1$ such that

$$\|g_0'(t)\| = L(C_0) = L_0, \qquad \|f_1'(t)\| = L(C_1) = L_1$$

(cf. Proposition 0.1). Define g_u by

$$g_u(t) = g_0(0) + \left[u \cdot \frac{L_1}{L_0} + (1-u)\right]\{g_0(t) - g_0(0)\}.$$

Then the family $\{g_u\}$ is a homotopy connecting g_0 and g_1. Further as $g_u'(t) \neq 0$, for each $t \in I$, the $\{g_u\}$ is actually a regular homotopy connecting g_0 and g_1. Set $f_0 = g_1$. We then have $\|f_0'(t)\| = \|g_1'(t)\| = L_1$.

We want to show that f_0 is regularly homotopic to f_1. Let K be the circle in $\mathbf{R}^2$ centered at the origin of radius L_1. Then f_0', $f_1' : I \to K \subset \mathbf{R}^2$. If $\hat{f}_0'$, $\hat{f}_1' : S^1 \to K$ are the natural maps corresponding to f_0' and f_1', we have $\deg(\hat{f}_0') = \deg(\hat{f}_1') = \gamma/2\pi$. Hence, $\hat{f}_0'$ and $\hat{f}_1'$ are homotopic. Now define

$$\theta : \mathbf{R} \longrightarrow K$$

by

$$\theta(t) = (L_1 \cos t,\, L_1 \sin t).$$

(i) For $\gamma \neq 0$ we have $\theta(0) = (L_1, 0)$. Without loss of generality we may assume that $f_0'(0) = f_1'(0) = \theta(0)$. As $f_i'(t) \in K$, $i = 0, 1$, denoting by $F_i(t)$ the argument of $f_i'(t)$, we have the following

$$F_i : I \longrightarrow \mathbf{R}, \qquad i = 0, 1,$$
$$f_i'(t) = \theta \circ F_i(t), \qquad F_i(0) = 0.$$

Then by the definition of and the assumption on γ we see that

$$F_i(1) = \gamma, \qquad i = 0, 1.$$

Now set

$$\begin{cases} F_u(t) = uF_1(t) + (1-u)F_0(t), & 0 \le u \le 1. \\ h_u(t) = \theta \circ F_u(t), \end{cases}$$

Then the $\{h_u\}$ is a homotopy connecting f_0' and f_1'. Set

$$\begin{cases} \varphi_u(t) = h_u(t) - \int_0^1 h_u(s)\,ds, \\ f_u(t) = f_0(0) + u[f_1(0) - f_0(0)] + \int_0^t \varphi_u(s)\,ds. \end{cases}$$

Evidently $\int_0^1 \varphi_u(t)\,dt = 0$, and so $f_u(0) = f_u(1)$. Moreover, we have that $f_u'(t) = \varphi_u(t)$. Since $F_u(0) = 0$, $F_u(1) = \gamma$, and γ is an integral multiple of 2π, we get

$$\begin{aligned} f_u'(1) - f_u'(0) &= \theta \circ F_u(1) - \theta \circ F_u(0) \\ &= \theta(\gamma) - \theta(0) = 0; \end{aligned}$$

therefore, $f_u'(1) = f_u'(0)$, for each $u \in I$.

Next we show that $f_u'(t) \neq 0$, $u \in [0, 1]$. We have

$$f_u'(t) = h_u(t) - \int_0^1 h_u(s)\,ds, \qquad h_u(t) \in K.$$

If $\gamma \neq 0$, then $\int_0^1 h_u(s)\,ds$ lies in the interior of K, because by Schwarz's inequality

$$\left\| \int_0^1 h_u(s)\,ds \right\|^2 \le \int_0^1 \|h_u(s)\|^2\,ds.$$

But $h_u(s)$ is not a constant number, and hence the above inequality must be a strict inequality. Moreover, $\|h_u(s)\|^2 = L_1$ implies that

$$\left\| \int_0^1 h_s(s)\,ds \right\|^2 < L_1.$$

Hence, $f_u'(t) \neq 0$. Thus, we have shown that f_u is a regular closed curve, and so the $\{f_u\}$ is a regular homotopy connecting f_0 and f_1.

(ii) Case $\gamma = 0$. Suppose we can change $F_u(t)$ so that for each $u \in [0, 1]$, $F_u(t)$ is not a constant map. Then $f_u'(t) \neq 0$ for each u and we have the proof. To make such a change take a point t_0 with $F_1(t_0) \neq 0$ and deform $F_0(t)$ to $F_1(t)$ in a sufficiently small neighborhood of t_0. Denoting by F_u the newly obtained deformation of F_0 to F_1 we repeat the above process. We then see that F_u is not a constant map for each u. □

Lemma 0.1 suggests a later development of our subject.

CHAPTER I

C^r Manifolds, C^r Maps, and Fiber Bundles

In this chapter we shall collect together the fundamental facts about C^r manifolds, C^r maps, and fiber bundles as well as other preparatory items necessary in the later chapters.

§1. C^∞ manifolds and C^∞ maps

Here we give a brief summary of C^∞ manifolds.

A. C^∞ manifolds. First we define a C^∞ manifold. Let $\mathbf{R}^n$ be n dimensional Euclidean space with a fixed coordinate system. Then a point x of $\mathbf{R}^n$ is represented by the n-tuple

$$x = (x_1, x_2, \dots, x_n).$$

Consider a function defined on an open subset U of $\mathbf{R}^n$

$$f : U \longrightarrow \mathbf{R}^1.$$

Let r be a natural number or ∞. We say that f is *differentiable of class* C^r, f is of *class* C^r, or simply f is C^r if at each point x of U all partial derivatives of f of the form

$$\left.\frac{\partial^s f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_s}}\right|_x, \qquad \begin{cases} 1 \le i_1 \le \dots \le i_s \le n, \\ 1 \le s \le r \end{cases}$$

exist and are continuous.

Consider a map $f : U \to \mathbf{R}^p$ from an open subset U of $\mathbf{R}^n$ to $\mathbf{R}^p$. Writing $f(x) = (f_1(x), \dots, f_p(x)) \in \mathbf{R}^p$, we say that f is *differentiable of class* C^r if for each $1 \le i \le p$, $f_i : U \to \mathbf{R}$ is a differentiable function of class C^r. The definition of a C^∞ map is similar. A real analytic map f is sometimes called C^ω.

DEFINITION 1.1. A topological space M^n is called an n-dimensional *topological manifold* if it satisfies the following:

(i) M^n is a Hausdorff space.

(ii) For each point x of M^n there exists a neighborhood $U(x)$ which is homeomorphic to $\mathbf{R}^n$.

(iii) M^n satisfies the second axiom of countability.

Now we define a differentiable structure on a topological manifold.

Let M^n be a topological manifold of dimension n. By a C^∞ *coordinate system* or an C^∞ *atlas* for M^n we mean a family $\mathscr{S} = \{(V_j, \varphi_j) \mid j \in J\}$ of pairs (V_j, φ_j) of open sets V_j in M^n and homeomorphisms $\varphi_j : V_j \to \mathbf{R}^n$ of V_j in $\mathbf{R}^n$ satisfying the following:

(i) $M^n = \bigcup_{j \in J} V_j$,
(ii) If $V_i \cap V_j \neq \varnothing$, then the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \longrightarrow \varphi_j(V_i \cap V_j)$$

from an open subset of $\mathbf{R}^n$ to an open subset of $\mathbf{R}^n$ is of class C^∞ (Figure 1.1).

The pair (V_j, φ_j) is a *chart* or *a system of local coordinates* and V_j is a *coordinate neighborhood.*

Two C^∞ atlases $\mathscr{S} = \{(V_j, \varphi_j) \mid j \in J\}$ and $\mathscr{S}' = \{(V'_k, \varphi'_k) \mid k \in K\}$ are *equivalent*, $\mathscr{S} \sim \mathscr{S}'$, if the combined family $\mathscr{S} \cup \mathscr{S}'$ of the two systems is also a C^∞ atlas for M^n. Evidently the relation $\sim$ is an equivalence relation. An equivalence class $\mathscr{D} = [\mathscr{S}]$ in M^n is a *differentiable structure* or a C^∞ *structure* for M^n, and the pair $(M^n, \mathscr{D})$ is a *differentiable* or C^∞ *manifold* with the *underlying topological manifold* M^n.

The above definition is known as Whitney's definition. More generally if the maps $\varphi_j \circ \varphi_i^{-1}$ in the definition of a C^∞ manifold are of class C^r, $0 \leq r \leq \omega$, we say that $(M^n, \mathscr{D})$ is a C^r *manifold.* A C^0 manifold is a topological manifold. Often a differentiable manifold is understood to be a C^1 manifold; however, in this book we agree for simplicity that a differentiable manifold is a C^∞ manifold, which is also called a *smooth manifold.*

Next we discuss orientations of a C^∞ manifold $(M^n, \mathscr{D})$. Let $\mathscr{D} = [\mathscr{S}]$, $\mathscr{S} = \{(V_j, \varphi_j) \mid j \in J\}$. For $x \in V_i \cap V_j$ let $a_{ji}(x)$ be the Jacobian matrix of $\varphi_j \circ \varphi_i^{-1}$ at $\varphi_i(x)$:

$$a_{ji}(x) = D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(x)}, \qquad x \in V_i \cap V_j.$$

Then it is easy to see that

$$a_{kj}(x) \cdot a_{ji}(x) = a_{ki}(x), \qquad x \in V_i \cap V_j \cap V_k.$$

If we set $k = i$, it follows that $a_{ji}(x)$ has an inverse. Hence $a_{ji}(x) \in \mathrm{GL}(n, \mathbf{R})$, where $\mathrm{GL}(n, \mathbf{R})$ denotes the general linear group of $\mathbf{R}^n$. Hence, we have a continuous map

$$a_{ji} : V_i \cap V_j \longrightarrow \mathrm{GL}(n, \mathbf{R}).$$

A differentiable atlas $\mathscr{S} = \{(V_j, \phi_j) \mid j \in J\}$ is *oriented* if for all i, j and all $x \in V_i \cap V_j$, the determinant $|a_{ij}(x)|$ is positive.

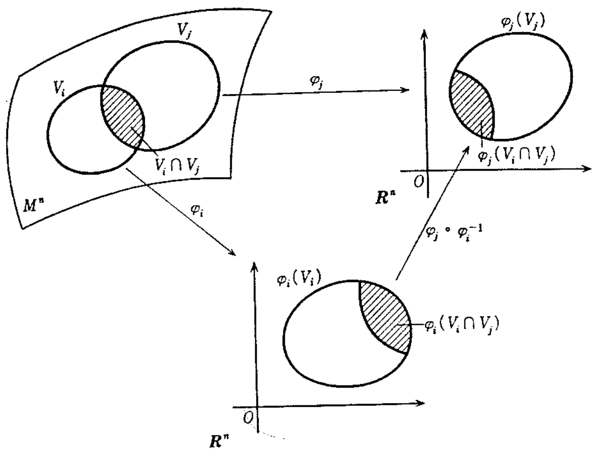


FIGURE 1.1

Let $\mathscr{S} = \{(V_j, \phi_j) \mid j \in J\}$ and $\mathscr{S}' = \{(V'_k, \phi'_k) \mid k \in K\}$ be oriented C^∞ atlases for M^n. For all $(j, k) \in J \times K$ and all $x \in V_j \cap V'_k$ with $V_j \cap V'_k \neq \varnothing$ the determinants of the Jacobian matrices of $\phi'_k \circ \phi_j^{-1}$ at $\phi_j(x)$ are either all positive or all negative, and we say that $\mathscr{S}$ and $\mathscr{S}'$ are *positively related* or *negatively related* accordingly. The oriented C^∞ atlases for M^n are divided into two classes according to the relation 'positively related'.

DEFINITION 1.2. An equivalence class of an oriented C^∞ atlas for M^n is called an *orientation* of M^n.

A C^∞ manifold $(M^n, \mathscr{D})$ is said to be *orientable* if it admits an oriented C^∞ atlas $\mathscr{S}$ such that $[\mathscr{S}] = \mathscr{D}$.

We say that an orientable manifold is *oriented* when we specify its orientation. The n dimensional sphere S^n, $n \geq 1$, is orientable.

We list some examples of differentiable manifolds. They will remind the reader that differentiable manifolds abound everywhere we look.

(1) n-dimensional Euclidian space is a C^∞ manifold.
(2) The n-dimensional sphere

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

with the relative topology as a subspace of $\mathbf{R}^{n+1}$ is a C^∞ manifold.

(3) Open submanifolds. Let $(M^n, \mathscr{D})$ be a C^∞ manifold and let U be an open subset of M^n. For an atlas $\mathscr{S} = \{(V_j, \varphi_j) \mid j \in J\}$,

$$\mathscr{S}_U = \{(V_j \cap U, \varphi_j | V_j \cap U) \mid j \in J\}$$

becomes an atlas of U. Set $\mathscr{D}_U = [\mathscr{S}_U]$ and say that $(U, \mathscr{D}_U)$ is an *open submanifold* of $(M^n, \mathscr{D})$. This definition does not depend on the choice of a representative $\mathscr{S}$.

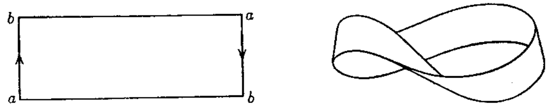


FIGURE 1.2

(4) Submanifolds. Let $(M^n, \mathscr{D})$ be a C^∞ manifold, and let A be a subset of M^n. Regard $\mathbf{R}^k$, $0 \leq k \leq n$, as a subspace of $\mathbf{R}^n$: $\mathbf{R}^k = \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_{k+1} = \cdots = x_n = 0\}$. Now assume that we can choose a representative $\mathscr{S} = \{(V_j, \varphi_j) | j \in J\}$ of $\mathscr{D}$ such that for each j with $V_j \cap A \neq \varnothing$

$$\varphi_j | V_j \cap A : V_j \cap A \longrightarrow \mathbf{R}^k \subset \mathbf{R}^n$$

is a homeomorphism onto an open subset of $\mathbf{R}^k$. Then evidently A is a topological manifold, and $\mathscr{S}_A = \{(V_j \cap A, \varphi_j | V_j \cap A) | j \in J\}$ defines an atlas of A. We say that $(A, \mathscr{S}_A)$ is a *submanifold* of M^n.

REMARK. A submanifold in Example 4 is different from a "submanifold" as used in differential geometry. Our submanifolds are submanifolds in differential geometry, but the converse is not true.

(5) Product manifolds. Let $(M, \mathscr{D})$ and $(M', \mathscr{D}')$ be C^∞ manifolds of dimensions n and n' respectively. Set $\mathscr{D} = [\mathscr{S}]$, $\mathscr{D}' = [\mathscr{S}']$, $\mathscr{S} = \{(V_j, \varphi_j) | j \in J\}$, $\mathscr{S}' = \{(V'_k, \varphi'_k) | k \in K\}$. Clearly, $M \times M'$ is an $n + n'$ topological manifold. Further, the set

$$\mathscr{S} \times \mathscr{S}' = \{(V_j \times V'_k, \varphi_j \times \varphi'_k) | (j, k) \in J \times K\}$$

turns out to be an atlas for $M \times M'$. We say that $(M \times M', [\mathscr{S} \times \mathscr{S}'])$ is the *product manifold* of $(M, \mathscr{D})$ and $(M', \mathscr{D}')$. When there is no confusion we simply write $M \times M'$.

EXAMPLE. The *torus* $T^2 = S^1 \times S^1$ is the product of two copies of the circle S^1.

(6) The Möbius strip. We obtain a *Möbius strip* by twisting a strip of a tape and pasting the edges as shown in Figure 1.2. More precisely the Möbius strip M^2 is defined by

$$M^2 = [0, 1] \times [0, 1] / \sim,$$
$$(0, t) \sim (1, 1 - t), \qquad t \in [0, 1].$$

The interior $\mathring{M}^2$ of the Möbius strip is a two-dimensional C^∞ manifold. This manifold is not orientable.

(7) Projective spaces. The *n-dimensional real projective space* $P_n(\mathbf{R}) = S^n / \sim$, $x \sim -x$, is an n-dimensional C^∞ manifold. We shall give a proof for the case $n = 2$. We may think of $P_2(\mathbf{R})$ as

$$P_2(\mathbf{R}) = \{[x_1, x_2, x_3] \mid \text{not all } x_1, x_2, x_3 \text{ are zero}, x_i \in \mathbf{R}, i = 1, 2, 3\}.$$

Set

$$U_i = \{[x_1, x_2, x_3] | x_i \neq 0\}, \qquad i = 1, 2, 3,$$

where $[x_1, x_2, x_3]$ is the equivalence class containing (x_1, x_2, x_3) (the condition $x_i \neq 0$ does not depend on the choice of a representative). Then $\{U_1, U_2, U_3\}$ is an open cover of $P_2(\mathbf{R})$:

$$P_2(\mathbf{R}) = U_1 \cup U_2 \cup U_3.$$

In addition $P_2(\mathbf{R})$ evidently satisfies the axiom of second countability. Next we define $\varphi_i : U_i \to \mathbf{R}^2$, $i = 1, 2, 3$ by

$$\varphi_1[x_1, x_2, x_3] = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right),$$
$$\varphi_2[x_1, x_2, x_3] = \left(\frac{x_3}{x_2}, \frac{x_1}{x_2}\right),$$
$$\varphi_3[x_1, x_2, x_3] = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

Clearly the above definition does not depend on the choice of a representative (x_1, x_2, x_3) of a point of $P_2(\mathbf{R})$. It is also obvious that each φ_i is a topological map of U_i onto $\mathbf{R}^2$. Hence we see that $P_2(\mathbf{R})$ is a topological manifold. Next for a point $[x_1, x_2, x_3]$ in the intersection $U_{12} = U_1 \cap U_2$ of U_1 and U_2 put

$$\varphi_1[x_1, x_2, x_3] = (u_1, u_2),$$
$$\varphi_2[x_1, x_2, x_3] = (\bar{u}_1, \bar{u}_2).$$

Then we have the expressions

$$\bar{u}_1 = \frac{u_2}{u_1}, \qquad \bar{u}_2 = \frac{1}{u_1},$$
$$u_1 = \frac{1}{\bar{u}_2}, \qquad u_2 = \frac{\bar{u}_1}{\bar{u}_2}.$$

Since $[x_1, x_2, x_3] \in U_{12}$, we have $x_1 \neq 0$, $x_2 \neq 0$. Hence, $u_1 \neq 0$ and $\bar{u}_2 \neq 0$. Therefore, $\bar{u}_1$, $\bar{u}_2$ are C^∞ functions of (u_1, u_2), and u_1, u_2 are C^∞ functions of $(\bar{u}_1, \bar{u}_2)$. The same statement holds for points of $U_2 \cap U_3$ and $U_3 \cap U_1$. Hence the family

$$\mathscr{D} = \{(U_i, \varphi_i) | i = 1, 2, 3\}$$

defines a C^∞ structure on $P_2(\mathbf{R})$. Hence, $P_2(\mathbf{R})$ is a C^∞ manifold. We may consider a natural C^∞ structure on $P_n(\mathbf{R})$ (verbatim as for the case $n = 2$ above) and thus conclude that $P_n(\mathbf{R})$ is a C^∞ manifold.

B. Differentiable maps. Let $(M_1, \mathscr{D}_1)$ and $(M_2, \mathscr{D}_2)$ be C^∞ manifolds of dimensions m and n, respectively.

DEFINITION 1.3. Consider a map $f : M_1 \to M_2$ of M_1 into M_2. For a point x of M_1 choose a chart V_j about x from a representative $\{(V_j, \varphi_j)|$

$j \in J\}$ of $\mathscr{D}_1$ and a chart V'_k about $f(x)$ from a representative $\{(V'_k, \varphi'_k) \mid k \in K\}$ of $\mathscr{D}_2$. Then we have the map

$$\varphi'_k \circ f \circ \varphi_j^{-1} : \varphi_j(V_j) \longrightarrow \varphi'_k(f(V_j) \cap V'_k)$$

from $\mathbf{R}^m$ onto an open subset of $\mathbf{R}^n$. We say that f is *differentiable at* x if $\varphi'_k \circ f \circ \varphi_j^{-1}$ is infinitely (continuously) differentiable at $\varphi_j(x)$. The map f is *differentiable* if it is differentiable at each point of M_1. We also say that f is a C^∞ *map*. Likewise we define a C^r *map* for any natural number r, $0 \leq r \leq \omega$.

From the definition of C^∞ structures, it is easy to see that the above definition is independent of the choice of representatives of $\mathscr{D}_1$ and $\mathscr{D}_2$ as well as V_j and V'_k.

Let M_1, M_2 be C^∞ manifolds, and let $f : A \to M_2$ be a map of a subset A of M_1. We say that f is *differentiable in* A if we can extend f to a C^∞ map of an open neighborhood U of A.

DEFINITION 1.4. Let M_1 and M_2 be C^∞ manifolds. We say that a map $f : M_1 \to M_2$ is a *diffeomorphism* if f satisfies the following:

(i) f is a homeomorphism of M_1 onto M_2, and

(ii) f, f^{-1} are C^∞ maps.

DEFINITION 1.5. Let M_1 and M_2 be C^∞ manifolds. We say that M_1 and M_2 are *diffeomorphic* and write $M_1 \underset{d}{\approx} M_2$ if there exists a diffeomorphism $f : M_1 \to M_2$.

Evidently $\underset{d}{\approx}$ is an equivalence relation. In differential topology we identify two manifolds which are diffeomorphic. According to Klein, differential topology is a field of mathematics where one studies properties of differentiable manifolds invariant under diffeomorphisms; however, this is too narrow a definition for contemporary differential topology.

We next define the ranks of differentiable maps, immersions, and embeddings of C^∞ manifolds.

DEFINITION 1.6. Let M_1 and M_2 be differentiable manifolds. Let $f : M_1 \to M_2$ be a differentiable map. For x in M_1 choose charts (U_1, h_1) and (U_2, h_2) about x and $f(x)$. We define the *rank* of f at x to be the rank of the Jacobian matrix of the map

$$h_2 \circ f \circ h_1^{-1} : h_1(U_1 \cap f^{-1}(U_2)) \longrightarrow h_2(U_2)$$

at $h_1(x)$.

Evidently Definition 1.6 does not depend on the chioce of charts.

DEFINITION 1.7. Let M^n and V^p be differential manifolds of dimensions n and p. A differentiable map $f : M^n \to V^p$ is an *immersion* if the rank of f at each point x of M^n is n. An immersion f is an *embedding* if f is a homeomorphism of M^n in V^p. We say that f is a *submersion* if the rank of f at each point x of M^n is p.

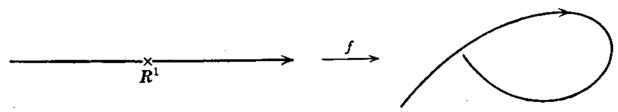


FIGURE 1.3

If $f : M^n \to V^p$ is an embedding, the image $f(M^n)$ is obviously a submanifold of V^p.

REMARK. An embedding f is an immersion but the converse is not true. Even when f is an immersion which is one-to-one into V^p, it may fail to be an embedding. Consider Figure 1.3.

DEFINITION 1.8. Let M^n and V^p be C^∞ manifolds of dimensions n and p, and let $f : M^n \to V^p$ be a C^∞ map. A point y of V^p is a *regular value* of f if the rank of f at each point x in $f^{-1}(y)$ is p; otherwise, y is a *critical value.*

According to the above definition points not in the image under f are regular values.

PROPOSITION 1.1. *Let M^n and V^p be C^∞ manifolds of dimensions n and p, and let $f : M^n \to V^p$ be a C^∞ map. If y is a regular value of f, then either f^{-1} is the empty set or an $n-p$ dimensional submanifold of M^n.*

The proposition follows easily from the definitions of submanifolds and of regular values.

C. Tangent spaces and the differentials of C^∞ maps.

DEFINITION 1.9. Let M^n be a C^∞ manifold, and let x be a point of M^n. A C^∞ map $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = x$ of an open interval $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$ (ε is sufficiently small), into M^n is called a *curve* at x. Suppose c_1 and c_2 are curves at x. For a chart $(U_\alpha, \varphi_\alpha)$ about x, $\varphi_\alpha \circ c_1$ and $\varphi_\alpha \circ c_2$ are C^∞ maps of $(-\varepsilon, \varepsilon)$ into $\mathbf{R}^n$. We say that c_1 and c_2 are equivalent and write $c_1 \sim c_2$ if

$$\left.\frac{d(\varphi_\alpha \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\varphi_\alpha \circ c_2)}{dt}\right|_{t=0}$$

(Figure 1.4).

By virtue of the definition of C^∞ structures the above definition does not depend on the choice of a chart. It is also clear that $\sim$ is an equivalence relation. Therefore, we can divide the set C_x of curves at x in M^n by $\sim$. We represent the class containing the curve c by $[c]_x$.

DEFINITION 1.10. Let M^n be a C^∞ manifold and let $x \in M^n$. We say that the set of equivalence classes of curves at x in M^n

$$T_x(M^n) = C_x/\sim = \{[c]_x \mid c \text{ is a curve at } x\}$$

is the tangent space of M^n at x.

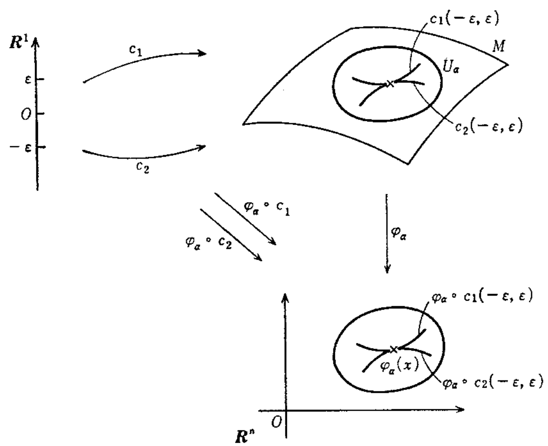


FIGURE 1.4

We define the operation of addition on $T_x(M^n)$ as follows.

DEFINITION 1.11. Let $[c_1]_x$ and $[c_2]_x$ be elements of $T_x(M^n)$, where $c_1, c_2 : (-\varepsilon, \varepsilon) \to M^n$ are C^∞ maps with $c_1(0) = c_2(0) = x$. For a chart $(U_\alpha, \varphi_\alpha)$ about x with $\varphi_\alpha(x) = 0$, we define the sum $\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}^n$ of the maps $\varphi_\alpha \circ c_1, \varphi_\alpha \circ c_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}^n$ by

$$(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)(t) = \varphi_\alpha \circ c_1(t) + \varphi_\alpha \circ c_2(t).$$

Hence, choosing a small enough ε' we have

$$(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)(-\varepsilon', \varepsilon') \subset \varphi_\alpha(U_\alpha).$$

Now we define $[c_1]_x + [c_2]_x$ by

$$[c_1]_x + [c_2]_x = [\varphi_\alpha^{-1}(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)]_x.$$

By the definition of C^∞ atlases, the above definition depends on neither the choice of a chart $(U_\alpha, \varphi_\alpha)$ nor the choice of $\varepsilon > 0$ for the domain of the curve.

For an element $[c]_x$ of $T_x(M^n)$ and a real number λ we define the scalar product $\lambda[c]_x$ in the natural manner.

LEMMA 1.1. *The space $T_x(M^n)$ with the operations of sum and scalar multiplication as above is an n-dimensional vector space.*

PROOF. It is trivial that $T_x(M^n)$ is a vector space. Thus, we only need to show that the dimension of $T_x(M^n)$ is n. Choose a chart $(U_\alpha, \varphi_\alpha)$ about x and consider the following n curves at x:

$$u_i : (-\varepsilon, \varepsilon) \longrightarrow M^n, \qquad i = 1, 2, \dots, n,$$

$$u_i(t) = \varphi_\alpha^{-1}(\underbrace{0, \dots, 0}_{i-1 \text{ zeros}}, t, 0, \dots, 0),$$

where $(0, \dots, 0, t, 0, \dots, 0)$ denotes an element of $\mathbf{R}^n$ whose components, except the ith one which is t, are zero. Then clearly the equivalence class $[c]_x$ of a curve at x is expressed as a linear combination of the $[u_1]_x, \dots, [u_n]_x$ which are easily seen to be linearly independent. □

We next define the differential of a differentiable map.

LEMMA 1.2. *Let M and N be C^∞ manifolds, and let $c_1, c_2 : (-\varepsilon, \varepsilon) \to M$ be curves at x in M with $c_1 \sim c_2$. For a C^∞ map $f : M \to N$, then the curves $f \circ c_1$ and $f \circ c_2$ at $f(x)$ satisfy $f \circ c_1 \sim f \circ c_2$.*

PROOF. Let $(U_\alpha, \varphi_\alpha)$ be a chart about x, and let $(V_\lambda, \psi_\lambda)$ be a chart about $f(x)$. Then

$$\psi_\lambda \circ f \circ c_i = (\psi_\lambda \circ f \circ \varphi_\alpha)^{-1} \circ (\varphi_\alpha \circ c_i), \qquad i = 1, 2.$$

Further, by assumption

$$\left.\frac{d(\varphi_\alpha \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\varphi_\alpha \circ c_2)}{dt}\right|_{t=0},$$

and hence we get

$$\left.\frac{d(\psi_\lambda \circ f \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\psi_\lambda \circ f \circ c_2)}{dt}\right|_{t=0}. \quad \square$$

DEFINITION 1.12. Let M and N be C^∞ manifolds and let $f : M \to N$ be a C^∞ map. For x in M define a map $(df)_x : T_x(M) \longrightarrow T_{f(x)}(N)$ by

$$(df)_x([c]_x) = [f \circ c]_{f(x)}.$$

By Lemma 1.2 this definition does not depend on the choice of a representative c of $[c]_x$. We say that $(df)_x$ is the *differential of f at x*.

LEMMA 1.3. *Let M and N be C^∞ manifolds, let $f : M \to N$ be a C^∞ map, and let $x \in M$. Then*

(1) *The differential $(df)_x : T_x(M) \longrightarrow T_{f(x)}(N)$ of f at x is a linear map.*

(2) *The rank of f at x equals the rank of $(df)_x$.*

We leave the proof to the reader.

Thus, a C^∞ map $f : M \to N$ is an immersion if and only if the map $(df)_x : T_x(M) \to T_{f(x)}(N)$ is injective at each point x of M.

§2. Fiber bundles

This section contains a summary of the facts about fiber bundles which are needed throughout our book. The material presented here is based largely on the work of Steenrod [A7].

A. Examples of fiber bundles. In order to enhance the reader's understanding of the subject we first give several examples.

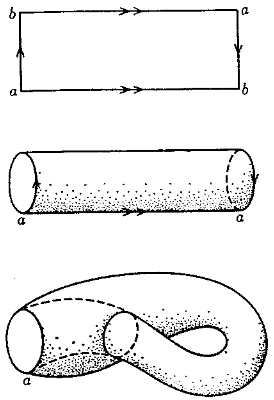


FIGURE 1.5

1. Product spaces Let X, Y be topological spaces, and set $B = X \times Y$. Define $p : B \to X$ by $p(x, y) = x$. Then p is a continuous map onto X and for each x in X, $p^{-1}(x) = Y_x$ is homeomorphic to Y. Fix a point y_0 of Y and define $f : X \to B$ by $f(x) = (x, y_0)$; f is continuous and $p \circ f(x) = x$.

2. The Möbius strip Recall that the Möbius strip is defined as follows:

$$M^2 = [0, 1] \times [0, 1]/\sim,$$
$$(0, t) \sim (1, 1-t), \qquad t \in [0, 1].$$

Setting $X = S^1 = [0, 1]/\sim$, $Y = [0, 1]$, and $B = M^2$ we define $p : B \to X$ by $p([(s, t)]) = [s] \in S^1$. Then p is a continuous map onto X, and for a point x of X, $p^{-1}(x) = Y_x$ is homeomorphic to Y. Further, there exists a neighborhood $V(x)$ of x such that $p^{-1}(V(x))$ is homeomorphic to $V(x) \times Y$. In addition the map $f : X \to B$ defined by $f(x) = [(x, 1/2)]$ is continuous and satisfies $p \circ f(x) = x$.

3. The Klein bottle The *Klein bottle* is the surface K^2 which we obtain by pasting one pair of facing edges of the rectangular $I \times J$, $I = J = [0, 1]$, in the same direction and the other pair in the opposite direction (Figure 1.5). That is,

$$K^2 = I \times J/\sim, \qquad (0, t) \sim (1, 1-t), \qquad t \in J,$$
$$(s, 0) \sim (s, 1), \qquad s \in I.$$

Put $B = K^2$, $X = S^1 = I/\sim$, and $Y = S^1 = J/\sim$. Define $p : B \to X$ by $p([(s, t)]) = [s] \in X$, which is continuous onto X. For each x of X $p^{-1}(x)$ is homeomorphic to $Y = S^1$. For a point x of X there exists a

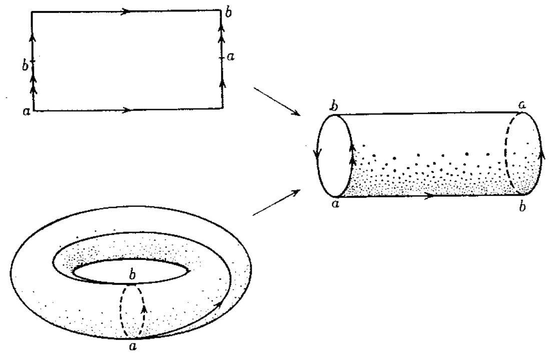


FIGURE 1.6

neighborhood $V(x)$ of x with $p^{-1}(V(x))$ homeomorphic to $V(x) \times Y$. The map $f : X \to B$ defined by $f(x) = [(x, 0)]$ is continuous, and $p \circ f(x) = x$.

4. Covering spaces Suppose B is a covering space of X and $p : B \to X$ is its covering map. Then p is evidently a continuous map onto X, and for each x in X the set $Y_x = p^{-1}(x)$ is discrete. In the case where X is arcwise connected, the Y_x are homeomorphic for all x in X. Further, for each x of X, there exists a neighborhood $V(x)$ of x and a homeomorphism between $p^{-1}(V(x))$ and $V(x) \times Y_x$ (B is a *covering space* of X, or (B, p, X) is a covering space, if (0) X is arcwise connected and locally arcwise connected, (i) $p : B \to X$ is a continuous surjection, (ii) for each $x \in X$ there exists an arcwise connected neighborhood V of x such that each connected component $\tilde{V}_\lambda$ is open in X and $p|\tilde{V}_\lambda : \tilde{V}_\lambda \to V$ is a homeomorphism onto V).

5. The twisted torus Consider $[0, 1] \times S^1$, and paste $\{0\} \times S^1$ and $\{1\} \times S^1$ with a $180°$ twist. The resulting surface T_W is the *twisted torus*:

$$T_W = [0, 1] \times S^1/\sim, \qquad (0, e^{2\pi i\theta}) \sim (1, e^{2\pi i(\theta+\pi)}).$$

Define $p : T_W \to S^1 = [0, 1]/\sim$ by $p([t, e^{2\pi i\theta}]) = [t]$. Then p is a continuous map onto X and for each point $[t]$ of S^1 there is a neighborhood V of $[t]$ in S^1 such that $p^{-1}(x)$ is homeomorphic to $V \times S^1$ (Figure 1.6).

B. The definition of a fiber bundle.

DEFINITION 1.13. Let G be a topological group, and let Y be a topological space. Suppose there is a continuous map $\eta : G \times Y \to Y$ satisying:

(i) For the unit e of G $\eta(e, y) = y$.
(ii) For all g_1, $g_2 \in G$ and $y \in Y$, $\eta(gg', y) = \eta(g, \eta(g', y))$.

Then we say that G is a *topological transformation group* of Y (with respect to η) and that G *acts* or *operates on* Y.

We shall write $g \cdot y$ for $\eta(g, y)$. By Definition 1.13 the map $g : Y \to Y$ which associates to each element y of Y the element $g \cdot y$ is a homeomorphism of Y. Hence, the map η induces a homomorphism $\bar{\eta} : G \to H(Y)$ from G to the group $H(Y)$ of homeomorphisms of Y.

DEFINITION 1.14. Let G be a topological transformation group of Y. If the homomorphism $\bar{\eta}$ above is injective, that is, $g \cdot y = y$ for all $y \in Y$ implies $g = e$, we say that G is *effective*.

For now unless otherwise stated our topological transformation groups are assumed effective.

DEFINITION 1.15. A *coordinate bundle* $\mathscr{B} = \{B, p, X, Y, G\}$ is a collection of topological spaces and continuous maps with structures satisfying the following:

(1) B and X are topological spaces; B is the *bundle space* or the *total space* and X is the *base space*. $p : B \to X$ is a continuous map called the *projection map* of $\mathscr{B}$.

(2) Y too is a topological space; Y is the *fiber* of $\mathscr{B}$. G is a topological transformation group called the *structural group* of $\mathscr{B}$.

(3) The base X has an open covering $\{V_j \mid j \in J\}$, and for each $j \in J$ there is a homeomorphism

$$\phi_j : V_j \times Y \longrightarrow p^{-1}(V_j);$$

the V_j's are *coordinate neighborhoods* and the ϕ_j's are *coordinate functions.*

(4) The coordinate functions satisfy the following:

(i) $p \circ \phi_j(x, y) = x$, $x \in V_j$, $y \in Y$, $j \in J$.

(ii) The map $\phi_{j,x} : Y \to p^{-1}(x)$ defined by

$$\phi_{j,x}(y) = \phi_j(x, y), \qquad y \in Y$$

gives a homeomorphism of Y,

$$\phi_{j,x}^{-1} \circ \phi_{i,x} : Y \to Y$$

for $x \in V_i \cap V_j$, which agrees with the action of an element $g_{ji}(x)$ of G.

(iii) Define a map

$$g_{ji} : V_i \cap V_j \longrightarrow G$$

by $g_{ji}(x) = \phi_{j,x}^{-1} \circ \phi_{i,x}$. Then g_{ji} is continuous; we say that the g_{ji} are *coordinate transformations* or a *transition functions* of $\mathscr{B}$.

Roughly speaking a coordinate bundle is a family $\{\bigcup_j V_j \times Y\}$ patched by the $\{g_{ji}\}$.

We write Y_x for $p^{-1}(x)$; Y_x is the *fiber over* x.

LEMMA 1.4. *Let $\mathscr{B} = \{B, p, X, Y, G\}$ be a coordinate bundle with coordinate transformations $\{g_{ji}\}$. Then*

(i) $g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x)$, $x \in V_i \cap V_j \cap V_k$,

(ii) $g_{ii}(x) = e$, $x \in V_i$, *where e is the unit element of G, and*

(iii) $g_{jk}(x) = [g_{kj}(x)]^{-1}$, $x \in V_j \cap V_k$.

The lemma follows readily from the definition of coordinate bundles.

We next define an equivalence relation in the strict sense between two coordinate bundles.

DEFINITION 1.16. We say that bundles $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X', Y', G'\}$ are *equivalent in the strict sense* and write $\mathscr{B} \approx \mathscr{B}'$ if they satisfy the following:

(i) $B = B'$, $X = X'$, $p = p'$.

(ii) $Y = Y'$, $G = G'$.

(iii) Their coordinate functions $\{\phi_j\}$, $\{\phi'_k\}$ satisfy the conditions that

$$\bar{g}_{kj}(x) = (\phi'_{k,x})^{-1} \circ \phi_{j,x}, \qquad x \in V_j \cap V'_k$$

coincides with the action of an element of G, and that the map

$$\bar{g}_{kj} : V_j \cap V'_k \longrightarrow G$$

is continuous.

It is easy to see that $\approx$ is an equivalence relation.

DEFINITION 1.17. An equivalence class of coordinate bundles is called a *fiber bundle*.

DEFINITION 1.18. We say that G is a *Lie group* if

(i) G is a topological group,

(ii) G is a C^∞ manifold, and

(iii) the group operations on G

$$\varphi_1 : G \times G \longrightarrow G, \qquad \varphi_2 : G \longrightarrow G,$$
$$\varphi_1(g, h) = gh, \qquad \varphi_2(g) = g^{-1}$$

are smooth.

EXAMPLE. $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{SO}(n)$ are Lie groups.

Here $\mathrm{SO}(n)$ is the group of n-dimensional orthogonal matrices whose determinants are of the value one, which is called the *n-dimensional rotation group.* There are natural inclusions $\mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbf{R}) \subset \mathbf{R}^{n^2}$. With the relative topology of $\mathrm{GL}(n, \mathbf{R})$, $\mathrm{SO}(n)$ is a topological group.

DEFINITION 1.19. Consider a coordinate bundle $\mathscr{B} = \{B, p, X, Y, G\}$ satisfying:

(i) B, X, Y are C^∞ manifolds.

(ii) G is a Lie group and its action on Y is C^∞.

(iii) The maps p, ϕ_j, ϕ_j^{-1}, g_{ij} are all C^∞.

We say that $\mathscr{B}$ is a *smooth coordinate bundle* and its equivalence class $\{\mathscr{B}\}$ is a *smooth fiber bundle*.

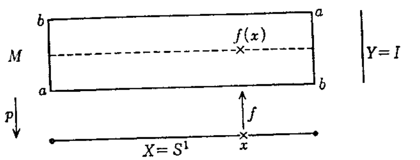


FIGURE 1.7

If X, Y are C^∞ manifolds and if G is a Lie group acting smoothly on Y, then we can make $\{\mathscr{B}\} = \{B, p, X, Y, G\}$ into a smooth fiber bundle by introducing a suitable C^∞ structure on B.

REMARK. In the above definition we often take G to be a subgroup, not necessarily finite dimensional, of the group $\operatorname{Diff} Y$ of diffeomorphisms of Y.

DEFINITION 1.20. Let $\mathscr{B} = \{B, p, X, Y, G\}$ be a coordinate bundle. By a *cross section* of $\mathscr{B}$ we mean a continuous map $f : X \longrightarrow B$ with $p \circ f = 1$. A smooth map f which is a cross section of a smooth coordinate bundle $\mathscr{B}$ is called a *smooth cross section.*

EXAMPLE. The center line of the Möbius strip $\{B, p, X, Y, G\}$ gives a cross section. See Figure 1.7.

DEFINITION 1.21. Let $\mathscr{B} = \{B, p, X, Y, G\}$ be a coordinate bundle, and let A be a subset of X. Then the restriction

$$p|p^{-1}(A) : p^{-1}(A) \longrightarrow A$$

inherits naturally a coordinate bundle structure from $\mathscr{B}$, which is called the *restriction of* $\mathscr{B}$ *to* the portion of A and denoted by $\mathscr{B}|A$.

C. Bundle maps.

DEFINITION 1.22. Let $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X', Y', G'\}$ be coordinate bundles such that $Y = Y'$ and $G = G'$. A continuous map h : $B \to B'$ is called a *bundle map* and is denoted by $h : \mathscr{B} \to \mathscr{B}'$ if it satsfies the following:

(i) h maps a fiber Y_x of B homeomorphically onto a fiber $Y'_{x'}$ over $x' \in X'$; thus, putting $\bar{h}(x) = x'$ we have a continuous map $\bar{h} : B \to B'$ and the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\bar{h}} & X' \end{array}$$

(ii) For a point x of $V_j \cap \bar{h}^{-1}(V'_k)$ and the homeomorphism $h_x : Y_x \to Y_{x'}$ defined by (i), the map

$$\bar{g}_{kj}(x) = (\phi'_{k,x'})^{-1} \circ h_x \circ \phi_{j,x}$$

is a homeomorphism of Y, which coincides with the action of an element of G.

(iii) The above correspondence defines a continuous map

$$\bar{g}_{kj} : V_j \cap \bar{h}^{-1}(V_k) \longrightarrow G.$$

We say that $\bar{h}$ is the *map induced by* h.

Evidently the identity map $1_B : B \to B$ defines a bundle map $\mathscr{B} \to \mathscr{B}'$. Further the composition of bundle maps is also a bundle map.

The maps $\{\bar{g}_{kj}\}$ defined in (iii) are called the *mapping transformations* of h and satisfy the following:

$$\left.\begin{array}{ll} \bar{g}_{kj}(x) g_{ji}(x) = \bar{g}_{ki}(x), & x \in V_i \cap V_j \cap \bar{h}^{-1}(V'_k), \\ g'_{lk}(\bar{h}(x)) \bar{g}_{kj}(x) = \bar{g}_{lj}(x), & x \in V_j \cap \bar{h}^{-1}(V'_k \cap V'_l). \end{array}\right\} \qquad (1)$$

These relations follow easily from Definition 1.22.

LEMMA 1.5. *Let* $\mathscr{B} = \{B, p, X, Y, G\}$ *and* $\mathscr{B}' = \{B', p', X', Y', G'\}$ *be coordinate bundles with* $Y = Y'$ *and* $G = G'$. *Let* $\bar{h} : X \to X'$ *be a continuous map. Suppose continuous maps*

$$\bar{g}_{kj} : V_j \cap \bar{h}^{-1}(V'_k) \longrightarrow G$$

satisfy the relations (1). *Then there exists a unique bundle map* $h : \mathscr{B} \to \mathscr{B}'$ *such that*

(i) h *induces* $\bar{h}$, *and*
(ii) *the* $\bar{g}_{jk}$ *are the mapping transformations of* h.

PROOF. Existence. Put $p(b) = x \in V_j \cap \bar{h}^{-1}(V'_k)$ and

$$h_{kj}(b) = \phi'_k(\bar{h}(x), \bar{g}_{kj}(x) \cdot p_j(b)).$$

Then h_{kj} is continuous in b. Here $p_j(b) = \phi_{j,x}{}^{-1}(b)$, $p(b) = x \in V_j$, and $p'_0 h_{kj}(b) = h(p(b))$. Now suppose that

$$x \in V_i \cap V_j \cap \bar{h}^{-1}(V'_k \cap V'_l);$$

that is, b is contained in each of the domains of h_{kj} and h_{li}. Then by Lemma 1.4,

$$\begin{aligned} h_{kj}(b) &= \phi'_k(x', \bar{g}_{kj}(x) \cdot g_{ji}(x) \cdot p_i(b)), \qquad x' = \bar{h}(x) \\ &= \phi'_k(x', \bar{g}_{ki}(x) \cdot p_i(b)) = h_{ki}(b) \\ &= \phi'_l(x', g'_{lk}(x') \cdot \bar{g}_{ki}(x) \cdot p_i(b)) \\ &= \phi'_l(x', \bar{g}_{li}(x) \cdot p_i(b)) = h_{li}(b). \end{aligned}$$

Hence, we have

$$B = \bigcup_{p^{-1}} (V_j \cap \bar{h}^{-1}(V'_k)),$$

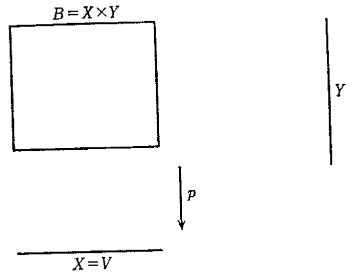


FIGURE 1.8

and the family $\{h_{jk}\}$ of continuous maps

$$h_{jk}: p^{-1}(V_j \cap \bar{h}^{-1}(V'_k)) \longrightarrow B', \qquad j \in J, k \in J',$$

which agree on the intersection on any two domains, defines a continuous map $B \to B'$. Further by the definition of h_{jk} we have $p' \circ h = \bar{h} \circ p$, that is, h induces $\bar{h}$. We also note that

$$\begin{aligned} p'_k \circ h \circ \phi_{j,x}(y) &= p'_k \circ \phi'_k(x', \bar{g}_{kj}(x) \cdot p_j \circ \phi_{j,x}(y)) \\ &= \bar{g}_{kj}(x) \cdot y. \end{aligned}$$

Hence, h is a bundle map and the $\{\bar{g}_{kj}\}$ are its mapping transformations. Here $p'_k(b') = \phi'^{-1}_{k,x'}(b')$ and $p'(b') = x' \in V'_k$.

Uniqueness. Suppose $h : B \to B'$ is a bundle map satisfing (i) and (ii). Then if

$$p(b) = x \in V_j \cap \bar{h}^{-1}(V'_k),$$

we have

$$h(b) = \phi'_k(\bar{h}(x), \bar{g}_{kj}(x) \cdot p_j(b)),$$

and so the uniqueness follows. □

DEFINITION 1.23. Let $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X', Y', G'\}$ be coordinate bundles with $X = X'$, $Y = Y'$, and $G = G'$. We say that $\mathscr{B}$ and $\mathscr{B}'$ are *equivalent* and write $\mathscr{B} \sim \mathscr{B}'$ if there exists a bundle map $h : \mathscr{B} \to \mathscr{B}'$ such that $\bar{h} = 1_X$.

The relation $\sim$ is an equivalence relation. If $\mathscr{B}$ and $\mathscr{B}'$ are equivalent in the strict sense they are also equivalent.

DEFINITION 1.24. Two fiber bundles $\{\mathscr{B}\}$ and $\{\mathscr{B}'\}$ are *equivalent* if their representatives $\mathscr{B}$ and $\mathscr{B}'$ are equivalent; in this case we write $\{\mathscr{B}\} \sim \{\mathscr{B}'\}$.

That two coordinate bundles are equivalent corresponds to the fact that they are "equal" with respect to their differential structures.

Let X and Y be topological space. The *product bundle* consists of the total space $B = X \times Y$, the projection $p : B \to X$ onto the first component, $G = \{e\}$, and a single coordinate neighborhood $V = X$. A coordinate bundle equivalent to the product bundle is *trivial*. See Figure 1.8.

LEMMA 1.6. *Let $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X', Y', G'\}$ be coordinate bundles with $X = X'$, $Y = Y'$ and $G = G'$. Then $\mathscr{B} \sim \mathscr{B}'$ if and only if there exist continuous maps*

$$\bar{g}_{kj} : V_j \cap V'_k \longrightarrow G$$

such that

$$\begin{aligned} \bar{g}_{ki}(x) &= \bar{g}_{kj}(x) g_{ji}(x), & x \in V_i \cap V_j \cap V'_k, \\ \bar{g}_{lj}(x) &= g'_{lk}(x) \bar{g}_{kj}(x), & x \in V_j \cap V'_k \cap V'_l, \end{aligned} \tag{2}$$

where the primed terms belong to the bundle $\mathscr{B}'$.

PROOF. To prove the 'only if' part suppose $\mathscr{B} \sim \mathscr{B}'$. Then there exists a bundle map $h : \mathscr{B} \to \mathscr{B}'$ with $\bar{h} = 1_X$; so the maps $\bar{g}_{kj}$ defined by

$$\bar{g}_{kj} = \phi'_{k,x} \circ h_x \circ \phi_{j,x}, \qquad x \in V_j \cap V'_k$$

satisfy the relations (1), and hence they satisfy the relations (2).

To prove the 'if' part suppose we have maps $\bar{g}_{kj}$ satisfying condition (2). This requirement is the same as condition (1) when $\bar{h} = 1$. Hence, Lemma 1.5 will construct the desired bundle map $h : \mathscr{B} \to \mathscr{B}'$. □

LEMMA 1.7. *Let $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X', Y', G'\}$ with $X = X'$, $Y = Y'$, $G = G'$. We assume in addition that their coordinate neighborhood systems are equal; $\{V_j \mid j \in J\} = \{V'_k \mid k \in J'\}$. Then $\mathscr{B}$ and $\mathscr{B}'$ are equivalent if and only if there exists a family of continuous maps*

$$\lambda_j : V_j \longrightarrow G, \qquad j \in J$$

satisfying the relations:

$$g'_{ji}(x) = \lambda_j(x)^{-1} g_{ji}(x) \lambda_i(x), \qquad x \in V_i \cap V_j. \tag{3}$$

PROOF. To prove the 'only if' part suppose $\mathscr{B} \sim \mathscr{B}'$, then by Lemma 1.6 we have maps $\bar{g}_{kj}$ which satisfy requirement (2); therefore, we define continuous maps λ_j by $\lambda_j(x) = (\bar{g}_{jj}(x))^{-1}$, which satisfy requirement (3).

To prove the 'if' part, assume conversely that there exists a family $\{\lambda_j : j \in J\}$ of maps satisfying condition (3). Putting

$$\bar{g}_{kj}(x) = \lambda_k(x)^{-1} g_{kj}(x), \qquad x \in V_j \cap V_k,$$

we get requirement (2) from (3). Hence, by Lemma 1.6 we get $\mathscr{B} \sim \mathscr{B}'$. □

D. Steenrod's structure theorem.

DEFINITON 1.25. Let X be a topological space and let G be a topological group. By a *system of coordinate transfomations in X with values in G* we mean a pair of families $(\{V_j\}, \{g_{ji}\})$ which satisfy the following:

(i) The $\{V_j \mid j \in J\}$ is an open covering of X.
(ii) The maps $g_{ji} : V_i \cap V_j \to G$ are continuous and satisfy

$$g_{kj}(x) g_{ji}(x) = g_{ki}(x), \qquad x \in V_i \cap V_j \cap V_k. \tag{4}$$

It follows immediately that the pair $(\{V_j\}, \{g_{ji}\})$ of coordinate neighborhoods and coordinate transformations of a coordinate bundle $\mathscr{B}$ is a system of coordinate transformations in its base space with values in its structure group. The converse also holds.

Theorem 1.1 (Steenrod [A7]). *Let X, Y be topological spaces, let G be a topological transformation group of Y and let $(\{V_j\}, \{g_{ji}\})$ be a system of coordinate transformations of X with values in G. Then*

(i) *There exists a coordinate bundle $\mathscr{B}$ with base space X, fiber Y, coordinate neighborhoods $\{V_j\}$, and coordinate transformations $\{g_{ji}\}$.*

(ii) *Two coordinate bundles satisfying requirement* (i) *are equivalent.*

Proof. (i) Introduce the discrete topology on the index set J in $\{V_j \mid j \in J\}$ and set

$$T = \{(x, y, j) \in X \times Y \times J \mid x \in V_j\}.$$

Then T is a topological space which is the union of pairwise disjoint open sets $V_j \times Y \times j$. Define a relation $\sim$ in T as follows:

$$(x, y, j) \sim (x', y', k) \quad \text{if and only if } x = x' \text{ and } g_{kj}(x)y = y',$$
$$(x, y, j), (x', y', k) \in T.$$

By (4), $\sim$ is an equivalence relation. Let B denote the quotient space of T by $\sim$; $B = T/\sim$. Let $q : T \to B$ be the natural projection map; then q is continuous (B is given the quotient topology: $U \subset B$ is open if $q^{-1}(U)$ is open in T). Define a map $p : B \to X$ by $p(\{(x, y, j)\}) = x$. Then the following commutative diagram shows that p is continuous:

$$\begin{array}{ccc} X \times Y \times J \supset T & \xrightarrow{\ q\ } & B = T/\sim \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p} & \\ & X & \end{array}$$

where p_1 is the projection onto the first component.

We next define coordinate functions $\phi_j : V_j \times Y \to p^{-1}(V_j)$ by $\phi_j(x, y) = q(x, y, j)$. As q is continuous so is each ϕ_j. Further, $p \circ q(x, y, j) = x$ implies $p \circ \phi_j(x, y) = x$, and so $\phi_j : V_j \times Y \to p^{-1}(V_j)$. Further, each ϕ_j maps $V_j \times Y$ onto $p^{-1}(V_j)$, for if $b = \{(x, y, k)\} \in p^{-1}(V_j)$ then $x \in V_j \cap V_k$ and $(x, y, k) \sim (x, g_{jk}(x) \cdot y, j)$; hence, we can write $b = \phi_j(x, g_{jk}(x) \cdot y)$.

To show that ϕ_j is one-to-one, suppose $\phi_j(x, y) = \phi_j(x', y')$; that is, $(x, y, j) \sim (x', y', j)$. Then $x = x'$ and $g_{jj}(x) \cdot y = y'$. But $g_{jj}(x) = e$, and so $y = y'$. Hence, ϕ_j is one-to-one.

We now prove that ϕ_j^{-1} is continuous. Suppose that W is an open subset of $V_j \times Y$; we want to show that $\phi_j(W)$ is open in B. To do so it is enough to show that $q^{-1}(\phi_j(W))$ is open in T. But T is a pairwise disjoint union of open sets $V_k \times Y \times k$; hence, it suffices to show that the intersection of $q^{-1}(\phi_j(W))$ and $V_k \times Y \times k$ is an open subset of $V_k \times Y \times k$. Now the set $q^{-1}(\phi_j(W)) \cap (V_k \times Y \times k)$ is contained in $(V_j \cap V_k) \times Y \times k$, and so we decompose q as follows:

$$\begin{array}{ccc} (V_j \cap V_k) \times Y \times k & \subset T \subset & X \times Y \times J \\ {\scriptstyle r} \swarrow & & \searrow {\scriptstyle q} \\ V_j \times Y & \xrightarrow{\ \phi_j\ } & p^{-1}(V_j) \subset B = T/\sim \\ \cup & & \cup \\ W & & \phi_j(W) \end{array}$$

Here $r(x, y, k) = (x, g_{jk}(x) \cdot y)$.

Since r is continuous, $r^{-1}(W)$ is an open set. Thus, ϕ_j^{-1} is continuous.

Now the map $\phi_{j,x}^{-1} \circ \phi_{i,x}$, $x \in V_i \cap V_j$, is a homeomorphism of Y. Putting $y' = \phi_{j,x}^{-1} \circ \phi_{i,x}(y)$, we get $\phi_j(x, y') = \phi_i(x, y)$; that is, $q(x, y', j) = q(x, y, i)$ or equivalently $(x, y', j) \sim (x, y, i)$. Hence, $y' = g_{ji}(x) \cdot y$, and so

$$\phi_{j,x}^{-1} \circ \phi_{i,x}(y) = g_{ji}(x) \cdot y, \qquad y \in Y.$$

We have constructed the desired coordinate bundle $\mathscr{B} = \{B, p, X, Y, G\}$ whose coordinate transformations are the $\{g_{ji}\}$.

(ii) Putting $\lambda_j(x) = e$, $x \in V_j$ in Lemma 1.7, we see that two coordinate bundles with equal coordinate transformations are equivalent. Thus, the coordinate bundle constructed above is unique up to equivalence classes. □

E. Tangent bundles of differentiable manifolds. We take an n-dimensional manifold M^n for the X, Euclidean n-space $\mathbf{R}^n$ for the Y, and the general linear group $\mathrm{GL}(n, \mathbf{R})$ for the G in Steenrod's structure theorem–Theorem 1.1. In this case G acts smoothly on Y. Choose a C^∞ atlas $\mathscr{S} = \{(U_j, \varphi_j) \mid j \in J\}$ on M^n and consider the following system of coordinate transformations $(\{U_j\}, \{a_{ij}\})$ of M^n with values in $\mathrm{GL}(n, \mathbf{R})$:

$$a_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(n, \mathbf{R}),$$
$$a_{ij}(x) = \text{the Jacobian matrix of } \varphi_j \circ \varphi_i^{-1} \text{ at } \varphi_i(x).$$

The fiber bundle constructed from this system via Steenrod's structure theorem is called the *tangent bundle* of M^n and is denoted by

$$\tau(M^n) = \{T(M^n), p, M^n, \mathbf{R}^n, \mathrm{GL}(n, \mathbf{R})\}.$$

It turns out that

$$T(M^n) = \bigcup_{x \in M^n} T_x(M^n)$$

with suitable differentiable structures.

DEFINITION 1.26. Let M^n be a C^∞ manifold and let $\tau(M^n)$ be the tangent bundle of M^n. A cross section of $\tau(M^n)$ is called a *vector field* on M^n.

DEFINITION 1.27. Let M^n be a C^∞ manifold. We say that M^n is *parallelizable* if the tangent bundle $\tau(M^n)$ is trivial.

EXAMPLE 1. A Lie group is parallelizable.

EXAMPLE 2. The n-sphere S^n is parallelizable if and only if $n = 1, 2$, or 7 (cf. J. Adams, On the nonexistence of elements of Hopf invariant one, Ann. of Math., **72** (1960)).

F. Reductions of structure groups.

DEFINITION 1.28. Let G be a topological group, let H be a subgroup of G, and let $i : H \to G$ be the inclusion map. Consider the coordinate bundle $\mathscr{B} = \{B, p, X, Y, H; \{V_j\}, \{g_{ij}\}\}$ whose structural group, coordinate neighborhoods, and coordinate transformations are H, $\{V_j\}$, and $\{g_{ij}\}$. We assume that G acts on Y extending the given action of H on Y. Using these Y, G, X, $\{V_j\}$, and $\{i \circ g_{ij}\}$, we construct a coordinate bundle $\mathscr{B}'$ according to Theorem 1.1, which is called a *G-image* of $\mathscr{B}$ or a coordinate bundle obtained by *enlarging* the structural group of $\mathscr{B}$ to G. Conversely if coordinate bundles $\mathscr{B}$ and $\mathscr{B}'$ are related as above we say that $\mathscr{B}$ is a coordinate bundle whose structural group is a *reduction* of the structural group G to H.

The structural group of the tangent bundle $\tau(M^n)$ of a C^∞ manifold M^n is $GL(n, \mathbf{R})$; $O(n)$ is a closed subgroup of $GL(n, \mathbf{R})$.

DEFINITION 1.29. A reduction of the structural group of the tangent bundle $\tau(M^n)$ of a C^∞ manifold M^n to $O(n)$ is called a *Riemannian metric* on M^n.

When a Riemannian metric is defined on a C^∞ manifold M^n we may assume a Euclidean metric $\langle\ ,\ \rangle_x$ on the fiber $T_x(M^n)$ over each $x \in M^n$, of $\tau(M^n)$. Note that this metric varies smoothly in x. The converse is also true.

THEOREM 1.2. *A C^∞ manifold M^n admits a Riemannian metric.*

PROOF. Let $M^n = (M^n, \mathscr{D})$, $\mathscr{D} = [\mathscr{S}]$, and $\mathscr{S} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$. Then the tangent bundle $\tau(M^n)$ of M^n is trivial over each U_α. Take a partition of unity $\{\lambda_i\}$ subordinate to $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$. Set $V_i = \lambda_i^{-1}(0, 1)$, then $\{V_i\}$ is a locally finite refinement of $\{U_\alpha\}$. Since $\tau(M^n)|V_i$ is a trivial vector bundle, it gives rise to a Euclidean metric $\langle\ ,\ \rangle_i$ on $T_x(M^n)$. In fact, we may define $\langle\ ,\ \rangle_{i,x}$ by

$$\langle u, v\rangle_{i,x} = \langle \phi_{i,x}^{-1}(u), \phi_{i,x}^{-1}(v)\rangle, \qquad u, v \in T_x(M^n),$$

where $\phi_i : V_i \times \mathbf{R}^n \to p^{-1}(V_i)$, $\phi_{i,x}(y) = \phi_i(x, y)$, and $\langle\ ,\ \rangle$ on the right-hand side of the equation is the usual inner product in $\mathbf{R}^n$. The desired

Euclidean metric may be defined by

$$\langle u, v\rangle_x = \sum_i \lambda_i(x)\langle u, v\rangle_{i,x}, \qquad u, v \in T_x(M^n).$$

We can readily check that $\langle\ ,\ \rangle_x$ is symmetric and positive definite. □

REMARK. We may also carry out our proof using the reducibility of the quotient $GL(n, \mathbf{R})/O(n)$.

The n-dimensional unitary group $U(n)$ sits naturally in the rotation group $SO(2n)$: consider the map $\rho : U(n) \to SO(2n)$ defined by

$$\rho(C) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad C \in U(n),$$

$$C = (c_{ij}), \qquad c_{ij} = a_{ij} + \sqrt{-1}\, b_{ij}, \qquad A = (a_{ij}), \qquad B = (b_{ij}).$$

Then ρ is a continuous isomorphism of $U(n)$ in $SO(2n)$.

DEFINITION 1.30. Let M^{2n} be a $2n$-dimensional C^∞ manifold. By Theorem 1.2 we may take $O(2n)$ for the structural group of the tangent bundle $\tau(M^n)$ of M^n. A reduction of the structural group $O(2n)$ of M^n to $U(n)$ is called an *almost complex structure* of M^n. An *almost complex manifold* M^{2n} is a manifold with an almost complex structure.

A complex manifold is almost complex. It is also evident that an almost complex manifold is orientable.

Since $SO(2) = U(1)$, a two-dimensional smooth orientable manifold has an almost complex structure.

G. Induced bundles.

DEFINITION 1.31. Let $\mathscr{B}' = \{B', p', X', Y, G\}$ be a coordinate bundle. Let X be a topological space, and let $\eta : X \to X'$ be a continuous map. For a system of coordinate neighborhoods $\{V_j' \mid j' \in J'\}$, the family $\{\eta^{-1}(V_j') \mid j \in J'\}$ is an open cover of X. Setting

$$g_{ji}(x) = g_{ji}'(\eta(x)), \qquad x \in V_i \cap V_j,$$

we obtain a system of coordinate transformations $(\{V_j\}, \{g_{ji}\})$ in X with values in G. We define the *pullback* or the *induced bundle* $\eta^*(\mathscr{B}')$ of $\mathscr{B}'$ *over* X by η to be the coordinate bundle as constructed in Theorem 1.1 from $\{Y, G; X, \{V_j\}, \{g_{ji}\}\}$.

The following is an alternative definition of an induced bundle over X. Let $\mathscr{B}' = \{B', p', X', Y, G\}$ be a coordinate bundle and let $\eta : X \to X'$ be a continuous map. Consider the following subspace B of $X \times B'$:

$$B = \{(x, b') \in X \times B' \mid \eta(x) = p'(b')\}.$$

We then have the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\eta} & X' \end{array}$$

where $\pi_1 : X \times B' \to X$ and $\pi_2 : X \times B' \to B'$ are the projections onto X and B' respectively, $p = \pi_1|B$ and $h = \pi_2|B$.

Putting $V_j = \eta^{-1}(V'_j)$ and defining $\phi_j : V_j \times Y \to p^{-1}(V_j)$ by

$$\phi_j(x, y) = (x, \phi'_j(\eta(x), y)),$$

we obtain the coordinate bundle $\{B, p, X, Y, G\}$, which is equivalent to the pullback $\eta^*\mathscr{B}$.

Induced bundles have the following properties.

PROPOSITION 1.2. (i) *Let $\mathscr{B}'_1$, $\mathscr{B}'_2$ be coordinate bundles over X', and let $\eta : X \to X'$ be a continuous map. Then*

$$\mathscr{B}'_1 \sim \mathscr{B}'_2 \Longrightarrow \eta^*\mathscr{B}'_1 \sim \eta^*\mathscr{B}'_2.$$

(ii) *For a coordinate bundle $\mathscr{B}$ over X we have $1_X{}^*\mathscr{B} \sim \mathscr{B}$, where $1_X : X \to X$ is the identity map.*

(iii) *For a coordinate bundle $\mathscr{B}'$ over X' and a constant map $c : X \to X$, the induced bundle $c^*\mathscr{B}'$ is trivial.*

(iv) *For a coordinate bundle $\mathscr{B}''$ over X'' and continuous maps $\eta : X \to X'$, $\eta' : X' \to X''$,*

$$(\eta' \circ \eta)^*\mathscr{B}'' \sim \eta^*(\eta'^*\mathscr{B}'').$$

(v) *For continuous maps $f, g : X \to X'$, which are homotopic,*

$$f^*\mathscr{B}' \sim g^*\mathscr{B}'.$$

(vi) *If $\mathscr{B}'$ is trivial so is $\eta^*\mathscr{B}'$.*

H. Associated bundles, principal bundles.

DEFINITION 1.32. A bundle $\{B, p, X, Y, G\}$ is called a *principal bundle* if $Y = G$ and G acts on Y by left translations.

EXAMPLE. Suppose B is a Lie group and G is a closed subgroup of B. Then the natural projection $p : B \to B/G$ is a principal bundle (cf. Steenrod[A7]).

DEFINITION 1.33. Let $\mathscr{B} = \{B, p, X, Y, G\}$ be a coordinate bundle with coordinate neighborhoods $\{V_j\}$ and coordinate transformations $\{g_{ji}\}$. The *associated principal bundle* $\tilde{\mathscr{B}}$ of $\mathscr{B}$ is the bundle given by Theorem 1.1 using $(G, G; X, \{V_j\}, \{g_{ij}\})$ where G acts on G by left translations.

In short, we obtain $\tilde{\mathscr{B}}$ from $\mathscr{B}$ by replacing Y by G.

DEFINITION 1.34. Suppose bundles $\mathscr{B} = \{B, p, X, Y, G\}$ and $\mathscr{B}' = \{B', p', X, Y', G\}$ have the same base space and the same structural group. We say that $\mathscr{B}$ and $\mathscr{B}'$ are *associated* if the associated principal bundles $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{B}}'$ of $\mathscr{B}$ and $\mathscr{B}'$ respectively are equivalent. We also say that $\mathscr{B}'$ is the associated bundle of $\mathscr{B}$ with the fiber Y'.

In particular, a bundle $\mathscr{B}$ and its associated principal bundle $\tilde{\mathscr{B}}$ are associated.

In other words $\mathscr{B}'$ is associated with $\mathscr{B}$ if $\mathscr{B}'$ is a replacement of $\mathscr{B}$ in which the new fiber is Y'.

EXAMPLE. The Möbius strip and the Klein bottle are associated as bundles over S^1.

I. Quotient spaces of Lie groups. An ordered k-tuple $v^k = (u_1, u_2, \dots, u_k)$ of linearly independent vectors in n-dimensional Euclidean space $\mathbf{R}^n$ is called an *orthonormal* **k**-*frame*. The set of all orthonormal k-frames is denoted by $V_{n,k}$. Since $V_{n,k}$ is a subset of $\underbrace{\mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{k}$, we give $V_{n,k}$ the relative topology of $\underbrace{\mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{k}$. We say that $V_{n,k}$ is the *Stiefel manifold* of orthonormal k-frames in n-dimensional Euclidean space $\mathbf{R}^n$.

Evidently $O(n)$ acts transitively on $V_{n,k}$. Now pick an orthonormal k-frame v_0^k. We may think of $O(n-k)$ as the isotropy group of v_0^k. Hence,

$$V_{n,k} \approx O(n)/O(n-k).$$

In particular, we have $V_{n,n} = O(n)$ and $V_{n,1} = S^{n-1}$.

The natural map

$$V_{n,k} = O(n)/O(n-k) \longrightarrow V_{n,k-1} = O(n)/O(n-k+1)$$

is a bundle whose fiber and structure group are S^{n-k} and $O(n-k+1)$, respectively (cf. Steenrod [A7]); thus, we see that $V_{n,k}$ is a C^∞ manifold.

Let $\mathbf{R}_{m,n}$ denote the set of n-dimensional subspaces of $m+n$-dimensional Euclidean space $\mathbf{R}^{m+n}$. Consider the map $p : V_{m+n,m} \to \mathbf{R}_{m,n}$ which associates to each element v^n of the Stiefel manifold $V_{m+n,n}$ the n-dimensional subspace spanned by v^n in $\mathbf{R}^{m+n}$. This map is clearly surjective. We give $\mathbf{R}_{m,n}$ the quotient topology; a subset U of $\mathbf{R}_{m,n}$ is open if $p^{-1}(U)$ is open. We say that $\mathbf{R}_{m,n}$ is the *Grassmann manifold* of n-dimensional subspaces in $\mathbf{R}^{m+n}$. The standard action of the orthogonal group $O(m+n)$ on $\mathbf{R}_{m,n}$ makes $O(m+n)$ a transitive topological transformation group of $\mathbf{R}_{m,n}$. Denote by $\mathbf{R}_0^n$ the following subset of $\mathbf{R}^{m+n}$:

$$\mathbf{R}_0^n = \{(x_1, \dots, x_{m+n}) \in \mathbf{R}^{m+n} | x_{n+1} = x_{n+2} = \cdots = x_{m+n} = 0\}.$$

The isotropy group of $\mathbf{R}_0^n$ is $O(n) \times O'(m)$, where $O'(m)$ is a subspace of $O(m+n)$,

$$O'(m) = \left\{ \begin{pmatrix} E_n & 0 \\ 0 & A \end{pmatrix} \in O(m+n) \,|\, A \in O(m), \; E_n \text{ is the } n\text{-dimensional identity matrix} \right\}.$$

Hence,

$$\mathbf{R}_{m,n} \approx O(m+n)/O(n) \times O'(m).$$

Further, the natural map

$$p : V_{m+n,n} = \mathrm{O}(m+n)/\mathrm{O}'(m) \longrightarrow \mathrm{O}(m+n)/\mathrm{O}(n) \times \mathrm{O}'(m) = \mathbf{R}_{m,n}$$

is a principal bundle whose structural group is $\mathrm{O}(n)$ (cf. Steenrod [A7]). Thus, we have the following

PROPOSITION 1.3. *The Grassmann manifold* $\mathbf{R}_{m,n}$ *is a* C^∞ *manifold of dimension* mn.

J. Classifying spaces. With a fixed n we have the following natural map between Grassmann manifolds:

$$\begin{array}{ccc} \rho_m : \mathbf{R}_{m,n} & \longrightarrow & \mathbf{R}_{m+1,n} \\ \| & & \| \\ \mathrm{O}(m+n)/\mathrm{O}(n) \times \mathrm{O}'(m) & & \mathrm{O}(m+1+n)/\mathrm{O}(n) \times \mathrm{O}'(m+1) \end{array}$$

which give us an inductive system $\{\mathbf{R}_{m,n}, \rho_m : m \in N\}$. The inductive limit of this system denoted by $B_{\mathrm{O}(n)}$,

$$B_{\mathrm{O}(n)} = \varinjlim_m \mathrm{O}(m+n)/\mathrm{O}(n) \times \mathrm{O}'(m),$$

is called the *classifying space for* $\mathrm{O}(n)$.

In general we think of a compact Lie group G as a closed subgroup of $\mathrm{O}(k)$ for sufficiently large k, and define the classifying space B_G for G in a similar manner as above.

If G and H are compact Lie groups it readily follows that a continuous homomorphism $\rho : G \to H$ naturally induces a map $\rho : B_G \to B_H$.

K. Vector bundles.

DEFINITION 1.35. A bundle $\mathscr{B} = \{B, p, X, Y, G\}$ in which $Y = \mathbf{R}^n$ and $G = \mathrm{GL}(n, \mathbf{R})$ acts on Y by the usual linear transformations is called an n-dimensional *vector bundle*.

EXAMPLE. The tangent bundle $\tau(M^n)$ of a smooth n-manifold is an n-dimensional vector bundle.

The natural map

$$p : V_{m+n,m} = \mathrm{O}(m+n)/\mathrm{O}'(m) \longrightarrow \mathrm{O}(m+n)/\mathrm{O}(n) \times \mathrm{O}'(m) = \mathbf{R}_{m,n}$$

is a principal bundle whose fiber is $\mathrm{O}(n)$, and we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{O}(m+n)/\mathrm{O}'(m) & \xrightarrow{\tilde{\rho}_n} & \mathrm{O}(m+1+n)/\mathrm{O}'(m+1) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \mathrm{O}(m+n)/\mathrm{O}(n) \times \mathrm{O}'(m) & \xrightarrow{\rho_n} & \mathrm{O}(m+1+n)/\mathrm{O}(n) \times \mathrm{O}'(m+1) \end{array}$$

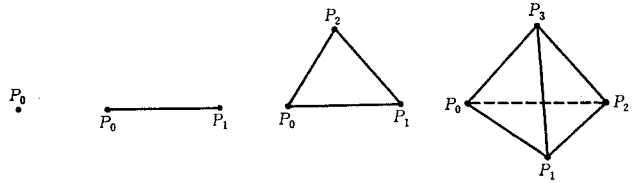


FIGURE 1.9

whose inductive limit with respect to m

$$\begin{array}{c} E_{\mathrm{O}(n)} \\ \downarrow \\ B_{\mathrm{O}(n)} \end{array}$$

is also a principal bundle with fiber $\mathrm{O}(n)$. The latter is called the *universal bundle* for $\mathrm{O}(n)$ and its associated bundle γ_n with the fiber $\mathbf{R}^n$ is called the *universal vector bundle.*

Now we define simplicial complexes and polyhedra. In what follows we shall carry out our discussions in N-dimensional Euclidean space $\mathbf{R}^N$ for a sufficiently large natural number N. The points $P_0, \ldots, P_m$ in $\mathbf{R}^N$ are *linearly independent* or *in general position* if the vectors $\overrightarrow{P_0P_1}, \cdots, \overrightarrow{P_0P_m}$ are linearly independent. It is routine to see that this definition does not depend on the order of the points $P_0, \ldots, P_m$.

DEFINITION 1.36. Let $P_0, \ldots, P_m$ be linearly independent points in $\mathbf{R}^N$. We call the set

$$|P_0P_1 \cdots P_n| = \left\{ X \in \mathbf{R}^N \;\middle|\; \begin{array}{l} \overrightarrow{OX} = \lambda_0\overrightarrow{OP_0} + \cdots + \lambda_n\overrightarrow{OP_n}, \\ \lambda_0 + \cdots + \lambda_n = 1, \quad l_i \geq 0 \end{array} \right\}$$

an *n-simplex.* The n is the *dimension* of the simplex $|P_0P_1 \cdots P_n|$.

A zero-simplex $|P_0|$ is the point P_0, a one-simplex $|P_0P_1|$ is the line $\overline{P_0P_1}$, a two-simplex $|P_0P_1P_2|$ is the triangle with vertices P_0, P_1, P_2, and a three-simplex $|P_0P_1P_2P_3|$ is the tetrahedron with vertices P_0, P_1, P_2, and P_3. See Figure 1.9.

DEFINITION 1.37. Any set of $q+1$ points $P_{i_0}, P_{i_1}, \ldots, P_{i_q}$ among the vertices $P_0, P_1, \ldots, P_n$ of an n-simplex $\sigma = |P_0P_1 \cdots P_n|$, $0 \leq q \leq n$, are again linearly independent; hence, they define a q-simplex

$$\tau = |P_{i_0}P_{i_1} \cdots P_{i_q}|,$$

called a *q-face* of σ. If τ is a face of σ, we write

$$\tau \prec \sigma \quad \text{or} \quad \sigma \succ \tau.$$

DEFINITION 1.38. A finite set K of simplices in N dimensional Euclidean space $\mathbf{R}^N$ is called a *simplicial complex* if it satisfies the following:

(i) If $\sigma \in K$ and $\sigma \succ \tau$, then $\tau \in K$.
(ii) If $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau \prec \sigma$ and $\sigma \cap \tau \prec \tau$.

The *dimension* of a simplicial complex K is the maximum value among the dimensions of simplices belonging to K and is denoted by $\dim K$.

DEFINITION 1.39. Let K be a simplicial complex. The union of all simplices belonging to K is a *polyhedron* of K denoted by $|K|$:

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbf{R}^N.$$

THEOREM 1.3 (classification of vector bundles). *Suppose that P is a polyhedron. Then there is a one-to-one correspondence between the set of equivalence classes of n-dimensional vector bundles over P and the set $[P, B_{O(n)}]$ of homotopy classes of continuous maps from P to $B_{O(n)}$, the correspondence being $\{f\} \mapsto \{f^*\gamma_n\}$.*

AN OUTLINE OF THE PROOF. The classification of equivalence classes of n-dimensional vector bundles over P reduces to the classification of equivalence classes of principal $O(n)$ bundles over P which is readily seen to be in one-to-one correspondence with $[P, B_{O(n)}]$; this follows from

$$B_{O(n)} = \lim_{m\to\infty} O(n+m)/O(n) \times O(m)$$

and

$$\pi_i(O(n+m)/O(m)) = 0, \qquad 0 < i < m,$$

where $\pi_i(X)$ is the homotopy group of X in dimension i. For a more detailed discussion see Steenrod [A7].

Let $\xi = \{E(\xi), P_\xi, X, \mathbf{R}^n, O(n)\}$ and $\eta = \{E(\eta), P_\eta, Y, \mathbf{R}^m, O(m)\}$ be vector bundles of respective dimensions n and m. Suppose that a continuous map $h: E(\xi) \to E(\eta)$ maps each fiber $\mathbf{R}^n_x$ of ξ, $x \in X$, homomorphically into a fiber $\mathbf{R}^m_y$ of η. We then say that h is a *homomorphism* of ξ in η and write $h: \xi \to \eta$. Such an h is called a *vector bundle homomorphism*.

EXAMPLE. Let M^n, V^p be smooth manifolds and let $f: M^n \to V^p$ be a C^∞ map. Then the differential df of f defines a homomorphism of $\tau(M^n)$ in $\tau(V^p)$.

§3. Jet bundles

A. Jets. Denote by $C^r(n, p)$ the set of all C^r maps from $\mathbf{R}^n$ to $\mathbf{R}^p$ sending the origin 0 to the origin 0, $r \geq 1$. We introduce the following relation in $C^r(n, p)$: f and g in $C^r(n, p)$ are *r-equivalent at* 0 if the partial derivatives of f and g agree at 0 in each order up to r,

$$\left.\frac{\partial^s f_i}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_s}}\right|_0 = \left.\frac{\partial^s g_i}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_s}}\right|_0, \qquad \begin{array}{l} s = 1, 2, \ldots, r, \\ i = 1, 2, \ldots, p, \\ 1 \leq j_1 \leq \cdots \leq j_s \leq n. \end{array}$$

We write $f \underset{r}{\sim} g$ if f and g are r-equivalent. It is clear that $\underset{r}{\sim}$ is an equivalence relation.

DEFINITION 1.40. We denote by $J^r(n, p)$ the set of all equivalence classes in $C^r(n, p)$ under $\underset{r}{\sim}$. An element of $J^r(n, p)$ is called an *r-jet*. We write $J^r(f)$, $j^r(f)$, or $f^{(r)}$ for the equivalence class which contains $f \in C^r(n, p)$.

In the case where f is a C^∞ map the r-jet $J^r(f)$ of f is just the truncated Taylor expansion of order r. We notice, in particular, that $J^1(n, p) = M(p, n; \mathbf{R})$, where $M(p, n)$ denotes the set of (p, n)-matrices over $\mathbf{R}$.

The set $J^r(n, p)$ is in one-to-one correspondence with $\mathbf{R}^N$, where $N = ({}_nH_1 + \cdots + {}_nH_r) \times p$, ${}_nH_s = (n+s-1)!/s!(n-1)!$; we associate to each element of J^r its partial derivatives of order up to r. This correspondence defines a natural topology in $J^r(n, p)$.

Now consider C^r maps

$$f: (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0), \qquad g: (\mathbf{R}^p, 0) \longrightarrow (\mathbf{R}^q, 0).$$

We define a map

$$J^r(p, q) \times J^r(n, p) \longrightarrow J^r(n, q)$$

by $(g^{(r)}, f^{(r)}) \mapsto (g \circ f)^{(r)}$. This definition is independent of the choice of representatives. Further as any partial derivative of $g \circ f$ can be written as a polynomial in partial derivatives of f and g, the above map is algebraic.

In particular for $n = p = q$, the above map defines a product in $J^r(n, n)$, and the r-jet $(1_{\mathbf{R}^n})^{(r)}$ of the identity map $1_{\mathbf{R}^n}$ of $\mathbf{R}^n$ is both the right and left unit of $J^r(n, n)$ with respect to this product. Denote by $L^r(n)$ a subset of $J^r(n, n)$ consisting of all invertible elements.

For $1 \leq s \leq r$, define a map

$$\pi_{r,s}: J^r(n, p) \longrightarrow J^s(n, p)$$

by $\pi_{r,s}(f^{(r)}) = f^{(s)}$.

PROPOSITION 1.4. (i) $L^1(n) = GL(n, \mathbf{R})$.

(ii) $L^r(n) = (\pi_{r,1})^{-1}(L^1(n))$.
(iii) $L^r(n)$ *is a Lie group for each* r, $1 \leq r < \infty$.
(iv) $L^r(n)$ *has the homotopy type of* $GL(n, \mathbf{R})$ (cf. Thom and Levine [B11]).

B. Singular sets. We write $L^r(n, p)$ for $L^r(p) \times L^r(n)$ and define an action of $L^r(n, p)$ on $J^r(n, p)$ as follows:

$$L^r(n, p) \times J^r(n, p) \longrightarrow J^r(n, p),$$
$$((a^{(r)}, b^{(r)}), f^{(r)}) \longmapsto (a^{-1} \circ f \circ b)^{(r)}.$$

DEFINITION 1.41. A jet $f^{(r)} \in J^r(n, p)$ is *regular* if its representative f in $C^r(n, p)$ has the maximal rank at 0. We shall denote by ${}^\rho J^r(n, p)$ the set of all regular jets in $J^r(n, p)$.

Let $q = \min(n, p)$. We identify $J^1(n, p)$ with $M(p, n; \mathbf{R})$ and set

$$S_k(n, p) = \{A \in M(p, n; \mathbf{R}) | \text{rank of } A \text{ is } q-k\}, \qquad 0 \le k \le q.$$

In what follows we simply write $S_k = S_k(n, p)$.

PROPOSITION 1.5. (0) ${}^{\rho}J^r(n, p)$ *is dense in* $J^r(n, p)$.

(i) $S_0 = {}^{\rho}J^1(n, p)$.

(ii) $J^1(n, p) = S_0 \cup S_1 \cup \cdots \cup S_q$, *pairwise disjoint union.*

(iii) S_k *is the orbit space of* $L^1(n, p)$.

(iv) $\bar{S}_k = S_k \cup S_{k+1} \cup \cdots \cup S_q$.

(v) S_k *is a codimension* $(n-q+k)(p-q+k)$ *submanifold of* $J^1(n, p)$.

PROOF. (i), (ii), and (iii) are immediate.

We want to show (iv). We shall first prove $\bar{S}_k \subset S_k \cup \cdots \cup S_q$. Let $A \in \bar{S}_k$. In an arbitrary neighborhood $U(A)$ of A there is a B whose rank is $q-k$. Hence, the rank of A must be less than or equal to $q-k$.

Next show $S_k \cup \cdots \cup S_q \subset \bar{S}_k$. Let $A \in S_k \cup \cdots \cup S_q$. Then the rank of A is $q-k-i$, $0 \le i \le q-k$. Hence, we can transform A by an element of $L^1(n, p)$ to

$$E_{q-k-i} = \begin{pmatrix} I_{q-k-i} & 0 \\ 0 & 0 \end{pmatrix},$$

where I_{q-k-i} stands for the $(q-k-i) \times (q-k-i)$ identity matrix. Hence, it is enough to show that E_{q-k-i} is in $\bar{S}_k$. But this is obvious.

(v) follows readily from the following

LEMMA 1.8. *Let* $M(n, p; \mathbf{R})$ *be the set of* (p, n)*-matrices over* $\mathbf{R}$. *Since* $M(p, n; \mathbf{R})$ *is in one-to-one correspondence with* $\mathbf{R}^{pn}$, *we give the usual topology of* $\mathbf{R}^{pn}$ *to* $M(p, n; \mathbf{R})$; *thus* $M(p, n; \mathbf{R})$ *becomes a smooth manifold. Let* $M(p, n; k)$ *be the set of all* (p, n)*-matrices of rank* k.

If $k \le \min(p, n)$, $M(p, n; k)$ *is a* $k(p+n-k)$*-dimensional submanifold of* $M(p, n; \mathbf{R})$.

PROOF. Let E_0 be an element of $M(p, n; k)$. Without loss of generality we may assume that $E_0 = \begin{pmatrix} A_0 & C_0 \\ B_0 & D_0 \end{pmatrix}$, A_0 is a (k, k)-matrix and $|A_0| \neq 0$. Then there exists an $\varepsilon > 0$ such that $|A| \neq 0$ if the absolute value of each entry of $A - A_0$ is less than ε.

Now let $U \subset M(p, n; \mathbf{R})$ consist of all (p, n)-matrices of the form $E = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, where A is a (k, k)-matrix such that the absolute value of each entry of $A - A_0$ is less than ε.

Then we have

$$E \in M(p, n; k) \quad \text{if and only if} \quad D = CA^{-1}B,$$

because the rank of

$$\begin{pmatrix} I_k & 0 \\ X & I_{p-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ XA + C & XB + D \end{pmatrix}$$

is equal to the rank of E for an arbitrary $(p-k, k)$-matrix X. Now setting $X = -CA^{-1}$ we see that the above matrix becomes

$$\begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix}.$$

The rank of this matrix is k if $D = CA^{-1}B$. The converse also holds, since if $-CA^{-1}B + D \neq 0$, then the rank of the above matrix becomes greater than k.

Let W be an open subset of $pn - (p-k)(n-k) = k(p+n-k)$-dimensional Euclidean space:

$$W = \left\{ \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in M(p, n; \mathbf{R}) \middle| (*) \right\}$$

$(*)$: the absolute value of each entry in $A - A_0$ does not exceed ε.

Then the correspondence

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix}$$

defines a diffeomorphism between W and the neighborhood $U \cap M(p, n; k)$ of E_0 in $M(p, n; k)$. Therefore, $M(p, n; k)$ is a $k(p+n-k)$-submanifold of $M(p, n; \mathbf{R})$. □

C. Jet bundles. Let V^n and M^p be C^s manifolds of respective dimensions n and p, $s \ge 1$. For $x \in V^n$ and $y \in M^p$, $1 \le r \le s$, set

$$C^r_{x,y}(V^n, M^p) = \{f : V^n \to M^p, C^r \text{map} | f(x) = y\}.$$

Elements f and g of $C^r_{x,y}(V^n, M^p)$ are *r-equivalent at* x, $f \underset{x,r}{\sim} g$, if the partial derivatives of f and g at x in some local coordinate system agree up to order r. The relation $\underset{x,r}{\sim}$ is well defined and is an equivalence relation.

Set

$$J^r_{x,y}(V^n, M^p) = C^r_{x,y}(V^n, M^p) / \underset{x,r}{\sim}.$$

We write $J^r_x(f)$ for the equivalence class containing f and we say that $J^r_x(f)$ is the *r-jet of* f *at* x. Set

$$J^r(V^n, M^p) = \bigcup_{x \in V^n, y \in M^p} J^r_{x,y}(V^n, M^p).$$

Using the atlases of V^n and M^p, we turn $J^r(V^n, M^p)$ into the total space of a bundle over $V^n \times M^p$ with fiber $J^r(n, p)$ and structure group $L^r(n, p)$:

$$\begin{array}{ccccc} J^r_{x,y}(V^n, M^p) & \subset & J^r(V^n, M^p) & \longleftarrow & J^r(n, p) \\ \downarrow & & \downarrow & & \\ (x, y) & \in & V^n \times M^p & & \end{array}$$

This is called a *jet bundle.*

An alternative definition of a jet bundle comes from the structure theorem of Steenrod (Theorem 1.1). Set $Y = J^r(n, p)$ and $G = L^r(n, p)$; G acts on Y. Set $X = V^n \times M^p$. Take a C^r atlas $\mathcal{S} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of V^n and an atlas $\mathcal{S}' = \{(W_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$ of M^p. Then with $X_{\alpha,\lambda} = U_\alpha \times W_\lambda$, the family $\{X_{\alpha,\lambda} \mid \alpha \in A, \lambda \in \Lambda\}$ is an open cover of X. For $X_{\alpha,\lambda} \cap X_{\beta,\mu} \neq \varnothing$, we define a map

$$g_{(\alpha,\lambda),(\beta,\mu)} : X_{\alpha,\lambda} \cap X_{\beta,\mu} \longrightarrow L^r(n, p)$$

by

$$g_{(\alpha,\lambda),(\beta,\mu)}(x, y) = (b^{(r)}_{\lambda,\mu}, a^{(r)}_{\alpha,\beta}) \in L^r(p) \times L^r(n),$$

where

$$a^{(r)}_{\alpha,\beta} = J^r_{\varphi_\beta(x)}(\varphi_\alpha \circ \varphi_\beta^{-1}), \qquad b^{(r)}_{\lambda,\mu} = J^r_{\psi_\mu(y)}(\psi_\lambda \circ \psi_\mu^{-1}).$$

Then the $\{X_{\alpha,\lambda}, g_{(\alpha,\lambda),(\beta,\mu)} \mid \alpha, \beta \in A, \lambda, \mu \in \Lambda\}$ is a system of coordinate transformations in $V^n \times M^p$ with values in G. Hence, we construct, by Theorem 1.1, a fiber bundle, which turns out to be the above jet bundle.

The total space $J^r(V^n, M^p)$ may be regarded as a C^{s-r} manifold when $r < \infty$.

Let $f : V^n \to M^p$ be a C^s map. Then we call the map

$$J^r(f) : V^n \longrightarrow J^r(V^n, M^p), \qquad x \longmapsto J^r_x(f)$$

the *r-extension* of f. The r-extension $J^r(f)$ is a C^{s-r} map making the following diagram commute:

$$\begin{array}{ccc} & & J^r(V^n, M^p) \longleftarrow J^r(n, p) \\ & {}^{J^r(f)}\nearrow & \downarrow \\ V^n & \xrightarrow{1\times f} & V^n \times M^p. \end{array}$$

In the following we take $r = 1$. The submanifold $S_k(n, p) \subset J^1(n, p)$ is invariant under the action of $L^1(n, p)$. Hence, we consider the associated bundle (this is a subbundle) of the jet bundle $(J^1(V^n, M^p), p, V^n \times M^p)$, whose fiber is $S_k(n, p)$:

$$\begin{array}{ccccc} J^1(V^n, M^p) & \supset & S_k(V^n, M^p) & \longleftarrow & S_k(n, p) \\ \downarrow & & \downarrow & & \\ V^n \times M^p & = & V^n \times M^p & & \end{array}$$

Since $S_k(n, p)$ is a codimension $(n - q + k)(p - q + k)$ submanifold of $J^1(n, p)$, the set $S_k(V^n, M^p)$ is also a codimension $(n - q + k)(p - q + k)$ submanifold of $J^1(V^n, M^p)$.

Let $f : V^n \to M^p$ be a C^1 map and set

$$S_k(f) = \{x \in V^n \mid \text{rank of } f \text{ at } x \text{ is } q - k\}.$$

We have the following routine

PROPOSITION 1.6.

$$S_k(f) = (J^1(f))^{-1}(S_k(V^n, M^p)).$$

D. Mapping spaces. Let V^n and M^p be C^s manifolds of respective dimensions n and p. Let $C^s(V^n, M^p)$ be the set of C^s maps from V^n to M^p, and introduce the C^r topology in $C^s(V^n, N^p)$, $1 \le r \le s$.

DEFINITION 1.42. Let $f \in C^s(V^n, M^p)$. Let K be a compact subset of V^n and let O be an open subset of M^p such that $f(K) \subset O$. For $p \in V^n$, $q = f(p) \in M^p$, choose a chart (U, φ) about x such that $K \subset U$ and a chart (W, ψ) about q. Let $\varepsilon > 0$ and $0 \le r \le s$. Write $\varphi(p) = (x_1, \dots, x_n)$ and define the set $N^r(f; x, y; K, O; \varepsilon)$ as follows:

$$N^r(f; x, y; K, O; \varepsilon) = \{g \in C^s(V^n, M^p) \mid \text{(i),(ii),(iii)}\},$$

where

(i) $g(K) \subset O$,

(ii) $|\psi_i \circ f(p) - \psi_j \circ g(p)| < \varepsilon$, for each $p \in K$, $1 \le i \le n$,

(iii) $\left| \dfrac{\partial^m(\psi_i \circ f \circ \varphi^{-1})(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_m}} - \dfrac{\partial^m(\psi_i \circ g \circ \varphi^{-1})(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_m}} \right| < \varepsilon$,

$x = \varphi(p)$, for each $p \in K$, $1 \le m \le r$, $1 \le i \le n$, $1 \le j_1 \le \cdots \le j_m \le n$.

The C^r *topology* of $C^s(V^n, M^p)$ has for its basis $\mathcal{V}(f)$ in the neighborhood system of $f \in C^s(V^n, M^p)$ the family of $N^r(f; x, y; K, O; \varepsilon)$ where we fix f and let K, O, ε, and charts about x, y move around.

That is, in this topology we consider two maps close if their partial derivatives in each order up to r are close in some compact sets.

PROPOSITION 1.7. *The C^r topology of $C^r(V^n, M^p)$ is the weakest topology making the map*

$$J^r : C^r(V^n, M^p) \longrightarrow C^0(V^n, J^r(V^n, M^p))$$

continuous with respect to the compact-open topology in $C^0(V^n, J^r(V^n, M^p))$.

§4. Morse functions

DEFINITION 1.43. Let M be an n-dimensional C^∞ manifold, and let $f : M \to \mathbf{R}$ be a C^∞ function. Further let $p \in M^p$ be a critical point of f. Choose a local coordinate system (U, φ), $\varphi : U \to \mathbf{R}^n$, $\varphi(p) = 0$. The square n-matrix

$$H(f)_p = \left[\left. \frac{\partial^2(f \circ \varphi)}{\partial x_i \partial x_j} \right|_{x=0} \right]$$

is called the *Hessian* of f at the critical point p. If the Hessian $H(f)_p$ is a regular matrix we say that p is a *nondegenerate critical point*. If p is a nondegenerate critical point we define the *index of* p to be the index (the number of the negative eigenvalues) of $H(f)_p$.

It is readily seen that these definitions do not depend on the choice of local coordinate systems.

THEOREM 1.4 (the lemma of Morse). *Let M be an n dimensional C^∞ manifold and let $f : M \to \mathbf{R}$ be a C^∞ function. Suppose $p_0 \in M$ is a nondegenerate critical point of f. Then there exists a chart (U, φ) about p_0 satisfying the following*:

(i) $\varphi(p_0) = 0$.

(ii) *If* $\varphi(x) = (x_1, \dots, x_n)$, $x \in U$, *then we have*

$$f(x) = f(p_0) - x^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2,$$

where r is the index of p_0.

LEMMA 1.9. *Let M be a C^∞ manifold and let (U, φ) be a chart about $p_0 \in M$, where $\varphi(p) = (x_1, \dots, x_n)$, $p \in U$. Let $f : U \to \mathbf{R}$ be a C^∞ function. Then there exist a neighborhood W of p_0 in W and a family of C^∞ functions $h_{ij} : W \to \mathbf{R}$ such that on W, f has the following expression*:

$$f = f(p_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p_0)(x_i - x_i(p_0)) + \sum_{i,j=1}^n h_{ij} \cdot (x_i - x_i(p_0))(x_j - x_j(p_0)).$$

We leave the proof to the reader. Hint: a Taylor expansion.

PROOF OF THEOREM 1.4. Using $f - f(p_0)$ in place of f we may assume without loss of generality that $f(p_0) = 0$. Choose a chart (U, φ) about p_0 such that

$$\varphi(p_0) = 0 \in \mathbf{R}^n. \tag{5}$$

By Lemma 1.9 there exists a sufficiently small neighborhood W of p_0, on which f has the following expression:

$$f = \sum_{i=1}^n \frac{\partial f}{\partial y_i}(p_0) y_i + \sum_{i,j=1}^n h_{ij} y_i y_j,$$

where $\varphi(p) = (y_1, \dots, y_n)$, $p \in U$. Since p_0 is a critical point of f, that is,

$$\frac{\partial f}{\partial y_i}(p_0) = 0, \qquad i = 1, \dots, n,$$

the expression for f reduces to

$$f = \sum_{i,j=1}^n h_{ij} y_i y_j.$$

Define $a_{ij} : W \to \mathbf{R}$ by $a_{ij} = 1/2(h_{ij} + h_{ji})$, then we have

$$a_{ij} = a_{ji}, \qquad f = \sum_{i,j=1}^n a_{ij} y_i y_j. \tag{6}$$

The cofficient matrix $A = (a_{ij})$ is regular at the point p_0. In fact, by differentiating (6), we obtain

$$\frac{\partial f}{\partial y_k} = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial y_k} y_i y_j + 2\sum_{j=1}^n a_{kj} y_j,$$

$$\frac{\partial^2 f}{\partial y_k \partial y_l} = \sum_{i,j=1}^n \frac{\partial^2 a_{ij}}{\partial y_k \partial y_l} y_i y_j + 2\sum_{j=1}^n \frac{\partial a_{lj}}{\partial y_k} y_j + 2\sum_{j=1}^n \frac{\partial a_{kj}}{\partial y_l} y_j + 2a_{kl};$$

thus, by (5) at the point p_0 we have

$$\frac{\partial^2 f}{\partial y_k \partial y_l}(p_0) = 2a_{kl}(p_0).$$

But p_0 is a nondegenerate critical point and so the matrix

$$\left(\frac{\partial^2 f}{\partial y_k \partial y_l}(p_0)\right)$$

is regular. Hence, the matrix $A(p_0) = (a_{ij}(p_0))$ is also regular. Now as the functions a_{ij} are continuous, we can find a small enough neighborhood V_1 of p_0 on which the matrix $A = (a_{ij})$ is regular. Hence, by selecting a sufficiently small neighborhood $U_1 \subset V_1$ of the point p_0 we have the following:

$${}^tP(p)A(p)P(p) = \left(\begin{array}{ccc|ccc} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right) \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} r \\ \vphantom{\begin{matrix}1\\1\\1\end{matrix}} \end{matrix}, \qquad p \in U_1.$$

Denote by $Q = (q_{ij})$ the inverse matrix of P, and define functions $x_i : U_i \to \mathbf{R}$, $i = 1, \dots, n$, by

$$x_i = \sum_{k=1}^n q_{ik} y_k; \tag{7}$$

then $(x_1, \dots, x_n)$ is local coordinates about p_0: $(U; x_1, \dots, x_n)$. In fact, by differentiating (7), we get

$$\frac{\partial x_i}{\partial y_j} = \sum_{k=1}^n \frac{\partial q_{ik}}{\partial y_j} y_k = q_{ij},$$

which gives us

$$\frac{\partial x_i}{\partial y_j}(p_0) = q_{ij}(p_0),$$

and hence we have

$$\left.\frac{D(x_1, \ldots, x_n)}{D(y_1, \ldots, y_n)}\right|_{p_0} = \det\left(\frac{\partial x_i}{\partial y_j}(p_0)\right) = \det(q_{ij}(p_0)) \neq 0.$$

Further, by (5) we have

$$x_1(p_0) = \cdots = x_n(p_0) = 0,$$

and by (6), (7), we have

$$f = \sum_{i,j=1}^{n} a_{ij} y_i y_j = {}^t\mathbf{y}A\mathbf{y} = {}^t\mathbf{x}PAP\mathbf{x}$$

$$= (x_1, \ldots, x_n)\begin{pmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2. \tag{8}$$

Finally we show that the index of f at p_0 is r. By differentiating equation (8) twice, we obtain

$$\frac{\partial^2 f}{\partial x_i^2} = \begin{cases} -2, & i = 1, \ldots, r, \\ 2, & i = r+1, \ldots, n, \end{cases}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = 0, \quad i \neq j.$$

Hence, the Hessian of f at p_0 is

$$H(f)_{p_0} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0)\right) = \left(\begin{array}{ccc|ccc} -2 & & & & & \\ & \ddots & & & 0 & \\ & & -2 & & & \\ \hline & & & 2 & & \\ & 0 & & & \ddots & \\ & & & & & 2 \end{array}\right) \Big\} r,$$

and thus its index is r. We now have proved the theorem. □

DEFINITION 1.44. Let M be a C^∞ manifold. We say that a C^∞ function $f: M \to \mathbf{R}$ is a *Morse function* if it satisfies the following:

(1) The critical points of f are nondegenerate.

(2) If p and q are critical points of f such that $p \neq q$, then $f(p) \neq f(q)$.

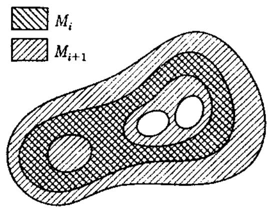


FIGURE 1.10

THEOREM 1.5. *Let M be a C^∞ manifold. Then there exists a Morse function $f: M \to \mathbf{R}$. Further, if M is open, we can choose f to be proper with the index of each critical point less than n.*

A continuous map $f: X \to Y$ from a topological space X to a topological space Y is *proper* if for each compact set K in Y, $f^{-1}(K)$ is compact in X.

The case of an open manifold is due to Phillips [C 14].

PROOF OF THEOREM 1.5. We know that the set $\mathscr{M}$ of Morse functions defined on M is dense in the set $C^\infty(M, \mathbf{R})$ of C^∞ functions of M (cf. Milnor [A2] for a proof). Hence, a Morse function of M exists.

For the case of an open manifold M, first express M as a union of an expanding sequence of compact manifolds with boundary:

$$M = \bigcup_{i=1}^{\infty} M_i, \qquad M_i \subset \operatorname{int} M_{i+1},$$

(see Figure 1.10). Next we denote by M_i' the union of M_i and compact components of $M - \operatorname{int} M_i$. Then M is again a union of an expanding sequence of compact manifolds with boundary:

$$M = \bigcup_{i=1}^{\infty} M_i', \qquad M_i' \subset \operatorname{int} M_{i+1}'.$$

To see this first notice that $M_i \subset \operatorname{int} M_{i+1} \subset \operatorname{int} M_{i+1}'$. Next we note that if A is a compact connected component of $M - \operatorname{int} M_i$, ∂A is a union of connected components of ∂M_i. Hence, either $A \subset \operatorname{int} M_{i+1}$ or A contains some connected component of ∂M_{i+1}. In the second case $A - (A \cap \operatorname{int} M_{i+1})$ is a union of compact components of $M - \operatorname{int} M_{i+1}$. Hence, $A \subset \operatorname{int} M_{i+1}'$.

Consider now the compact manifold with boundary $M_{i+1}' - \operatorname{int} M_i'$. Since $M_i' \subset \operatorname{int} M_{i+1}'$, $\partial(M_{i+1}' - \operatorname{int} M_i')$ falls into two disjoint parts corresponding to $\partial M_{i+1}'$ and $\partial M_i'$ (Figure 1.11). Then there exists a Morse function f: $M_{i+1}' - \operatorname{int} M_i' \to \mathbf{R}$ such that

(0) the range of $f_i \subset [0, 1]$,

(i) $f_i^{-1}(0) = \partial M_i'$, $f_i^{-1}(1) = \partial M_{i+1}'$, and

(ii) f_i has no critical points in $\partial(M_{i+1}' - \operatorname{int} M_i')$

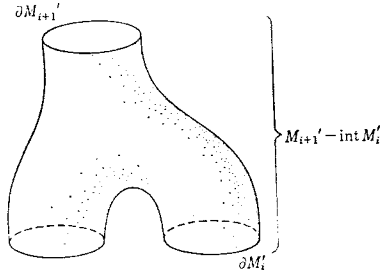


FIGURE 1.11

(Milnor [A4], Theorem 2.5; this might be thought of as a relative case of what is described above). Fitting the f_i together we obtain a proper Morse function f on M.

We next eliminate the critical points of index n. By the construction of M_i' we have

$$H_0(M_{i+1}' - \operatorname{int} M_i', \partial M_{i+1}') = 0.$$

In fact, if a connected component of $M_{i+1}' - \operatorname{int} M_i'$ cannot be joined to ∂M_{i+1} by a curve, it must be a compact component of $\operatorname{int} M_{i+1}' - \operatorname{int} M_i'$, and hence it must be a compcact component of $M - \operatorname{int} M_i'$. But this is a contradiction. Thus, we can remove critical points of index n (cf. Milnor [A4], Theorem 8.1).

§5. The transversality theorem of Thom

This section presents the transversality theorem of Thom, one of the most fundamental theorems in the theory of differential topology, which will be used in Chapter VII.

DEFINITION 1.45. Let M^n, N^p be C^∞ manifolds of dimensions n and p. Let $f : M^n \to N^p$ be a continuous map and let W^{p-q} be a $p-q$-dimensional submanifold of N^p. We say that f is *t-regular* on W^{p-q} or that f is *transverse* to W^{p-q} if for an arbitrary point y of W^{p-q} and an arbitrary point x of $f^{-1}(y)$ the composite map

$$T_x(M^n) \xrightarrow{(df)_x} T_y(N^p) \xrightarrow{\pi} T_y(N^p)/T_y(W^{p-q})$$

is a surjection, where π is the natural projection.

The concept of t-regularity was first established by R.Thom.

LEMMA 1.10. *Let M^n and N^p be C^∞ manifolds, and let $f : M^n \to N^p$ be a smooth map. Let W^{p-q} be a $(p-q)$-dimensional submanifold of N^p. If f is t-regular on W^{p-q}, $f^{-1}(W^{p-q})$ is either an $(n-q)$-submanifold of M^n or the empty set.*

The lemma routinely follows from the definition.

THEOREM 1.6 (the transversality theorem of Thom). *Let M^n be an n-dimensional compact C^∞ manifold, let N^p be a p-dimensional C^∞ manifold, and let W^{p-q} be a $(p-q)$-dimensional submanifold of N^p. Let $C^r(M^n, N^p)$ be the space of C^∞ maps from M^n to N^p with the C^r topology, $r \geq 1$. Then the set*

$$T_W = \{ f \in C^r(M^n, N^p) \mid f \text{ is } t\text{-regular on } W^{p-q} \}$$

is open and dense in $C^r(M^n, N^p)$.

We omit the proof (cf. Thom and Levine [B11]).

CHAPTER II

Embeddings of C^∞ Manifolds

Our discussions in the present chapter will revolve around theorems of Whitney, which were later generalized by Haefliger. We shall save Haefliger's works for Chapter VII.

The main problem here is the existence of embeddings of V in M and their classification up to isotopy for a given pair of manifolds V and M.

Throughout this chapter we shall work in the C^∞ category.

§1. Embeddings and isotopies

We first define isotopies of embeddings and then state the theorems of Whitney and Haefliger.

Let V be an n-dimensional C^∞ manifold and let M be an m-dimensional manifold.

DEFINITION 2.2. Let f and g be embeddings. We say that f and g are *isotopic* and write $f \simeq g$ if there exists a C^∞ map

$$F : V \times I \longrightarrow M, \qquad I = [0, 1]$$

which satisfies the following, with the notation $f_t(x) = F(x, t)$:

(i) $f_0 = f$, $f_1 = g$.
(ii) $f_t : V \to M$ is an embedding for each $t \in [0, 1]$.

The map F or the family of maps $\{f_t\}$ is called an *isotopy* between f and g.

It is a routine that $\simeq$ is an equivalence relation (the transitivity requires a bit of thinking).

The basic problem concerning embeddings is whether given V and M there are embeddings of V in M, and if so can we classify these embeddings by isotopies?

We stated in Chapter I that if $f : V \to M$ is an embedding, then the image $f(V)$ of V under f is a submanifold of M (the converse is false). Hence, when M is a Euclidean space of low dimension we can fairly readily grasp the $f(V)$ and so V. Hence, we always try to embed V in Euclidean space of the lowest possible dimension.

The question of isotopic classifications is called the *Problem der lage*; that of classifying topological spaces, polyhedra, manifolds and so on by

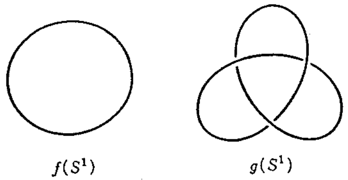


FIGURE 2.1

homeomorphisms (or diffeomorphisms) is called the *Problem der gestalt.* There has been considerable progress on the Problem der gestalt in the last half century but progress on the Problem der lage seems to have stagnated somewhat.

EXAMPLE. Consider two embeddings f and g of the circle S^1 in Euclidean space $\mathbf{R}^3$ as shown in Figure 2.1.

Here $f(S^1)$ and $g(S^1)$ are homeomorphic but not isotopic.

The problem of classifying embeddings of S^1 in $\mathbf{R}^3$ is called knot theory and forms one of many active fields of research in topology (for the interested reader we recommend *Introduction to knot theory,* by Crowell and Fox ([1]); strictly speaking isotopies in knot theory differ slightly from our isotopies).

Another expression for isotopies comes in the following

THEOREM 2.1. *Let V be a closed manifold and let $f, g : V \to M$ be embeddings. Then f and g are isotopic if and only if there exists a homotopy $h_t : M \to M$, $t \in [0, 1]$ such that*

(i) *for each $t \in [0, 1]$, h_t is a diffeomorphism of M, and*
(ii) $h_0 = 1_M$, $g = h_1 \circ f$.

As we have no occasion to use this theorem in our book we omit the proof (cf. Thom [B10]).

In 1944 Whitney showed the following

THEOREM 2.2 (Whitney's embedding theorem). *Let V^n be an n-dimensional C^∞ manifold, $n \geq 3$. Then we can embed V^n in $\mathbf{R}^{2n}$, i.e., there exists an embbedding $f : V^n \to \mathbf{R}^{2n}$.*

DEFINITION 2.2. We say that a topological space X is *k-connected* if it is arcwise connected and

$$\pi_i(X) = 0, \qquad 0 < i \leq k.$$

Later Haefliger discussed and honed Whitney's work and in 1961 obtained the following

THEOREM 2.3 (Haefliger's embedding theorem). *Let V^n be an n-dimensional C^∞ manifold.*

([1]) *Introduction to knot theory,* Ginn, Needham Heights, MA, 1963; Springer-Verlag, New York, 1977.

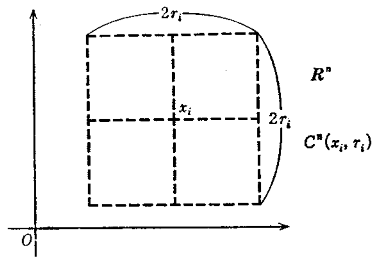


FIGURE 2.2

(a) *Suppose V^n is k connected, $2k + 3 \leq n$. Then we can embed V^n in $\mathbf{R}^{2n-k}$.*

(b) *Let V^n be k-connected, $2k + 2 \leq n$. Then two embeddings of V^n in $\mathbf{R}^{2n-k+1}$ are isotopic.*

Taking $k = 0$ in Haefliger's theorem we obtain the embedding theorem of Whitney. In this chapter we discuss Whitney's theorem and in Chapter VII we will focus on Haefliger's work.

§2. Two approximation theorems

In this section as well as in the next we follow closely Milnor's lecture notes [B6].

We present two approximation theorems fundamental for the proof of Whitney's embedding theorem. First some preparations.

DEFINITION 2.3. We say that a subset A of Euclidean n-space $\mathbf{R}^n$ has *measure zero* if for an arbitrary $\epsilon > 0$ there exists an open cover of A such that

$$A \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i), \qquad \sum_{i=1}^{\infty} r_i^n < \epsilon,$$

where $C^n(x_i, r_i)$ is the n-cube centered at x_i in $\mathbf{R}^n$, whose edges are each of length $2r_i$ (Figure 2.2); when no confusion arises we might abbreviate $C^n(x_i, r_i)$ by $C(x_i, r_i)$.

PROPOSITION 2.1. *If a subset A of $\mathbf{R}^n$ has measure zero, then the complement $R^n - A$ of A is everywhere dense.*

LEMMA 2.1. *Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^n$ be a C^∞ map. If a subset A of U has measure zero, so does $f(A)$.*

PROOF. Let C be an n-cube whose closure $\bar{C}$ is contained in U. Put

$$b = \max_{x \in \bar{C}, i, j} \left| \left(\frac{\partial f_i}{\partial x_j} \right)_x \right|,$$

then by the mean value theorem we get

$$\|f(x) - f(y)\| \leq b \cdot n \cdot \|x - y\|, \qquad x, y \in \bar{C}.$$

As A has measure zero by assumption, $A \cap C$ also has measure zero, i.e., for an arbitrary $\epsilon > 0$, we can choose an open cover of $A \cap C$:

$$A \cap C \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i), \qquad \sum_{i=1}^{\infty} r_i^n < \epsilon.$$

But the above inequality implies that

$$f(C^n(x_i, r_i)) \subset C^n(f(x_i), bnr_i).$$

Hence, we get

$$f(A \cap C) \subset f\left(\bigcup_{i=1}^{\infty} C^n(x_i, r_i)\right) = \bigcup_{i=1}^{\infty} f(C^n(x_i, r_i)) \subset \bigcup_{i=1}^{\infty} C^n(f(x_i), bnr_i).$$

But the sum of volumes of cubes in the above open cover of $A \cap C$ is

$$\sum_{i=1}^{\infty} 2^n b^n r_i^n = 2^n b^n n^n \sum_{i=1}^{\infty} r_i^2 < 2^n b^n n^n \epsilon.$$

Hence, $f(A \cap C)$ has measure zero. Since we can cover A with a countable number of such C we see that the measure of $f(A)$ is also zero. □

Corollary 2.1. *Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^p$ be a C^∞ map, $n < p$. Then $f(U)$ has measure zero.*

Proof. We may think of $U \times \mathbf{R}^{p-n}$ as an open subset of $\mathbf{R}^n$. Hence, we define $g : U \times \mathbf{R}^{p-n} \to \mathbf{R}^p$ by $g = f \circ p_1$:

$$g : U \times \mathbf{R}^{p-n} \xrightarrow{p_1} U \xrightarrow{f} \mathbf{R}^p,$$

where p_1 is the projection onto the first component. Then g is evidently a C^∞ map. But $U = U \times 0 \subset U \times \mathbf{R}^{p-n}$ has measure zero, and so $f(U) = g(U \times 0)$ has measure zero in $\mathbf{R}^p$. □

We next define a concept of measure zero for a subset of a C^∞ manifold.

Definition 2.4. Let $(M^n, \mathscr{D})$ be an n-dimensional manifold, and let A be a subset of M^n. We say that A has *measure zero* if $\varphi_\alpha(U \cap A) \subset \mathbf{R}^n$ has measure zero for an arbitrary chart $(U_\alpha, \varphi_\alpha)$ of $\mathscr{D}$.

By Lemma 2.1 the above definition does not depend on the choice of a chart. This definition agrees with the definition of measure zero for subsets of $\mathbf{R}^n$.

Corollary 2.2. *Let V^n and M^m be C^∞ manifolds of respective dimensions n and m, $n < m$. Let $f : V^n \to M^M$ be a C^∞ map. Then $f(V^n)$ has measure zero in M^m.*

Theorem 2.4. *Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^p$ be a C^∞ map, $2n \le p$. Then for an arbitrary $\epsilon > 0$ there exists an (p, n)-matrix $A = (a_{ij})$ such that*

(i) *$|a_{ij}| < \epsilon$, $i = 1, \ldots, p$, $j = i, \ldots, n$, and*

(ii) *the map $g : U \to \mathbf{R}^p$ defined by $g(x) = f(x) + Ax$ is an immersion.*

Proof. Notice that

$$Dg(x) = Df(x) + A.$$

Hence, we need to choose a suitable A so that the rank of $Dg(x)$ at each point x of U is n, *i.e.*, A must be of the form $Q - Df$ where Q is a (p, n)-matrix of rank n.

Define maps $F_k : M(p, n; k) \times U \to M(p, n)$ by

$$F_k(Q, x) = Q - Df(x).$$

Then Lemma 1.5 implies that $M(p, n; k) \times U$ is a differentiable manifold of dimension $k(p + n - k) + n$ and that the F_k are C^∞ maps. We note further that as long as $k < n$, $k(p + n - k) + n$ is monotone increasing with respect to k. Hence, the dimension of the domain of F_k does not exceed $(n-1)(p + n - (n-1)) + n = (2n - p) + pn - 1$ when $k < n$. Now our assumption that $p \ge 2n$ implies that this dimension is less than $pn = \dim M(p, n)$.

Hence, by Corollary 2.2 the image of $M(p, n; k) \times U$ under F_k has measure zero in $M(p, n)$. Thus, we can find an element A sufficiently close to the zero matrix in $M(p, n)$, which is not contained in any of the F_k, $k = 0, 1, \ldots, n-1$. This is the desired A. □

§3. An immersion theorem

We show in the present section that we can immerse an n dimensional C^∞ manifold in $\mathbf{R}^{2n}$. Set $\mathbf{R}_+ = \{x \in \mathbf{R} | x > 0\}$.

Definition 2.5. Let $f : X \to Y$, where X is a topological space and Y is a metric space, and let $\delta : X \to \mathbf{R}_+$ be a continuous function. We say that g is a *δ-approximation* of f if $d(f(x), g(x)) < \delta(x)$ for all x in X, where d is the metric in Y.

Theorem 2.5. *Let M^n be an n dimensional C^∞ manifold, and let $f : M^n \to \mathbf{R}^p$ be a C^∞ map, $2n \le p$. Given a continuous function $\delta : M^n \to \mathbf{R}_+$, there exists an immersion $g : M^n \to \mathbf{R}^p$ which is a δ-approximation of f.*

In addition, if the rank of f on a closed subset N of M^n is n, we may choose g such that

(i) *$g|N = f|N$, and*

(ii) *g is homotopic to f relative to N.*

Before proving the theorem we need the following

Lemma 2.2. *A locally compact topological space X with a countable basis is paracompact.*

Proof. By assumption we may take open sets

$$U_1, U_2, \ldots; \qquad \overline{U}_i \text{ compact}, \ i = 1, 2, \ldots,$$

as a basis of X. We construct inductively a sequence of compact sets

$$A_1, A_2, \dots; \qquad X = \bigcup_{i=1}^{\infty} A_i, \qquad A_i \subset \operatorname{int} A_{i+1}.$$

Put $A_1 = \overline{U}_1$. Assuming that the A_j up to $i = j$ have been constructed, we define A_{i+1}. Let k be the smallest natural number such that $A_i \subset U_1 \cup \cdots \cup U_k$, and put

$$A_{i+1} = (\overline{U_1 \cup \cdots \cup U_k}) \cup \overline{U}_{i+1}.$$

Then the $\{A_i\}$ obviously satisfy the above condition.

Let $\mathscr{W} = \{W_j\}$ be an open covering of X. Since the set $A_{i+1} - \operatorname{int} A_i$ is compact, it can be covered by finitely many W_j, and we can find a finite number of open sets V_{i_r} satisfying:

$$A_{i+1} - \operatorname{int} A_i \subset \bigcup_{r=1}^{s} V_{i_r},$$

(i) $V_{i_r} \subset W_j$ for some j,
(ii) $V_{i_r} \subset \operatorname{int} A_{i+2} - A_{i-1}$.

Setting $P_i = \{V_{i_1}, \dots, V_{i_s}\}$ and $\mathscr{P} = P_0 \cup P_1 \cup \cdots$, we see that $\mathscr{P}$ is a locally finite refinement of $\mathscr{W}$. □

From Lemma 2.2 it follows that all our topological or differential manifolds are paracompact.

LEMMA 2.3. *Let $\{U_\alpha\}$ be an open covering of a C^∞ manifold M^n. Then M^n has an atlas $\{(V_j, h_j) \mid j \in J\}$ with the following properties:*

(i) *J has cardinality $\aleph_0$.*
(ii) *$\{V_j \mid j \in J\}$ is a locally finite refinement of $\{U_\alpha\}$.*
(iii) *$h_j(V_j) = C(3)$.*
(iv) *$M^n = \bigcup_{j\in J} W_j$, where $W_j = h_j^{-1}(C^n(1))$.*

Here $C^n(r) = C^n(0, r)$ stands for the n-cube centered at 0 with edges of length $2r$.

PROOF. We select a sequence of compact sets as in the proof of Lemma 2.2,

$$A_1, A_2, \dots; \qquad M^n = \bigcup_{i=1}^{\infty} A_i, \qquad A_i \subset \operatorname{int} A_{i+1},$$

and construct a locally finite refinement of the $\{U_\alpha\}$; keeping in mind that we must have (1) $h_j(V_j) = C^n(3)$ and (2) $A_{i+1} - \operatorname{int} A_i \subset \cup_j h_j^{-1}(C^n(1))$. □

LEMMA 2.4. *There exists a C^∞ function $\varphi : \mathbf{R}^n \to \mathbf{R}$ satisfying the following:*

(i) $\varphi(x) = 1$, $x \in \overline{C^n(1)}$.
(ii) $0 < \varphi(x) < 1$, $x \in C^n(2) - \overline{C^n(1)}$.
(iii) $\varphi(x) = 0$, $x \in \mathbf{R}^n - C^n(2)$.

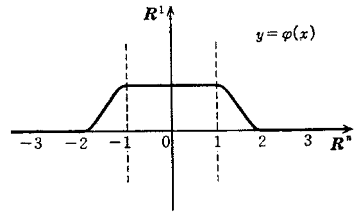


FIGURE 2.3

PROOF. Define a function $\lambda : \mathbf{R} \to \mathbf{R}$ by

$$\lambda(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and set

$$\psi(x) = \frac{\lambda(2+x)\lambda(2-x)}{\lambda(2+x)\lambda(2-x) + \lambda(x-1) + \lambda(-x-1)}.$$

Now define φ by

$$\varphi(x_1, \dots, x_n) = \prod_{i=1}^{n} \psi(x_i)$$

(see Figure 2.3). □

We say that φ is a *bell-shaped function.*

PROOF OF THEOREM 2.5. Assume the rank of f is n on N, that is, the rank of f is n on some open neighborhood U of N. Then the family $\{U, M - N\}$ is an open cover of M. We select an atlas $\mathscr{D}_0 = \{(V_j, h_j) | j \in J\}$ according to Lemma 2.3, which is a locally finite refinement of $\{U, M - N\}$ with $h_i(W_i) = C^n(1)$ and $h_i(V_i) = C^n(3)$ (see Figure 2.4). We next set $h_i(U_i) = C^n(2)$ and reindex the $\{(V_i, h_i) | i \in J\}$ so that

$$i \le 0 \text{ if and only if } V_i \subset U,$$
$$i > 0 \text{ if and only if } V_i \subset M - N.$$

Noticing that $\overline{U}_i$ is compact, we set

$$\epsilon_i = \min_{x \in \overline{U}_i} \delta(x).$$

Now we construct the desired g by induction. Put $f_0 = f$; then the rank of f_0 is n on U, and so it is n on $\bigcup_{j \le 0} \overline{W}_j$. Assume next that $f_{k-1} : M^n \to \mathbf{R}^p$ is a C^∞ map having rank n on $N_{k-1} = \bigcup_{j<k} \overline{W}_j$. We now construct, as a $\delta/2^k$-approximation of f_{k-1}, a C^∞ map $f_k : M^n \to \mathbf{R}^p$ whose rank is n on N_k as follows. Consider the map $f_{k-1} \circ h_k^{-1} : C^n(3) \to \mathbf{R}^p$ and the bell-shaped function $\varphi : \mathbf{R}^n \to \mathbf{R}$ of Lemma 2.4 for $C^n(1) \subset C^n(2)$. Choose a (p, n)-matrix A in such a way that if we define $F_A : C^n(3) \to \mathbf{R}^p$ by

$$F_A(x) = f_{k-1} \circ h_k^{-1}(x) + \varphi(x) A x,$$

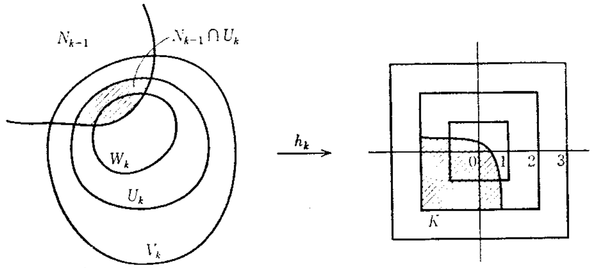


FIGURE 2.4

the following conditions (i), (ii), and (iii) are met.

(i) F_A is of rank n on $K = h_k(N_{k-1} \cap \overline{U}_k)$. By assumption the rank of $f_{k-1} \circ h_k^{-1}$ is n on K. The Jacobian matrix of F_A gives

$$DF_A(x) = D(f_{k-i} \circ h_k^{-1}(x)) + D\varphi(x)Ax + \varphi(x)A.$$

The map

$$\Phi : K \times M(p, n; \mathbf{R}) \to M(p, n; \mathbf{R})$$

which assigns $DF_A(x)$ to (x, A) is continuous and the subset $M(p, n; n)$ consisting of matrices of rank n is open in $M(p, n; \mathbf{R})$. But since $\Phi(K \times 0) \subset M(p, n; n)$ we have $\Phi(K \times A) \subset M(p, n; n)$ for a small enough A (for an A close enough to 0).

(ii) The desired A should also be sufficiently small that

$$\|Ax\| < \frac{\epsilon_k}{2^k}, \qquad x \in C^n(3).$$

(iii) Finally, by Theorem 2.4 we may take A so small that $f_{k-1} \circ h_k^{-1}(x) + Ax$ has rank n on $C^n(2)$.

Having chosen the desired A as above, we define for each k a map $f_k : M^n \to \mathbf{R}^p$ by

$$f_k(x) = \begin{cases} f_{k-1}(x) + \varphi(h_k(x))Ah_k(x), & x \in V_k, \\ f_{k-1}(x), & x \in M - \overline{U}_k. \end{cases}$$

The map f_k is well defined, *i.e*, for a point x of $V_k - \overline{U}_k$,

$$f_{k-1}(x) + \varphi(h_k(x))Ah_k(x) = f_{k-1}(x).$$

The rank of f_k is by (i) n on N_{k-1} and by (iii) n on U_k; therefore, the rank of f_k is n on $N_k = \bigcup_{j<k+1} \overline{W}_j$. Further, by (ii) the map f_k is a $\delta/2^k$-approximation of f_{k-1}.

Define $g : M^n \to \mathbf{R}^p$ by $g(x) = \lim_{k\to\infty} f_k(x)$. This means the following.

Recall that

$$M^n = \bigcup_i W_i,$$

$$N_0 \subset N_1 \subset N_2 \subset \dots, \qquad N_{k-1} = \bigcup_{j<k} \overline{W}_j,$$

$$M^n = \bigcup_k N_k.$$

Since the $\{V_i\}$ is locally finite, for an arbitrary point x of M^n there exists a neighborhood $U(x)$ of x such that $U(x) \cap V_j \neq \varnothing$ for a finitely many j's, the largest among which we call k; then

$$g(x) = f_k(x) = f_{k+1}(x) = \cdots.$$

Clearly, the map g is C^∞ and of the rank n on M^n. Moreover, g is a δ-approximation of f. The construction of f_k implies immediately that f_k is homotopic to f_{k-1} while N is kept fixed. Hence, f is homotopic to g with N fixed. □

§4 Whitney's embedding theorem I: $M^n \subset \mathbf{R}^{2n+1}$

In this section we show that we can embed an n-dimensional C^∞ manifold M^n in $2n+1$-dimensional Euclidean space $\mathbf{R}^{2n+1}$.

DEFINITION 2.6. A map $f : M \to N$ from a C^∞ manifold M to C^∞ manifold N is a *one-to-one immersion* if f is an immersion and a one-to-one map.

A one-to-one immersion is not necessarily an embedding.

EXAMPLE. Reflect upon $f : \mathbf{R} \to \mathbf{R}^2$ in Figure 2.5.

We define a regular homotopy of immersions as used in Chapter 0 and which we shall discuss in detail in the next chapter.

DEFINITION 2.7. Let M^n and V^p be C^∞ manifolds of dimensions n and p. The immersions $f, g : M^n \to V^p$ are *regularly homotopic*, $f \underset{r}{\simeq} g$, if there exists a C^∞ map $F : M^n \times I \to V^p$ which with $f_t(x) = F(x, t)$ satisfies the following:

(i) $f_0 = f$, $f_1 = g$.
(ii) f_t is an immersion for $t \in I$.

We say that F or the $\{f_t\}$ is a *regular homotopy* connecting f and g.

Notice that the $\underset{r}{\simeq}$ is an equivalence relation.

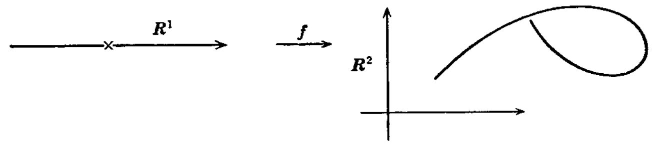


FIGURE 2.5

Lemma 2.5. *Let $f: M^n \to \mathbf{R}^p$ be an immersion of a C^∞ manifold M^n, $2n < p$. Then for an arbitrary continuous function $\delta: M^n \to \mathbf{R}_+$, there exists a one-to-one immersion g which is a δ-approximation of f. Further, if f is one-to-one in an open neighborhood U of a closed subset N of M^n, we can choose g such that*

(i) *$g|N = f|N$, and*
(ii) *g is regularly homotopic to f relative to N.*

Proof. On one hand, because f is an immersion, there exists an open covering $\{U_\alpha\}$ of M^n such that $f|U_\alpha$ is an embedding for each α, on the other hand, the set $\{U, M^n - N\}$ is another covering of M^n, and together these coverings form a third covering with respect to which we consider an atlas $\{(V_i, h_i)|i \in J\}$ as given in Lemma 2.3. Further, using the bell-shaped function $\varphi: \mathbf{R}^n \to \mathbf{R}$ of Lemma 2.4, we define $\varphi_i: M^n \to \mathbf{R}$ by

$$\varphi_i(y) = \begin{cases} \varphi \circ h_i(y), & y \in V_i, \\ 0, & y \notin V_i. \end{cases}$$

Then it follows immediately that φ_i is a C^∞ function. For the sake of convenience we take

$$V_i \subset U \quad \text{if and only if} \quad i \le 0.$$

Now we define the desired g by induction. First put $f_0 = f$. Assuming next that we have an immersion $f_{k-1}: M^n \to \mathbf{R}^p$ which is one-to-one on $\bigcup_{j<k} V_j$, we define $f_k: M^n \to \mathbf{R}^p$ by

$$f_k(x) = f_{k-1}(x) + \varphi_k(x) b_k,$$

where b_k is a point of $\mathbf{R}^p$, which we select as follows:

(i) b_k is small enough to make f_k an immersion.
(ii) b_k is so small that f_k is a $\delta/2^k$-approximation of f_{k-1}.
(iii) Let N^{2n} be a subset of $M^n \times M^n$,

$$N^{2n} = \{(x, y) \in M^n \times M^n | \varphi_k(x) \ne \varphi_k(y)\}.$$

Evidently N^{2n} is open in $M^n \times M^n$. Now consider the following smooth map $\Phi: N^{2n} \to \mathbf{R}^p$:

$$\Phi(x, y) = \frac{-(f_{k-1}(x) - f_{k-1}(y))}{\varphi_k(x) - \varphi_k(y)}.$$

By assumption $2n < p$, and so the measure of $\Phi(N^{2n})$ in $\mathbf{R}^p$ is zero. Hence, we want b_k to be outside $\Phi(N^{2n})$ (if b_k is small enough to satisfy the conditions (i) and (ii) it will work for the condition (iii)).

The map f_k satisfies

$$f_k(x) - f_k(y) = 0 \quad \text{if and only if} \quad \begin{cases} \varphi_k(x) - \varphi_k(y) = 0, \\ f_{k-1}(x) - f_{k-1}(y) = 0. \end{cases}$$

The 'if part' is clear.

To show the converse notice that

$$0 = f_k(x) - f_k(y) = (f_{k-1}(x) - f_{k-1}(y)) + (\varphi_k(x) - \varphi_k(y)) b_k;$$

hence, if $\varphi_k(x) - \varphi_k(y) \ne 0$, then $b_k \in \Phi(N^{2n})$. But this is a contradiction. Hence, we must have $\varphi_k(x) - \varphi_k(y) = 0$. Therefore, we also get $f_{k-1}(x) - f_{k-1}(y) = 0$.

Define $g: M^n - \mathbf{R}^p$ by

$$g(x) = \lim_{k\to\infty} f_k(x).$$

This means the following. Since $f_k(x) = f_{k-1}(x) + \varphi_k(x) b_k$ we must have $f_k|V_k^c = f_{k-1}|V_k^c$. But the $\{V_j\}$ is locally finite and so for some neighborhood $U(x)$ of x, $U(x) \cap V_j \ne \varnothing$ for only finitely many j. Denoting by $j_0(x)$ the largest such j we get $g(x) = f_{j_0(x)}(x)$.

The definition of $g(x)$ readily implies that g is a C^∞ map which is an immersion as well. It is also clear that $g|N = f|N$. It remains to show that g is one-to-one. Suppose now $g(x) = g(x_0)$ and $x \ne x_0$. Then from what was discussed above we have $f_{k-1}(x) = f_{k-1}(x_0)$, $\varphi_k(x) = \varphi_k(x_0)$ for all integers $k > 0$. From the former equation we get $f(x) = f(x_0)$. Therefore, x and x_0 do not belong to the same V_j. But from the latter equation we see that if $x \in V_k$ for $k > 0$, then x_0 must also be in V_k, which cannot happen; hence, both x and x_0 must be in U (the V_j were reindexed this way). This is a contradiction, however, as f is one-to-one on U.

Finally, a close look at the definition of f_k reveals that f_k and f_{k-1} are regularly homotopic relative to N, and hence the same is true for f and g. □

Definition 2.8. Let M^n be an n-dimensional C^∞ manifold and let $f: M^n \to \mathbf{R}^p$ be a continuous map. We say that the set

$$L(f) = \left\{ y \in \mathbf{R}^p \;\middle|\; \begin{array}{l} y = \lim f(x_n) \\ \text{for some sequence } (x_1, x_2, \dots) \\ \text{which has no limit point in } M^n \end{array} \right\}$$

is the *limit set* of f.

Lemma 2.6. *Let M^n be a C^∞ manifold, and let $f: M^n \to \mathbf{R}^p$ be a continuous map.*

(i) *$f(M^n)$ is a closed subset of $\mathbf{R}^p$ if and only if $L(f) \subset f(M^n)$.*
(ii) *f is a topological embedding, i.e., f is a topological map, if and only if f is one-to-one and $L(f) \cap f(M^n) = \varnothing$.*

Proof. (i) Suppose $f(M^n)$ is a closed subset of $\mathbf{R}^p$. Let $y \in Y$. Then $y = \lim f(x_n)$. But $f(x_n) \in f(M^n)$, and as $f(M^n)$ is closed, $y \in f(M^n)$.

Conversely suppose $L(f) \subset f(M^n)$. Let y be a point in the closure of $f(M^n)$. For each natural number n there exists a point $f(x_n)$ in $C^p(y, 1/n)$.

Consider a sequence $x_1, x_2, \dots$ in M^n. Set $x = \lim x_n$ if $\lim x_n$ exists. Then

$$f(x) = f(\lim x_n) = \lim f(x_n) = y,$$

and hence $y \in f(M^n)$. If the sequence $\{x_n\}$ has no limit point, we have $y \in L(f) \subset f(M^n)$.

(ii) Evidently a topological embedding f is one-to-one. If $L(f) \cap f(M^n) \neq \emptyset$, there exists a point y of $L(f) \cap f(M^n)$ with $y = \lim x_n$, where the sequence $\{x_1, x_2, \dots\}$ has no limit point. On the other hand, we can write $y = f(x)$. Since f is a topological map, f^{-1} is continuous, and hence the $\{x_1, x_2, \dots\}$ has limit point x. This is a contradiction.

Suppose now f were not a topological map, *i.e*, $f^{-1} : f(M^n) \to M^n$ were not continuous. Then there would exist a point x of M^n such that $f(x) \in L(f)$. But this contradicts $f(M^n) \cap L(f) = \emptyset$. □

LEMMA 2.7. *Let M^n be an n-dimensional C^∞ manifold. Then there exists a C^∞ map $f : M^n \to \mathbf{R}^p$ with $L(f) = \emptyset$.*

PROOF. Let $\{(V_j, h_j)\}$ be a C^∞ atlas of Lemma 2.3 with respect to the open covering $\{M^n\}$ of M^n, and let φ be the C^∞ function of Lemma 2.5. For each $j > 0$ let $\varphi_j : M^n \to \mathbf{R}$ be the C^∞ function given in the proof of Lemma 2.5. Set

$$f(x) = \sum_{j>0} j\varphi_j(x).$$

The right-hand side of this equation makes sense because the $\{V_j\}$ is locally finite. It follows that f is continuous.

We want to show $L(f) = \emptyset$. Let $\{x_1, x_2, \dots\}$ be a sequence in M^n, which has no limit point. Then for any integer $m > 0$ there exists an integer $i > 0$ with $x_i \notin \overline{W}_1 \cup \dots \cup \overline{W}_m$, and so $x_i \in \overline{W}_j$ for some $j > m$. Hence, $f(x_i) > m$; hence, the sequence $\{f(x_1), f(x_2), \dots\}$ has no limit point. □

Now we are ready to prove

THEOREM 2.6 (Whitney's embedding theorem). *Let M^n be an n-dimensional C^∞ manifold. Then we can embed M^n in $\mathbf{R}^{2n+1}$ as a closed subset.*

PROOF. Let $f : M^n \to \mathbf{R}^1 \subset \mathbf{R}^{2n+1}$ be a C^∞ map given in Lemma 2.7; $L(f) = \emptyset$. Let $\delta : M^n \to \mathbf{R}_+$ be the constant map $\delta(x) \equiv \epsilon > 0$. Then by Theorem 2.5 there exists an immersion $g : M^n \to \mathbf{R}^{2n+1}$ which is a δ-approximation of f. Further, by Lemma 2.5 there exists a one-to-one immersion $h : M^n \to \mathbf{R}^{2n+1}$ which is a δ-approximation of g. We have $L(f) = \emptyset$ if the $\epsilon > 0$ is small enough. Hence, Lemma 2.6 impies that h is an embedding, and so $h(M^n)$ is a closed subset of $\mathbf{R}^{2n+1}$. □

The embedding theorem of Whitney allows us to regard an n-dimensional C^∞ manifold as a submanifold of $\mathbf{R}^{2n+1}$.

Whitney proved the above theorem in 1936.

In the following sections we shall show that we can actually embed an n-dimensional manifold in $2n$ dimensional Euclidean space $\mathbf{R}^{2n}$.

§5. The theorem of Sard

We prove Sard's theorem which we will need for the proof of Whitney's theorem on completely regular immersions in the next section. Sard's theorem is one of the most fundamental in differential topology. In general the set of critical values of a C^∞ map is meager as we see in the following

THEOREM 2.7 (Sard's theorem). *Let U be an open subset of n-dimensional Euclidean space, let $f : U \to \mathbf{R}^p$ be a C^∞ map, and set*

$$C = \{ x \in U \mid \text{rank of } f \text{ at } x < p \}.$$

Then $f(U)$ has measure zero in $\mathbf{R}^p$.

We follow the proof in Milnor's book [A3].

PROOF. Our proof is given by induction on n.

When $n = 0$ the theorem is obvious.

Let $n \geq 1$ and assume that the theorem is true up to $n - 1$.

For $i = 1, 2, \dots$ put

$$C_i = \left\{ x \in U \;\middle|\; \left.\frac{\partial^k f_j}{\partial x_{r_1} \partial x_{r_2} \cdots \partial x_{r_k}}\right|_x = 0, \begin{array}{l} j = 1, 2 \dots, p, \\ 1 \leq k \leq i, \\ 1 \leq r_1 \leq \dots \leq r_k \leq n \end{array} \right\}.$$

Evidently we have

$$C \supset C_1 \supset C_2 \dots \supset C_k \supset C_{k+1} \supset \cdots.$$

We shall carry out our proof in three stages:

Step 1. The measure of $f(C - C_1)$ is zero.

Step 2. The measure of $f(C_i - C_{i+1})$ is zero, $i = 1, 2, \dots$.

Step 3. For a sufficiently large k the measure of $f(C_k)$ is zero.

Combining these steps we obtain Sard's theorem.

Proof of Step 1. If $p = 1$, we have $C = C_1$. Hence, $C - C_1 = \emptyset$, and so the set $f(C - C_1)$ has measure zero. For $p \geq 2$ we will need Fubini's theorem.

FUBINI'S THEOREM (2). *Let A be a measurable set in $\mathbf{R}^p = \mathbf{R}^1 \times \mathbf{R}^{p-1}$. For any point t of $\mathbf{R}^1$ if the measure of $A \cap \{t\} \times \mathbf{R}^{p-1}$ is zero in $\{t\} \times \mathbf{R}^{p-1}$, the measure of A is also zero in $\mathbf{R}^p$.*

It is enough to show that for an arbitrary x in $C - C_1$ there exists a neighborhood $V(x)$ of x in U such that the measure of $f(V(x) \cap C)$ is zero. This is for the following reason. Since $\mathbf{R}^n$ is locally compact and

(2) Sternberg [A8]; also consult any introductory book on real analysis.

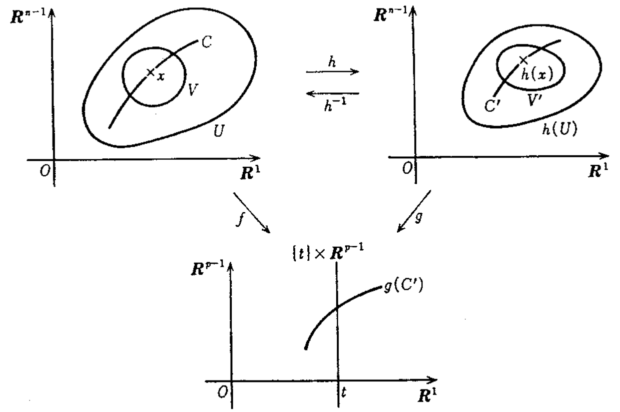


FIGURE 2.6

paracompact it is Lindelöf (any open cover of $\mathbf{R}^n$ has a countable subcover). Hence,

$$C - C_1 \subset \bigcup_{i=1}^{\infty} V_i, \qquad f(V_i \cap C) \text{ has measure zero.}$$

Thus, $C - C_1 = \bigcup_{i=1}^{\infty}[V_i \cap (C - C_1)]$ and $f(C - C_1) = \bigcup_{i=1}^{\infty} f[V_i \cap (C - C_1)]$. Therefore, $f(C - C_1)$ as a countable union of sets of measure zero has measure zero.

We continue our proof. For $x \in C - C_1$ we may assume without loss of generality that $\left.\frac{\partial f_1}{\partial x_1}\right|_x \neq 0$. Define a map $h : U \longrightarrow \mathbf{R}^n$ by $h(x) = (f_1(x), x_2, \dots, x_n)$. Then h is regular at x since

$$\left(\left.\frac{\partial h_i}{\partial x_j}\right|_x\right) = \begin{pmatrix} \partial f_1/\partial x_1|_x & \partial f_1/\partial x_2|_x & \cdots & \partial f_1/\partial x_n|_x \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Hence, h maps some neighborhood V of x diffeomorphically onto the corresponding neighborhood V' of $h(x)$. The restriction of $g \circ h^{-1}$ to V' is a map from V' to $\mathbf{R}^p$, and the set C' of critical points of g is $h(V \cap C)$ (see Figure 2.6). Hence,

$$g(C') = g \circ h(V \cap C) = (f \circ h^{-1}) \circ h(V \cap C) = f(V \cap C).$$

On the other hand, for each point $(t, x_2, \dots, x_n)$ of V', $g(t, x_2, \dots, x_n)$ belongs to the hyperplane $\{t\} \times \mathbf{R}^{p-1} \subset \mathbf{R}^p$. So we have the restriction of g

$$g^t : \{t\} \times \mathbf{R}^{n-1} \cap V' \longrightarrow \{t\} \times \mathbf{R}^{p-1},$$
$$g^t = g|\{t\} \times \mathbf{R}^{n-1}.$$

Now

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} 1 & 0 \\ * & \partial g_i^t/\partial x_j \end{pmatrix}.$$

By the induction hypothesis the set of critical values $g^t(C^t)$ of g^t has measure zero in $\{t\} \times \mathbf{R}^{p-1}$ (C^t denotes the set of critical points of g^t in $\{t\} \times \mathbf{R}^{p-1}$). But we have

$$g(C') \cap \{t\} \times \mathbf{R}^{p-1} = g^t(C^t).$$

Hence, by Fubini's theorem $g(C')$ has measure zero. Thus, $f(V \cap C)$ has measure zero. We have proved the first step.

Proof of Step 2. Pick $x_0 \in C_k - C_{k+1}$ and assume

$$\left.\frac{\partial^{k+1} f_r}{\partial x_{s_1} \cdots \partial x_{s_{k+1}}}\right|_{x_0} \neq 0.$$

Define $w : U \to \mathbf{R}$ by

$$w(x) = \frac{\partial^k f_r(x)}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}}.$$

Then w is a C^∞ map and $\partial w/\partial x_{s_1}|_{x_0} \neq 0$. Here without loss of generality we may assume that $s_1 = 1$. Define a map $h : U \longrightarrow \mathbf{R}^n$ by

$$h(x_1, \dots, x_n) = (w(x), x_2, \dots, x_n).$$

Then h is a C^∞ map satisfying

$$\left(\frac{\partial h_i}{\partial x_j}\right) = \begin{pmatrix} \partial w/\partial x_1 & \partial w/\partial x_2 & \cdots & \partial w/\partial x_n \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix};$$

therefore, h is a homeomorphism of some neighborhood of x_0. Hence, we can show that $f(C_k - C_{k+1})$ has measure zero following the procedure of Step 1.

Proof of Step 3. We show that if $k > n/p - 1$, the set $f(C_k \cap I^n)$ has measure zero in $\mathbf{R}^p$. Here I^n denotes an n-cube of edge δ. Then as in the first step we can write

$$f(C_k) = \bigcup_{i=1}^{\infty} f(C_k \cap I_i^n)$$

to finish the proof.

Let $x \in C_k$ and consider the Taylor expansion of f at x:

$$f(x + h) = f(x) + \frac{h^{k+1}}{(k+1)!} D^{k+1} f(x + \theta h), \qquad 0 < \theta < 1$$
$$= f(x) + R(x, h).$$

Then for $x \in C_k \cap I^n$, $x + h \in I^n$, there exists a constant C such that

$$\|R(x, h)\| \le C\|h\|^{k+1}.$$

Subdivide the cube I^n into r^n n-cubes of edge δ/r, and let I_1 be the subcube which contains the point x. Then each point of I_1 can be expressed as

$$x + h, \qquad \|h\| \le \sqrt{n}\left(\frac{\delta}{r}\right).$$

Hence, from the above equation we get

$$\|f(x+h) - f(x)\| = \|R(x, h)\| \le c\|h\|^{k+1}$$
$$\le c\left(\sqrt{n}\left(\frac{\delta}{r}\right)\right)^{k+1} = c\frac{(\sqrt{n}\delta)^{k+1}}{r^{k+1}} = \frac{a}{r^{k+1}},$$
$$a = c(\sqrt{n}\delta)^{k+1}.$$

Hence, we may cover $f(C_k \cap I^n)$ by a family of small cubes whose volumes add up to V such that

$$V \le r^n\left(\frac{2a}{r^{k+1}}\right)^p = 2a^p r^{n-p(k+1)}.$$

Recall however that $k > n/p - 1$ and so $n - p(k+1) < 0$. Hence, we have proved Step 3. □

REMARK. We may use Sard's theorem to prove Theorem 2.4.

COROLLARY 2.3. *Let M^n and V^p be C^∞ manifolds of respective dimensions n and p, and let $f : M^n \to V^p$ be a C^∞ map. Put*

$$C = \{x \in M^n | \text{rank of } f \text{ at } x < p\}.$$

Then $f(C)$ has measure zero in V^p.

This follows routinely from Sard's theorem.

§6. Whitney's theorem on completely regular immersions

In this section we prove Whitney's theorem on completely regular immersions, which offers a stepping stone for the proof of Whitney's embedding theorem "an n-dimensional C^∞ manifold can be embedded in $\mathbf{R}^{2n}$".

DEFINITION 2.9. Let M^n be an n dimensional C^∞ manifold and let $f : M^n \to \mathbf{R}^{2n}$ be an immersion. We say that f is *completely regular* if it satisfies the following:

(i) f has no triple points.
(ii) For p_1 and p_2, $p_1 \ne p_2$, $f(p_1) = f(p_2) = q$,

$$(df)_{p_1}(T_{p_1}(M^n)) \oplus (df)_{p_2}(T_{p_2}(M^n)) = T_q(\mathbf{R}^{2n}).$$

We say that f *intersects transversely* at q when f satisfies (ii).

Here by "f has no triple points" includes "f has no quadraple points, quintaple points, etc."

THEOREM 2.8. *Let M^n be an n-dimensional C^∞ manifold, and let $f : M^n \to \mathbf{R}^{2n}$ be a C^∞ map. For any continuous function $\delta : M^n \to \mathbf{R}_+$ there exists a completely regular immersion $g : M^n \to \mathbf{R}^{2n}$ which is a δ-approximation of f.*

Further, if the restriction of f to some open neighborhood of a compact set N is an completely regular immersion, we may choose g so that $g|N = f|N$.

In addition if f is an immersion, then g may be chosen to be regularly homotopic to f relative to N.

PROOF. By Theorem 2.5 there exists an immersion $\bar{f}$ which is a $\delta/2$-approximation of f with $\bar{f}|N = f|N$. $\bar{f}$ is completely regular on some open neighborhood of N. Select an atlas of $\mathbf{R}^{2n}$, $\mathscr{S}' = \{(C^{2n}(x_i, 1), \psi_i) | i = 1, 2, \dots\}$, where $\psi_i : C^{2n}(x_i, 1) \to C^{2n}(1)$. The family $\mathscr{U} = \{U, (M^n - N) \cap \bar{f}^{-1}(C^{2n}(x_i, 1)) | i = 1, 2, \dots\}$ is an open covering of M^n. We take an atlas $\{(V_j, h_j) | j \in J\}$ of M^n given by Lemma 2.3 with respect to $\mathscr{U}$, such that for each V_j the following conditions hold.

($*$) $\bar{f}|V_j : V_j \to \mathbf{R}^{2n}$ is an embedding.

($**$) For some λ_j such that $f(V_j) \subset C^{2n}(x_{\lambda_j}, 1)$, there exists a diffeomorphism $\varphi_j : C^{2n}(1) \to \mathbf{R}^{2n}$ with $\varphi_j \circ \psi_{\lambda_j} \circ \bar{f}(V_j) \subset C^{2n}(1) \cap \mathbf{R}^n$. (We think of $\mathbf{R}^n$ as $\mathbf{R}^n = \{(x_1, \dots, x_n, 0, \dots, 0) \in \mathbf{R}^{2n}\} \subset \mathbf{R}^{2n}$.)

Reindexing the $\{(V_i, h_i) | i \in J\}$, as we did in the proof of Theorem 2.5, we assume

$$i \le 0 \text{ iff } V_i \subset U, \qquad i > 0 \text{ iff } M^n - N.$$

We write $\varphi'_j = \varphi_j \circ \psi_{\lambda_j}$.

We shall construct g inductively. Put $g_0 = \bar{f}$, and assume that we have defined $g_j : M^n \to \mathbf{R}^{2n}$ with $g_j|N = \bar{f}|N$.

(a) Replacing the φ_i we may assume that g_j satisfies the conditions ($*$) and ($**$) in place of $\bar{f}$.

(b) We assume that if $N_j = N \cup \left(\bigcup_{i \le j} \overline{W}_i\right)$, then for any point p of $g_j(N_j)$ the set $(g_j|N_j)^{-1}(p)$ contains at most two points and in case it contains two points $g_j|N_j$ intersects transversely at p.

Now we construct $g_{j+1} : M^n \to \mathbf{R}^{2n}$. Consider the map

$$\varphi'_{j+1} \circ g_j \circ (h_{j+1})^{-1} : C^n(3) \longrightarrow C^{2n}(1).$$

Define a projection $\pi : \mathbf{R}^{2n} \to \mathbf{R}^n$ by $\pi(x_1, \dots, x_{2n}) = (x_{n+1}, \dots, x_{2n})$. Then by the choice of φ'_{j+1} we have

$$\varphi'_{j+1} \circ g_j \circ (h_{j+1})^{-1}(C^n(3)) \subset \pi^{-1}(0, 0, \dots, 0),$$

and by Sard's theorem the set of critical values of the C^∞ map

$$\pi \circ \varphi'_{j+1} \circ g_j : [U \cup (\cup_{i \le j} W_i)] \cap g_j^{-1}(U_{\lambda_j}) \longrightarrow C^n(1) \subset \mathbf{R}^n$$

has measure zero. Hence, we may select a point $c_q \in \mathbf{R}^n$ satisfying the following:

(i) c_q is sufficiently close to $(0, 0, \dots, 0)$.
(ii) c_q is a regular value of $\pi \circ \varphi'_{j+1} \circ g_j$.
(iii) If the points p_1 and p_2, $p_1 \neq p_2$, satisfy $g_j(p_1) = g_j(p_2) \in C^{2n}(x_{\lambda_j}, 1)$, then

$$\varphi'_{j+1} \circ g_j(p_1) = \varphi'_{j+1} \circ g_j(p_2) \notin \pi^{-1}(c_q).$$

Using the c_q, we define a C^∞ map $g_{j+1} : M^n \to \mathbf{R}^{2n}$ by

$$g_{j+1}(x) = \begin{cases} g_j(x), & x \in M^n - h_{j+1}^{-1}(C^n(2)), \\ (\varphi'_{j+1})^{-1}\{\varphi'_{j+1} \circ g_j(x) + c_q \varphi(|h_{j+1}(x)|)\}, & x \in V_j, \end{cases}$$

where φ is our bell-shaped function. Then the conditions (i), (ii), and (iii) for c_q guarantee that g_{j+1} is an immersion and that g_{j+1} is a $\delta/2^{q+1}$-approximation of g_j. Further setting $N'_{j+1} = N'_j \cup \overline{W}_j$ and noting N'_{j+1} is compact, we see that for any point p of $g_{j+1}(N'_{j+1})$, the set $(g_{j+1}|N'_{j+1})^{-1}(p)$ contains at most two points, and if it contains two points, then g_{j+1} intersects transversely at p. It is also routine that $g_{j+1}|N = \bar{f}|N$. In addition, by readjusting the φ_i we see that g_{j+1} replacing $\bar{f}$ satisfies the conditions $(*)$ and $(**)$.

Finally, by examining the definition of g_{j+1} carefully we see that g_{j+1} and g_j are regularly homotopic relative to N.

Now define a map $g : M^n \to \mathbf{R}^{2n}$ by

$$g(x) = \lim_{i \to \infty} g_i(x),$$

which is what we wanted. □

§7. Special self-intersections

Our aim is to prove that a compact C^∞ manifold M^n of dimension n can be embedded in $\mathbf{R}^{2n}$; to this end we construct a model of an immersion with a *self-intersection* (double point which intersects transversely).

Consider $n = 1$. Define a map $\alpha : \mathbf{R}^1 \to \mathbf{R}^2$ by

$$y = x - x/(1 + x^2), \qquad z = 1/(1 + x^2).$$

This is a C^∞ immersion as shown in Figure 2.7, with one self-intersection. From this we get an immersion $\beta : \mathbf{R}^1 \to \mathbf{R}^2$ with exactly one self-intersection such that for a sufficiently large $r > 0$, β is the identity map outside $D'(0, r)$, the one-disk centered at 0 with radius r.

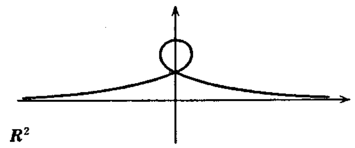


FIGURE 2.7

More generally, for n, $n \ge 2$, define a C^∞ map $\alpha : \mathbf{R}^n \to \mathbf{R}^{2n}$ by

$$\alpha(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_{2n}), \qquad (x_1, x_2, \dots, x_n) \in \mathbf{R}^n,$$
$$y_1 = x_1 - \frac{2x_1}{u},$$
$$y_i = x_i, \qquad i = 2, \dots, n,$$
$$y_{n+1} = \frac{1}{u}, \qquad y_{n+i} = \frac{x_1 x_i}{u}, \qquad i = 2, \dots, n,$$

where $u = (1 + x_1^2)(1 + x_2^2) \cdots (1 + x_n^2)$. We show that the map α is an immersion. The Jacobian matrix $(D\alpha)(x)$ of α is

$$\begin{pmatrix} 1 - \dfrac{2(1 - x_1^2)}{u(1 + x_1^2)} & \dfrac{4x_1x_2}{u(1 + x_2^2)} & \cdots & \dfrac{4x_1x_n}{u(1 + x_n^2)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 0 \\ \dfrac{-2x_1}{u(1 + x_1^2)} & \dfrac{-2x_2}{u(1 + x_2^2)} & \cdots & \dfrac{-2x_n}{u(1 + x_n^2)} \\ \dfrac{x_2(1 - x_1^2)}{u(1 + x_1^2)} & \dfrac{x_1(1 - x_2^2)}{u(1 + x_2^2)} & \cdots & \dfrac{-2x_1x_2x_n}{u(1 + x_n^2)} \\ \vdots & \vdots & & \vdots \\ \dfrac{x_n(1 - x_1^2)}{u(1 + x_1^2)} & \dfrac{-2x_1x_2x_n}{u(1 + x_2^2)} & \cdots & \dfrac{x_1(1 - x_n^2)}{u(1 + x_n^2)} \end{pmatrix}.$$

Notice that not all entries of the first column are zero, for if $-2x_1^2/\{u(1 + x_1^2)\}$ is zero then $x_1 = 0$ so that $1 - 2(1 - x_1^2)/\{u(1 + x_1^2)\} = 1 - 2/u$. The last equation is zero when $u = 2$, and hence we have some numbers different from zero among the $x_2, x_3, \dots, x_n$. Hence, at least one of the $x_i(1 - x_1^2)/\{u(1 + x_1^2)\}$, $i = 2, 3, \dots, n$, is different from zero. Therefore, the rank of $D(\alpha)(x)$ is n, and so α is an immersion.

Now we look for double points of α; we want $x = (x_1, x_2, \dots, x_n)$ and

$x' = (x'_1, x'_2, \ldots, x'_n)$ in $\mathbf{R}^n$ such that $\alpha(x) = \alpha(x')$ and $x \neq x'$. Put

$$u' = (1 + (x'_1)^2)(1 + (x'_2)^2) \cdots (1 + (x'_n)^2),$$
$$\alpha(x) = (y_1, y_2, \ldots, y_{2n}), \qquad \alpha(x') = (y'_1, y'_2, \ldots, y'_{2n}).$$

Since $y'_i = y_i$, $i = 2, 3, \ldots, n$, we must have $x'_i = x_i$, $i = 2, 3, \ldots, n$. Further, $y'_{n+1} = y_{n+1}$ implies $u' = u$, and so $(x'_1)^2 = x_1^2$, that is, $x'_1 = -x_1$. Hence, from $y'_{n+i} = y_{n+i}$, $i = 2, 3, \ldots, n$, we get $x_i = 0$, $i = 2, 3, \ldots, n$. Further, from $y'_1 = y_1$, we have

$$x_1 - \frac{2x_1}{u} = -x_1 + \frac{2x_1}{u}, \qquad u = 1 + x_1^2 = 2;$$

therefore, $x_1 = \pm 1$. Hence, the only intersecting point in the image of α is

$$\alpha(1, 0, \ldots, 0) = \alpha(-1, 0, \ldots, 0).$$

The Jacobian matrix $D(\alpha)(x)$ at $x = (\pm 1, 0, \ldots, 0)$ is

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 1 \\ \mp 1/2 & 0 & \cdots & \cdots & 0 \\ 0 & \pm 1/2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 & \pm 1/2 \end{pmatrix}.$$

The ith column represents the components of the vector $(d\alpha)(\partial/\partial x_i) = \partial\alpha/\partial x_i$ at $(\pm 1, 0, \ldots, 0)$. We make an $(2n, 2n)$-matrix by combining the above two $(2n, n)$-matrices:

$$\left(\begin{array}{cccc:cccc} 1 & & & & 1 & & & \\ & 1 & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 1 \\ -1/2 & & & & 1/2 & & & \\ & 1/2 & & & & -1/2 & & \\ & & \ddots & & & & \ddots & \\ & & & 1/2 & & & & -1/2 \end{array}\right),$$

which is evidently regular. Hence, the map α intersects transversely at $\alpha(\pm 1, 0, \ldots, 0)$.

From this we see that an immersion $\beta : \mathbf{R}^n \to \mathbf{R}^{2n}$ with exactly one self-intersection exists and that β is the identity map outside $D^n(0, r)$ for a sufficiently large $r > 0$.

§8. The intersection number of a completely regular immersion

In this section we define the intersection number I_f of a completely regular immersion f of an n-dimensional manifold M^n in $\mathbf{R}^{2n}$, and using the model constructed in the previous section we show the existence of a completely regular immersion with arbitrary intersection number.

DEFINITION 2.10. Let $f : M^n \to \mathbf{R}^{2n}$ be a completely regular immersion of an n-dimensional compact C^∞ manifold M^n.

(i) The case where M^n is orientable and n is even. Choose an orientation in M^n. Suppose that $f(p) = f(q)$, $p \neq q$. Let $u_1, \ldots, u_n \in T_p(M^n)$ and $v_1, \ldots, v_n \in T_q(M^n)$ be ordered sets of linealy independent vectors, which define the orientations of $T_p(M^n)$ and $T_q(M^n)$. We say that the self-intersection at $f(p)$ has *positive type* or *negative type* depending on whether the ordered set of vectors

$$(df)_p(u_1), \ldots, (df)_p(u_n), (df)_q(v_1), \ldots, (df)_q(v_n) \in T_{f(p)}(\mathbf{R}^{2n})$$

defines the positive orientation or the negative orientation in $\mathbf{R}^{2n}$, and we define the intersection number of this self-intersection to be $+1$ or -1 accordingly. The *intersection number* I_f of f is the sum of intersection numbers of self-intersections of f: $I_f \in \mathbf{Z}$.

(ii) The case where M^n is nonorientable or n is odd.

In this case we define the intersection number $I_f \in \mathbf{Z}_2$ in the same way as above.

REMARK. For $n = 1$ compare the I_f in this section with the I_f defined in Chapter 0.

THEOREM 2.9. *Let M^n be an n dimensional compact C^∞ manifold.*

(i) *If M^n is orientable and n is even, then for an arbitrary integer m there exists a completely regular immersion $f : M^n \to \mathbf{R}^{2n}$ with $I_f = m$.*

(ii) *If M^n is nonorientable or n is odd, then for any $m \in \mathbf{Z}_2$, there exists a completely regular immersion $f : M^n \to \mathbf{R}^{2n}$ with $I_f = m$.*

PROOF. By Theorem 2.8 there exists a completely regular immersion f_0 of M^n in $\mathbf{R}^{2n}$. Take a point x_0 of M^n and some neighborhood U of x_0 to replace f_0 by the map β with exactly one self-intersection defined in §7 or by the composite of β with the map $r : \mathbf{R}^{2n} \longrightarrow \mathbf{R}^{2n}$ defined by $r(x_1, x_2, \ldots, x_{2n}) = (-x_1, x_2, \ldots, x_{2n})$. The intersection number of the new completely regular immersion f is $I_{f_0} + 1$ or $I_{f_0} - 1$. We repeat this process till we get an immersion with intersection number m. $\square$

§9. Whitney's embedding theorem II: $M^n \subset \mathbf{R}^{2n}$

In this section we prove Whitney's theorem "An n-dimensional compact manifold M^n can be embedded in $\mathbf{R}^{2n}$".

THEOREM 2.10. *Let $f : M^n \to \mathbf{R}^{2n}$ be an immersion, $n \geq 3$. Then there exists a regular homotopy $\{f_t \mid t \in [0, 1]\}$ of f satisfying the following:*

(0) *$f_0 = f$, f_1 is a completely regular immersion.*
(1) *The number of self-intersections of f_1 is two more than that of f.*

If (i) M^n is orientable, n is even, and the number of self-intersections of f is greater than $|I_f|$, or (ii) M^n is nonorientable or n is odd, and f has at least one self-intersection, then there exists a regular homotopy $\{f_t\}$ of f such that the number of self-intersections of f_1 is two less than that of $f = f_0$.

(For more on regular homotopies see §1, Chapter III).

PROOF OF THE FIRST HALF. Notice first that by Theorem 2.8 we may assume f is a completely regular immersion. Take a sufficiently small n-disk D^n in $f(M^n)$, $D^n \subset f(M^n) \subset \mathbf{R}^{2n}$, and choose distinct points p and q in D^n.

(a) We try to grasp a whole picture with $n = 1$. We take an embedding of D^1 in $\mathbf{R}^2$ as shown in Figure 2.8 and make two self-intersections by moving the disk around by a regular homotopy, say, $q = 0$, $p \in D^1 \subset \mathbf{R}^1 \subset \mathbf{R}^2$.

(b) Suppose D^n is embedded in $\mathbf{R}^n \subset \mathbf{R}^{2n}$. Let $q = 0$ and p be points of D^n with $p \neq q$. Assume p sits on the x_1-axis. Pull up p to the (x_1, x_{n+1})-plane in the positive direction on the x_{n+1}-axis. We then position some neighborhood of x parallel to the $(x_1, x_{n+2}, \dots, x_{2n})$-plane. Next we move this neighborhood in the $(x_1, x_{n_2}, \dots, x_{2n})$-plane through the origin 0 (at this point the disk crosses itself) making sure not to create any other intersections (see Figure 2.9). The only intersections are on the x_1-axis; we have created two self-intersections.

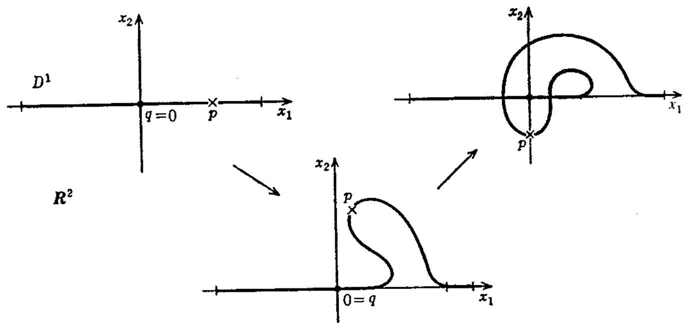


FIGURE 2.8

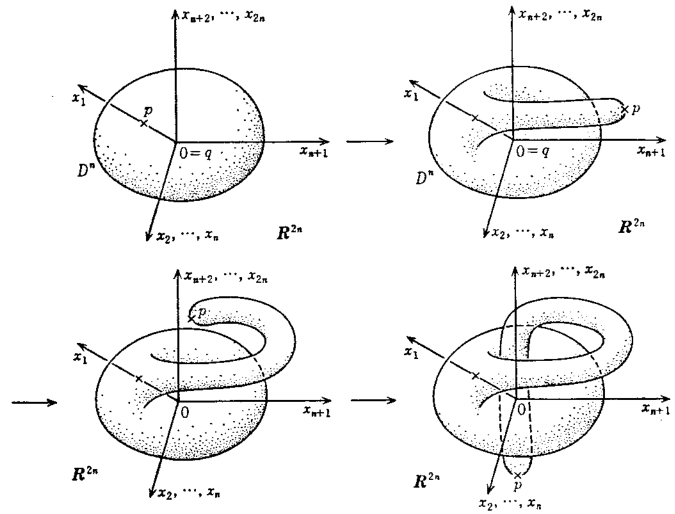


FIGURE 2.9

A sketch (of a portion) of the proof for the latter half. We show how to eliminate two self-intersections in case (ii) where either M^n is nonorientable or n is odd, and how to eliminate two self-intersections of distinct types in case (i) where M^n is orientable and n is even, in each case through a regular homotopy.

Let $f(p_1) = f(p_2) = q$ and $f(p_1') = f(p_2') = q'$, and assume that these self-intersections have opposite types. Let C_1, C_2 be disjoint curves in M^n, C_i connecting p_i and p_i' and not passing through any other self-intersections of f, $i = 1, 2$. Then $B_i = f(C_i)$ is a curve connecting q and q', and $B = B_1 \cup B_2$ is a simple closed curve in $f(M^n)$. Take a two-cell σ^2 with boundary B such that $F(M^n) \cap \sigma^2 = B$ (Lemma 2.9).

We next deform f through σ^2 in a neighborhood of C_2 in M^n so that the new image of C_2 will not meet B_1. In this way we will remove the two self-intersections.

The detail of this proof appears later. For now, using Theorem 2.9, we shall show the following

THEOREM 2.11 (Whitney's embedding theorem). *Let M^n be a closed C^∞ manifold of dimension n. Then we can embed M^n in $\mathbf{R}^{2n}$.*

PROOF. The theorem is routine for $n = 1$ as M^1 is a finite union of S^1.

Case $n = 2$. We can embed the sphere S^2, the projective space $\mathbf{R}P^2$ and the Klein bottle K^2 in $\mathbf{R}^4$. By the classification theorem for closed surfaces, M^2 is a connected sum of a finite number of copies of the above

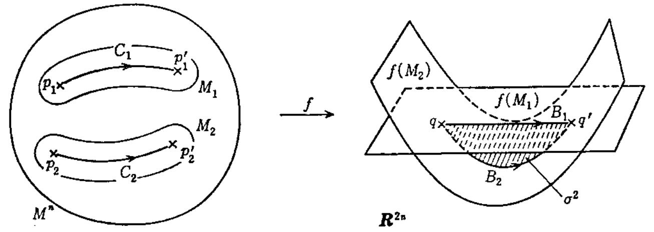


FIGURE 2.10

surfaces. Hence, we can embed M^2 in $\mathbf{R}^4$ (cf. any elementary text covering the classification of surfaces.) (3)

Let $n \geq 3$. By Theorem 2.9 there exists a completely regular immersions $f_0 : M^n \to \mathbf{R}^{2n}$ with $I_{f_0} = 0$. By Theorem 2.10 we can remove all self-intersections. Thus, we have obtained an embedding $f_1 : M^n \to \mathbf{R}^{2n}$. □

In order to complete the proof of Theorem 2.10 we need the following three lemmas.

LEMMA 2.8. *Let* $\mathscr{B} = \{B, p, |K|; \mathbf{R}^m, \mathrm{O}(m)\}$ *be an m-vector bundle over a polyhedron* $|K|$. *Let* K' *be a subcomplex of* K. *Assume that* (i) $\zeta_1, \dots, \zeta_{i-1}$ *are cross sections over* $|K|$ *such that for each* $p \in |K|$, $\zeta_1(p), \dots, \zeta_{i-1}(p)$ *form an orthonormal system, and* (ii) ζ_i *is a cross section of* $\mathscr{B}$ *over* $|K'|$ *such that for each point* $p \in |K'|$, $\zeta_1(p), \dots, \zeta_{i-1}(p)$, $\zeta_i(p)$ *form an orthonormal system. Then if* $\dim K \leq m - i$, *we can extend* ζ_i *over* $|K|$ *in such a way that the* $\zeta_1(p), \dots, \zeta_{i-1}(p)$, $\zeta_i(p)$ *is an orthonormal system.*

PROOF. Fix a point p of $|K|$ and let $\mathbf{R}^m_p$ denote the fiber over p. Then the desired $\zeta_i(p)$ may be a unit vector in the orthogonal complement of the $(i-1)$-subspace spanned by $\zeta_1(p), \dots, \zeta_{i-1}(p)$, *i.e.*, we want $\zeta_i(p) \in S^{m-i}_p \subset \{\{\zeta_1(p), \dots, \zeta_{i-1}(p)\}\}^\perp \subset \mathbf{R}^m_p$, where S^{m-i}_p is the unit sphere in the orthogonal complement of the space $\{\{\zeta_1(p), \dots, \zeta_{i-1}(p)\}\}$ spanned by $\zeta_1(p), \dots, \zeta_{i-1}(p)$. Hence our problem reduces to the problem of extending a cross section over $|K'|$ to $|K|$ in the $(m-i)$-sphere bundle over $|K|$. But the obstructions for this are in

$$H^j(K, K'; \pi_{j-1}(S^{m-i}))$$

(see for instance Steenrod [A7]), which is empty since we have $\dim K \leq m - i$, and so we can extend our cross section. □

In what follows assume $n \geq 3$. Let C_i, $i = 1, 2$, be the curves given in (b) of the proof of Theorem 2.10 (Figure 2.10). Let M_1 and M_2 be suitable

(3) H. Seifert and W. Threlfall, *A textbook of topology*, Academic Press, San Diego, CA, 1980; S. Lefschetz, *Introduction to topology*, Princeton University Press, Princeton, NJ, 1949, *etc.*

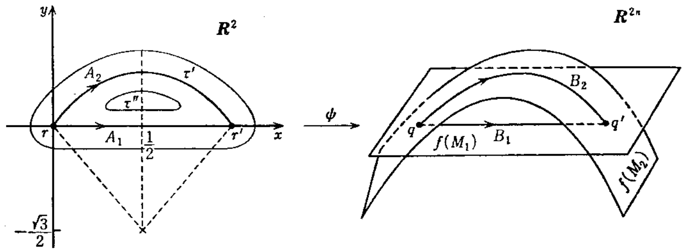


FIGURE 2.11

neighborhoods in M^n of C_1 and C_2, respectively. Put

$$C_i = \{p_{it} | 0 \leq t \leq 1\}, \qquad p_{i0} = p_i,\ p_{i1} = p_i'.$$

We take the curves $p_i : [0, 1] \to M^n$, $p_i(t) = p_{it}$, to be embeddings. Then the curves $q_i : [0, 1] \to \mathbf{R}^{2n}$, $q_i(t) = q_{it}$, where $f(p_{it}) = q_{it}$, are embeddings as well.

LEMMA 2.9. *Consider* $\mathbf{R}^2 = \{(x, y) | x, y \in \mathbf{R}\}$. *Set* $A_1 = \{(x, y) \in \mathbf{R}^2 | 0 \leq x \leq 1, y = 0\}$. *Denote by* A_2 *the arc of the circle of radius one centered at* $(1/2, -\sqrt{3}/2)$, *connecting the endpoints* r *and* r' *of* A_1 *in the halfplane* $y \geq 0$. *Let* $A = A_1 \cup A_2$. *Let* τ' *be a small neighborhood of* A *in* $\mathbf{R}^2$, *and let* τ *be the union of* τ' *and the finite region* τ'' *bounded by* A (Figure 2.11). *Then there exists an embedding* $\psi : \tau \to \mathbf{R}^{2n}$ *satisfying the following*:

(i) $\psi(r) = q$, $\psi(r') = q'$, $\psi(A_i) = B_i$, $i = 1, 2$.
(ii) $\psi(\tau) \cap f(M^n) = B$.
(iii) $T_{q^*}(\psi(\tau)) \not\subset T_{q^*}(f(M^n))$, *for all* $q^* \in B$.

PROOF. We first show that an embedding ψ as specified above exists with respect to the set τ'. Put

$$T_1^n = T_q(f(M_1)), \qquad T_2^n = T_q(f(M_2)),$$
$$T^2 = T_q(B_1) \oplus T_q(B_2) \subset T_q(\mathbf{R}^{2n}).$$

T^2 intersects T_1^n and T_2^n in straight lines:

$$T_1^1 = T^2 \cap T_2^n, \qquad T_2^1 = T^2 \cap T_2^n.$$

Changing f slightly we may assume that f maps some neighborhood of p_i in M_i onto the exponential image of some neighborhood of the origin of T_i^n in $f(M_i)$. We may assume further that f maps some neighborhood of C_i in M_i onto the exponential image of some neighborhood of T_i^1 in T_i^n (valid only very near q), $i = 1, 2$.

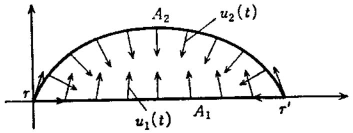


FIGURE 2.12

Now there exist a closed neighborhood $\overline{V(r)}$ of r in τ and a linear map $\phi : \overline{V(r)} \to T^2$ mapping $\overline{V(r)}$ to a closed neighborhood of the origin in T^2. It follows that the ϕ, when restricted to $\overline{V(r)} \cap A_i$, maps a closed neighborhood of r in A_i to a closed neighborhood of the origin in T_i^1. Similarly, we define a linear map ϕ in some closed neighborhood of r'.

Let

$$r_i : [0, 1] \longrightarrow A_i \subset \mathbf{R}^2, \qquad i = 1, 2$$

be a parametric representation of the curve A_i such that $\phi(r_i(t)) = q_i(t)$ wherever ϕ is defined.

For $t \in [0, 1]$ let $u_i(t)$ be a smooth cross section in the restriction $T(\tau)|A_i$ of the tangent bundle $T(\tau)$ of τ to A_i, subject to the following:

(0) $\exp(u_i(t)) \subset \tau$, $i = 1, 2$.

(i) $\begin{cases} u_1(0) \in T_r(A_2),\ u_1(0) & \text{is pointed forward along } A_2, \\ u_1(1) \in T_{r'}(A_2),\ u_1(1) & \text{is pointed backward along } A_2, \end{cases}$

$\begin{cases} u_2(0) \in T_r(A_1),\ u_2(0) & \text{is pointed forward along } A_1, \\ u_2(1) \in T_{r'}(A_1),\ u_2(1) & \text{is pointed backward along } A_1. \end{cases}$

(ii) $u_i(t) \in T_{r_i(t)}(\tau)$, $u_i(t) \notin T_{r_i(t)}(A_i)$, $i = 1, 2$.

(iii) $u_1(t)$ rotates counterclockwise as t goes from 0 to 1; $u_2(t)$ rotates clockwise as t moves from 0 to 1 (Figure 2.12).

Denote by $R_i(t)$ the line segment in the direction of $u_i(t)$ centered at $r_i(t)$ of length ρ. Here we readjust $\overline{V(r)}$ and $\overline{V'(r')}$ so that each $R_i(t)$ forms a part of their boundaries. We may also choose ρ small enough to have $R_1(t_1) \cap R_2(t_2) = \varnothing$ if $r_1(t), r_2(t) \notin \overline{V(r)} \cup \overline{V'(r')}$.

For $r_i(t) \in [\overline{V(r)} \cup \overline{V'(r')}] \cup A_i$, set $v_i(t) = (d\phi)_{r_i(t)}(u_i(t))$. Then $v_i(t)$ is a vector field in $f(\overline{V(r)} \cap \overline{V'(r')})$, which we want to extend to B_i without making it touch $f(M^n)$ at $q_i(t)$. But this can be done by Lemma 2.8.

Since ϕ is linear on $\overline{V(r)} \cup \overline{V'(r')}$, we have

$$\phi(r_i(t) + \alpha u_i(t)) = q_i(t) + \alpha v_i(t), \qquad r_i(t) \in \overline{V(r)} \cup \overline{V'(r')},\ |\alpha| \le \rho.$$

Using this identity we can extend ϕ to the closed neighborhood $\overline{\tau'}$ of A.

Let ψ' be a continuous extension of ϕ over τ. By taking a smooth approximation, we may assume that ψ' is a smooth map. Since $2n \ge 5$, by Theorem 2.6, there exists an embedding ψ which is a C'-approximation of ψ'. Hence, ψ is an embedding and for $r^* \in A$ we have

$$T_{\psi(r^*)}(\psi(\tau)) \not\subset T_{\psi(r^*)}(f(M)).$$

Further, because $n + 2 < 2n$ we may assume $\psi(\tau) \cap f(M) = B$. Hence, $\sigma = \psi(\tau)$ is the desired 2-cell. □

LEMMA 2.10. *Suppose q and q' are self-intersections of f of opposite types. Then there exist cross sections $w_1, \dots, w_{2n}$ in $T(\mathbf{R}^{2n})|\sigma$, the restriction of the tangent bundle $T(\mathbf{R}^{2n})$ of $\mathbf{R}^{2n}$ to σ, satisfying the following:*

(0) *For each q^* in σ, $w_1(q^*), \dots, w_{2n}(q^*)$ are linearly independent.*

(i) *For $q^* = \psi(r^*)$, $r^* \in A$,*

$$w_1(q^*) = (d\psi)_{r^*}(e_1), \qquad w_2(q^*) = (d\psi)_{r^*}(e_2).$$

(ii) *For $q^* \in B_1$,*

$$w_3(q^*), \dots, w_{n+1}(q^*) \in T_{q^*}(f(M_1)).$$

(iii) *For $q^* \in B_2$,*

$$w_{n+1}(q^*), \dots, w_{2n}(q^*) \in T_{q^*}(f(M_2)).$$

This lemma clarifies how the σ, the $f(M_1)$, and the $f(M_2)$ are positioned relative to each other in $\mathbf{R}^{2n}$.

PROOF. Since ψ is an embedding, $w_1(q^*)$ and $w_2(q^*)$ are linearly independent. Set

$$V_1^{n-1} = \{\{e_3, \dots, e_{n+1}\}\}, \qquad V_2^{n-1} = \{\{e_{n+2}, \dots, e_{2n}\}\},$$

where $\{\{\cdots\}\}$ denotes the subspace of $\mathbf{R}^{2n}$ spanned by $\{\cdots\}$. For each point q^* of B_1 put

$$V_1^{n-1}(q^*) = \{v \in T_{q^*}(f(M_1)) | v \perp T_{q^*}(B_1)\}.$$

Setting $\mathscr{B}_1 = \bigcup_{q^* \in B_1} V_1^{n-1}(q^*)$, we obtain an $(n-1)$-dimensional vector bundle $\mathscr{B}_1 = \{\mathscr{B}_1, \pi_1, B_1\}$ over B_1. As the base space B_1 is contractible, the bundle $\mathscr{B}_1$ is trivial. Hence, there exist linearly independent vector fields, say, $w_1, w_3, \dots, w_{n+1}$, in $\mathscr{B}_1$, which define an orientation of $f(M_1)$ at each point q^* of B_1.

Similarly for $q^* \in B_2$ setting

$$V_2^{n-1}(q^*) = \{v \in T_{q^*}(f(M_2)) | v \perp T_{q^*}(B_2)\},$$

we obtain an $(n-1)$-dimensional vector bundle $\mathscr{B}_2 = \{\mathscr{B}_2, \pi_2, B_2\}$, where $\mathscr{B}_2 = \bigcup_{q^* \in B_2} V_2^{n-1}(q^*)$. Again there exist linearly independent vector fields $w_2', w_{n+2}, \dots, w_{2n}$ in B_2, which define an orientation of $f(M_2)$ at each point q^* of B_2. Here we assume $w_2'(q^*)$ to be an element of $T_{q^*}(B_2)$, which points in the positive direction along B_2.

Let $\mathscr{B} = \{\sigma \times \mathbf{R}^{2n}, p_1, \sigma\}$ be an $2n$-dimensional trivial bundle over σ. The restriction $\mathscr{B}|B_2$ of $\mathscr{B}$ over B_2 has $n + 1$ linearly independent cross sections $w_1, w_2, w_{n+2}, \dots, w_{2n}$. Further, if $q^* = q$ or $q^* = q'$, the $w_1(q^*), \dots, w_{2n}(q^*)$ are linearly independent. By Lemma 2.8 we can extend the $w_3, \dots, w_n$ over B_2 and for each $q^* \in B_1$ the $2n-1$ vectors $w_1(q^*)$, $w_2(q^*)$, $w_3(q^*), \dots, w_n(q^*)$, $w_{n+2}(q^*), \dots, w_{2n}(q^*)$ are linearly independent.

Recall that the self-intersections q and q' have opposite types. By suitable choices of $w_1, w_3, \dots, w_{n+1}$ in B_1 and w_2', $w_{n+2}, \dots, w_{2n}$ in B_2 we may assume that for $q^* = q$ or $q^* = q'$, the vectors

$$w_1(q^*), w_3(q^*), \dots, w_{n+1}(q^*), w_2'(q^*), w_{n+2}(q^*), \dots, w_{2n}(q^*)$$

define the orientation opposite from the preassigned orientation in $\mathbf{R}^{2n}$.

Now we can deform $w_2(q)$ and $w_2(q')$ to $w_2'(q)$ and $w_2'(q')$ keeping them in $T(\sigma)$ and out of $T(B_1)$; thus, these vectors remain linearly independent of the above vectors. Hence, the vectors

$$w_1(q^*), w_2(q^*), \dots, w_{2n}(q^*)$$

define the positive orientation of $\mathbf{R}^{2n}$ at $q^* = q$ or $q^* = q'$. Therefore, we can extend the cross section w_{n+1} over B_2 while keeping its linear independence with the other vectors.

Using Lemma 2.8 again we extend the sections $w_3, \dots, w_{n+1}$ over σ so that $w_1, \dots, w_{n+1}$ are linearly independent at each point of σ.

Finally, we extend the cross sections $w_{n+2}, \dots, w_{2n}$ defined over B_2 to σ while keeping their linear independence. This is possible since we may think of B_2 as a smooth deformation retract of σ. Hence, we have shown the lemma is valid. □

The w_i, as constructed above, have clarified the situation in a small neighborhood of σ in $\mathbf{R}^{2n}$.

PROOF OF THEOREM 2.10. Consider $\tau \subset \mathbf{R}^2 \subset \mathbf{R}^{2n}$. For each point $r = (a_1, a_2, \dots, a_{2n})$ of $\mathbf{R}^{2n}$ set $r^* = (a_1, a_2, 0, \dots, 0)$ and

$$\psi(r) = \psi(r^* + \sum_{i=3}^{2n} a_i e_i) = \psi(r^*) + \sum_{i=3}^{2n} a_i w_i(\psi(r^*)).$$

For each point q^* of σ, the vectors $w_1(q^*), \dots, w_{2n}(q^*)$ are linearly independent and they are C^∞ in q^*. Therefore, ψ when considered as a map from a neighborhood of σ to $\mathbf{R}^{2n}$ has the nonzero Jacobian matrix at each point in its domain. Hence, we have the inverse ψ^{-1}. Setting

$$N_1 = \psi^{-1}(f(M_1)), \qquad N_2 = \psi^{-1}(f(M_2)),$$

we see that N_1 and N_2 are contained in a neighborhood U of τ in $\mathbf{R}^{2n}$. If we can deform N_2 in U so N_2 does not intersect N_1 (*i.e*, there exists an isotopy $\{i_t : t \in [0, 1]\}$ of the inclusion map $i: N_2 \to U$ such that $i_0 = i$, $i_1(N_2) \cap N_1 = \varnothing$), the family $\{\psi \circ i_t\}$ defines a deformation of f and the number of self-intersections of f decreases by two. Hence, in this case the proof will be complete. Set

$$\pi(x_1, x_2, \dots, x_{2n}) = (x_1, 0, x_3, \dots, x_{2n}).$$

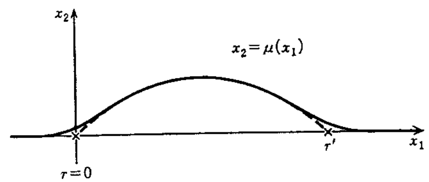


FIGURE 2.13

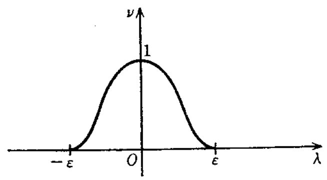


FIGURE 2.14

By the conditions (i), (ii), and (iii) of Lemma 2.10 and the definition of $\psi(r)$ as given above for $r^* \in A_1$, $T_{r^*}(N_1)$ is in the $(x_1, x_3, \dots, x_{n+1})$-plane. Hence $T_{r^*}(\pi(N_1))$ is also in the $(x_1, x_3, \dots, x_{n+1})$-plane. Similarly $T_{r^*}(\pi(N_2))$ is in the $(x_1, x_{n+2}, \dots, x_{2n})$-plane. Hence, $\pi(N_1) \cap \pi(N_2)$ is on the x_1-axis.

Let $\mu(x_1)$ be a C^∞ function whose graph $x_2 = \mu(x_1)$ is the union of the set A_2 and the x_1-axis minus the set A_1 smoothed out at the points r and r' (Figure 2.13).

Take $\epsilon > 0$ such that the interior of N_2 consists of points whose distances from the (x_1, x_2)-plane are each less than ϵ. Consider a C^∞ function ν : $\mathbf{R}^1 \to \mathbf{R}$ as follows (Figure 2.14):

$$|\nu(\lambda)| \le 1, \qquad \nu(0) = 1,$$
$$\nu(\lambda) = 0 \quad \text{if } |\lambda| \ge \epsilon^2.$$

Now define a map $\theta_t : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ by

$$\theta_t(r) = r - t\nu(x_3^2 + \dots + x_{2n}^2)\mu(x_1)e_2, \qquad r = (x_1, \dots, x_{2n}) \in \mathbf{R}^{2n}.$$

By the definition of ν, θ_t is the identity map outside N_2. Evidently the $\{\theta_t : \mathbf{R}^{2n} \to \mathbf{R}^{2n}\}$ is a regular homotopy with $\theta_0 = 1$. When $t = 1$, θ_1 maps the portion of N_2 on A_2 to the halfplane $x_1 < 0$. But $\pi(\theta_t(N_2)) = \pi(N_2)$ and $\pi(N_1) \cap \pi(N_2)$ is on the x_1-axis, and so $N_1 \cap \theta_1(N_2)$ must be on the x_1-axis. However, $\theta_1(N_2)$ does not intersect the x_1-axis, and so $\theta_1(N_2)$ must be empty.

Further, since $\psi(\tau) \cap f(M) = B$ no new self-intersection will arise if we take ϵ sufficiently small. This concludes the proof for the case that M^n is orientable and n is even.

In other cases too we can remove pairs of self-intersections by regular homotopies in just about the same way as above. To do this we only need to show that the C_1 and C_2 can be chosen so that q and q' have opposite types.

The case M is not orientable. If we can take C_1 and C_2 as above and q and q' are of opposing types we follow the previous argument. If q and q' is of the same type, we choose a curve C_2' from p_2 to p_2' so that $C_2 \cup C_2'$ reverses the orientation in M. Then q and q' are of opposing type with respect to C_1 and C_2'.

The case n is odd and M is orientable. Suppose q and q' have the same type for C_1 and C_2. Then choose a curve C_1' from p_1 to p_2' and a curve C_2' from p_2 to p_1' as follows. The curve C_i' agrees with C_i near the starting point and with C_j, $i \neq j$, near the endpoint. The neighborhood M_i' of C_i' is chosen in such a way that M_i agrees with M_i' near the point p_i and with M_j' near p_i', $j \neq i$. Orient M_i and M_i' with the preassigned orientations near p_i and p_i'. Then q and q' have the opposing orientations with respect to (M_1', M_2'). □

CHAPTER III

Immersions of C^∞ Manifolds

In this chapter we discuss classifications of immersions by regular homotopies, a weaker version of classifications of embeddings by isotopies (with the Smale-Hirsch theorem as the central feature) which reduces the problem of immersions to the homotopy theory.

§1. Immersions and regular homotopies

Let M^n and V^p be C^∞ manifolds of dimensions n and p respectively. Denote by $C^\infty(M^n, V^p)$ the set of C^∞ maps from M^n to V^p with the C^∞ topology. Write $\mathrm{Imm}(M^n, V^p)$ for the set of immersions of M^n in V^p. Since $\mathrm{Imm}(M^n, V^p) \subset C^\infty(M^n, V^p)$, we give $\mathrm{Imm}(M^n, V^p)$ the relative topology.

DEFINITION 3.1. Let f and g be immersions of M^n in V^p. We say that f and g are *regularly homotopic* and write $f \underset{r}{\simeq} g$ if they belong to the same arcwise connected component of $\mathrm{Imm}(M^n, V^p)$.

Evidently the $\underset{r}{\simeq}$ is an equivalence relation.

REMARK 1. Write $\mathrm{Imm}^1(M^n, V^p)$ for $\mathrm{Imm}(M^n, V^p)$ with the relative C^1 topology of $C^\infty(M^n, V^p)$. Then the identity map

$$1 : \mathrm{Imm}(M, V) \longrightarrow \mathrm{Imm}^1(M, V)$$

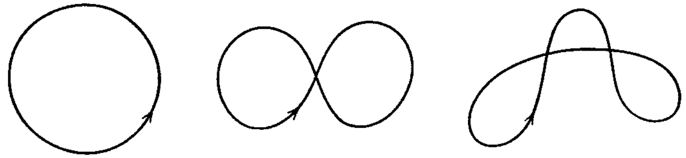
is continuous and induces the bijection

$$1_* : \pi_0(\mathrm{Imm}(M, V)) \longrightarrow \pi_0(\mathrm{Imm}^1(M, V))$$

(the surjectivity is clear from the definition and the injectivity follows from the approximation theorem). Hence, we may define $f, g \in \mathrm{Imm}(M, V)$ to be homotopic if they can be connected by a curve in $\mathrm{Imm}^1(M, V)$; this was our definition in Chapter II.

REMARK 2. The approximation theorem in Remark 1 goes as follows.

THEOREM 3.1. *Let M and V be C^s manifolds, $1 \leq s \leq \infty$, and let $f : M \to V$ be a C^r map, $0 \leq r \leq s$. Suppose that f is of class C^s on A, $A \subset M$ (f is C^s on some open neighborhood U of A). Then there exists a C^s map $g : M \to V$ which approximates f in the C^r topology such that $g|W = f$, $A \subset W \subset U$.*



$f(S^1)$ $g(S^1)$ $h(S^1)$

FIGURE 3.1

We give our proof in the next section.

REMARK 3. To show 1_* to be injective (Remark 1) we apply the approximation theorem with $A = M \times \partial I$, $I = [0, 1]$, to the regular homotopy $F : M \times I \to V$ in the C^1 topology.

From the above remarks we obtain the following

PROPOSITION 3.1. *Let f and g be immersions of M^n in V^p. Then f and g are regularly homotopic if and only if there exists a homotopy $\{f_t\}$ with $f_0 = f$ and $f_1 = g$ satisfying:*

(i) *For each $t \in [0, 1]$, $f_t : M^n \to V^p$ is an immersion.*
(ii) *Set $F_t = d(f_t) : T(M) \to T(M)$ and define a map $F : T(M) \times [0, 1] \to T(M)$ by $F(v, t) = F_t(v)$. Then F is continuous.*

EXAMPLE. Look at the three immersions $f, g, h : S^1 \to \mathbf{R}^2$ in Figure 3.1. While f and h are regularly homotopic, f and g are not.

PROPOSITION 3.2. *Let f and g be embeddings of M^n in V^p. If f and g are isotopic, they are regularly homotopic.*

The proof is left to the reader.

The basic problem in the immersion theory is the following: Given C^∞ manifolds M and V, classify immersions of M in V by regular homotopies. A special case of this problem is the question of the immersibility of M in V; in Chapter II we stated the Smale-Hirsch theorem which gives a complete answer to this case in the language of the homotopy theory. In §3 we discuss the Smale-Hirsch theorem.

§2. Spaces of maps: The approximation theorem

Here we put together facts about spaces of maps and reformulate the approximation theorem.

Let M and N be C^r manifolds, and let $C^r(M, N)$ be the set of C^r maps from M to N. In part D of §3 in Chapter I, we defined the C^r topology in $C^r(M, N)$, which is also called the *weak C^r topology* or the *compact-open C^r topology*. We denote by $C^r_W(M, N)$ the space $C^r(M, N)$ with the weak topology.

The space $C^r_W(M, N)$ is a complete metric space and satisfies the second axiom of countability. If M is compact, $C^r_W(M, N)$ is locally contractible; in particular, $C^r_W(M, \mathbf{R}^m)$ becomes a Banach space.

When M is not compact, the weak topology does not regulate the behavior of maps "at a distance". For this reason a stronger topology is sometimes desired. We define a strong topology below which is also called the *Whitney topology* or the *fine topology*. Let $\Phi = \{(U_i, \varphi_i) | i \in \Lambda\}$ be a locally finite C^r atlas of M; that is, for each point x in M there exists a neighborhood $U(x)$ such that only finitely many U_i intersect $U(x)$. Let $K = \{K_i | i \in \Lambda\}$ be a family of compact sets K_i in M with $K_i \subset U_i$. Let $\Psi = \{(V_i, \psi_i) | i \in \Lambda\}$ be a C^r atlas of N, and let $\varepsilon = \{\varepsilon_i | i \in \Lambda\}$ be a set of positive numbers. For $f \in C^r(M, N)$ with $f(K_i) \subset V_i$, set

$$\mathscr{N}^r(f; \Phi, \Psi, K, \varepsilon) = \{g \in C^r(M, N) | \text{(i), (ii)}\},$$

(i) $g(K_i) \subset V_i$, $i \in \Lambda$,
(ii) $\|D^k(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^k(\psi_i \circ g \circ \varphi_i^{-1})(x)\| < \varepsilon_i$, $\forall x \in \varphi_i(K_i)$, $k = 0, 1, \ldots, r$.

For a basis of the *strong topology* in $C^r(M, N)$ we take the family of sets of the form $\mathscr{N}^r(f; \Phi, \Psi, K, \varepsilon)$ (moving f, Φ, Ψ, K and ε). Denote by $C^r_S(M, N)$ the space $C^r(M, N)$ with the strong topology.

If M is compact, the spaces $C^r_S(M, N)$ and $C^r_W(M, N)$ are identical. When M is not compact and $\dim N > 0$, $C^r_S(M, N)$ does not satisfy the second axiom of countability.

These two topologies are induced on the space $C^\infty(M, N)$ of smooth maps by the inclusion maps $C^\infty(M, N) \to C^r_W(M, N)$ and $C^\infty(M, N) \to C^r_S(M, N)$.

The strong topology has the advantage that important subspaces in differential topology are often open with respect to this topology.

THEOREM 3.2. *The set* $\mathrm{Imm}^r(M, N)$ *of C^r immersions of M in N is open in $C^r_S(M, N)$, $r \geq 1$.*

PROOF. Since $\mathrm{Imm}^r(M, N) = \mathrm{Imm}^1(M, N) \cap C^r(M, N)$, it suffices to show the case $r = 1$. If $f : M \to N$ is a C^1 immersion, we choose a neighborhood $N^1(f; \Phi, \Psi, K, \varepsilon)$ of f as follows. Let $\Psi^0 = \{(V_\beta, \psi_\beta) | \beta \in B\}$ be a C^r atlas of N. Choose a C^r atlas $\Phi = \{(U_i, \varphi_i) | i \in \Lambda\}$ of M satisfying

(i) $\overline{U}_i$ is compact.
(ii) For each $i \in N$ there exists $\beta(i) \in B$ such that $f(U_i) \subset V_{\beta(i)}$.

Set $V_i = V_{\beta(i)}$, $\psi_i = \psi_{\beta(i)}$, $\Psi = \{(V_i, \psi_i) | i \in \lambda\}$. Consider a compact covering $K = \{K_i | i \in \Lambda\}$ of M, with $K_i \subset U_i$.

Then for each i in Λ, the set

$$A_i = \{D(\psi_i \circ f \circ \varphi_i^{-1})(x) | x \in \varphi_i(K_i)\}$$

is compact in the space of the one-to-one linear maps from $\mathbf{R}^m$ to $\mathbf{R}^n$. But the set of one-to-one linear maps are open in the space $L(\mathbf{R}^m, \mathbf{R}^n)$ of linear maps from $\mathbf{R}^m$ to $\mathbf{R}^n$. So there exists $\varepsilon_i > 0$ such that $T \in L(\mathbf{R}^m, \mathbf{R}^n)$ is one-to-one if $\|T - S\| < \varepsilon_i$ for $S \in A_i$. Set $\varepsilon = \{\varepsilon_i | i \in \Lambda\}$. Then each element of $N^1(f; \Phi, \Psi, K, \varepsilon)$ is a C^1 immersion. □

A similar proof applies to the following

THEOREM 3.3. *The set* $\mathrm{Emb}^r(M, N)$ *of* C^r *embeddings of* M *in* N *is open in* $C^r_S(M, N)$, $r \geq 1$.

Now we proceed to the following approximation

THEOREM 3.4. *Let* $U \subset \mathbf{R}^m$ *and* $V \subset \mathbf{R}^n$ *be open. Then* $C^\infty(U, V)$ *is dense in* $C^r_S(U, V)$, $0 \leq r < \infty$.

PROOF. $C^r_S(U, V)$ is open in $C^r_S(U, \mathbf{R}^n)$, so we need only to prove the theorem for $V = \mathbf{R}^n$.

Let $f \in C^r(U, \mathbf{R}^n)$. For a neighborhood basis of f in $C^r_S(U, \mathbf{R}^n)$, we may choose the following sets

$$N(f, K, \varepsilon) = \left\{ g : U \to \mathbf{R}^n,\ C^r \text{ map} \,|\, \|g - f\|_{r, K_i} < \varepsilon_i, \forall i \in \Lambda \right\},$$

where $K = \{K_i | i \in \lambda\}$ is a locally finite family of compact sets covering U, $\varepsilon = \{\varepsilon_i | i \in \Lambda\}$ is a family of numbers. We must show that with the fixed f, K and ε,

$$C^\infty(U, \mathbf{R}^n) \cap N(K, f, \varepsilon) \neq \varnothing.$$

Let $\{\lambda_i | i \in \Lambda\}$ be a partition of unity of class C^∞ on U such that $\mathrm{Supp}(\lambda_i)$ is compact and $K_i \subset \mathrm{Supp}(\lambda_i)$.

Given a family $\{\alpha_i | i \in \Lambda\}$ of positive numbers, there exist C^∞ maps $g_i : U_i \to \mathbf{R}$ such that

$$\|g_i - f\|_{r, K_i} < \alpha_i.$$

Define $g : U \longrightarrow \mathbf{R}$ by $g(x) = \sum_i \lambda_i(x) g_i(x)$. Then g is of class C^∞. To evaluate $\|D^k g(x) - D^k f(x)\|$ we notcice that if $\lambda : U \to \mathbf{R}$ and $\varphi : U \to \mathbf{R}$ are C^k functions and if we define a map ψ by $\psi(x) = \lambda(x)\varphi(x)$, then $D^k\psi(x)$ is a linear function of $D^p\lambda(x)$ and $D^q\varphi(x)$, $p, q = 0, 1, \ldots, k$, which does not depend on x, λ, or φ. Hence for some constant A_k we have

$$\|D^k(\lambda\varphi)(x)\| \leq A_k \max_{0 \leq p \leq k} \|D^p\lambda(x)\| \cdot \max_{0 \leq q \leq k} \|D^q\varphi(x)\|.$$

Write

$$A = \max\{A_0, \ldots, A_r\},$$

and for each i in Λ set

$$\Lambda_i = \{j \in \Lambda | K_i \cap K_j \neq \varnothing\}.$$

Then Λ_i is a finite set, say Λ_i has m_i elements. Setting

$$\mu_i = \max\{\|\lambda_j\|_{r, K_i} | j \in \Lambda_i\},$$
$$\beta_i = \max\{\alpha_j | j \in \Lambda_i\},$$

we obtain the following inequalities for $x \in K_i$, $0 \leq k \leq r$:

$$\begin{aligned} \|D^k g(x) - D^k f(x)\| &= \left\| \sum_{j \in \Lambda_i} D^k(\lambda_j g_j - \lambda_j f)(x) \right\| \\ &\leq \sum_{j \in \Lambda_i} \|D^k(\lambda_j(g_j - f))\| \\ &\leq m_i A \mu_i \beta_i. \end{aligned}$$

Evidently we may choose α_i so that

$$m_i A \mu_i \beta_i < \varepsilon_i.$$

With such a choice of α_i we have

$$\|g - f\|_{r, K_i} < \varepsilon_i,$$

for each i in Λ_i. □

From the above theorem it is not difficult to derive the following approximation

THEOREM 3.5. *let* M *and* N *be* C^s *manifolds,* $1 \leq s \leq \infty$. *Then* $C^s(M, N)$ *is dense in* $C^r_S(M, N)$, $0 \leq r < s$.

§3. Characteristic classes

This section is based on Wu [C23]. We discuss characteristic classes as a preparation for the further developement of the text (see Milnor and Stasheff [A5] for details).

A. Cell divisions of Grassmann manifolds. Let $\mathbf{R}^{m+n}$ be $n+m$-dimensional Euclidean space. Let $\mathbf{R}_{n,m}$ denote the set of m-dimensional subspaces of $\mathbf{R}^{m+n}$ and let $\widehat{\mathbf{R}}_{n,m}$ denote the set of m-dimensional oriented subspaces of the same. A similar definition holds for $\mathbf{C}_{n,m}$ when we take $\mathbf{C}$ instead of $\mathbf{R}$ for scalars.

DEFINITION 3.2. We call $\mathbf{R}_{n,m}$, $\widehat{\mathbf{R}}_{n,m}$, and $\mathbf{C}_{n,m}$ *Grassmann manifolds.*

We topologize these sets as follows. We will do only $\mathbf{R}_{n,m}$, but the rest are similar.

Let $V_{n+m,m}$ be the Stiefel manifold of orthonormal m-frames in $\mathbf{R}^{n+m}$. It is well known that

$$V_{n+m,m} \approx \mathrm{O}(n+m)/\mathrm{O}(n).$$

We have a surjection

$$\pi : V_{n+m,m} \longrightarrow \mathbf{R}_{n,m},$$

which sends an element $\{\wp_1, \dots, \wp_m\}$ of $V_{n+m,m}$ to the subspace $\{\{\wp_1, \dots, \wp_m\}\}$ of $\mathbf{R}^{n+m}$ spanned by $\{\wp_1, \dots, \wp_m\}$. We give $R_{n,m}$ the quotient topology of $V_{n+m,m}$ by π. Then it is easy to see that

$$\mathbf{R}_{n,m} \approx \mathrm{O}(n+m)/\mathrm{O}(n) \times \mathrm{O}'(m),$$

where $\mathrm{O}'(m)$ is a subgroup of $\mathrm{O}(n+m)$ of the form

$$\mathrm{O}' = \left\{ \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & A \end{array} \right) \in \mathrm{O}(n-m) \,\middle|\, A \in \mathrm{O}(m) \right\}.$$

Definition 3.3. For natural numbers m and n, a map

$$\omega : \{1, 2, \dots, m\} \longrightarrow \{0, 1, 2, \dots, n\},$$

which is monotone increasing in weak sense,

$$0 \le \omega(1) \le \omega(2) \le \cdots \le \omega(m),$$

is called an $(m, n; \omega)$-function or a *Schubert function* of type (m, n). The set of $(m, n; \omega)$-functions is denoted by $\Omega(m, n)$.

Definition 3.4. Define the *dimension* $d(\omega)$ of an element ω in $\Omega(m, n)$ by

$$d(\omega) = \sum_{i=1}^{m} \omega(i).$$

Special attention should be paid to the following notation for Schubert functions. We often write $\omega = (\omega(1), \omega(2), \dots, \omega(m))$ for an element ω in $\Omega(m, n)$. We also write

$$\omega_k^m = (0, 0, \dots, 0, \underbrace{1, \dots, 1}_{k}), \qquad 0 \le k \le m,$$

$$\bar{\omega}_k^m = (0, 0, \dots, 0, k), \qquad 0 \le k \le n,$$

$$\omega_{2k,2k}^m = (0, \dots, 0, \underbrace{2, \dots, 2}_{2k}), \qquad 0 \le 2k \le m,$$

$$\bar{\omega}_{2k,2k}^m = (0, \dots, 0, 2k, 2k), \qquad 0 \le 2k \le n.$$

Henceforth, we put $\omega(0) = 0$, $\omega(m+1) = n$ for convenience.

Definition 3.5. Let $\omega \in \Omega(m, n)$. We say that the number i is a *jump point* if $\omega(i) < \omega(i+1)$.

Given an element ω of $\Omega(m, n)$ we define the following new functions:

$\omega_{(i)} = (\omega(1), \dots, \omega(i-1), \omega(i) - 1, \omega(i+1), \dots, \omega(m))$,
(here $i-1$ is a jump point of ω),

$\omega^{(i)} = (\omega(1), \dots, \omega(i-1), \omega(i) + 1, \omega(i+1), \dots, \omega(m))$,
(here i is a jump point of ω),

$\omega^* = (n - \omega(m), n - \omega(m-1), \dots, n - \omega(m-i+1), \dots, n - \omega(1))$,

that is, $\omega^*(i) = n - \omega(m-i+1)$.

Now we are ready to do cell division of the Grassmann manifold $\widehat{\mathbf{R}}_{n,m}$. Let $\mathbf{R}^k$ be the subspace of $\mathbf{R}^{n+m} = \{(x_1, \dots, x_{n+m}) \mid x_i \in \mathbf{R}\}$,

$$\mathbf{R}^k = \{x_{k+1} = \cdots = x_{n+m} = 0\}.$$

Definition 3.6. For an element ω of $\Omega(n, m)$, set

$$\overline{U}_\omega = \{X \in \mathbf{R}_{n,m} \mid \dim(X \cap \mathbf{R}^{\omega(i)+i}) \ge i,\ i = 1, 2, \dots, m\}.$$

We say that $\overline{U}_\omega$ is the *Schubert variety* of ω.

The interior U_ω of $\overline{U}_\omega$ is a $d(\omega)$-cell. For $\omega \in \Omega(n, m)$ set

$$\{\bar{i}(\omega)\} = \{\omega(1) + 1, \omega(2) + 2, \dots, \omega(m) + m\},$$

that is, $\bar{i}(\omega) = \omega(i) + i$. Put

$$X_\omega = \{\{e_{\bar{1}(\omega)}, e_{\bar{2}(\omega)}, \dots, e_{\bar{m}(\omega)}\}\}. \tag{1}$$

Here $\{\{\cdots\}\}$ is the subspace of $\mathbf{R}^{n+m}$ spanned by $\{\cdots\}$. Further set

$$\{\tilde{j}(\omega)\} = \{1, \dots, \omega(1), \omega(1) + 2, \dots, \omega(1) + m, \omega(2) + 1, \omega(2) + 3, \dots, \omega(m) + m + 1, \dots, n + m\},$$

that is,

$$\{\tilde{j}(\omega)\} = \{1, 2, \dots, m + n\} - \{\bar{i}(\omega)\} \quad \text{(in this order)}.$$

Then X_ω is given by the equation

$$x\tilde{j}(\omega) = 0, \qquad j = 1, \dots, n.$$

We denote by $\overset{+}{X}_\omega$ the space X_ω oriented with the basis ordered as in (1) and by $\overset{-}{X}_\omega$ the X_ω with the opposite orientation.

Let $\overset{+}{N}_\omega$ be the set of all elements of $\widehat{\mathbf{R}}_{n,m}$ whose normal projections to $\overset{+}{X}_\omega$ are nondegenerate and orientation preserving. The definition of $\overset{-}{N}_\omega$ is similar. Then we have $d(\omega)$-dimensional open cells

$$\overset{+}{U}_\omega = \overset{+}{N}_\omega \cap \overline{U}_\omega, \qquad \overset{-}{U}_\omega = \overset{-}{N}_\omega \cap \overline{U}_\omega.$$

Further, we have

$$(\overset{+}{U}_\omega \cup \overset{-}{U}_\omega)^a = \overline{U}_\omega.$$

Without orientation we can characterize $\overset{+}{N}_\omega$ or $\overset{-}{N}_\omega$ by

$$x_{\bar{j}} = \sum_{i=1}^{m} \xi_{ji} x_{\bar{i}}, \qquad j = 1, \dots, n,$$
$$\bar{j} = \bar{j}(\omega), \qquad \bar{i} = \bar{i}(\omega), \tag{2}$$

and we have

$$\xi_{ji} = 0, \qquad j > \omega(i), \tag{3}$$

for subsets of $\overset{\pm}{N}_\omega$ contained in $\overline{U}_\omega$. The above fact follows from (2) and (3).

THEOREM 3.6 (Pontrjagin). *The set* $\{\overset{+}{U}_\omega, \overset{-}{U}_\omega | \omega \in \Omega(n, m)\}$ *gives a cell division of* $\widehat{\mathbf{R}}_{n,m}$. □

We denote this cell division by $\widehat{K}_{(x)}$.

Just as avove we obtain a cell division of $\mathbf{R}_{n,m}$ by letting N_ω be the set of all elements of $\mathbf{R}_{n,m}$ whose normal projection to X_ω is nondegenerate and setting

$$U_\omega = N_\omega \cap \overline{U}_\omega;$$

we see that U_ω is an open $d(\omega)$-cell.

THEOREM 3.6′ (Pontrjagin). *The set* $\{U_\omega | \omega \in \Omega(n, m)\}$ *gives a cell division for* $\mathbf{R}_{n,m}$.

We denote this cell division $K_{(x)}$.

Henceforth, for simplicity we only discuss $\mathbf{R}_{n,m}$, the case for $\mathbf{C}_{n,m}$ being similar. The case $\widehat{\mathbf{R}}_{n,m}$ is slightly more complicated with signs (for detail, see Wu [C23]).

PROPOSITION 3.3. *The coboundary* δU_ω *of the cell* U_ω *is given by*

$$\delta U_\omega = \sum \eta_{\omega,i}(1 + (-1)^{\omega(i)+i+m+1}) U_{\omega(i)},$$

where the sum runs through all i *with* $\omega(i) \in \Omega(n, m)$ *and* $\eta_{\omega,i}$ *is* $+1$ *or* -1 *depending on* ω *and* i.

Now for ω with $\delta U_\omega = 0$ we denote by $\{\omega\}$ the cohomology class of $K_{(x)}$ represented by U_ω.

If $\delta U_\omega \equiv 0(2)$ we denote the mod 2 cohomology class represented by U_ω $\{\omega\}_2$.

B. Characteristic classes. Here we define characteristic classes of vector bundles and of C^∞ manifolds. We shall first define characteristic classes of Grassmann manifolds $\mathbf{R}_{n,m}$ and $\widehat{\mathbf{R}}_{n,m}$.

DEFINITION 3.7. We call

$$W^k = \{\omega_k^m\}_2 \in H^k(\mathbf{R}_{n,m}, \mathbf{Z}_2) \quad \text{or} \quad \in H^k(\widehat{\mathbf{R}}_{n,m}, \mathbf{Z}_2),$$
$$\overline{W}^k = \{\bar{\omega}_k^m\}_2 \in H^k(\mathbf{R}_{n,m}, \mathbf{Z}_2) \quad \text{or} \quad \in H^k(\widehat{\mathbf{R}}_{n,m}, \mathbf{Z}_2),$$

the k th *Stiefel-Whitney class* and the k th *dual Stiefel-Whitney class* respectively.

We call $X^m = \{\omega_m^m\} \in H^m(\mathbf{R}_{n,m}, \mathbf{Z})$ or $H^m(\widehat{\mathbf{R}}_{n,m}, \mathbf{Z}_2)$ the *Euler-Poincaré class*. The cohomology classes

$$P^{4k} = \{\omega_{2k,2k}^m\} \in H^{4k}(\mathbf{R}_{n,m}, \mathbf{Z}) \quad \text{or} \quad \in H^{4k}(\widehat{\mathbf{R}}_{n,m}, \mathbf{Z}),$$
$$\overline{P}^{4k} = \{\bar{\omega}_{2k,2k}^m\} \in H^{4k}(\mathbf{R}_{n,m}, \mathbf{Z}) \quad \text{or} \quad \in H^{4k}(\widehat{\mathbf{R}}_{n,m}, \mathbf{Z})$$

are called the $4k$-th *Pontrjagin class* and $4k$-th dual *Pontrjagin class* respectively. Further $C^{2i} = \{\omega_i^m\} \in H^{2k}(\mathbf{C}_{n,m}, \mathbf{Z})$ is called the ith Chern class.

NOTE. The above definitions agree with their usual definitions such as given in Milnor and Stasheff [A5], except Pontrjagin classes are off by two components. For instance the Stiefel-Whitney classes defined here satisfy the axioms for the Stiefel Whitney classes in the book of Milnor and Stasheff and so they must be the same by the uniqueness axiom. The same goes for our definition of Chern classes, and so our Pontrjagin classes agree essentially with Milnor's Pontrjagin classes.

Now we are ready to define characteristic classes of vector bundles. Before doing so we need the following

PROPOSITION 3.4. *Let* K *be a* k*-dimensional locally finite complex, and let* $P = |K|$ *be its polyhedron. Recall the natural map defined in Chapter* I,

$$i_n : \mathbf{R}_{n,m} \longrightarrow B_{\mathrm{O}(m)} = \varinjlim_n \mathrm{O}(n+m)/\mathrm{O}(m) \times \mathrm{O}'(n),$$

which induces the map $(i_n)_* : [P, \mathbf{R}_{n,m}] \to [P, B_{\mathrm{O}(m)}]$. *If* $k < n$, *the map* $(i_n)_*$ *is a bijection.*

PROOF. This follows from the fact that the Stiefel manifold $\mathrm{O}(n+1)/\mathrm{O}(n)$ is homeomorphic to the n-sphere S^n. □

COROLLARY 3.1. *Let* K *be a locally finite* k*-complex, and let* $P = |K|$ *be its polyhedron. If* $k < n$, *there is a one-to-one correspondence between equivalence classes of* m*-vector bundles* ξ *over* P *and* $[P, \mathbf{R}_{n,m}]$.

The above correspondence is given as follows. Recalling that $\mathbf{R}_{n,m}$ consists of m dimensional subspaces of $\mathbf{R}^{n+m}$, we define $E_{n,m}$ by

$$E_{n,m} = \{(X, u) | X \in \mathbf{R}_{n,m}, u \in X\} \subset \mathbf{R}_{n,m} \times \mathbf{R}^{n+m},$$

and define $p : E_{n,m} \to \mathbf{R}_{n,m}$ by $p(X, u) = X$. We now have an m-vector bundle which we will denote by $\gamma_{n,m}$. We then assign to each $\{f\} \in [P, \mathbf{R}_{n,m}]$ the equivalence class of $f^*\gamma_{n,m}$.

Let $P = |K|$ where K is a locally finite k-complex, and let ξ be an m-dimensional vector bundle over P.

DEFINITION 3.8. By Corollary 3.1 the bundle ξ is induced by a map $f: P \to \mathbf{R}_{n,m}$ for a sufficiently large n, *i.e.*, $\xi \sim f^*\gamma_{n,m}$. Put

$$W^k(\xi) = f^*W^k \in H^k(P, \mathbf{Z}_2), \qquad \overline{W}^k(\xi) = f^*\overline{W}^k \in H^k(P, \mathbf{Z}_2),$$
$$X^m(\xi) = f^*X^m \in H^m(P, \mathbf{Z}), \qquad P^{4k}(\xi) = f^*P^{4k} \in H^{4k}(P, \mathbf{Z}).$$

We call $W^k(\xi)$ the *kth Stiefel Whitney class*, $\overline{W}^k(\xi)$ the *dual kth Stiefel-Whitney class*, $X^m(\xi)$ the *Euler-Poincaré class* and $P^{4k}(\xi)$ the *4kth Pontrjagin class* of ξ. They are *characteristic classes* of the bundle ξ.

It is easy to see that these definitions do not depend on the choice of n. If we consider for example the natural embedding

$$i_{n,n+1}: \mathbf{R}_{n,m} \longrightarrow \mathbf{R}_{n+1,m},$$

we have $(i_{n,n+1})^*\{\omega_k^m\}_2 = \{\omega_k^m\}_2$.

We next define characteristic classes of C^∞ manifolds.

DEFINITION 3.9. Let M be an m-dimensional C^∞ manifold. By a *characteristic class of* M we mean a characteristic class of the tangent bundle $\tau(M)$ of M:

$$W^k(M) = W^k(\tau(M)), \qquad \overline{W}^k(M) = \overline{W}^k(\tau(M)),$$
$$X^m(M) = X^m(\tau(M)), \qquad P^{4k}(M) = P^{4k}(\tau(M)).$$

These characteristic classes have the following properties.

PROPOSITION 3.5. *Let K be a locally finite k-complex, and let $P = |K|$. Let ξ and η be vector bundles over P.*

1. *If $\xi \sim \eta$, then*

$$W^k(\xi) = W^k(\eta), \qquad \overline{W}^k(\xi) = \overline{W}^k(\eta),$$
$$P^{4k}(\xi) = P^{4k}(\eta), \qquad X(\xi) = X(\eta).$$

2. *If ε is the trivial vector bundle, then*

$$W^i(\varepsilon) = 0, \qquad i > 0,$$
$$P^{4k}(\varepsilon) = 0, \qquad k > 0,$$
$$X(\varepsilon) = 0, \qquad \dim \varepsilon > 0.$$

3. *For an m-bundle ξ, we have*

$$W^i(\xi) = 0, \qquad i > m,$$
$$P^{4k}(\xi) = 0, \qquad 4k > m.$$

4. *For an m-vector bundle ξ, define the total Stiefel-Whitney class $W(\xi)$ by*

$$W(\xi) = 1 + W^1(\xi) + \cdots + W^m(\xi) \in H^*(P, \mathbf{Z}_2).$$

Then

$$W(\xi \oplus \eta) = W(\xi)W(\eta),$$

where $\xi \oplus \eta$ is the Whitney sum of ξ and η. This is called the Whitney duality theorem.

5. *If a k-frame field exists for ξ, that is, ξ admits k linearly independent cross sections at every point, then*

$$W^{m-k+1}(\xi) = W^{m-k+2}(\xi) = \cdots = W^m(\xi) = 0.$$

The proofs of 1–3 are immediate from the definitions. We omit the proofs of 4 and 5. See, for example, Milnor and Stasheff [A5].

Property 5 represents a geometric significance of the Stiefel-Whitney class. We may think of characteristic classes as algebraic expressions for the geometric properties of C^∞ manifolds.

§4. Immersions and characteristic classes

In this section we give a necessary condition for the immersibility of an m-dimensional C^∞ manifold M^m in Euclidean $(m+k)$-space $\mathbf{R}^{m+k}$ in terms of characteristic classes.

THEOREM 3.7. *If we can immerse an m-dimensional C^∞ manifold M^m in $\mathbf{R}^{m+k}$, then we have*

$$\overline{W}^i(M^m) = 0, \qquad i > k,$$
$$\overline{P}^{4j}(M^m) = 0, \qquad 2j > k.$$

PROOF. Suppose that M^m is immersed in $\mathbf{R}^{m+k}$. Let $f: M \to \mathrm{BO}(m)$ denote the classifying map for the tangent bundle $\tau(M^m)$ of M^m. Here $\mathrm{BO}(m)$ is the classifying space for the orthogonal group $\mathrm{O}(m)$:

$$\mathrm{BO}(m) = \varinjlim \mathrm{O}(m+n)/\mathrm{O}(m) \times \mathrm{O}(n),$$

(cf. §5, Chapter I). Consider $\mathbf{R}_{k,m}$, the Grassmann manifold of m-dimensional subspaces in $\mathbf{R}^{m+k}$. We may think of $\mathbf{R}_{k,m}$ as

$$\mathbf{R}_{k,m} = \mathrm{O}(m+k)/\mathrm{O}(m) \times \mathrm{O}(k).$$

Let ι_k be the natural map from $\mathbf{R}_{k,m}$ to $\mathrm{BO}(m)$. Let $g: M^m \to \mathbf{R}_{k,m}$ be the map which assigns to each point x of M^m the tangent space $T_x(M^m)$ to M^m at x. Then we get the following homotopy commutative diagram (cf. Proposition 3.4):

$$\begin{array}{ccc} M^m & \xrightarrow{f} & \mathrm{BO}(m) \\ & g \searrow \quad \nearrow \iota_k & \\ & \mathbf{R}_{k,m} & \end{array}$$

and we have

$$\overline{W}^i(M^m) = f^*\{\bar{\omega}_i^m\}_2, \qquad \overline{P}^{4j}(M^m) = f^*\{\bar{\omega}_{2j,2j}^m\}_0,$$

(see the previous section concerning $\{\omega_i^m\}_2$, $\{\omega_{2j,2j}^m\}_0$). But for $\mathbf{R}_{k,m}$ we have

$$\{\bar{\omega}_i^m\}_2 = 0, \qquad i > k,$$
$$\{\bar{\omega}_{2j,2j}^m\}_0 = 0, \qquad 2j > k.$$

Hence, the theorem follows. □

COROLLARY 3.2. *Let* $\mathbf{R}P^n$ *be the* n*-dimensional real projective space. We cannot immerse* $\mathbf{R}P^n$ *in* $\mathbf{R}^{2n-2}$ *if* $n = 2^s$.

PROOF. Suppose we immersed $\mathbf{R}P^n$ in $\mathbf{R}^{2n-2}$. Then by Theorem 3.7 $\overline{W}^i(\mathbf{R}P^n) = 0$, $i > n-2$. But we know that the Stiefel Whitney classes of $\mathbf{R}P^n$ are as follows:

$$H^*(\mathbf{R}P^n, \mathbf{Z}_2) = \mathbf{Z}_2[x], \qquad \deg x = 1, \ x^{n+1} = 0,$$
$$W(\mathbf{R}P^n) = (1+x)^{n+1}$$

(see for example Milnor and Stasheff [A5]). So

$$\overline{W}(\mathbf{R}P^n) = (1+x)^{-n-1}.$$

Since $n = 2^s$ we get $\overline{W}^{n-1}(\mathbf{R}P^n) \neq 0$ which is a contradiction. □

The corollary is due to R. Thom and was incorporated in his thesis of 1952 (cf. [C18]).

On the other hand, H. Whitney whose research in singularities of C^∞ maps dates back to 1930, obtained the following

THEOREM 3.8. *Let* $n \geq 2$. *Then we can immerse an* n*-dimensional* C^∞ *manifold* M^n *in* $\mathbf{R}^{2n-1}$.

The proof will be given in the next section.

Combined with Corollary 3.2, Whitney's theorem offers the best result in a general setting.

§5. The Smale-Hirsch theorem and its applications

In this section we state the Smale-Hirsch theorem and its applications. This is the most fundamental result in the theory of immersions. The proof will be given in §9.

DEFINITION 3.10. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map satisfying the following:

(i) f induces a one-to-one correspondence between the arcwise-connected components of X and the arcwise-connected components of Y.

(ii) For each x_0 of X, the induced homomorphism

$$f_* : \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$$

is an isomorphism for all $i \geq 1$.

We then say that f is a *weak homotopy equivalence*—to be abbreviated by w.h.e.

Let M and V be C^∞ manifolds of dimension n and p respectively. Let $\mathrm{Imm}(M, N)$ be the space of immersions of M in V with the C^∞ topology. Let $\mathrm{Mon}(T(M), T(V))$ be the space of monomorphisms of the tangent bundle $T(M)$ of M into the tangent bundle $T(V)$ of V with the compact-open topology. Here by a *monomorphism of vector bundles* we mean a bundle map whose restriction to each fiber $T_x(M)$ is a vector space monomorphism (see §2, Chapter 1).

THEOREM 3.9. *Let* M *and* V *be as above with* $n < p$. *Then the map*

$$d : \mathrm{Imm}(M, V) \longrightarrow \mathrm{Mon}(T(M), T(V)),$$
$$f \to df$$

sending a C^∞ *map* f *to its differential* df *is a weak homotopy equivalence.*

This theorem which we will refer as the *Smale-Hirsch Theorem* was first proved by S. Smale in 1959 for the case $M = S^n$, $V = \mathbf{R}^p$, and later in 1960 by M. Hirsch for the general case. Starting with Smale's work, Hirsch proved the theorem by induction on each simplex over each skeleton using C^∞ triangulations (to be explained in the next section) of C^∞ manifolds.

We save the proof for §9. For now we discuss some applications and corollaries of the theorem.

DEFINITION 3.11. Let M and V be C^∞ manifolds of dimensions n and p, $n < p$. Let $f : M \to V$ be an immersion. By a *normal r-frame field* of f or $f(M)$ we mean a cross section of the bundle associated with the normal bundle of $f(M)$ in V, whose fiber is the Stiefel manifold $V_{p-n,r}$.

THEOREM 3.10. *Let* $n < p$.

(i) *If we can immerse* M^n *in* $\mathbf{R}^{p+r}$ *with a normal r-frame field, then we can immerse* M^n *in* $\mathbf{R}^p$.

(ii) *Conversely, if we can immerse* M^n *in* $\mathbf{R}^p$ *then we can immerse* M^n *in* $\mathbf{R}^{p+r}$ *with a normal r-frame field.*

PROOF. (ii) is clear.

(i) Suppose that $f : M^n \to \mathbf{R}^{p+r}$ is an immersion of M^n in $\mathbf{R}^{p+r}$ which admits a normal r-frame bundle. Then we have

$$f^*T(\mathbf{R}^{p+r}) \sim T(M^n) \oplus f^*\nu^{p+r-n},$$

where ν^{p+r-n} is the normal bundle of $f(M^n)$ in $\mathbf{R}^{p+r}$. But the existence of a normal r-frame field implies that

$$\nu^{p+r-n} \sim \xi^{p-n} \oplus \varepsilon^r_{f(M)},$$

where $\varepsilon^r_{f(M)}$ denotes the r-dimensional trivial vector bundle over $f(M)$. On the other hand, since $n < p$ the natural map

$$[M^n, \mathrm{BO}(p)] \longrightarrow [M^n, \mathrm{BO}(p+r)]$$

is a bijection. Hence, we have

$$T(M^n) \oplus f^*\xi^{p-r} \sim \varepsilon^p_M.$$

Thus, $\mathrm{Mon}(T(M), T(\mathbf{R}^p))$ is not empty. Hence, the Smale-Hirsch theorem implies that $\mathrm{Imm}(M^n, \mathbf{R}^p)$ is not empty. □

From these theorems we obtain various solid examples.

DEFINITION 3.12. We say that M^n is *parallelizable* if its tangent bundle $T(M^n)$ is trivial.

COROLLARY 3.3. *If M^n is parallelizable, it can be immersed in* $\mathbf{R}^{n+1}$.

This is obvious from Theorem 3.10.

COROLLARY 3.4. *We can immerse a three dimensional closed manifold in Euclidean four space* $\mathbf{R}^4$.

PROOF. Let M^3 be a closed three manifold. By Whitney's embedding theorem, we can imbed M^3 in $\mathbf{R}^6$. It is enough to show that this embedding has a normal two-frame field. But the obstructions to the existence of a normal two-frame field lie in $H^i(M^3, \pi_{i-1}(V_{3,2}))$. The first obstruction is the dual Stiefel-Whitney class $\overline{W}^2(M^3)$, which is known to be zero (see Remark 1). The second obstruction is also zero since $\pi_2(V_{3,2}) = 0$ (see Remark 2). Thus, there is a normal two-frame field. □

REMARK 1. The two dimensional dual Stiefel Whitney class $\overline{W}^2(M^3)$ of a three dimensional closed manifold is zero; we can show this as follows. Imbed M^3 in $\mathbf{R}^6$ with the normal bundle ν. Then

$$\tau(M^3) \oplus \nu \sim \varepsilon^6.$$

By the Whitney duality theorem we have $W(\tau(M^3))W(\nu) = W(\varepsilon^6) = 1$. But

$$W^i(\nu) = \overline{W}^i(M^3).$$

Therefore,

$$\overline{W}^2(M^3) = W^2(M^3) + (W^1(M^3))^2.$$

But $(W^1)^2 = W^2$ holds for any three-manifold. Hence, we get $\overline{W}^2(M^3) = 0$.

REMARK 2. $V_{3,2} \approx SO(3) \approx \mathbf{R}P^3$ implies $\pi_2(V_{3,2}) = 0$.

REMARK 3. A three-dimensional open manifold can be immersed in $\mathbf{R}^4$ [1]. This fact follows easily from a theorem of Phillips which we shall state in §8.

DEFINITION 3.13. A manifold M is *open* if each of its connected component is noncompact.

COROLLARY 3.5. *Let M^n be an n-dimensional closed manifold. Assume that $n \equiv 1(4)$. Then we can immerse M^n in* $\mathbf{R}^{2n-2}$.

[1] J. H. C. Whitehead, *The immersion of an open 3-manifold in Euclidean 3-space*, Proc. London Math Soc., **11** (1960), 81-90.

PROOF. Immerse M^n in $\mathbf{R}^{2n}$ and consider constructing a normal two-frame field. The first obstruction to do so is $\overline{W}^{n-1}(M^n)$, which turns out to be zero by Remark 1. We also have $\pi_{n-1}(V_{n,2}) = 0$ for $n \equiv 1(4)$ (see Remark below). Hence, the second obstruction is also zero, and so we can construct a normal two-frame field. □

REMARK. We see this by considering the homotopy exact sequence of the bundle $p : \mathrm{O}(n)/\mathrm{O}(n-2) \to \mathrm{O}(n)/\mathrm{O}(n-1) = S^{n-1}$.

Now let us prove the theorem of H. Whitney which was stated in the previous section.

PROOF OF THEOREM 3.8. (1) The case where M^n is open. By the Smale-Hirsch theorem it is enough to show that $\mathrm{Mon}(T(M^n), T(\mathbf{R}^{2n-1})) \neq \varnothing$. To do this we only need to show that the associated bundle of $T(M^n)$, whose fiber is $V_{2n-1,n}$ admits a cross section (see the remark below). Now the obstructions to do so sit in

$$H^i(M^n, \pi_{i-1}(V_{2n-1,n})).$$

But $\pi_{i-1}(V_{2n-1,n}) = 0$, $i < n$. Hence, we obtain the desired cross section.

(2) The case where M^n is compact and n is odd. As before we can immerse M^n in $\mathbf{R}^{2n}$. Then the normal Stiefel-Whitney class $\overline{W}^n(M^n)$ of M^n is zero. Hence, by Theorem 3.10 we can immerse M^n in $\mathbf{R}^{2n-1}$.

(3) The case where M^n is compact and n is even. In this case also we can immerse M^n in $\mathbf{R}^{2n}$ so that the normal Stiefel-Whitney class is zero (see Theorem 8.2, Hirsch [C12]). Thus, we can immerse M^n in $\mathbf{R}^{2n-1}$. □

REMARK. $\mathrm{O}(n)$ acts on $V_{2n-1,n}$ as follows. For $g = (a_{ij}) \in \mathrm{O}(n)$ and $\{X_1, \dots, X_n\} \in V_{2n-1,n}$,

$$g \cdot \{X_1, \dots, X_n\} = \{Y_1, \dots, Y_n\},$$

$$Y_i = \sum_{j=1}^{n} a_{ji} X_j, \qquad i = 1, \dots, n.$$

ADDENDUM. A. Phillips in his paper "Turning a sphere inside out" turns S^2 inside out in $\mathbf{R}^3$ through smooth deformations allowing self-intersections (see A. Phillips [C14]).

This fact was also made into a movie[2]. The article of Phillips together with the movie give a concrete demonstration for one particular case of the statement "immersions of S^2 in $\mathbf{R}^3$ are all regularly homotopic". The quotation follows from the Smale-Hirsch theorem and the equalities

$$\pi_2(V_{3,2}) = \pi_2(\mathrm{SO}(3)) = 0.$$

§6. C^r triangulations of a C^r manifold

In this section we discuss C^r triangulations of C^r manifolds, a tool with which Hirsch generalized Smale's theorem on immersions of the sphere.

[2] N. L. Max, *Turning a sphere inside out*, International Film Bureau Inc., Chicago.

DEFINITION 3.14. Let K be a locally finite simplicial complex, and let $|K|$ be the polyhedron of K. Let X be a topological space. We say that (K, f) is a *triangulation* of X if $f: |K| \to X$ is a homeomorphism.

Let (K, f), (K_1, f_1) be triangulations of X. We say that (K_1, f_1) is a *subdivision* of (K, f) if the map

$$f^{-1} \circ f_1 : |K_1| \longrightarrow |K|$$

sends each simplex of $|K_1|$ linearly into a simplex of $|K|$. We may also say that the *triangulated space* $(X; K_1, f_1)$ is a *subdivision* of the triangulated space $(X; K, f)$.

DEFINITION 3.15. If simplicial complexes K_1 and K_2 are isomorphic, we say that the triangulated spaces $(X_1; K_1, f_1)$ and $(X_2; K_2, f_2)$ are *isomorphic*, and write $(X_1; K_1, f_1) \cong (X_2; K_2, f_2)$ (cf. Spanier([3])).

We say $(X_1; K_1, f_1)$ and $(X_2; K_2, f_2)$ are *combinatorially equivalent* when each has a subdivision isomorphic to a subdivision of the other.

DEFINITION 3.16. A triangulated space $(X; K, f)$ is called a *combinatorial n-cell* if it is combinatorially equivalent to an n-simplex.

DEFINITION 3.17. A triangulated space $(X; K, f)$ is an n-dimensional *combinatorial manifold* if the star at each vertex of K is a combinatorial n-cell.

We are now ready to define C^r triangulations.

DEFINITION 3.18. Let M^n be an n-dimensional C^r manifold and let (K, f) be a triangulation of M^n. We say that (K, f) is a C^r *triangulation* of M^n if for each n-simplex σ, $f|\sigma : \sigma \to M^n$ is a C^r map (this means that with K considered as a simplicial complex in Euclidian space $\mathbf{R}^N$ of a high enough dimension we can extend f to a C^r map in some neighborhood $U(\sigma)$ of σ in $\mathbf{R}^N$), and the degree of f at each point of K is n. If (K, f) is a C^r triangulation of $M^n = (M^n, \mathscr{D})$, the C^r structure $\mathscr{D}$ is *compatible* with the tringulation (K, f).

In 1940 J. H. C. Whitehead proved the following([4])

THEOREM 3.11. *Assume* $1 \le r \le \infty$. *Then*:

(i) *A closed* C^r *manifold admits a* C^r *triangulation* (K, f).

(ii) *If* (K, f) *is a* C^r *triangulation of a* C^r *manifold* M^n, *the triangulated space* $(M^n; K, f)$ *is a combinatorial n-manifold.*

(iii) *Let* (K_1, f_1) *and* (K_2, f_2) *be triangulations of a* C^r *manifold* M^n. *Then the triangulated manifolds* $(M^n; K_1, f_1)$ *and* $(M^n; K_2, f_2)$ *are combinatorially equivalent.*

We omit the proof([5]).

([3])E. H. Spanier, *Algebraic topology*, MacGraw-Hill, New York, 1966.

([4])J. H. C. Whitehead, *On* C^1*-complexes*, Ann. of Math. **41** (1940), 809–824.

([5])See for instance Munkres, *Elementary differential topology*, Ann. of Math. Stud., Princeton Univ. Press, revised ed., Princeton, NJ, 1966.

§7. Gromov's theorem

The thesis of Gromov,(*) 1969, established a generalization encompassing the Smale-Hirsch theorem and the Phillips submersion theorem. The idea of the proof is essentially similar to the method Poénaru exploited in his proof of the Smale-Hirsch theorem([6]), that is, the idea of handlebody decompositions.

In this section we first state Gromov's theorem. In §8, we shall cite some corollaries of Gromov's theorem: the Phillips theorem, the Gromov-Phillips theorem, and others. In §9 we prove Gromov's theorem.

Let M be a C^∞ manifold. We define a *local diffeomorphism* of M to be a diffeomorphism $f: U \to V$, where U and V are open subsets of M. We may compose two local diffeormorphisms $f: U \to W$ and $g: W \to T$ only when $f(U) \subset W$. The set $\mathscr{D}(M)$ of local diffeormophisms of M is a pseudogroup, which we shall call the *pseudogroup of local diffeomorphisms* of M (see Chapter IV for the definition of pseudogroups).

DEFINITION 3.19. Let (E, p, M) be a smooth fiber bundle. We say that a map $\Phi: \mathscr{D}(M) \to \mathscr{D}(E)$ is an *extension* of $\mathscr{D}(M)$ to $\mathscr{D}(E)$ if it satisfies:

(1) For $f \in \mathscr{D}(M)$, $f: U \to V$, the map $\Phi(f): p^{-1}(U) \to p^{-1}(V)$ is a diffeomorphism and the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi(f)} & p^{-1}(V) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[f]{} & V \end{array}$$

(2) $\Phi(1_U) = 1_{p^{-1}(U)}$.

(3) $\Phi(f \circ g) = \Phi(f)\Phi(g)$, whenever $f \circ g$ is defined.

EXAMPLE 1. For $(E, p, M) = (T(M), p, M)$, $\Phi: \mathscr{D}(M) \to \mathscr{D}(T(M))$ defined by $\Phi(f) = df$ is an extension.

EXAMPLE 2. For a smooth fiber bundle (E, p, M), let (E^r, p^r, M) be the r-jet bundle of germs of local cross sections of E. Define a map Φ^r: $\mathscr{D}(M) \to \mathscr{D}(E^r)$ by $\Phi^r(f)(J^r_x(g)) = J^r_{f(x)}(\tilde{f} \circ g \circ f^{-1})$, $f \in \mathscr{D}(M)$, $f: U \to V$, and $x \in U$. Here $\tilde{f}$ is the induced bundle map by f:

$$\begin{array}{ccc} (p^r)^{-1}(U) & \xrightarrow{\Phi^r(f)} & (p^r)^{-1}(V) \\ {\scriptstyle J^r(g)}\nearrow\ {\scriptstyle p^r}\downarrow & & \downarrow{\scriptstyle p^r} \\ U & \xrightarrow{f} & V \end{array} \qquad \begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tilde{f}} & p^{-1}(V) \\ {\scriptstyle g}\nearrow\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

Then Φ^r is an extension.

(*)Editor's note. For Gromov's thesis see M. L. Gromov, Sm̄aduuskoe om̄odrazhelukh seoekuū v ukoioodrayukh. Uzv. Akad. Nauk serukh Mam̄, T. 33. H-4, 1969, pp. 707–734.

([6])V. Poénaru, *Regular homotopy and isotopy*, mimeographed, Harvard Univ., 1964.

DEFINITION 3.20. Let (E, p, M) be a smooth fiber bundle. For an open set U in M let $\mathrm{Diff}^\infty(U)$ be the set of diffeomorphisms of U onto itself. We give $\mathrm{Diff}^\infty(U)$ the compact-open topology. An extension Φ of $\mathscr{D}(M)$ to $\mathscr{D}(E)$ is *continuous* if for an arbitrary open set U of X the map

$$\Phi : \mathrm{Diff}^\infty(U) \longrightarrow \mathrm{Diff}^\infty(p^{-1}(U))$$

is continuous with respect to the compact-open topology.

In each of the above examples we have a continuous extension. In what follows, all our extensions will be continuous. Let (E, p, M) be a smooth fiber bundle, and denote by $\Gamma^\infty(p)$ (or $\Gamma^\infty(E)$) the set of C^∞ cross sections in (E, p, M). Let E^r_ω be a subbundle of the r-jet bundle (E^r, p^r, M) and write $p^r_\omega = p^r|E^r_\omega$. Denote by $\Gamma^\infty_\omega(p)$ (or $\Gamma^\infty_\omega(E)$) the set of elements f of $\Gamma^\infty(p)$ with the property that $J^r f(x) \in E^r_\omega$ for each $x \in M$. We give $\Gamma^\infty(p)$ the C^∞ topology and $\Gamma^\infty_\omega(p)$ the relative topology. Then the map

$$J^r : \Gamma^\infty_\omega(p) \longrightarrow \Gamma^0(p^r_\omega)$$

is continuous, where $\Gamma^0(p^r_\omega)$ (also written $\Gamma^0(E^r_\omega)$) is the set of C^0 cross sections of $(E^r_\omega, p^r_\omega, M)$ with the compact-open topolgy.

THEOREM 3.12 (GROMOV'S THEOREM). *Let (E, p, M) be a smooth fiber bundle and let (E^r, p^r, M) be the r-jet bundle of germs of C^∞ cross sections of E. Assume*

(a) *M is open,*

(b) *E^r_ω is an open subset of E^r, and*

(c) *E^r_ω is invariant under the action of $\mathscr{D}(M)$ by the extension Φ^r; that is, for $f \in \mathscr{D}(M)$ we have $\Phi^r(f)(E^r_\omega) \subset E^r_\omega$.*

Then the map

$$J^r : \Gamma^\infty_\omega(p) \longrightarrow \Gamma^0(p^r_\omega)$$

is a weak homotopy equivalence.

REMARK. This theorem does not work in general if M is a closed manifold.

For example, consider $M = S^1$ and $E = T(S^1) = S^1 \times \mathbf{R}^1$. Then $E^1 = J^1(S^1, \mathbf{R}^1)$, and the fiber is $J^1(1, 1) = M(1, 1; \mathbf{R}) = \mathbf{R}$. Let E^1_0 be the associated bundle of E^1, whose fiber is $\mathrm{GL}(1, \mathbf{R})$; E^1_0 is open in E^1 and is invariant under the action of $\mathscr{D}(S^1)$.

Now $\Gamma^\infty_0(E) = \mathrm{Imm}(S^1, \mathbf{R}^1) = \varnothing$ but $\Gamma^0(E^1_0) \neq \varnothing$.

§8. Submersions: The Phillips theorem

This section features the Phillips theorem. Let M and V be C^∞ manifolds of dimensions n and p respectively. Denote by $\mathrm{Sub}(M, V)$ the subspace of $C^\infty(M, V)$ consisting of all submersions of M in V. Denote by $\mathrm{Epi}(T(M), T(V))$ the set of epimorphisms of the tangent bundle $T(M)$

of M in the tangent bundle $T(V)$ with the compact open topology. Here $\phi : T(M) \to T(V)$ is an *epimorphism* if ϕ is a homomorphism of vector bundles, which is surjective on each fiber $T_x(M)$ (see §2, Chapter II). If $f : M \to V$ is a submersion, its differential $df : T(M) \to T(V)$ of f is an epimorphism.

THEOREM 3.13 (PHILLIPS). *Let M be an open manifold. Then the map*

$$d : \mathrm{Sub}(M, V) \longrightarrow \mathrm{Epi}(T(M), T(V)),$$
$$f \longmapsto df$$

is a weak homotopy equivalence.

PROOF. Let n and p be the respective dimensions of M and V with $n \geq p$. Set $E = M \times V$, $\pi = p_1$. Then we get the one-jet bundle $E^1 = J^1(M, V)$ whose fiber is $J^1(n, p) \times V = M(p, n; \mathbf{R}) \times V$. Denote by Σ the subspace of all matrices in $M(p, n; \mathbf{R})$ each having rank less than or equal to $p - 1$. Then Σ is closed in $M(p, n; \mathbf{R})$. Put

$$\Omega = M(p, n; \mathbf{R}) - \Sigma.$$

The subbundle E^1_0 of E^1 corresponding to $\Omega \times V$ is invariant under the action of $\mathscr{D}(M)$ by the extension $\Phi^1 : \mathscr{D}(M) \to \mathscr{D}(E^1)$ defined by $\Phi^1(f) = df$. Hence by Gromov's theorem,

$$j^1 : \Gamma^\infty_0(E) \longrightarrow \Gamma^0(E^1_0)$$

is a weak homotopy equivalence. A careful reflection on this fact yields natural isomorphisms ϕ, ψ making the following diagram commute:

$$\begin{array}{ccc} \Gamma^\infty_0(E) & \xrightarrow{j^1} & \Gamma^0(E^1_0) \\ \phi \big\downarrow \approx & & \approx \big\downarrow \psi \\ \mathrm{Sub}(M, V) & \xrightarrow{d} & \mathrm{Epi}(T(M), T(V)). \end{array}$$

The theorem is obvious for $n < p$. □

§9. Proof of the Smale-Hirsch theorem

The proof given here is due to Phillips [C14]. Let M be an n-dimensional C^∞ manifold and let V be a p-dimensional C^∞ manifold. Assume $n < p$. To an immersion $f : M \to V$, there corresponds a normal bundle ν. The normal bundles ν_0 and ν_1 corresponding to immersions $f_0, f_1 : M \to V$ are equivalent if f_0 and f_1 are regularly homotopic. We consider a fixed $p - n$ dimensional vector bundle ν over M, and let $\mathrm{Imm}_\nu(M, N)$ be the space of immersions whose normal bundles are equivalent to ν. Denote by $\mathrm{Mon}_\nu(T(M), T(N))$ the set of monomorphisms ϕ of $T(M)$ in $T(V)$ such that $\bar\phi^*(T(V)|\bar\phi(M)/\phi(T(M)))$ is equivalent to ν. Here $\bar\phi$ is induced by ϕ.

Letting $E(\nu)$ be the total space of ν we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sub}(E(\nu), V) & \xrightarrow{d} & \mathrm{Epi}(T(M) \oplus \nu, T(V)) \\ \downarrow & & \downarrow \\ \mathrm{Imm}_\nu(M, V) & \xrightarrow{d} & \mathrm{Mon}_\nu(T(M), T(V)) \end{array}$$

where the vertical maps are naturally induced. The first line is a weak homotopy equivalence by the Phillips theorem. Each vertical line is a homotopy equivalence. Thus, the second line is also a homotopy equivalence.

Since the above statement is valid for every ν, we have proved the Smale-Hirsch theorem. □

In Chapter IV we shall give another proof of the Smale-Hirsch theorem without using the Phillips theorem.

§10. The Gromov-Phillips theorem

Now we examine the Gromov-Phillips theorem which was independently proved by Gromov and Phillips at about the same time, and which later attracted much attention when used by Haefliger in the proof of his classification theorem for foliations of open manifolds.

Let M be an m-dimensional and N an n-dimensional C^∞ manifolds. Consider a k-plane field on N, that is, a k-dimensional subbundle η of the tangent bundle $T(N)$ of N. Let ν be the quotient bundle $T(N)/\eta$, and let π be the natural projection

$$\pi : T(N) \longrightarrow \nu.$$

Denote by $\mathrm{Epi}(T(M), \nu)$ the space of the epimorphisms between $T(M)$ and ν with the compact-open topology, and denote by $\mathrm{Tr}(M, \eta)$ the space (with C^1 topology) of all C^1 maps $f : M \to N$ each $\pi \circ df$ belonging to $\mathrm{Epi}(T(M), \nu)$. Then we have the following

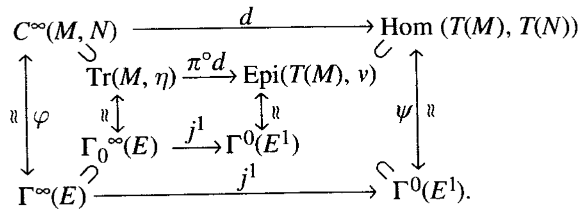
THEOREM 3.14 (GROMOV AND PHILLIPS). *Let M be an open manifold. Then the map*

$$\pi \circ df : \mathrm{Tr}(M, \eta) \longrightarrow \mathrm{Epi}(T(M), \nu), \qquad f \longmapsto \pi \circ df$$

is a weak homotopy equivalence.

PROOF. Set $E = M \times N$ and $\pi = p_1$ (the projection onto the first factor). Then we have $E^1 = J^1(M, N)$. Let E_0^1 be the subbundle of one-jets which are transverse to the given k-plane field η. Then E_0^1 is an open subbundle invariant under the action of $\mathscr{D}(M)$ by Φ', where $\Phi' : \mathscr{D}(M) \to \mathscr{D}(E^1)$ was defined in Example 2 in §4. The $\Gamma^0(E_0^1)$ corresponds to $\mathrm{Epi}(T(M), \nu)$.

This is shown in the diagram below



Hence, the theorem follows from Gromov's theorem. □

§11. Handlebody decompositions of C^∞ manifolds

In preparation for proving Gromov's theorem, this section offers a brief discussion of handlebody decompositions of C^∞ manifolds (for further details, see Smale [C17a]).

Let M be an n-dimensional compact C^∞ manifold, and let Q be a connected component of the boundary ∂M of M. Consider embeddings

$$f_i : \partial D_i^s \times D_i^{n-s} \longrightarrow Q, \qquad i = 1, \dots, k, \quad 0 \le s \le n,$$

whose images are mutually disjoint. Introduce an n-dimensional compact C^∞ manifold $V = \chi(M, Q; f_1, \dots, f_k; s)$ as follows: the underlying manifold of V is

$$M^n \cup \left(\bigcup_{i=1}^k D_i^s \times D_i^{n-s} \right) \Big/ \sim,$$

$$(x, y) \sim f(x, y), \quad (x, y) \in \partial D_i^s \times D_i^{n-s}, \quad f(x, y) \in Q \subset M^n,$$

which has the usual smooth structure everywhere outside $\partial D_i^s \times D_i^{n-s}$. By "smoothing out" the "corners" along $\partial D_i^s \times D_i^{n-s}$ we will have the C^∞ manifold V (Figure 3.2).

We often write $V = \chi(M; f_i, \dots, f_k; s)$ in case $\partial M^n = Q$. The embedded $D_i^s \times D_i^{n-s}$ in V are called *s-handles*.

DEFINITION 3.21. Let $V = \chi(M, Q; f_1, \cdots, f_k; s)$. We say that $\sigma = (M, Q; f_1, \dots, f_k; s)$ is the *presentation* of V. A manifold with a presentation $(D^n; f_1, \dots, f_k; s)$ is called a *handlebody*. We obtain the *manifold* $V = \chi(M, Q; f_1, \dots, f_k; s)$ by *attaching s-handles* $D_1^s \times D_1^{n-s}, \dots, D_k^i \times D_k^{n-s}$ to M.

More generally, we say that $V = \chi(M, Q; f_1, \dots, f_k; s)$ is a *handlebody* if M is a handlebody.

THEOREM 3.15. *Let W be a compact manifold and let $f : W \to \mathbf{R}^1$ be a C^∞ function such that the only singularities in $f^{-1}(-\varepsilon, \varepsilon)$ are nondegenerate of index λ lying in $f^{-1}(0)$. Further assume that $N \cap \partial W = \varnothing$ and that*

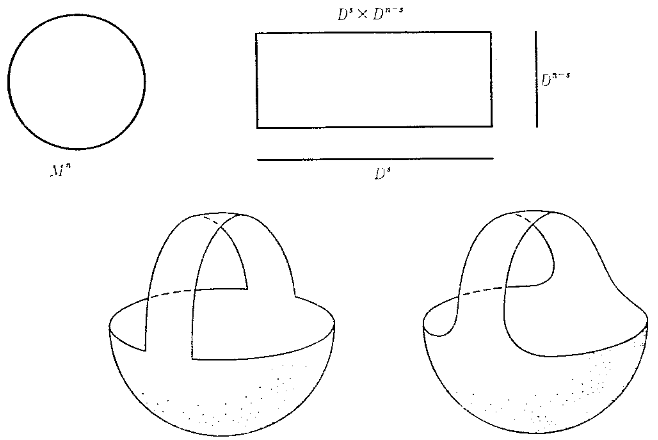


FIGURE 3.2

$f^{-1}(-\varepsilon)$ is connected. Then $f^{-1}[-\infty, \varepsilon]$ has a presentation of the form

$$(f^{-1}[-\infty, -\varepsilon], f^{-1}(-\varepsilon); f_1, \dots, f_k; \lambda).$$

OUTLINE OF THE PROOF. Let $\beta_1, \dots, \beta_k$ be the singularities of f in $f^{-1}(0)$, and choose neighborhoods V_i of β_i, $i = 1, \dots, k$, which are mutually disjoint. Then in V_i, f can be written

$$f(x) = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2,$$

with respect to the local coordinates $x = (x_1, \dots, x_n)$, $\|x\| < \delta$, $\delta > 0$. Denote by E_1 the $(x_1, \dots, x_\lambda)$-plane and by E_2 the $(x_{\lambda+1}, \dots, x_n)$-plane in V_i. Then for a sufficiently small $\varepsilon_1 > 0$, $E_1 \cap f^{-1}[-\varepsilon_1, \varepsilon_1]$ and D^λ are diffeomorphic (Figure 3.3). There exists a diffeomorphism

$$\phi : T' = T \cap f^{-1}[-\varepsilon_1, \varepsilon_1] \longrightarrow D^\lambda \times D^{n-\lambda},$$
$$\phi(T \cap f^{-1}(-\varepsilon_1)) = \partial D^\lambda \times D^{n-\lambda},$$

where T is some small enough tubular neighborhood of E_1.

Now comparing $f^{-1}[-\infty, -\varepsilon_1]$ and $f^{-1}[-\infty, \varepsilon_1]$ we see $f^{-1}[-\infty, -\varepsilon_1]$ with a λ-handle attached to it for each i agrees with $f^{-1}[-\infty, \varepsilon_1]$. Hence, we have proven the theorem. □

DEFINITION 3.22. Let M^n be an n-dimensional manifold and let $f : M^n \to \mathbf{R}$ be a C^∞ function. We say that f is a *nice function* if every critical point β of f is nondegenerate and the index of β is equal to $f(\beta)$.

THEOREM 3.16. *Let M^n be an n-dimensional C^∞ manifold. Then M^n admits a nice function.*

We omit the proof.

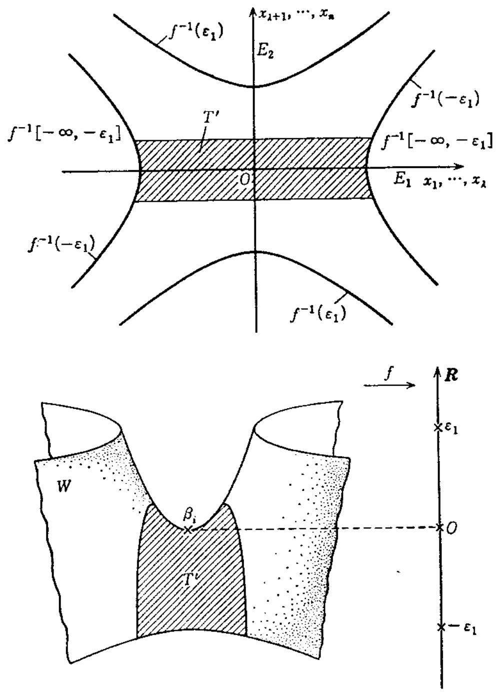


FIGURE 3.3

From Theorems 3.15 and 3.16 we see that an n-dimensional C^∞ manifold without boundary is obtained by attaching handles to the n-dimensional disk D^n one by one.

§12. Proof of Gromov's Theorem

In this section, following Haefliger [B3], we prove Gromov's theorem(7). The basic tool is handlebody decompositions of C^∞ manifolds.

Let M be an m-dimensional C^∞ manifold. If M is open, there exists a unique Morse function $f : M \to [0, \infty)$ such that the index of each critical point of f is less than m (see Chapter I). We arrange the critical points of f as follows (Figure 3.4):

$$a_1, a_2, \dots,$$
$$c_1 = f(a_i), \quad i = 1, 2, \dots$$
$$c_1, c_2, \dots \text{are monotone increasing.}$$

(7)See also V. Poénaru, *Homotopy theory and differentiable singularities*, Lecture Notes in Mathematics, vol. 197, Springer-Verlag, Berlin and New York, 1970.

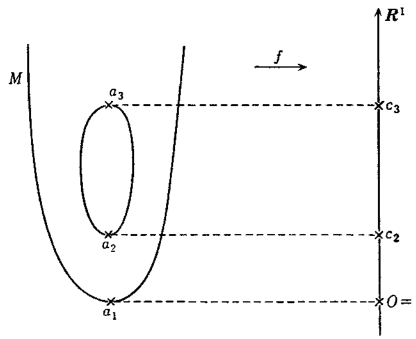


FIGURE 3.4

There exist local coordinates $(x_i, \dots, x_m)$ about each a_i (there is a local chart $(U_\alpha, \varphi_\alpha)$ with $a_i \in U_\alpha$, $\varphi_\alpha(x) = (x_1, \dots, x_m)$ for $x \in U_\alpha$ and $\varphi_\alpha(a_i) = 0$) in which f has the form

$$f = c_i - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2,$$

where k is the index of a_i (in other words, for each x in $\varphi_\alpha(U_\alpha)$, $f \circ \varphi_\alpha^{-1}(x)$ has the above form). We often adopt the above expression. Notice that $k < m$ (see §4, Chapter I).

For each i, put $M_i = f^{-1}([0, c_i + \varepsilon_i])$, where $0 < \varepsilon_i < c_{i+1} - c_i$ and ε_i is sufficiently small. Take a neighborhood $U(a_i) \subset U_\alpha$ of a_i and for a sufficiently small $\delta_i > 0$ set

$$W_i = \left\{ x \in U(a_i) | \varphi_\alpha(x) = (x_1, \dots, x_m), x_1^2 + \dots + x_k^2 < \frac{\delta_i}{2} \right\}.$$

Put $M_{i-1}^{\neg} = M_i - W_i$. In Figure 3.5 we identify $U(a_i)$ with $\phi_\alpha(U(a_i)) \subset \mathbf{R}^m$.

Now $M_{i-1}^{\neg}$ is a manifold with boundary with an edge diffeomorphic to $S^{k-1} \times S^{m-k-1}$. We obtain $M_{i-1}^{\neg}$ by attaching a collarlike neighborhood to M_{i-1} along its boundary $\partial M_{i-1} = f^{-1}(c_{i-1} + \varepsilon_{i-1})$ (Figure 3.5); by a collarlike neighborhood we mean a subset of $\partial M_{i-1} \times [0, 1]$ of the form

$$\{(x, t) | t \le g_i(x)\},$$

for some C^∞ function $g_i : \partial M_{i-1} \to (0, 1]$. Hence M_i is diffeomorphic to the union of $M_{i-1}^{\neg}$ and A_k, where A_k is diffeomorphic to $D^k \times D^{m-k}$ and $M_{i-1}^{\neg} \cap A_k$ is diffeomorphic to a collarlike neighborhood B of $\partial D^k \times D^{m-k}$ (cf. Figure 3.5).

We say that M_i is obtained by attaching a k-handle to M_{i-1} (see Figures 3.5 and 3.6).

Now M is the union of the following expanding sequence of compact

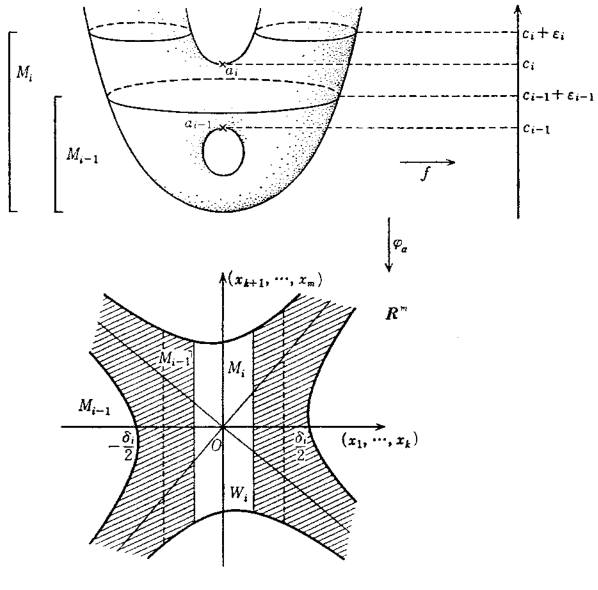


FIGURE 3.5

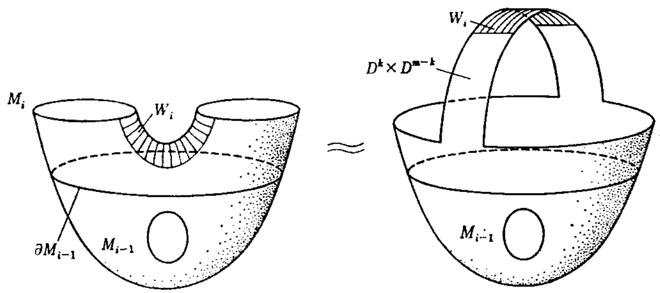


FIGURE 3.6

manifolds with boundary:

$$M_1 \subset \dots \subset M_{i-1} \subset M_{i-1}^{\neg} \subset M_i \subset M_i^{\neg} \subset \cdots. \tag{0}$$

The three propositions below will yield Gromov's theorem.

In what follows we write $(E(M), p, M)$ for (E, p, M), $(E^r(M), p^r, M)$ for (E^r, p^r, M), and a similar adaptation applies for $(E_\omega^r(M), p^r, M)$. For $A \subset M$, $(E(A), p, A)$ is the restriction of $(E(M), p, M)$ to A. Similarly $E^r(A)$ is the restriction of $E^r(M)$ to A.

PROPOSITION 3.6. *Gromov's theorem is true if M is the m-dimensional disk D^m; that is*

$$j^r : \Gamma_\omega^\infty(E(D^m)) \longrightarrow \Gamma^0(E_\omega^r(D^m))$$

is a weak homotopy equivalence.

DEFINITION 3.23. Let E and B be topological spaces, and let $p : E \to B$ be a continuous map of E onto B. We say that (E, p, B) is a *fibration* or a fiber space if for any finite polyhedron P and continuous maps $F : P \times [0, 1] \to B$ and $f : P \to E$ with $p \circ f(x) = F(x)$, there exists a continuous map $\tilde{F} : P \times [0, 1] \to E$ satisfying

(i) $p \circ \tilde{F}(x, t) = F(x, t)$,
(ii) $\tilde{F}(x, 0) = f(x)$.

$$\begin{array}{ccc} P & \xrightarrow{f} & E \\ & \overset{\tilde{F}}{\nearrow} & \downarrow p \\ P \times I & \xrightarrow{F} & B \end{array}$$

We also say that (E, p, B) has the *covering homotopy property* (CHP).

Evidently a fiber bundle is a fibration (cf. Steenrod [A7]).

PROPOSITION 3.7. *Suppose that a C^∞ manifold M^- is diffeomorphic to a C^∞ manifold M with a collarlike neighborhood attached along its boundary ∂M. Then the restriction maps*

$$\rho_\omega : \Gamma^\infty_\omega(E(M^-)) \longrightarrow \Gamma^\infty_\omega(E(M)),$$
$$\rho : \Gamma^0(E^r_\omega(M^-)) \longrightarrow \Gamma^0(E^r_\omega(M))$$

are both homotopy equivalences and define fibrations.

PROPOSITION 3.8. *Let $k < m$. Set $A = D^k \times D^{m-k}$, $B = D^k_{1/2} \times D^{m-k}$, where*

$$D^k_{1/2} = \{ x \in D^k | 1/2 \le |x| \le 1 \}.$$

Then the restriction maps

(i) $\rho_\omega : \Gamma^\infty_\omega(E(A)) \longrightarrow \Gamma^\infty_\omega(E(B))$,
(ii) $\rho : \Gamma^0(E^r_\omega(A)) \longrightarrow \Gamma^0(E^r_\omega(B))$

are fibrations.

Let (E, p, B) and (E', p', B') be fibrations. A continuous map $g : E \to E'$ mapping each fiber of E into some fiber of E' is called a *fiber map*. In this case, the corresponding map $\bar{g} : B \to B'$ between the base spaces is continuous and the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ p\downarrow & & \downarrow p' \\ B & \xrightarrow{\bar{g}} & B' \end{array}$$

LEMMA 3.1. *Consider fibrations (E, p, B) and (E', p', B'), a fiber map $g : E \to E'$, and the corresponding map $\bar{g} : B \to B'$ of base spaces. Assume*

that $\bar{g}$ is a weak homotopy equivalence. Then the following statements are true.

(a) *If g is a weak homotopy equivalence, then the restriction of g to each fiber*

$$g_x : F_x \longrightarrow F'_{g(x)}, \qquad x \in B,$$

is a weak homotopy equivavence. Here $F_x = p^{-1}(x)$, $F'_{g(x)} = p'^{-1}(g(x))$.

(b) *Conversely, if for each $x \in B$ the map $g_x : F_x \longrightarrow F'_{g(x)}$ is a weak homotopy equivalence, then g too is a weak homotopy equivalence.*

PROOF. We consider the homotopy exact sequences of fibrations:

$$\begin{array}{ccccccccccc} \longrightarrow \pi_i(F) & \xrightarrow{i_*} & \pi_i(E) & \xrightarrow{p_*} & \pi_i(B) & \longrightarrow & \pi_{i-1}(F) & \xrightarrow{i_*} & \pi_{i-1}(E) \longrightarrow \cdots \\ \downarrow & & g_*\downarrow & & \bar{g}_*\downarrow & & \downarrow & & g_*\downarrow \\ \longrightarrow \pi_i(F') & \xrightarrow{i_*} & \pi_i(E') & \xrightarrow{p_*} & \pi_i(B') & \longrightarrow & \pi_{i-1}(F') & \xrightarrow{i_*} & \pi_{i-1}(E') \longrightarrow \cdots \end{array}$$

This diagram is commutative (Steenrod [A7]). Hence, the proposition follows from the "five-lemma". □

THE FIVE-LEMMA. *Suppose two exact sequences of abelian groups give the following commutative diagram:*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ h_1\downarrow & & h_2\downarrow & & h_3\downarrow & & h_4\downarrow & & h_5\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

If h_1, h_2, h_4, and h_5 are isomorphism, then so is h_3.

We leave the proof to the reader.

PROOF OF GROMOV'S THEOREM. The proof is by induction. Assume first that the theorem is true for a compact manifold M which consists of handles of indices less than k; that is, Proposition 3.6 is the starting point of our induction.

Step 1. Let $\dim M = m$, and construct a manifold M' by attaching a handle of index k to M, $k < m$. Assuming that the theorem holds for M, we shall prove that it is also true for M'. As before we let M^- be the manifold obtained from M by attaching a collarlike neighborhood along ∂M, that is,

$$M' = M^- \cup A, \qquad A \underset{d}{\approx} D^k \times D^{m-k},$$
$$M^- \cap A = B, \qquad B \underset{d}{\approx} D^k_{1/2} \times D^{m-k} \subset D^k \times D^{m-k}.$$

Consider the following commutative diagram:

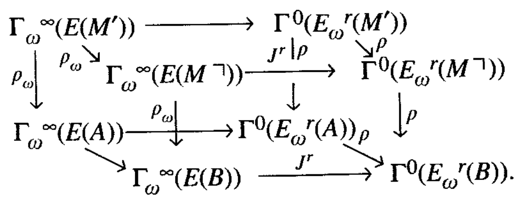
$$\begin{array}{ccc} \Gamma^\infty_\omega(E(A)) & \xrightarrow{J^r} & \Gamma^0(E^r_\omega(A)) \\ {\scriptstyle\rho_\omega}\downarrow & & \downarrow{\scriptstyle\rho} \\ \Gamma^\infty_\omega(E(B)) & \xrightarrow{J^r} & \Gamma^0(E^r_\omega(B)) \end{array} \tag{1}$$

By Proposition 3.8, ρ_ω and ρ are fibrations. By Propositions 3.6 and 3.7, the maps $J^r : \Gamma^\infty_\omega(E(A)) \to \Gamma^0(E^r_\infty(A))$ and $J^r : \Gamma^\infty_\omega(E(B)) \to \Gamma^0(E^\infty_\omega(B))$ are weak homotopy equivalences, for we may think of B as the union of a 0-handle and a $(k-1)$-handle. Thus by Proposition 3.1, J^r is a weak homotopy equivalence on each fiber.

Look at the following commutative diagram.

$$\begin{array}{ccc} \Gamma^\infty_\omega(E(M')) & \xrightarrow{J^r} & \Gamma^0(E^r_\omega(M')) \\ {\scriptstyle\rho_\omega}\downarrow & & \downarrow{\scriptstyle\rho} \\ \Gamma^\infty_\omega(E(M^-)) & \xrightarrow{J^r} & \Gamma^0(E^r_\omega(M^-)) \end{array} \tag{2}$$

By restriction to A and B we get a morphism from diagram (2) to diagram (1).



(3)

In (2) ρ_ω and ρ are fibrations (cf. Proposition 3.7), so we have the homotopy exact sequences of fibrations. But by the induction hypothesis $J^r : \Gamma^\infty_\omega(E(M^-)) \to \Gamma^0(E^r_\omega(M^-))$ is a weak homotopy equivalence. On the other hand the restriction of $J^r : \Gamma^\infty_\omega(E(M')) \to \Gamma^0(E^r_\omega(M'))$ to each fiber can be thought of via (3) as the restriction of the map $J^r : \Gamma^\infty_\omega(E(A)) \to \Gamma^0(E^\infty_\omega(A))$ in (1) to each fiber. The latter was a weak homotopy equivalence. Hence, it follows from Lemma 3.1 that $J^r : \Gamma^\infty_\omega(E(M')) \to \Gamma^0(E^r_\omega(M'))$ is a weak homotopy equivalence.

Step 2. Since M is an open manifold, it has in general infinitely many handles. Decompose M into the sequence (0),

$$M_1 \subset \cdots \subset M_{i-1} \subset M^-_{i-1} \subset M_i \subset M^-_i \subset \cdots , \tag{0}$$

which gives the following sequences of restriction maps

$$\Gamma^\infty_\omega(E(M_1)) \leftarrow \cdots \leftarrow \Gamma^\infty_\omega(E(M^-_{i-1})) \leftarrow \Gamma^\infty_\omega(E(M_i)) \leftarrow \Gamma^\infty_\omega(E(M^-_i)) \leftarrow \cdots ,$$
$$\Gamma^0(E^r_\omega(M_1)) \leftarrow \cdots \leftarrow \Gamma^0(E^r_\omega(M^-_{i-1})) \leftarrow \Gamma^0(E^r_\omega(M_i)) \leftarrow \Gamma^0(E^r_\omega(M^-_i)) \leftarrow \cdots ,$$

whose projective limits are

$$\varprojlim \Gamma^\infty_\omega(E(M_i)) = \Gamma^\infty_\omega(E(M)),$$
$$\varprojlim \Gamma^0(E^r_\omega(M_i)) = \Gamma^0(E^r_\omega(M)).$$

Hence, the theorem follows from Step 1 and the following lemma. □

LEMMA 3.2. *We consider the following commutative diagram for projective systems of topological spaces:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & A_{i+1} & \longrightarrow & A_i & \longrightarrow & A_{i-1} & \longrightarrow \cdots \longrightarrow & A_1 \\ & {\scriptstyle j_{i+1}}\downarrow & & {\scriptstyle j_i}\downarrow & & {\scriptstyle j_{i-1}}\downarrow & & \downarrow{\scriptstyle j_1} \\ \cdots \longrightarrow & B_{i+1} & \longrightarrow & B_i & \longrightarrow & B_{i-1} & \longrightarrow \cdots \longrightarrow & B_1 \end{array}$$

where each vertical map $j_i : A_i \to B_i$ is a fibration. If each j_i is a weak homotopy equivalence, then the projective limit

$$j = \varprojlim j_i : \varprojlim A_i \longrightarrow \varprojlim B_i$$

is also a weak homotopy equivalence.

We leave the proof to the reader (cf. Phillips [C 14]).

It remains to prove the above three propositions.

PROOF OF PROPOSITION 3.6. The fiber bundle $E(D^m)$ is the product bundle $D^m \times F$ because D^m is contractible; therefore, we identify cross sections in this bundle with maps from D^m to F and the fiber of $E^r(D^m)$ over $0 \in D^m$ with

$$J^r_0(D^m, F) = \bigcup_{y \in F} J^r_{0,y}(D^m, F).$$

Evidently the restriction map

$$\rho : \Gamma^0(E^r_\omega(D^m)) \longrightarrow \Gamma^0(E^r_\omega(0))$$

over the center 0 of D^m is a weak homotopy equivalence. Hence, it is enough to show that

$$\rho \circ j^r : \Gamma^\infty_\omega(E(D^m)) \longrightarrow \Gamma^0(E^r_\omega(0))$$

is a weak homotopy equivalence.

To show that

$$(\rho \circ j^r)_* : \pi_i(\Gamma^\infty_\omega(E(D^m))) \longrightarrow \pi_i(\Gamma^0(E^r_\omega(0)))$$

is surjective, we think of the fiber F of E as a closed submanifold of Euclidean space $\mathbf{R}^N$ (Whitney's embedding theorem). Let $\pi : W \to F$ be a retraction, where W is a tubular neighborhood of F in $\mathbf{R}^N$. Let

$$f : S^i \longrightarrow \Gamma^\infty(E^r_\omega(0)) \subset J^r_0(D^m, F) \subset J^r_0(D^m, \mathbf{R}^N)$$

be a continuous map. We represent each jet of $J_0^r(D^m, F)$ by a polynomial of degree r and obtain a continuous map

$$F : S^i \times D^m \longrightarrow \mathbf{R}^N,$$

where for each $s \in S^i$, the map $F_s : D^m \longrightarrow \mathbf{R}^N$ defined by $F_s(x) = F(s, x)$ is C^r differentiable, $j^r(F_s)$ is continuous in s, and $j^r(F_s)(0) = f(s)$.

Since F is continuous, there exists a neighborhood $V(0)$ of $0 \in D^m$ such that $F(S^i \times V(0)) \subset W$. Set $F_s' = \pi \circ F_s | V(0)$.

Since $E_\omega^r(D^m)$ is an open subbundle, there exists a neighborhood $U(0)$ of 0 in $V(0)$ such that $F_s'|U(0) \in \Gamma^0(E_\omega^r(U(0)))$. Let $h : D^m \to U(0)$ be an embedding which is the identity map in some neighborhood of 0. Define a map $g : S^i \longrightarrow \Gamma^0(E_\omega^r(D^m))$ by $g(s) = \tilde{h}^{-1} \circ F_s' \circ h$. Then we have $j^r(g)(0) = f$. Here $\tilde{h}$ is the induced map of h. Thus, we have shown that $(\rho \circ j^r)_*$ is a surjection.

The proof of injectivity is similar. □

The proof of Theorem 3.7 is not very hard and is left to the reader. Hint: One can extend a cross section of M to a neighborhood U of M in M^-. There exists an isotopy $\{f_t\}$ of the identity map 1_{M^-} of M^- satisfying the following:

(i) $f_0 = 1_{M^-}$,

$$f_1 : M^- \longrightarrow U \text{ is an embedding;}$$

(ii) there exists a neighborhood V of M in M^- such that

$$f_t(x) = x, \qquad x \in V, \ t \in I.$$

PROOF OF PROPOSITION 3.8. In the following we simply write $\Gamma^0(M)$ for $\Gamma_\omega^r(E(M))$ and $\Gamma(M)$ for $\Gamma^0(E_\omega^r(M))$. Set

$$D_{[a,b]}^k = \{x \in \mathbf{R}^k \mid a \le |x| \le b\},$$
$$D_a^k = \{x \in \mathbf{R}^k \mid |x| \le a\},$$
$$S_a^{k-1} = \{x \in \mathbf{R}^k \mid |x| = a\}.$$

Further, set $A = D_2^k \times D^{m-k}$, $\quad B = D_{[1,2]}^k \times D^{m-k}$.

The fiber bundle $E(\mathbf{R}^m)$ is the product bundle $\mathbf{R}^m \times F$ (however, the action of the pseudogroup $D(\mathbf{R}^m)$ of local diffeomorphisms of $\mathbf{R}^m$ is not necessarily trivial). Cross sections in this bundle are identified with maps from $\mathbf{R}^m$ to F.

Let P be a polyhedron, and let U be a subspace of $\mathbf{R}^m \times P$. We say that a map $f : U \to F$ is *admissible* if

(a) f is of class C^r on $U \cap (\mathbf{R}^m \times \{p\})$, $\forall p \in P$,
(b) $j^r(f)$ is continuous, and
(c) $j^r(f)(U) \subset E_\omega^r(U)$.

PROOF OF (ii). To show that $\Gamma(A) \to \Gamma(B)$ is a fibration, follow the argument that a fiber bundle has the covering homotopy property, CHP (Definition 3.23).

PROOF OF (i). The statement that $\Gamma_0(A) \to \Gamma_0(B)$ is a fibration means the following. Given a polyhedron P and continuous maps

$$f : P \times I \longrightarrow \Gamma_0(D_{[1,2]}^k \times D^{m-k}),$$
$$g_0 : P \times (0) \longrightarrow \Gamma_0(D_2^k \times D^{m-k}),$$
$$f(x, 0) = \rho \circ g_0(x, 0),$$

there exists a continuous map

$$g : P \times I \longrightarrow \Gamma_0(D_2^k \times D^{m-k}),$$

which extends g_0 with $\rho \circ g = f$;

$$\begin{array}{ccc} P \times 0 & \xrightarrow{g_0} & \Gamma_0(D_2^k \times D^{m-k}) \\ & \overset{g}{\nearrow} & \downarrow \rho \\ P \times I & \xrightarrow{f} & \Gamma_0(D_{[1,2]}^k \times D^{m-k}) \end{array}$$

Equivalently we can say that: given two admissible maps

$$f : D_{[1,2]}^k \times D^{m-k} \times P \times [0, 1] \longrightarrow F,$$
$$g_0 : D_2^k \times D^{m-k} \times P \times \{0\} \longrightarrow F,$$

which agree on their common domains, there exists an admissible map

$$g : D_2^k \times D^{m-k} \times P \times [0, 1] \longrightarrow F$$

which extends both f and g_0.

We shall construct g in three stages.

(a) We first extend f just a bit; that is, we construct an admissible map f' as follows.

$$f' : D_{[\alpha,2]}^k \times D^{m-k} \times P \times [0, 1] \longrightarrow F, \qquad \alpha < 1,$$
$$f'|D_{[1,2]}^k \times D^{m-k} \times P \times [0, 1] = f,$$
$$f'|D_2^k \times D^{m-k} \times P \times \{0\} = g_0.$$

(b) We can find an increasing sequence $0 = t_0 < t_1 < \cdots < t_s = 1$ such that for each n, $0 \le n < s$, there is an admissible map μ_n defined in some neighborhood of $D_{[\alpha,2]}^k \times D^{m-k} \times P \times [t_n, t_{n+1}]$ satisfying

(i) $\mu_n(x, y, p, t) = f'(x, y, p, t)$, $t = t_n$ or $x \in U(D_{[1,2]}^k)$, where $U(D_{[1,2]}^k)$ is some neighborhood of $D_{[1,2]}^k$.

(ii) $\mu_n(x, y, p, t) = \mu_n(x, y, p, t_n)$, $x \in U(S_\alpha^{k-1})$,

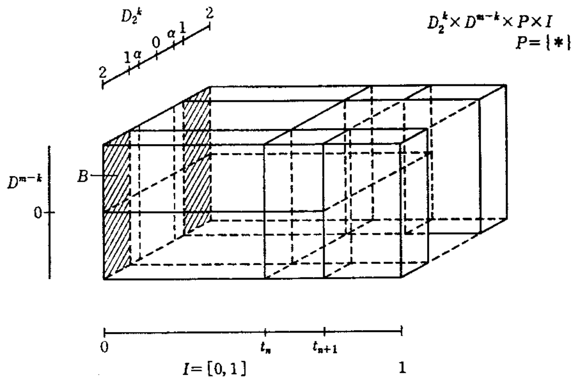


FIGURE 3.7

where $U(S_\alpha^{k-1})$ is some neighborhood of S_α^k.

If we do not require admissibility, such a map μ_n can be uniformly constructed for each t. Since admissibility is an open condition, μ_n is also admissible for a sufficiently small change in t. Using the compactness of $[0, 1]$, we may choose t_i to construct the desired μ_n. (See Figure 3.7.)

Using the above μ_0, we can extend g to $D_2^k \times D^{m-k} \times P \times [0, t_1]$ as follows.

$$g(x, y, p, t) = \begin{cases} \mu_0(x, y, p, t), & x \in D^k_{[\alpha, 2]}, \\ g_0(x, y, p, 0), & x \notin D^k_{[\alpha, 2]}. \end{cases}$$

(c) Now we shall extend g_0 to g_n inductively. Assume that we have constructed the admissible map

$$g_n : D_2^k \times D^{m-k} \times P \times [0, t_n] \longrightarrow F,$$
$$g_n | D_2^k \times D^{m-k} \times P \times [0, t_0] = g_0,$$
$$g_n | U(D^k_{[\beta, 2]} \times D^{m-k} \times P \times [0, t_n]) = f', \qquad \alpha < \beta < 1,$$

where $U(D^k_{[\beta, 2]} \times \cdots)$ is some neighborhood of $D^k_{[\beta, 2]} \times D^{m-k} \times P \times [0, t_n]$.

Suppose μ_n and f' are defined on some neighborhood $U \subset D_2^k \times D^{m-k}$ of $D_{[\alpha, \beta]} \times \{0\}$, and suppose $U \cap (D^k_{[1, 2]} \times D^{m-k}) = \varnothing$.

Since $k < m$ (this is becase M is an open manifold; see §4, Chapter I), $m - k > 0$, there is an isotopy Δ_t of $D_2^k \times D^{m-k}$, $0 \le t \le t_n$, such that

(1) Δ_t is the identity outside U, in some neighborhoods of $S_\beta \times 0$ and $S_1 \times 0$ respectively, and for all t, $t \le t_n/2$;

(2) $\Delta_{t_n}(S_\gamma \times 0) = S_\alpha \times 0$ for γ satisfying $\beta < \gamma < 1$.

We first construct g_{n+1} in a sufficiently small neighborhood of the core

$$C = [(D_2^k \times 0) \cup (D^k_{[1, 2]} \times D^{m-k})] \times P \times [1, t_{n+1}]$$

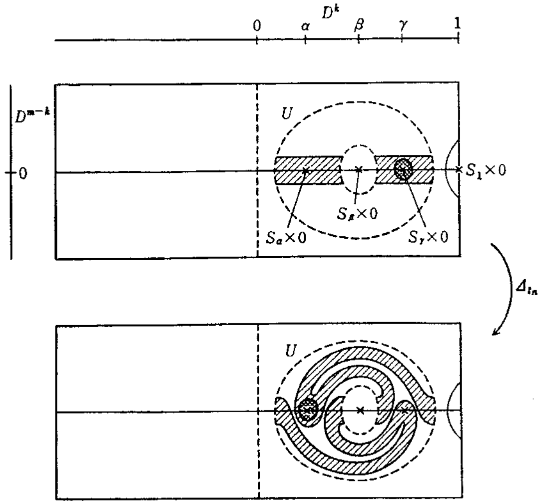


FIGURE 3.8

as follows.

In region I ($\|x\| < \beta$, $0 \le t \le t_n$), $g_{n+1} = g_n$.

In region II ($\beta \le \|x\| \le 2$, $0 \le t \le t_n$),

$$g_{n+1}(x, y, p, t) = (\bar{\Delta}_t)^{-1} f'(\Delta_t(x, y), p, t).$$

In region III ($\gamma \le \|x\| \le 2$, $t_n \le t \le t_{n+1}$),

$$g_{n+1}(x, y, p, t) = (\bar{\Delta}_{t_n})^{-1} \mu_n(\Delta_{t_n}(x, y), p, t),$$

where $\bar{\Delta}_t$ and $\bar{\Delta}_{t_n}$ represent the actions of $\Phi(\Delta_t)$ and $\Phi(\Delta_{t_n})$ on the fiber F; $\Phi(\Delta_t)$ and $\Phi(\Delta_{t_n})$ are extensions of the diffeomorphisms Δ_t and Δ_{t_n}.

In region IV ($\|x\| \le \gamma$, $t_n \le t \le t_{n_1}$),

$$g_{n+1}(x, y, p, t) = g_{n+1}(x, y, p, t_n).$$

In a small enough neighborhood V of the core C, the g_{n+1} agree in the intersection of their domains. (See Figure 3.8.) Let h_t be an isotopy of $D_2^k \times D^{m-k}$ such that

(i) $h_0 = 1$,

(ii) $h_t = 1$ on $U(D^k_{[1, 2]} \times D^{m-k})$,

(iii) $h_t(D_2^k \times D^{m-k}) \subset V$, $t \ge t_n/2$, where $U(D^k_{[1, 2]} \times D^{m-k})$ denotes some neighborhood of $D^k_{[1, 2]} \times D^{m-k}$. Then it is enough to define the desired g_{n+1} by

$$g_{n+1}(x, y, p, t) = (\bar{h}_t)^{-1} g_{n+1}(h_t(x, y), p, t),$$

where $\bar{h}_t$ is the action of the extension $\Phi(h_t)$ of the diffeomorphism h_t on the fiber F. □

§13. Further applications of Gromov's theorem

(A) Symplectic structures. Let M be an $2n$-dimensional C^∞ manifold.

DEFINITION 3.24. We say that a closed differential two-form ω on M is a *symplectic structure* when $\omega^n \neq 0$. Here $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$.

A symplectic structure on M defines naturally an almost complex structure. In the following we shall explain this concept.

Let $\mathrm{Sp}(2n, \mathbf{R})$ be an $2n$-dimensional real symplectic group. Then we have

$$\begin{aligned} O(2n) \cap \mathrm{Sp}(2n, \mathbf{R}) &= \mathrm{Sp}(2n, \mathbf{R}) \cap \mathrm{GL}(n, \mathbf{C}) \\ &= \mathrm{GL}(n, \mathbf{C}) \cap O(2n) = U(n). \end{aligned}$$

DEFINITION 3.25. Let M^{2n} be an $2n$-dimensional C^∞ manifold. An *almost symplectic structure* (or an *almost Hamiltonian structure*) on M^{2n} is a reduction of the structural group of the tangent bundle $\tau(M^{2n})$ to $\mathrm{Sp}(2n, \mathbf{R})$.

DEFINITION 3.26. We say that a nondegenarate skew-symmetric bilinear two-form $[\ ,\ ] : \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}$ on $\mathbf{R}^{2n}$ is a *linear symplectic structure* and that $[\ ,\ ]$ is its *skew scalar product.*

Here $[\ ,\ ]$ is *nondegenerate* means that if $[\xi, \eta] = 0$, $\forall \eta \in \mathbf{R}^{2n}$, then $\xi = 0$.

EXAMPLE. Set $\mathbf{R}^{2n} = \{(p_1, \dots, p_n, q_1, \dots, q_n)\}$, and define ω^2 by

$$\omega^2 = p_1 \wedge q_1 + \cdots + p_n \wedge q_n.$$

Then ω^2 is a nondegenerate skew symmetric two-form; so the scalar product $[\ ,\ ]$ defined by $[\xi, \eta] = \omega^2(\xi, \eta)$ is skew. The two-form ω^2 is the *standard symplectic structure on* $\mathbf{R}^{2n}$, which henceforth we will assume on $\mathbf{R}^{2n}$ unless otherwise stated.

DEFINITION 3.27. A linear transformation $S : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is *symplectic* if the skew scalar product is invariant under S; that is

$$[S(\xi), S(\eta)] = [\xi, \eta], \qquad \forall \xi, \eta \in \mathbf{R}^{2n}.$$

The set of all symplectic transformations on $\mathbf{R}^{2n}$ form a group, the $2n$ dimensional real symplectic group.

If M^{2n} has a symplectic structure ω, then for each point x of M^{2n} the tangent space $T_x(M^{2n})$ enjoys a symplectic structure, and the structural group for the tangent bundle $\tau(M^{2n})$ of M^{2n} reduces to $\mathrm{Sp}(2n, \mathbf{R})$; that is, M^{2n} has an almost symplectic structure.

On the other hand, since the above symplectic structure ω on M^{2n} defines a symplectic structure on each tangent space $T_x(M^{2x})$, the map $I : T_x(M) \to T_x(M)$ defined on each $T_x(M^{2n})$ by

$$[\xi, \eta] = (I\xi, \eta),$$

where $(\ ,\ )$ is the inner product of $T_x(M)$, satisfies $I^2 = -1$. Hence, $T_x(M)$ has a complex structure, which implies that we can reduce the structural group for the tangent bundle $\tau(M^{2n})$ of M^{2n} to $\mathrm{GL}(n, \mathbf{C})$. This also follows from the fact that $U(n)$ and $\mathrm{Sp}(2n, \mathbf{R})$ have the same homotopy type.

EXAMPLE. Let V be an n-dimensional C^∞ manifold. The cotangent bundle $T^*(V) = \mathrm{Hom}(T(V), \mathbf{R})$ of V is a $2n$-dimensional manifold which admits a symplectic structure. Taking local coordinates $(x_1, \dots, x_n)$ at each point x in V, we use the local coordinates $(x_1, \dots, x_n, dx_1, \dots, dx_n)$ at each point of $T^*(V)$, and setting $(x_1, \dots, x_n, dx_1, \dots, dx_n) = (p_1, \dots, p_n, q_1, \dots, q_n)$ we define

$$\omega = d\mathbf{p} \wedge d\mathbf{q} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

THEOREM 3.17. *Let M be an $2n$-dimensional open manifold. Then M has a symplectic structure if and only if M has an almost complex structure.*

PROOF. Put $E = T^*(M) = \mathrm{Hom}(T(M), \mathbf{R})$. Let E^1_ω be the subbundle of E^1 consisting of one-jets of one-forms α with $(d\alpha)^n \neq 0$. Then E^1_ω is an open subbundle invariant under the natural action of $\mathscr{D}(M)$. Now $\Gamma^\infty_\omega(E)$ is the space of one-forms with $(d\alpha)^n \neq 0$. We may regard $\Gamma^0(E^1_\omega)$ as the space of two-forms β with $\beta^n \neq 0$.

On the other hand, we saw earlier that the set of two-forms β with $\beta^n \neq 0$ corresponds to an almost complex structure on M. Hence, the theorem follows from Gromov's theorem. □

NOTE. This proposition is not valid if M is a closed manifold. One can define an almost complex structure on S^6 which has no symplectic structures.

(B) Contact structures. Let M be a $(2n+1)$-dimensional C^∞ manifold.

DEFINITION 3.28. A one-form ω on M is a *contact form* if

$$\omega \wedge (d\omega)^n \neq 0.$$

A one-form α is a *contact structure* on M if $\alpha \wedge (d\alpha)^n$ is an volume element.

A reduction of the structural group $\mathrm{GL}(2n+1, \mathbf{R})$ of the tangent bundle $T(M)$ of M to $U(n)$ is an *almost contact structure.*

In the manner of (A) we can show the following

THEOREM 3.18. *Let M be a $(2n+1)$-dimensional open manifold. A necessary and sufficient condition for M to have a contact structure is that M admit an almost contact structure.*

NOTE. Martinet showed that the theorem is true for three-dimensional closed manifolds([8]).

([8])J. Martinet, *Formes de contact sur variétés de dimension* 3, Lecture Notes in Math., vol. 209, Springer-Verlag, Berlin and New York, 1971, 142–163.

The theorem leads to a natural consideration: Is the same statement valid for almost complex structures and complex structures on open manifolds? We deal with this question in Chapter VI.

We discuss applications of the Gromov-Phillips theorem in foliation theory in Chapter IV.

ADDENDUM. In an historical perspective, Gromov's theorem and its applications are based on the covering homotopy method of Smale [17].

CHAPTER IV

The Gromov Convex Integration Theory

This chapter is based on Gromov's work [C5]. We discuss a natural generalization of the Smale-Hirsch theory, which we shall refer to as the Gromov convex integration theory. We feel that there are various applications of this theory; in Chapter VI we will discuss one of them.

§1. Fundamental theorem

Let X and B be n and q dimensional C^∞ manifolds. Let $p : X \to V$ be a smooth fiber bundle. Let

$$p^r : X^r \longrightarrow V$$

be the fiber bundle formed by the r-jets of germs of smooth cross sections in (X, p, V). We say that a subset Ω of X^r is a *differential relation of order* r or a *differential equation of order* r. If the r-jet $J^r(f) : V \to X^r$ of a smooth cross section $f : V \to X$ of (X, p, V) satisfies $J^r(f)(V) \subset \Omega$, we say that f is a solution of the differential relation Ω.

If Ω is an open set we are considering a family of partial differential inequalities of order r, and if Ω is a closed set we will be thinking of a family of partial differential equations of order r.

We denote the set of cross sections in (X, p, V) by Sect(X) ([1]). We give Sect(X) the relative topology of $C^\infty(V, X)$. We write Sect(X^r) for the set of cross sections of (X^r, p^r, V), with the relative topology of $C^0(V, X^r)$. Then the map

$$J^r : \mathrm{Sect}(X) \longrightarrow \mathrm{Sect}(X^r),$$
$$f \mapsto J^r(f)$$

assigning to each section its r-jet becomes continuous.

Set

$$\mathrm{Sect}(X^r, \Omega) = \{\sigma \in \mathrm{Sect}(X^r) | \sigma(V) \subset \Omega\},$$
$$\mathrm{Sect}(X, \Omega) = (J^r)^{-1}(\mathrm{Sect}(X^r, \Omega)).$$

Then Sect(X, Ω) is the solution space of Ω.

([1]) We wrote $\Gamma(X)$ in Chapter III; however, we will use this for something else in Chapter V. Thus, this new notation.

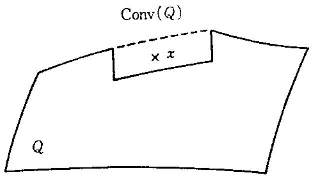


FIGURE 4.1

DEFINITION 4.1. We say that the *w.h.e-principle* holds for a differential relation Ω if the map $J^r : \mathrm{Sect}(X, \Omega) \longrightarrow \mathrm{Sect}(X^r, \Omega)$ is a weak homotopy equivalence, in which case

(i) J^r induces a bijection between arcwise connected components,

(ii) J^r induces an isomorphism of homotopy groups

$$(J^r)_* : \pi_i(\mathrm{Sect}(X, \Omega)) \xrightarrow{\cong} \pi_i(\mathrm{Sect}(X^r, \Omega)), \qquad \text{for each } i, \quad i \geq 1.$$

The fundamental theorem of Gromov gives a sufficient condition for the relation Ω to satisfy the w.h.e.-principle. For the moment we shall establish some notation for stating this fundamental theorem.

DEFINITION 4.2. Let L be an affine space and $Q \subset L$. We say that Q *envelopes* an element x of L if there exists a neighborhood $U(x)$ of x, which is contained in the convex hull $\mathrm{Conv}(Q)$ of Q.

Here the *convex hull* of Q is the smallest convex set containing Q (Figure 4.1).

If L is a Banach space and $Q \subset \Omega$, the closure of the convex hull of Ω is the smallest closed convex set containing Q.

DEFINITION 4.3. Q is *ample* if each arcwise-connected component of Q envelopes every point of L.

We agree that the empty set is ample.

EXAMPLE. (i) Let $L = \mathbf{R}^n$, $\mathbf{R}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n = 0\}$. Then Q defined by $Q = \mathbf{R}^n - \mathbf{R}^{n-1}$ is not ample.

(ii) Let $L = \mathbf{R}^n$, $\mathbf{R}^{n-2} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_{n-1} = x_n = 0\}$. Let $Q = \mathbf{R}^n - \mathbf{R}^{n-2}$; then Q is ample.

Let A_n be an affine transformation group of $\mathbf{R}^n$. A fiber bundle whose fiber is $\mathbf{R}^n$ with the structural group A_n is called an *affine bundle*.

DEFINITION 4.4. Let (Z, p, K) be an affine bundle. Suppose that the structual group A_n is reducible to $A_{n-q} \times A_q$. Further, we decompose the fiber Z_k at each point $k \in K$ into

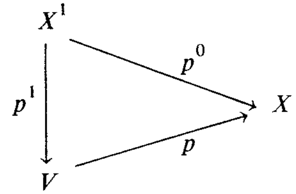
$$Z_k = \mathbf{R}_k^{n-q} \times \mathbf{R}_k^q = \bigcup_{\lambda \in \mathbf{R}_k^{n-q}} \{\lambda\} \times \mathbf{R}_k^q.$$

Henceforth we shall write $\mathbf{R}_{k,\lambda}^q = \{\lambda\} \times \mathbf{R}_k^q$. A reduction of A to $A_{n-q} \times A_q$ together with a decomposition of each fiber as above is called a *direction of dimension* q of (Z, p, K).

DEFINITION 4.5. Let (Z, p, K) be an affine bundle. By an *affine embedding* $\alpha : \mathbf{R}^q \to Z$ we mean a linear embedding of $\mathbf{R}^q$ in the corresponding fiber. Further, suppose (Z, p, K) has a direction of dimension q. We then say that an affine embedding $\alpha : \mathbf{R}^q \to Z$ is *parallel to the given direction* if α embeds $\mathbf{R}^q$ in one of the $\mathbf{R}_{k,\lambda}^q$.

DEFINITION 4.6. Let (Z, p, K) be an affine bundle with an q dimensional direction and $Q \subset Z$, $K_0 \subset K$. We say that Q is *ample over* K_0 if for any affine embedding $\alpha : \mathbf{R}^q \to Z_k \subset Z$, $k \in K_0$, parallel to the given direction, $\alpha^{-1}(Q) \subset \mathbf{R}^q$ is ample.

Let (X, p, V) be a smooth fiber bundle whose base space and fiber are n- and q-dimensional C^∞ manifolds. Let (X^1, p^1, V) be the fiber bundle of one-jets of germs of sections of the bundle (X, p, V), and let (X^1, p^0, X) be the natural fiber bundle associated with (X^1, p^1, V). The bundle (X^1, p^0, X) is affine. We have the following commutative diagram:



DEFINITION 4.7. Let V_0 be a codimension one submanifold of V. The submanifold V_0 defines a q-dimensional direction as follows. Consider the fiber $(p^0)^{-1}(x) \subset X^1$ over a point $x \in X$, $p(x) \in V_0 \subset V$. We say $J^1(f_0)$ and $J^1(f_1) \in (P^0)^{-1}(x)$ are equivalent if $f_0|V_0$ and $f_1|V_0$ define the same one-jet at $p(x)$. This equivalence relation decomposes the fiber $(p^0)^{-1}(x)$ into equivalences classes, which define a q-dimensional direction called a *principal direction*. We say that an affine embedding $\alpha : \mathbf{R}^q \to X$ parallel to a principal direction is *principal*.

DEFINITION 4.8. Let (X, p, V) be a smooth fiber bundle as above. Let (U, φ) be a chart of V, $\varphi : U \to \mathbf{R}^n$, $\varphi(x) = (u_1, \ldots, u_n)$. Set $V_i = U \cap \varphi^{-1}\{u_i = 0\}$. Then V_i is a codimension one submanifold which defines a q dimensional principal direction called a *coordinate direction* for the natural affine bundle over $p^{-1}U$, $i = 1, 2, \ldots, n$.

Now we are ready to state the fundamental

THEOREM 4.1 (Gromov's fundamental theorem). *Let (X, p, V) be a fiber bundle, and let $\Omega \subset X^1$ be an open set. Assume that for each x in X there exist a coordinate neighborhood N of x and a chart (U, φ) of $p(x) \in V$ such that*

(i) *$p(N) \subset U$,*

(ii) *Ω is ample in every coordinate direction for (U, φ).*

Then the w.h.e.-principle holds for Ω.

We postpone the proof of the theorem and present some simple consequences. Notice that we do not assume V to be open, whereas Gromov's theorem in Chapter 3 required that V be open; however, the condition on Ω was weaker in Chapter 3 than it is in the fundamental theorem in this section.

COROLLARY 4.1. *Let (X, p, V) be a smooth fiber bundle, and let Ω be an open subset of X^1. If for an arbitrary principal affine embedding $\alpha : \mathbf{R}^q \to X^1$, $\alpha^{-1}(\Omega)$ is ample in $\mathbf{R}^q$, then the w.h.e.-principle holds for Ω.*

This corollary is immediate from Gromov's fundamental theorem.

COROLLARY 4.2. *Let (X, p, V) be a smooth fiber bundle and let Σ be a closed subset of X^1. Suppose for an arbitrary principal affine embedding $\alpha : \mathbf{R}^q \to X^1$ either $\alpha^{-1}(\Sigma) \subset \mathbf{R}^q$ coincides with $\mathbf{R}^q$, or it is nowhere dense and its compliment is connected. Then the w.h.e.-principle holds for $\Omega = \Sigma^c$.*

Corollary 4.2 follows immediately from Corollary 4.1.

DEFINITION 4.9. Let (X^1, p^1, V) be the bundle of one-jets of germs of smooth sections of (X, p, V). The fiber of this bundle is $J^1(n, q) \times Y$. Consider a closed subset $\Sigma \times Y$ of $J^1(n, q)$ invariant under $L^1(n, q)$, and let J_Σ be the associated bundle of (X^1, p^1, V) whose fiber is $\Sigma \times Y$. Such a closed subset J_Σ of X^1 is called a *typical singularity.*

COROLLARY 4.3. *Let (X, p, V) be a smooth fiber bundle in which the base space V and the fiber are C^∞ manifolds of dimensions n and q, $n \leq q$. Let Σ be a typical singularity of codimension greater than or equal to two in X^1. Set $\Omega = \Sigma^c$. Then the w.h.e.-principle holds for Ω.*

This corollary follows from Corollary 4.2.

§2. Proofs of the Smale-Hirsch theorem and Feit's thorem

In this section we prove the Smale-Hirsch theorem and the Feit theorem.

A. The Smale-Hirsch theorem.

THEOREM 4.2. (the Smale-Hirsch theorem). *Let V and W be C^∞ manifolds of dimensions n and q respectively, $n < q$. Then the map assigning to each immersion of V in W its differential*

$$d : \mathrm{Imm}(V, W) \longrightarrow \mathrm{Mon}(T(V), T(W)),$$
$$f \mapsto df$$

is a weak homotopy equivalence.

PROOF. Set $X = V \times W$ and consider a map $p : X \to V$ which is the projection of X onto its first component. The bundle (X, p, V) is smooth and there exist natural isomorphisms φ and ψ making the following diagram commute:

$$\begin{array}{ccc} C^\infty(V, W) & \xrightarrow{d} & \mathrm{Hom}(T(V), T(W)) \\ \varphi \downarrow \approx & & \psi \downarrow \approx \\ \mathrm{Sect}(X) & \xrightarrow{J^1} & \mathrm{Sect}(X^1) \end{array}$$

The bundle (X^1, p^1, V) is equal to the following jet bundle.

$$\begin{array}{ccc} X^1 = J^1(V, W) & \longleftarrow & J^1(n, q) \\ p^1 \downarrow & & \downarrow \pi \\ X = V \times W & & \\ \downarrow & & \downarrow p_1 \\ V = V & & \end{array}$$

where $J^1(n, q) = M(q, n; \mathbf{R}) = \{(q, n)\text{-matrices over } \mathbf{R}\}$. Further, the structural group of $(J^1(V, W), \pi, V \times W)$ is $L^1(n, q) = L^1(q) \times L^1(n) = \mathrm{GL}(q, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$. Let M_0 denote the set of (q, n)-matrices whose ranks are n. Then M_0 is an open subset of $M(q, n; \mathbf{R})$, which is invariant under the action of $L^1(n, q)$. Hence, we may consider the subbundle whose fiber is M_0 as follows:

$$\begin{array}{ccc} \Omega & \longleftarrow & M_0 \\ \downarrow & & \\ V \times W & & \end{array}$$

Now Ω is an open subset of X^1, and by definition we obtain a subcommutative diagram of the above commutative diagram:

$$\begin{array}{ccc} \mathrm{Imm}(V, W) & \xrightarrow{d} & \mathrm{Mon}(T(V), T(W)) \\ \varphi \downarrow \approx & & \psi \downarrow \approx \\ \mathrm{Sect}(X, \Omega) & \xrightarrow{J^1} & \mathrm{Sect}(X^1, \Omega). \end{array}$$

Letting $\Sigma = \Omega^c \subset X^1$ we see that Σ satisfies the hypothesis of Corollary 4.2, because if $\Sigma_0 = M_0^c \subset M(q, n; \mathbf{R})$, then

$$\mathrm{codim}\Sigma_0 = q - n + 1 \geq 2.$$

Hence the theorem follows from Corollary 4.3. □

B. The Theorem of Feit. An analogous proof using Gromov's fundamental theorem works for the theorem of Feit.

Let V and W be C^∞ manifolds whose respective dimensions are n and q. Let $f : V \to W$ be an C^∞ map. For the natural number k and for each point x of V, if the rank of f at x is greater than or equal to 0, we ay that f is a k-*mersion*. Let k-mer(V, W) denote the topological space vhose underlying set is the set of all k-mersions of V in W, with the relative opology of $C^\infty(V, W)$. Let k-mor$(T(V), T(W))$ be the space of homo-norphisms of $T(V)$ in $T(W)$, whose restrictions to each fiber are of rank reater than or equal to k, with the relative topology of $\mathrm{Hom}(T(V), T(W))$ the compact-open topology).

THEOREM 4.3 (the theorem of Feit). *Let $k < q$. Then the map*

$$d : k\text{-}mer(V, W) \longrightarrow k\text{-}mor(T(V), T(W)),$$
$$f \mapsto df$$

s a weak homotopy equivalence.

REMARK. We cannot prove Phillips's theorem of Chapter 3 in this way.

§3. Convex hulls in Banach spaces

In this section we discuss convex hulls in certain Banach spaces as a reparatory step in proving Gromov's fundamental theorem.

1. Let $P = \{p_1, \dots, p_n\}$ be a set of nonnegative real numbers satisfying

$$\sum_{i=1}^{n} p_i = 1.$$

For P and a number ϵ, $0 < \epsilon < 1$, define a weakly monotone increasing iecewise-linear function

$$\theta = \theta_\epsilon^P : [0, 1] \longrightarrow [0, 1]$$

s follows. Let

$$0 < t_1 \le t_1' \le t_2 \le t_2' \le \cdots \le t_n \le t_n' < 1,$$
$$t_i' - t_i = (1-\epsilon)p_i,$$
$$t_1 = t_2 - t_1' = \cdots = t_{i+1} - t_i' = 1 - t_n' = \frac{\epsilon}{n+1}.$$

Then θ takes on the value $i/(n+1)$ in the interval $[t_i, t_i']$ and $\theta(0) = 0$, $\theta(1) = 1$ (see Figure 4.2).

2. Let B be a Banach space with the norm $\|\ \|$. Let Γ be the set of continuous curves $\gamma : [0, 1] \to B$, and define $I : \Gamma \longrightarrow B$ by

$$I(\gamma) = \int_0^1 \gamma(t)\, dt, \qquad \gamma \in \Gamma.$$

For results on the integration and differentiation of functions with values n a Banach space consult, for example, Riess & Sz.-Nagy (2).

In the following, $\mathrm{Conn}(Q, b)$ denotes the arcwise connected component n a topological space Q, which contains $b \in Q$.

(2) F. Riess & B. Sz.-Nagy, *Lecons d'analyse fonctionnelle*, Akadémai Kiadó, 1952.

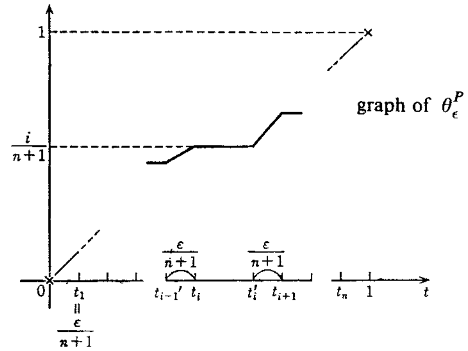


FIGURE 4.2

3. Let $B \supset Q$, and let $\gamma_0 : [0, 1] \longrightarrow Q$ be a continuous map. We consider some fixed Q for now. Define $\Gamma_0 \subset \Gamma$ as follows:

$$\Gamma_0 = \{\gamma : [0, 1] \to Q \mid \text{(i) } \gamma \text{ continuous, (ii) } \gamma(0) = \gamma_0(0),\ \gamma(1) = \gamma_0(1),$$
$$\text{(iii) } \gamma \simeq \gamma_0,\ \mathrm{rel}\{0, 1\},\ \text{in } Q\}.$$

LEMMA 4.1. *The set $I(\Gamma_0) \subset B$ has the following properties:*

(a) *$I(\Gamma_0)$ is convex.*
(b) *$I(\Gamma_0)$ is contained in the closure of the convex hull of* $\mathrm{Conn}(Q, \gamma_0(0))$.
(c) *$I(\Gamma_0)$ is dense in the closure of the convex hull of* $\mathrm{Conn}(Q, \gamma_0(o))$.
(d) *If Q is open in B, so is $I(\Gamma_0)$.*

PROOF OF (a). It is enough to show that for $\gamma_1, \gamma_2 \in \Gamma_0$, $p > 0$, $q > 0$, $p + q = 1$, there exists a $\gamma \in \Gamma_0$ such that

$$\int_0^1 \gamma(t)\, dt = p\int_0^1 \gamma_1(t)\, dt + q\int_0^1 \gamma_2(t)\, dt.$$

Setting $u = pt$ we get

$$\int_0^1 p\gamma_1(t)\, dt = \int_0^p \gamma_1\left(\frac{u}{p}\right) du.$$

Setting $v = qt + 1 - q$ we get

$$\int_0^1 q\gamma_2(t)\, dt = \int_p^1 \gamma_2\left(\frac{v+q-1}{q}\right) dv.$$

Hence, we get

$$p\int_0^1 \gamma_1(t)\, dt + q\int_0^1 \gamma_2(t)\, dt = \int_0^p \gamma_1\left(\frac{t}{p}\right) dt + \int_p^1 \gamma_2\left(\frac{t+q-1}{q}\right) dt.$$

On the other hand, we have

$$\int_0^p \gamma_1\left(\frac{t}{p}\right) dt = \int_0^{p/2} \gamma_1\left(\frac{2t}{p}\right) dt + \int_{p/2}^p \gamma_1\left(2 - \frac{2t}{p}\right) dt.$$

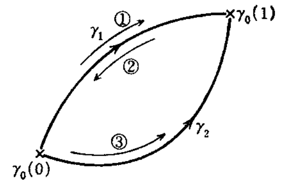


FIGURE 4.3

This is because, setting $2s/p = 2 - 2t/p$, we have

$$\int_{p/2}^{p} \gamma_1\left(2 - \frac{2t}{P}\right) dt = \int_{p/2}^{0} \gamma_1\left(\frac{2s}{p}\right)(-ds) = \int_0^{p/2} \gamma_1\left(\frac{2s}{p}\right) ds,$$

and setting $w/p = 2t/p$ we get

$$\int_0^{p/2} \gamma_1\left(\frac{2t}{p}\right) dt = \int_0^{p} \gamma_1\left(\frac{w}{p}\right)\frac{1}{2} dw = \frac{1}{2}\int_0^{p} \gamma_1\left(\frac{w}{p}\right) dw.$$

Hence, we have

$$p\int_0^1 \gamma_1(t)\,dt + q\int_0^1 \gamma_2(t)\,dt$$
$$= \int_0^{p/2} \gamma_1\left(\frac{2t}{p}\right) dt + \int_{p/2}^{p} \gamma_1\left(\frac{2-2t}{p}\right) dt + \int_p^1 \gamma_2\left(\frac{t+q-1}{q}\right) dt.$$

Therefore, if we set

$$\gamma(t) = \begin{cases} \gamma_1(2t/p), & 0 \le t \le p/2, \\ \gamma_1(2 - 2t/p), & p/2 \le t \le p, \\ \gamma_2((t+q-1)/q), & p \le t \le 1, \end{cases}$$

we get $\gamma \simeq \gamma_2$, rel$\{0, 1\}$ in Q (Figure 4.3), and because $\gamma \in \Gamma_0$, the proof of (a) is complete.

PROOF OF (b). Approximate the integral by a Riemann sum (cf. the proof of Proposition 4.2).

PROOF OF (c). The point $b = \sum_{i=1}^n p_i b_i$ of Conv(Conn($Q, \gamma_0(0)$)), $\sum p_i = 1$, $b_i \in$ Conn(Q, $\gamma_0(0)$) can be estimated as follows. Choose $\gamma \in \Gamma_0$ such that

$$\gamma\left(\frac{i}{n+1}\right) = b_i, \qquad i = 1, \ldots, n,$$

and put $\gamma_\epsilon = \gamma \circ \theta_\epsilon^P$, $P = \{p_1, \ldots, p_n\}$. Then we have

$$\int_0^1 \gamma_\epsilon(t)\,dt \longrightarrow b \qquad (\epsilon \to 0).$$

The proof of (d) is evident. □

4. We have the following well-known

LEMMA 4.2. *Let B be a Banach space, and let Q_1 and Q_2 be convex open sets in B. If the closures of Q_1 and Q_2 agree, then $Q_1 = Q_2$.*

HINT. This follows from the separability of convex sets in a vector space. See Riesz & Sz.-Nagy ([3]).

The lemma also follows from *For an open convex set Q in a Banach space B, we have $(\bar{Q})^0 = Q$* ([4]).

From Lemmas 4.1 and 4.2 we get the following

PROPOSITION 4.1. *Let B be a Banach space and let Q be an open subset of B. Then $I(\Gamma_0)$ agrees with the convex hull of* Conn(Q, $\gamma_0(0)$).

5. Now we want to present a basic lemma in Gromov's convex integration theory.

Let Ω be an open subset of $[0, 1] \times B$. For $t \in [0, 1]$ we denote by Ω_t the set $\Omega \cap (\{t\} \times B)$; $\Omega_t \subset B$.

LEMMA 4.3 (One-dimensional Lemma). *Let $\varphi_0 : [0, 1] \to B$ be a continuous map whose graph is contained in Ω. Let $f_0 : [0, 1] \to B$ be a C^1 map such that*

(i) $\frac{df_0}{dt}(0) = \varphi_0(0)$, $\frac{df_0}{dt}(1) = \varphi_0(1)$, *and*

(ii) *for* $t_0 \in [0, 1]$, Conn(Ω_{t_0}, $\varphi_0(t_0)$) *envelopes* $\frac{df_0}{dt}(t_0)$.

Then for an arbitrary $\epsilon > 0$ there exists a C^1 map $f : [0, 1] \to B$ satisfying the following:

(a) $f(0) = f_0(0)$, $f(1) = f_0(1)$,

$$\frac{df}{dt}(0) = \frac{df_0}{dt}(0), \quad \frac{df}{dt}(1) = \frac{df_0}{dt}(1).$$

(b) $\|f - f_0\| \le \epsilon$.

(c) *The graph of* $\frac{df}{dt} : [0, 1] \to B$ *is contained in* Ω.

(d) *There exists a homotopy* $\{\psi_\tau : [0, 1] \to B \mid \tau \in [0, 1]\}$ *such that*

(α) $\psi_0 = \varphi_0$, $\psi_1 = df/dt$,

(β) $\psi_\tau(0) = \varphi_0(0)$, $\psi_\tau(1) = \varphi_0(1)$, *and*

(γ) *the graph of* ψ_τ *is contained in* Ω.

PROOF. (I) Case $\Omega = [0, 1] \times Q$, Q is bounded open in B. Here we divide $[0, 1]$ uniformly into $n + 1$ intervals, apply Proposition 4.1 to each interval $[i/(n+1), (i+1)/(n+1)]$, and construct a path $\varphi_1 : [0, 1] \to Q$ such that

([3]) Riess & Sz,-Nagy, *Lecons d'analyse fonctionelle*, Akadémai Kiadó, 1952.

([4]) H. Eggleston, "Convexity", Cambridge Univ. Press, London and New York, 1958, Chapter I.

(1) $\varphi_1 \simeq \varphi_0$ in Q,

(2) $\varphi_1\left(\frac{i}{n+1}\right) = \varphi_0\left(\frac{i}{n+1}\right) \qquad i = 0, 1, \dots, n$,

(3) $\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \varphi_1(t)\,dt = f_0\left(\frac{i+1}{n+1}\right) - f_0\left(\frac{i}{n+1}\right), \quad i = 0, 1, \dots, n$.

Sublemma. *Let B be a Banach space and let Q be a convex open subset of B. Let $f : I \to Q$ be a continuous map, where $I = [0, 1]$. Then*

$$\int_0^1 f\,dt \in Q.$$

By assumption (ii) we may consider

$$\frac{df_0}{dt} : \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right] \longrightarrow \operatorname{Conv}\left(\operatorname{Conn}\left(Q, \varphi_0\left(\frac{i}{n+1}\right)\right)\right).$$

Hence, by the sublemma we have

$$f_0\left(\frac{i+1}{n+1}\right) - f_0\left(\frac{i}{n+1}\right) = \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \frac{df_0}{dt}(t)\,dt \in \operatorname{Conv}\left(\operatorname{Conn}\left(Q, \varphi_0\left(\frac{i}{n+1}\right)\right)\right).$$

On the other hand, since (Proposition 4.1)

$$\operatorname{Conv}\left(\operatorname{Conn}\left(Q, \varphi_0\left(\frac{i}{n+1}\right)\right)\right) = I(\Gamma_0),$$

there exists some $\varphi_1 \in \Gamma_0$ such that

$$f_0\left(\frac{i+1}{n+1}\right) - f_0\left(\frac{i}{n+1}\right) = \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \varphi_1(t)\,dt.$$

Notice that we have taken $\left[\frac{i}{n+1}, \frac{i+1}{n+1}\right]$ in place of $[0, 1]$ for defining Γ_0. Now set

$$f(t) = f_0(0) + \int_0^t \varphi_1(\theta)\,d\theta.$$

Then f satisfies (a), (c), and (d). We see that f will satisfy (b) for a large enough n.

(II) General case. We estimate Ω from inside; that is,

$$\Omega' = \bigcup_{i=0}^{m} \left[\frac{i}{m}, \frac{i+1}{m}\right] \times Q_i \subset \Omega \subset [0, 1] \times B,$$
$$Q_i \subset B, \qquad Q_i \text{ is as in (I).}$$

Then by (I) the lemma is true for Ω', and so it is enough to consider $\Omega' \to \Omega$, i.e. we take Ω' so that $\varphi_0([0, 1]) \subset \Omega' \subset B$. □

6. Proposition 4.2. *Let Q be an open subset of $\mathbf{R}^q$, let K be a compact space, and let F be the space of continuous maps from K to Q. If Q is arcwise connected, then the convex hull of F in $C^0(K, \mathbf{R}^q)$ agrees with the space of continuous maps from K to the convex hull* $\operatorname{Conv}(Q)$ *of Q in $\mathbf{R}^q$:* $\operatorname{Conv}(C^0(K, Q)) = C^0(K, \operatorname{Conv}(Q))$.

Proof. (i) We show $\operatorname{Conv}(C^0(K, Q)) \subset C^0(K, \operatorname{Conv}(Q))$. Suppose we have $f \in \operatorname{Conv}(C^0(K, Q))$. Then $f = pf_1 + qf_2$, $p + q = 1$, $0 \le p, q \le 1$, $f_1, f_2 \in C^0(K, Q)$. Here we have

$$f_1, f_2 : K \longrightarrow Q,$$
$$f : K \longrightarrow \mathbf{R}^q,$$

but for $x \in K$, $f(x) = pf_1(x) + qf_2(x) \in \operatorname{Conv}(Q)$, so $f \in C^0(K, \operatorname{Conv}(Q))$.

(ii) To show that $C^0(K, \operatorname{Conv}(Q)) \subset \operatorname{Conv}(C^0(K, Q))$, let $\varphi \in C^0(K, \operatorname{Conv}(Q))$. For the map $\varphi : K \to \operatorname{Conv}(Q)$ it suffices to construct a family of maps $\psi_\epsilon : K \times [0, 1] \longrightarrow Q$ such that as $\epsilon \to 0$

$$\int_0^1 \psi_\epsilon(k, t)\,dt \longrightarrow \varphi(k).$$

The argument for this goes as follows. Since K is compact, $B = C^0(K, \mathbf{R}^q)$ is a Banach space and $C^0(K, Q)$ is open in B. For the ψ_ϵ as above, define $\varphi_\epsilon : [0, 1] \longrightarrow C^0(K, Q) \subset B$ by

$$\varphi_\epsilon(t)(k) = \psi_\epsilon(k, t), \qquad k \in K, \ t \in [0, 1],$$

and define $\lambda_\epsilon : K \longrightarrow B$ by

$$\lambda_\epsilon(k) = \int_0^1 \psi_\epsilon(k, t)\,dt, \qquad \epsilon > 0, \ k \in K.$$

Then we get

$$I(\varphi_\epsilon)(k) = \lambda_\epsilon(k).$$

Hence, as $\epsilon \to 0$ we have $\lambda_\epsilon \to \varphi$ and so $I(\varphi_\epsilon) \to \varphi$. Thus, by Lemma 4.1 we have $\varphi \in \overline{I(\Gamma_0)}$ (for some γ_0). But

$$\varphi \in \overline{I(\Gamma_0)} = \overline{\operatorname{Conv}(\operatorname{Conn} C^0(K, Q), \varphi_0)} \subset \overline{\operatorname{Conv}(C^0(K, Q))}.$$

Hence, $C^0(K, \operatorname{Conv}(Q)) \subset \overline{\operatorname{Conv}(C^0(K, Q))}$, and so by Lemma 4.2 we get $C^0(K, \operatorname{Conv}(Q)) \subset \operatorname{Conv}(C^0(K, Q))$.

Now let us construct ψ_ϵ. First choose $x_1, \dots, x_n \in Q$ such that $\varphi(K) \subset \operatorname{Conv}\{x_1, \dots, x_n\}$. Then using a partition of unity, we construct continuous functions $p_1, \dots, p_n : K \longrightarrow [0, 1]$ satisfying:

$$p_i \ge 0, \qquad \sum_{i=1}^n p_i = 1, \qquad \sum_{i=1}^n p_i(k)x_i = \varphi(k), \qquad k \in K$$

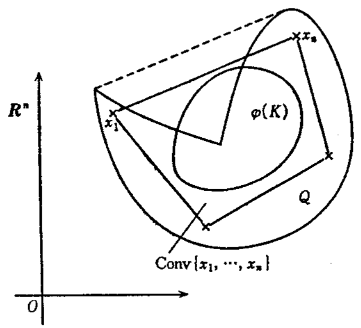


FIGURE 4.4

(visualize the situation with a two-dimensional picture, Figure 4.4).

We then choose a path $\gamma : [0, 1] \to Q$ such that $\gamma(i/(n+1)) = x_i$, $i = 1, \dots, n$, and put

$$\psi_\epsilon = \gamma(\theta_\epsilon^{P(k)}(t)),$$

where $P(k) = \{p_1(k), \dots, p_n(k)\}$ and θ is the function defined in §3.1. Now by following the proof of Lemma 4.1, (c) we can show that the $\{\psi_\epsilon\}$ is the desired family. □

7. Consider a q-dimensional vector bundle (Z, π, K). Assume K to be compact. We write $\pi^{-1}(k) = Z_k$. Let $\Omega \subset Z$ be open, and let $\varphi_0 : K \to Z$ be a cross section of (Z, π, K) such that $\varphi_0(K) \subset \Omega$. Put $\Omega_k = \Omega \cap Z_k$. In each fiber Z_k we consider the convex hull of the connected component $\mathrm{Conn}(\Omega_k, \varphi_0(k))$ of Ω_k containing $\varphi_0(k)$, which we shall denote by Ω_k^0. Set

$$\Omega^0 = \bigcup_{k \in K} \Omega_k^0 \subset Z.$$

Fix a closed subset K_0 of K for the moment, and let Γ_0 be the following space of sections of (Z, π, K):

$$\Gamma_0 = \{\varphi : K \to Z \mid \pi \circ \varphi = 1,\ \varphi|K_0 = \varphi_0|K_0,\ \varphi(K) \subset \Omega,\ \varphi \simeq \varphi_0 \ \mathrm{rel}\, K_0\}.$$

Then we have the following

LEMMA 4.4. *The convex hull of Γ_0 in the space of sections of (Z, π, K) agrees with*

$$\{\varphi : K \to \Omega^0 \mid \pi \circ \varphi = 1,\ \varphi|K_0 = \varphi_0|K_0\}.$$

PROOF. We show that our set is contained in the convex hull of Γ_0. Let $\varphi : K \to \Omega^0$ be a given section with $\varphi|K_0 = \varphi_0|K_0$. Just as in Proposition 4.2 in subsection 6 we start with sections $x_i : K \to \Omega$ homotopic to φ_0 relative to K_0 and continuous maps $p_i : K \to [0, 1]$, $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n p_i x_i = \varphi$. Choose a path $\gamma(t)$, $t \in [0, 1]$, in the space of cross sections $K \to \Omega$ which

agree with φ_0 on K_0 with the property $\gamma(i/(n+1)) = x_i$. Now define $\psi_\epsilon : K \times I \longrightarrow \Omega$ by $\psi_\epsilon(k, t) = \gamma(k, \theta_\epsilon^{P(k)}(t))$. Then letting $\epsilon \to 0$ we get

$$\int_0^1 \psi_\epsilon(k, t)\, dt \longrightarrow \varphi.$$

The reverse inclusion is obvious. □

§4. Proof of the fundamental theorem

In this section we prove the fundamental theorem of Gromov's convex integration theory.

1. Let V_0 be a compact C^∞ manifold and let $V = V_0 \times [0, 1]$. Let $Y = V \times \mathbf{R}^q \to V$ be the product bundle. Cover V with finitely many charts. We represent a chart by $(u_1, \dots, u_{n-1}, t)$, $t \in [0, 1]$.

For a C^1 map $f : V \to \mathbf{R}^q$, set

$$\|f\|^{\hat{n}} = \max_{\substack{v \in V \\ 1 \le i \le n-1}} \left\{ \|f(v)\|,\ \left\| \frac{\partial f}{\partial u_i}(v) \right\| \right\},$$

where max is the maximum with respect to all v in V, all charts about v, and $i = 1, \dots, n-1$. In other words $\|\ \|^{\hat{n}}$ differs from the C^1 norm in that one disregards $\partial f/\partial t$ for the former.

2. Main lemma.

LEMMA 4.5. *Let Q be an open subset of Y and consider*

$$Q = \bigcup_{v \in V} Q_v, \qquad Q_v = Q \cap (v \times \mathbf{R}^q) \subset \mathbf{R}^q.$$

Suppose that C^∞ maps $f_0, \varphi_0 : V \to \mathbf{R}^q$ satisfy the following:

(i) *The graph of φ_0 is contained in Q.*

(ii) $\left.\dfrac{\partial f_0}{\partial t}\right| \partial V = \varphi_0 | \partial V$.

(iii) $\mathrm{Conn}(Q_v, \varphi_0(v)) \subset \mathbf{R}^q$ *envelopes* $\dfrac{\partial f_0}{\partial t}(v)$.

Then for an arbitrary $\epsilon > 0$ there exists a C^1 map $f : V \to \mathbf{R}^q$ such that

(a) $f|\partial V = f_0|\partial V$,

$$\left.\frac{\partial f}{\partial u_i}\right| \partial V = \left.\frac{\partial f_0}{\partial u_i}\right| \partial V, \qquad i = 1, \dots, n-1,$$

$$\left.\frac{\partial f}{\partial t}\right| \partial V = \left.\frac{\partial f_0}{\partial t}\right| \partial V,$$

(b) $\|f - f_0\|^{\hat{n}} < \epsilon$,

(c) *the graph of $\partial f/\partial t$ is contained in Q, and*

(d) φ_0 *has a deformation* $\{\psi_\tau : \tau \in [0, 1]\}$ *which satisfies the following:*

(α) $\psi_0 = \varphi_0$, $\psi_1 = \dfrac{\partial f}{\partial t}$,

(β) $\psi_\tau|\partial V = \varphi_0|\partial V$, $\tau \in [0, 1]$,

(γ) *the graph of* ψ_τ *is contained in* Q, $\tau \in [0, 1]$.

PROOF. There is no loss of generality in assuming that $f_0(\partial V) = 0$ and $\varphi_0(\partial V) = 0$. (5) Let B be the space of all C^1 maps $g : V_0 \to \mathbf{R}^q$ such that

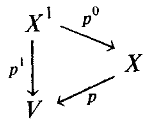
$$g(\partial V_0) = 0, \qquad \frac{\partial g}{\partial u_i}(\partial V_0) = 0, \qquad i = 1, \dots, n-1.$$

Then we may think of C^1 maps $V \to \mathbf{R}^q$ as continuous maps $[0, 1] \to B$. Let

$$\Omega = \bigcup_{t\in[0,1]} \Omega_t \subset [0, 1] \times B,$$

$\Omega_t = \{g : V_0 \to \mathbf{R}^q \mid g \text{ continuous, the graph of } g \subset Q \cap (V_0 \times \{t\} \times \mathbf{R}^q)\}$. Now the lemma follows from Lemma 4.4 about convex hulls together with One-dimensional Lemma. □

3. Singularities in cubes. Let V be an n-dimensional C^∞ manifold and let $p : X \to V$ be a C^∞ fiber bundle whose fiber is of dimension q. Let $p^1 : X^1 \to V$ be the fiber bundle of one-jets of germs of C^∞ sections in (X, p, V). Let $p^0 : X^1 \to X$ be the fiber bundle naturally defined by $p^1 : X^1 \to V$. In this case (X^1, p^0, X) is an affine bundle making the following diagram commute.



Now we assume that the base space V can be written as the product $V_0 \times V_1$, where V_1 is a one-dimensional manifold. Then the fiber bundle (X^1, p^0, X) has the direct sum decomposition $X^1 = X_0^1 \oplus X_1^1$, where $p_0 : X_0^1 \to X$ and $p_1 : X_1^1 \to X$ are affine bundles of dimensions $q(n-1)$ and q respectively.

Let $\pi_1 : X^1 \to X_0^1$ denote the natural projection. Then (X^1, π_1, X_0^1) is evidently an affine bundle whose fiber is of dimension q.

PROPOSITION 4.3. *Let V be the n dimensional cube I^n with coordinates $(u_1, \dots, u_n)$. Let $X = V \times \mathbf{R}^q \to V$ be the trivial bundle. Let Ω be open in*

(5) Actually we should define $B = \{g : V_0 \to \mathbf{R}^q | g(\partial V_0) = f_0(\partial V), \partial g/\partial u_i(\partial V_0) = \varphi_0(\partial V_0)\}$; however the proof will remain the same if we assume this.

X^1 and ample in each coordinate direction. Let $f_0 : V \to X$ and $\varphi_0 : V \to \Omega$ be C^∞ sections and assume $J^1(f_0) = \varphi_0$ on ∂I^n.

Then for any $\epsilon > 0$, there exists a C^1 section $f : V \to X$ which satisfies the following:

(a) *$J^1(f)$ and φ_0 agree on ∂I^n.*

(b) $\|f - f_0\| < \epsilon$.

(c) $J^1(f)(V) \subset \Omega$.

(d) *There exists a deformation $\{\psi_\tau | \tau \in [0, 1]\}$ of sections in (X^1, p^1, V) such that*

(α) $\psi_0 = \varphi_0$, $\psi_1 = J^1(f)$,

(β) $\psi_\tau(V) \subset \Omega$,

(γ) $\psi_\tau|\partial I^n = \varphi_0|\partial I^n$,

(δ) $|p^0 \circ \psi_\tau - f_0| \le \epsilon$.

PROOF. We might construct the desired section f as follows. In the ith step we consider the i th coordinate of $V = I^n$ and represent V as the product $V = V_0 \times [0, 1]$ with coordinates $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$; $u_i = t$, $i = 1, \dots, n$. Then for the fiber bundle $X = V \times \mathbf{R}^q \to V$ consider the decomposition of $X^1 = X_0^1 \oplus X_1^1$ as discussed above. The bundle $X \to V$ is identified with the subbundle of the bundle $X^1 \to X_0^1$, whose fibers intersect $\varphi_0(V) \subset X^1$. We then let $Q \subset V \times \mathbf{R}^q$ correspond to Ω under this identification. By the assumption on Ω, the set Q satisfies the hypothesis of the main lemma.

By the above method we first construct $f_1 \to X$, $\varphi_1 : V \to \Omega$ from the given f_0, φ_0. Next from f_1, φ_1, construct $f_2 : V \to X$, $\varphi_2 : V \to \Omega$, ..., and eventually we will come to f_n, φ_n. Then we have

$$\|f_i - f_{i-1}\|^i < \epsilon_i, \qquad i = 1, 2, \dots, n.$$

Further φ_i is obtained from φ_{i-1} through a deformation leaving $\pi_1 \circ \varphi_{i-1}$ unchanged (recall $\pi_1 : X^1 \to X_0^1$). So setting $f = f_0$ we have the desired section. □

4. Restatement of the assumptions of the fundamental theorem.

We may assume

$$X = \bigcup_j p^{-1}(I_j^n),$$

with the following.

(i) There exists a chart (U, φ) of V such that I_j^n is contained in U and $\varphi(I_j^n)$ is a cube in $\mathbf{R}^q$.

(ii) U is contained in some chart of (X, p, V).

By assumption Ω is ample with respect to each coordinate direction of I_j^n in the fiber Y_j.

5. Proof of the fundamental theorem. Now the fundamental theorem follows from Proposition 4.3. We perform cubic subdivisions on the C^∞ manifolds X and V as we do triangulation. We apply Proposition 4.3 to each cube in each skeleton to obtain the fundamental theorem. See § 6, Chapter 3, about C^∞ triangulations.

REMARK. We can do without cubic subdivisions in step 5. This is done by using the theorem in Gromov-Eliashberg, *Removal of singularities of smooth mappings*, Izv. Akad. Nauk SSSR. **35** (1971), 600-627.

CHAPTER V

Foliations of Open Manifolds

As an application of the Gromov-Phillips transversality theorem as discussed in Chater III we present in this chapter Haefliger's classification theorem for foliations of open manifolds, which brought the limelight back to the Smale-Hirsch theorem.

We will follow closely Haefliger's papers [B2, C10].

A detailed discussion of foliations is found in *The Topology of Foliations* [A9]. Research in the field of foliations continues to be active at present.

§1. Topological groupoids

In this section we define the classifying space B_Γ of a groupoid Γ.

DEFINITION 5.1. Suppose that a set M is divided into the cosets M_{ij}, i, $j = 1, 2, \ldots$ and satisfies the following:

(i) If $a \in M_{ij}$ and $b \in M_{jk}$, $ab \in M_{ik}$ is defined.

(ii) If $a \in M_{ij}$ and $b \in M_{ik}$, $a^{-1}b \in M_{jk}$ is defined and $a(a^{-1}b) = b$.

(iii) If $a \in M_{ij}$ and $b \in M_{kj}$, $ab^{-1} \in M_{ik}$ is defined and $(ab^{-1})b = a$.

(iv) For $a \in M_{ij}$, $b \in M_{jk}$, and $c \in M_{kl}$, we have $ab(c) = a(bc)$.

We then say that M is a *groupoid*.

DEFINITION 5.2. Suppose that a topological space Γ has a groupoid structure. We say that Γ is a *topological groupoid* if the groupoid operations (i), (ii), and (iii) are continuous.

EXAMPLE. A topological group G is a groupoid.

DEFINITION 5.3. Let X be a topological space. Let $\Gamma = \{(U_f, f, V_f)\}$ be a set of triples (U_f, f, V_f) of homeomorphisms f between open subsets U_f and V_f of X. We say that $\Gamma = \{(U_f, f, V_f)\}$ is a *pseudogroup* (of local homeomorphisms) of X if it satisfies the following:

(1) $(X, 1_X, X) \in \Gamma$.

(2) If $(U_f, f, V_f) \in \Gamma$ and U is open in U_f, then $(U, f|U, f(U)) \in \Gamma$.

(3) $(U_f, f, V_f) \in \Gamma$, $(U_g, g, V_g) \in \Gamma$, $V_f \subset U_g \Rightarrow (U_f, g \circ f, g \circ f(U_f)) \in \Gamma$.

(4) $(U_f, f, V_f) \in \Gamma \Rightarrow (V_f, f^{-1}, U_f) \in \Gamma$.

Similarly, we define a pseudogroup of local diffeomorphisms of a C^∞ manifold X.

Let B be a topological space, and let $\mathscr{G} = \{(U_f, f, V_f)\}$ be a pseudogroup of local homeomorphisms of B. Denote by Γ_b the set of germs of $f \in \mathscr{G}$ at b, and write $\Gamma = \bigcup_{b\in B}\Gamma_b$. We topologize Γ as follows (recall the sheaf topology): A basis for the open sets in Γ may consist of the sets $\{[f]_x | x \in U\}$, where each $[f]_x$ is the germ of f at $x \in U$, $(U, f, V) \in \mathscr{G}$, and U runs through open subsets of B. Then Γ becomes a topological groupoid; we say that Γ is the *topological groupoid* (or simply groupoid) *associated with* $\mathscr{G}$.

Denote by Γ_q the groupoid associated with the pseudogroup of local diffeomorphisms of $\mathbf{R}^q$ and by $\Gamma_q^{\mathbf{C}}$ the groupoid associated with the pseudogroup of analytic automorphisms of $\mathbf{C}^q$.

Assigning each element $[f]_x$ of Γ_q to the differential of f at x, we obtain a homomorphism of topological groupoids, $\nu : \Gamma_q \longrightarrow GL(q, \mathbf{R})$. We also have the verbatim definition of the homomorphism $\nu : \Gamma_q^{\mathbf{C}} \longrightarrow GL(q, \mathbf{C})$.

§2. Γ-structures

Let Γ be a topological groupoid. Suppose X is a topological space with an open covering $\mathscr{U} = \{U_i | i \in J\}$. We define a *one-cocycle* to be a system $\{U_i, \gamma_{ij} | i, j \in J\}$ of open sets and continuous maps with the property that to each pair i and j in J, there corresponds a continuous map

$$\gamma_{ij} : U_i \cap U_j \longrightarrow \Gamma$$

such that for each $x \in U_i \cap U_j \cap U_k$, $\gamma_{ik}(x) = \gamma_{ij}(x)\gamma_{jk}(x)$.

Two one-cocycles $\{U_i, \gamma_{ij} | i, j \in J\}$ and $\{U'_k, \gamma'_{k,l} | k, l \in K\}$ are *equivalent* if there is a family of continuous maps $\delta_{ik} : U_i \cap U'_k \longrightarrow \Gamma$ satisfying:

$$\delta_{ik}(x)\gamma'_{kl}(x) = \delta_{il}(x), \qquad x \in U_i \cap U'_k \cap U'_l,$$
$$\gamma_{ji}(x)\delta_{ik}(x) = \delta_{jk}(x), \qquad x \in U_i \cap U_j \cap U'_k.$$

This is obviously an equivalence relation.

An equivalence class of one-cocycles is a Γ-*structure* on X. We denote the set of Γ-structures on X by $\tilde{\Gamma}(X)$ or $H^1(X, \Gamma)$. When Γ is a topological group, each Γ-structure on X corresponds to an equivalence class of a principal G-bundle over X (cf. §2, Chapter I).

Let $f : Y \to X$ be a continuous map, and let $\sigma = \{U_i, \gamma_{ij} | i, j \in J\}$ be a Γ-structure on X. Then f induces a Γ-structure on Y;

$$f^*\sigma = \{f^{-1}(U_i), \gamma_{ij} \circ f | i, j \in J\}.$$

§3. Vector bundles associated with Γ_q-structures

Let Γ_q be the groupoid associated with the pseudogroup $\mathscr{G}$ of local diffeomorphisms of $\mathbf{R}^q$ as defined in §1.

The following is another interpretation of a Γ_q-structure on a topological space X. We consider the collection $\mathscr{C}$ of systems $\mathscr{F} = \{(U_\alpha, f_\alpha) | \alpha \in A\}$ of open sets and continuous maps satisfying:

(i) $\{U_\alpha | \alpha \in A\}$ is an open covering of X.

(ii) Each $f_\alpha : U_\alpha \to \mathbf{R}^q$ is continuous, $\alpha \in A$.

(iii) If $U_\alpha \cap U_\beta \neq \varnothing$, there exists a continuous map $f_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma_q$ such that for $x \in U_\alpha \cap U_\beta$, $f_\alpha(x) = f_{\alpha\beta} \cdot f_\beta(x)$.

We say families $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{C}$ are equivalent and write $\mathscr{F} \sim \mathscr{F}'$ if $\mathscr{F} \cup \mathscr{F}'$ belongs to $\mathscr{C}$. Evidently $\sim$ is an equivalence relation. An equivalence class under $\sim$ is nothing but a Γ_q-structure on X.

In §1 we introduced the homomorphism $\nu : \Gamma_q \longrightarrow GL(q, \mathbf{R})$ of groupoids by associating the germ of f at x to the differential of f at x. Hence, through the map ν we obtain a natural correspondence between Γ_q-structures on X and q-dimensional vector bundles over X (cf. §4, Chapter I). When the image $\nu(\Gamma_q)$ is contained in a subgroup G of $GL(q, \mathbf{R})$, the structural group of the corresponding vector bundle can be reduced to G.

§ 4. Homotopies of Γ-structures

Suppose we have two Γ-structures σ_0 and σ_1 on a topological space X.

DEFINITION 5.4. We say σ_0 and σ_1 are *homotopic* and write $\sigma_0 \simeq \sigma_1$, if there exists a Γ-structure σ on $X \times I$ such that $i_0^*\sigma = \sigma_0$, $i_1^*\sigma = \sigma_1$, where

$$i_0 : X \longrightarrow X \times 0 \subset X \times I,$$
$$i_1 : X \longrightarrow X \times 1 \subset X \times I$$

are inclusion maps. The relation $\simeq$ is an equivalence relation.

The set of all equivalence classes under homotopies of Γ-structures on X is denoted by $\Gamma(X)$, $\Gamma(X) = \tilde{\Gamma}(X)/\simeq$.

According to the definition of $\Gamma(X)$, Γ is a homotopy functor in the sense of Brown from the categoty of CW-complexes and continuous maps to the category of sets and maps. By the "representability theorem" of E. Brown, there exists a CW-complex $B\Gamma$ with a Γ-structure ω such that the functors $X \mapsto \Gamma(X)$ and $X \mapsto [X, B\Gamma]$ are equivalent (cf. E. Brown ([1])).

This $B\Gamma$ is difficult to understand in concrete terms and so in the following section we shall construct $B\Gamma$ according to Buffet and Lor (cf. Buffet and Lor ([2])).

§ 5. The construction of the classifying space for Γ-structures

Now we define the classifying space for Γ-structures.

Let Γ be a topological groupoid, and let B be the set of units of Γ. Let $\beta : \Gamma \to B$ be a map which sends each element γ of Γ to its left

([1]) *Abstract homotopy theory*, Trans. Amer. Math. Soc., **119** (1965), 79-85.

([2]) *Une construction d'un universal pour classe assez large de Γ-structures*, C.R. Acad. Sci. Paris, **270** (1970), 640-642.

unit $\beta(\gamma)$. We define $E\Gamma$ to be the set of equivalence classes of infinite sequences $(t_0, x_0, t_1, x_1, \dots, t_n, x_n, \dots)$,

$$\begin{cases} t_i \in [0, 1], & i = 1, 2, \dots, \\ t_i = 0 & \text{for all but finitely many } i, \\ \sum_i t_i = 1. \end{cases}$$

$$\begin{cases} x_i \in \Gamma, & i = 0, 1, \dots, \\ \beta(x_i) = \beta(x_j), & i, j = 1, 2, \dots, \end{cases}$$

as follows:

$$(t_0, x_0, t_1, x_1, \dots, t_n, x_n, \dots) \sim (t'_0, x'_0, t'_1, x_1{}', \dots,)$$

$$\sum \text{ iff } \begin{cases} t_i = t'_i, & i = 0, 1, \dots, \\ t_i \neq 0 & \text{implies } x_i = x'_i. \end{cases}$$

Henceforth we shall denote the equivalence class of $(t_0, x_0, t_1, x_1, \dots)$ by $(t_0x_0, t_1x_1, \dots)$.

We define maps $t_i : E\Gamma \longrightarrow [0, 1]$ by $(t_0x_0, t_1x_1, \dots) \longmapsto t_i$, and maps $x_i : t_i^{-1}(0, 1] \longrightarrow \Gamma$ by $(t_0x_0, t_1x_1, \dots) \longmapsto x_i$.

These maps are well defined by the above definition of equivalence classes. We give $E\Gamma$ the weakest topology making the t_i and the x_i continuous.

Consider the natural action of Γ on $E\Gamma$, and denote by $B\Gamma$ the quotient space of $E\Gamma$ by this action. More precisely we have the following:

$$B\Gamma = E\Gamma / \sim;$$

$$(t_0x_0, t_1x_1, \dots) \sim (t'_0x'_0, t'_1x'_1, \dots)$$

iff

(i) $t_i = t'_i$, $i = 0, 1, 2, \dots$.

(ii) $\beta(x_i) = \beta(x'_i)$, $i = 0, 1, 2, \dots$.

(iii) There exists $\gamma \in \Gamma$ such that for all i with $t_i \neq 0$ we have

$$x_i = \gamma x'_i.$$

Sometimes we simply write $(x, t) \sim (x', t')$ iff $(x, t) = (\gamma x', t)$. We give $B\Gamma$ the quotient topology and denote the natural projection by

$$p : E\Gamma \longrightarrow B\Gamma.$$

We call $B\Gamma$ the *classifying space for* Γ*-structures*

The projection $t_i : E\Gamma \to [0, 1]$ maps the equivalent elements under $\sim$ to the same element; hence, it defines the projection $u_i : B\Gamma \longrightarrow [0, 1]$, and we have $u_i \circ p = t_i$.

The classifying space $B\Gamma$ has a natural Γ-structure as follows: Set

$$V_i = u_i^{-1}(0, 1], \qquad i = 0, 1, 2, \dots,$$

$$\gamma_{ij} : V_i \cap V_j \longrightarrow \Gamma,$$

$$(t_0x_0, t_1x_1, \dots) \longmapsto x_i x_j^{-1}.$$

Then $\omega = \{ V_i, \gamma_{ij} | i, j = 0, 1, 2, \dots \}$ is a Γ-structure on $B\Gamma$, called *the universal* Γ*-structure.*

§ 6. Numerable Γ-structures

DEFINITION 5.5. An open covering $\{ U_j | j \in J \}$ of a topological space X is *numerable* if there exists a locally finite partition of unity $\{u_i | i \in I\}$ with $u_i^{-1}(0, 1] \subset U_i$.

Any open covering of a paracompact space is numerable.

DEFINITION 5.6. A Γ-structure $[\sigma]$ on X, $\sigma = \{ U_j, \gamma_{ij} | i, j \in J \}$, is *numerable* if its representative σ can be chosen so that the $\{U_j\}$ is numerable.

We say that two numerable Γ-structures are *numerably homotopic* if they are connected by a numerable homotopy.

PROPOSITION 5.1. *Let* $B\Gamma$ *be the classifying space and let* ω *be its* Γ*-structure as defined in* §5. *Then*

(a) ω *is numerable,*

(b) *for a numerable* Γ*-structure* σ *on a topological space* X, *there exists a continuous map* $f : X \to B\Gamma$ *such that* $f^*\omega = \sigma$, *and*

(c) *two continuous maps* f_0 *and* f_1 *of* X *to* $B\Gamma$ *are homotopic if and only if the* Γ*-structures* $f_0^*\omega$ *and* $f_1^*\omega$ *are numerably homotopic.*

PROOF. (a) We cite Milnor's proof verbatim (J. Milnor (3)). We construct a locally finite partition of unity $\{v_i\}$ on $B\Gamma$ as follows. Define a map $w_i : B\Gamma \longrightarrow [0, 1]$ by

$$w_i(b) = \max \left\{ 0, u_i(b) - \sum_{j<i} u_j(b) \right\},$$

then we have $w_i^{-1}(0, 1] \subset V_i$. For $b \in B\Gamma$, let m be the smallest i such that $u_i(b) \neq 0$. We then have

$$\sum_{m \leq i \leq n} u_i(b) = 1, \qquad u_m(b) = w_m(b),$$

and $B\Gamma$ has an open covering $B\Gamma = \bigcup_i w_i^{-1}(0, 1]$. Since $u_i(b) = 0$ for $n < i$, we have

$$\sum_{0 \leq i \leq n} u_i(b') > 1/2 \quad \text{implies} \quad w_i(b') = 0.$$

Therefore,

$$N_n(b) = \left\{ b' \middle| \sum_{0 \leq i \leq n} u_i(b') > 1/2 \right\}$$

(3) *Construction of universal bundles* II, Ann. of Math., **63** (1956), 430-436. One will also find the proof on p. 54 in the book of Husemoller, *Fiber bundles*, McGraw Hill, New York, 1966.

is a neighborhood of b, and $N_n(b) \cap w_i^{-1}(0, 1] = \emptyset$ if $n < i$. Hence, the open covering $\{w_i^{-1}(0, 1]\}$ of $B\Gamma$ is locally finite, and so if we set

$$v_i = \frac{w_i}{\sum_j w_j},$$

the $\{v_i\}$ is the desired partition of unity.

(b) We need the following

Lemma 5.1. *Let $\{U_i | \subset\in J\}$ be a numerable covering of X. Then there exists a locally finite countable partition of unity $\{t_n | n \in N\}$ such that each open set $V_n = t_n^{-1}(0, 1]$ is a union of mutually disjoint open subsets V_{ni} of U_i.*

Again the verbatim proof is in the book *Fiber bundles* by D. Husemoller.

Proof. Let $\{v_i | i \in T\}$ be a locally finite patrtion of unity on X with $v_i^{-1}(0, 1] \subset U_i$. Then for each $b \in X$, the set $S(b) = \{i \in T | v_i(b) > 0\}$ is finite. Further, for each finite subset S of T,

$$W(S) = \{b \in X | v_i(b) > v_j(b), \ \forall i \in S, \forall j \in T - S\}$$

is an open subset of X. The map $u_S : B \to [0, 1]$ defined by

$$u_S(b) = \max\left\{0, \min_{i\in S, j \in T-S}(v_i(b) - v_j(b))\right\}$$

is continuous, and we have $W(S) = u_S^{-1}(0, 1]$.

Let $\operatorname{Card} S$ denote the cardinality of S. We claim that if $\operatorname{Card} S = \operatorname{Card} S'$ and $S \neq S'$, then $W(S) \cap W(S') = \emptyset$. Let $i \in S - S'$, $j \in S' - S$. Then $v_i(b) > v_j(b)$ for $b \in W(S)$, and $v_j(b) > v_i(b)$ for $b \in W(S')$. But these two relations cannot occur at the same time, so the claim is true. Now put

$$W_m = \bigcup_{\operatorname{Card} S = m} W(S), \qquad w_m(b) = \sum_{\operatorname{Card} S = m} u_S(b).$$

Then $w_m^{-1}(0, 1] = W_m$. Set

$$t_m(b) = \frac{w_m(b)}{\sum_{n\geq 0} w_n(b)}.$$

Then $t_n^{-1}(0, 1] = W_n$; so the $\{t_n\}$ is the desired partition of unity. □

To complete the proof of (b), let σ be a numerable Γ-structure on X. By Lemma 5.1 we may assume that σ is defined by one-cocycles γ_{nm} on a countable open covering $\{U_i | n = 0, 1, \ldots\}$. Here $U_n = t_n^{-1}(0, 1]$, and the $\{t_n\}$ is a partion of unity. The desired map $f : X \to B\Gamma$ may be defined by

$$f(x) = [(t_0(x)\gamma_{m0}(x), t_1(x)\gamma_{m1}(x), \ldots)], \qquad x \in U_m,$$

where $[(t_0(x)\gamma_{m0}(x), \ldots)]$ denotes the equivalence class of $(t_0(x)\gamma_{m0}(x), \ldots)$.

Proof of (c). Denote by $B\Gamma^{\text{od}}$ a subset of $B\Gamma$, which is the image under $p : E\Gamma \to B\Gamma$ of the points $(t_0x_0, t_1x_1, \ldots)$ in $E\Gamma$ with $t_n = 0$ for n odd. Similarly we define $B\Gamma^{\text{ev}}$ (replacing 'odd' by 'even' in the above notation). We define h^{od}, $h^{\text{ev}} : B\Gamma \to B\Gamma$ by

$$h^{\text{od}}[(t_0x_0, t_1x_1, \ldots)] = [(t_0x_0, 0, t_1x_1, 0, \ldots)],$$
$$h^{\text{ev}}[(t_0x_0, t_1x_1, \ldots)] = [(0, t_0x_0, 0, t_1x_1, 0, \ldots)].$$

Lemma 5.2. *The maps h^{od} and h^{ev} are homotopic to the identity map. Further, we have $(h^{\text{od}})^*\omega = (h^{\text{ev}})^*\omega = \omega$.*

Proof. Define a linear function $\alpha_n : [1 - (1/2)^n, 1 - (1/2)^{n+1}] \longrightarrow [0, 1]$ by $\alpha_n(t) = 2^{n+1}t - 2^{n+1} + 2$. Clearly we have

$$\alpha_n(1 - (1/2)^n) = 0, \qquad \alpha_n(1 - (1/2)^{n+1}) = 1.$$

Define a homotopy $h_s^{\text{od}} : E\Gamma \to E\Gamma$ as follows:

$$\begin{cases} h_s^{\text{od}}(x, t) = (t_0x_0, \ldots, t_nx_n, \alpha_n(s)t_{n+1}x_{n+1}, (1 - \alpha_n(s))t_{n+1}x_{n+1}, \\ \qquad \alpha_n(s)t_{n+2}x_{n+2}, (1 - \alpha_n(s))t_{n+2}x_{n+2}, \ldots), \\ \qquad 1 - (1/2)^n \leq s \leq 1 - (1/2)^{n+1}, \\ h_1^{\text{od}}(x, t) = (x, t). \end{cases}$$

Then we have

$$h_s^{\text{od}}(x, t)y = h_s^{\text{od}}(xy, t).$$

The maps h_s^{od} are continuous because they are continuous on a locally finite open covering of $v_i^{-1}(0, 1]$, where the $\{v_i\}$ is a partition of unity. The corresponding maps $g_s^{\text{od}} : B\Gamma \to B\Gamma$ of the base space define a homotopy connecting h^{od} and the identity map. The proof for h^{ev} follows verbatim the case for h^{od}. □

Now back to the proof of (c). Suppose that $f_0, f_1 : X \to B\Gamma$ are continuous and that the Γ-structures $f_0^*\omega$ and $f_1^*\omega$ are numerably homotopic. We want to show that in this case f_0 and f_1 are homotopic. By (b) we may assume $f_0^*\omega = f_1^*\omega$. By Lemma 5.2 we may assume $f_0(X) \subset B\Gamma^{\text{od}}$ and $f_1(X) \subset B\Gamma^{\text{ev}}$. If i is odd, the set $f_0^{-1}(V_i) = f_0^{-1}(u_i^{-1}(0, 1])$ is empty. Therefore, we may consider the pullback by f_0 of the cocycle defining ω as defined on the covering

$$\{U_i | i \text{ is an even natural number}, U_i = f_0^{-1}(V_i)\}.$$

Similarly, $f_1^*\omega$ is defined by a cocycle on the covering

$$\{U_i | i \text{ is an odd natural number}, U_i = f_1^{-1}(V_i)\}.$$

Let $s_i : V_i \to E\Gamma$ be the continuous map defined by

$$s_i[t_0x_0, t_1x_1, \ldots] = (t_0x_i^{-1}x_0, t_1x_i^{-1}x_1, \ldots);$$

then we can write

$$s_i \circ f_0(x) = (t_0(x)\gamma_{0i}(x), 0, t_2(x)\gamma_{2i}(x), 0, \ldots), \qquad x \in U_i, \ i \text{ even},$$
$$s_i \circ f_1(x) = (0, t_1(x)\gamma_{1i}(x), 0, t_3(x)\gamma_{3i}(x), \ldots), \qquad x \in U_i, \ i \text{ odd}.$$

Since $f_0^*\omega = f_1^*\omega$, if $i \not\equiv j (\mathrm{mod}\ 2)$ there exists a family of continuous maps $\gamma_{ij} : U_i \cap U_j \to \Gamma$ such that the $\{\gamma_{ij} | i, j \geq 0\}$ is a cocycle on the $\{U_i | i \geq 0\}$.

Now a homotopy connecting f_0 and f_1 may be given by the following maps:

$$s_i f_s(x) = [(1-s)t_0(x)\gamma_{0i}(x), st_1(x)\gamma_{1i}(x), (1-s)t_2(x)\gamma_{2i}(x), st_3(x)\gamma_{3i}(x), \ldots], \qquad x \in U_i.$$

The proof of Proposition 1 is now complete. □

§7. Γ-foliations

Let Γ_q^r be the topological pseudogroup of germs of local C^r diffeomorphisms of $\mathbf{R}^q$, where r is one of $0, 1, 2, \ldots, \infty, \omega$. Let Γ be an open pseudosubgroup of Γ_q^r.

DEFINITION 5.7. Let X be a C^r manifold. By a *Γ-foliation* of X we mean a Γ-structure on X represented by $\mathscr{F} = \{(U_i, f_i), \gamma_{ij} | i, j \in J\}$, such that each $f_i : U_i \to \mathbf{R}^q$ is a C^r submersion.

In other words, a Γ-foliation $\mathscr{F}$ is a system $\{(U_i, f_i) | i \in J\}$ of open sets and C^r maps satisfying:

(i) $\{U_i | i \in J\}$ is an open covering of X.
(ii) $f_i : U_i \to \mathbf{R}^q$ is a C^r submersion.
(iii) If $U_i \cap U_j \neq \varnothing$, then for every point x of $U_i \cap U_j$, there exists a C^r diffeomorphism g_{ji} of some neighborhood of $f_i(x)$ onto the corresponding neighborhood of $f_j(x)$ such that
  (a) the germ of g_{ji} at $f_i(x)$ belongs to Γ,
  (b) $f_j = g_{ji} \circ f_i$ in some neighborhood of x.

DEFINITION 5.8. Let X be a C^r manifold and let $\mathscr{F}_0$, $\mathscr{F}_1$ be Γ-foliations of X. We say that $\mathscr{F}_0$ and $\mathscr{F}_1$ are *integrably homotopic* and write $\mathscr{F}_0 \underset{i}{\simeq} \mathscr{F}_1$ if there exists a Γ-foliation $\mathscr{F}$ of $X \times [0, 1]$ such that the maps $i_t : X \longrightarrow X \times [0, 1]$ defined by $i_t(x) = (x, t)$ satisfy the following:

(i) $i_0^*\mathscr{F} \sim \mathscr{F}_0$, $\quad i_1^*\mathscr{F} \sim \mathscr{F}_1$.
(ii) For each $t \in [0, 1]$, i_t is transverse to $\mathscr{F}$.

Here the map $i_t : X \to X \times [0, 1]$ is *transverse* to $\mathscr{F}$ if for any point x of X, the composite

$$\pi \circ (df)_x : T_x(X) \xrightarrow{(df)_x} T_{f(x)}(X \times [0, 1]) \xrightarrow{\pi} T_{f(x)}(X \times [0, 1])/T_{f(x)}(L_{f(x)})$$

is a surjection, where $L_{f(x)}$ is the leaf of $\mathscr{F}$ passing through $f(x)$, and π is the natural projection.

Evidently the relation $\underset{i}{\simeq}$ is an equivalence relation.

If X is a closed manifold, the Γ-foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ of X are integrably homotopic if and only if there exists a C^r diffeomorphism f of X such that

(i) $f^*\mathscr{F}_1 \sim \mathscr{F}_0$,
(ii) f is isotopic to 1_X.

We omit the proof (cf. Tamura [A9]).

§ 8. The graphs of Γ-structures

Let Γ be a subset of Γ_q.

DEFINITION 5.9. By a q-dimensional *Γ-foliated microbundle* over X we mean an ordered triple $(E, \xi, \mathscr{E})$ such that

(i) ξ consists of topological spaces and continuous maps,

$$\xi : X \xrightarrow{i} E \xrightarrow{p} X, \qquad p \circ i = 1,$$

(ii) $\mathscr{E}$ is a Γ-structure on E, $\mathscr{E} = \{(U_\alpha, f_\alpha) | \alpha \in A\}$, such that for each $\alpha \in A$, the map $(p|U_\alpha) \times f_\alpha : U_\alpha \longrightarrow X \times \mathbf{R}^q$ is a homeomorphism of U_α onto the corresponding open subset of $X \times \mathbf{R}^q$.

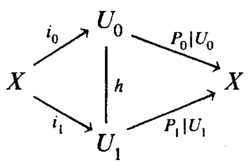
For a C^r manifold X, we may take the total space E to be a C^r manifold and the maps i and p to be of class C^r. Further, $\mathscr{E}$ may be thought of as a Γ-foliation.

PROPOSITION 5.2. *Let X be locally compact and paracompact. Then for a Γ-structure $\mathscr{F}$ on X, there exists a q dimensional Γ-foliated microbundle $(E, \xi, \mathscr{E})$,*

$$\xi : X \xrightarrow{i} E \xrightarrow{p} X$$

such that $\mathscr{F} \sim i^*\mathscr{E}$. Further the germ of this microbundle $(E, \xi, \mathscr{E})$ is unique up to isomorphisms. In other words, if we have two such microbundles $(E_0, \xi_0, \mathscr{E}_0)$ and $(E_1, \xi_1, \mathscr{E}_1)$, there exist neighborhoods U_0 and U_1 of $i_0(X)$ and $i_1(X)$ together with a homeomorphism $h : U_0 \to U_1$ such that

(i) *the diagram*



commutes, and
(ii) $h^*\mathscr{E}_1 \sim \mathscr{E}_0|U_0$.

We call the microbundle $(E, \xi, \mathscr{E})$ as constructed above the *graph* of a Γ-structure $\mathscr{F}$. This idea was formulated by Haefliger.

PROOF OF PROPOSITION 5.2. Suppose we have a Γ-structure $\mathscr{F}$ represented by $\mathscr{F} = \{(U_\alpha, \varphi_\alpha), g_{\alpha\beta} | \alpha, \beta \in A\}$. Consider the graph $G(\varphi_\alpha) \subset$

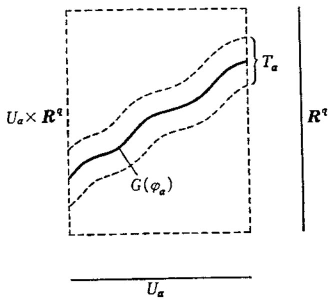


FIGURE 5.1

$U_\alpha \times \mathbf{R}^q$ of $\varphi_\alpha : U_\alpha \to \mathbf{R}^q$. Choose a neighborhood T_α of $G(\varphi_\alpha)$ in $U_\alpha \times \mathbf{R}^q$ (Figure 5.1). We construct E by pasting T_α together

$$E = \bigcup_\alpha T_\alpha / \sim : U_\alpha \cap U_\beta \neq \varnothing, \ (x, y) \in T_\alpha, \ (x', y') \in T_\beta,$$

$$(x, y) \sim (x', y') \quad \text{iff} \quad \begin{cases} x = x', \\ y = g_{\alpha\beta}(x) y'. \end{cases}$$

Define $i : X \to E$ by $i(x) = [(x, \varphi_\alpha(x))]$, $x \in U_\alpha$. The construction of E implies the map i is well defined. Define $p : E \to X$ by $p(x, y) = x$. Notice that $p \circ i = 1$. To define a Γ-structure on E, set $W_\alpha = \pi(T_\alpha)$, where $\pi : \bigcup_\alpha T_\alpha \to E$ is a natural projection. Define $\phi_\alpha : W_\alpha \to \mathbf{R}^q$ by $\phi_\alpha(x, y) = y$. Then the system $\mathscr{E} = \{(W_\alpha, \phi_\alpha) | \alpha \in A\}$ is a Γ-structure on E, and it is a routine to see $i^*\mathscr{E} \sim \mathscr{F}$. □

§9. The Gromov-Phillips transversality theorem

We mentioned in Chapter III that the Gromov-Phillips transversality theorem is a major tool for the classification of Γ-foliations of open manifolds.

Let X be a C^r manifold and let $\mathscr{F} = \{(U_i, f_i), g_{ij} | i, j \in J\}$ be a Γ-foliation of M.

Denote by $\nu\mathscr{F}$ the normal bundle of $\mathscr{F}$; that is, if $f : M \to B\Gamma$ is the classifying map of $\mathscr{F}$, $\nu : B\Gamma \to BGL(q, \mathbf{R})$ is the map as defined in § 1, and γ_q is the universal q-vector bundle, then $\nu\mathscr{F} \sim (\nu \circ f)^*\gamma_q$. Another interpretation goes as follows. $\nu\mathscr{F}|U_i = f_i^*(T(\mathbf{R}^q))$ and if $U_i \cap U_j \neq \varnothing$, $U_i \times \mathbf{R}^q$ and $U_j \times \mathbf{R}^q$ are pasted by dg_{ij} along $U_i \cap U_j$: $(x, y) \sim (x', y')$ iff $x' = x$, $y' = (dg_{ij})_x(y)$.

Let X and M be C^r manifolds, $r \geq 1$. Let $\mathscr{F}$ be a Γ-foliation of M, $\mathscr{F} = \{(U_i, f_i) | i \in J\}$.

DEFINITION 5.10. We say that a C^r map $f : X \to M$ is *transverse* to the Γ-foliation $\mathscr{F}$ if for each $i \in J$ the composite $f_i \circ f : f^{-1}(U_i) \to \mathbf{R}^q$ is a submersion.

Denote by $\mathrm{Tr}(X, \mathscr{F})$ the set of C^r maps from X to M which are transverse to $\mathscr{F}$. We give $\mathrm{Tr}(X, \mathscr{F})$ the C^1 topology. Let $\tau(X)$ be the tangent bundle of X, and $\nu\mathscr{F}$ the normal bundle of $\mathscr{F}$. Denote by $\mathrm{Epi}(\tau(X), \nu\mathscr{F})$ be the set of bundle maps of $\tau(X)$ into $\nu\mathscr{F}$ whose restrictions to each fiber are each surjective (such a map is called an *epimorphism* of vector bundles). We give $\mathrm{Epi}(\tau(X), \nu\mathscr{F})$ the compact-open topology. Let $\tau(M)$ be the tangent bundle of M and let $\pi : \tau(M) \to \nu\mathscr{F}$ be the natural epimorphism.

THEOREM 5.1 (Gromov and Phillips). *Let X be an open C^r manifold, $r = 1, 2, \ldots, \infty, \omega$. Let M be a C^r manifold and let $\mathscr{F}$ be a Γ-foliation of M. Then the map*

$$\Phi : \mathrm{Tr}(X, \mathscr{F}) \longrightarrow \mathrm{Epi}(\tau(X), \nu\mathscr{F})$$

defined by $\Phi(f) = \pi(df)$ is a weak homotopy equivalence. □

Theorem 5.1 follows routinely from Theorem 3.13 (the Gromov-Phillips theorem). As we remarked earlier the assumption that X is open is essential.

§10. The classification theorem for Γ-foliations of open manifolds

Let ω be the universal Γ-structure of the classifing space $B\Gamma$ for Γ-structures. Suppose $\Gamma \subset \Gamma_q^r$, $r \geq 1$. Let $\nu\omega$ be the q dimensional vector bundle associated with ω.

THEOREM 5.2 (the classification theorem for foliations of open manifolds). *Let X be an open C^r manifold, $r = 1, 2, \ldots, \infty, \omega$. Then there exists a one-to-one correspondence between integrable homotopy classes of Γ-foliations of X and homotopy classes of the epimorphisms of $\tau(X)$ onto the normal bundle $\nu\omega$ of ω.*

PROOF. The desired correspondence may be given as follows. Let $\mathscr{F}$ be a Γ-foliation of X. Let $\pi : \tau(X) \to \nu\mathscr{F}$ be a natural epimorphism. A priori, the foliation $\mathscr{F}$ is a Γ-structure, and so $\mathscr{F}$ is induced from the universal Γ-structure ω: $f^*\omega \sim \mathscr{F}$ by some continuous map $f : X \to B\Gamma$. This defines a bundle map $\varphi : \nu\mathscr{F} \longrightarrow \nu\omega$, and the map $\Psi = \varphi \circ \pi : \tau(X) \to \nu\omega$ is a bundle epimorphism. We assign to the integrable homotopy class of $\mathscr{F}$ the homotoppy class of Ψ. This correspondence is well defined. We want to show that this is a surjection.

Let $\psi : \tau(X) \to \nu\omega$ be an epimorphism of vector bundles, and let $\bar{\psi} = f : X \to B\Gamma$ be the corresponding map of the base spaces. Let $\sigma = f^*\omega$; σ is a Γ-structure on X. Let $(E, \xi, \mathscr{E})$ be the Γ-foliated microbundle over X corresponding to σ, $\xi : X \xrightarrow{i} E \xrightarrow{p} X$, where E is a C^r manifold, $\mathscr{E}$ is a Γ-foliation of E, and $i^*\mathscr{E} \sim \sigma$. Then $i^*\nu\mathscr{E} \sim f^*\nu\omega$. Here $\nu\mathscr{E}$ is the normal bundle of $\mathscr{E}$. Hence, the given epimorphism ψ defines an epimorphism $\phi_1 : \tau(X) \to i^*\nu\mathscr{E}$ and hence an epimorphism $\phi : \tau(X) \to \nu\mathscr{E}$,

$$\begin{array}{ccccccc} \phi : \tau(X) & \xrightarrow{\phi_1} & i^*\nu\mathscr{E} & \xrightarrow{\phi_2} & \nu\mathscr{E}, & & \phi = \phi_2 \circ \phi_1 \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \xrightarrow{1} & X & \xrightarrow{i} & E. & & \end{array}$$

Therefore, by the Gromov-Phillips transversality theorem there exists a C^r map $j : X \to E$ homotopic to $i : X \to E$ such that j is transverse to $\mathscr{E}$ and $\pi \circ dj : \tau(X) \to \nu\mathscr{E}$ is homotopic to ϕ. Thus, $\mathscr{F} = j^*\mathscr{E}$ is the desired Γ-foliation, $\{\mathscr{F}\} \mapsto \{\psi\}$. Similarly we can show that the correspondence is injective. □

We restate the above classification theorem in such a way that one can actually calculate. Recall $\Gamma \subset \Gamma_q^r$. Also recall the differential maps

$$\nu : \Gamma_q \longrightarrow \mathrm{GL}(q, \mathbf{R})$$
$$\nu : B\Gamma_q \longrightarrow B\mathrm{GL}(q, \mathbf{R}).$$

Now assume that $\nu(\Gamma) \subset G$ for some subset of $\mathrm{GL}(q, \mathbf{R})$. Then we have the induced map $\nu : B\Gamma \longrightarrow BG$ and the following commutative diagram

$$\begin{array}{lll} B\Gamma \times B\,\mathrm{GL}(n-q, \mathbf{R}) & & \\ \quad\big| & \searrow^{\nu \times 1} & \\ \quad\big| & & BG \times B\,\mathrm{GL}(n-q, \mathbf{R}) \\ \sigma\big| & & \quad\downarrow \rho \times 1 \\ \quad\big| & & B\,\mathrm{GL}(q, \mathbf{R}) \times B\,\mathrm{GL}(n-q, \mathbf{R}) \\ \quad\downarrow & \swarrow^{\oplus} & \\ B\,\mathrm{GL}(n, \mathbf{R}) & & \end{array}$$

where $\oplus$ represents the Whitney sum and ρ is the induced map of $G \subset \mathrm{GL}(q, \mathbf{R})$.

COROLLARY 5.1. *Let X be an open C^r manifold of dimension n, $r \geq 1$, and let $\tau : X \to B\,\mathrm{GL}(n, \mathbf{R})$ be the classifying map for the tangent bundle $\tau(X)$ of X. Then there is a one-to-one correspondence between integrable homotopy classes of Γ-foliations of X and homotopy classes of lifts of τ in $B\Gamma \times B\,\mathrm{GL}(n-q, \mathbf{R})$:*

$$\begin{array}{ccc} & & B\Gamma \times B\mathrm{GL}(n-q, \mathbf{R}) \\ & \nearrow & \downarrow \sigma \\ X & \xrightarrow{\tau} & B\mathrm{GL}. \end{array}$$

Here by a lift of τ in $B\Gamma \times B\,\mathrm{GL}(n-q, \mathbf{R})$ we mean a coninuous map $X \to B\Gamma \times B\,\mathrm{GL}(n-q, \mathbf{R})$ making the above diagram commute.

PROOF. There is a one-to-one correspondence between homotopy classes of lifts of τ in $B\Gamma \times B\,\mathrm{GL}(n-q, \mathbf{R})$ and the triples (f, η, ϕ) consisting of

$f : X \longrightarrow B\Gamma$, a continuous map,
η, an $(n-q)$ dimensional vector bundle over X,
$\phi : f^*\nu\omega \oplus \eta \longrightarrow \tau(X)$, an isomorphism of vector bundles.

But there is a one-to-one correspondence between the set of triples as specified above and the set of homotopy classes of epimorphisms of $\tau(X)$ onto $\nu\omega$. Hence, the corollary follows from Theorem 5.2. □

Because of Corollary 5.1, it becomes necessary to investigate topological properties of the classifying space $B\Gamma$ in order to classify Γ-foliations of open manifolds. See Bott [4] concerning this subject.

NOTE. Thurston showed that a similar classification holds for the foliations of closed manifolds [5].

([4]) R. Bott, *Lectures on characteristic classes and foliations*, Lecture Notes in Math., Vol. **279** (1972), Springer-Verlag, Berlin and New York, pp. 1-94.

([5]) W. Thurston, *The theory of foliations of codimension greater than one*, Comment. Math. Helv. **49** (1974) 214-231; *Existence of codimension one foliations*, Ann. of Math. **104** (1976), 249-268.

CHAPTER VI

Complex Structures on Open Manifolds

In this chapter we discuss complex structures of open manifolds as applications of the Gromov-Phillips theorem of Chapter III and the Gromov convex integration theory of Chapter IV.

§1. Almost complex structures and complex structures

Let X be a $2q$-dimensional C^∞ manifold. We may think of the structural group of the tangent bundle $\tau(X)$ of X, as the orthogonal group $O(2q)$. We may regard the n-dimensional unitary group $U(q)$ as a subgroup of $O(2q)$ as follows:

$$\begin{aligned} &\rho : U(q) \longrightarrow O(2q), \\ &A = (a_{ij}) \in U(n), \\ &a_{ij} = b_{ij} + \sqrt{-1}c_{ij}, \qquad b_{ij},\ c_{ij} \in \mathbf{R}, \\ &\rho(A) = \begin{pmatrix} B & C \\ -C & B \end{pmatrix}, \qquad B = (b_{ij}),\ C = (c_{ij}). \end{aligned}$$

DEFINITION 6.1. By an *almost complex structure* on a $2q$-dimensional C^∞ manifold X we mean a reduction of the structural group $O(2q)$ of the tangent bundle $\tau(X)$ to $U(n)$. The manifold X together with an almost complex structure is called an *almost complex manifold.*

REMARK 1. Here is another way to define an almost complex structure. Consider the following diagram:

$$\begin{array}{ccc} & & BU(q) \\ & \overset{\tilde{\tau}}{\nearrow} & \downarrow \rho \\ X & \xrightarrow{\ \tau\ } & BO(2q) \end{array}$$

where $BO(2q)$ and $BU(q)$ denote the classifying spaces of compact Lie groups $O(2q)$ and $U(q)$, respectively, and τ is the classifying map of the tangent bundle $\tau(X)$. Further, ρ is the continuous map naturally induced by the standard homomorphism $\rho : U(q) \to O(2q)$. Then an almost complex structure on X is a lift of τ to $BU(q)$ in the above diagram, that is a continuous map $\tilde{\tau} : X \to BU(q)$ such that $\rho \circ \tilde{\tau} = \tau$. This definition an easy consequence of the classification theorem for fiber bundles (cf. ıapter I).

REMARK 2. Another interpretation. A vector bundle homomorphism J : $X) \to \tau(X)$ with $J^2 = -1$ of the tangent bundle $\tau(X)$ of a $2q$-dimensional anifold X is an almost complex structure on X. The works by W. Wu 23], A. Borel, F. Hirzebruch, T. Heaps, and so on include problems conrning the existence as well as the classification of almost complex structures ı $2q$-manifolds.

DEFINITION 6.2. Let X be a $2q$-dimensional topological space. Suppose $J_\lambda \mid \lambda \in \Lambda\}$ is an open cover of X and $\{\varphi_\lambda : U_\lambda \to \mathbf{C}^q \mid \lambda \in \Lambda\}$ is the family homeomorphisms between the U_λ and the corresponding open subsets of such that for $U_\lambda \cap U_\mu \neq \varnothing$,

$$\varphi_\lambda \circ \varphi_\mu^{-1} : \varphi_\mu(U_\lambda \cap U_\mu) \longrightarrow \varphi_\lambda(U_\lambda \cap U_\mu)$$

holomorphic. Then the family $\mathscr{C} = \{(U_\lambda, \varphi_\lambda) \mid \lambda \in \Lambda\}$ is called a *complex ructure* of X. The pair $(X, \mathscr{C})$ is a *complex manifold.*

A complex structure $\mathscr{C}$ on X defines naturally an almost complex strucre on X, which is called the *underlying almost complex structure of* $\mathscr{C}$.

DEFINITION 6.3. We say that an almost complex structure σ of a $2q$-mensional manifold X is *integrable* if it is the underlying almost complex ructure of some complex structure on X.

The question 'Is the given almost complex structure integrable?' was instigated in the case of open manifolds by Grauert, Brender, etc. mainly rough functional analytic methods; however, recently P. Landweber made uch progress in this area using a geometric approach via the Gromov-aillips theorem stated in Chapter III. M. Adachi also made some contriıtions towards the solution of this problem using the convex integration ıeory of Chapter IV. We shall discuss these in the following sections.

§2. Complex structures on open manifolds

Let X be a q-dimensional C^∞ manifold.

DEFINITION 6.4. Two almost complex structures σ_0, σ_1 on X are *homoıpic* if they are homotopic in the sense of Remark 1 or Remark 2, and in ıis case we write $\sigma_0 \simeq \sigma_1$.

Evidently $\simeq$ is an equivalence relation; we call an equivalence class by $\simeq$ *homotopy class.*

DEFINITION 6.5. Suppose that X has two complex structures $\mathscr{C}_0$ and $\mathscr{C}_1$ hich induce the integrable almost complex structures σ_0 and σ_1, respecvely. We say that $\mathscr{C}_0$ and $\mathscr{C}_1$ are *integrably homotopic* if σ_0 and σ_1 are omotopic, and if $\{\sigma_t \mid t \in [0, 1]\}$ is the homotopy connecting them, then ər each $t \in [0, 1]$ σ_t is integrable.

This is an equivalence relation.

THEOREM 6.1. *Let X be a $2q$-dimensional open manifold. If*

$$H^i(X, \mathbf{Z}) = 0, \qquad i > q+1,$$

then every almost complex structure of X is homotopic to an integrable almost complex structure.

THEOREM 6.2. *Let X be a $2q$-dimensional open manifold. If*

$$H^i(X, \mathbf{Z}) = 0, \qquad i \geq q+1,$$

then there is a natural one-to-one correspondence between integrable homotopy classes of complex structures of X and homotopy classes of almost complex structures of X.

Let P be a polyhedron and let ϵ_P^n be the trivial n-vector bundle over P. Let ξ_0, ξ_1 be vector bundles over P. We say that ξ_0 and ξ_1 are *stably equivalent* and write $\xi_0 \underset{s}{\sim} \xi_1$ if there exist natural numbers r, s, such that

$$\xi_0 \oplus \epsilon_P^r \sim \xi_1 \oplus \epsilon_P^s.$$

Clearly $\underset{s}{\sim}$ is an equivalence relation. We let $[\xi]$ denote the equivalence class containing ξ.

We consider the inductive limits of the following classifying spaces:

$$B_{\mathrm{O}} = \varinjlim B_{\mathrm{O}(n)}, \qquad B_{\mathrm{U}} = \varinjlim B_{\mathrm{U}(n)}.$$

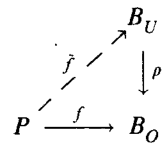
To a homotopy class $[\xi]$ corresponds the homotopy class of some continuous map

$$f : P \longrightarrow B_{\mathrm{O}}.$$

On the other hand, we can define a continuous map

$$\rho : B_{\mathrm{U}} \longrightarrow B_{\mathrm{O}}$$

as the inductive limit of $\rho : B_{\mathrm{U}(n)} \to B_{\mathrm{O}(2n)}$. Thus, we call the lift $\tilde{f}$ of f in the following diagram the *complex structure* of a stably equivalent class $[\xi]$.



THEOREM 6.3. *Let $n > 2$. Let M^n be an n-dimensional C^∞ manifold and assume that the stably equivalence class $[\tau(M^n)]$ of the tangent bundle $\tau(M^n)$ admits a complex structure. Then $M^n \times \mathbf{R}^{n-2}$ admits a complex structure.*

COROLLARY 6.1. *Let M^n be an n-dimensional orientable C^∞ manifold, $3 \leq n \leq 6$. Assume that $u \bmod 2 = W^2(M^n)$ for some integral cohomology class $u \in H^2(M^n, \mathbf{Z})$, where $W^2(M^n)$ is the two-dimensional Stiefel-Whitney class of M^n. Then $M^n \times \mathbf{R}^{n-2}$ has a complex structure.*

The corollary follows routinely from Theorem 6.3 and the following

Proposition 6.1. *Let K be an n-dimensional polyhedron, $3 \leq n \leq 6$. The stable equivalence class $[\xi]$ of a vector bundle ξ over K has a complex structure if and only if $u \bmod 2 = W^2(\xi)$ for some integral cohomology class $u \in H^2(K, \mathbf{Z})$. Here $W^2(\xi)$ is the Stiefel-Whitney class in dimension two.*

See Milnor [A5] for the proof of this proposition.

Now we shall give outlines of proofs of the theorems. To this end we first recall topological groupoids as discussed in Chapter V.

Let $\Gamma_q^{\mathbf{C}}$ be the groupoid of germs of local analytic isomorphisms of $\mathbf{C}^q$, and let $B\Gamma_q^{\mathbf{C}}$ be the classifying space of $\Gamma_q^{\mathbf{C}}$-structures. Taking differentials yields a homomorphism of topological groupoids

$$\nu : \Gamma_q^{\mathbf{C}} \longrightarrow \mathrm{GL}(q, \mathbf{C}).$$

Further, we have the continuous map between classifying spaces,

$$\nu : B\Gamma_q^{\mathbf{C}} \longrightarrow B\mathrm{GL}(q, \mathbf{C}),$$

induced by the first ν. We are using the same symbol for both of the maps. We regard the second ν as a fibration (moving it through homotopies), and denote by $F\Gamma_q^{\mathbf{C}}$ its homotopy fiber.

The following construction in a more general setting may simplify our discussion. Given topological spaces X and Y and a continuous map f: $X \to Y$, there exist a fibration (E, p, B) and homotopy equivalences φ: $X \to E$ and $\psi : Y \to B$ such that the diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{\varphi} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi} & B \end{array}$$

; homotopy commutative. The homotopy type of the fiber F of the fibraion (E, p, B) is well defined. The construction is as follows. We may ssume f to be the inclusion map $X \subset Y$ by passing to the mapping cylin-er C_f of f. Then for (E, p, B) we take the fibration $(\Omega_{X,Y}(Y), p_2, Y)$, 'here $\Omega_{X,Y}$ is the path space

$$\Omega_{X,Y}(Y) = \{u : [0, 1] \to Y \mid u \text{ is continuous}, u(0) \in X,\ u(1) \in Y\}$$

'ith the compact-open topology and $p_2(u) = u(1)$ (cf. Spanier [A6]). Now iking this construction back to the realm of groupoids and classifying spaces, e obtain the fibration $\nu : B\Gamma_q^{\mathbf{C}} \to B\mathrm{GL}(q, \mathbf{C})$ whose fiber is denoted by $\Gamma_q^{\mathbf{C}}$.

The groups $\mathrm{GL}(q, \mathbf{C})$ and $\mathrm{GL}(2q, \mathbf{R})$ have the following Iwasawa de-)mpositions:

$$\mathrm{GL}(q, \mathbf{C}) \approx \mathrm{U}(q) \times \mathbf{R}^{q^2}, \qquad \mathrm{GL}(2q, \mathbf{R}) \approx \mathrm{O}(2q) \times \mathbf{R}^{q(2q+1)},$$

which commute with the standard homomorphism ρ.

$$\begin{array}{ccc} \mathrm{U}(q) & \xrightarrow{\rho} & \mathrm{O}(2q) \\ \downarrow & & \downarrow \\ \mathrm{GL}(q, \mathbf{C}) & \xrightarrow{\rho} & \mathrm{GL}(2q, \mathbf{R}) \end{array}$$

So we use the same letter ρ to represent the induced map

$$\rho : B\mathrm{GL}(q, \mathbf{C}) \longrightarrow B\mathrm{GL}(2q, \mathbf{R}).$$

Let X be a $2q$-dimensional open manifold and consider the diagram below

$$\begin{array}{ccc} & & B\Gamma_q^{\mathbf{C}} \longleftarrow F\Gamma_q^{\mathbf{C}} \\ & \nearrow & \downarrow{\scriptstyle \nu} \\ & \overset{\tilde\tau}{\nearrow} & B\mathrm{GL}(q, \mathbf{C}) \\ & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\tau} & B\mathrm{GL}(2q, \mathbf{R}) \end{array}$$

where τ is the classifying map of the tangent bundle $\tau(X)$ of X. By definition there is a one-to-one correspondence between homotopy classes of lifts of τ in $B\mathrm{GL}(q, \mathbf{C})$ and homotopy classes of almost complex structures on X. A $\Gamma_q^{\mathbf{C}}$-foliation of the $2q$-manifold X is nothing but a complex structure on X. By Haefliger's classification theorem, as discussed in Chapter V, there is a one-to-one correspondence between homotopy classes of lifts of τ in $B\Gamma_q^{\mathbf{C}}$ and integrable homotopy classes of complex structures on X (the assumption 'X is open' is necessary for Haefliger's classification). Hence, the study of the fiber $F\Gamma_q^{\mathbf{C}}$ becomes vital for the investigation of complex structures on open manifolds as well as their integrable homotopy classes.

Theorem 6.4. $\pi_i(F\Gamma_q^{\mathbf{C}}) = 0,\ 0 < i \leq q.$

We assume Theorem 6.4 for the moment and prove Theorems 6.1, 6.2, and 6.3.

Proof of Theorem 6.1. Consider the above diagram. An almost complex structure on X is a lift $\tilde\tau$ of τ in $B\mathrm{GL}(q, \mathbf{C})$. We want to lift the $\tilde\tau$ to $B\Gamma_q^{\mathbf{C}}$. Notice that $B\mathrm{GL}(q, \mathbf{C})$ is simply connected. Consider a C^1 triangulation K of X and try to construct $\tilde\tau$ skeletonwise. The obstructions in doing so lie in

$$H^i(X, \pi_{i-1}(F\Gamma_q^{\mathbf{C}})), \qquad i = 1, 2, \ldots$$

(cf. Spanier [A6], Steenrod [A7]). Hence, Theorem 6.4 together with the hypothesis of Theorem 6.1 completes our proof. □

PROOF OF THEOREM 6.2. The proof proceeds in the same way as the proof Theorem 6.1; however, here the obstructions for $\tilde{\tau}_0$ and $\tilde{\tau}_1$ to be homo- ›ic as lifts lie in

$$H^i(X, \pi_i(F\Gamma_q^{\mathbf{C}})), \qquad i = 1, 2, \ldots.$$

Theorem 6.2 follows. □

PROOF OF THEOREM 6.3. Set $X = M^n \times \mathbf{R}^{n-2}$. Then because $n > 2$, X ı $2(n-1)$-dimensional open manifold, and $H^i(X, Z) = 0$, $i > n$. But by r assumption, the stable equivalence class of the tangent bundle $\tau(M^n)$ of ^{n} has a complex structure; hence, $n-2 > 0$ implies that $M^n \times \mathbf{R}^{n-2}$ has an nost complex structure. To see this we turn to the following commutative ıgram:

$$\begin{array}{ccc} \pi_i(B\mathrm{U}(n-1)) & \xrightarrow[i_*]{\simeq} & \pi_i(B\mathrm{U}) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \qquad i \leq n. \\ \pi_i(B\mathrm{O}(2n-2)) & \xrightarrow[i_*]{\simeq} & \pi_i(B\mathrm{O}) \end{array}$$

:nce, the theorem follows from Theorem 6.1. □

We shall prove Theorem 6.4. in the following three sections.

;3. Holomorphic foliations of complexifications of real analytic manifolds

In this section we consider holomorphic foliations of complexifications of ıl analytic manifolds. Let M be a real analytic manifold.

DEFINITION 6.6. By a *complexification* of the real analytic manifold M : mean a pair $(i, \mathbf{C}M)$ of a complex manifold $\mathbf{C}M$ and a real analytic ıbedding $i: M \to \mathbf{C}M$, with a complex structure $\mathscr{C} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ $\mathbf{C}M$ satisfying $\phi_\alpha(i(M) \cap U_\alpha) = \phi_\alpha(U_\alpha) \cap \mathbf{R}^n$.

In other words, the pair $(\mathbf{C}M, M)$ is locally the pair $(\mathbf{C}^n, \mathbf{R}^n)$. We state me facts about complexifications of a real analytic manifold in the follow- ʒ

THEOREM 6.5. *Let M be a real analytic manifold.*

(1) *M has a complexification $(i, \mathbf{C}M)$.*

(2) *Let $f: M \to W$ be a real analytic map from M to a complex man- ›ld W. Then we can extend f to a holomorphic map $\mathbf{C}f: \mathbf{C}M \to W$ of me complexification $\mathbf{C}M$ of M into W.*

(3) *Let $(i, \mathbf{C}M)$ and $(i', \mathbf{C}'M)$ be complexifications of M. Then there ists an analytic isomophism h between an open neighborhood U in $i(M)$ d the corresponding open neighborhood U' in $i'(M)$ such that the following*

diagram commutes.

$$\begin{array}{ccc} & \overset{i}{\nearrow} & U \subset \mathbf{C}M \\ M & & \downarrow{\scriptstyle h} \\ & \underset{i'}{\searrow} & U' \subset \mathbf{C}'M \end{array}$$

(4) *For a complexification $(i, \mathbf{C}M)$ of M we have an isomorphism of complex vector bundles*

$$T(\mathbf{C}M)|M \sim T(M) \otimes \mathbf{C}.$$

We omit the proof.

The above theorem implies that the germ of a complexification of a real analytic manifold is well defined. Henceforth, $\mathbf{C}M$ denotes either a complexification of M or its germ.

DEFINITION 6.7. Let M be a C^∞ manifold, let W be a complex manifold, and let $f: M \to W$ be a C^∞ map. We say that f is a $\mathbf{C}$-*submersion* if the differential of f, $df: T(M) \to T(W)$, induces an epimorphism of vector bundles

$$\mathbf{C}(df): T(M) \otimes \mathbf{C} \longrightarrow T(W).$$

REMARK 1. Let E be a real vector bundle and F a complex vector bundle. Let $\phi: E \to F$ be a homomorphism of real vector bundles. Then there is a naturally induced complex vector bundle

$$\mathbf{C}(\phi): E \otimes \mathbf{C} \longrightarrow F.$$

REMARK 2. Let M be a real analytic manifold, W a complex manifold, and $f: M \to W$ a real analytic map. If f is a $\mathbf{C}$-submersion, then the extension of f, $\mathbf{C}f: \mathbf{C}M \longrightarrow W$, is a holomorphic submersion, that is, $\mathbf{C}f$ is a holomorphic map of the maximal rank at each point of $\mathbf{C}M$.

DEFINITION 6.8. Let W be a complex manifold. By a codimension q *holomorphic foliation* of W, we mean a family $\mathscr{F} = \{(U_\alpha, f_\alpha), f_{\alpha\beta} \mid \alpha, \beta \in A\}$ of pairs (U_α, f_α), where $\{U_\alpha \mid \alpha \in A\}$ is an open covering of W and the $f_\alpha: U_\alpha \to \mathbf{C}^q$ are holomorphic submersions, and the

$$f_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow \Gamma_q^{\mathbf{C}}$$

are continuous maps satisfying

$$f_\alpha(x) = f_{\alpha\beta} \circ f_\beta(\,, x), \qquad x \in U_\alpha \cap U_\beta.$$

The definition of the equivalence of codimension q holomorphic foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ is verbatim that of the equivalence of foliations in Chapter V.

Further, let M be a real analytic manifold and $\mathbf{C}M$ a complexification of M. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be codimension q foliations of $\mathbf{C}M$. Suppose that there exists a neighborhood U of M in $\mathbf{C}M$ such that $\mathscr{F}_1|U$ and $\mathscr{F}_2|U$ are ıuivalent. Then we say $\mathscr{F}_1$ and $\mathscr{F}_2$ are germ-wise equivalent in M and rite $\mathscr{F}_1 \underset{M}{\sim} \mathscr{F}_2$. Clearly, this is an equivalence relation.

DEFINITION 6.9. Let M be a C^∞ manifold. By a codimension q C-*liation* is meant a family $\mathscr{G} = \{(V_\lambda, g_\lambda), g_{\lambda\mu} | \lambda, \mu \in \Lambda\}$, where $\{V_\lambda | \lambda \in$ $\}$ is an open cover of M and the maps $g_\lambda : V_\lambda \to \mathbf{C}^q$ are C-submersions gether with continuous maps

$$g_{\lambda\mu} : V_\lambda \cap V_\mu \longrightarrow \Gamma_q^{\mathbf{C}}$$

tisfying

$$g_\lambda(x) = g_{\lambda\mu} \circ g_\mu(x), \qquad x \in U_\lambda \cap U_\mu.$$

In particular, if M is a real analytic manifold and for each λ, g_λ is a C^ω ap, we call $\mathscr{G}$ an *analytic* C*-foliation*. The 'equivalence' of C-foliations ıd the equivalence of analytic C-foliations are defined much the same way the equivalence of foliations.

Let M be a C^ω manifold, and let $\mathscr{F}$ be a codimension q holomorphic ·liation of $\mathbf{C}M$. Then $\mathscr{F}|M$ is a codimension q analytic C-foliation of r.

LEMMA 6.1. *Let M be a C^ω manifold. The map which assigns to each ›lomorphic foliation $\mathscr{F}$ of* C*M the restriction $\mathscr{F}|M$ defines a one-to-one ·rrespondence between germ-wise equivalence classes in M of codimension q ›lomorphic foliations of* C*M and equivalence classes of codimension q an- ytic* C*-foliations of M.*

PROOF. (i) Injectivity. Suppose we have two holomorphic codimension q liations $\mathscr{F}_1$ and $\mathscr{F}_2$ of $\mathbf{C}M$ with $\mathscr{F}_1|M \sim \mathscr{F}_2|M$. Then by the uniqueness analytic continuations, $\mathscr{F}_1$ and $\mathscr{F}_2$ are germ-wise equivalent in M.

(ii) Surjectivity. Let $\mathscr{G}$ be a codimension q real analytic foliation of M; $= \{(V_\alpha, g_\alpha), g_{\alpha\beta} | \alpha, \beta \in A\}$. We want to extend $\mathscr{G}$ to a holomorphic liation of $\mathbf{C}M$. Take a C^ω atlas $\{(V_\alpha, \phi_\alpha) | \alpha \in A\}$ in M (we can use ›ordinate neighborhoods of $\mathscr{G}$). If $n = \dim M$, we think of $\phi_\alpha : V_\alpha \to \mathbf{R}^n$ $\phi_\alpha : V_\alpha \to \mathbf{C}^n$ and obtain the C^ω map

$$\phi_\alpha \times g_\alpha : V_\alpha \longrightarrow \mathbf{C}^n \times \mathbf{C}^q.$$

the same way as we constructed the graph of a Γ-structure in Chapter V, we ıste tubular neighborhoods of the $(\phi_\alpha \times g_\alpha)(V_\alpha)$ in $\mathbf{C}^n \times \mathbf{C}^q$ together to get an $+q)$-dimensional complex manifold W^{n+q}, the holomorphic foliation $\mathscr{F}'$ W^{n+q} defined by the projection onto $\mathbf{C}^q$, and a C^ω embedding $j : M \to$ ${}^{n+q}$ defined by $\phi_\alpha \times g_\alpha$. Let $\mathbf{C}j : \mathbf{C}M \to W^{n+q}$ be a complexification of Since g_α is a C-submersion we may assume $\mathbf{C}j$ to be transverse to $\mathscr{F}'$. ıen the pullback $(\mathbf{C}j)^*\mathscr{F}'$ is the desired extention of $\mathscr{G}$. □

By virtue of Lemma 6.1 we may identify a codimension q holomorphic liation of $\mathbf{C}M$ with its restriction to M.

DEFINITION 6.10. Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be codimension q holomorphic foliations of $\mathbf{C}M$. We say that $\mathscr{F}_0$ and $\mathscr{F}_1$ are *integrably homotopic* and write $\mathscr{F}_0 \underset{i}{\simeq} \mathscr{F}_1$ if there exists a codimension q holomorphic foliation $\mathscr{F}$ of $\mathbf{C}(M \times [0, 1])$ satisfying the following:

(i) $\mathscr{F}|\mathbf{C}(M \times 0) \sim \mathscr{F}_0$, $\mathscr{F}|\mathbf{C}(M \times 1) \sim \mathscr{F}_1$.

(ii) For each $t \in [0, 1]$, $\mathscr{F}$ is transverse to $\mathbf{C}(M \times t)$.

REMARK. A C^ω manifold with boundary $M \times [0, 1]$ has a complexification $\mathbf{C}(M \times [0, 1])$ which contains $\mathbf{C}(M \times t)$ as a complex submanifold for each t.

Evidently, $\underset{i}{\simeq}$ is an equivalence relation; we call an equivalence class of this relation an integrable homotopy class.

THEOREM 6.6 (classification theorem for foliations of $\mathbf{C}M$). *Let M^n be an n-dimensional open C^ω manifold. Consider the following diagram*

$$\begin{array}{ccc} & & B\Gamma_q^{\mathbf{C}} \times B\mathrm{GL}(n-q, \mathbf{C}) \\ & \nearrow & \downarrow \nu \times 1 \\ & & B\mathrm{GL}(q, \mathbf{C}) \times B\mathrm{GL}(n-q, \mathbf{C}) \\ & \nearrow & \downarrow \oplus \\ M^n & \xrightarrow{\tau\otimes\mathbf{C}} & B\mathrm{GL}(n, \mathbf{C}) \end{array} \tag{6.1}$$

where $\tau\otimes\mathbf{C}$ is the classifying map of the complexification $\tau(M)\otimes\mathbf{C}$ of the tangent bundle $\tau(M)$. There is a one-to-one correspondence between homotopy classes of lifts of $\tau\otimes\mathbf{C}$ in $B\Gamma_q^{\mathbf{C}} \times B\mathrm{GL}(n-q, \mathbf{C})$ and integrable homotopies of codimension q holomorphic foliations of C*M.*

THEOREM 6.7. *Let M be an n-dimensional compact C^ω manifold, and consider the following diagram*

$$\begin{array}{ccc} & B\Gamma_n^{\mathbf{C}} & \longleftarrow F\Gamma_n^{\mathbf{C}} \\ \nearrow & \downarrow \nu & \\ M \xrightarrow{\tau\otimes\mathbf{C}} & B\mathrm{GL}(n, \mathbf{C}) & \end{array} \tag{6.2}$$

where $\tau\otimes\mathbf{C}$ is the classifying map of the complexification $\tau(M)\otimes\mathbf{C}$ of the tangent bundle $\tau(M)$ of M. Then there exists a map from the set of integrable homotopy classes of holomorphic codimension n foliations of C*M onto the set of homotopy classes of lifts of $\tau\otimes\mathbf{C}$ to $B\Gamma_n^{\mathbf{C}}$.*

We postpone the proofs of the two theorems above and prove Theorem 6.4.

PROOF OF THEOREM 6.4. (i) First we show $\pi_i(F\Gamma_q^{\mathbf{C}}) = 0$, $i < q$. Set $M = S^i \times \mathbf{R}^{q-i}$. As $i < q$, M is a q dimensional open manifold. We have

$= q$; so the commutative diagram (6.2) becomes:

$$\begin{array}{ccc} & B\Gamma_q^{\mathbf{C}} & \longleftarrow \quad F\Gamma_q^{\mathbf{C}} \\ & \nearrow \quad \downarrow & \\ M = S^i \times \mathbf{R}^{q-i} & \xrightarrow{\tau\otimes\mathbf{C}} & BGL(q, \mathbf{C}) \end{array}$$

ince $q - i > 0$, M is parallelizable; hence, we can take $\tau \otimes \mathbf{C}$ as a constant ıap. Therefore each lift of $\tau \otimes \mathbf{C}$ in $B\Gamma_q^{\mathbf{C}}$ corresponds to some map of $^i \times \mathbf{R}^{q-i}$ into $F\Gamma_q^{\mathbf{C}}$. On the other hand, $\mathbf{C}M$ have only one codimension q olomorphopic foliation, and it comes from the complex structure. Hence heorem 6.6 implies that continuous functions from S^i to $F\Gamma_q^{\mathbf{C}}$ has only ne homotopy class, and so $\pi_i(F\Gamma_q^{\mathbf{C}}) = 0$, $i < q$.

(ii) To show $\pi_q(F\Gamma_q^{\mathbf{C}}) = 0$, we utilise Theorem 6.7. Let $M = S^n$. Then ſ is a compact C^ω manifold. We consider the following commutative iagram:

$$\begin{array}{ccc} \{\tau\} \in \pi_n(BO(n)) & \xrightarrow{\sigma_*} & \pi_n(BU(n)) \\ i_* \downarrow & & \simeq \downarrow i_* \\ \pi_n(BO) & \xrightarrow{\sigma_*} & \pi_n(BU), \end{array}$$

'here i_* is the homomorphism induced by $i : U(n) \to \mathbf{U}$ and σ_* is the omomorphism induced by the natural map $\sigma : O(n) \to U(n)$. Here the ıap i_* on the right is an isomorphism. Let τ be the classifying map of the ıngent bundle $\tau(M)$ of M. Since S^n is parallelizable, $i_*\{\tau\} = 0$. Hence, ·om the commutativity of the above diagram we have

$$\{\tau \otimes \mathbf{C}\} = \sigma_*\{\tau\} = 0.$$

'herefore, we may take $\tau \otimes \mathbf{C}$ to be a constant map. So there corresponds ı a lift of $\tau \otimes \mathbf{C}$ in $B\Gamma_n^{\mathbf{C}}$ a continuous map $S^n \to F\Gamma_n^{\mathbf{C}}$. On the other and, $\mathbf{C}M$ has only one holomorphic codimension n foliation, namely the omplex structure, and so by Theorem 6.7, we get $\pi_n(F\Gamma_n^{\mathbf{C}}) = 0$, $n \geq 1$. □

§4. The C-transversality theorem

Let M, W be C^∞ manifolds and let $\mathcal{F}$ be a $\Gamma_q^{\mathbf{C}}$-foliation of W. Then he normal bundle $\nu\mathcal{F}$ of $\mathcal{F}$ is a complex vector bundle. Let $\mathbf{C}_\pi : \tau(W) \otimes$ ː $\to \nu\mathcal{F}$ be the natural projection. Let E and F be vector bundles. Given vector bundle homomorphism $\phi : E \to F$, denote by $\mathbf{C}(\phi) : E \otimes \mathbf{C} \to F \otimes \mathbf{C}$ ts complexification.

DEFINITION 6.11. A C^∞ map $f : M \to W$ is **C**-*transversal* to $\mathcal{F}$ if he composite $\tau(M) \otimes \mathbf{C} \xrightarrow{\mathbf{C}(df)} \tau(W) \otimes \mathbf{C} \to \nu\mathcal{F}$ of complex vector bundle omomorphisms is a surjection.

We denote by $\mathbf{C}\mathrm{Tr}(M, \mathcal{F})$ the space of C^∞ maps $f : M \to W$ which re **C**-transversal to $\mathcal{F}$ with C^1 topology.

Let $\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{F})$ together with the compact-open topology be the space of vector bundle homomorphisms $\phi : \tau(M) \to \nu\mathcal{F}$ whose complexifications $\mathbf{C}(\phi)$ are surjective. For an element f of $\mathbf{C}\mathrm{Tr}(M, \mathcal{F})$, the composite

$$\tau(M) \xrightarrow{df} \tau(W) \xrightarrow{\pi} \nu\mathcal{F}$$

sits in $\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{F})$. Here π is the natural projection.

THEOREM 6.8. *If M is an open manifold, the map*

$$\phi : \mathbf{C}\mathrm{Tr}(M, \mathcal{F}) \longrightarrow \mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{F})$$

sending f to $\pi \circ df$ is a weak homotopy equivalence.

This theorem follows from Theorem 3.14. We leave the details of the proof to the reader. See Theorem 6.9 to fill the gap between C^1 and C^ω.

Theorem 6.8 implies Theorem 6.6. The proof is verbatim the proof of the classification theorem for foliations of open manifolds in Chapter V.

We proceed to the proof of Theorem 6.7, which is slightly more cumbersome. Here we apply Theorem 4.1 of Chapter IV. Let M be an n-dimensional C^ω manifold. Let $(E, \xi, \mathcal{E})$ be a $\Gamma_n^{\mathbf{C}}$-foliated microbundle over M. Then we may think of E as a $3n$-dimensional C^ω manifold and $\mathcal{E}$ as an analytic **C**-foliation of E. Let $\pi : \tau(E) \to \nu\mathcal{E}$ be the natural projection.

Denote by $\mathbf{C}\mathrm{Tr}^\omega(M, \mathcal{E})$ the space, with the C^1 topology, of C^ω maps $f : M \to E$ such that the composite of complex vector bundle homomorphisms

$$\tau(M) \otimes \mathbf{C} \xrightarrow{(df)\otimes\mathbf{C}} \tau(E) \otimes \mathbf{C} \xrightarrow{\mathbf{C}(\pi)} \nu\mathcal{E}$$

is surjective. If $f \in \mathbf{C}Tr^\omega(M, \mathcal{E})$, then $f^*\mathcal{E}$ is a codimension n holomorphic foliation of $\mathbf{C}M$.

Let $\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{E})$ be the space, with the compact-open topology, of vector bundle homomorphisms $\phi : \tau(M) \to \nu\mathcal{E}$ whose complexifications $\mathbf{C}(\phi) : T(M) \otimes \mathbf{C} \to \nu\mathcal{E}$ are each surjections of complex vector bundles.

THEOREM 6.9 (the **C**-transversality theorem). *Let M be a C^ω manifold. The the map which associates to each f the composite $\pi \circ df$,*

$$\pi \circ d : \mathbf{C}\mathrm{Tr}^\omega(M, \mathcal{E}) \longrightarrow \mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{E}),$$

induces the following surjection

$$\pi_0(\mathbf{C}\mathrm{Tr}^\omega(M, \mathcal{E})) \longrightarrow \pi_0(\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathcal{E})).$$

Let $\mathbf{C}\mathrm{Tr}^1(M, \mathcal{E})$ be the space of C^1 maps $f : M \to E$ transverse to $\mathcal{E}$, with the C^1 topology. Then $\mathbf{C}\mathrm{Tr}^\omega(M, \mathcal{E})$ is a subspace of $\mathbf{C}\mathrm{Tr}^1(M, \mathcal{E})$. By the approximation theorem (cf. §2, Chapter III) we have the following

Proposition 6.2. *Let M be a compact C^ω manifold. Then the inclusion map $i : \mathbf{C}\mathrm{Tr}^\omega(M, \mathscr{E}) \to \mathbf{C}\mathrm{Tr}^1(M, \mathscr{E})$ induces the surjection:*

$$i_* : \pi_0(\mathbf{C}\mathrm{Tr}^\omega(M, \mathscr{E})) \longrightarrow \pi_0(\mathbf{C}\mathrm{Tr}^1(M, \mathscr{E})).$$

We omit the proof. See the approximation theorem in Chapter III.

Now Theorem 6.9 results from Proposition 6.2 and the following

Theorem 6.10 (**C**-transversality theorem). *Let M be a compact C^ω manifold. Then the map*

$$\pi \circ d : \mathbf{C}\mathrm{Tr}^1(M, \mathscr{E}) \longrightarrow \mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathscr{E}),$$

sending f to $\pi \circ df$ induces a surjection

$$\pi_0(\mathbf{C}\mathrm{Tr}^1(M, \mathscr{E})) \longrightarrow \pi_0(\mathbf{C}\,\mathrm{Epi}(\tau(M), \mathscr{E})).$$

Proof. Set $X = M \times E$, and let $p : X \to E$ be the projection onto the first factor. Then (X, p, M) is a smooth fiber bundle.

Denote by $\mathrm{Sect}(X)$ the space of C^1 sections in (X, p, M) with C^1 topology. Let (X^1, p^1, M) be the one-jet bundle of germs of C^1 sections in (X, p, M). Let $S(X^1)$ be the space with the compact-open topology of continuous sections in (X^1, p^1, M). Then taking one-jets we get the continuous map $J^1 : \mathrm{Sect}(X) \to \mathrm{Sect}(X^1)$.

Let $C^1(M, E)$ be the space of C^1 maps from M to E with the C^1 topology, and let $\mathrm{Hom}(\tau(M), \tau(E))$ be the space of homomorphisms of the tangent bundle $\tau(M)$ of M into the tangent bundle $\tau(E)$ of E with the compact-open topology. Likewise we consider $\mathrm{Hom}(\tau(M), \tau(\mathscr{E}))$. Then we have the commutative diagram

$$\begin{array}{ccccc}
C^1(M, E) & \xrightarrow{d} & \mathrm{Hom}(\tau(M), \tau(E)) & \xrightarrow{\pi} & \mathrm{Hom}(\tau(M), \nu\mathscr{E}) \\
\varphi \uparrow \approx & & \psi \uparrow \approx & & \cup \\
\mathrm{Sect}(X) & \xrightarrow{J^1} & \mathrm{Sect}(X^1) & & \mathbf{C}\mathrm{Epi}(\tau(M), \nu\mathscr{E})
\end{array}$$

where d associates to a map f its differential df, π associates to ϕ the composite $\pi \circ \phi$, and φ and ψ are natural isomorphisms.

Put $\heartsuit = \pi^{-1}(\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathscr{E}))$. Then we have the following commutative diagram

$$\begin{array}{ccccc}
C^1(M, E) & \xrightarrow{d} & \mathrm{Hom}(\tau(M), \tau(E)) & \xrightarrow{\pi} & \mathrm{Hom}(\tau(M), \nu\mathscr{E}) \\
\cup \uparrow & & \cup \uparrow & & \cup \uparrow \\
\mathbf{C}\mathrm{Tr}^1(M, \mathscr{E}) & \xrightarrow{d} & \heartsuit & \xrightarrow{\pi|\heartsuit} & \mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathscr{E})
\end{array}$$

where $\pi|\heartsuit$ is a surjection. Hence, the map

$$\pi_* : \pi_0(\heartsuit) \longrightarrow \pi_0(\mathbf{C}\,\mathrm{Epi}(\tau(M), \nu\mathscr{E}))$$

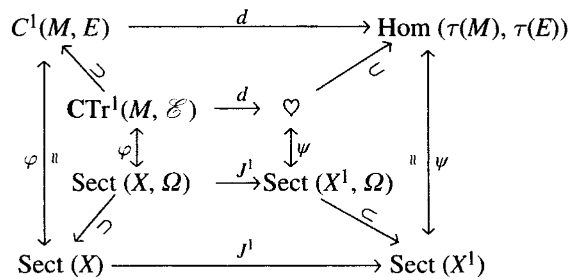
is onto. Thus, to prove the theorem it is enough to show that the map

$$d_* : \pi_0(\mathbf{C}\mathrm{Tr}^1(M, \mathscr{E})) \longrightarrow \pi_0(\heartsuit)$$

is a surjection. Now choose an open subset Ω of X^1 which gives the correspondence by the isomorphism ψ as follows

$$\begin{array}{ccc}
\mathrm{Hom}(\tau(M), \tau(E)) \supset & & \heartsuit \\
\psi \uparrow \approx & & \psi \uparrow \\
\mathrm{Sect}(X^1) \supset & & \mathrm{Sect}(X^1, \Omega)
\end{array}$$

We then get the following commutative diagram:



where $\mathrm{Sect}(X, \Omega) = (J^1)^{-1}(\mathrm{Sect}(X^1, \Omega))$. Hence, it suffices to show that Gromov's theorem of Chapter IV is applicable to our Ω.

Now X^1 is a jet bundle as shown below

$$\begin{array}{lcl}
X^1 = J^1(M, E) & \longleftarrow & J^1(n, 3n) = M(3n, n; \mathbf{R}) \\
p^1 \downarrow & & \downarrow \\
X = M \times E & & \\
p \downarrow & & \downarrow p_1 \\
M = M & &
\end{array}$$

Here $M(3n, n; \mathbf{R})$ consists of the $(3n, n)$-matrices over $\mathbf{R}$, whose structural group is $L^1(n, 3n) = L^1(3n) \times L^1(n) = \mathrm{GL}(3n, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$. However, since $M \times E$ is locally compact and paracompact, we may assume its structural group to be $\mathrm{O}(3n) \times \mathrm{O}(n)$.

Further, because $(E, \xi, \mathscr{E})$ is a $\Gamma_q^{\mathbf{C}}$-foliated microbundle over the C^ω manifold M, we can choose a C^ω atlas in E as follows: set

$$\{(U_\lambda, \varphi_\lambda) \mid \lambda \in \Lambda\},$$

$$\varphi_\lambda : U_\lambda \longrightarrow \mathbf{R}^{3n} = \mathbf{R}^n \oplus \mathbf{R}^{2n},$$

$$\varphi_\lambda \circ \varphi_\mu^{-1}(x, y) = (x', y') \quad \text{if } U_\lambda \cap U_\mu \neq \varnothing$$

so that

$$\begin{cases} x' = \bar{g}_{\lambda\mu}(\bar{x}), & \bar{x} = p(x), \\ y' = g_{\lambda\mu}(x)y, \end{cases}$$

where $\mathscr{E} = \{(U_\lambda, \psi_\lambda), g_{\lambda\mu} \mid \lambda, \mu \in \Lambda\}$, and the maps

$$g_{\lambda\mu} : U_\lambda \cap U_\mu \longrightarrow \Gamma_q^{\mathbf{C}};$$
$$\bar{g}_{\lambda\mu} : p(U_\lambda) \cap p(U_\mu) \longrightarrow \mathrm{GL}(n, \mathbf{R})$$

take on values in the Jacobian matrices of coordinate transformations of M.

Let F be a subgroup of $\mathrm{O}(3n)$ consisting of elements of the following form:

$$\underbrace{\begin{pmatrix} A & O \\ B & C \end{pmatrix}}_{n \quad 2n}\begin{matrix}]n \\ ]2n \end{matrix} \qquad \begin{cases} A \in \mathrm{O}(n), \\ C \in \rho(\mathrm{U}(n)) \subset \mathrm{O}(2n), \end{cases}$$

where $\rho : \mathrm{U}(n) \to \mathrm{O}(2n)$ is the standard map. Then we can reduce the structural group of the above jet bundle to $F \times \mathrm{O}(n)$. Now we define a closed subset Σ of $M(3n, n; \mathbf{R})$ as follows. Consider the natural correspondence:

$$\phi : M(2n, n; \mathbf{R}) \longrightarrow M(n, n; \mathbf{C})$$
$$\begin{matrix} n[\\ n[\end{matrix}\begin{pmatrix} A \\ B \end{pmatrix} \longmapsto (A + iB).$$

Set $\Sigma_1 = M(n, n; \mathbf{C}) - \mathrm{GL}(n, \mathbf{C})$. Then Σ_1 is a closed subset of real codimension two. Write $\Sigma_2 = \phi^{-1}(\Sigma_1)$. The set Σ_2 is also a closed subset of $M(2n, n; \mathbf{R})$ of real codimension two. Put

$$M(3n, n; \mathbf{R}) \supset \Sigma = M(n, n; \mathbf{R}) \oplus \Sigma_2.$$

Then Σ is a closed set of codimension two. Further Σ is invariant under the action of $F \times \mathrm{O}(n)$. Hence, we may consider a sub-jet bundle J_Σ of the jet bundle $J^1(M, E)$, whose fiber is Σ. Then J_Σ is also a closed codimension two subset of $J^1(M, E)$. Put

$$\Omega = J^1(M, E) - J_\Sigma.$$

The definition of Ω implies that $\mathrm{Sect}(X^1, \Omega)$ corresponds to $\heartsuit$ under ψ. Therefore, $\mathrm{Sect}(X, \Omega)$ corresponds to $\mathrm{CTr}^1(M, \mathscr{E})$ under φ.

Now Theorem 4.1 applied for Ω confirms that

$$J^1 : \mathrm{Sect}(X, \Omega) \longrightarrow \mathrm{Sect}\, X^1, \Omega)$$

is a weak homotopy equivalence. Hence, we have proved the theorem. □

§5. Notes

1. Almost complex structures on closed manifolds are not in general integrable; Fröhlicher showed the existence of an nonintegrable almost complex structure on S^6. See A. Fröhlicher, *Zur Differentialgeometrie der komplexen Structuren*, Math. Ann. **129** (1955), 50–95.

Later Van de Ven showed that there exists an almost complex structure which is not integrable on a four-dimensional manifold. See A. Van de Ven, *On the Chern numbers of certain complex and almost complex manifolds*, Proc. Nat. Acad. Sci. USA. **55** (1966), 1624–1627.

Recently S. T. Yau showed the existence of a four-dimensional closed parallelizable manifold (hence, an almost complex manifold) without complex structure. Also N. Brotherton constructed an example similar but different from the Yau's. See S. T. Yau, *Parellelizable manifolds without complex structure*, Topology **15** (1976), 51–53, and N. Brotherton, *Some parallelizable four manifolds not admitting almost complex structure*, Bull. London Math. Soc. **10** (1978), 303–304.

The above works by Van de Ven, Yau, and Brotherton are all based on Kodaira's work (1).

2. Almost complex structures of open manifolds of dimensions less than or equal to six, on the other hand, can be made integrable through homotopies by Theorem 6.1 (Adachi [C1, C2]).

So far no examples of open manifolds with almost complex structures not admitting complex structures are known.

(1) K. Kodaira, *On the structures of compsct complex analytic surfaces* I, Amer. J. Math. **86** (1964), 751–793; II, Amer. J. Math. **88** (1966), 687–721; III, Amer. J. Math. 90 (1968), 55–83; IV, Amer. J. Math. **90** (1968), 1048–1066.

CHAPTER VII

Embeddings of C^∞ Manifolds (continued)

In this chapter we discuss embeddings centered around Haefliger's theorem which is the most fundamental in the theory of embeddings. Haefliger's theorem contains the classical theorem of Whitney given in Chapter II as a special case. Haefliger gave two proofs for his theorem. Here we take up the second proof and introduce briefly a generalization of Whitney's method of eliminating double points of completely regular immersions given in Chapter II and the works on embeddings of complexes by van Kampen ([1]), Wu [C24], Shapiro [C16], etc.

§1. Embeddings in Euclidean spaces

Let V be an n dimensional C^∞ manifold and let Δ_V be the diagonal set of $V \times V$, that is,

$$V \times V \supset \Delta_V = \{(x, x) \mid x \in V\}.$$

DEFINITION 7.1. We say that a continuous map

$$F : V \times V - \Delta_V \longrightarrow S^{m-1}$$

is *equivariant* or *$\mathbf{Z}_2$-equivariant* if F satisfies the relation

$$F(x, y) = -F(y, x), \qquad x, y \in V, \ x \neq y.$$

A homotopy $\{F_t\}$, $F_t : V \times V - \Delta_V \to S^{m-1}$ is *equivariant* if for each $t \in [0, 1]$, F_t is an equivariant map. Two equivariant maps F and G are *equivariantly homotopic* if there is an equivariant homotopy connecting F and G; in this case we write $F \underset{e}{\simeq} G$ or $F \underset{\mathbf{Z}_2}{\simeq} G$. Evidently $\underset{e}{\simeq}$ is an equivalence relation.

Let $f : V \to \mathbf{R}^m$ be an embedding, and define the *associated map* $\bar{f}$ as follows:

$$\bar{f} : V \times V - \Delta_V \longrightarrow S^{m-1},$$
$$\bar{f}(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

([1]) E. van Kampen, *Komplexe in euklidischen Raumen*, Hamburg Abh. **9** (1933), 72–78.

n $\bar f$ is clearly an equivariant map. Further, if two maps $f, g : V \to \mathbf{R}^m$ isotopic, their associated equivariant maps $\bar f$ and $\bar g$ are equivariantly ıotopic.

'HEOREM 7.1. *Let V be an n-dimensional C^∞ manifold. The assignment ›ach embbedding $f : V \to \mathbf{R}^m$ to its associated equivariant map $\bar f$: $V - \Delta_V \to S^{m-1}$ gives rise to a map Φ of equivalence classes:*

$$f : V \to \mathbf{R}^m,\ \textit{embedding}\}/\cong$$
$$\xrightarrow{\Phi} \{F : V \times V - \Delta_V \to S^{m-1},\ \textit{equivariant map}\}/\underset{e}{\simeq}.$$

· map Φ enjoys the following properties:
(a) If $3(n+1) \le 2m$, Φ is a surjection.
(b) If $3(n+1) < 2m$, Φ is a bijection.

Ienceforth the range for (n, m) with $3(n+1) < 2m$ will be referred ıs the *stable range*. We postpone the proof of this theorem till in a later ·ion; here we state some facts which follow routinely from the theorem.

)EFINITION 7.2. We define an equivalence relation $\sim$ in $V \times V - \Delta_V$ by

$$(x, y) \sim (y, x), \qquad x, y \in V,\ x \neq y,$$

. denote by V^* the quotient space of $V \times V - \Delta_V$ under $\sim$. We say that is the *reduced symmetric square* of V.

:onsider the following relation in $(V \times V - \Delta_V) \times S^{m-1}$:

$$(x, y; s) \sim (y, x; -s), \qquad x, y \in V,\ x \neq y,\ s \in S^{m-1}.$$

E be the quotient space of $(V \times V - \Delta) \times S^{m-1}$. Define $p : E \to V^*$ by

$$p([x, y; s]) = [x, y];$$

n (E, p, V^*) is a fiber bundle whose fiber is S^{m-1} with the structural up $\mathbf{Z}_2$, which is an associated bundle of the double covering space $V \times$ $- \Delta_V \to V^*$.

Γhe following lemma is trivial.

ˍEMMA 7.1. *There is a one-to-one correspondence between equivariant homotopy classes of equivariant maps of $V \times V - V_\Delta$ into S^{m-1} and homotopy ısses of sections in the fiber bundle (E, p, V^*).*

Γhis lemma together with Theorem 7.1 reduces the question of the existence of embeddings of an n-dimensional C^∞ manifold V in $\mathbf{R}^m$ and their ssification by isotopies within the stable range, to the question of the existence of cross sections of the fiber bundle (E, p, V^*) and their classification homotopies.

COROLLARY 7.1. *Within the stable range the classification of the embed- 'gs of an n-dimensional C^∞ manifold V in Euclidean space $\mathbf{R}^m$ does not ›end on the C^∞ structure of V.*

Γhis is evident from the above comment.

COROLLARY 7.2. *Within the stable range any two embeddings of the n-dimensional sphere S^n in Euclidean space $\mathbf{R}^m$ are isotopic.*

LEMMA 7.2. *The reduced symmetric space $(S^n)^*$ of the n-dimensional sphere S^n has the homotopy type of the real projective space $\mathbf{RP}^n$.*

PROOF. Consider the following commutative diagram. □

$$\begin{array}{ccccccc}
x & \in & S^n & \longrightarrow & S^n \times S^n - \Delta & \ni & (x, -x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[x] & \in & \mathbf{RP}^n & \longrightarrow & (S^n)^* & \ni & [(x, -x)]
\end{array}$$

PROOF OF COROLLARY 7.2. We show that if $V = S^n$ any pair of cross sections s_0 and s_1 of (E, p, V^*) are homotopic in the stable range. The obstructions for this homotopy are in the cohomology groups

$$H^i(V^*, \Pi_i(\mathscr{S}^{m-1})), \qquad i = 1, 2, \dots,$$
$$(\Pi_i(\mathscr{S}^{m-1}) = \pi_i(S^{m-1}) \otimes \mathbf{Z}_T)$$

with local coefficients associated with the covering $V \times V - \Delta_V \to V^*$ (Steenrod [A7]). We have $n < m - 1$ in the stable range $3(n+1) < 2m$; hence, by Lemma 7.2 $H^i(V^*, \Pi_i(\mathscr{S}^{m-1})) = 0$ for all $i > 0$. Now Corollary 7.2 follows from Theorem 7.1 and Lemma 7.1. □

PROOF OF THEOREM 2.3.

LEMMA 7.3. *If V is k-connected, the cohomology groups with local coefficients associated with the double covering $V \times V - \Delta_V \to V^*$ satisfy the following:*

$$\left.\begin{array}{l} H^i(V^*, \Pi_i(\mathscr{S}^{m-1})) = 0, \\ H^i(V^*, \Pi_{i-1}(\mathscr{S}^{m-1})) = 0, \end{array}\right\} \quad i \ge 2n - k.$$

REMARK. Note that the lemma is independent of the choice of m.

PROOF. Let $\mathbf{Z}'$ be one of $\mathbf{Z}$ and $\mathbf{Z}_T$, the twisted integer system associated with the double cover $V \times V - \Delta_V \to V^*$, and let $\mathbf{Z}''$ be the other. Since $\pi_j(S^{m-1})$ is a finitely generated Abelian group it is a direct sum of cyclic groups. Hence, it suffices to show $H^i(V^*, \mathbf{Z}' \otimes \mathbf{Z}_p) = 0$, $i \ge 2n - k$, for every prime number p. Looking at the cohomology exact sequence whose coefficients are the exact sequence

$$0 \longrightarrow \mathbf{Z}' \longrightarrow \mathbf{Z}' \longrightarrow \mathbf{Z}' \otimes \mathbf{Z}_p \longrightarrow 0,$$

we see that it is sufficient to show $H^i(V^*, \mathbf{Z}') = 0$, $i \ge 2n - k$.

Consider the Thom-Gysin exact sequence for $V \times V - \Delta_V \to V^*$ regarded as a sphere bundle (cf. Spanier [A6], Steenrod [A7])

$$\cdots \longrightarrow H^i(V \times V - \Delta) \longrightarrow H^i(V^*, \mathbf{V}') \longrightarrow H^{i+1}(V^*, \mathbf{Z}'')$$
$$\longrightarrow H^{i+1}(V \times V - \Delta) \longrightarrow \cdots.$$

ıe proof will be complete when we show $H^i(V\times V-\Delta)=0$, $i\geq 2n-k$, ıce in this case we have by the exactness $H^i(V^*, \mathbf{Z}')\cong H^{i+1}(V^*, \mathbf{Z}'')$, $\geq 2n-k$, and these groups are 0 for $i>2n$, where $\dim V^*=2n$.
However, by the Lefschetz duality theorem (Spanier [A6]) we have

$$H^i(V\times V-\Delta_V)\cong H_{2n-i}(V\times V, \Delta_V).$$

ıing the Künneth formula and the fact that V is k-connected, we see that right-hand side is zero for $2n-k\leq i\leq 2n$. □

Proof of (a). We assume that V is a closed n-manifold which is k-nnected. If $m=2n-k$ then by the assumption $2k+3\leq n$, the ir (n,m) satisfies the inequality $3(n+1)\leq 2m$. The obstructions to existence of cross sections of (E,p,V^*) lie in the cohomology groups

$$H^i(V^*, \Pi_{i-1}(\mathscr{S}^{m-1})), \qquad i=1,2,\ldots$$

th local coefficients associated with the covering $V\times V-\Delta_V\to V^*$ (cf. eenrod[A7]). But by Lemma 7.3 we have

$$H^j(V^*, \Pi_{j-1}(\mathscr{S}^{m-1}))=0, \qquad j\geq 2n-k.$$

ence, the above cohomology groups are 0 for all $i>0$, and so we have oved part (a) of the theorem.

Proof of (b). The obstructions for cross sections s_0 and s_1 of (E,p,V^*) be homotopic lie in the cohomology groups

$$H^i(V^*, \Pi_i(\mathscr{S}^{m-1})), \qquad i=1,2\ldots$$

th local cofficients associated with the covering space $V\times V-\Delta_V\to V^*$. ıt by Lemma 7.3 these groups are zero for all $i>0$. Hence, we have oved part (b) of the theorem. □

§2. Embeddings in manifolds

In the remaining sections of this chapter, we follow Haefliger's paper [C9].

Definition 7.3. Let X and Y be topological spaces, and let $F: X\times X\to$ $\times Y$ be a continuous map. We say that F is *equivariant* or $\mathbf{Z}_2$-*equivariant* for each point (x_1,x_2) of $X\times X$, we have

$$F(x_1,x_2)=(y_1,y_2)\in Y\times Y,$$
$$F(x_2,x_1)=(y_2,y_1).$$

e say that F is an *isovariant map* if F is equivariant and satisfies

$$F^{-1}(\Delta_Y)=\Delta_X,$$

here Δ_X and Δ_Y are the diagonal sets of X and Y respectively. A ho-otopy $F_t: X\times X\to Y\times Y$ is *equivariant* if for each $t\in[0,1]$, F_t an equivariant map. Two equivariant maps F and G are *equivariantly* ımotopic, $F\underset{e}{\simeq}G$, if there is an equivariant homotopy connecting them.

A homotopy $F_t: X\times X\to Y\times Y$ is *isovariant*, if F_t is isovariant for each $t\in[0,1]$. Two isovariant maps F and G are *isovariantly homotopic*, $F\underset{i}{\simeq}G$, if there is an isovariant homotopy between them. Evidently the $\underset{e}{\simeq}$ and the $\underset{i}{\simeq}$ are equivalence relations.

Definition 7.4. Let $f,g: X\to Y$ be continuous maps, and suppose there are two homotopies $\{h_\tau\}$ and $\{h'_\tau\}$ between them. By a *homotopy connecting* these two homotopies we mean a continuous map

$$H: X\times I\times I\longrightarrow Y, \qquad I=[0,1],$$

such that if $h_{\tau,t}(x)=H(x,\tau,t)$, we have the following

(i) $h_{\tau,0}=h_\tau$, $h_{\tau,1}=h'_\tau$,
(ii) $h_{0,t}=h_0=h'_0=f$, $h_{1,t}=h_1=h'_1=g$.

When such a homotopy exists we say that the homotopies $\{h_\tau\}$ and $\{h'_\tau\}$ are *homotopic* and write $\{h_\tau\}\simeq\{h'_\tau\}$. Clearly this is an equivalence relation.

Theorem 7.2. *Let V be an n-dimensional closed C^∞ manifold, and let M be an m-dimensional C^∞ manifold.*

(a) *Suppose $3(n+1)\leq 2m$. Then a continous map $f: V\to M$ is homotopic to an embedding g if and only if there exists an equivariant homotopy*

$$H_t: V\times V\longrightarrow M\times M,$$

satisfying the following:

(i) $H_0=f\times f$.
(ii) H_1 *is an isovariant map.*

Further, we can choose g so that $g\times g$ is isovariantly homotopic to H_1.

(b) *Suppose $3(n+1)<2m$. Let $f,g: V\to M$ be embeddings. Then a homotopy $\{f_\tau\}$ between f and g is homotopic to an isotopy if and only if there exists an equivariant homotopy*

$$H_{\tau,t}: V\times V\longrightarrow M\times M, \qquad \tau,t\in[0,1]$$

such that

(i) $H_{\tau,0}=f_\tau\times f_\tau$,
(ii) $H_{\tau,1}$ *is an isovariant homotopy.*

The proof of Theorem 7.2 shall be given in the next section; here we state one of its easy consequences in the following

Corollary 7.3. *The classification of the embeddings of V in M in the stable range does not depend on the C^∞ structures of V and M.*

The corollary is obvious by Theorem 7.2.

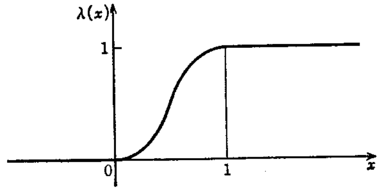


FIGURE 7.1

PROOF OF THEOREM 7.1. We look at the following diagram; $M = \mathbf{R}^m$:

$$\{f : V \to \mathbf{R}^m,\ \text{embedding}\}/\cong \xrightarrow{\Psi} \{H : V \times V \to \mathbf{R}^m \times \mathbf{R}^m,\ \text{isovariant}\}/\underset{i}{\simeq}$$

$$\Phi \searrow \qquad \swarrow \Lambda$$

$$\{\phi : V \times V - \Delta_V \to S^{m-1},\ \text{equivariant}\}/\underset{e}{\simeq}$$

Here we defined Φ in Theorem 7.1, and we define Ψ via Theorem 7.2: $\Psi(f) = f \times f$. Define Λ by $\Lambda(H) = \bar{H}$;

$$\bar{H}(x_1, x_2) = \frac{h_1(x_1, x_2) - h_2(x_1, x_2)}{|h_1(x_1, x_2) - h_2(x_1, x_2)|},$$

where $H(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$.

Then $\bar{H}$ is an equivariant map. Further, the above diagram commutes. Hence, if Λ is a bijection, Theorem 7.1 follows from Theorem 7.2.

We show that Λ is surjective. Consider an equivariant map

$$\phi : V \times V - \Delta_V \longrightarrow S^{m-1},$$

and define

$$H_\phi : V \times V \longrightarrow \mathbf{R}^m \times \mathbf{R}^m$$

as follows. Choose a sufficiently small tubular neighborhood U of the diagonal set Δ_V in $V \times V$, $\Delta_V \subset U \subset V \times V$. Let $\lambda : \mathbf{R}^1 \to \mathbf{R}^1$ be a C^∞ function such that

(i) λ is monotone increasing,
(ii) $\lambda(x) = 0$, $x \le 0$, $\lambda(x) = 1$, $x \ge 1$,
(iii) $0 \le \lambda(x) \le 1$, $x \in \mathbf{R}^1$

(Figure 7.1).

Define a C^∞ function μ on U, $\mu : U \to \mathbf{R}$, by

$$\mu(x, y) = \lambda(\rho((x, y), \Delta_V)),$$

where ρ is a Riemannian metric in U. Now set

$$H_\phi(x_1, x_2) = \begin{cases} \mu(x_1, x_2)(\phi(x_1, x_2), -\phi(x_1, x_2)), & x_1 \ne x_2, \\ (0, 0), & x_1 = x_2. \end{cases}$$

Then H_ϕ is an equivariant map which satisfies $H_\phi^{-1}(\Delta_{\mathbf{R}^m}) = \Delta_V$. Set $\Lambda(H_\phi) = \bar{H}_\phi$; we then have

$$\bar{H}_\phi(x_1, x_2) = \frac{\phi(x_1, x_2) + \phi(x_1, x_2)}{|\phi(x_1, x_2) + \phi(x_1, x_2)|} = \phi(x_1, x_2).$$

In a similar manner we can show that Λ is injective. □

§3. The proof of Theorem 7.2

As in Chapter 2, for the proof of Theorem 7.2 we first consider immersions and then go on to embeddings.

Let V and M be manifolds of respective dimensions n and m, and let $f_0, f_1 : V \to M$ be embeddings. It is evident that if f_0 and f_1 are isotopic, then they are regularly homotopic. We have the following routine

LEMMA 7.4. *Let $f : V \to M$ be an immersion. For $f \times f : V \times V \to M \times M$, the set Δ_V is open in $(f \times f)^{-1}(\Delta_M)$.*

THEOREM 7.3. *Let V be an n-dimensional compact C^∞ manifold, and let M be an m-dimensional C^∞ manifold.*

(a) *Assume $3(n+1) \le 2m$. An immersion $f : V \to M$ is regularly homotopic to an embedding f_1 if and only if there exists an equivariant homotopy $H_t : V \times V \to M \times M$ such that*

(i) *$H_0 = f \times f$, H_1 is isovariant, and*
(ii) *for all t in $[0, 1]$, Δ_V is an open subset of $H_t^{-1}(\Delta_M)$.*

Further, we can take $f_1 \times f_1$ to be isovariantly homotopic to H_1.

(b) *Assume $3(n+1) < 2m$. Let $f_0, f_1 : V \to M$ be embeddings. A regular homotopy $\{f_t\}$ connecting f_0 and f_1 is regularly homotopic to an isotopy $\{f_t'\}$ between f_0 and f_1 if and only if there exists an equivariant homotopy $H_{\tau,t} : V \times V \to M \times M$ satisfying the following:*

(i) *$H_{\tau,0} = f_\tau \times f_\tau$, $H_{\tau,1}$ is an isovariant homotopy.*
(ii) *$H_{0,t} = f_0 \times f_0$, $H_{1,t} = f_1 \times f_1$.*
(iii) *For all t and τ in $[0, 1]$, Δ_V is an open subset of $H_{\tau,t}^{-1}(\Delta_M)$.*

Here by a regular homotopy connecting $\{f_t\}$ and $\{f_t'\}$ we mean a homotopy $\{F_{\tau,t}\}$ satisfying

(i) $F_{\tau,0} = f_\tau$, $F_{\tau,1} = f_\tau'$, $F_{0,t} = f_0$, $F_{1,t} = f_1$.
(ii) For each τ and t in $[0, 1]$, $F_{\tau,t}$ is an immersion.

We shall prove Theorem 7.2 assuming Theorem 7.3. But first we give a preparatory discussion on the Haefliger-Hirsch theorem.

Denote by $\mathrm{Mon}(T(V), T(M))$ the set of all bundle monomorphisms of $T(V)$ in $T(M)$ with the compact-open topology. We say that a continuous map $\varphi : T(V) \to T(M)$ sending a fiber to the corresponding fiber is *isovariant* if for each $x \in V$ the restriction of φ to $T_x(V)$ satisfies

$$\varphi(-\xi) = -\varphi(\xi), \qquad \varphi^{-1}(0) = 0.$$

Denote the set of all isovariant maps from $T(V)$ to $T(M)$ by $I(T(v), T(M))$. This set is also given the compact-open topology.

Let f_0 and f_1 be equivariant maps from neighborhoods U_0 and U_1 of Δ_V in $V \times V$ to $M \times M$:

$$f_0 : U_0 \longrightarrow M \times M,$$
$$f_1 : U_1 \longrightarrow M \times M.$$

We say that f_0 and f_1 are *equivalent* if there exists a neighborhood U of Δ_V, $\Delta_V \subset U \subset U_0 \cap U_1$, such that $f_0|U = f_1|U$. Let $I(\Delta_V, \Delta_M)$ denote the set of equivalence classes of isovariant maps from neighborhoods of Δ_V to $M \times M$. We give $I(\Delta_V, \Delta_M)$ the compact-open topology.

Let $\mathrm{Imm}(V, M)$ be the set of all immersions of V in M endowed with the C^1-topology. Denote the set of topological immersions of V in M by TOP-Imm(V, M); this set too is given the compact-open topology.

We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{Imm}(V, M) & \xrightarrow{d} & \mathrm{Mon}(T(V), T(M)) & \searrow^{j} & \\
\downarrow i & & \downarrow \theta & & I(T(V), T(M)) \\
\text{TOP-Imm}(V, M) & \xrightarrow{\delta} & I(\Delta_V, \Delta_M) & \swarrow_{\Theta} &
\end{array}$$

Here d is a continuous map which assigns to an immersion f its differential df; δ assigns to f the germ of $f \times f : V \times V \to M \times M$ in Δ_V; i and j are inclusion maps. We define Θ as follows. Give V and M each a complete Riemannian metric.(²) Then we have the map $\exp : T(V) \to V$ sending $X \in T_x(V)$ to the end point of the geodesic starting at x of length $\|X\|$. Define $e_V : T(V) \to V \times V$ by

$$e_V(X) = (\exp X, \exp(-X)).$$

The restriction of the map e_V to some neighborhood U of the zero section V of $T(V)$ defines a diffeomorphism between U and some neighborhood of Δ_V. We have a parallel situation for $e_M : T(M) \to M \times M$. Now define Θ on $I(T(V), T(M))$ by

$$\Theta(\psi) = e_M \circ \psi \circ e_V^{-1}, \qquad \psi \in I(T(V), T(M)),$$

and let θ be the the restriction of Θ to $\mathrm{Mon}(T(V), T(M))$.

(²) We can always do this. See Nomizu and Ozeki, *The existence of complete Riemannian metrics*, Proc. Amer. Math. Soc. **12** (1961), 889–891.

Theorem 7.4. (i) *The diagram above induces the following commutative diagram:*

$$\begin{array}{ccccc}
\pi_i(\mathrm{Imm}(V, M)) & \xrightarrow{d_*} & \pi_i(\mathrm{Mon}(T(V), T(M))) & \searrow^{j_*} & \\
\downarrow i_* & & \downarrow \theta_* & & \pi_i(I(T(V), T(M))) \\
\pi_i(\text{TOP-Imm}(V, M)) & \xrightarrow{\delta_*} & \pi_i(I(\Delta_V, \Delta_M)) & \swarrow_{\Theta_*} &
\end{array}$$

(ii) $\delta_* \circ i_*$ *is a bijection for* $3n + 1 < 2m$,
$\delta_* \circ i_*$ *is a surjection for* $3n + 1 \le 2m$.

Proof. (i) The commtativity in the triangle is obvious by definition. The commutativity in the rectangle on the left follows from the fact that the tangent bundle of V is equivalent to the normal bundle of Δ_V in $V \times V$

$$\begin{array}{ccc}
T(V) & \xrightarrow{d*} & U \subset V \times V \\
\downarrow & & \downarrow \\
V & \xrightarrow{d} & \Delta_V
\end{array}$$

where $\Delta(x) = (x, x)$. (δ assigns to a map $f : M \to V$ its topological differential df.(³)

(ii) This follows from the three facts.

(a) Θ is a bijection.

(b) j^* is a surjection when $3n + 1 \le 2m$, and it is a bijection when $3n + 1 < 2m$.

(Haefliger and Hirsch [C11]; this is profoundly related to the homotopy groups of Stiefel manifolds).

(c) d_* is surjective when $n < m$; this was the Smale-Hirsch theorem (cf. Chapter III).

Proof of Theorem 7.2. (a) the if part is obvious. We want to show the converse. Give each of V and M a Riemannian metric, and consider the product metric on $V \times V$ and $M \times M$. Let $\delta : \Delta_V \to \mathbf{R}$ be a positive-valued C^∞ function.

The hypothesis $3(n + 1) \le 2m$ implies $3n + 1 < 2m$; therefore, by Theorem 7.4, f is an immersion and further we may assume that when we restrict f to some δ-tubular neighborhood U_δ (for a very small δ) of Δ_V there exists an isovariant homotopy $H_t^\delta : U_\delta \longrightarrow M \times M$ between $f \times f$ and H_1.

The δ-tubular neighborhood U_δ of Δ_V is the set of endpoints of geodesics in $V \times V$, starting at points (x, x) of Δ, orthogonal to Δ_V, and of length

(³) Milnor, *Microbundles* I, Topology **3** Suppl. (1964), 53-80.

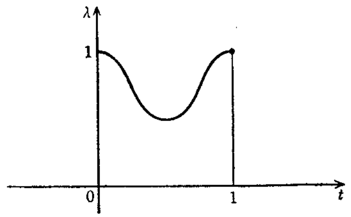


FIGURE 7.2

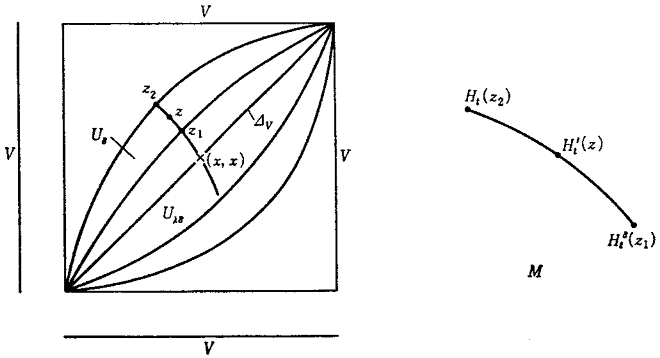


FIGURE 7.3

ss than or equal to $\delta(x)$. If we take δ small enough, these geodesics arting at different points of Δ_V do not intersect.

Let δ be sufficiently small so that we may assume our isovariant homotopy ıtisfies the following:

a) $H_t^\delta = H_t | U_\delta$ near $t = 0, t = 1$.

b) $H_t^\delta = H_t$ on Δ_V.

et $\lambda : [0, 1] \to \mathbf{R}$ be a C^∞ function (Figure 7.2) such that

(i) $\lambda(0) = 1$, $\lambda(1) = 1$,

(ii) $0 < \lambda(t) < 1$, $0 < t < 1$.

We now define a new equivariant homotopy H_t' by

$$H_t' = \begin{cases} H_t & \text{on } (U_\delta)^c, \\ H_t^\delta & \text{on } U_{\lambda\delta}. \end{cases}$$

On $U_\delta - U_{\lambda\delta}$ we define H_t' as follows. Suppose that a point z of $U_\delta - U_{\lambda\delta}$ on a geodesic starting at (x, x) in Δ_V and orthogonal to Δ_V. Suppose ırther that this geodesic intersects $\partial U_{\lambda\delta}$ and ∂U_δ at z_1 and z_2, respecvely. Then $H_t'(z)$ is defined to be the point on the geodesic (unique if δ

is small enough) connecting $H_t^\delta(z_1)$ and $H_t(z_2)$ in $M \times M$ such that

$$z_1 z : z z_2 = H_t^\delta(z_1) H_t'(z) : H_t'(z) H_t(z_2)$$

(see Figure 7.3). The map H_t' is an equivariant homotopy which is equal to $f \times f$ for $t = 0$ and to H_1 for $t = 1$. Further, Δ_V is open in $H_t'^{-1}(\Delta_M)$. Hence we have shown part (a) of Theorem 7.2. The proof of (b) is similar. □

§4. Proof of Theorem 7.3

Now we shall outline the proof of Theorem 7.3; the method we use here is a generalization of the way we eliminated double points of a completely regular immersion.

A. Construction of a standard model for a deformation eliminating double points. We construct a model having the following properties: L, L' are C^∞ manifolds, $\Phi : L \to L'$ is an immersion and $\Phi_t : L \to L'$ is a regular homotopy with $\Phi_0 = \Phi$ such that

(1) Φ_1 is an embedding.

(2) $\Phi_t | K^c = \Phi_0 | K^c$, K a compact subset of L.

We start with the following three objects (Figure 7.4):

(i) A compact manifold D with a fixed-point free involution $J : D \to D$, $J^2 = 1$ $J(x) \neq x$ for all $x \in D$.

(ii) A C^∞ function $\lambda : D \to [-1, 1)$ such that

(a) $\lambda \circ J = \lambda$;

(b) $\lambda^{-1}(-1) = \partial D$,

(c) $d\lambda \neq 0$ on $\lambda^{-1}(0)$.

(iii) A vector bundle $\xi = (L, p, D)$.

Consider $I = [-1, +1]$ and let $D' = D \times I / \sim$, where

$$(d, t) \sim (J(d), -t), \qquad d \in D, t \in I.$$

Then D' is a fiber bundle with the base space D/J and fiber I.

We construct a vector bundle L' over D' as follows. Let $\alpha : D \times I \to D \times D$ be a map defined by $\alpha(d, t) = (d, J(d))$, and let

$$\tilde{\xi} = (\tilde{L}, \tilde{p}, D \times I), \qquad \tilde{\xi} = \alpha^*(\xi \times \xi);$$

that is, $\tilde{L} = \{(l_d, l_{J(d)}, t) | l_d \in L_d, l_{J(d)} \in L_{J(d)}, t \in I, d \in D\}$, where L_d denotes the fiber over d. Define L' by $L' = \tilde{L}/\sim$,

$$(l_d, l_{J(d)}, t) \sim (-l_{J(d)}, -l_d, -t).$$

We denote the elements of D' and L' containing (d, t) and $(l_d, l_{J(d)}, t)$, respectively, by $[d, t]$ and $[l_d, l_{J(d)}]$. Then

$$L' \longrightarrow D',$$
$$[l_d, l_{J(d)}, t] \mapsto [d, t]$$

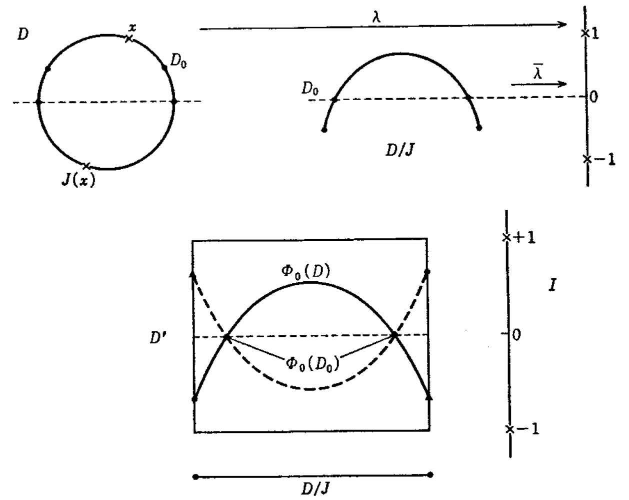


FIGURE 7.4

is the desired vector bundle. Define an immersion $\varphi : D \longrightarrow D'$ by $\varphi(d) = [d, \lambda(d)]$. Identifying D and D' with the zero-sections of L and L', we extend φ to an immersion

$$\Phi : L \longrightarrow L',$$
$$\Phi(l_d) = [l_d, 0, \lambda(d)].$$

The set of pairs of double points of Φ is $\{(d, J(d)) \mid d \in D_0\}$ where $D_0 = \lambda^{-1} + (0) \subset D \subset L$. Since λ has no critical points in D_0, two local components of $\Phi(L)$ are in general position and their intersection is $\Phi(D_0)$.

Now to construct a regular homotopy $\Phi_t : L \to L'$ we first consider a C^∞ function $\mu : D \to \mathbf{R}$ satisfying

$$\begin{cases} 0 \le \mu(d) < \lambda(d) + 1, \\ \mu(d) + \mu(J(d)) > 2\lambda(d), \\ \mu(d) = 0 \text{ if } \quad \lambda(d) \le -1/2. \end{cases}$$

A function μ satisfying the above specifications exists. For instance consider a C^∞ function $\beta : D \to \mathbf{R}$ such that

$$\begin{cases} 0 \le \beta(d) \le 1, \\ \beta(d) = 0 \quad \text{if} \quad \lambda(d) \le -1/2, \\ \beta(d) = 1 \quad \text{if} \quad \lambda(d) \ge 0, \end{cases}$$

and define μ by $\mu(d) = (\lambda(d) + 1/2)\beta(d)$.

Define $\varphi_t : D \to D'$ by

$$\varphi_t(d) = [d, \lambda(d) - t\mu(d)], \qquad t \in [0, 1].$$

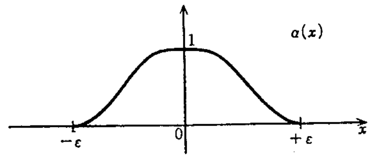


FIGURE 7.5

Then the φ_t constitute a regular homotopy such that $\varphi_0 = \varphi$ and φ_1 is an embedding. If $[d, \lambda(d) - \mu(d)] = [d', \lambda(d') - \mu(d')]$, we have $(d, \lambda(d) - \mu(d)) \sim (d', \lambda(d') - \mu(d'))$. Hence, by definition $d = d'$ or

$$d' = J(d), \qquad \lambda(d') - \mu(d') = -(\lambda(d) - \mu(d)).$$

Since $\lambda(d') = \lambda(d)$, we must have $2\lambda(d) = \mu(d) + \mu(J(d))$, and hence, $d = d'$.

We assume the structural group of the vector bundle L is an orthogonal group, *i.e.*, we think of L as a Euclidean vector bundle so that each fiber of L has a Euclidean metric. We denote by $\|l_d\|$ the length of an element l_d of the fiber L_d over d. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a bell-shaped function (Figure 7.5):

(i) $\alpha(\mathbf{R}) = [0, 1]$.
(ii) $\alpha(0) = 1$, $\alpha(x) = 0$, $|x| > \epsilon$, $\epsilon > 0$.

Now define $\Phi_t : L \to L'$ by

$$\Phi_t(l_d) = [l_d, 0, \lambda(d) - \alpha(|l_d|)t\mu(d)].$$

Then the $\{\Phi_t\}$ is a regular homotopy such that $\Phi_0 = \Phi$ and Φ_1 is an embedding. Further, Φ_t does not move outside some compact set.

B. Selection of a model. In the hypothesis of Theorem 7.3 (a) we may assume that $f : V \to M$ is in general position, that is, the map $f \times f : V \times V \to M \times M$ is transverse to Δ_M at each point of $V \times V - \Delta_V$. This fact follows from the Thom transversality theorem (cf. §5, Chapter I). We may also assume that no triple points exist since $3n < 2m$. This can be shown in a similar way as in the proof of Theorem 2.8.

We may assume that an equivariant homotopy H_t is defined for $t \in [-1, 1] = I$ and satisfies the following:

(1) The C^∞ map

$$H : V \times V \times I \longrightarrow M \times M,$$
$$H(v_1, v_2, t) = H_t(v_1, v_2)$$

is transverse to Δ_M outside $\Delta_V \times I$ in which case $\Delta = H^{-1}(\Delta_M) - \Delta_V \times I$ is a closed submanifold of $V \times V \times I$.

(2) Let $p_1, p_2 : V \times V \times I$ be the projections to the first and second components. Then $\mathbf{p}_1 = p_1|\Delta : \Delta \to V$ is an embedding.

(3) Let $t : V \times V \times I \to V$ be the projection onto the third component. Then 0 is not a critical value of $t|\Delta : \Delta \to I$.

The conditions (1) and (3) are satisfied by Thom's transversality theorem (§5, Chapter I). Here we do not require any restrictions on dimensions.

As for condition (2) notice that $3(n+1) \le 2m$ implies $2\dim\Delta < \dim V$, and hence, we may use Whitney's embedding theorem (§4, Chapter II) (we can redefine H_t so that $p_1|\Delta$ is an embedding). Now to construct the model $\Phi_t : L \to L'$ suggested in **A** we take for (i), (ii), and (iii) the following:

(i) $D = p_1(\Delta) = p_2(\Delta)$, $J = p_2 \circ \mathbf{p}_1^{-1}$.

(ii) $\lambda : D \longrightarrow I$, $\lambda = t \circ \mathbf{p}_1^{-1}$.

(iii) The normal bundle of D in V for the $\xi = (L, p, D)$. We have $p_1(\Delta) = p_2(\Delta)$ and $J \circ J = 1$ since the H_t are equivariant.

C. Application of the model. We shall construct diffeomorphisms

$$\Psi : L_\epsilon \longrightarrow V, \qquad \Psi' : L'_\epsilon \longrightarrow M,$$

where L_ϵ (respectively, L'_ϵ) is an ϵ-neighborhood (respectively, ϵ'-neighborhood) of D (respectively, D') in L (respectively, L') such that

(a) $f \circ \Psi = \Psi' \circ \Phi$,

(b) $f^{-1}(\Psi'(L'_\epsilon)) = \Psi(L_\epsilon)$.

Next we set

$$f_t(v) = \begin{cases} \Psi' \circ \Phi_t \circ \Psi^{-1}(v), & v \in \Psi(L_\epsilon), \\ f(v), & v \notin \Psi(L_\epsilon). \end{cases}$$

Then the $\{f_t\}$ is a regular homotopy such that $f_0 = f$ and f_1 is an embedding. This shows the "sufficiency" part of Theorem 7.3 (a). The "necessity" part is obvious.

Constructing these Ψ and Ψ' constitutes the hardest part of the proof. We shall accomplish this in three steps.

Step 1. We construct embeddings

$$\psi : D \longrightarrow V, \qquad \psi' : D' \longrightarrow M$$

such that

(a_0) $f \circ \psi = \psi' \circ \varphi$,

(b_0) $f^{-1} \circ \psi'(D') = \psi(D)$,

(c_0) $\psi'(D')$ is transverse to $f(V)$ along $f(\psi(D))$, *i.e.*, for each point d in D, $(df)_{\psi(d)}(T_{\psi(d)}(V))$ and $(d\psi')_{\varphi(d)}(T_{\varphi(d)}(D'))$ intersect transversely in the commutative diagram

$$\begin{array}{ccc} T_{\psi(d)}(V) & \xrightarrow{(df)_{\psi(d)}} & T_{f\circ\psi(d)}(V') \\ {\scriptstyle (d\psi)_d}\uparrow & & \uparrow{\scriptstyle (d\psi')_{\varphi(d)}} \\ T_d(D) & \xrightarrow{(d\varphi)_d} & T_{\varphi(d)}(D') \end{array}$$

and $f(V) \cap \psi'(D') = f(\psi(D))$.

As before we let p_1, $p_2 : V \times V \times I \to V$ be the projections onto the first and second components. Similarly, we define p'_1 and $p'_2 : M \times M \to M$.

The map ψ shall be the inclusion of D in V. As for the ψ' the restriction $\psi' = \psi'|\varphi(D)$ is defined by the condition (a_0). Set

$$A = \{[d, t] \in D' | 0 \le t \le \lambda(d)\},$$

and

$$\bar{\psi}'_0[d, t] = p'_1 \circ H(d, J(d), \lambda(d) - t), \qquad [d, t] \in A.$$

Then $\bar{\psi}'_0$ is a continuous extension of ψ'_0 over A. Since A is a deformation retract of D', we get an extension $\bar{\psi}' : D' \to M$ of ψ'_0.

On the other hand, we have $\dim D < m - n$, and so, we can extend $\psi'_0 : \varphi(D) \to M$ to an embedding ψ'_1 of a small neighborhood U of $\varphi(D)$ in D' such that

$$\psi'_1(U) \text{ and } f(V) \text{ are transveral along } f(D),$$
$$\psi'_1(U) \cap f(V) = f(D)$$

(this follows from Lemma 5.2 in Shapiro [C 16]). Finally, applying the Whitney embedding theorem with $2\dim D' < \dim M$ we obtain an embedding $\psi' : D' \to M$ satisfying

(α) $\psi' \simeq \bar{\psi}'$.

(β) $\psi'|U(D) = \psi'_1|U(D)$, where $U(D)$ is some neighborhood of D.

(γ) $f(V - \psi(D)) \cap \psi'(D') = p$.

The condition (γ) follows from $\dim D' + \dim V < \dim M$. The condition ($\beta$) implies the condition (c_0), and the condition (γ) implies the condition (b_0).

Step 2. We construct vector bundle monomorphisms

$$\dot{\psi} : L \longrightarrow T(V), \qquad \dot{\psi}' : L' \longrightarrow T(M)$$

such that

(i) $\dot{\psi}|D = \psi$, $\dot{\psi}'|D' = \psi'$,

(ii) $(df) \circ \dot{\psi} = \dot{\psi}' \circ \Phi$,

(iii) $\dot{\psi}(L)$ (respectively, $\dot{\psi}'(L')$) is the normal bundle of $T(D)$ (respectively, $T(\psi'(D'))$) in $T(V)|D$ (respectively, $T(M)|\psi'(D')$).

Recalling that L is the normal bundle of D in V we simply take the inclusion map for $\dot{\psi}$.

To construct $\dot{\psi}'$ we first define a C^{∞} map $\xi : D \longrightarrow M$ by $\xi(d) = \psi'[d, 0]$. The map satisfies $\xi \circ J = \xi$.

LEMMA 7.5. *Denote by $N(V, D)$ the normal bundle of D in V. Then there exists a vector bundle monomorphism*

$$\dot{\xi} : N(V, D) \oplus_J N(V, D) \longrightarrow T(M)$$

such that

(0) $\dot{\xi}$ *induces* ξ,

(1) $\dot{\xi}(l_d, l_{J(d)}) = -\dot{\xi}(l_{J(d)}, l_d)$,

(2) $\dot{\xi}(l_d, l_{J(d)}) = (df)(l_d) - (df)(l_{J(d)})$, $d \in D_0 = \lambda^{-1}(0)$.

Here $N(V, D) \oplus_J N(V, D)$ means $(1 \times J)^*(N(V, D) \times N(V, D))$.

PROOF. Let π_1 and π_2 be the restrictions of p_1 and p_2 to Δ. Since π_2 is an embbedding of Δ in V, the normal bundle $N(V \times V \times I, \Delta)$ is the Whitney sum of $T(V)|D(D = p_1(\Delta))$, $N(V, D)(D = p_2(\Delta))$, and the trivial bundle ϵ:

$$N(V \times V \times I, \Delta) \sim \pi_1^* N(V, D) \oplus \pi_2{}^* N(V, D) \oplus \pi_1^* T(D) \oplus \epsilon.$$

Hence, we get a vector bundle isomorphism

$$\begin{array}{ccc} N(V \times V \times I, \Delta) & \longrightarrow & N(V, D) \oplus_J N(V, D) \oplus T(D) \oplus \epsilon \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\pi_1} & D \end{array}$$

On the other hand, H is tranverse to Δ_M on Δ; therefore,

$$N(V \times V \times I, \Delta) \sim (H|\Delta)^* N(M \times M, \Delta_M).$$

But we have

$$N(M \times M, \Delta_M) \sim (p_1')^* T(M).$$

From these bundle isomorphisms we get a bundle monomorphism

$$\begin{array}{ccc} \Xi : N(V, D) \oplus_J N(V, D) \oplus T(D) \oplus \epsilon & \longrightarrow & T(M) \\ \downarrow & & \downarrow \\ D & \xrightarrow{p_1' \circ H \circ \pi_1^{-1}} & M \end{array}$$

Define a map $\sigma : N(V, D) \oplus_J N(V, D) \oplus T(D) \oplus \epsilon \hookleftarrow$ by

$$\sigma(l_d, l_{J(d)}, t_d, e) = (l_{J(d)}, l_d, t_{J(d)}, e),$$

where l_d (respectively, $l_{J(d)}$) is a vector normal to D at d (respectively, $J(d)$); $t_d \in T_d(D)$ and $t_{J(d)} = (dJ)_d(t_d)$. Then σ is a vector-bundle involution and satisfies $\Xi \circ \sigma = -\Xi$ because H_t is equivariant.

Finally, Ξ, restricted to the portion over D_0 in $N(V, D) \oplus_J N(V, D) \oplus 0$ reduces to $(df) \oplus_J (-df) \oplus 0$.

The construction of ψ' in the first step implies the existence of a homotopy connecting $p_1' \circ H \circ \pi_1^{-1}$ and $\xi : D \to M$, which is invariant under J and which is left fixed in D_0. Therefore, Ξ is homotopic to a monomorphism Ξ' satisfying:

(i α) Ξ' induces ξ.

(ii β) $\Xi' \circ \sigma = -\Xi'$.

(iii γ) $\Xi'|D_0 = \Xi|D_0$.

Now we want to construct the desired $\dot{\xi}$. Before doing so we need the following

SUBLEMMA. *Let B be a complex, and let A be a subcomplex of B. Let $E = E_1 \oplus E_2$ and $E' = E_1' \oplus E_2'$ be Whitney sums of vector bundles over B. Suppose that $\sigma : E \to E'$ is a bundle isomorphism whose restriction over A is the direct sum $\sigma_1^0 \oplus \sigma_2^0$ of bundle isomorphisms*

$$\sigma_1^0 : E_1|A \longrightarrow E_1'|A, \qquad \sigma_2^0 : E_2|A \longrightarrow E_2'|A.$$

We assume that σ_1^0 can be extended to an isomorphism $\sigma_1 : E_1 \to E_1'$ and that

$$\dim B < \dim(\textit{the fiber of } E_2).$$

Then there exists a bundle homomorphism $\sigma_2 : E_2 \to E_2'$ such that

(i) σ_2 is an extension of σ_2^0, and

(ii) there exists a homotopy connecting σ and $\sigma_1 \oplus \sigma_2$, which is left fixed on A.

We leave the proof to the reader.

Now back to the proof of Lemma 7.5. In the sublemma we take

$$E_1 = T(D) \oplus \epsilon / \sim, \qquad (t_d, e) \sim (-t_{J(d)}, -e),$$
$$E_2 = N(V, D) \oplus_J N(V, D)/\sim, \qquad (l_d, l_{J(d)}) \sim (-l_{J(d)}, -l_d).$$

Then E_1 and E_2 are fiber bundles over D/J. Denote by $\xi/J : D/J \to \bar{\psi}'(D') \subset M$ the induced map of $\xi : D \to M$. Set

$$E_1' = (\xi/J)^* T(\bar{\psi}'(D')),$$
$$E_2' = (\xi/J)^* N(M, \bar{\psi}'(D')).$$

Then as $\dim D < 2(n - \dim D)$ we may apply the sublemma to obtain Lemma 7.5. □

We next construct $\psi' : L' \to T(M)$. Recall that L' is the quotient of the bundle

$$(L \oplus_J L) \times [0, 1] \to D \times [0, 1]$$

by the equivalence relation $(l_d, l_{J(d)}, t) \sim (-l_{J(d)}, -l_d, -t)$. Hence, it suffices to construct a bundle monomorphism

$$\chi : (L \oplus_J L) \times [0, 1] \longrightarrow N(M, \psi'(D'))$$

satisfying the following:

(1) χ induces $\chi' : D \times [0, 1] \to D'$, $\chi'(d, t) = \psi'[d, t]$.

(2) $\chi|(L \oplus_J L) \times 0 = \dot{\xi}$.

(3) (a) $\chi(l_d, 0, \lambda(d)) = df(l_d)$, $\lambda(d) \geq 0$.

(b) $\chi(0, -l_d, -\lambda(d)) = df(l_d)$, $\lambda(d) \leq 0$.

We first construct the restriction χ_1 of χ to $(L \oplus_J 0) \times [0, 1]$. By the requirements (2) and (3a) χ_1 has already been defined on $D \times \{0\}$ and on the set $(d, \lambda(d))$, $\lambda(d) \geq 0$. The obstructions to extending χ_1 over $D \times [0, 1]$ are in the homotopy groups

$$\pi_i(V_{m-\dim D', n-\dim D}), \qquad i \leq \dim D,$$

where $V_{p,q}$ is the Stiefel manifold of q-frames in $\mathbf{R}^p$. But we know

$$\pi_i(V_{p,q}) = 0, \qquad i < p - q.$$

In our present case these homotopy groups all vanish if

$$\dim D < m - \dim D' - (n - \dim D),$$
$$0 < m - n - \dim D',$$

that is, if $3n + 1 < 2m$ since $\dim D' = \dim D + 1 = 2n - m + 1$. But $3n + 2 < 2m$ by our hypothesis. Thus, the above homotopy groups are zero for $i \leq \dim D = 2n - m$. Hence, we can extend χ_1 over $D \times [0, 1]$.

We next extend χ_1 to χ so that the requirements (2) and (3)(b) will be satisfied. This is not difficult because the set $D \times [0, 1]$ is a deformation retract of the set

$$D \times \{0\} \cup \{(d, \lambda(d)) | \lambda(d) \leq 0\}.$$

Now define

$$\dot\psi'[l_d, l_{J(d)}, t] = \begin{cases} \chi(l_d, l_{J(d)}, t), & t \geq 0, \\ \chi(-l_{J(d)}, -l_d, -t), & t \leq 0. \end{cases}$$

Constructions of Ψ and Ψ'. First, we construct Ψ in L_ϵ for a sufficiently small $\epsilon > 0$ such that

$$(d\Psi)_x(T_x(L_\epsilon)) = \dot\psi(T_x(L_\epsilon)), \qquad x \in D$$

(consider the exponential map exp). On the other hand, for $\varphi(x) = x' \in D'$ there exists a map ψ'_x from some neighborhood U'_x of x' in L' to M such that

$$(d\psi'_x)_{x'}(T_{x'}(U'_x)) = \dot\psi'(T_{x'}(U'_x)),$$
$$f \circ \Psi = \psi'_x \circ \Phi$$

(we can do this by choosing a suitable chart to apply the implicit function theorem). From this we construct a C^∞ map Ψ' of some neighborhood U' of D' in L' into M such that

$$(d\Psi')_{x'}(T_{x'}(U')) = \dot\psi'(T_{x'}(U')), \qquad x' \in D',$$
$$f \circ \Psi = \Psi' \circ \Phi.$$

We can choose ϵ so small that $\Psi' | L'_\epsilon$ is a diffeomorphism of L'_ϵ in U'.

Thus, we have shown the sufficiency of (a) in Theorem 7.3. We leave the rest of the proof for the reader to continue in the original paper of Haefliger C9].

THE PROOF OF THEOREM 7.3 (b). The method of proof is identical to that of the above. Replace V by $V \times [0, 1]$, M by $M \times [0, 1]$, and f by $F : V \times [0, 1] \to M \times [0, 1]$ defined by $F(v, t) = f_t(v)$. Then the condition $3(n + 2) \leq 2(m + 1)$ becomes $3(n + 1) < 2m$. □

Afterword

Chapter 0 featured Whitney [C20] as a means for giving the reader an intuitive preview of the book.

In Chapter I we dealt with the fundamentals of differential topology— C^r manifolds, C^r maps, fiber bundles, and other related concepts; we limited our discussion both in selection and scope to the parts essential for this book. For more in-depth information on the subject, we recommend *Differentiable manifolds* by Y. Matsushima.([1])

In Chapter II we discussed embeddings of manifolds, our focal point being Whitney's embedding theorems [C19], [C21]. Here we used, among other references, the book of Tamura.([2]) Our topics in this chapter included a method for eliminating double points in completely regular immersions. We saved Haefliger's generalization of this method for Chapter VII.

The theme of Chapter III was immersions of C^∞ manifolds. Here our discussion centered around the Smale-Hirsch theory—a natural generalization of (the topic in) Chapter 0—and included the theorems of Phillips and Gromov. The theorem of Gromov encompasses the submersion theorem of Phillips and is a generalization of the Smale-Hirsch theorem. Consequently, we presented the proofs of these theorems as corollaries of Gromov's theorem. For the proof of Gromov's theorem we followed Haefliger's presentation of the subject [B3]. We also mentioned that Gromov founded his theorem on Smale's "homotopy covering technique" [C17], "taking an idea out of an old wise man's paper".([*])

In Chapter IV we introduced yet another generalization of the Smale-Hirsch theory. Gromov's integration theory [C5], unlike his theorem in Chapter III, does not require openness for the base spaces of jet bundles. This constitutes an essential difference between these two works. Here we used a report paper of Shigeo Kawai. In our opinion this chapter points to a promising future direction for the subject of this book.

We presented in Chapter V an application—Haefliger's classification the-

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([1]) Y. Matsushima, *Differentiable manifolds*, Marcel Dekker, New York, 1972.
([2]) I.Tamura, *Differential topology*, Iwanami Shoten, Tokyo, 1978. (Japanese)
([*]) Editor's note. The phrase in quotes was added in translation by the author.

orem for foliations of open manifolds—of the theorem of Gromov in Chapter III. We recommend Tamura [A9] for the fundamentals of foliation theory.

We devoted Chapter VI to complex structures on open manifolds as an application of the theorems of Gromov in Chapters III and IV. The integrability of almost complex structures on open manifolds of arbitrary dimensions remains unsolved to date.

We gave an outline of a proof of Haefliger's embedding theorem in Chapter VII. As we mentioned above, we had wished to shed some light on the fact that Haefliger's proof is a natural extension of Whitney's method as used in eliminating double points of immersions; we are somewhat dubious as to what extent we succeeded in doing so. We feel that finding a sufficient condition, independent of connectivity, for the existence of embeddings of manifolds is a major open problem in this field.

We did not mention Haefliger's alternative proof for his embedding theorem [C8]. This is based on the so-called Whitney-Thom theory concerning singular sets of differentiable maps.

We also omitted any solid application of the Smale-Hirsch theory to manifolds; for this we refer the reader to Smale [B9] and James [B4].

From the historical perspective we marvel at the evolution of the simple problem of Chapter 0 as developed throughout our book to its present stage, and we expect further progress in the future.

Addendum. A videotaped version of the main topic of Chapter 0 is available.(3)

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