Whittaker Colloquium, University of Edinburgh, 2 October 2017

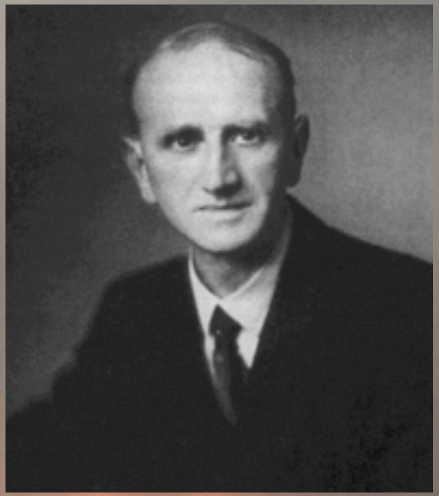
Topological Fluid Dynamics

Keith Moffatt

DAMTP and Trinity College
Cambridge

Turbulent vortex ring emitted by the 2000 eruption of Mount Etna

Edmund Whittaker
1873-1956

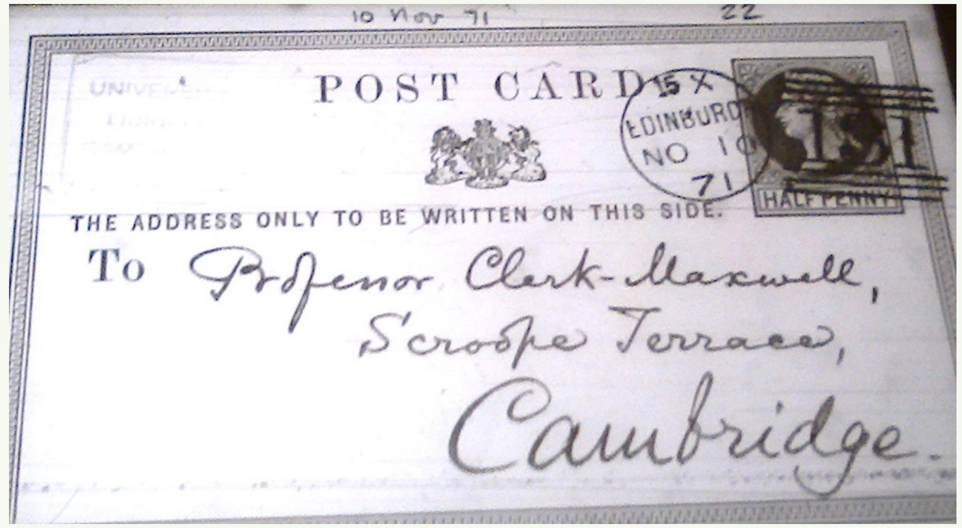


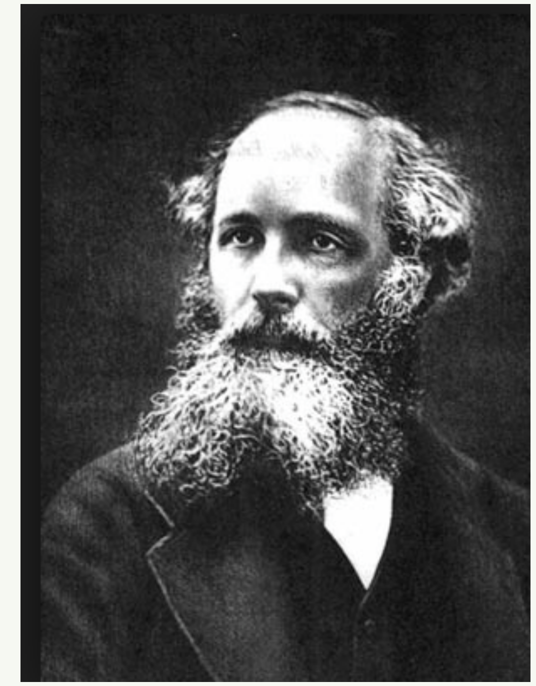
Alex. C. Aitken
1895-1967
Scintillating

W.L.Edge
1904-1997
Redoubtable

Robin Schlapp
1899-1991
Benevolent

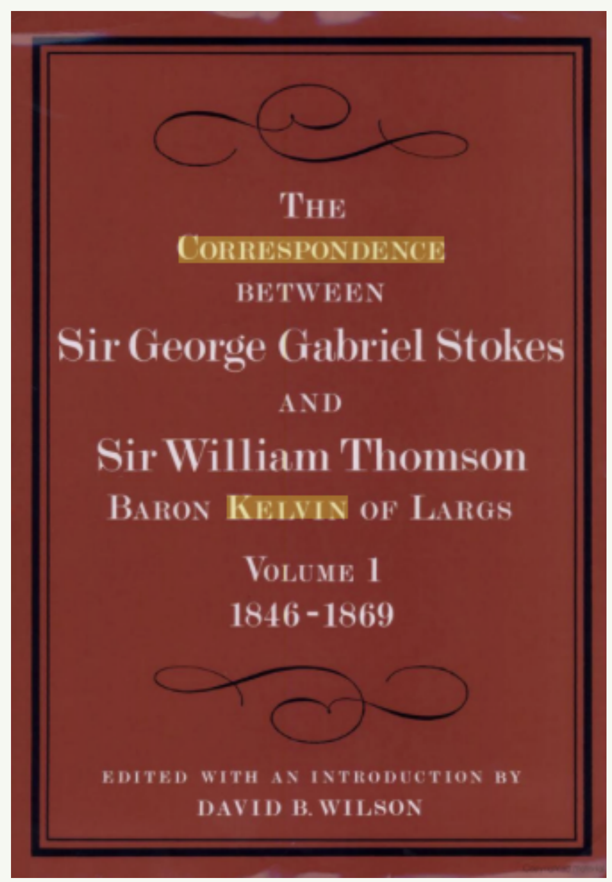
Peter Guthrie Tait
1831-1901

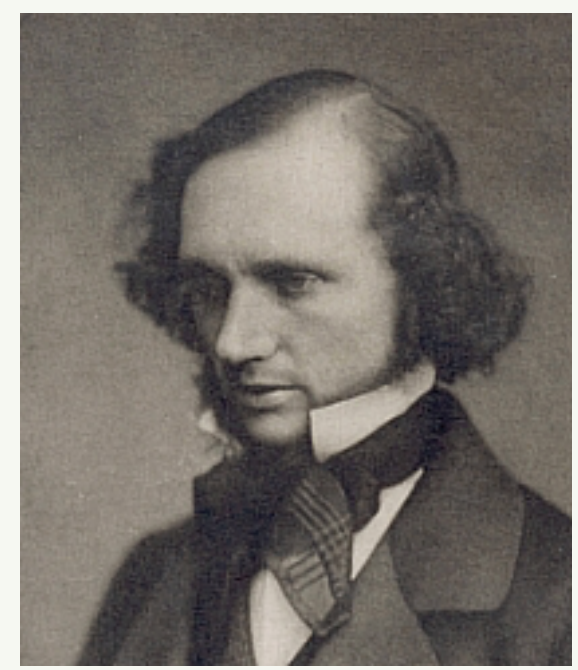




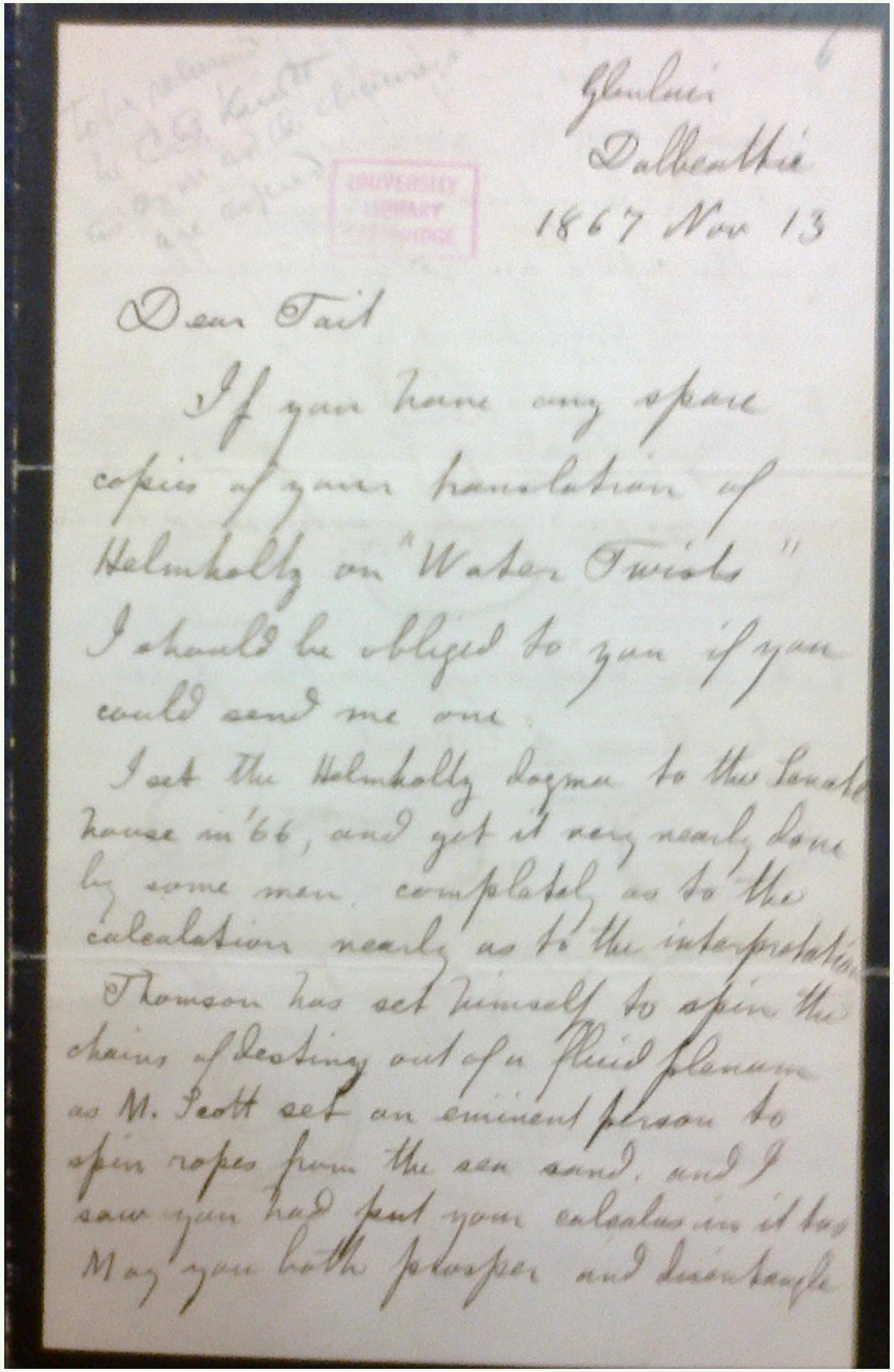
James Clerk Maxwell
1831-1879

George Gabriel Stokes
1819-1903





Wm Thomson, Lord Kelvin
1824-1907



6

Glenlair
Dalbeattie
1867 Nov 13

Dear Tait

If you have any spare copies of your translation of Helmholtz on "Water Twirls" I should be obliged to you if you could send me one.

I set the Helmholtz dogma to the Senate house in '66, and got it very nearly done by some men, completely as to the calculation nearly as to the interpretation.

Thomson has set himself to spin the chains of destiny out of a fluid plenum as M. Scott set an eminent person to spin ropes from the sea sand, and I saw you had put your calculus in it too. May you both prosper and disentangle

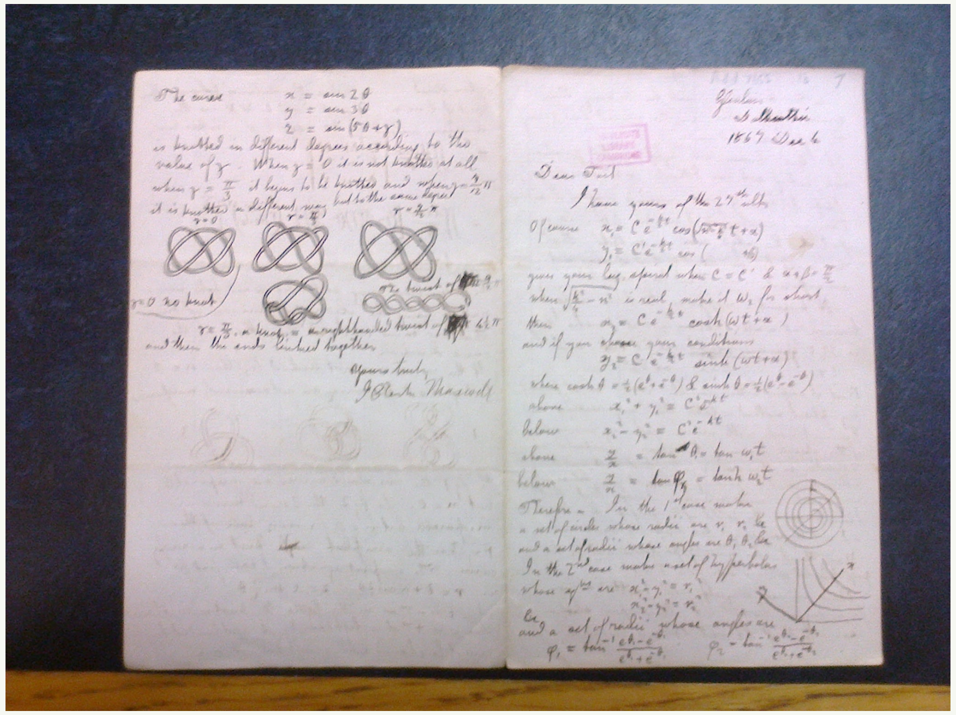
Letter dated 13 Nov 1867

Maxwell requests a copy of Tait's translation of Helmholtz 1858

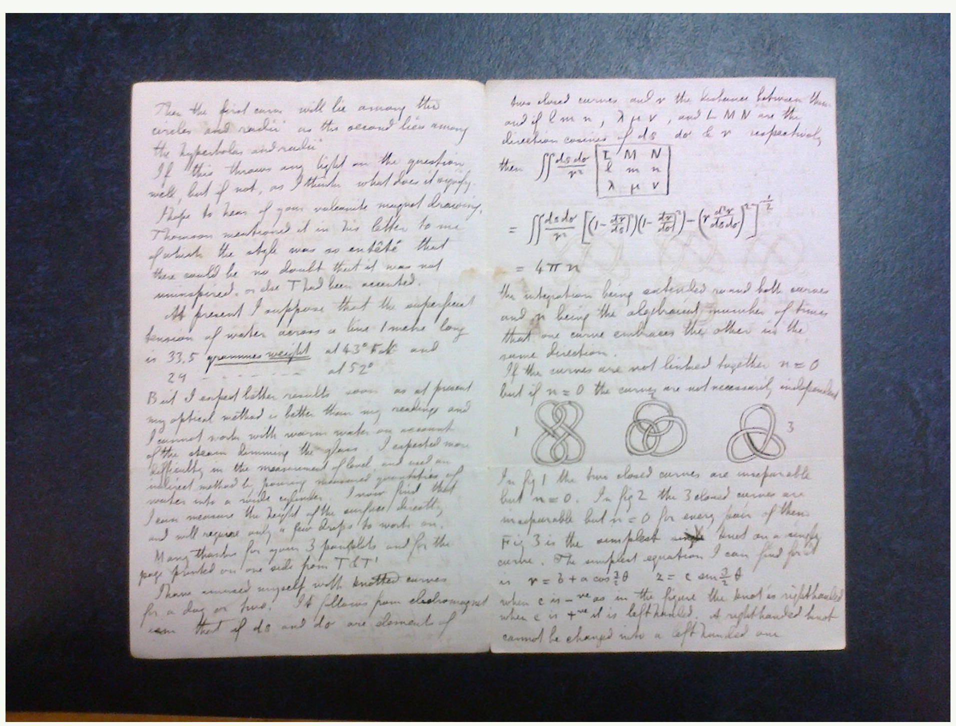
Then explains how he had set some of the 'Helmholtz dogma' for student exams in 1866! They had solved it "completely as to the calculation, nearly as to the interpretation" — Maxwell was much concerned with physical understanding!

"Thomson has set himself to spin the chains of destiny out of a fluid plenum . . . and I saw that you had put your calculus in it too. May you both prosper and disentangle your formulas in proportion as you entangle your worbles."

Three weeks later: Maxwell to Tait 4 December 1867



p4 p1



p2 p3

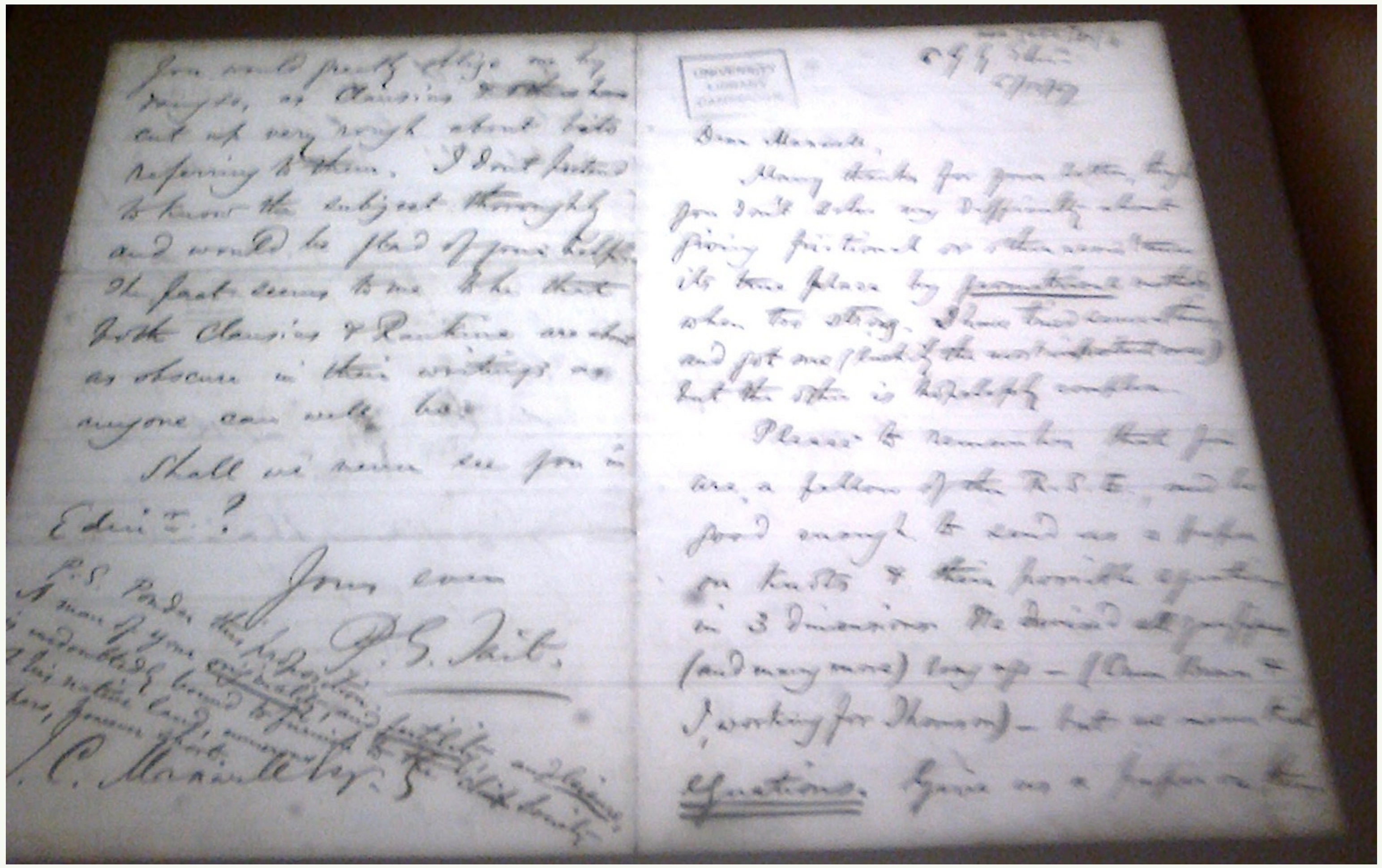
Maxwell discusses knotted and linked vortices (he called them 'worbles')

This letter alone is enough to reveal Maxwell's extraordinary genius

It was motivated by Kelvin's 'vortex atom theory' proposed earlier in 1867

The sketches in this letter are reproduced in Kelvin's paper (PRSE,1869) where his famous 'Circulation Theorem' is proved!

Two days later: Tait to Maxwell, 6 December 1867

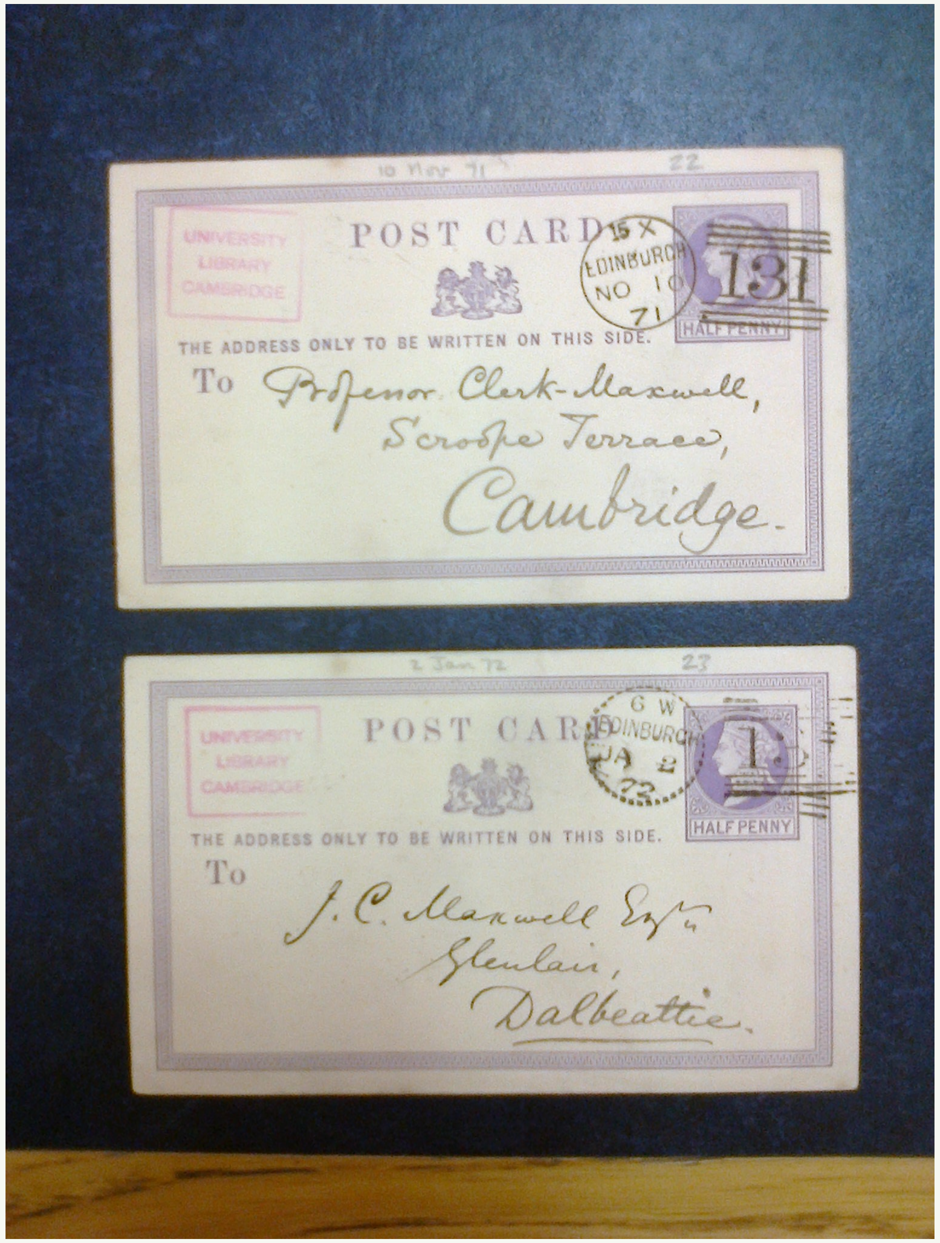


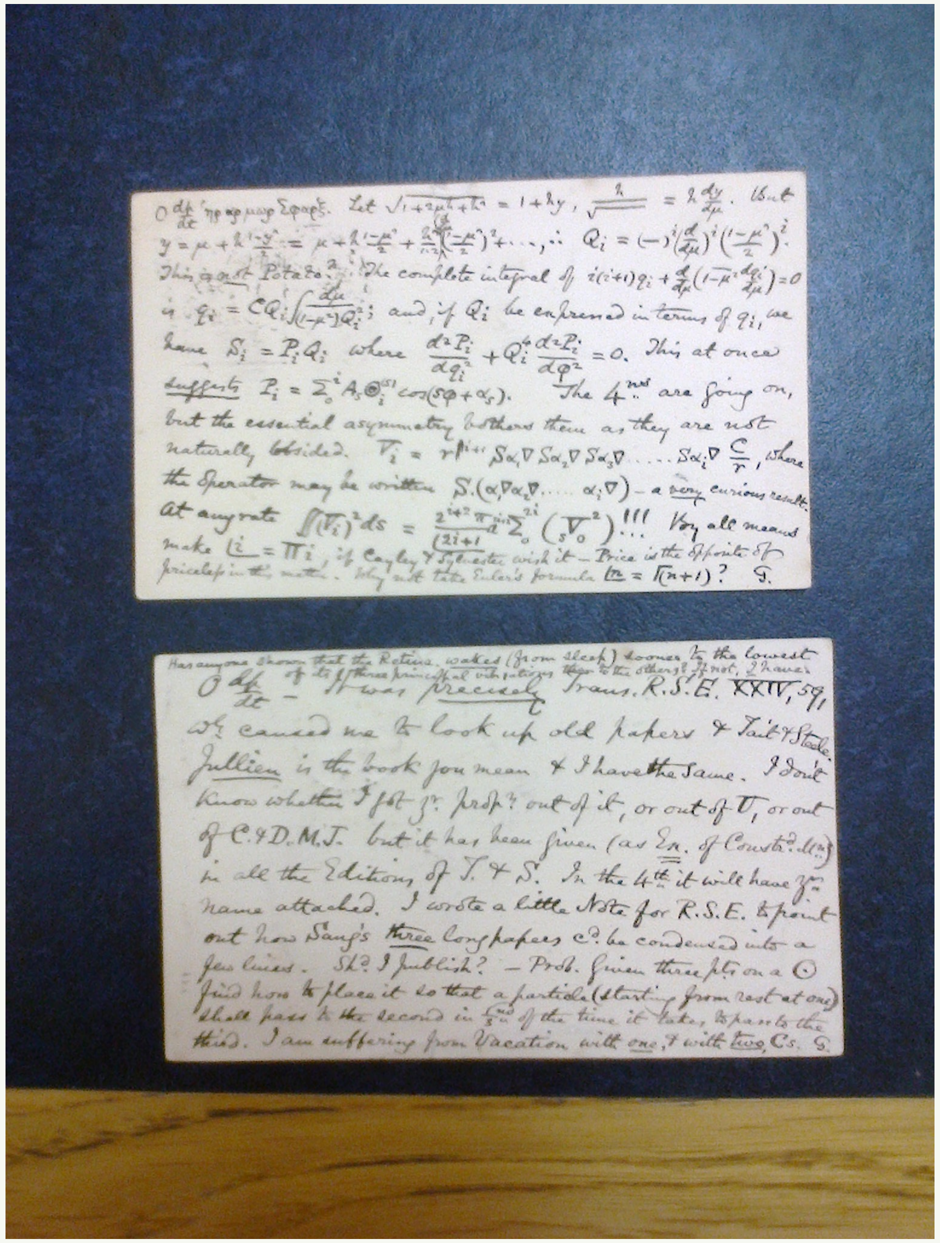
The Postscript reads:

P.S. Ponder this proposition: A man of your originality and fertility is undoubtedly bound to furnish to the chief Society of his native land, numerous papers, however short.

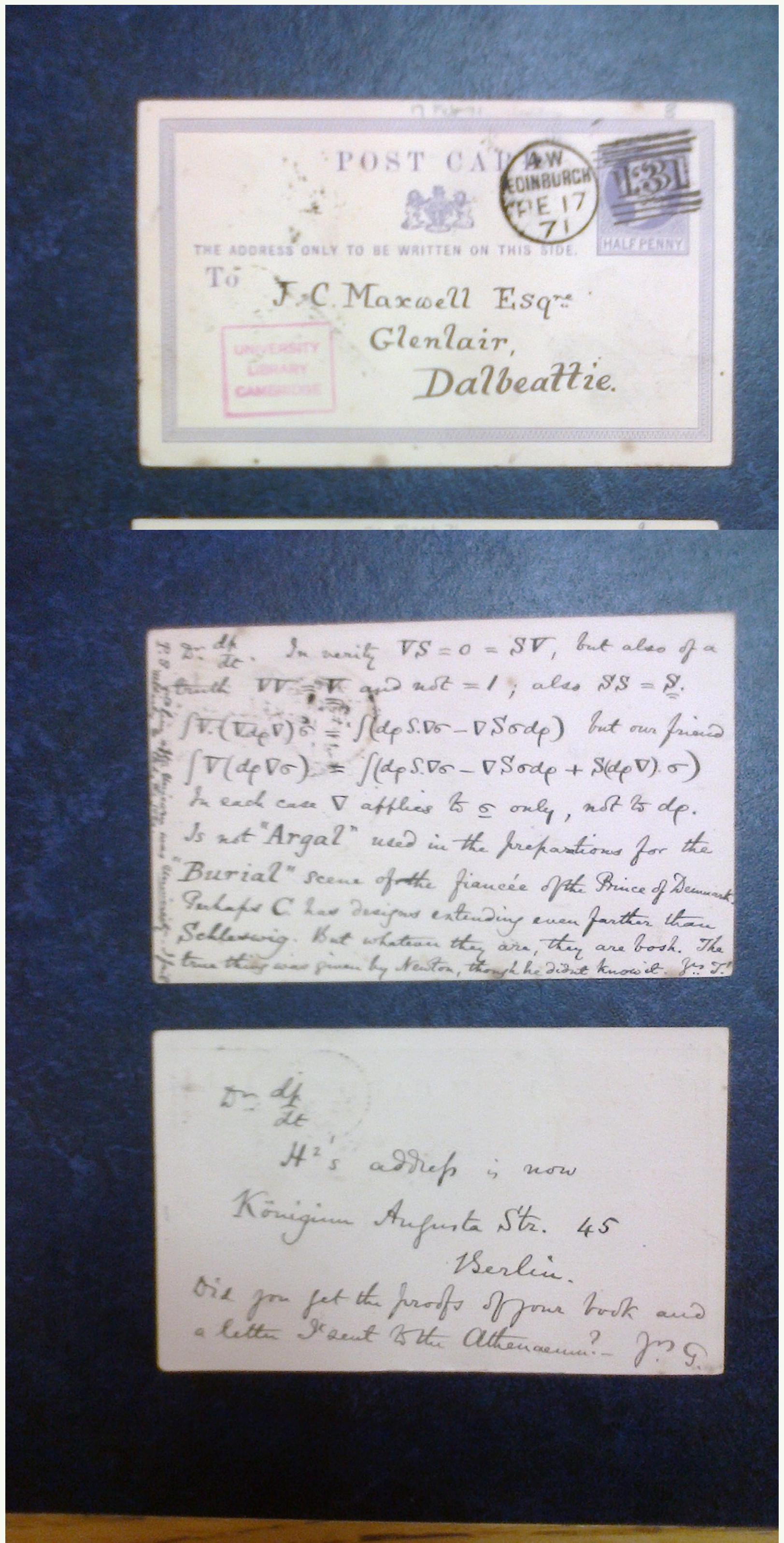
The halfpenny postcard was how Maxwell and Tait communicated in the1870's — the e-mail of the day!

Maxwell commutes between Cambridge and Glenlair, 1871





Here the symbol ∇ makes an early appearance



POST CARD

EDINBURGH DE 17 71

131

THE ADDRESS ONLY TO BE WRITTEN ON THIS SIDE.

HALF PENNY

To J. C. Maxwell Esq^re^
Glenlair,
Dalbeattie.

$\frac{dp}{dt}$. In verity $\nabla S = 0 = S\nabla$, but also of a truth $\nabla\nabla = \nabla$ and not $= 1$; also $SS = S$.

$\int \nabla . (\nabla d\rho \nabla)\sigma = \int (d\rho S.\nabla\sigma - \nabla S\sigma d\rho)$ but our friend

$\int \nabla (d\rho \nabla \sigma) = \int (d\rho S.\nabla\sigma - \nabla S \sigma d\rho + S(d\rho\nabla).\sigma)$

In each case ∇ applies to σ only, not to $d\rho$.

Is not "Argal" used in the preparations for the "Burial" scene of the fiancée of the Prince of Denmark. Perhaps C. has designs extending even farther than Schleswig. But whatever they are, they are bosh. The true thing was given by Newton, though he didnt know it. J. T.

$\frac{dp}{dt}$

H's address is now
Königinn Augusta Str. 45
Berlin.
Did you get the proofs of your book and a letter I sent to the Athenaeum? – J. T.

Dynamo (increasing energy) *vs* Relaxation (decreasing energy)

Consider the induction equation of magnetohydrodynamics (MHD):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \qquad \text{or} \qquad \frac{\partial \mathbf{B}}{\partial t} = \mathcal{L}_{\mathbf{u}}(\mathbf{B}) + \ldots$$

if you prefer,

where $\mathbf{u}(\mathbf{x},t)$ is a velocity field, and to avoid boundary complications, $\mathbf{B}(\mathbf{x}, t)$ is a localised field in an infinite fluid; η is the magnetic diffusivity of the fluid.

With current $\mathbf{j} = \nabla \times \mathbf{B}$, and Lorentz force $\mathbf{j} \times \mathbf{B}$, we may easily deduce the equation for rate of change of magnetic energy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \tfrac{1}{2}\mathbf{B}^2 \, \mathrm{d}V = - \int \mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) \, \mathrm{d}V - \eta \int \mathbf{j}^2 \, \mathrm{d}V$$

For a **dynamo**, $\mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) < 0$ on average, i.e. fluid ***pushes against*** the Lorentz force

For **relaxation**, $\mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) > 0$ on average, i.e. the Lorentz force ***drives*** the fluid

We should regard these as complementary processes. In a dynamic (as opposed to a kinematic) regime, both processes are present in a statistically steady state.

Here I digress:

My wife Linty and I were married on 17th December 1960:

Nine months and one day later, Linty gave birth to our first child!

Edge sent a telegram from Edinburgh to Cambridge which read

Congratulations on having attained the minimum within epsilon

In magnetic relaxation, we are concerned with attaining a minimum of magnetic energy, subject to the topological constraint of a ‘frozen-in’ field.

I spent the first half of my life on

DYNAMO THEORY

I spent the first half of my life on DYNAMO THEORY

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Energy increasing!
New Year's Day 1963

with second son, born 23 December 1962

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DYNAMO THEORY

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. . .and the second half
RELAXING

I spent the first half of my life on
DYNAMO THEORY

Energy increasing!
New Year's Day 1963
with second son, born 23 December 1962

. . .and the second half
RELAXING

I spent the first half of my life on
DYNAMO THEORY

. . .and the second half
RELAXING

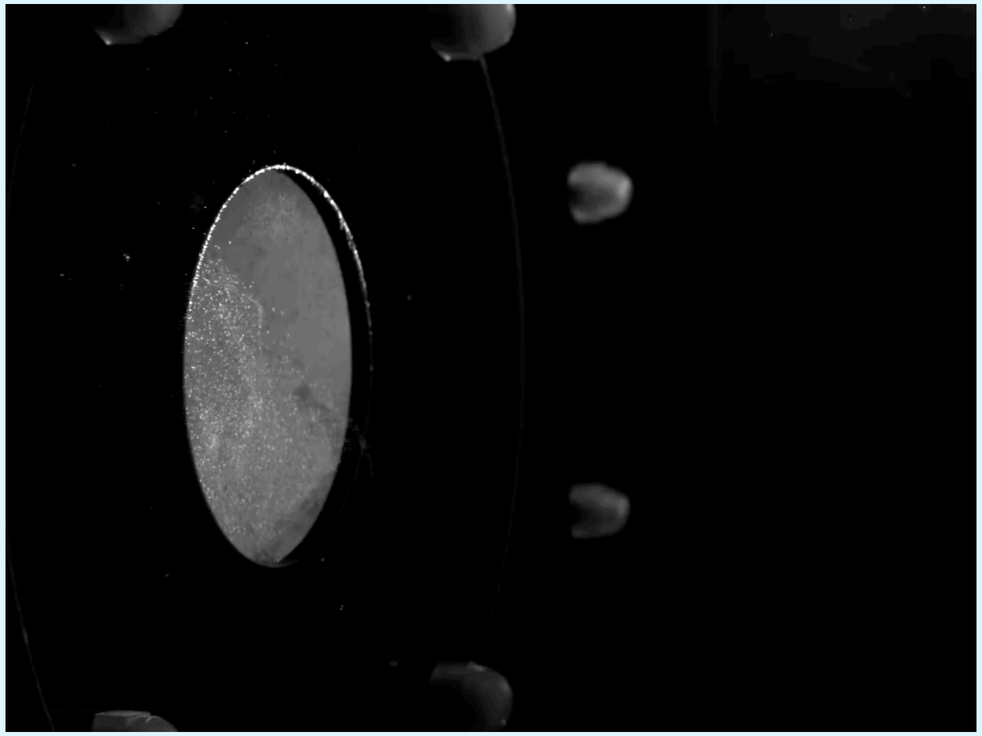
Energy increasing!
New Year's Day 1963
with second son, born 23 December 1962

Energy decreasing!
Every other day

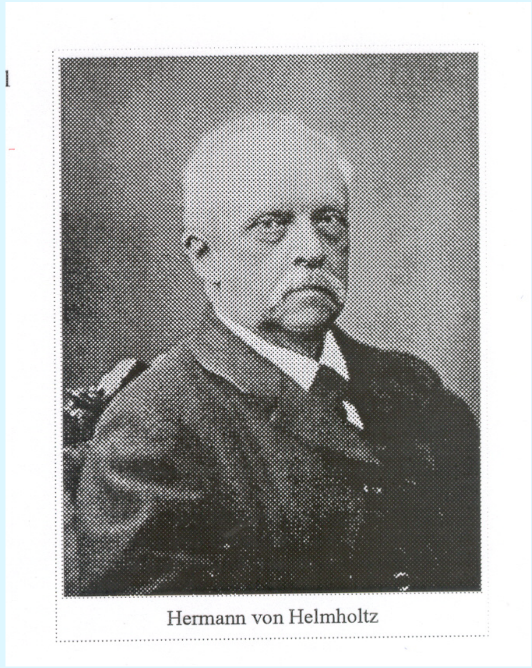
Vortex Rings

A vortex ring in water, visualised by air bubbles that migrate to the low pressure 'vortex core'

Leonhard Euler
1707-1783
Portrait by J.E.Handman (1753)
Euler equations (1756) of 'ideal' fluid mechanics



Note how the vortex 'moves' with the flow



Hermann von Helmholtz
1821-1894
He derived the laws of vortex motion (1858)

In a turbulent flow, 'vorticity' (i.e. spin) is randomly distributed throughout the fluid

but still VORTICITY IS CONVECTED WITH THE FLOW

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A vortex ring in water, visualised by air bubbles that migrate to the low pressure 'vortex core'

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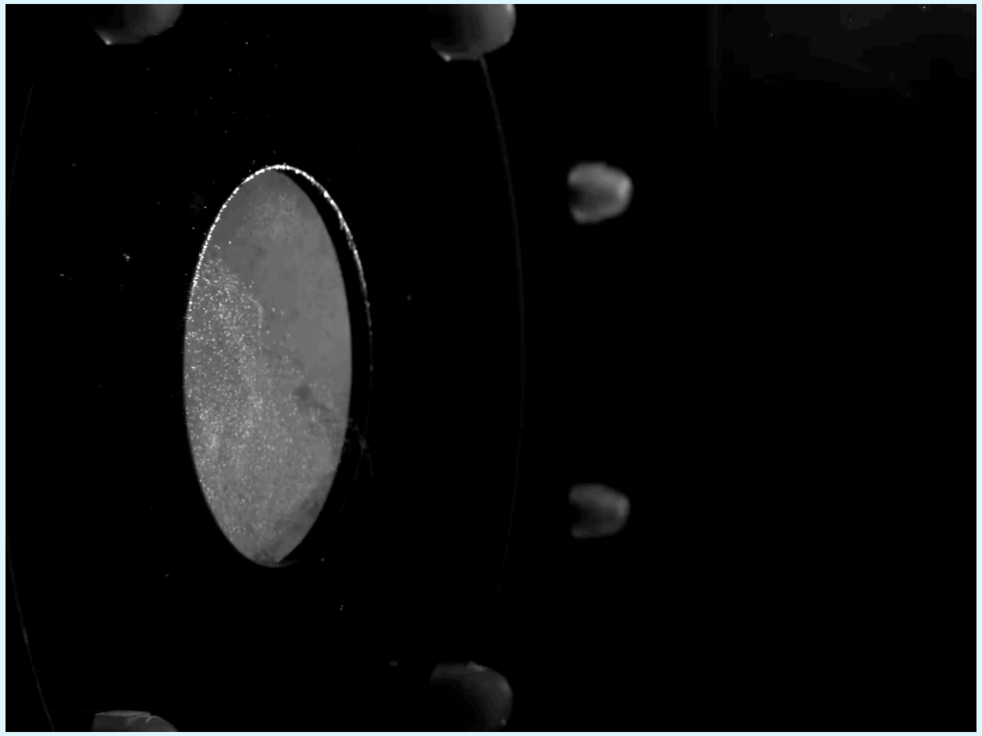

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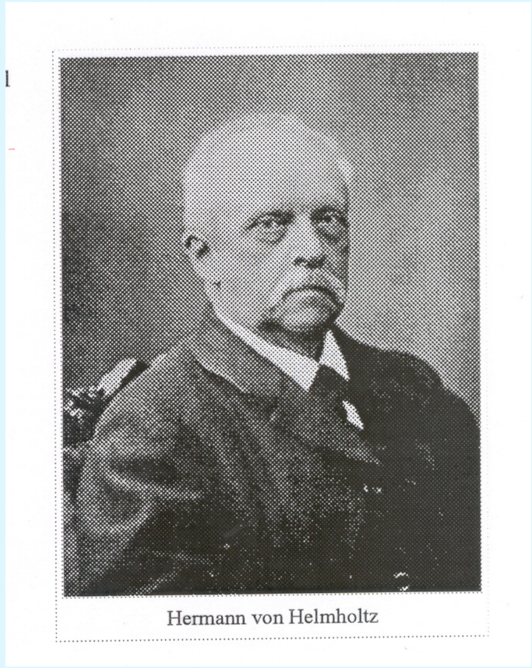

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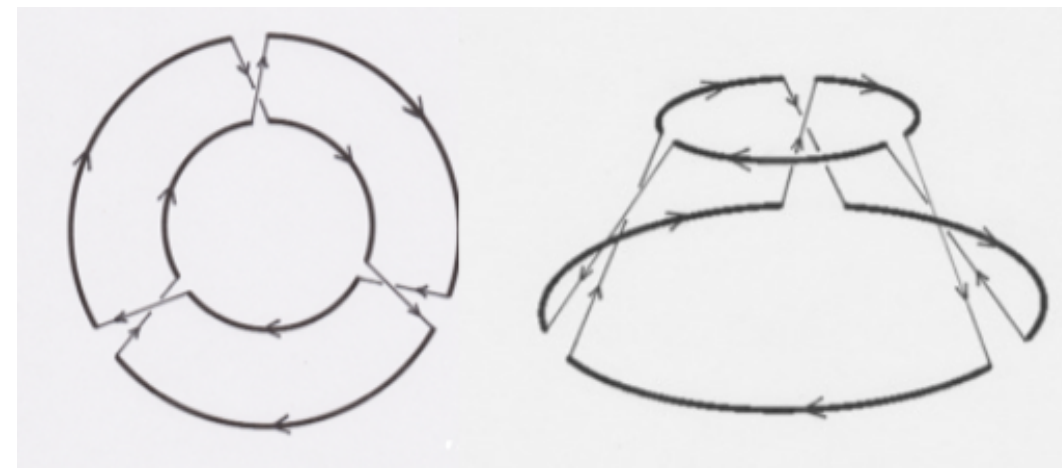
A trefoil vortex in water

Kleckner & Irvine [Nature Physics 2013]

Generated by jerking a 'knotted airfoil' into motion



Note (i) the knotted structure that is 'convected' with the flow; and (ii) the reconnection caused by 'diffusion' that changes the topology; all described in principle by the **Navier-Stokes equations**

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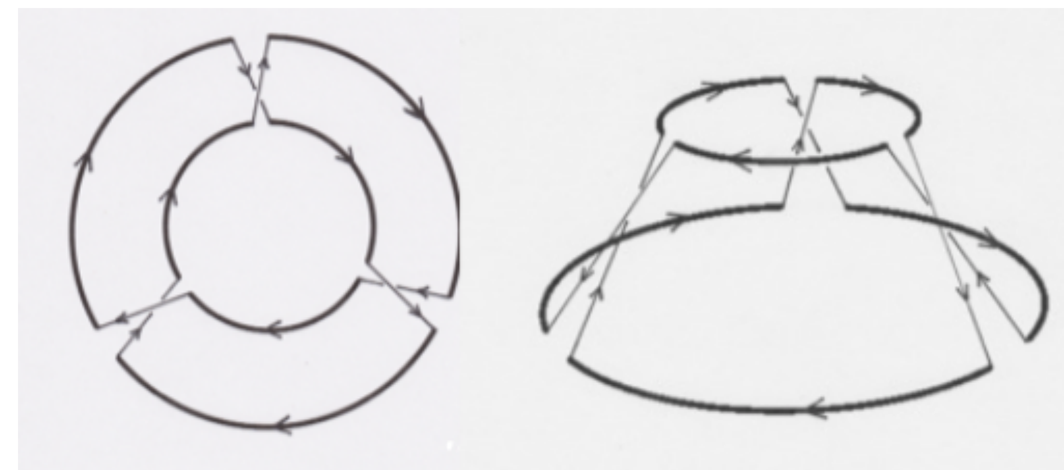

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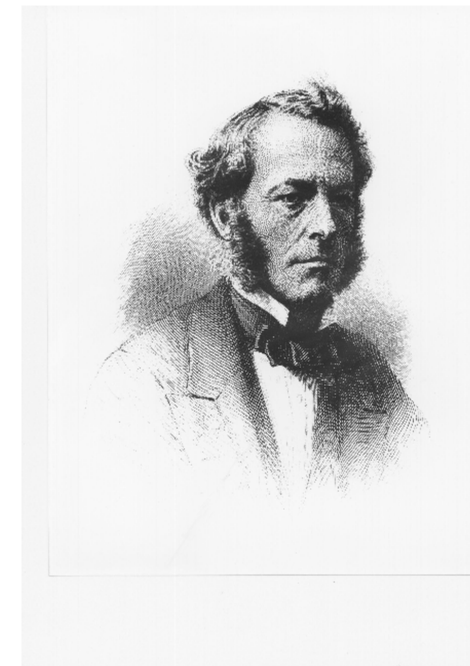
The same for the 'Hopf link'

Kleckner & Irvine Nature Physics 2013

Claude-Louis **Navier**
(1785-1836)
École des Ponts et
Chaussées, Paris



George Gabriel **Stokes**
(1819-1903)
Lucasian Professor
Cambridge

The Navier-Stokes (vorticity) equation $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \wedge \boldsymbol{u}(\mathbf{x}, t)$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

convection **diffusion**

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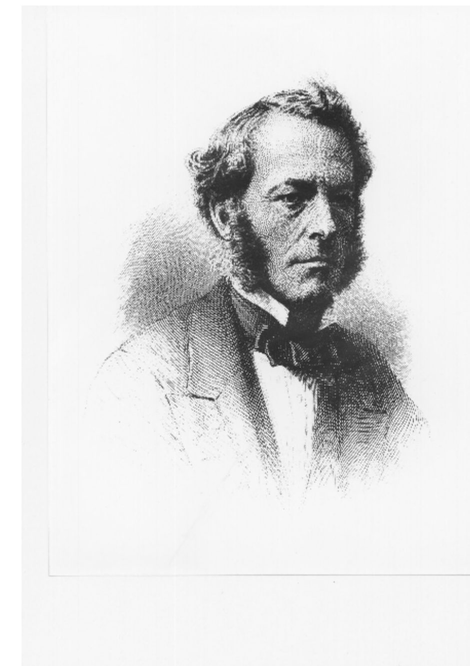

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convection **diffusion**

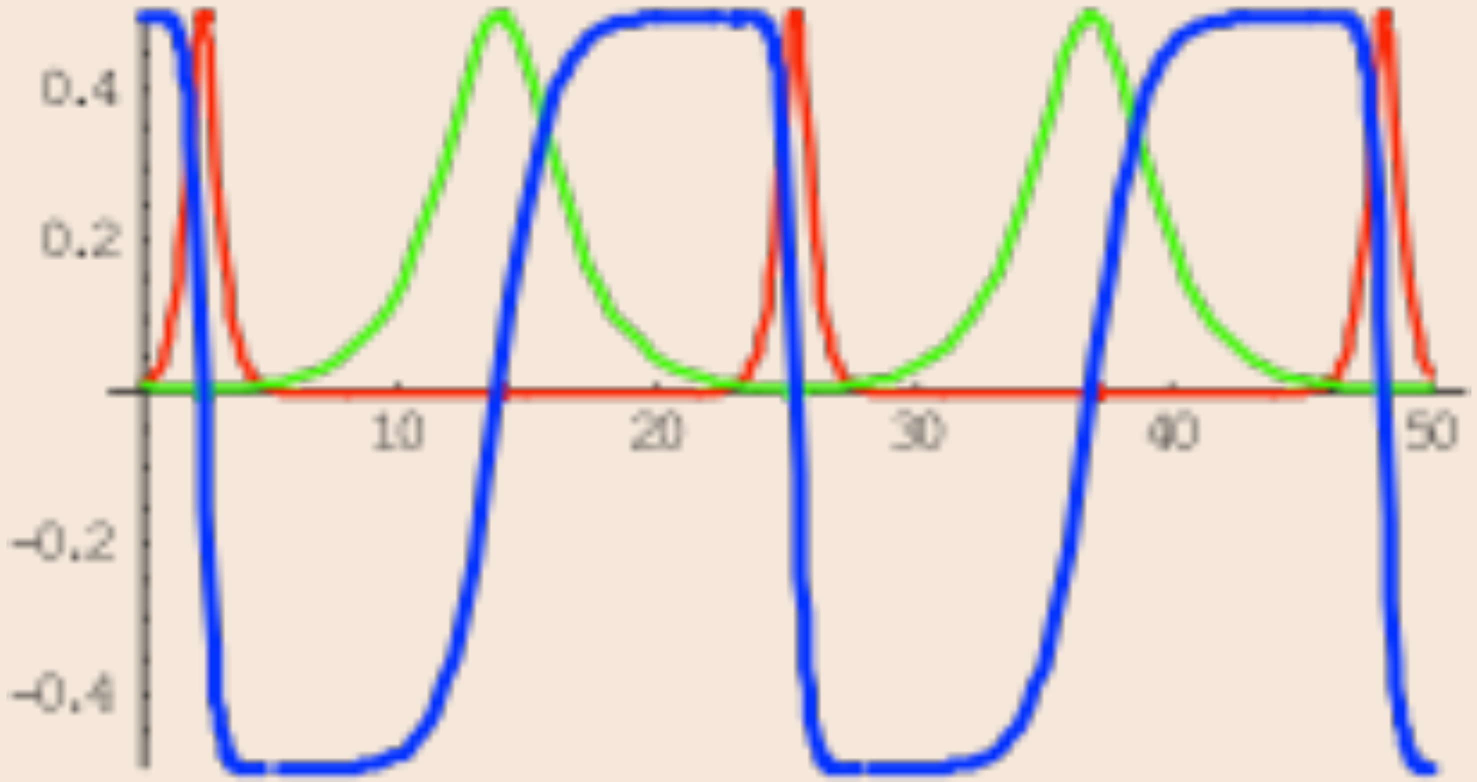
Chirality: lack of reflexion symmetry,
exemplified by the Rattleback (or Celt)

The simplest measure of chirality in fluid dynamics is helicity, and nonzero helicity results in dynamo instability

Chirality: lack of reflexion symmetry,
exemplified by the Rattleback (or Celt)

The simplest measure of chirality in fluid dynamics is helicity, and nonzero helicity results in dynamo instability

Here's what the solutions of 'mean field equations' look like if all dissipative effects are ignored (totally unrealistic!): time periodic but asymmetric wrt plus/minus

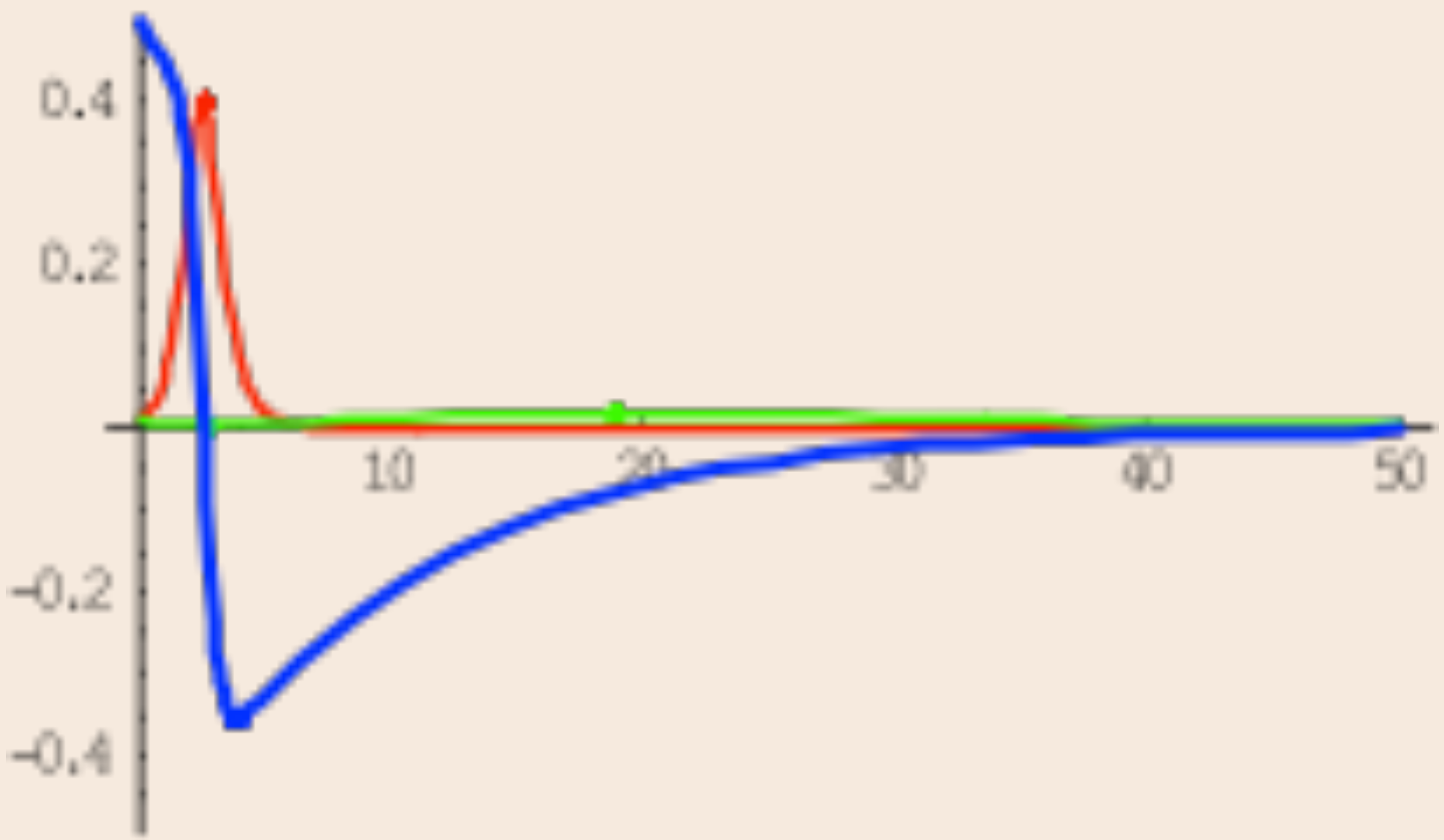


The blue curve shows N as a function of t, the red curve is $A(t)$ (pitching amplitude), and the green curve is $B(t)$ (rolling amplitude)

hkm & Tokieda (2008) Celt reversals: a prototype of chiral dynamics.

Proc. Roy. Soc. Edinb. A **138 (2), 361–368.**

…and here's the same but with appropriate friction parameters switched on:



The pitching instability (red) still causes one spin reversal,but the rolling instability (green) is too weak to cause a second reversal.

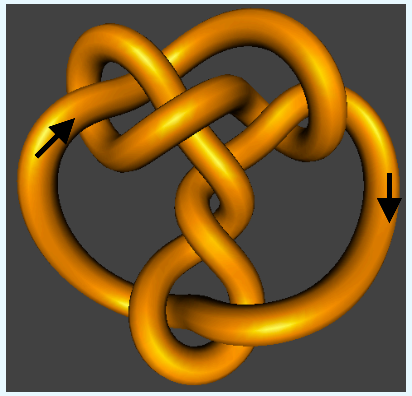
This is just what is observed with the toy rattleback

Some notation

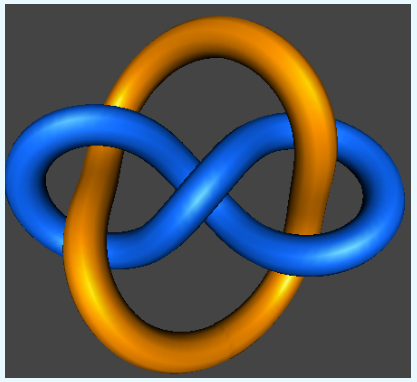
Fluid velocity $\boldsymbol{u}(\mathbf{x}, t)$, a function of position $\mathbf{x}$ and time t (random if the flow is turbulent)

Vorticity $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \wedge \boldsymbol{u}(\mathbf{x}, t)$

Helicity $\mathcal{H} = \langle \boldsymbol{u}(\mathbf{x}, t) \,.\, \boldsymbol{\omega}(\mathbf{x}, t) \rangle$ the correlation between $\boldsymbol{u}$ and $\boldsymbol{\omega}$, *a pseudo-scalar* *(think of a box of screws, vigorously shaken)* *(the flow is 'right-handed' if* $\mathcal{H} > 0$, *'left-handed' if* $\mathcal{H} < 0$) *(it being conventional to use a right-handed coordinate system).*

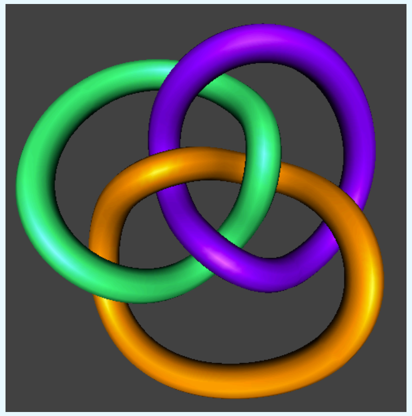


Oriented alternating 9-crossing knot



Whitehead link (chiral)

Helicity is an 'invariant' (constant in time) of the Euler equations of fluid mechanics (cf energy) (J-J.Moreau 1961); it is a measure of the conserved *'degree of knottedness'* of tangled vortex lines — ***ergo*** topological in character (hkm 1969)

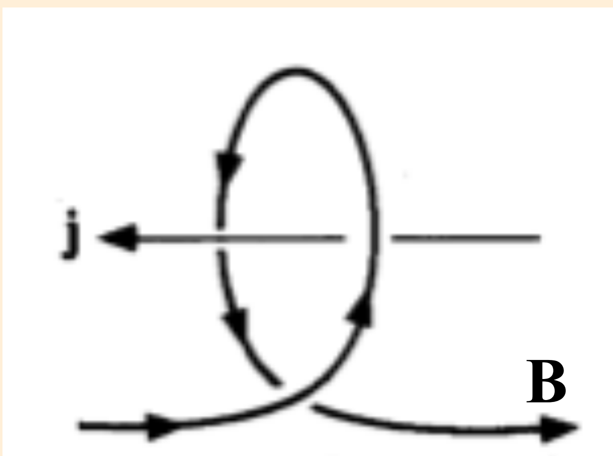


Borromean rings (achiral)

Just the sort of result that Kelvin needed!

Helicity
1960s theory breakthrough
'mean field electrodynamics'
The α-effect: $\langle J \rangle = \alpha \langle B \rangle$

E.N.Parker; Chicago, USA
S.I. Braginski; Moscow, USSR
Steenbeck, Krause & Rädler; Potsdam, DDR



ω

$\mathcal{H} > 0$ u

$\alpha \sim \mathcal{H}$

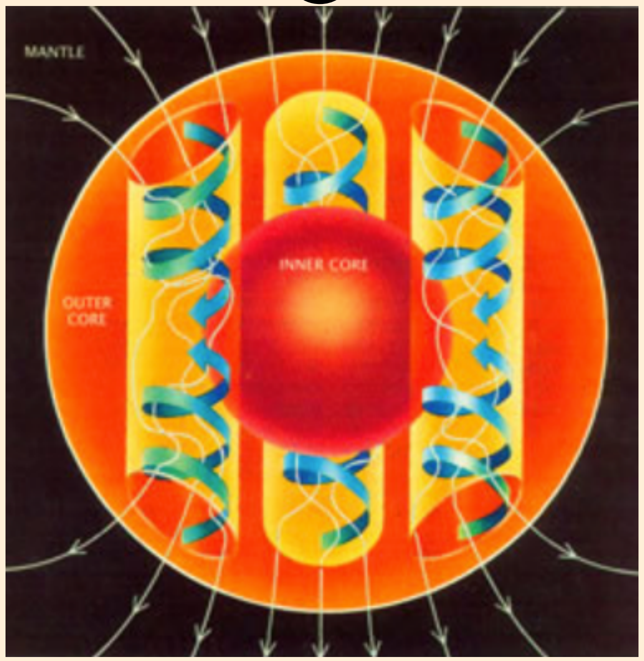
both pseudo-scalars

AN AMAZING RESULT

If $\mathcal{H} \neq 0$ in a sufficient expanse of electrically conducting fluid in turbulent motion, then a large-scale magnetic field will grow spontaneously from an infinitesimal level:

Order emerges out of Chaos!

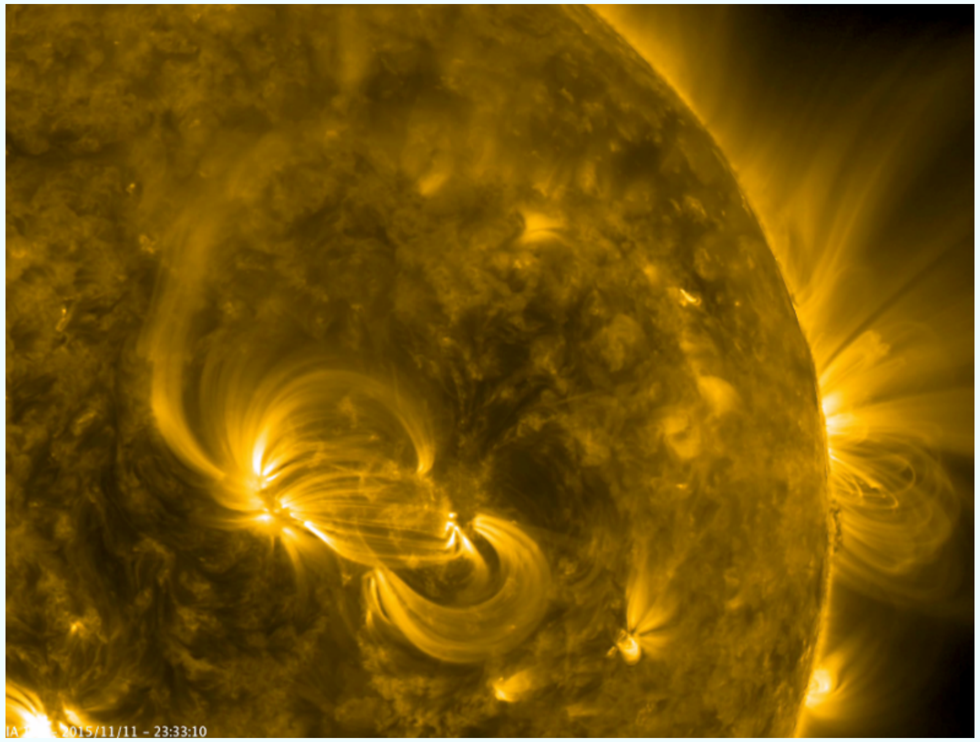
$\Omega = 2\pi/1\text{day}$



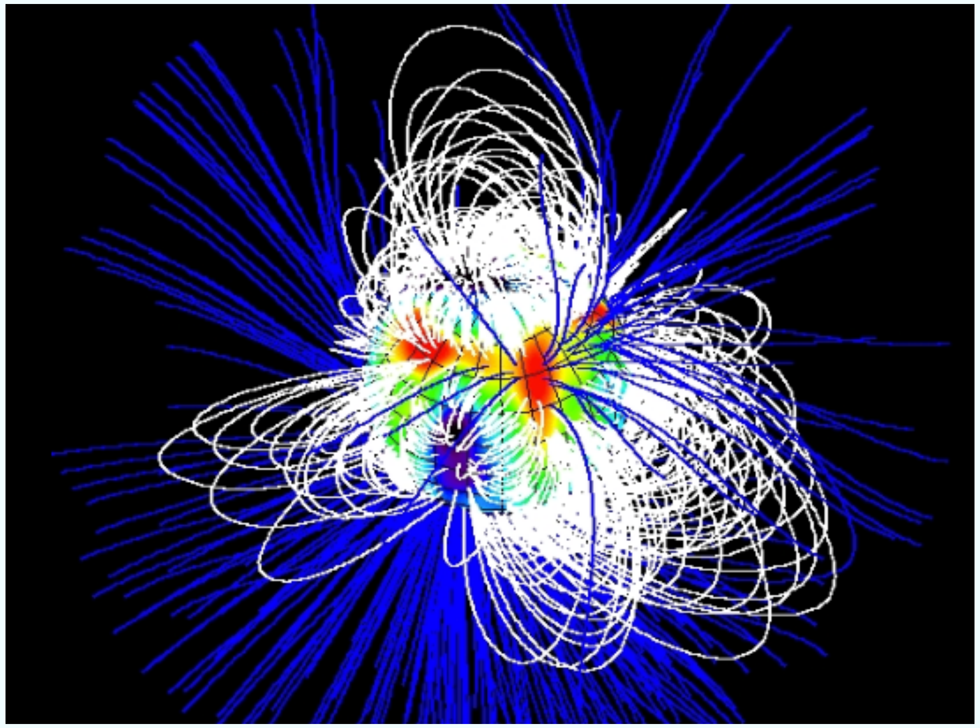
$\mathbf{g} \cdot \mathbf{\Omega} \neq 0$

The Earth certainly *is* large enough; and $\mathcal{H}$ is non-zero because Coriolis forces due to the Earth's rotation control the turbulent convection in the liquid core; this is why the magnetic axis ends up so near to the rotation axis of the Earth.

It is an even more amazing fact that the same dynamo mechanism operates in all self-gravitating, rotating, celestial bodies with conducting fluid interiors — most other planets, and the Sun, stars and galaxies, . . . [fully ionised gas, or ‘plasma’ in the controlled fusion context]



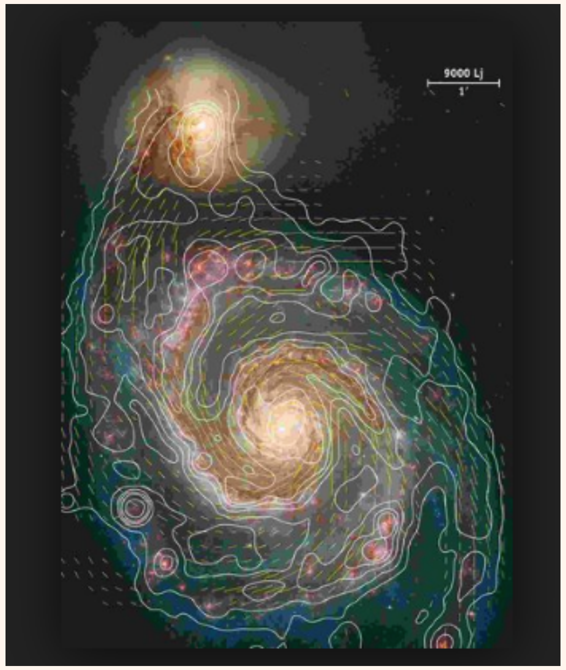
Nov. 11, 2015 The Sun
NASA’s Solar Dynamics Observatory (SDO):
picture taken in ultraviolet,
wavelength ~ 171 Å, coloured here in gold



Surface magnetic field of the young star SU Aur
(Zeeman-Doppler imaging, plus creative processing!)

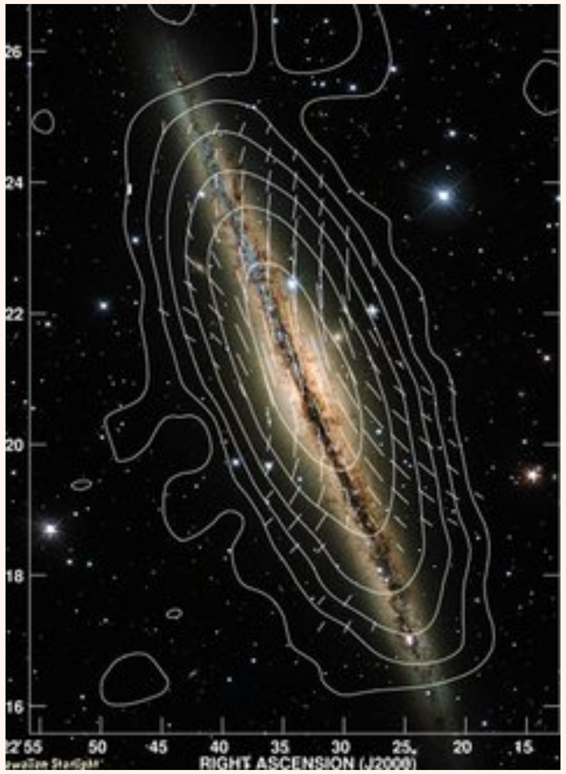
Spiral Galaxies

M 51



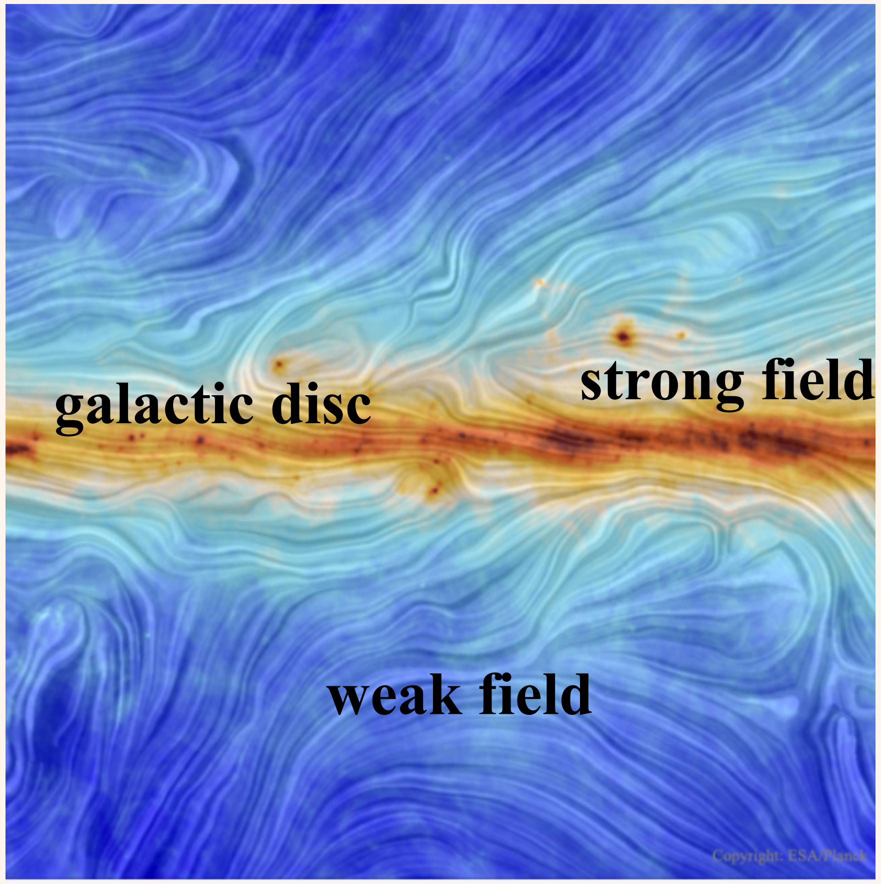
Spiral galaxy M 51 with magnetic field data. Radio polarisation measurements.
© MPIfR (R. Beck) and Newcastle Univ. (A. Fletcher)

NGC891



Magnetic field in galactic halo
© MPIfR, M. Krause & CFHT/ Coelum Astronomia
Canada-France-Hawaii Telescope

Our own galaxy, the Milky Way



© ESA/Planck [27 January 2015]
M-A. Miville-Deschênes, CNRS-IAS Univ.Paris-XI

"What caused many of the details in this and similar Planck maps, and how magnetism in general affected our Galaxy's evolution, will likely remain topics of research for years to come."

Convection and Diffusion

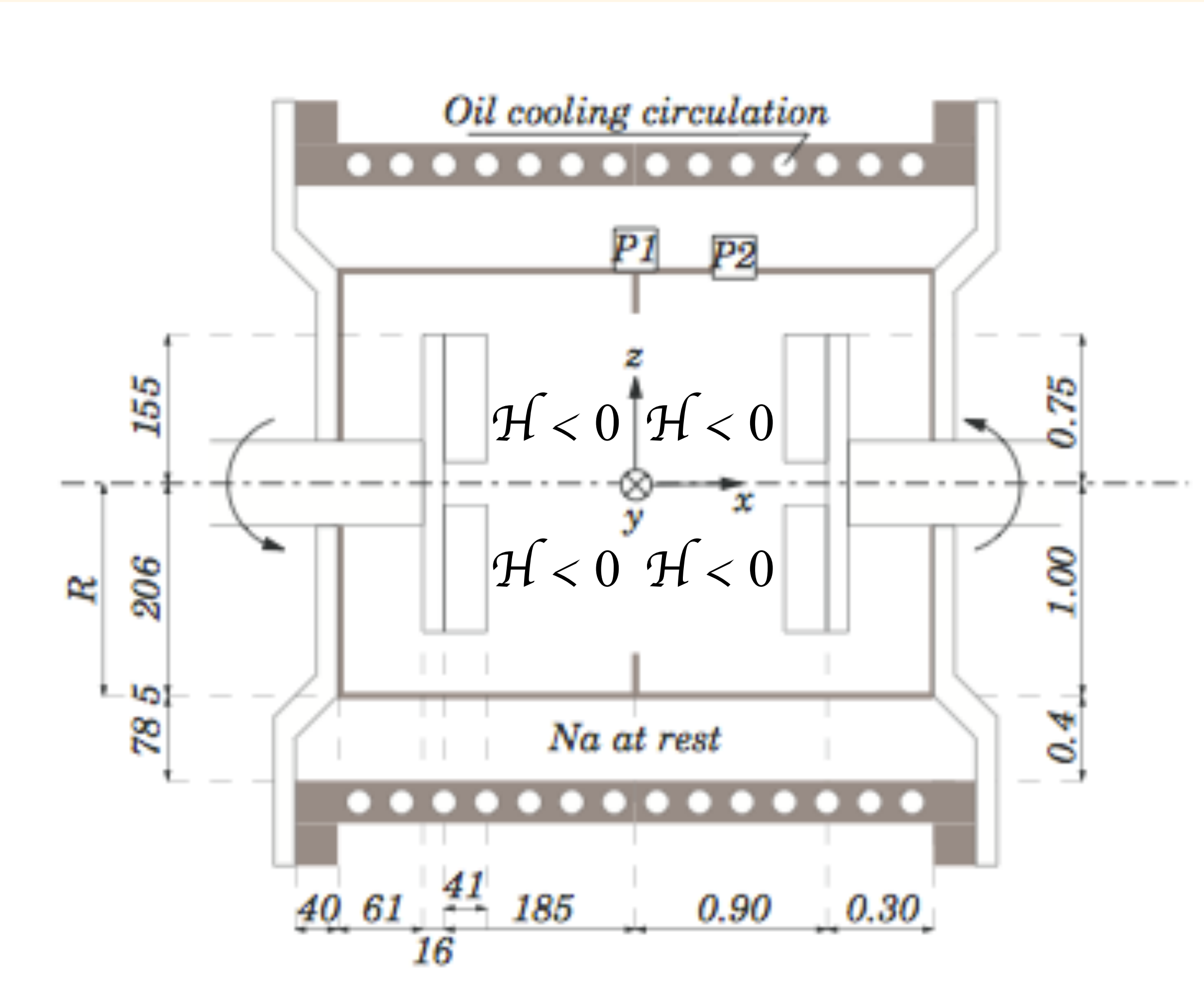
In turb'lence with helicity

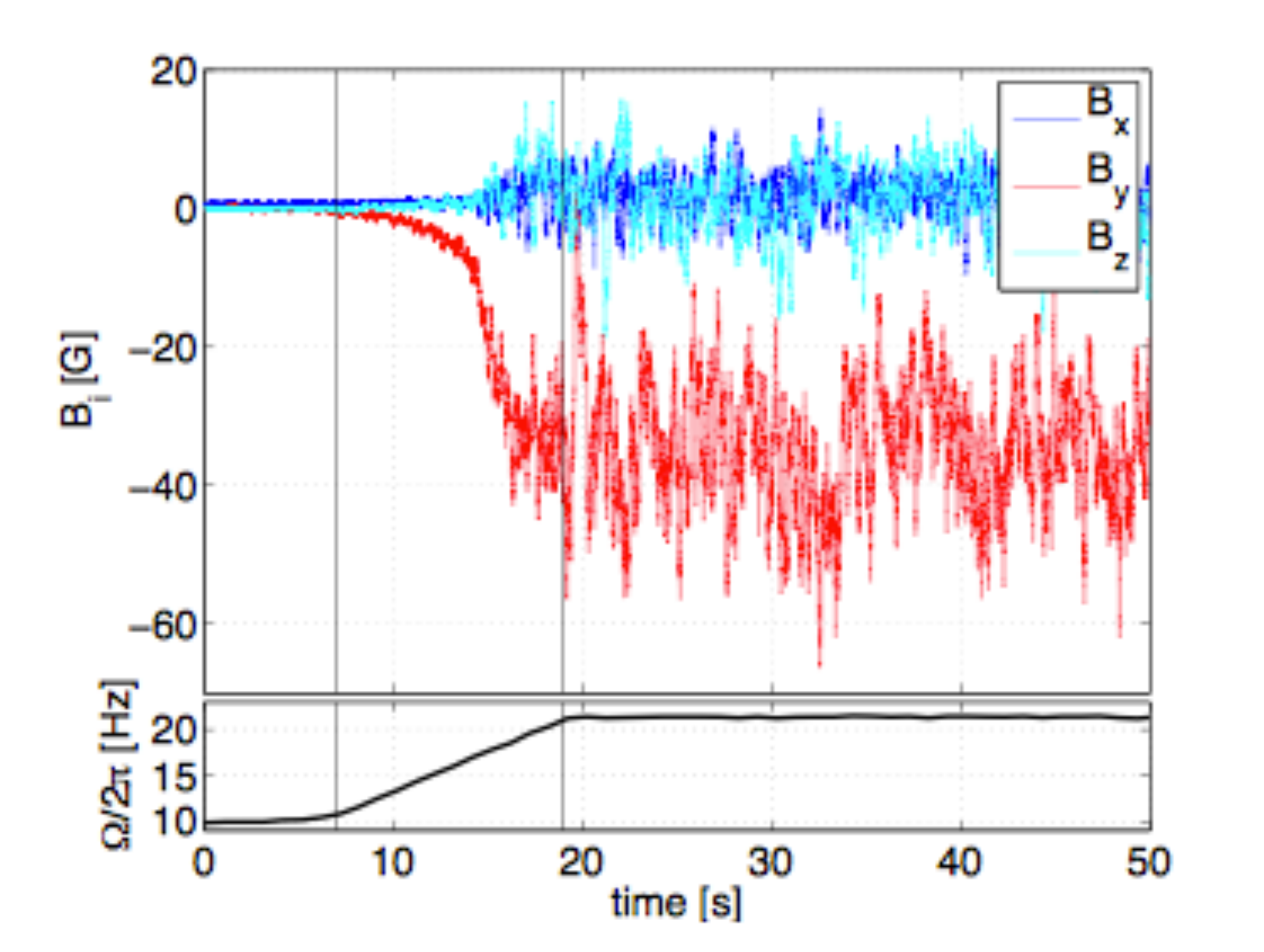
Yields order from confusion

In cosmic electricity

The best laboratory demonstration (so far) of turbulent dynamo action:

Monchaux et al (2007) at the CEA site, Cadarache, France





Liquid sodium (highly inflammable!) in turbulent flow driven by two counter-rotating propellors inside a copper cylindrical container about the size of a washing machine.

When the propellor speed is increased, the magnetic field 'bursts into life' at a critical value of this speed, and grows until a saturation value is reached — 'an αω-dynamo'

The most serious problem facing dynamo theory: 'α-quenching' at high R_m

When a mean field grows exponentially by dynamo instability, it reacts back on the turbulence and so tends to 'quench' the α-effect that is responsible for the instability: a simple model (forced helical Alfvén waves) gives

$$\alpha = \frac{\alpha_0}{\left(1 + B_0^2/\eta\nu k^2\right)^2}$$

No problem if $\mathrm{R_m} = \hat{u}/\eta k \ll 1$: the mean field grows till $B_0^2 = \mathrm{O}(\eta\nu k^2)$
then α-effect is quenched

Big problem if $\mathrm{R_m} = \hat{u}/\eta k \gg 1$: particularly in the 'solar limit', η —> 0.

The mean field apparently saturates at an extremely low level.

cf Cattaneo & Hughes (1996) who found saturation behaviour compatible with

$$\alpha = \frac{\alpha_0}{1 + \mathrm{R_m} B_0^2}$$

Turbulent diffusivities

I believe this quenching problem can be resolved only by taking small-scale turbulence into account

Then η can be replaced by turbulent diffusivity η_e and (ν by ν_e)

So then $R_{me} = O(1)$ and saturation is now at the ***equipartition level***:

$B_0^2 = O(\eta_e \nu_e k^2) = O(<\boldsymbol{u}^2>)$, and this looks much more reasonable!

However, the emphasis now shifts to the need to seek a better understanding of small-scale turbulent processes; in particular the precise nature of magnetic-flux-tube and vortex-tube *reconnection*

RECONNECTION: A VORTEX TUBE MODEL

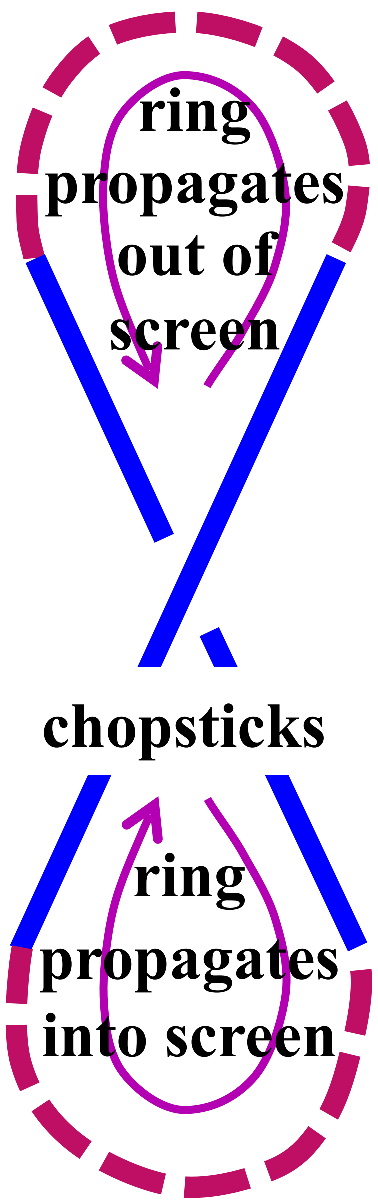


Figure-of-eight twisted ring at $t = 0$:

$$\begin{cases} x = 0.5 \sin 2\theta \\ y = 2.5 \sin \theta \\ z = 0.05 \cos \theta \end{cases} \qquad (0 \le \theta < 2\pi)$$

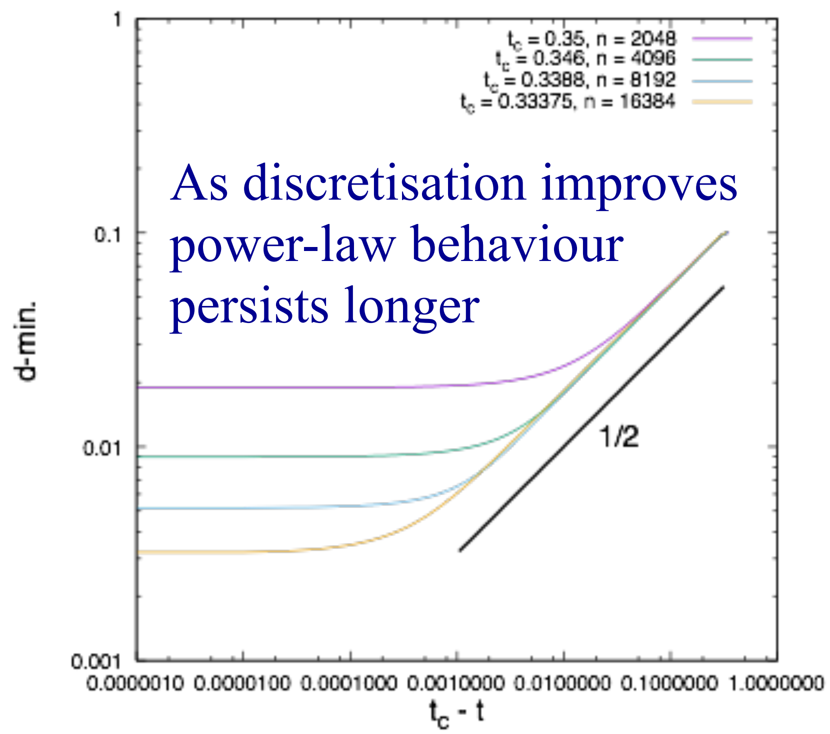
and use Biot-Savart law for a closed curve C

$$\mathbf{u}(\mathbf{r}) \;=\; \frac{\Gamma}{4\pi} \int_C \frac{\mathbf{t}(s) \times (\mathbf{r} - \mathbf{x}(s))}{|\mathbf{r} - \mathbf{x}(s)|^3}\, ds$$

(1) To calculate $\mathbf{t}(s)$, FFT is used for spectral accuracy

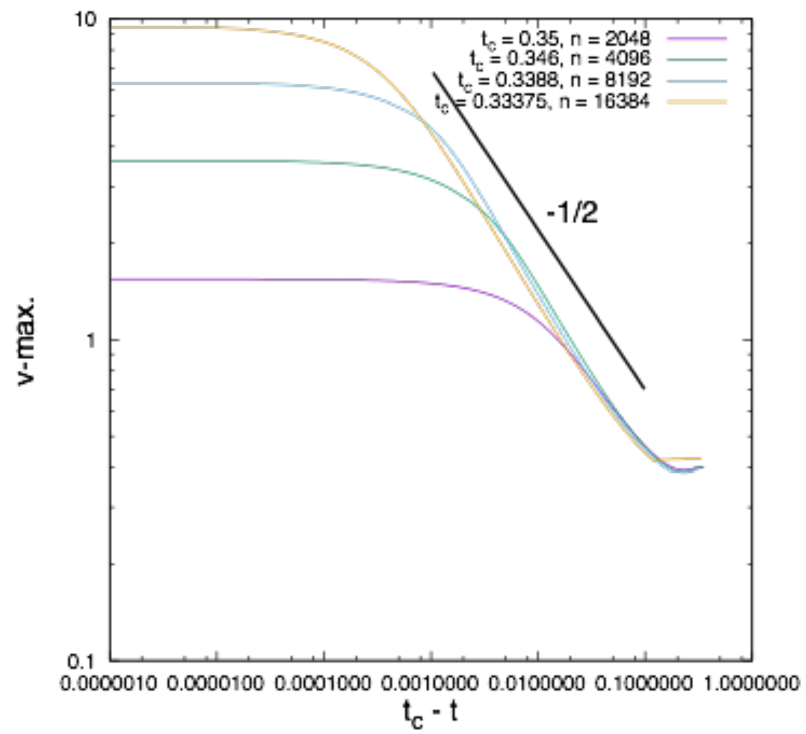
(2) For de-singularization of the Biot-Savart integral, the ‘cut-off’ method is used.

Scaling properties of the reconnection



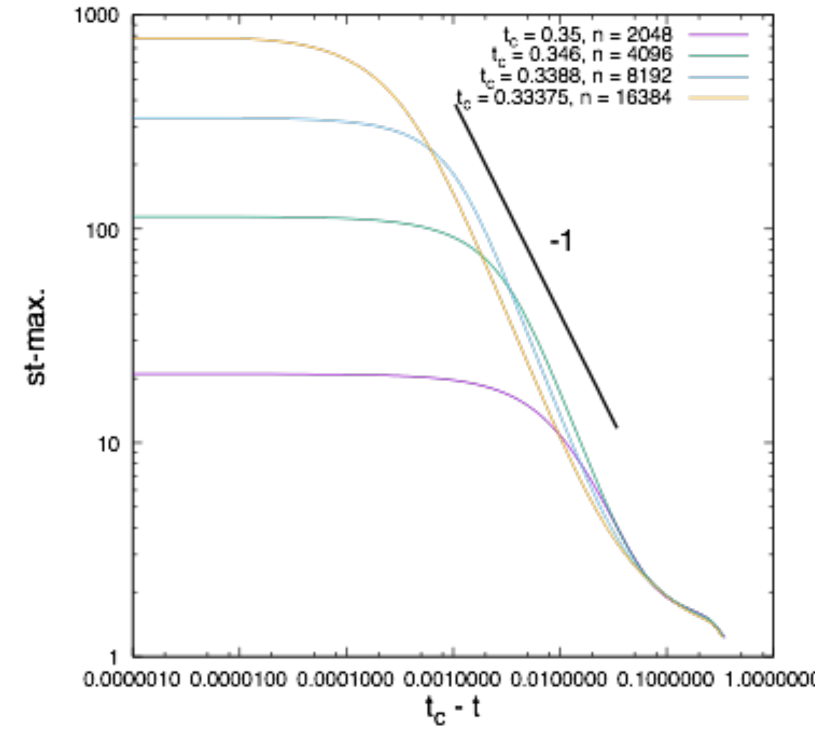
minimum distance
between the chopsticks

$$d \sim (t_c - t)^{1/2}$$



maximum velocity at the
reconnection point

$$|\mathbf{u}| \sim (t_c - t)^{-1/2}$$



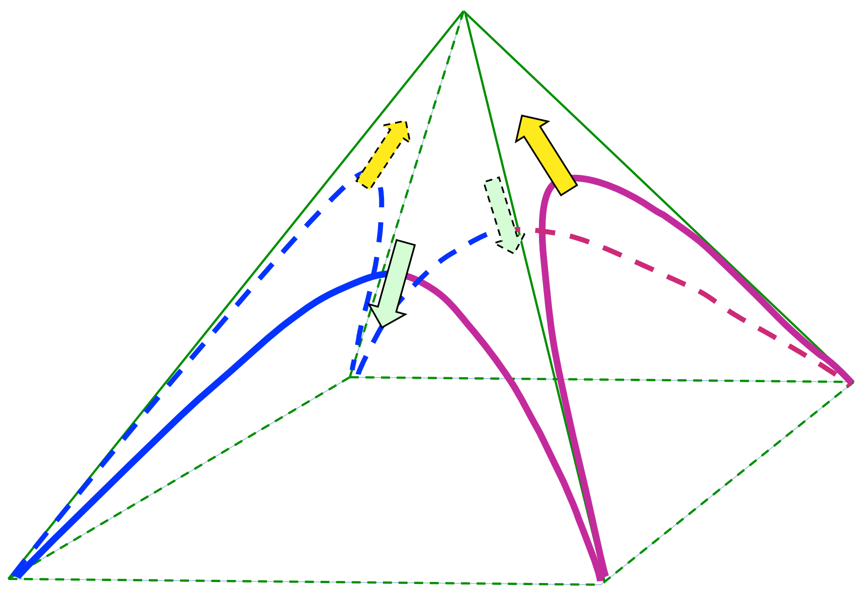
maximum axial strain rate
near the reconnection point

$$\sigma \sim (t_c - t)^{-1}$$

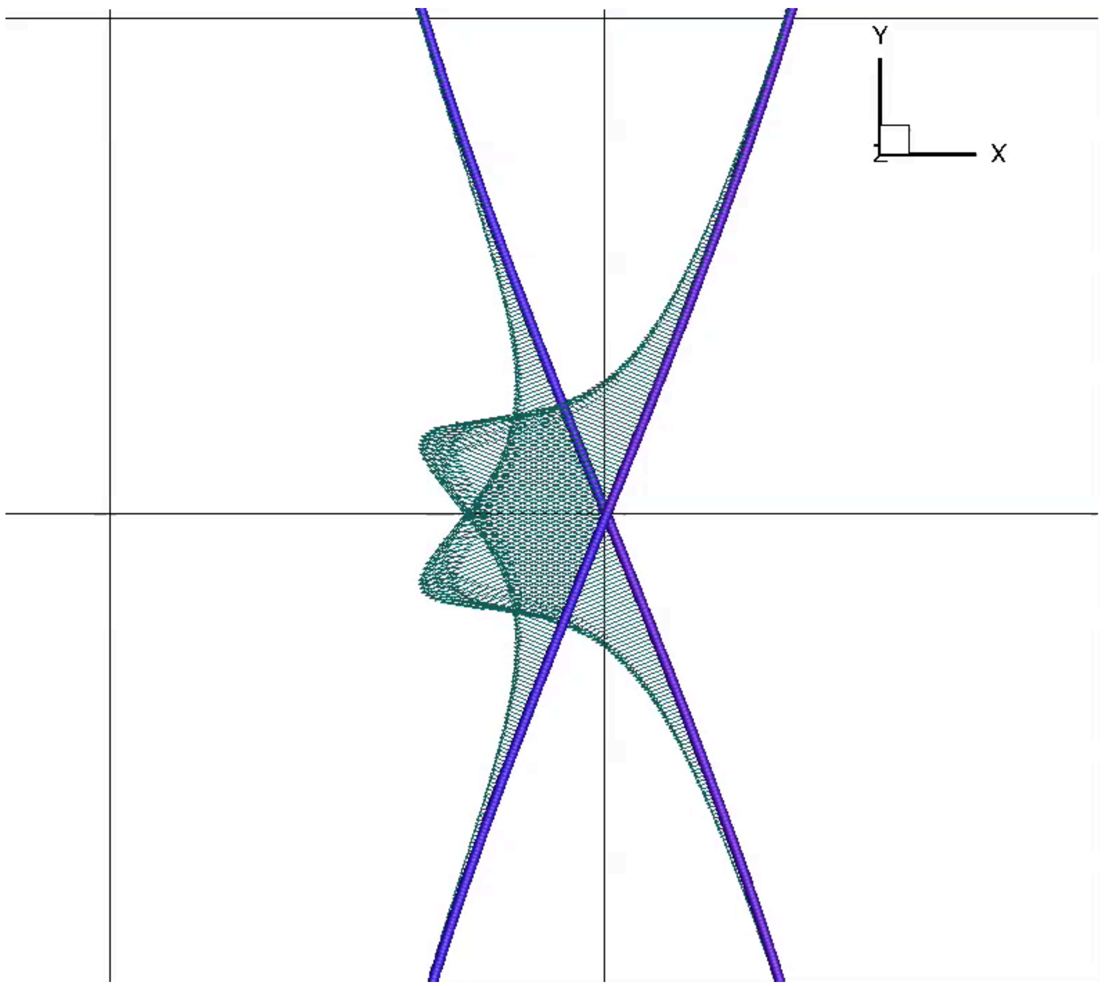
The actual scalings deviate very slightly from these Leray (self-similar) scaling

cf: Siggia (1985) Phys. Fluids **28**, 794.
de Waele & Aarts (1994) Phys. Rev. Lett. **72**, 482

Pyramid model (tilted hyperbolae)



Reconnection occurs at the singular point at the apex of the pyramid



For validity of Biot-Savart approach, the vortex core must remain compact despite the local rate-of-strain which tends to stretch it to 'sheet-like' form

Kimura & M (2017) Scaling properties towards vortex reconnection under the Biot-Savart law. Fluid Dyn. Res., doi.org/10.1088/1873-7005/aa710c.

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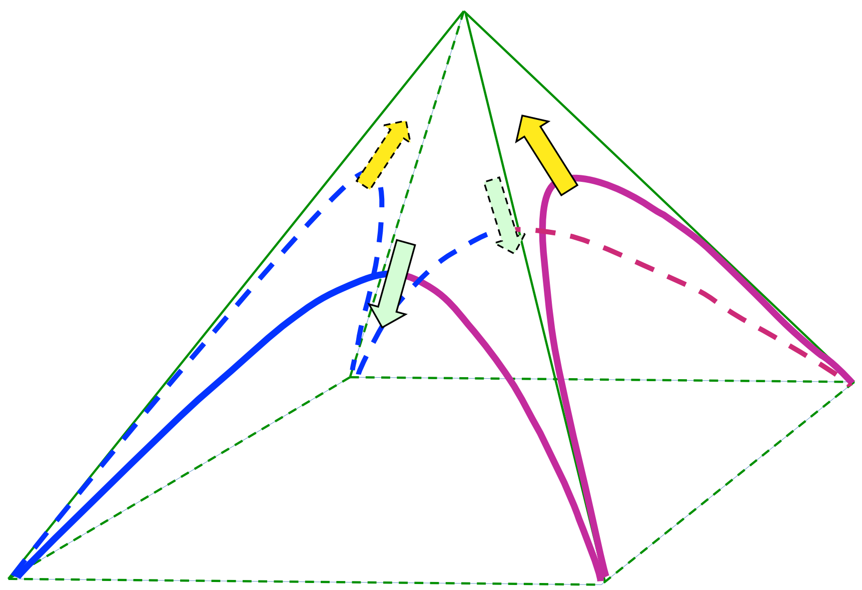

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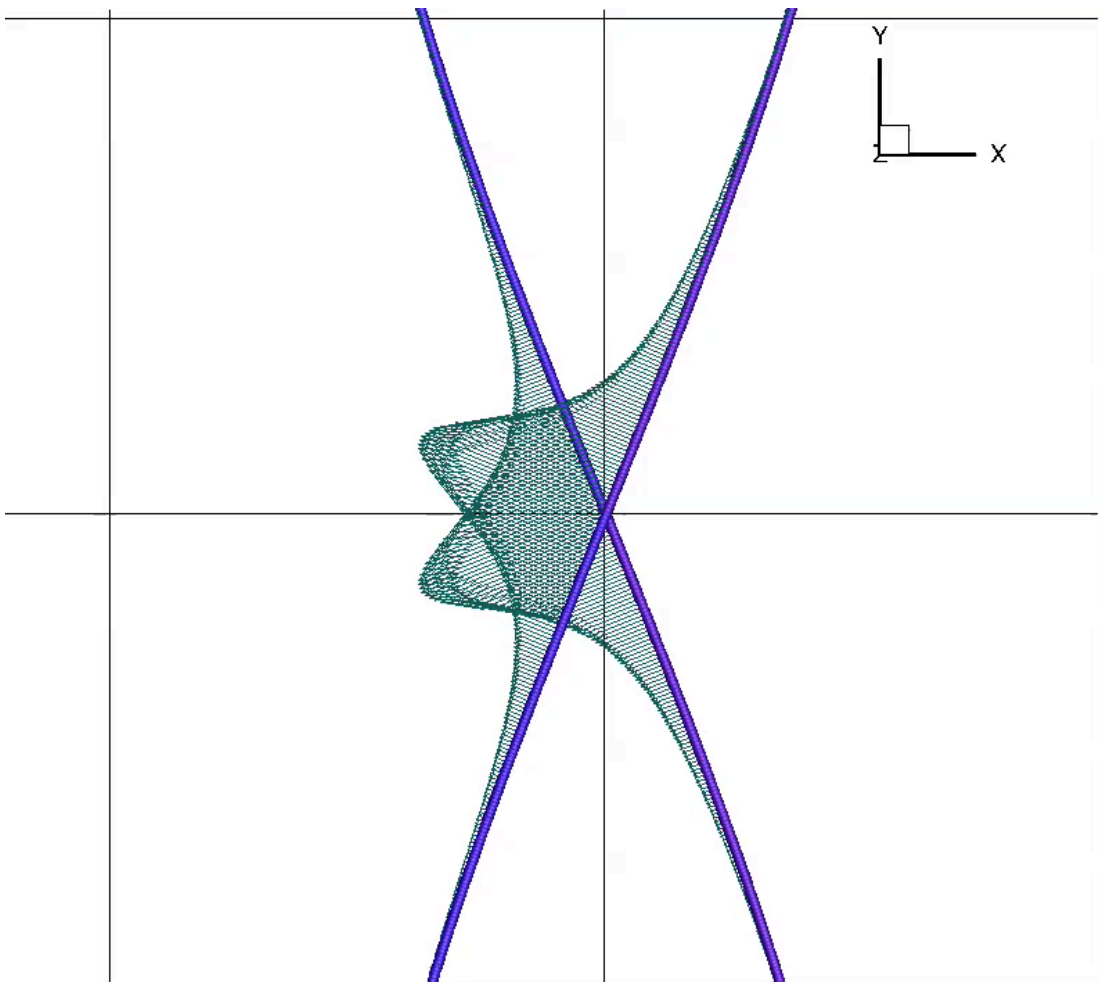

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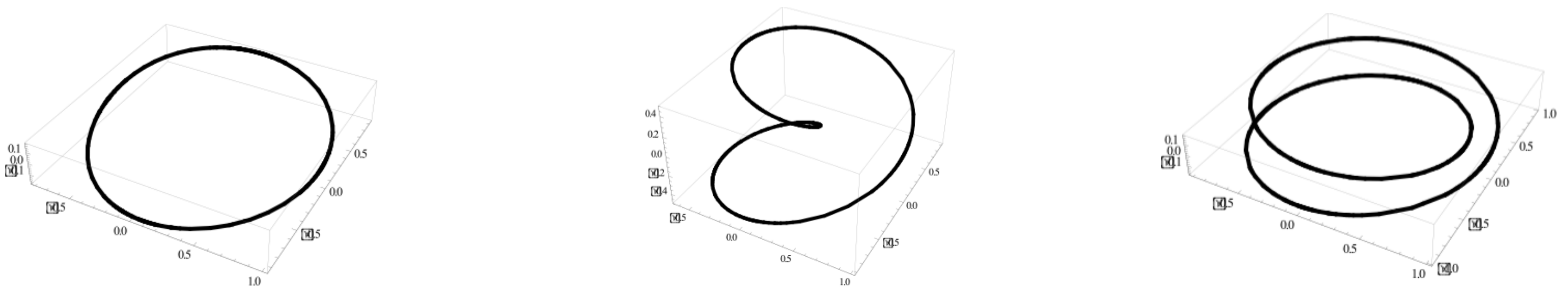
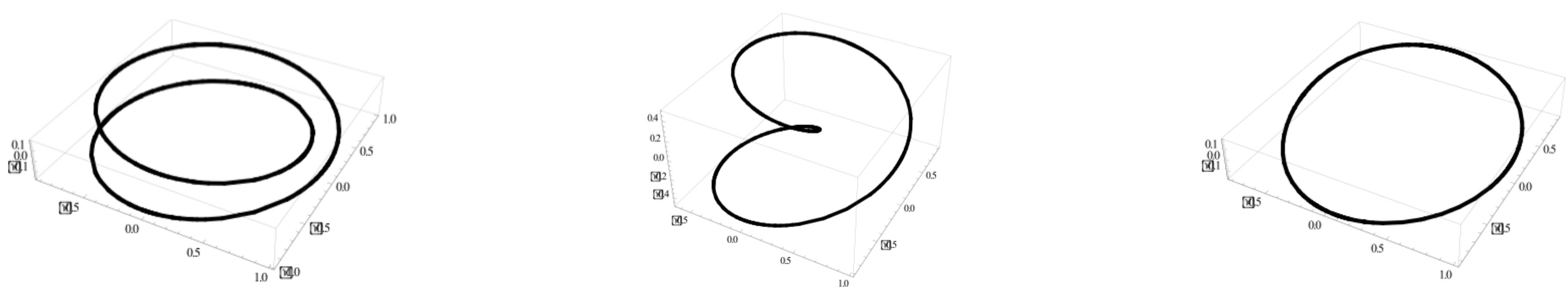
Stretch- Twist- Fold
fundamental to the dynamo process

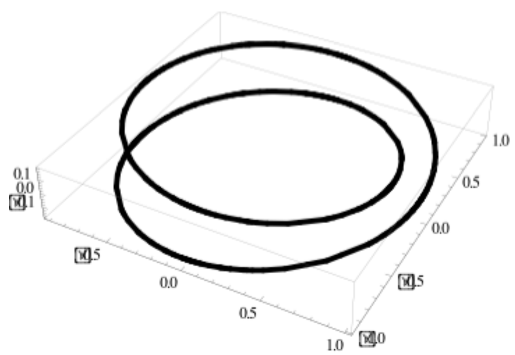


Unfold- Untwist-Relax!

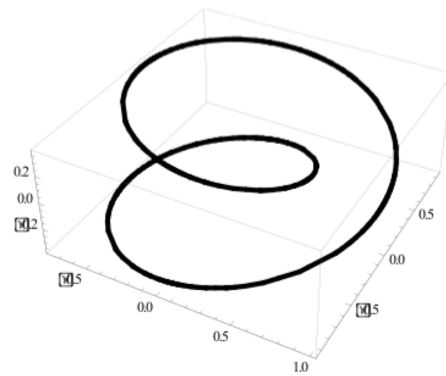


INVERSE PROCESSES

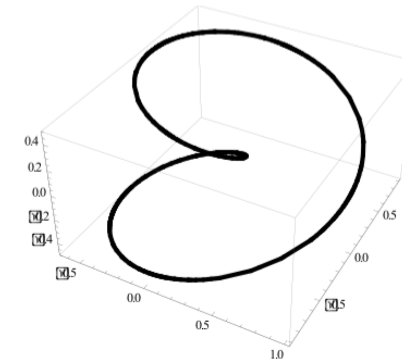
What about the surface spanning an untwisting wire?



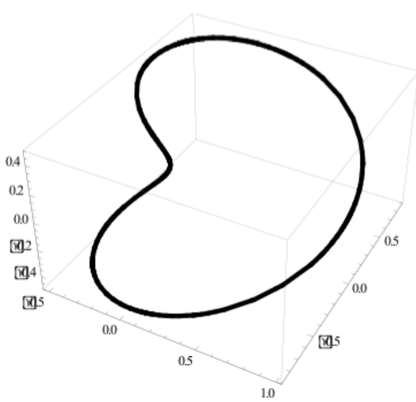
$t = 0.1$



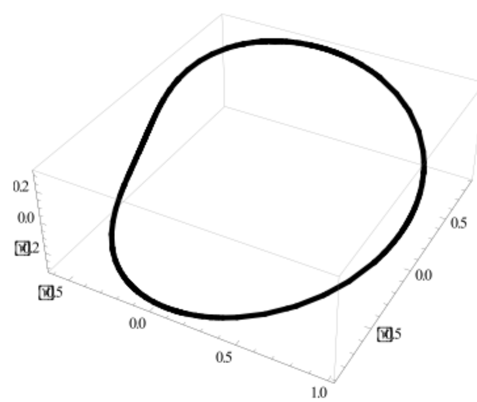
0.25



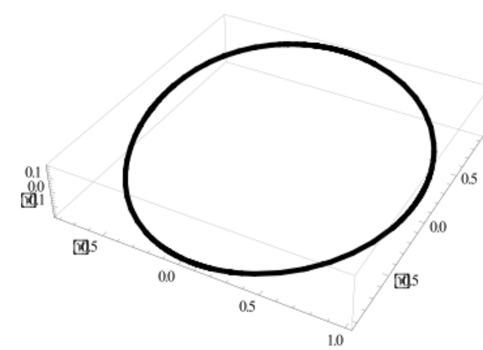
0.4



$t = 0.6$



0.8



0.9

Parametric equations of an untwisting wire

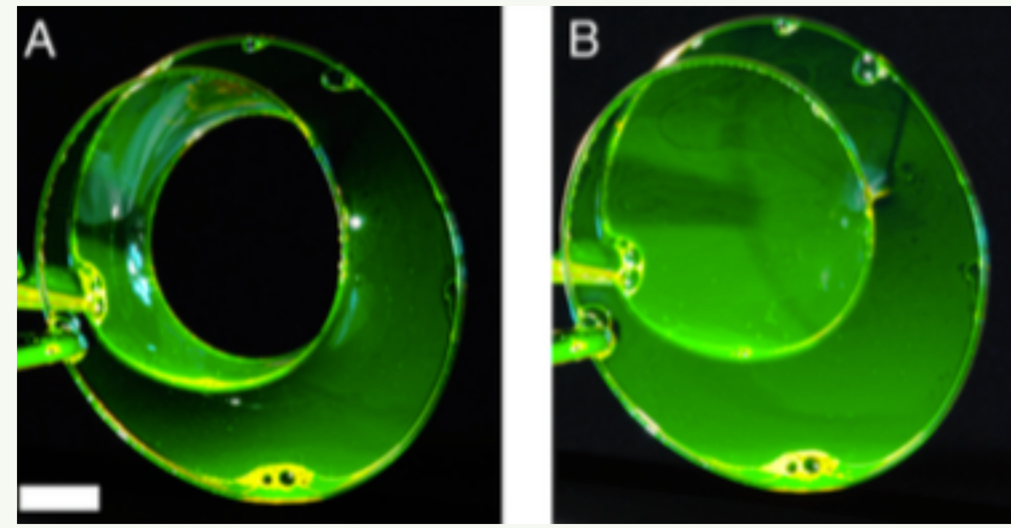
$$\mathbf{x}(s) = (x,y,z) = [-t\cos s + (1-t)\cos 2s,\ -t\sin s + (1-t)\sin 2s,\ -2t(1-t)\sin s]/l(t),$$

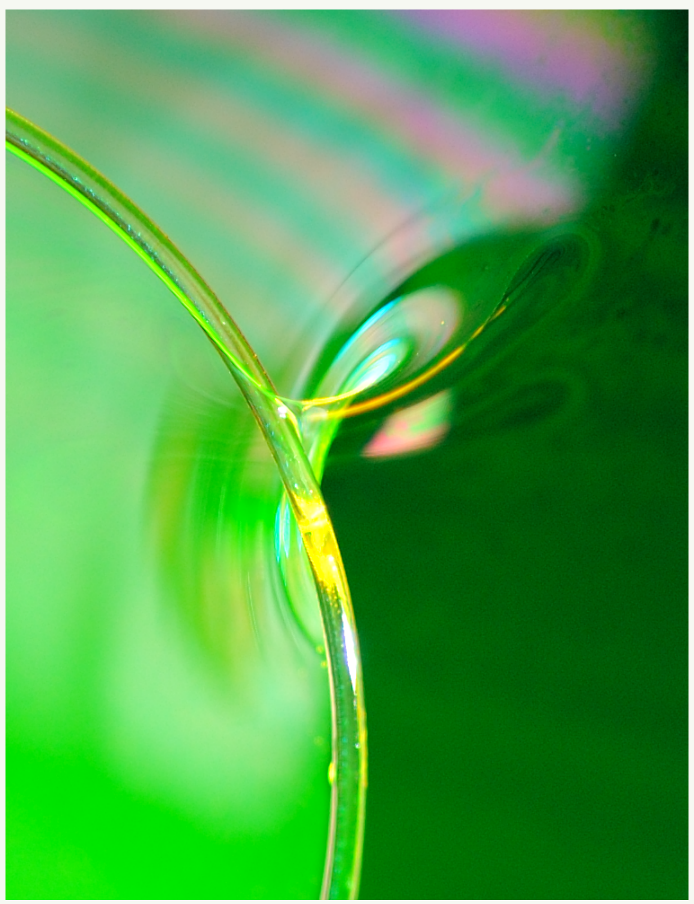
$$0 \le s \le 2\pi,$$

A One-Sided Soap-Film

a study originally motivated by dynamo theory

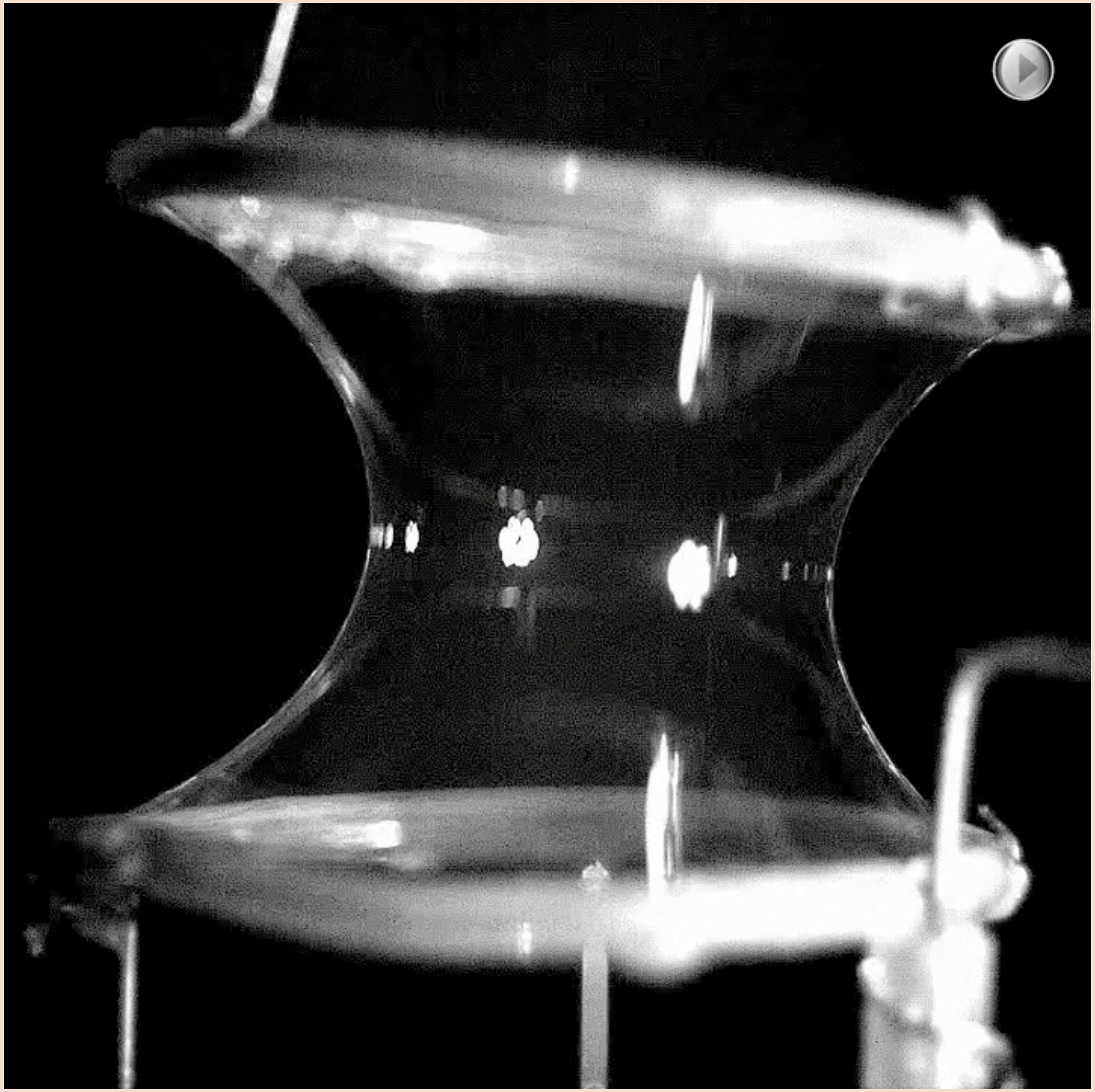
More topology! Take a circular wire, twist it and fold it back on itself; it is now the boundary of a Möbius strip. Dip this in soap solution and extract it; you thus create a one-sided soap-film.





Now untwist and unfold the wire. The soap film suddenly jumps to a two-sided disc-like topology. How does this topological transition occur?

Goldstein, R. E., HKM., Pesci, A. I. & Ricca, R. L. 2010 **Soap-film Möbius strip changes topology with a twist singularity.** *Proc. Natl. Acad. Sci.* **107** (51), 21979 – 21984.



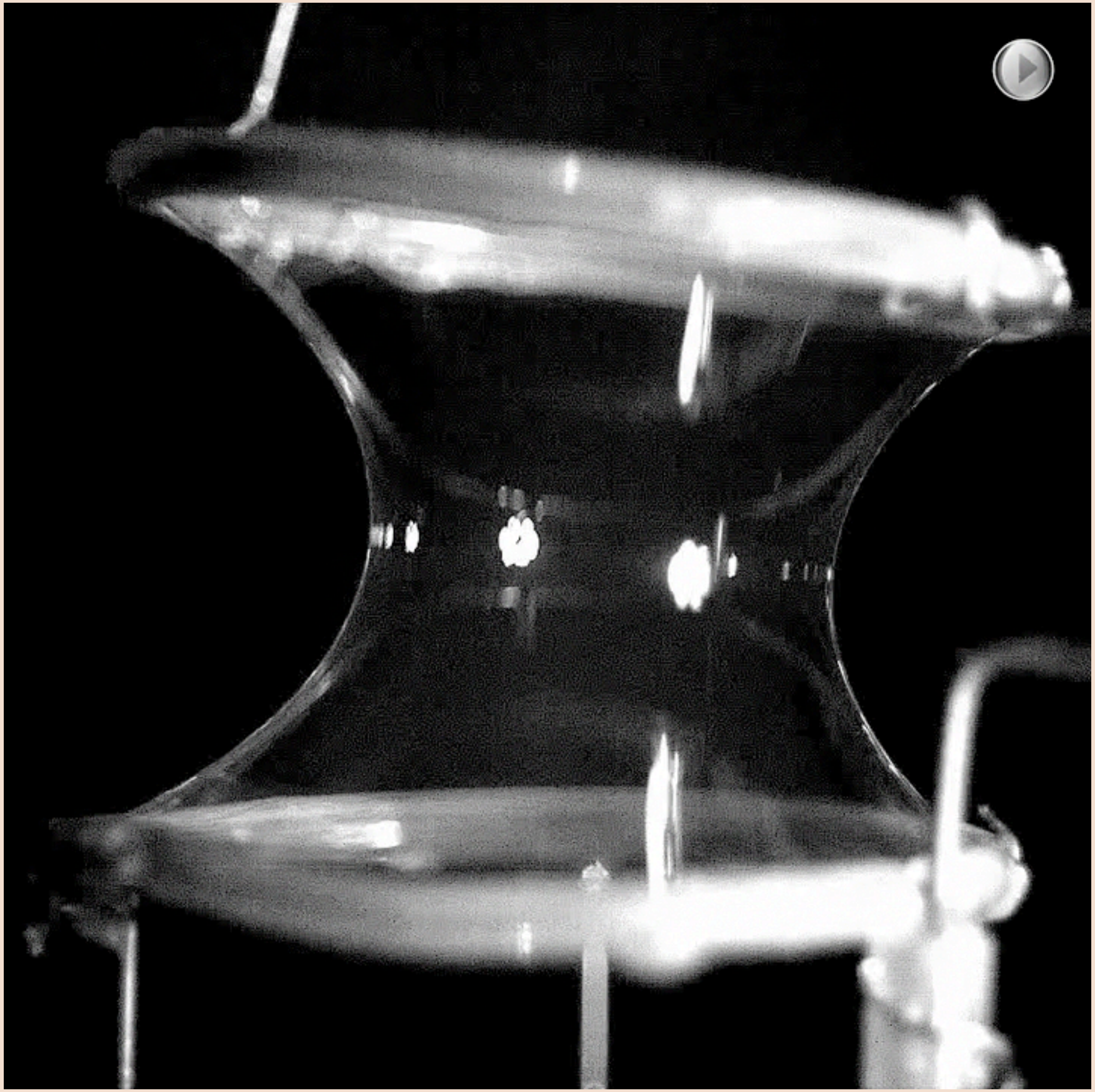
Under a slight rotation of the upper wire, the catenoid collapses and splits at an interior singularity, a process driven by surface tension, and 'resolved' by film inertia.

Robinson, N.D. & Steen, P.H. 2001
Observations of singularity formation during the capillary collapse and bubble pinch-off of a soap film bridge. *J. Coll. Int. Sci.* 241, 448-458

Nitsche, M. & Steen, P.H. 2004
Numerical simulations of inviscid capillary pinchoff. *J. Comp. Phys.* 200, 299-324

cf Couder's bouncing droplets!

There's enough going on here for five PhD theses!



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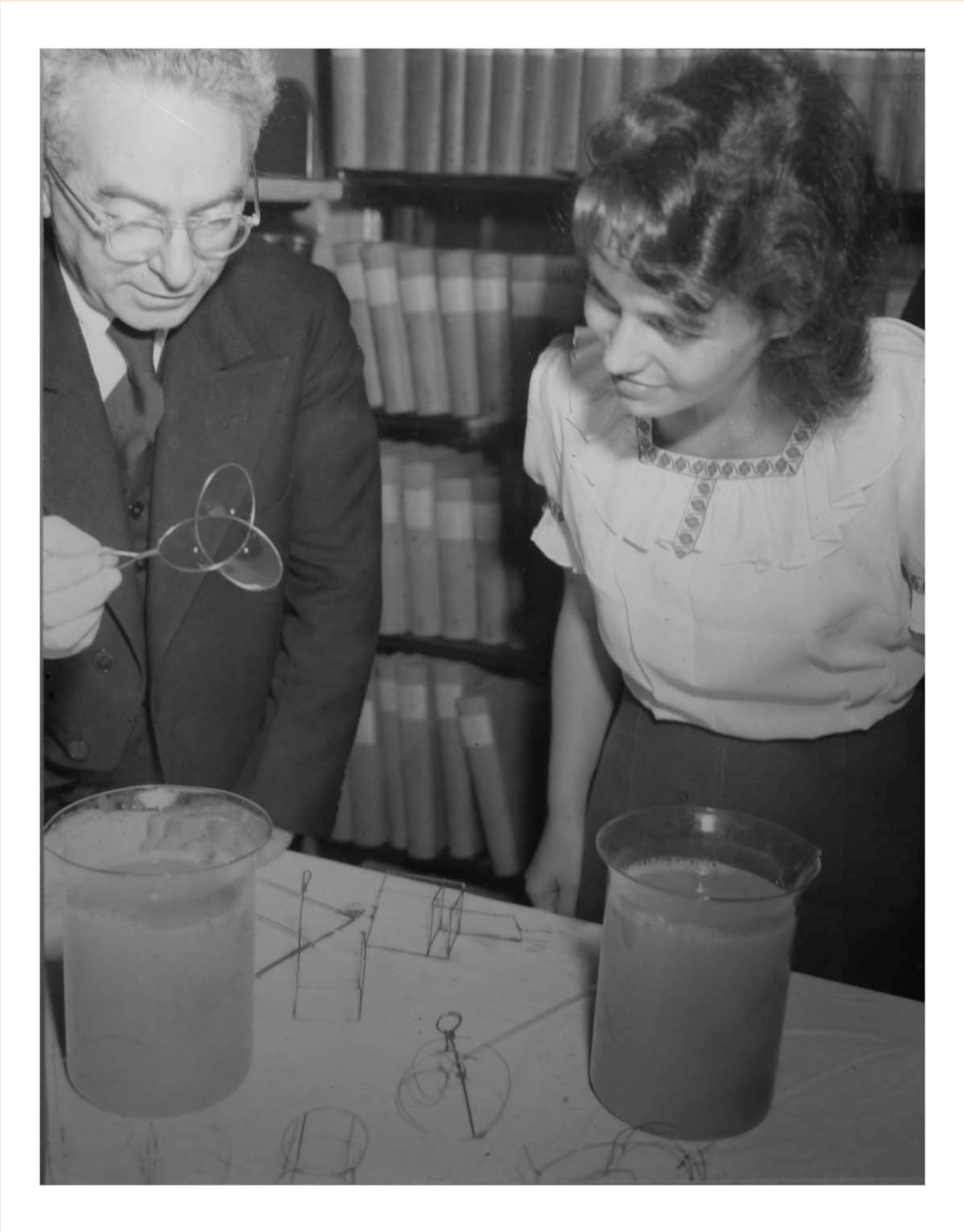
A distinguished precedent: **Richard Courant (1888 – 1972)**

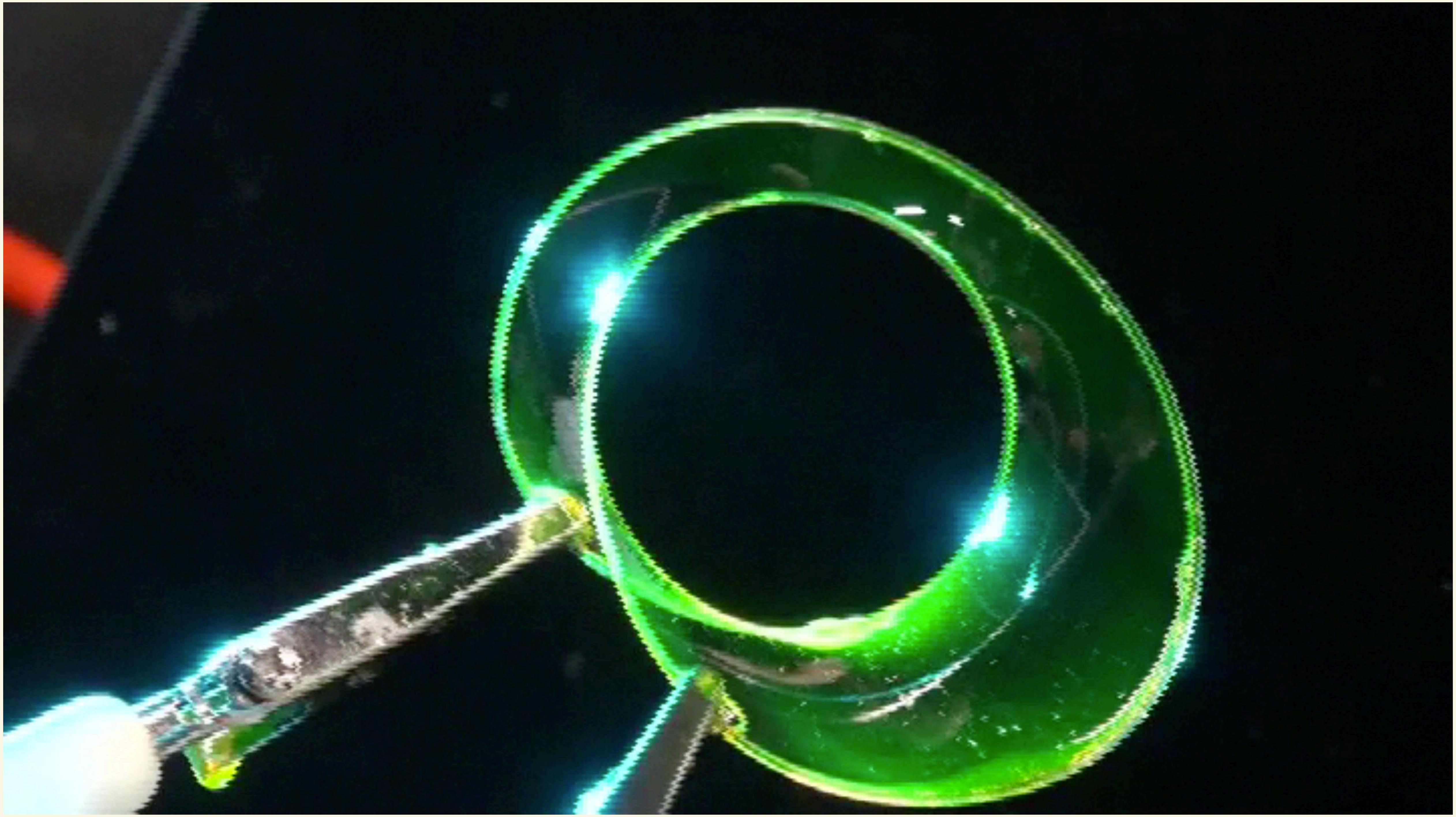


Soap film experiments with minimal surfaces.
Courant, R. (1940) *Amer.Math.Mon.* **47**, 167–174.

In this informal article, Courant mentions that a one-sided soap-film in the form of a Möbius strip is easily formed, and that it can jump to a two-sided film if the wire boundary is suitably distorted.

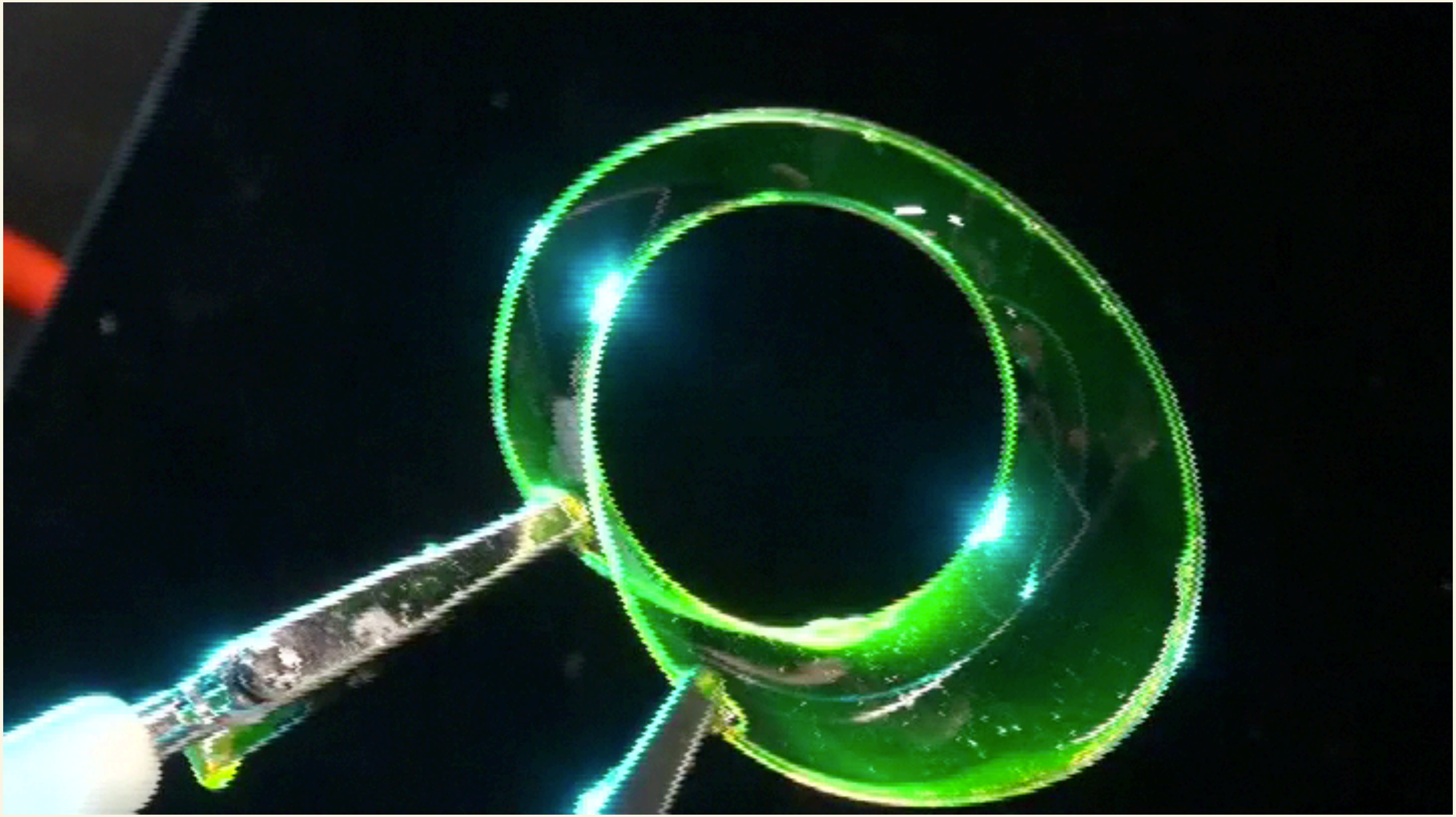
We decided to repeat this experiment . . . .

Return to the Möbius soap-film



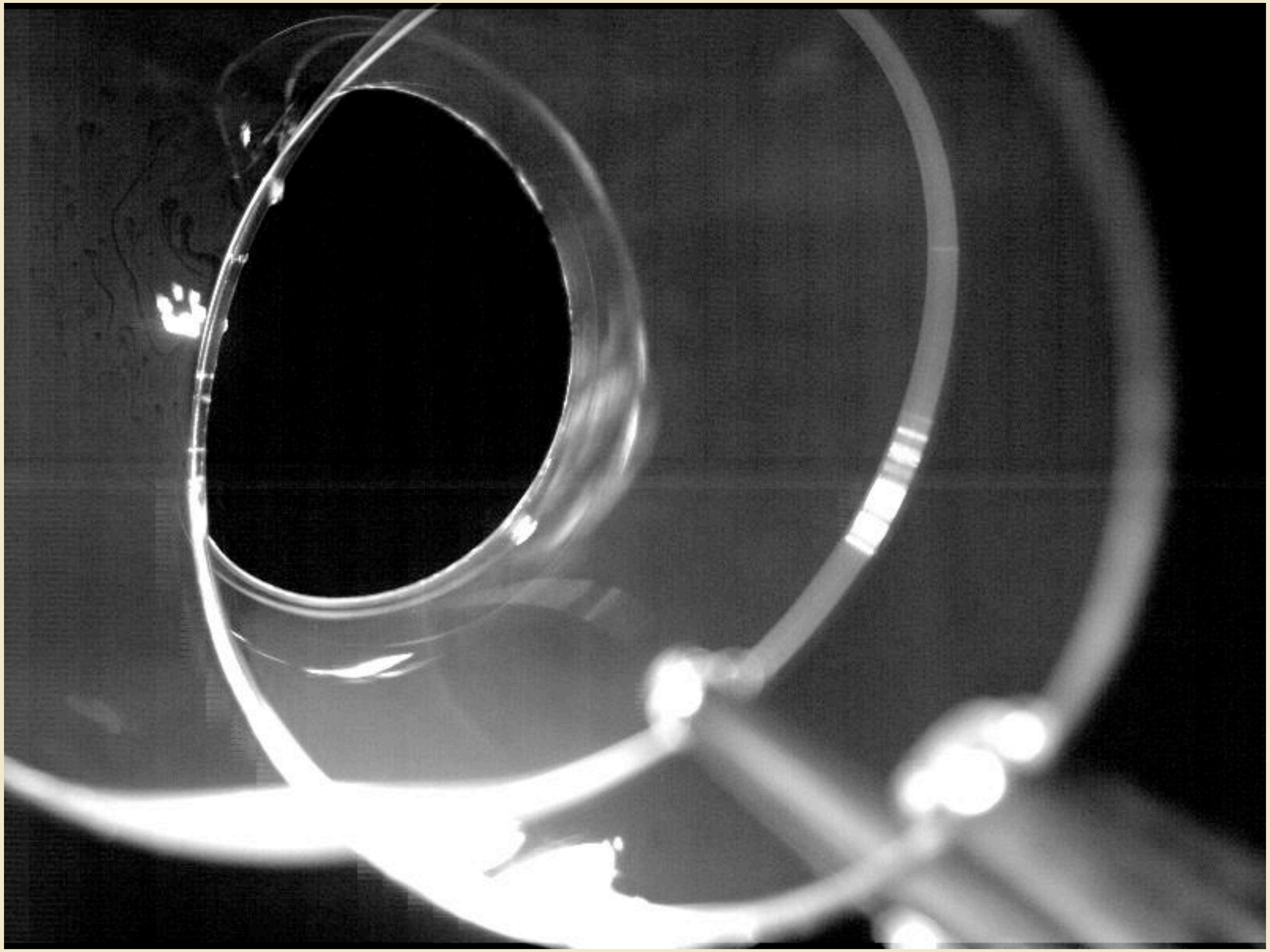
Real time collapse: the singularity occurs at the boundary

Return to the Möbius soap-film



Real time collapse: the singularity occurs at the boundary

High-speed close-up of the singularity



5600 frames/sec

Soap-film Möbius strip changes topology with a twist singularity
Goldstein, Moffatt, Pesci & Ricca (2010) *PNAS* **107**, 21979

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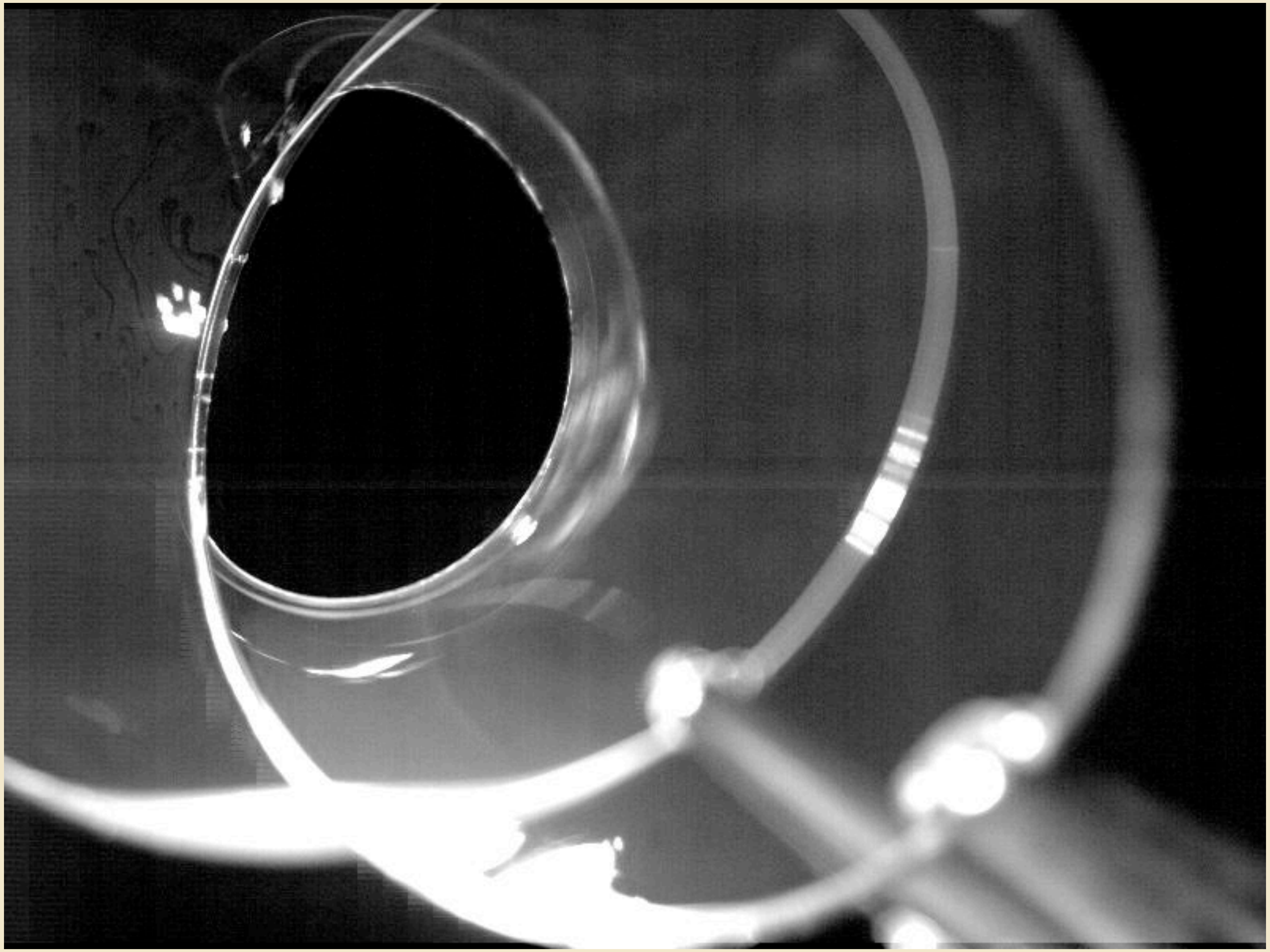

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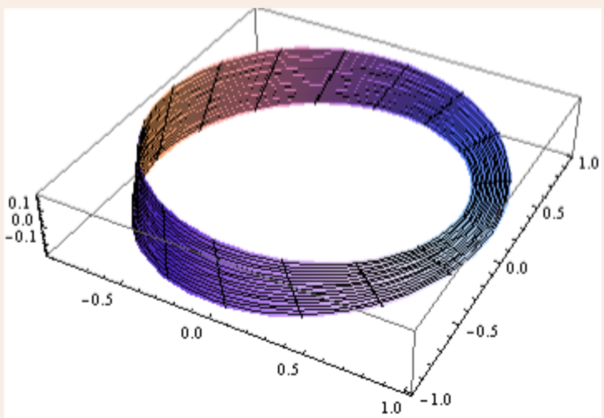
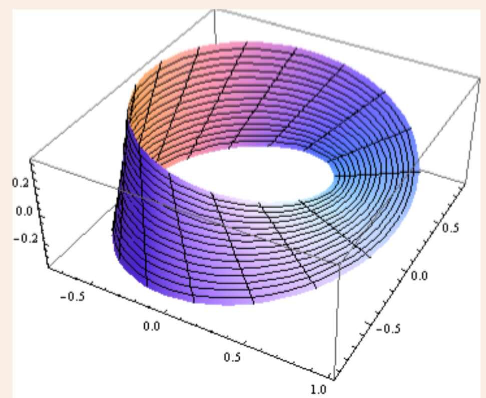
We get a clue from the family of ruled surfaces:

$$\mathbf{x}(s, \mu; t) = [\mu\, t \cos s + (1-t) \cos 2s,\ \ \mu\, t \sin s + (1-t) \sin 2s,\ \ 2\,\mu\, t\,(1-t) \sin s]$$

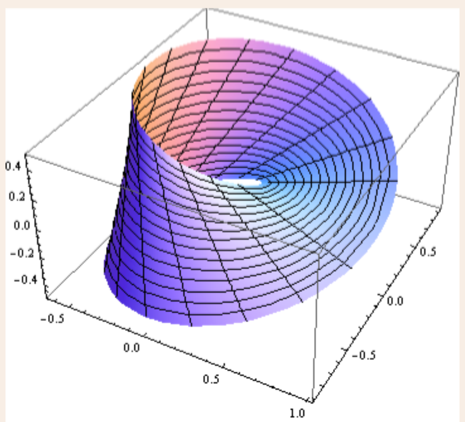
$(-1 \leq \mu \leq 1)$ The centreline $\mu = 0$ is a circle of radius $1 - t$ in the plane $z = 0$



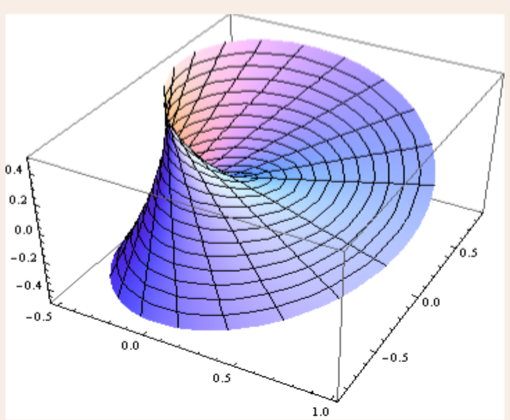
***t* = 0.1**



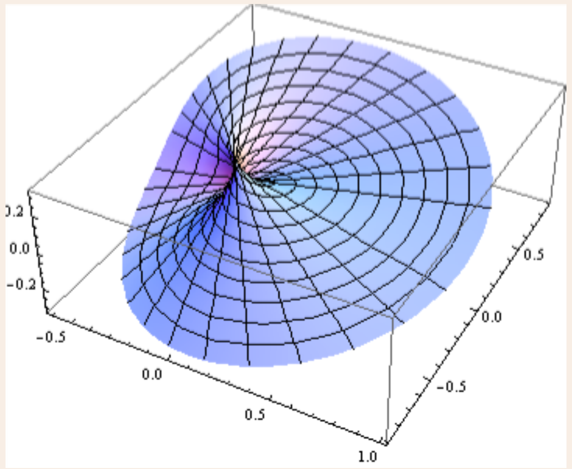
0.25



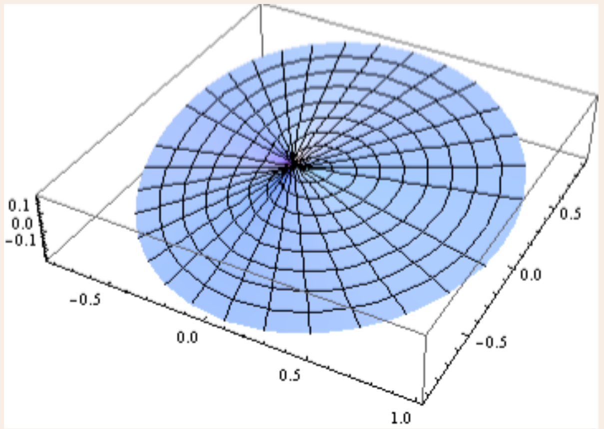
0.4



***t* = 0.6**



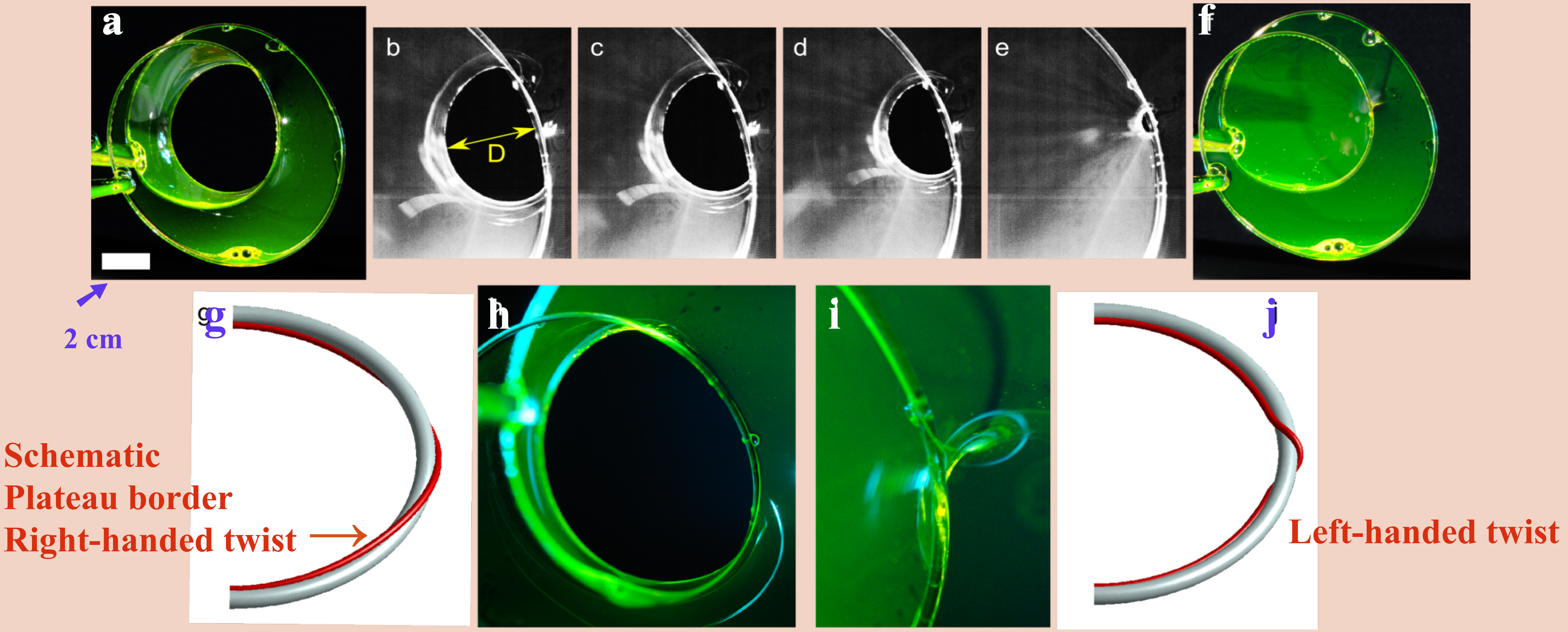
0.8



0.9

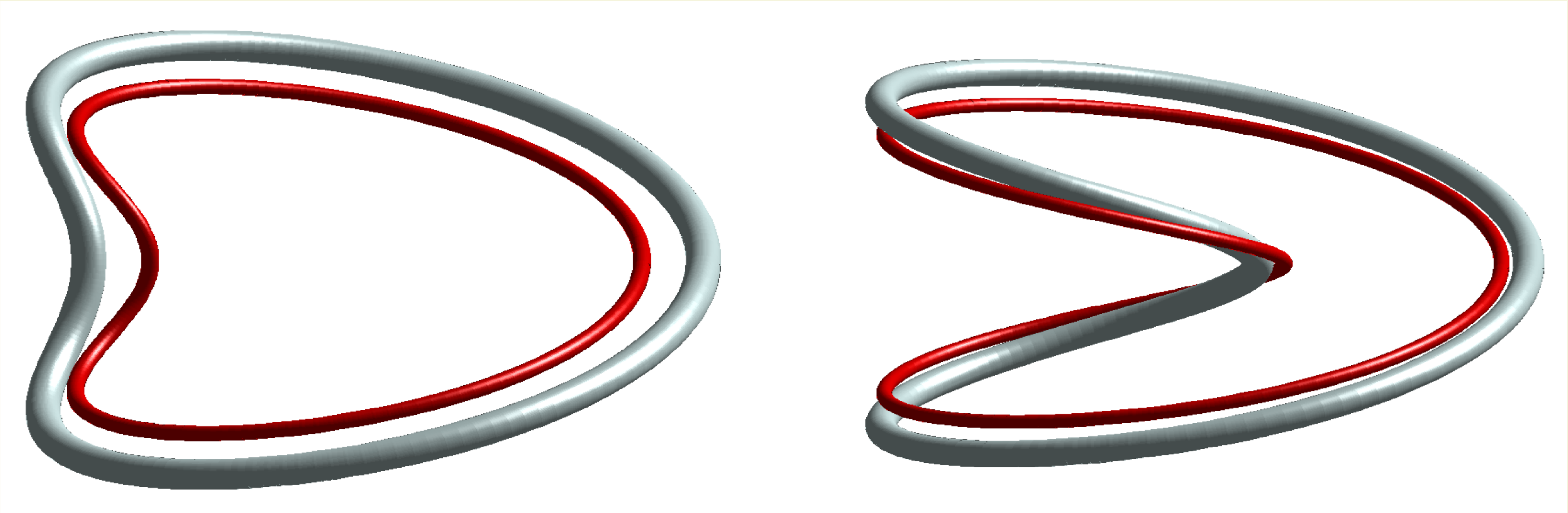
Topology changes (the hole disappears) somewhere between *t* = 0.6 and 0.8, actually at 2/3

Topological transition of a soap film Möbius strip



As the wire is gradually distorted, the one-sided film (a) jumps to a two-sided film (f).
(b)-(e): Frames from high-speed movie at intervals of 5.4 ms showing the collapse process.
After the collapse (i) the Plateau border is twisted round the frame in the opposite sense (!) as shown in (j).
The caustic in (i) is a consequence of the locally concentrated twist of the Plateau border.

Linking of the Plateau border and the wire frame



After the jump:
Two-sided surface:
PB and wire are unlinked

Before the jump:
One-sided surface:
PB and wire are doubly linked

over over under under ⬅️ **over under over under**

The linking number changes from 2 to 0 at the jump!

Euler's Disc — a Prototypical Example of a Finite-Time Singularity

Question: how is the energy dissipated?

My answer: Viscous dissipation in lubricating layer of air

Hotly disputed! The alternative is solid rolling friction

But all agree that dissipation* causes *the finite-time singularity

hkm (2000) Nature 404 (6780), 833–834.

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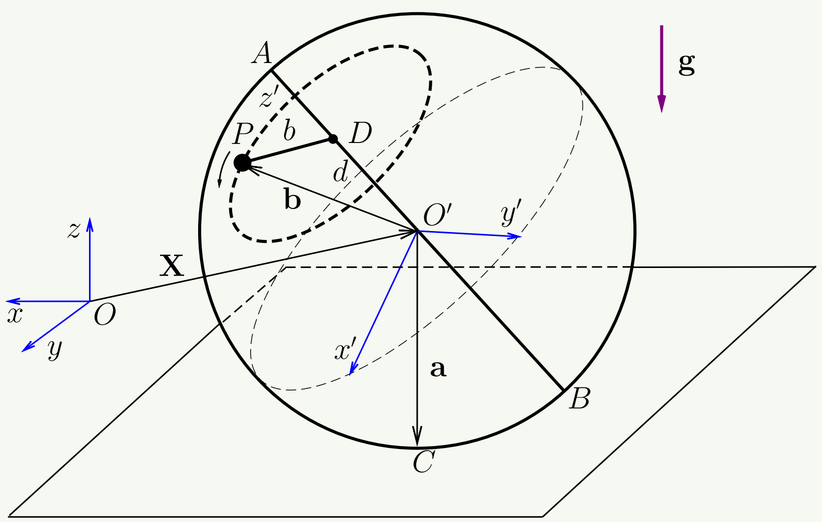
The Dance of the Beaver Ball
(in the anti-chapel of Trinity College, Cambridge)



Chaos in a dynamical system; the ball is driven by an internal rotor;
note how the ball keeps time with the music!
Ilin, hkm & Vladimirov 2017, PNAS, to appear

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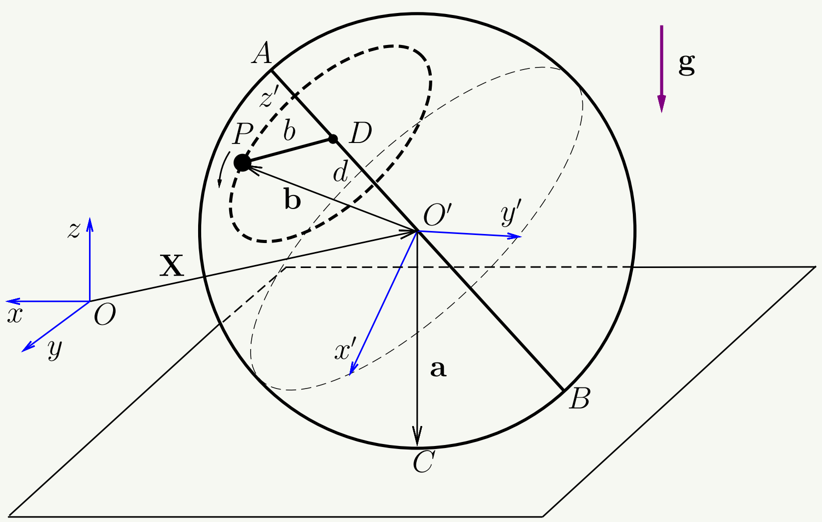

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