ALGEBRAIC K-THEORY OVER THE INFINITE DIHEDRAL GROUP

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ABSTRACT. We study the algebraic K-theory of $R[G]$, for any ring R and any group G with a surjection $p : G \to D_\infty$ onto the infinite dihedral group D_∞.

INTRODUCTION

For any ring R we establish isomorphisms between the codimension 1 splitting obstruction nilpotent class groups which arise in the decompositions of the algebraic K-theory of the R-coefficient group ring $R[G]$ of a group G with a surjection $p : G \to D_\infty$ onto the infinite dihedral group D_∞ and the α-twisted polynomial ring $R[F]_\alpha[t]$, with $F = \ker(p)$ and $\alpha : F \to F$ the automorphism such that $F \times_\alpha \mathbb{Z} = \ker(G \to \mathbb{Z}_2)$.

The infinite dihedral group D_∞ is such that:

(A) D_∞ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ of two cyclic groups of order 2, whose generators will be denoted t_1, t_2.

(B) D_∞ contains the infinite cyclic group $\mathbb{Z} = \langle t \rangle$ as a subgroup of index 2 with $t = t_1 t_2$

$$\{1\} \to \mathbb{Z} \to D_\infty \to \mathbb{Z}_2 \to \{1\} \ .$$

More generally, if G is a group with a surjection $p : G \to D_\infty$ then:

(A) $G = G_1 *_F G_2$ is a free product with amalgamation of two groups

$$G_1 \ = \ \ker(p_1 : G \to \mathbb{Z}_2) \ , \ G_2 \ = \ \ker(p_2 : G \to \mathbb{Z}_2) \subset G$$

with a common subgroup $F = \ker(p) = G_1 \cap G_2$ of index 2 in both G_1 and G_2.

(B) G has a subgroup $\bar{G}$ of index 2 which is an HNN extension

$$\{1\} \to F \to \bar{G} = F \times_\alpha \mathbb{Z} \to \mathbb{Z} \to \{1\}$$

for an automorphism $\alpha : F \to F$ such that $\alpha^2 = 1$.

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By Waldhausen [Wal78] the algebraic K-theory of $\mathbb{Z}[G]$ for any amalgamated free product $G = G_1 *_F G_2$ with $F \to G_1$, $F \to G_2$ injections decomposes as

$$K_*(\mathbb{Z}[G]) \;=\; K_*(\mathbb{Z}[F] \to \mathbb{Z}[G_1] \times \mathbb{Z}[G_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F]; \mathbb{Z}[G_1 \backslash F], \mathbb{Z}[G_2 \backslash F]) \;.$$

By Bass [Bas68] and Farrell-Hsiang [FH73] the algebraic K-theory of $\mathbb{Z}[\bar{G}]$ for any HNN extension of the type $\bar{G} = F \times_\alpha \mathbb{Z}$ decomposes as

$$K_*(\mathbb{Z}[\bar{G}]) \;=\; K_*(1-\alpha : \mathbb{Z}[F] \to \mathbb{Z}[F]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F], \alpha) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F], \alpha^{-1}) \;.$$

Given a ring R and an R-bimodule $\mathscr{B}$ let

$$T_R(\mathscr{B}) \;=\; R \oplus \mathscr{B} \oplus \mathscr{B} \otimes_R \mathscr{B} \oplus \dots$$

be the tensor algebra of $\mathscr{B}$ over R. The Nil-groups $\mathrm{Nil}_*(R, \mathscr{B})$ are defined to be the algebraic K-groups $K_*(\mathrm{Nil}(R, \mathscr{B}))$ of the exact category $\mathrm{Nil}(R, \mathscr{B})$ with objects pairs (P, ρ) with P a f.g. projective R-module and $\rho : P \to \mathscr{B} \otimes_R P$ a nilpotent R-module morphism. The reduced Nil-groups $\widetilde{\mathrm{Nil}}_*$ are such that

$$\mathrm{Nil}_*(R, \mathscr{B}) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R, \mathscr{B}) \;.$$

Waldhausen [Wal78] proved that if $\mathscr{B}$ is a f.g. projective R-module and free as a right R-module

$$K_*(T_R(\mathscr{B})) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R, \mathscr{B}) \;.$$

The Nil-groups of a ring automorphism $\alpha : R \to R$ are defined by

$$\mathrm{Nil}_*(R, \alpha) \;:=\; \mathrm{Nil}_*(R, R_\alpha)$$

with R_α the R-bimodule defined by the additive group of R with

$$R \times R_\alpha \times R \to R_\alpha \;;\; (r, x, s) \mapsto r.x.\alpha(s) \;,$$

and the reduced Nil-groups $\widetilde{\mathrm{Nil}}_*$ are such that

$$\mathrm{Nil}_*(R, \alpha) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R, \alpha) \;.$$

The tensor algebra in this case is the α-twisted polynomial extension

$$T_R(R_\alpha) \;=\; R_\alpha[t]$$

with t an indeterminate over R such that $rt = t\alpha(r)$ $(r \in R)$, and

$$\begin{aligned} K_*(R_\alpha[t]) \;&=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R, \alpha) \;, \\ K_*(R_\alpha[t, t^{-1}]) \;&=\; K_*(1-\alpha : R \to R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R, \alpha) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R, \alpha^{-1}) \;. \end{aligned}$$

The Nil-groups $\mathrm{Nil}_*(R, \mathscr{B}_1, \mathscr{B}_2)$ are defined for any ring R and R-bimodules $\mathscr{B}_1, \mathscr{B}_2$ to be the algebraic K-groups $K_*(\mathrm{Nil}(R, \mathscr{B}_1, \mathscr{B}_2))$ of the exact category $\mathrm{Nil}(R, \mathscr{B}_1, \mathscr{B}_2)$ with objects quadruples $(P_1, P_2, \rho_1, \rho_2)$ with P_1, P_2 f.g. projective R-modules and

$$\rho_1 \;:\; P_1 \to \mathscr{B}_1 \otimes_R P_2 \;,\; \rho_2 \;:\; P_2 \to \mathscr{B}_2 \otimes_R P_1$$

R-module morphisms such that $\rho_2\rho_1 : P_1 \to \mathscr{B}_1 \otimes_R \mathscr{B}_2 \otimes_R P_1$ is nilpotent. The reduced Nil-groups $\widetilde{\mathrm{Nil}}_*$ are such that

$$\mathrm{Nil}_*(R, \mathscr{B}_1, \mathscr{B}_2) \;=\; K_*(R) \oplus K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R, \mathscr{B}_1, \mathscr{B}_2) \;.$$

Theorem 0.1. *For any group G with a surjection $G \to D_\infty$ the $\widetilde{\mathrm{Nil}}$-groups associated to the HNN extension $\overline{G} = F \times_\alpha \mathbb{Z}$ are isomorphic to the $\widetilde{\mathrm{Nil}}$-groups associated to the amalgamated free product decomposition $G = G_1 *_F G_2$*

$$\widetilde{i} \ : \ \widetilde{\mathrm{Nil}}_*(\mathbb{Z}[F],\alpha) \ \cong \ \widetilde{\mathrm{Nil}}_*(\mathbb{Z}[F];\mathbb{Z}[G_1\backslash F],\mathbb{Z}[G_2\backslash F]) \ .$$

□

Remark 0.2. The inclusion $\mathbb{Z}[F]_\alpha[t] \to \mathbb{Z}[G]$ sending t to t_1t_2 for any $t_1 \in G_1\backslash F$, $t_2 \in G_2\backslash F$ induces morphisms

$$\begin{aligned} K_*(\mathbb{Z}[F]_\alpha[t]) \ &= \ K_*(\mathbb{Z}[F]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F],\alpha) \\ \to K_*(\mathbb{Z}[G]) \ &= \ K_*(\mathbb{Z}[F] \to \mathbb{Z}[G_1] \times \mathbb{Z}[G_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F];\mathbb{Z}[G_1\backslash F],\mathbb{Z}[G_2\backslash F]) \end{aligned}$$

which are the isomorphisms $\widetilde{i}$ of 0.1 on the $\widetilde{\mathrm{Nil}}$-groups. The transfer maps

$$\begin{aligned} K_*(\mathbb{Z}[G]) \ &= \ K_*(\mathbb{Z}[F] \to \mathbb{Z}[G_1] \times \mathbb{Z}[G_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F];\mathbb{Z}[G_1\backslash F],\mathbb{Z}[G_2\backslash F]) \\ \to K_*(\mathbb{Z}[\bar{G}]) \ &= \ K_*(1-\alpha : \mathbb{Z}[F] \to \mathbb{Z}[F]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F],\alpha) \oplus \widetilde{\mathrm{Nil}}_{*-1}(\mathbb{Z}[F],\alpha) \end{aligned}$$

are given by $\widetilde{i}^{-1} \oplus \widetilde{i}^{-1}$ on the $\widetilde{\mathrm{Nil}}$-groups.

□

Remark 0.3. Lafont and Ortiz [LO07] proved that for a virtually cyclic group G with a surjection $G \to D_\infty$ and $* = 0,1$ $\widetilde{\mathrm{Nil}}_*(\mathbb{Z}[F];\mathbb{Z}[G_1\backslash F],\mathbb{Z}[G_2\backslash F]) = 0$ if and only if $\widetilde{\mathrm{Nil}}_*(\mathbb{Z}[F],\alpha) = 0$.

□

In fact, Theorem 0.1 is the special case $R = \mathbb{Z}[F]$, $\mathscr{B}_1 = \mathbb{Z}[G_1\backslash F]$, $\mathscr{B}_2 = \mathbb{Z}[G_2\backslash F]$ of the following general result:

Theorem 0.4 (Algebraic semi-splitting)**.** *Let R be a ring and let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules such that $\mathscr{B}_2$ is a f.g. projective left R-module. The* Nil*-groups of $(R,\mathscr{B}_1,\mathscr{B}_2)$ are related to the* Nil*-groups of $(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2)$ by isomorphisms*

$$\begin{aligned} \mathrm{Nil}_*(R,\mathscr{B}_1,\mathscr{B}_2) \ &\cong \ \mathrm{Nil}_*(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2) \oplus K_*(R) \ , \\ \widetilde{\mathrm{Nil}}_*(R,\mathscr{B}_1,\mathscr{B}_2) \ &\cong \ \widetilde{\mathrm{Nil}}_*(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2) \ . \end{aligned}$$

In particular, for $* = 0$ there are defined inverse isomorphisms

$$\begin{aligned} &i \ : \ \mathrm{Nil}_0(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2) \oplus K_0(R) \to \mathrm{Nil}_0(R,\mathscr{B}_1,\mathscr{B}_2) \ ; \\ &([P_1,\rho_{12} : P_1 \to \mathscr{B}_1 \otimes_R \mathscr{B}_2 \otimes_R P_1],[P_2]) \mapsto [P_1, \mathscr{B}_2 \otimes_R P_1 \oplus P_2, \begin{pmatrix} \rho_{12} \\ 0 \end{pmatrix}, (1\ 0)] \ , \\ &j \ : \ \mathrm{Nil}_0(R,\mathscr{B}_1,\mathscr{B}_2) \to \mathrm{Nil}_0(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2) \oplus K_0(R) \ ; \\ &[P_1,P_2,\rho_1 : P_1 \to \mathscr{B}_1 \otimes_R P_2, \rho_2 : P_2 \to \mathscr{B}_2 \otimes_R P_1] \mapsto ([P_1,\rho_2 \circ \rho_1],[P_2] - [\mathscr{B}_2 \otimes_R P_1]) \ . \end{aligned}$$

The reduced versions are the inverse isomorphisms

$$\begin{aligned} &\widetilde{i} \ : \ \widetilde{\mathrm{Nil}}_0(R,\mathscr{B}_1 \otimes_R \mathscr{B}_2) \to \widetilde{\mathrm{Nil}}_0(R,\mathscr{B}_1,\mathscr{B}_2) \ ; \ [P_1,\rho_{12}] \mapsto [P_1, \mathscr{B}_2 \otimes_R P_1, \rho_{12}, 1] \ , \\ &\widetilde{j} \ : \ \widetilde{\mathrm{Nil}}_0(R,\mathscr{B}_1,\mathscr{B}_2) \to \widetilde{\mathrm{Nil}}_0(R,\mathscr{B}_1 \otimes_S \mathscr{B}_2) \ ; \ [P_1,P_2,\rho_1,\rho_2] \mapsto [P_1,\rho_2 \circ \rho_1] \ . \end{aligned}$$

□

The proof of Theorem 0.4 is motivated by the obstruction theory of Waldhausen [Wal69] splitting homotopy equivalences of finite CW complexes X along codimension 1 subcomplexes $Y \subset X$ with $\pi_1(Y) \to \pi_1(X)$ injective, and the subsequent algebraic K-theory decomposition theorems of Waldhausen [Wal78].

A *codimension 1 pair* $(X, Y \subset X)$ is a pair of spaces such that the inclusion $Y = Y \times \{0\} \subset X$ extends to an open embedding $Y \times \mathbb{R} \subset X$. A *map* of codimension 1 pairs $(f,g) : (M,N) \to (X,Y)$ has $g = f|_N : N = f^{-1}(Y) \to Y$.

Let (X,Y) be a codimension 1 finite CW pair, A homotopy equivalence $f : M \to X$ from a finite CW complex *splits at* $Y \subset X$ if f is simple homotopic to a map of pairs $(f,g) : (M,N) \to (X,Y)$ such that $g : N \to Y$ is also a homotopy equivalence.

A codimension 1 pair (X,Y) is *injective* if X, Y are connected and $\pi_1(Y) \to \pi_1(X)$ is injective. Let $\widetilde{X}$ be the universal cover of X. The cover of X

$$\bar{X} \;=\; \widetilde{X}/\pi_1(Y)$$

is such that $(\bar{X}, Y)$ is a codimension 1 pair with

$$\bar{X} \;=\; \bar{X}^+ \cup_Y \bar{X}^-$$

for connected subspaces $\bar{X}^+, \bar{X}^- \subset \bar{X}$ such that

$$\pi_1(\bar{X}) \;=\; \pi_1(\bar{X}^+) \;=\; \pi_1(\bar{X}^-) \;=\; \pi_1(Y) \ .$$

As usual, there are two cases:
(A) $X \backslash Y$ is disconnected, so

$$X \;=\; X_1 \cup_Y X_2$$

with X_1, X_2 connected. By the Seifert-van Kampen theorem

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

is the amalgamated free product, with $\pi_1(Y) \to \pi_1(X_1)$, $\pi_1(Y) \to \pi_1(X_2)$ injective. The labelling is to be chosen such that

$$\bar{X}_1 \;=\; \widetilde{X}_1/\pi_1(Y) \subset \bar{X}^+ \ , \ \bar{X}_2 \;=\; \widetilde{X}_2/\pi_1(Y) \subset \bar{X}^- \ .$$

(B) $X \backslash Y$ is connected, so

$$X \;=\; X_1/\{y = ty | y \in Y\}$$

for a connected space X_1 (a deformation retract of $X \backslash Y$) which contains two disjoint copies $Y, tY \subset X_1$ of Y. We shall only consider the case when $\pi_1(Y) \to \pi_1(X_1)$, $\pi_1(tY) \to \pi_1(X_1)$ are isomorphisms, so that

$$\pi_1(X) \;=\; \pi_1(Y) \times_\alpha \mathbb{Z}$$

for an automorphism $\alpha : \pi_1(Y) \to \pi_1(Y)$ and $\overline{X}$ is an infinite cyclic cover of X with a generating covering translation $t : \overline{X} \to \overline{X}$. The labelling is to be chosen such that $\bar{X}_1 \subset \bar{X}^+$, $t\bar{X}_1 \subset \bar{X}^-$.

In both cases $(\overline{X}, Y)$ is an injective codimension 1 pair of type (A).

The *kernel* $\mathbb{Z}[\pi_1(X)]$*-modules* of a map $f : M \to X$ are the relative homology $\mathbb{Z}[\pi_1(X)]$-modules

$$K_r(M) \;=\; H_{r+1}(\widetilde{f} : \widetilde{M} \to \widetilde{X})$$

with $\widetilde{X}$ the universal cover of X, $\widetilde{M} = f^*\widetilde{X}$ the pullback cover of M and $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ a $\pi_1(X)$-equivariant lift of f. For a map of injective codimension 1 CW pairs $(f, g) : (M, N) \to (X, Y)$ the kernel $\mathbb{Z}[\pi_1(Y)]$-modules fit into an exact sequence

$$\dots \to K_r(N) \to K_r(\bar{M}) \to K_r(\bar{M}^+, N) \oplus K_r(\bar{M}^-, N) \to K_{r-1}(N) \to \dots \ .$$

If f is a homotopy equivalence and $g_* : \pi_1(N) \to \pi_1(Y)$ is an isomorphism then g is a homotopy equivalence if and only if $K_*(N) = 0$, if and only if $K_*(\bar{M}^+, N) = K_*(\bar{M}^-, N) = 0$.

Theorem 0.5. (Waldhausen [Wal69] for (A), Farrell-Hsiang [FH73] for (B)) *Let (X, Y) be an injective codimension 1 finite CW pair, and let $f : M \to X$ be a homotopy equivalence from a finite CW complex.*

(i) *f is simple homotopic to a map of pairs $(f, g) : (M, N) \to (X, Y)$ with $g_* : \pi_1(N) \to \pi_1(Y)$ an isomorphism and for some $n \geqslant 2$*

$$K_r(N) \ = \ 0 \text{ for } r \neq n \ ,$$

so that $K_{n+1}(\bar{M}^+, N)$, $K_{n+1}(\bar{M}^-, N)$ are f.g. projective $\mathbb{Z}[\pi_1(Y)]$-modules such that

$$K_{n+1}(\bar{M}^+, N) \oplus K_{n+1}(\bar{M}^-, N) \ = \ K_n(N)$$

is stably f.g. free and

$$[K_{n+1}(\bar{M}^+, N)] \ = \ -[K_{n+1}(\bar{M}^-, N)] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \ .$$

(ii) *In case* (A) *there is defined an exact sequence*

$$\begin{aligned} \dots \to \mathrm{Wh}(\pi_1(X_1)) \oplus \mathrm{Wh}(\pi_1(X_2)) &\to \mathrm{Wh}(\pi_1(X)) \\ &\to \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \mathscr{B}_1, \mathscr{B}_2) \to \dots \ . \end{aligned}$$

The Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(X))$ has image

$$\begin{aligned} [\tau(f)] \ &= \ ([K_{n+1}(\bar{M}^+, N)], [K_{n+1}(\bar{M}^+, N), K_{n+1}(\bar{M}^-, N), \rho_1, \rho_2]) \\ &\in \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \mathscr{B}_1, \mathscr{B}_2) \end{aligned}$$

with

$$\begin{aligned} &\mathscr{B}_1 \ = \ \mathbb{Z}[\pi_1(X_1) \backslash \pi_1(Y)] \ , \ \mathscr{B}_2 \ = \ \mathbb{Z}[\pi_1(X_2) \backslash \pi_1(Y)] \ , \\ &\rho_1 \ : \ K_{n+1}(\bar{M}^+, N) \to K_{n+1}(\bar{M}^+, \bar{M}_1) \ = \ \mathscr{B}_1 \otimes_{\mathbb{Z}[\pi_1(Y)]} K_{n+1}(\bar{M}^-, N) \ , \\ &\rho_2 \ : \ K_{n+1}(\bar{M}^-, N) \to K_{n+1}(\bar{M}^-, \bar{M}_2) \ = \ \mathscr{B}_2 \otimes_{\mathbb{Z}[\pi_1(Y)]} K_{n+1}(\bar{M}^+, N) \ . \end{aligned}$$

The homotopy equivalence f splits along $Y \subset X$ (up to simple homotopy) if and only if $[\tau(f)] = 0$.

(iii) *In case* (B) *there is defined an exact sequence*

$$\begin{aligned} \dots \to \mathrm{Wh}(\pi_1(Y)) \xrightarrow{1-\alpha} \ &\mathrm{Wh}(\pi_1(Y)) \to \mathrm{Wh}(\pi_1(X)) \\ &\to \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \alpha) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \alpha^{-1}) \to \dots \ . \end{aligned}$$

The Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(X))$ has image

$$\begin{aligned} [\tau(f)] \ &= \ ([K_{n+1}(\bar{M}^+, N)], [K_{n+1}(\bar{M}^+, N), \rho_1], [K_{n+1}(\bar{M}^-, N), \rho_2]) \\ &\in \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \alpha) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], \alpha^{-1}) \ , \end{aligned}$$

with

$$\begin{aligned} &\rho_1 \ : \ K_{n+1}(\bar{M}^+, N) \to K_{n+1}(\bar{M}^+, \bar{M}_1) \ = \ \alpha K_{n+1}(\bar{M}^+, N) \ , \\ &\rho_2 \ : \ K_{n+1}(\bar{M}^-, N) \to K_{n+1}(\bar{M}^-, t\bar{M}_1) \ = \ \alpha K_{n+1}(\bar{M}^-, N) \ . \end{aligned}$$

The homotopy equivalence f splits along $Y \subset X$ (up to simple homotopy) if and only if $[\tau(f)] = 0$.

□

Let (X, Y) be an injective type (A) codimension 1 finite CW pair, so that $X = X_1 \cup_Y X_2$. A homotopy equivalence $f : M \to X$ from a finite CW complex is *semi-split at $Y \subset X$* if the restriction $N = f^{-1}(Y) \subset M$ is a subcomplex and $(f, g = f|_N) : (M, N) \to (X, Y)$ is a map of pairs such that $K_*(\bar{M}_2, N) = 0$, or equivalently such that the $\mathbb{Z}[\pi_1(Y)]$-module morphism

$$\rho_2 \ : \ K_*(\bar{M}^-, N) \to K_*(\bar{M}^-, \bar{M}_2) \ = \ \mathscr{B}_2 \otimes_{\mathbb{Z}[\pi_1(Y)]} K_*(\bar{M}^+, N)$$

is an isomorphism.

The proof of Theorem 0.5 (i) in [Wal69] was based on a one-one correspondence between the elementary operations in the algebraic K-theory of the nilpotent categories and the elementary operations ('surgeries' or cell-exchanges) for maps of injective codimension 1 pairs. The proof of our Theorem 0.4 shows that there is no algebraic obstruction to making a homotopy equivalence semi-split by elementary operations, and hence there is no geometric obstruction:

Corollary 0.6 (Topological semi-splitting)**.** *If (X, Y) is an injective type* (A) *codimension 1 finite CW pair such that $\pi_1(Y) \subset \pi_1(X_2)$ is a subgroup of finite index, then any homotopy equivalence $f : M \to X = X_1 \cup_Y X_2$ from a finite CW complex M is simple homotopic to a semi-split homotopy equivalence.*

□

1. Higher Nil-groups

Recall that D. Quillen defined the K-theory space $K\mathscr{E} := \Omega BQ(\mathscr{E})$ of an exact category $\mathscr{E}$ [Qui73]. The space $BQ(\mathscr{E})$ is the geometric realization of the simplicial set $N_\bullet Q(\mathscr{E})$, which is the nerve of a certain associated category $Q(\mathscr{E})$. Each of the two Nil-categories defined in the Introduction have the structure of exact categories.

We shall use the following short-hand notation. Let R be ring. For a right R-module M and a left R-module N, write $MN := M \otimes_R N$. For an R-bimodule $\mathscr{B}$ and $n \in \mathbb{N}$, write $\mathscr{B}^n := \underbrace{\mathscr{B} \otimes_R \cdots \otimes_R \mathscr{B}}_{r \text{ copies}}$ with $\mathscr{B}^0 := R$.

Theorem 1.1. *Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Suppose the left R-module structure on $\mathscr{B}_2$ is finitely generated and projective. Observe there is defined an exact functor i of exact categories of projective nil-objects:*

$$i : \mathrm{Nil}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \longrightarrow \mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2); \quad (Q, \sigma) \longmapsto (Q, \mathscr{B}_2 \otimes_R Q, \sigma, 1).$$

Then the induced map of K-theory spaces is a homotopy equivalence:

$$\bar{K}i : K\mathrm{Nil}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \longrightarrow K\mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2)/0 \times K(R).$$

In particular, for all $n \in \mathbb{N}$, there is an induced isomorphism of abelian groups:

$$i_* : \mathrm{Nil}_n(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \times K_n(R) \longrightarrow \mathrm{Nil}_n(R; \mathscr{B}_1, \mathscr{B}_2).$$

There exists an exact functor j such that $j \circ i = 1$, defined by

$$j : \mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2) \longrightarrow \mathrm{Nil}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2); \quad (P_1, P_2, \rho_1, \rho_2) \longmapsto (P_1, (1_{\mathscr{B}_1} \otimes \rho_2) \circ \rho_1).$$

Proof. Our setting is the exact category $\mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2)$. Consider objects

$$\begin{array}{rcl} x & = & (P_1, P_2, \rho_1, \rho_2) \\ x' & := & (P_1, \mathscr{B}_2 P_1 \oplus P_2, \left(\begin{smallmatrix} 0 \\ \rho_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \rho_2 \end{smallmatrix}\right)) \\ x'' & := & (P_1, \mathscr{B}_2 P_1, \rho_2 \circ \rho_1, 1) \\ a & := & (0, P_2, 0, 0) \\ a' & := & (0, \mathscr{B}_2 P_1, 0, 0). \end{array}$$

Define morphisms

$$\begin{array}{rcl} f & := & (1, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) : x \longrightarrow x' \\ f' & := & (1, \left(\begin{smallmatrix} 1 & \rho_2 \end{smallmatrix}\right)) : x' \longrightarrow x'' \\ g & := & (0, \left(\begin{smallmatrix} \rho_2 \\ -1 \end{smallmatrix}\right)) : a \longrightarrow x' \\ g' & := & (0, \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)) : x' \longrightarrow a' \\ h & := & (0, \rho_2) : a \longrightarrow a'. \end{array}$$

Observe the admissible exact sequences

$$0 \longrightarrow x \oplus a \xrightarrow{\left(\begin{smallmatrix} f & g \\ 0 & 1 \end{smallmatrix}\right)} x' \oplus a \xrightarrow{\left(\begin{smallmatrix} g' & h \end{smallmatrix}\right)} a' \longrightarrow 0$$

$$0 \longrightarrow a \xrightarrow{g} x' \xrightarrow{f'} x'' \longrightarrow 0.$$

Consider endofunctors

$$\begin{array}{rcl} F' & : & x \longmapsto x' \\ F'' & : & x \longmapsto x'' \\ G & : & x \longmapsto a \\ G' & : & x \longmapsto a'. \end{array}$$

Recall $j \circ i = 1$, and note $i \circ j = F''$. By Quillen's Additivity Theorem [Qui73, p. 98, Cor. 1], we obtain that $KF' \simeq 1 + KG'$ and $KF' \simeq KG + KF''$ are homotopic maps to infinite loop spaces. Then $Ki \circ Kj \simeq 1 + (KG' - KG)$. Observe $G, G' : \mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2) \to 0 \times \mathrm{Proj}(R)$. Therefore the functor i induces a homotopy equivalence of K-theory spaces:

$$\bar{K}i : K\mathrm{Nil}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \longrightarrow K\mathrm{Nil}(R; \mathscr{B}_1, \mathscr{B}_2)/0 \times K(R).$$

$\square$

2. Lower Nil-groups

2.1. **Cone and suspension rings.** Let us recall some additional structures on the tensor product of modules.

Definition 2.1. Let S be a commutative ring. Let R_1, R_2 be S-algebras, which means that R_i is a ring equipped with a ring map $S \to \mathrm{Center}(R_i)$. Then the tensor product $R_1 \otimes_S R_2$ is an S-algebra, with multiplication given by

$$(r_1 \otimes r_2) \cdot (r_1' \otimes r_2') := r_1 r_1' \otimes r_2 r_2'.$$

Let $\mathscr{B}_i$ be an R_i-bimodule. Then the S-bimodule $\mathscr{B}_1 \otimes_S \mathscr{B}_2$ has the structure of an $(R_1 \otimes_S R_2)$-bimodule:

$$(r_1 \otimes r_2) \cdot (b_1 \otimes b_2) \cdot (r_1' \otimes r_2') := r_1 b_1 r_1' \otimes r_2 b_2 r_2'.$$

Originating from ideas of Karoubi and Villamayor [KV69], the following concept was studied independently by S.M. Gersten [Ger72] and J.B. Wagoner [Wag72] in the construction of the non-connective K-theory spectrum of a ring.

Definition 2.2 (Gersten, Wagoner)**.** Let the **cone ring** $\Lambda\mathbb{Z}$ be the ring of $(\mathbb{N}\times\mathbb{N})$-matrices over $\mathbb{Z}$ such that each row and column has only a finite number of non-zero entries. Let the **suspension ring** $\Sigma\mathbb{Z}$ be the quotient ring of $\Lambda\mathbb{Z}$ by the two-sided ideal of matrices with only a finite number of non-zero entries. For each $n \in \mathbb{N}$, write $\Sigma^n\mathbb{Z} := \underbrace{\Sigma\mathbb{Z} \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \Sigma\mathbb{Z}}_{n \text{ copies}}$ with $\Sigma^0\mathbb{Z} = \mathbb{Z}$.

Let R be a ring. Let $\mathscr{B}$ be an R-bimodule. Consider the additional structures in Definition 2.1. Then, for each $n \in \mathbb{N}$, we obtain $\Sigma^n R := \Sigma^n\mathbb{Z} \otimes_\mathbb{Z} R$ is a ring and $\Sigma^n\mathscr{B} := \Sigma^n\mathbb{Z} \otimes_\mathbb{Z} \mathscr{B}$ is a $\Sigma^n R$-bimodule. Roughly speaking, the suspension should be regarded as the ring of "bounded modulo compact operators." Gersten and Wagoner showed that $K_i(\Sigma^n R)$ is naturally isomorphic to $K_{i-n}(R)$ for all $i, n \in \mathbb{N}$, in the sense of Quillen when the subscript is positive, in the sense of Grothendieck when the subscript is zero, and in the sense of Bass when the subscript is negative.

Lemma 2.3. *Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Then, for each $n \in \mathbb{N}$, there is a natural isomorphism of $\Sigma^n R$-bimodules:*

$$t_n : \Sigma^n(\mathscr{B}_1 \otimes_R \mathscr{B}_2) \longrightarrow \Sigma^n\mathscr{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathscr{B}_2; \quad s \otimes (b_1 \otimes b_2) \longmapsto (s \otimes b_1) \otimes (1_{\Sigma^n R} \otimes b_2).$$

Proof. By transposition of the middle two factors, note that

$$\Sigma^n\mathscr{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathscr{B}_2 = (\Sigma^n\mathbb{Z} \otimes_\mathbb{Z} \mathscr{B}_1) \otimes_{(\Sigma^n\mathbb{Z} \otimes_\mathbb{Z} R)} (\Sigma^n\mathbb{Z} \otimes_\mathbb{Z} \mathscr{B}_2)$$

is isomorphic to

$$(\Sigma^n\mathbb{Z} \otimes_{\Sigma^n\mathbb{Z}} \Sigma^n\mathbb{Z}) \otimes_\mathbb{Z} (\mathscr{B}_1 \otimes_R \mathscr{B}_2) = \Sigma^n\mathbb{Z} \otimes_\mathbb{Z} (\mathscr{B}_1 \otimes_R \mathscr{B}_2) = \Sigma^n(\mathscr{B}_1 \otimes_R \mathscr{B}_2).$$

$\square$

2.2. **Definition of lower Nil-groups.** Associated to the generalized polynomial extension $T_R(\mathscr{B})$, we define the lower K-groups of the Nil-category of the pair $(R; B)$, as follows.

Definition 2.4. Let R be a ring. Let $\mathscr{B}$ be an R-bimodule. For all $n \in \mathbb{N}$, define

$$\begin{aligned} \Sigma^n(R;\mathscr{B}) &:= (\Sigma^n R; \Sigma^n\mathscr{B}) \\ \mathrm{Nil}_{-n}(R;\mathscr{B}) &:= \mathrm{Nil}_0 \Sigma^n(R;\mathscr{B}) \\ &= K_{-n}(R) \times \widetilde{\mathrm{Nil}}_{-n}(R;\mathscr{B}). \end{aligned}$$

Associated to a pure amalgamated product $A_1 *_R A_2$ of rings, A. Bartels and W. Lück defined the non-connective K-theory Nil-spectrum of the triple $(R; \mathscr{B}_1, \mathscr{B}_2)$ [BL06, Defn. 9.4]. Their negative homotopy groups are given as follows.

Definition 2.5 (Bartels-Lück)**.** Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. For all $n \in \mathbb{N}$, define

$$\begin{aligned} \Sigma^n(R; \mathscr{B}_1, \mathscr{B}_2) &:= (\Sigma^n R; \Sigma^n \mathscr{B}_1, \Sigma^n \mathscr{B}_2) \\ \mathrm{Nil}_{-n}(R; \mathscr{B}_1, \mathscr{B}_2) &:= \mathrm{Nil}_0 \Sigma^n(R; \mathscr{B}_1, \mathscr{B}_2) \\ &= K_{-n}(R) \times K_{-n}(R) \times \widetilde{\mathrm{Nil}}_{-n}(R; \mathscr{B}_1, \mathscr{B}_2). \end{aligned}$$

The next two propositions follow from the definitions and [Wal78, Theorems 1,3].

Proposition 2.6. (Waldhausen) *Let R be a ring and $\mathscr{B}$ an R-bimodule. Consider the tensor ring*

$$A = T_R(\mathscr{B}) = R \oplus \mathscr{B} \oplus \mathscr{B}\mathscr{B} \oplus \mathscr{B}\mathscr{B}\mathscr{B} \oplus \cdots .$$

Suppose $\mathscr{B}$ is finitely generated projective as a left R-module and free as a right R-module. Then, for all $n \in \mathbb{N}$, there is a split monomorphism

$$\widetilde{\mathrm{Nil}}_{-n}(R; \mathscr{B}) \longrightarrow K_{1-n}(A); \quad [Q, \nu] \longmapsto [A \otimes_R Q, 1 + \nu].$$

Furthermore, there is a natural decomposition

$$K_{1-n}(A) = K_{1-n}(R) \oplus \widetilde{\mathrm{Nil}}_{-n}(R; \mathscr{B}).$$

□

Proposition 2.7. (Waldhausen) *Let R, A_1, A_2 be rings. Let $R \to A_i$ be ring monomorphisms such that $A_i = R \oplus \mathscr{B}_i$ for some R-bimodule $\mathscr{B}_i$. Consider the pure pushout of rings*

$$A = A_1 *_R A_2 = R \oplus (\mathscr{B}_1 \oplus \mathscr{B}_2) \oplus (\mathscr{B}_1\mathscr{B}_2 \oplus \mathscr{B}_2\mathscr{B}_1) \oplus (\mathscr{B}_1\mathscr{B}_2\mathscr{B}_1 \oplus \mathscr{B}_2\mathscr{B}_1\mathscr{B}_2) \oplus \cdots .$$

Suppose each $\mathscr{B}_i$ is free as a right R-module. Then, for all $n \in \mathbb{N}$, there is a split monomorphism

$$\widetilde{\mathrm{Nil}}_{-n}(R; \mathscr{B}_1, \mathscr{B}_2) \longrightarrow K_{1-n}(A); \quad [P_1, P_2, \rho_1, \rho_2] \longmapsto [A \otimes_R (P_1 \oplus P_2), \left(\begin{smallmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{smallmatrix}\right)].$$

Furthermore, there is a natural Mayer-Vietoris type exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & K_{1-n}(R) & \longrightarrow & K_{1-n}(A_1) \oplus K_{1-n}(A_2) & & \\ & \longrightarrow & \dfrac{K_{1-n}(A)}{\widetilde{\mathrm{Nil}}_{-n}(R; \mathscr{B}_1, \mathscr{B}_2)} & \xrightarrow{\partial} & K_{-n}(R) & \longrightarrow & \cdots . \end{array}$$

□

2.3. **The isomorphism for lower Nil-groups.**

Theorem 2.8. *Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Suppose the left R-module structure on $\mathscr{B}_2$ is finitely generated and projective. Then, for all $n \in \mathbb{N}$, there is an induced isomorphism of abelian groups:*

$$i_* \circ t_* : \mathrm{Nil}_{-n}(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2) \times K_{-n}(R) \longrightarrow \mathrm{Nil}_{-n}(R; \mathscr{B}_1, \mathscr{B}_2).$$

Proof. Let $n \in \mathbb{N}$. By Lemma 2.3 and Theorem 1.1, the composite $i_* \circ t_*$ consists of induced isomorphisms:

$$\begin{aligned}\mathrm{Nil}_{-n}(R;\mathscr{B}_1 \otimes_R \mathscr{B}_2) \times K_{-n}(R) &= \mathrm{Nil}_0\Sigma^n(R;\mathscr{B}_1 \otimes_R \mathscr{B}_2) \times K_0\Sigma^n(R)\\ &\xrightarrow{\mathrm{Nil}_0(\mathbf{1}_{\Sigma^n R};t_n)} \mathrm{Nil}_0(\Sigma^n R;\Sigma^n\mathscr{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathscr{B}_2) \times K_0\Sigma^n(R)\\ &\xrightarrow{\mathrm{Nil}_0(i\circ\Sigma^n)} \mathrm{Nil}_0\Sigma^n(R;\mathscr{B}_1,\mathscr{B}_2) = \mathrm{Nil}_{-n}(R;\mathscr{B}_1,\mathscr{B}_2).\end{aligned}$$

□

3. Applications to injective amalgams of groups

Natural examples of such bimodules originate from certain ring decompositions.

Corollary 3.1. *Let $A = A_- *_{A_0} A_+$ be a pure pushout of rings. Write $A_\pm = A_0 \oplus \mathscr{B}_\pm$ for some A_0-bimodules $\mathscr{B}_\pm$. Suppose the left A_0-module structure on $\mathscr{B}_-$ is finitely generated and projective. Then, for all $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups:*

$$\mathrm{Nil}_n(A_0;\mathscr{B}_- \otimes_{A_0} \mathscr{B}_+) \times K_n(A_0) \longrightarrow \mathrm{Nil}_n(A_0;\mathscr{B}_-,\mathscr{B}_+).$$

□

Here is our reduction for a certain class of group rings.

Corollary 3.2. *Let R be a ring. Suppose $G = G_- *_{G_0} G_+$ is an injective amalgam of groups such that $[G_- : G_0]$ is finite. Write $\mathscr{B}_\pm := R[G_\pm - G_0]$. Then, for all $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups:*

$$\widetilde{\mathrm{Nil}}_n(R[G_0];\mathscr{B}_-,\mathscr{B}_+) \longrightarrow NK_{n+1}(R[G_0];\mathscr{B}_- \otimes_{R[G_0]} \mathscr{B}_+).$$

□

The case of $G = D_\infty = \mathbb{Z}_2 *_1 \mathbb{Z}_2$ has a particularly simple form.

Corollary 3.3. *Let R be a ring. Then, for all $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups:*

$$\widetilde{\mathrm{Nil}}_n(R;R,R) \longrightarrow NK_{n+1}(R).$$

□

Our main application is the class of virtually cyclic groups of infinite dihedral type. Their K-theory is conjectured to be the one of the three building blocks of the K-theory of arbitrary groups [FJ93].

Corollary 3.4. *Let R be a ring. Consider any group extension*

$$1 \longrightarrow F \longrightarrow G \longrightarrow D_\infty \longrightarrow 1$$

where F is finite. There is an induced decomposition $G = G_- *_F G_+$ where $[G_\pm : F] = 2$. Uniquely up to inner automorphisms, there are induced group automorphisms $\alpha, \beta : F \to F$ such that as $R[F]$-bimodules:

$$R[G_-] = R[F] \oplus {}_\alpha R[F] \quad \text{and} \quad R[G_+] = R[F] \oplus R[F]_\beta.$$

Then, for all $n \in \mathbb{Z}$, there is an isomorphism of abelian groups:

$$H_n^G(E_{\mathrm{vc}}(G), E_{\mathrm{fin}}(G); \mathbf{K}_R) \longrightarrow NK_n({}_\alpha R[F]_\beta).$$

□

3.1. **Jim's application.** Let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3 = PSL_2(\mathbb{Z})$. The following theorem follows from applying our main theorem and the recent proof [BLR] of the K-theoretic Farrell-Jones conjecture in K-theory for word hyperbolic groups.

Theorem 3.5. *For any ring R and integer q,*

$$K_q(R\Gamma) = K_q R \oplus \widetilde{K}_q(R\mathbb{Z}_2) \oplus \widetilde{K}_q(R\mathbb{Z}_3) \oplus \bigoplus_{\mathcal{M}} \widetilde{\mathrm{Nil}}_{q-1}(R)$$

where the sum is over all the conjugacy classes of maximal infinite dihedral subgroups.

Proof. By Waldhausen's theorem (see also Davis [Dav]), the homology exact sequence of the pair $(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma)$

$$H_q^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \to H_q^\Gamma(E_{\mathrm{all}}\Gamma; \mathbf{K}_R) \to H_q^\Gamma(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R)$$

is short exact and split. Hence

$$K_q(R\Gamma) = H_q^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \oplus H_q^\Gamma(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \tag{3.5.1}$$

Note that $E_{\mathrm{fin}}\Gamma$ is constructed as a pullback of Γ-spaces

$$\begin{tikzcd}
\Gamma \sqcup \Gamma \arrow[r] \arrow[d] & \Gamma/\mathbb{Z}_2 \sqcup \Gamma/\mathbb{Z}_3 \arrow[d] \\
\Gamma \times D^1 \arrow[r] & E_{\mathrm{fin}}\Gamma
\end{tikzcd}$$

Then $E_{\mathrm{fin}}\Gamma$ is simply the Bass-Serre tree for $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$. Note that $H_*^\Gamma(\Gamma/H; \mathbf{K}_R) = K_*(RH)$. Hence there is a long exact sequence

$$\cdots \to K_q(R) \to K_q(R\mathbb{Z}_2) \oplus K_q(R\mathbb{Z}_3) \to H_q^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \to K_{q-1}(R) \to \cdots$$

Thus

$$H_q^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) = K_q R \oplus \widetilde{K}_q(R\mathbb{Z}_2) \oplus \widetilde{K}_q(R\mathbb{Z}_3)$$

Since Γ is a word hyperbolic group, the K-theoretic Farrell-Jones conjecture holds. By the reformulation of this conjecture by Davis-Lück, this means

$$H_q^\Gamma(E_{\mathrm{all}}\Gamma; \mathbf{K}_R) \cong H_q^\Gamma(E_{\mathrm{vc}}\Gamma; \mathbf{K}_R).$$

Thus

$$H_q^\Gamma(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) = H_q^\Gamma(E_{\mathrm{vc}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R)$$

By Lück-Weiermann [LW07] (if one insists),

$$H_q^\Gamma(E_{\mathrm{vc}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \cong \bigoplus_{\mathcal{M}} H_q^{D_\infty}(E_{\mathrm{vc}}D_\infty, E_{\mathrm{fin}}D_\infty; \mathbf{K}_R)$$

By Waldhausen again,[1]

$$H_q^{D_\infty}(E_{\mathrm{vc}}D_\infty, E_{\mathrm{fin}}D_\infty; \mathbf{K}_R) = \widetilde{\mathrm{Nil}}_{q-1}(R; R, R).$$

---

[1]Consider using Corollary 3.4.

Finally, by our main theorem

$$\widetilde{\mathrm{Nil}}_{q-1}(R;R,R) \cong \widetilde{\mathrm{Nil}}_{q-1}(R).$$

Remark 3.6. It is not difficult to show that $\mathcal{M}$ is countably infinite.

One then applies vanishing and non-vanishing results to show compute $K_q(R\Gamma)$ for some R and Γ. For example, even from Waldhausen's result one knows $\widetilde{\mathrm{Nil}}_*(R) = 0$ for R regular coherent. For a finite group G, Bass showed that $NK_{-q}(\mathbb{Z}G) = 0$ for $q > 0$. For G finite of square-free order, Harmon [Har87] showed that $NK_0(\mathbb{Z}G) = 0$. For finite abelian group G which is not of square-free order, Bass showed that $NK_0(\mathbb{Z}G)$ is infinitely generated torsion. Finally, a ring R is quasi-regular if there is a two-sided ideal I so that R/I is regular. Bass showed that $NK_q(R) = 0$ for $q \leq 0$. $\square$

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