

The Maslov Index

Seminar Series

Edinburgh, 2009/10

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Chapter 1

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The Maslov Index

1.1 Introduction: The landscape of the Maslov index

There are several different things called “the Maslov index”. See [CLM] for the connections between them. In this chapter we will consider only the “algebraic” version developed in [Th] and references.

Let (V, ω) be a vector space over any field (with characteristic $\neq 2$) with symplectic form ω . There is a set of Lagrangian subspaces

$$\text{Lagr}(V, \omega) := \{L < V : L = L^\perp\},$$

where $W^\perp = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$.

Idea. The Maslov index is an invariant associated to any n -tuple in $\text{Lagr}(V)$, more precisely a quadratic form. So it is a morphism

$$\tau : (\text{Lagr}(V, \omega))^n \longrightarrow \{\text{quadratic forms}\}.$$

(This will be made precise later.) Commonly we have $n = 3$, and τ is invariant (at least) under the action of the symplectic group.

Here are some contexts in which the Maslov index plays an important role.

Example 1.1.1. If M is any smooth manifold, T^*M is canonically a symplectic manifold with symplectic form $\omega = -d\lambda$, where λ is the canonical Liouville 1-form. So $T(T^*M)$ is a family of symplectic vector spaces. For example, the vertical tangent spaces are Lagrangians. This is the original setting for the Maslov index in the theory of Maslov-Arnold [MA], Leray [Le], Kashiwara-Schapira [KS], under the name of “micro-local analysis”.

Example 1.1.2. Let G be a Lie group with Lie algebra \mathfrak{g} . Then \mathfrak{g}^* is a disjoint union of coadjoint orbits (i.e. orbits under the coadjoint action of G on \mathfrak{g}^*). Each of these is again a symplectic manifold. Kirillov studied these structures in the context of representation theory, under the name “orbit method”. The role of the Maslov index was explained by Lion, Perrin [LP], and others. Something similar can also be done for finite nilpotent groups [KT].

Example 1.1.3. Let M be a real manifold of dimension $4n + 2$. Then $H^{2n+1}(M; \mathbb{k})$ is a symplectic vector space. The role of the Maslov index here was independently developed by Wall [Wa], and is related to surgery theory and higher cobordisms.

Example 1.1.4. Hermitian K -theory (Karoubi). The role of the Maslov index has not been satisfactorily developed yet.

1.1.1 Symplectic vs. projective geometry

Motto. The Maslov index is to symplectic geometry as the cross ratio is to projective geometry.

The projective linear group $PGL(2; \mathbb{k})$ acts transitively on $\mathbb{P}_{\mathbb{k}}^1$ and simply transitively on P_3 , the space of triples of distinct points. So consider the space P_4 of quadruples (L_1, L_2, L_3, L_4) . By the action of $PGL(2)$ we can map it to $(0, 1, \infty, L)$, and $L \in \mathbb{k} \setminus \{0, 1\}$ is the *cross ratio* of the quadruple. Thus two quadruples of distinct points are in the same $PGL(2; \mathbb{k})$ -orbit if and only if they have the same cross ratio.

Let us look at the symplectic group. Note that for any choice of symplectic form on \mathbb{k}^2 , we have $Sp(2; \mathbb{k}) \cong SL(2; \mathbb{k})$, and $Lagr(\mathbb{k}^2) \cong \mathbb{P}_{\mathbb{k}}^1$. In general, $PSL(2)$ is a proper subgroup of $PGL(2)$, so it does not act transitively on P_3 . Rather, the set of $PSL(2)$ -orbits of P_3 corresponds to the quotient $PGL(2)/PSL(2)$. Taking determinants gives an isomorphism

$$\det: PGL(2; \mathbb{k})/PSL(2; \mathbb{k}) \xrightarrow{\sim} \mathbb{k}^*/(\mathbb{k}^*)^2.$$

The right-hand side is the set of 1-dimensional quadratic forms up to isomorphism. This map $\tau: P_3 \rightarrow \mathbb{k}^*/(\mathbb{k}^*)^2$ is a version of the Maslov index (and it is independent of any choices).

Example 1.1.5. If $\mathbb{k} = \mathbb{R}$, then $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{-1, +1\}$, and $\mathbb{R}P^1 \cong S^1$. The Maslov index of L_1, L_2, L_3 is positive if the three lines are in cyclic order, and negative if they are not.

Theorem 1.1.6 ([RR, Theorem 2.11]). *Consider the action of $Sp(V, \omega)$ on $Lagr(V, \omega)^3$. Two triples $L, L' \in Lagr(V, \omega)^3$ are in the same $Sp(V, \omega)$ -orbit if and only if they are in the same $GL(V)$ -orbit and $\tau(L) = \tau(L')$.*

1.2 The Maslov index of three transverse Lagrangians

Here is a more somewhat general construction of the Maslov index. For any symplectic vector space (V, ω) , consider three transverse Lagrangians L_1, L_2, L_3 , so $V = L_1 \oplus L_3$. Think of L_2 as the graph of a linear map $f: L_1 \rightarrow L_3$. So we have maps

$$L_1 \xrightarrow{f} L_3 \xrightarrow{\sim} V/L_1 = L_1^*,$$

and we can think of L_2 as the graph of a map $\gamma: L_1 \rightarrow L_1^*$. Moreover, L_2 is Lagrangian if and only if γ is symmetric considered as a map $L_1 \otimes L_1 \rightarrow \mathbb{k}$. So we define $\tau(L_1, L_2, L_3)$ to be γ considered as a quadratic form. Explicitly, $\tau(L_1, L_2, L_3)$ is the symmetric form on L_1 given by

$$a \otimes b \mapsto \omega(a, f(b)).$$

This was Leray's definition [Le]. The full definition will consider arbitrary n -tuples of Lagrangians.

1.3 Categories of symplectic/symmetric spaces

Terminology. We will say “symplectic space” for a symplectic vector space, and “quadratic space” for a vector space with a non-degenerate, symmetric bilinear form. Since $2 \neq 0$, symmetric bilinear forms and quadratic forms determine each other uniquely.

We construct a category Q_+ , whose objects are quadratic spaces. If V_1, V_2 are quadratic spaces, then morphisms are given by

$$\mathrm{Hom}_{Q_+}(V_1, V_2) := \mathrm{Lagr}(V_1^\circ \oplus V_2),$$

where $(V, q)^\circ := (V, -q)$, and (just as in the symplectic case) $L < V$ is “Lagrangian” if $L = L^\perp$ with respect to the quadratic form. $\mathrm{Lagr}(V_1^\circ \oplus V_2)$ is sometimes called the space of “Lagrangian correspondences”. Composition of morphisms is like composition of general correspondences (think of composing functions $f: A \rightarrow B$ and $g: B \rightarrow C$ via the subsets of $A \times B$ and $B \times C$).

Example 1.3.1. If $f: V_1 \rightarrow V_2$ is an isometry, then the graph $\Gamma_f \subset V_1^\circ \oplus V_2$ is Lagrangian.

We construct another category Q_- in the same way, only that we take the objects to be symplectic spaces, and $(V, \omega)^\circ := (V, -\omega)$.

Definition 1.3.2. The *Witt group* of a field \mathbb{k} is $W(\mathbb{k}) := \pi_0(Q_+)$. To spell this out, we say that two quadratic spaces are *Witt equivalent* if there exists a Lagrangian correspondence between them. For example, (V, q) is equivalent to 0 if and only if it contains a Lagrangian. Then $W(\mathbb{k})$ is the group whose elements are Witt equivalence-classes of quadratic spaces, with addition induced by direct sum. Note that inverses are given by $-(V, q) = (V, q)^\circ$.

For example, $(\mathbb{k}^2, \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}) \equiv 0$. For $\mathbb{k} = \mathbb{R}$,

$$\mathrm{index}: W(\mathbb{R}) \xrightarrow{\sim} \mathbb{Z},$$

where the $\mathrm{index}(q)$ is the number of positive eigenvalues minus the number of negative eigenvalues. For $\mathbb{k} = \mathbb{C}$ (or any algebraically closed field) we have

$$\mathrm{dim}: W(\mathbb{C}) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}.$$

What we are going to do is to define a function $L \mapsto (T_L, q_L)$ from the set $\mathrm{Lagr}(V)^n$ of n -tuples of Lagrangians in a symplectic space, to the category Q_+ of quadratic spaces. The class of (T_L, q_L) in $W(\mathbb{k})$ will be, by definition, the Maslov index of L .

1.4 Signatures of manifolds

First we consider a parallel construction coming from topology.

There is a relation between Lagrangian correspondences and the cobordism category. Recall that the category of cobordisms \mathcal{Cob}_{4n+1} has as objects closed, oriented $4n$ -manifolds. For any such manifold M , write M° for M with the opposite orientation. A morphism from M_1 to M_2 is a cobordism (W, M_1, M_2) , where W is a $(4n+1)$ -dimensional manifold-with-boundary such that $\partial W = M_1^\circ \sqcup M_2$.

There is a functor $\mathrm{sig}: \mathcal{Cob}_{4n+1} \rightarrow Q_+$. On objects,

$$\mathrm{sig}(M) = (H^{2n}(M, \mathbb{k}), \cup).$$

Note that $\mathrm{sig}(M^\circ) = \mathrm{sig}(M)^\circ$. On morphisms,

$$\mathrm{sig}(W, M_1, M_2) = \mathrm{im}(H^{2n}(W) \rightarrow H^{2n}(M_1^\circ) \oplus H^{2n}(M_2)).$$

The point is that this image is Lagrangian. The traditional “signature of M ” is the class of $\mathrm{sig}(M)$ in $W(\mathbb{k})$. The existence of this functor shows that class to be “cobordism invariant”.

1.5 Definition and properties of the Maslov index

Following [Th], we will define the Maslov index as an n -ary map

$$\tau: \text{Lagr}(V, \omega)^{\mathbb{Z}/n\mathbb{Z}} \rightarrow \mathbf{ob}Q_+ \rightarrow W(\mathbb{k}).$$

Given an n -tuple $L = (L_1, \dots, L_n)$ of Lagrangians, we have a cochain complex

$$C_L := \bigoplus (L_i \cap L_{i+1}) \xrightarrow{\partial} \bigoplus L_i \xrightarrow{\Sigma} V,$$

concentrated in degrees 0, 1 and 2. Here Σ just sums the components, while ∂ takes $a \in L_i \cap L_{i+1}$ to $(a, -a) \in L_i \oplus L_{i+1}$. As Beilinson pointed out, one can think of this cochain complex topologically as a “decorated cell complex,” namely a n -gon with the face labelled by V , edges labelled by the L_i , and vertices labelled by $L_i \cap L_{i+1}$ (see Figure 1.1).

In analogy to the situation with signatures of manifolds, the “middle cohomology” $T_L := H^1(C_L) = \ker \Sigma / \text{im } \partial$ of the complex is equipped with a “cup product” symmetric form, given explicitly by

$$q_L(a, b) = \sum_{i>j} \omega(a_i, b_j).$$

Here we have lifted $a, b \in T_L$ to representatives $(a_i), (b_i) \in \bigoplus_i L_i$.

Proposition 1.5.1. *(T_L, q_L) is a quadratic space, i.e. q_L is well defined, symmetric, and non-degenerate. It is called the “Maslov form”.*

Definition 1.5.2. The Maslov index $\tau(L_1, \dots, L_n)$ is the quadratic space $(T_L, q_L) \in W(\mathbb{k})$.

Proposition 1.5.3. *The obvious identities*

$$\begin{aligned} T(L_1, \dots, L_n) &= T(L_n, L_1, \dots, L_{n-1}) \\ &= T(L_n, \dots, L_1)^\circ \end{aligned}$$

are isometries.

In our topological interpretation, Figure 1.1, these isometries come from the dihedral symmetry of the polygon.

Proposition 1.5.4. *There exist canonical Lagrangian correspondences*

$$T(L_1, \dots, L_k) \oplus T(L_1, L_k, \dots, L_n) \rightarrow T(L_1, \dots, L_n)$$

for each $k < n$. In particular, these give rise to equalities for τ in the Witt group.

Using this proposition, one can always reduce to the case of three Lagrangians. In the topological interpretation, these Lagrangian correspondences come from a “cobordism” (Figure 1.2). These pictures are given a precise meaning in [Th].

Propositions 1.5.3 and 1.5.4 show that “ τ is a cocycle”:

$$\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0. \quad (1.5.1)$$

This is the most important property of τ .

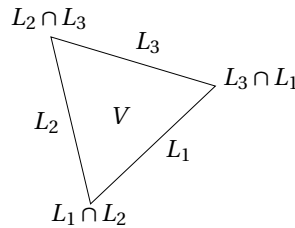


Figure 1.1: The cell complex when $n=3$.

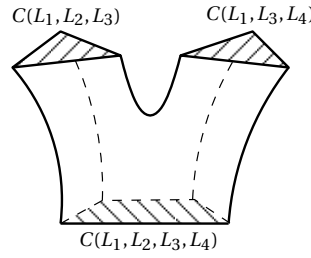


Figure 1.2: A “cobordism” between cell complexes defining the Lagrangian correspondence of Proposition 1.5.4 ($n = 4, k = 3$).

1.6 The topology of the Lagrangian Graßmannian

1.6.1 Over the real numbers

Recall the identities

$$W(\mathbb{R}) \cong \mathbb{Z} \cong \pi_1 \text{Lagr}(V, \omega).$$

Let $\pi: \widetilde{\text{Lagr}}(V, \omega) \rightarrow \text{Lagr}(V, \omega)$ be the universal cover of the Lagrangian Graßmannian.

Theorem 1.6.1 ([Le]). *There exists a function $m: (\widetilde{\text{Lagr}}(V))^2 \rightarrow \mathbb{Z}$ such that*

$$\tau(\pi(\tilde{L}_1), \dots, \pi(\tilde{L}_n)) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} m(\tilde{L}_i, \tilde{L}_{i+1}).$$

Remark 1.6.2. We can think of m as a function of homotopy classes of paths in the Lagrangian Graßmannian.

We now describe the construction of Leray’s function m . For the standard Graßmannian $Gr_k(V)$ of k -dimensional planes in V , the tangent spaces are

$$T_L Gr_k(V) = \text{Hom}(L, V/L).$$

In the Lagrangian case, $V/L = L^*$ and the tangent space consists of *symmetric* maps,

$$T_L \text{Lagr}(V) = \text{Sym}(L, L^*) = \{\text{quadratic forms on } L\}.$$

So we can speak of tangents being positive, negative, etc. A *path* is positive, etc., if all its tangents are. The identification $\pi_1 \text{Lagr}(V) \cong \mathbb{Z}$ is chosen so that $1 \in \mathbb{Z}$ is represented by a positive path.

Lemma 1.6.3. *Between any L_1 and L_2 there exists a unique homotopy class γ_{\pm} of paths $L_1 \rightarrow L_2$ such that*

- γ_+ is positive, γ_- is negative.
- $\gamma_{\pm}(t) \cap L_1 = \gamma_{\pm}(t) \cap L_2 = L_1 \cap L_2$ for $t \in (0, 1)$.

Definition 1.6.4. Suppose that $\gamma: L_1 \rightarrow L_2$ is a path. Then

$$m(\gamma) = \gamma_+(L_2, L_1) \circ \gamma + \gamma_-(L_2, L_1) \circ \gamma.$$

Here each summand is an element of $\pi_1(\text{Lagr}(V), L_1) \cong \mathbb{Z}$.

Remark 1.6.5. $\text{Lagr}(V)$ has a unique double cover $L^{(2)}$. For any pair $\tilde{L}_1, \tilde{L}_2 \in L^{(2)}$, the number $m(\tilde{L}_1, \tilde{L}_1)$ is well-defined modulo 4.

1.6.2 General ground fields

In the previous section, we studied real vector spaces. To generalise the notions to general ground fields, we observe the following correspondences.

- The double cover $L^{(2)}$ corresponds to the set Λ of oriented Lagrangians (L, o) , where $o \in \Lambda^{\text{top}} L / (\mathbb{k}^*)^2$ is the ‘orientation’.
- \mathbb{Z} corresponds to the general Witt group $W(\mathbb{k})$.
- The ideals $2\mathbb{Z}, 4\mathbb{Z}$, etc. correspond to ideals $I := \ker(\dim: W(\mathbb{k}) \rightarrow \mathbb{Z}/2), I^2$, etc. (Note that the Witt group is in fact a ring with multiplication given by \otimes .)

Theorem 1.6.6 ([PPS]). *There exists a function $m: \Lambda \times \Lambda \rightarrow W(\mathbb{k})$ such that*

$$\tau(L_1, \dots, L_n) = \sum_i m(L_i, L_{i+1}) \pmod{I^2}.$$

Moreover, m is invariant under the natural action of $Sp(V)$ on $\Lambda \times \Lambda$.

1.7 Postscript: The Metaplectic Group

The cocycle property (1.5.1) has the following consequence. Choose $L \in \text{Lagr}(V)$. Let $Mp_1(V)$ be the set $W(\mathbb{k}) \times Sp(V)$ equipped with the multiplication

$$(q, g)(q', g') := (q + q' + \tau(L, gL, gg'L), gg').$$

Then $Mp_1(V)$ is a group and gives a central extension $0 \rightarrow W(\mathbb{k}) \rightarrow Mp_1(V) \rightarrow Sp(V) \rightarrow 1$.

Moreover, Theorem 1.6.6 implies that, choosing $\tilde{L} \in \Lambda$ over $L \in \text{Lagr}(V)$, the subset

$$Mp_2(V) = \{(m(g\tilde{L}, \tilde{L}) + q, g) : q \in I^2, g \in Sp(V)\} \subset Mp_1(V)$$

is a subgroup, giving a central extension $0 \rightarrow I^2 \rightarrow Mp_2(V) \rightarrow Sp(V) \rightarrow 1$. Finally, it is traditional to quotient I^2 by I^3 to define a central extension

$$0 \rightarrow I^2/I^3 \rightarrow Mp(V) \rightarrow Sp(V) \rightarrow 1,$$

in which $Mp(V)$ is called the *metaplectic group*.

Over \mathbb{R} , $I^2/I^3 = \mathbb{Z}/2\mathbb{Z}$, so $Mp(V)$ is then the (unique) double cover of $Sp(V)$. In this case, $Mp_1(V)$ has four connected components, among which $Mp_2(V)$ is the identity. $Mp_2(V)$ is the universal covering group of $Sp(V)$.

Remark 1.7.1. Instead of choosing a Lagrangian $L \in \text{Lagr}(V)$, one can construct the metaplectic group more canonically by observing that $Sp(V)$ embeds into $\text{Lagr}(V^\circ \oplus V)$, by $g \mapsto \Gamma_g$, the graph of g . Then define multiplication on $Mp_2(V)$ by

$$(q, g)(q', g') = (q + q' + \tau(\Gamma_1, \Gamma_g, \Gamma_{gg'}), gg').$$

Moreover, Γ_g has a canonical orientation. But the first construction using L is convenient in many contexts.

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