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Localization in quadratic L-theory

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Introduction

Localization is an indispensable tool in the computation of the surgery obstruction groups $L_n(\pi) \equiv L_n(\mathbb{Z}[\pi])$ (n(mod 4)) of Wall [3], at least for finite groups π. The L-groups $L_n(A)$ of a ring with involution A are compared with the L-groups $L_n(S^{-1}A)$ of the localization $S^{-1}A$ inverting some multiplicative subset $S \subset A$, the difference being measured by certain L-groups $L_n(A,S)$ depending on the category of S-torsion A-modules. In particular, if $A = \mathbb{Z}[\pi]$, $S = \mathbb{Z} - \{0\} \subset A$ and π is finite then $S^{-1}A = \mathbb{Q}[\pi]$ is semi-simple, and it is comparatively easy to compute $L_n(\mathbb{Q}[\pi])$, $L_n(\mathbb{Z}[\pi],S)$ and hence $L_n(\mathbb{Z}[\pi])$.

Localization in algebraic L-theory has already been studied by many authors, including Wall [1],[2],[6], Passman and Petrie [1], Connolly [1], Milnor and Husemoller [1], Bak and Scharlau [1], Karoubi [1], Pardon [1],[2], Carlsson and Milgram [1], though not in the generality obtained here.

The behaviour of the L-groups under localization is governed by an exact sequence of the type

$$\ldots \longrightarrow L_n(A) \longrightarrow L_n(S^{-1}A) \longrightarrow L_n(A,S) \longrightarrow L_{n-1}(A) \longrightarrow L_{n-1}(S^{-1}A) \longrightarrow \ldots .$$

Our immediate aim in this paper is to obtain a precise statement of this sequence (Proposition 2.4). We shall go some way towards a proof, but the detailed account is deferred to a projected instalment of the series "The algebraic theory of surgery" (Ranicki [2]), where we shall also prove a localization exact sequence of this type for symmetric L-theory.

Apart from the localization exact sequence itself we shall discuss the following applications:

- Let $\hat{A} = \varprojlim_{s \in S} A/sA$ be the S-adic completion of A. There are defined excision isomorphisms

$$L_n(A,S) \longrightarrow L_n(\hat{A},\hat{S}) \quad (n(\text{mod } 4))$$

and a Mayer-Vietoris exact sequence of the type

$$\ldots \longrightarrow L_n(A) \longrightarrow L_n(\hat{A}) \oplus L_n(S^{-1}A) \longrightarrow L_n(\hat{S}^{-1}\hat{A}) \longrightarrow L_{n-1}(A) \longrightarrow \ldots$$

(Proposition 3.2).

- If the ring A is an R-module then the symmetric Witt group $L^0(R)$ acts on the localization exact sequence of (A,S). This $L^0(R)$-module structure is used to prove that natural maps of the type

$$L_n(\mathbb{Z}[\pi]) \longrightarrow L_n(\mathbb{Q}[\pi]) \quad (n(\mathrm{mod}\ 4))$$

are isomorphisms modulo 8-torsion, and that the L-groups $L_n(\mathbb{Z}_m[\pi])$ are of exponent 8 (Propositions 4.2, 4.4).

- If the ring A is an algebra over a Dedekind ring R and $S = R - \{0\} \subset A$ there are defined natural direct sum decompositions

$$L_n(A,S,\varepsilon) = \bigoplus_{\mathcal{P}} L_n(A,\mathcal{P}^{\infty},\varepsilon) \quad (n(\mathrm{mod}\ 4))$$

with $\mathcal{P}$ ranging over the non-zero prime ideals of R such that $\bar{\mathcal{P}} = \mathcal{P}$. The L-groups $L_n(A,\mathcal{P}^{\infty},\varepsilon)$ are defined using quadratic structures on $\mathcal{P}$-primary S-torsion A-modules. (Proposition 5.1).

We shall consistently use the language of forms and formations of Ranicki [1]. We shall omit the proofs of results of the following nature:

i) some relation, invariably called "cobordism", involving forms and formations is claimed to be an equivalence relation such that the equivalence classes define an abelian group with respect to the direct sum $\oplus$.

ii) some function between such cobordism groups is claimed to be an isomorphism.

The chain complex formulation of quadratic L-theory in Ranicki [2] lends itself more readily to proofs of such results, those of type i) being obtained by an algebraic mimicry of the cobordism of manifolds, and those of type ii) by identifying cobordism groups of forms and formations with cobordism groups of quadratic Poincaré complexes. From the point of view of Ranicki [2] the L-groups $L_n(A)$ are defined for $n \geqslant 0$ to be the algebraic cobordism groups of pairs (C,ψ) such that C is an n-dimensional f.g. projective A-module chain complex and ψ is a quadratic structure inducing Poincaré duality $H^{n-*}(C) = H_*(C)$. The groups $L_n(A,S,\varepsilon)$ are defined for $n \geqslant 0$ to be the algebraic cobordism groups of pairs (D,θ) such that D is an (n+1)-dimensional f.g. projective A-module chain complex which becomes chain contractible over $S^{-1}A$ and θ is a quadratic structure inducing Poincaré duality $H^{n+1-*}(D) = H_*(D)$. It is relatively easy to prove the exact sequence

$$\ldots \longrightarrow L_n(A) \longrightarrow L_n(S^{-1}A) \longrightarrow L_n(A,S) \longrightarrow L_{n-1}(A) \longrightarrow L^S_{n-1}(S^{-1}A) \longrightarrow \ldots,$$

so that to obtain a localization exact sequence for the surgery obstruction groups it remains only to identify the chain complex L-groups with the 4-periodic L-groups defined using forms and formations. Although this identification can be used to both state and prove the localization exact sequence in terms of forms and formations we find the chain complex approach more illuminating, at least as far as proofs are concerned.

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§1. Quadratic L-theory

We recall some of the definitions and results of Ranicki [1],[2].

Let A be an associative ring with 1, and with an involution

$$\bar{\ } : A \longrightarrow A \; ; \; a \longmapsto \bar{a}$$

such that

$$(\overline{ab}) = \bar{b}.\bar{a} \quad , \quad (\overline{a+b}) = \bar{a} + \bar{b} \quad , \quad \bar{1} = 1 \quad , \quad \bar{\bar{a}} = a \in A \qquad (a,b \in A).$$

A-modules will always be taken to have a left A-action.

The <u>dual</u> of an A-module M is the A-module

$$M^* = \mathrm{Hom}_A(M,A) \ ,$$

with A acting by

$$A \times M^* \longrightarrow M^* \; ; \; (a,f) \longmapsto (x \longmapsto f(x)\bar{a}) \quad .$$

The dual of an A-module morphism $f \in \mathrm{Hom}_A(M,N)$ is the A-module morphism

$$f^* : N^* \longrightarrow M^* \; ; \; g \longmapsto (x \longmapsto g(f(x))) \ .$$

If M is a f.g. projective A-module then so is the dual M*, and there is defined a natural A-module isomorphism

$$M \longrightarrow M^{**} \; ; \; x \longmapsto (f \longmapsto \overline{f(x)})$$

which we shall use to identify $M^{**} = M$.

Let $\varepsilon \in A$ be a central unit such that

$$\bar{\varepsilon} = \varepsilon^{-1} \in A$$

(for example, $\varepsilon = \pm 1$). Given a f.g. projective A-module M define the <u>ε-duality</u> involution

$$T_\varepsilon : \mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*) \; ; \; \varphi \longmapsto (\varepsilon\varphi^* : x \longmapsto (y \longmapsto \varepsilon\overline{\varphi(y)(x)})) \ ,$$

let

$$Q^\varepsilon(M) = \ker(1-T_\varepsilon : \mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*))$$

$$Q_\varepsilon(M) = \mathrm{coker}(1-T_\varepsilon : \mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*)) \ ,$$

and define a morphism of abelian groups

$$1+T_\varepsilon : Q_\varepsilon(M) \longrightarrow Q^\varepsilon(M) \; ; \; \psi \longmapsto \psi + \varepsilon\psi^* \ .$$

An $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ *form over A* $\begin{cases}(M,\varphi)\\ (M,\psi)\end{cases}$ is a f.g. projective A-module M together with an element $\begin{cases}\varphi\in Q^{\varepsilon}(M)\\ \psi\in Q_{\varepsilon}(M)\end{cases}$. A *morphism* (resp. *isomorphism*) of such forms

$$\begin{cases} f\ :\ (M,\varphi)\longrightarrow(M',\varphi') \\ f\ :\ (M,\psi)\longrightarrow(M',\psi') \end{cases}$$

is an A-module morphism (resp. isomorphism) $f\in \mathrm{Hom}_A(M,M')$ such that

$$\begin{cases} f^*\varphi' f = \varphi\in Q^{\varepsilon}(M) \\ f^*\psi' f = \psi\in Q_{\varepsilon}(M)\ . \end{cases}$$

The form $\begin{cases}(M,\varphi)\\ (M,\psi)\end{cases}$ is *non-singular* if $\begin{cases}\varphi\in \mathrm{Hom}_A(M,M^*)\\ \psi+\varepsilon\psi^*\in \mathrm{Hom}_A(M,M^*)\end{cases}$ is an isomorphism.

A *sublagrangian* of a non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(M,\varphi)\\ (M,\psi)\end{cases}$ is a direct summand L of M such that the inclusion $j\in \mathrm{Hom}_A(L,M)$ defines a morphism of forms

$$\begin{cases} j\ :\ (L,0)\longrightarrow(M,\varphi) \\ j\ :\ (L,0)\longrightarrow(M,\psi)\ . \end{cases}$$

The *annihilator* of a sublagrangian L is the direct summand $L^{\perp}$ of M defined by

$$\begin{cases} L^{\perp} = \ker(j^*\varphi : M\longrightarrow L^*) \\ L^{\perp} = \ker(j^*(\psi+\varepsilon\psi^*) : M\longrightarrow L^*)\ . \end{cases}$$

A *lagrangian* is a sublagrangian L such that

$$L^{\perp} = L\ .$$

A non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A is *hyperbolic* if it admits a lagrangian, or equivalently if it is isomorphic to the standard hyperbolic form

$$\begin{cases} H^{\varepsilon}(P,\theta) = (P\oplus P^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix} \in Q^{\varepsilon}(P\oplus P^*)) \\ H_{\varepsilon}(P) = (P\oplus P^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_{\varepsilon}(P\oplus P^*)) \end{cases}$$

for some $\begin{cases} \varepsilon\text{-symmetric form over } A\ (P^*,\theta\in Q^{\varepsilon}(P^*)) \\ \text{f.g. projective } A\text{-module } P \end{cases}$.

The $\begin{cases} \underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}} \end{cases}$ Witt group of A $\begin{cases} L^0(A,\varepsilon) \\ L_0(A,\varepsilon) \end{cases}$ is the abelian group with respect to the direct sum $\oplus$ of the equivalence classes of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A $\begin{cases} (M,\varphi) \\ (M,\psi) \end{cases}$ under the equivalence relation

$\begin{cases} (M,\varphi) \sim (M',\varphi') \\ (M,\psi) \sim (M',\psi') \end{cases}$ if there exists an isomorphism of forms

$$\begin{cases} f : (M,\varphi)\oplus H^{\varepsilon}(P,\theta) \longrightarrow (M',\varphi')\oplus H^{\varepsilon}(P',\theta') \\ f : (M,\psi)\oplus H_{\varepsilon}(P) \longrightarrow (M',\psi')\oplus H_{\varepsilon}(P') \end{cases}$$

for some $\begin{cases} \varepsilon\text{-symmetric forms over } A\ (P^*,\theta),(P'^*,\theta') \\ \text{f.g. projective } A\text{-modules } P,P' \end{cases}$.

The $\underline{\varepsilon\text{-symmetrization}}$ map of Witt groups

$$1+T_{\varepsilon} : L_0(A,\varepsilon) \longrightarrow L^0(A,\varepsilon) \ ; \ (M,\psi) \longmapsto (M,(1+T_{\varepsilon})\psi)$$

is an isomorphism modulo 8-torsion.

From now on we shall restrict attention to just those aspects of symmetric L-theory which we shall use in our treatment of quadratic L-theory. We refer to Part I of Ranicki [2] for a more thorough development of symmetric L-theory.

An $\underline{\varepsilon\text{-quadratic formation over } A}$ $(M,\psi;F,G)$ is a non-singular ε-quadratic form over A (M,ψ) together with a lagrangian F and a sublagrangian G. An isomorphism of formations

$$f : (M,\psi;F,G) \longrightarrow (M',\psi';F',G')$$

is an isomorphism of forms $f:(M,\psi)\longrightarrow(M',\psi')$ such that

$$f(F) = F' \ , \ f(G) = G' \ .$$

A <u>stable isomorphism</u> of formations

$$[f] : (M,\psi;F,G) \longrightarrow (M',\psi';F',G')$$

is an isomorphism of formations

$$f : (M,\psi;F,G)\oplus(H_\varepsilon(P);P,P^*) \longrightarrow (M',\psi';F',G')\oplus(H_\varepsilon(P');P',P'^*)$$

for some f.g. projective A-modules P,P'.

An ε-quadratic formation $(M,\psi;F,G)$ is <u>non-singular</u> if G is a lagrangian of (M,ψ).

The <u>boundary</u> of an ε-quadratic $\left\{\begin{matrix}\text{form}\\ \text{formation}\end{matrix}\right.$ over A $\left\{\begin{matrix}(M,\psi)\\ (M,\psi;F,G)\end{matrix}\right.$ is the non-singular $\left\{\begin{matrix}(-\varepsilon)\text{-}\\ \varepsilon\text{-}\end{matrix}\right.$ quadratic $\left\{\begin{matrix}\text{formation}\\ \text{form}\end{matrix}\right.$ over A

$$\left\{\begin{array}{l}\partial(M,\psi) = (H_{-\varepsilon}(M);M,\{(x,(\psi+\varepsilon\psi^*)(x))\in M\oplus M^*\,|\,x\in M\})\\ \partial(M,\psi;F,G) = (G^\perp/G,\psi^\perp/\psi)\quad .\end{array}\right.$$

An ε-quadratic $\left\{\begin{matrix}\text{form}\\ \text{formation}\end{matrix}\right.$ is non-singular if and only if its boundary $\left\{\begin{matrix}\text{formation}\\ \text{form}\end{matrix}\right.$ is $\left\{\begin{matrix}\text{stably isomorphic to } 0\\ 0\end{matrix}\right.$.

Non-singular ε-quadratic $\left\{\begin{matrix}\text{forms}\\ \text{formations}\end{matrix}\right.$ over A $\left\{\begin{matrix}(M,\psi),(M',\psi')\\ (M,\psi;F,G),(M',\psi';F',G')\end{matrix}\right.$ are <u>cobordant</u> if there exists $\left\{\begin{matrix}\text{an isomorphism}\\ \text{a stable isomorphism}\end{matrix}\right.$ of $\left\{\begin{matrix}\text{forms}\\ \text{formations}\end{matrix}\right.$

$$\left\{\begin{array}{l}f : (M,\psi)\oplus(M',-\psi') \longrightarrow \partial(N,\varphi;H,K)\\ [f] : (M,\psi;F,G)\oplus(M',-\psi';F',G') \longrightarrow \partial(N,\varphi)\end{array}\right.$$

for some $\left\{\begin{matrix}\varepsilon\text{-}\\ (-\varepsilon)\text{-}\end{matrix}\right.$ quadratic $\left\{\begin{matrix}\text{formation}\\ \text{form}\end{matrix}\right.$ over A $\left\{\begin{matrix}(N,\varphi;H,K)\\ (N,\varphi)\end{matrix}\right.$.

<u>Proposition 1.1</u> Cobordism is an equivalence relation on the set of non-singular ε-quadratic $\left\{\begin{matrix}\text{forms}\\ \text{formations}\end{matrix}\right.$ over A, such that the equivalence classes define an abelian group $\left\{\begin{matrix}L_0(A,\varepsilon)\\ L_1(A,\varepsilon)\end{matrix}\right.$ with respect to the direct sum $\oplus$.

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The cobordism group of forms $L_0(A,\varepsilon)$ is just the Witt group of ε-quadratic forms over A, as defined previously.

Define abelian groups $L_n(A,\varepsilon)$ for n(mod 4) by

$$L_n(A,\varepsilon) = \begin{cases} L_0(A,(-)^i\varepsilon) \\ L_1(A,(-)^i\varepsilon) \end{cases} \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases} .$$

For $\varepsilon = 1 \in A$ we shall write

$$L_n(A,1) = L_n(A) \quad , \quad L^0(A,1) = L^0(A) .$$

In the terminology of Part I of Ranicki [1]

$$L_n(A) = U_n(A) .$$

Given a subgroup $X \subseteq \widetilde{K}_0(A)$ (resp. $X \subseteq \widetilde{K}_1(A)$) which is preserved as a set by the duality involution

$$* : \widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(A) \; ; \; [P] \longmapsto [P^*]$$

$$(\text{resp. } * : \widetilde{K}_1(A) \longrightarrow \widetilde{K}_1(A) \; ; \; \tau(f:P \longrightarrow Q) \longmapsto \tau(f^*:Q^* \longrightarrow P^*))$$

let $L_n^X(A,\varepsilon)$ (n(mod 4)) be the L-groups defined as in Proposition 1.1, but using only forms and formations involving f.g. projective A-modules P such that $[P] \in X \subseteq \widetilde{K}_0(A)$ (resp. based f.g. free A-modules such that all isomorphisms $f \in \mathrm{Hom}_A(P,Q)$ have torsion $\tau(f) \in X \subseteq \widetilde{K}_1(A)$). In particular, for $X = \widetilde{K}_0(A)$

$$L_n^{\widetilde{K}_0(A)}(A,\varepsilon) = L_n(A,\varepsilon) .$$

For $\varepsilon = 1 \in A$ we shall write

$$L_n^X(A,1) = L_n^X(A) .$$

In the terminology of Part III of Ranicki [1]

$$L_n^X(A) = U_n^X(A) \text{ for } X \subseteq \widetilde{K}_0(A) \quad (\text{resp. } L_n^X(A) = V_n^X(A) \text{ for } X \subseteq \widetilde{K}_1(A)) .$$

<u>Proposition 1.2</u> Given $*$-invariant subgroups $X \subseteq Y \subseteq \widetilde{K}_m(A)$ (m = 0 or 1) there is defined an exact sequence of abelian groups

$$\ldots \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;Y/X) \longrightarrow L_n^X(A,\varepsilon) \longrightarrow L_n^Y(A,\varepsilon) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Y/X) \longrightarrow L_{n-1}^X(A,\varepsilon) \rightarrow \ldots$$

with the Tate $\mathbb{Z}_2$-cohomology groups defined by

$$\hat{H}^n(\mathbb{Z}_2;Y/X) = \{g \in Y/X \mid g^* = (-)^n g\} / \{h + (-)^n h^* \mid h \in Y/X\} .$$

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(In dealing with based A-modules it is convenient to assume that A is such that the rank of a f.g. free A-module is well-defined and $\tau(\varepsilon:A \rightarrow A) = 0 \in \widetilde{K}_1(A)$).

In order to define even-dimensional relative L-groups we shall need the following refinement of the notion of formation.

A <u>split ε-quadratic formation over A</u> $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is an ε-quadratic formation over A $(H_\varepsilon(F);F,G)$, where $\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G\longrightarrow F\oplus F^*$ is the inclusion, together with a <u>hessian</u> (-ε)-quadratic form over A $(G,\theta\in Q_{-\varepsilon}(G))$ such that

$$\gamma^*\mu = \theta - \varepsilon\theta^* : G \longrightarrow G^* .$$

Such a split formation will normally be written as (F,G).

An <u>isomorphism</u> of split ε-quadratic formations

$$(\alpha,\beta,\psi) : (F,G)\longrightarrow(F',G')$$

is defined by A-module isomorphisms $\alpha\in \mathrm{Hom}_A(F,F')$, $\beta\in \mathrm{Hom}_A(G,G')$ together with a (-ε)-quadratic form $(F^*,\psi\in Q_{-\varepsilon}(F^*))$ such that

i) $\alpha\gamma + (\psi-\varepsilon\psi^*)^*\mu = \gamma'\beta : G \longrightarrow F'$

ii) $\alpha^{*-1}\mu = \mu'\beta : G \longrightarrow F'^*$

iii) $\theta + \mu^*\psi\mu = \beta^*\theta'\beta \in Q_{-\varepsilon}(G)$.

A <u>stable isomorphism</u> of split ε-quadratic formations

$$[\alpha,\beta,\psi] : (F,G)\longrightarrow(F',G')$$

is an isomorphism of the type

$$(\alpha,\beta,\psi) : (F,G)\oplus(P,P^*)\longrightarrow(F',G')\oplus(P',P'^*) ,$$

for some f.g. projective A-modules P,P' with $(P,P^*) = (P,(\begin{pmatrix}0\\ 1\end{pmatrix},0)P^*)$.

An isomorphism of split ε-quadratic formations $(\alpha,\beta,\psi):(F,G)\longrightarrow(F',G')$ determines an isomorphism of the underlying ε-quadratic formations

$$\begin{pmatrix}\alpha & \alpha(\psi-\varepsilon\psi^*)^* \\ 0 & \alpha^{*-1}\end{pmatrix} : (H_\varepsilon(F);F,G)\longrightarrow(H_\varepsilon(F');F',G') .$$

Conversely, every isomorphism of ε-quadratic formations

$$f : (H_\varepsilon(F);F,G)\longrightarrow(H_\varepsilon(F');F',G')$$

can be refined to an isomorphism of split ε-quadratic formations $(\alpha,\beta,\psi):(F,G)\longrightarrow(F',G')$. Similarly for stable isomorphisms.

The <u>split boundary</u> of an ε-quadratic form over A $(M,\psi\in Q_\varepsilon(M))$ is the non-singular split (-ε)-quadratic formation over A

$$\partial(M,\psi) = (M,(\begin{pmatrix}1\\ \psi+\varepsilon\psi^*\end{pmatrix},\psi)M) .$$

A morphism of rings with involution is a function

$$f : A \longrightarrow B$$

such that

$$f(a_1+a_2) = f(a_1)+f(a_2) \ , \ f(a_1a_2) = f(a_1)f(a_2) \ , \ f(\bar{a}) = \overline{f(a)} \ , \ f(1) = 1 \in B$$
$$(a_1, a_2, a \in A) \ .$$

Given such a morphism regard B as a (B,A)-bimodule by

$$B \times B \times A \longrightarrow B \ ; \ (b,x,a) \longmapsto b.x.f(a) \ .$$

A f.g. projective A-module M induces a f.g. projective B-module $B\otimes_A M$, and there is defined a natural B-module isomorphism

$$B\otimes_A M^* \longrightarrow (B\otimes_A M)^* \ ; \ b\otimes f \longmapsto (c\otimes x \longmapsto c.f(x).\bar{b})$$

which we shall use to identify $(B\otimes_A M)^* = B\otimes_A M^*$. Given a central unit $\varepsilon \in A$ such that $\bar{\varepsilon} = \varepsilon^{-1}$ (as above) we have that $\overline{f(\varepsilon)} = f(\varepsilon)^{-1} \in B$, and it will be assumed that $f(\varepsilon)$ is central in B. It is convenient to also denote $f(\varepsilon) \in B$ by ε. An ε-quadratic $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ over A $\begin{Bmatrix} (M,\psi) \\ (M,\psi;F,G) \end{Bmatrix}$ induces an ε-quadratic $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ over B

$$\begin{cases} B\otimes_A (M,\psi) = (B\otimes_A M, 1\otimes\psi) \\ B\otimes_A (M,\psi;F,G) = (B\otimes_A M, 1\otimes\psi; B\otimes_A F, B\otimes_A G) \ , \end{cases}$$

and there are induced morphisms in the L-groups

$$f : L_n(A,\varepsilon) \longrightarrow L_n(B,\varepsilon) \ ; \ x \longmapsto B\otimes_A x \qquad (n(\text{mod } 4)) \ .$$

We shall now define relative L-groups $L_n(f,\varepsilon)$ $(n(\text{mod } 4))$ to fit into an exact sequence

$$\ldots \longrightarrow L_n(A,\varepsilon) \xrightarrow{f} L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots \ .$$

A <u>relative ε-quadratic form over $f:A \longrightarrow B$</u> $((F,G),(M,\psi),h)$ is a triple consisting of a non-singular split $(-\varepsilon)$-quadratic formation over A (F,G), an ε-quadratic form over B (M,ψ), and a stable isomorphism of non-singular split $(-\varepsilon)$-quadratic formations over B

$$h : B\otimes_A (F,G) \longrightarrow \partial(M,\psi) \ .$$

The relative forms $((F,G),(M,\Psi),h),((F',G'),(M',\Psi'),h')$ are <u>cobordant</u> if there exist a $(-\varepsilon)$-quadratic form over A (L,φ) and a stable isomorphism of non-singular split $(-\varepsilon)$-quadratic formations over A

$$k : \partial(L,\varphi) \longrightarrow (F',G') \oplus -(F,G) \quad (\text{where } -(F,G) = (F,(\begin{pmatrix}-\gamma\\ \mu\end{pmatrix},-\theta)G))$$

such that the non-singular ε-quadratic form over B obtained by glueing

$$(N,\nu) = B\otimes_A(L,\varphi) \cup_{(h'\oplus h)(1\otimes k)} ((M',-\Psi')\oplus(M,\Psi))$$

is null-cobordant, that is

$$(N,\nu) = 0 \in L_0(B,\varepsilon) \ .$$

The glueing operation was introduced in the proof of Theorem 4.3 of Part I of Ranicki [1], and it has also been described in Wall [6],[7]. We shall not repeat its definition here.

A <u>relative ε-quadratic formation over $f:A\longrightarrow B$</u> $((P,\theta),Q,h)$ is a triple consisting of a non-singular ε-quadratic form over A (P,θ), a f.g. projective B-module Q, and an isomorphism of non-singular ε-quadratic forms over B

$$h : B\otimes_A(P,\theta) \longrightarrow H_\varepsilon(Q) \ .$$

The relative ε-quadratic formations $((P,\theta),Q,h),((P',\theta'),Q',h')$ are <u>cobordant</u> if there exist an ε-quadratic formation over A $(M,\Psi;F,G)$ and an isomorphism of non-singular ε-quadratic forms over B

$$k : \partial(M,\Psi;F,G) \longrightarrow (P',\theta')\oplus(P,-\theta)$$

such that the non-singular ε-quadratic formation over B

$$\begin{aligned}(N,\nu;H,K) = (&B\otimes_A(M,-\Psi)\oplus H_\varepsilon(Q);(B\otimes_A F)\oplus Q,\\ &\{(x+y,(h'\oplus h)(1\otimes k)(y))\in B\otimes_A M\oplus(Q\oplus Q^*)\,|\,x\in B\otimes_A G, y\in B\otimes_A(G^\perp/G)\})\end{aligned}$$

is null-cobordant, that is

$$(N,\nu;H,K) = 0 \in L_1(B,\varepsilon) \ .$$

Proposition 1.3 Cobordism is an equivalence relation on the set of relative ε-quadratic $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over $f:A \longrightarrow B$, such that the equivalence classes define an abelian group $\begin{cases}L_0(f,\varepsilon)\\ L_1(f,\varepsilon)\end{cases}$ with respect to the direct sum $\oplus$.

The L-groups defined for n(mod 4) by

$$L_n(f,\varepsilon) = \begin{cases} L_0(f,(-)^i\varepsilon) \\ L_1(f,(-)^i\varepsilon)\end{cases} \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

fit into an exact sequence of abelian groups

$$\dots \longrightarrow L_n(A,\varepsilon) \xrightarrow{f} L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \dots ,$$

with

$$L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \ ; \ x \longmapsto (0,x,0)$$

$$L_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \ ; \ (y,x,g) \longmapsto y \quad .$$

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In the case $\varepsilon = 1$ we shall write

$$L_n(f,1) = L_n(f) \ .$$

Relative L-groups $L_n(f)$ were first defined by Wall [3] (for n odd) and Sharpe [1] (n even), in the case when all the modules involved are f.g. free.

The above definition of the relative ε-quadratic L-groups $L_n(f,\varepsilon)$ generalizes immediately to the intermediate ε-quadratic L-groups. Given $*$-invariant subgroups $X \subseteq \widetilde{K}_m(A)$, $Y \subseteq \widetilde{K}_m(B)$ ($m = 0$ or 1) such that $B \otimes_A X \subseteq Y$ there are defined L-groups $L_n^{X,Y}(f,\varepsilon)$ (n(mod 4)) which fit into an exact sequence of abelian groups

$$\dots \longrightarrow L_n^X(A,\varepsilon) \xrightarrow{f} L_n^Y(B,\varepsilon) \longrightarrow L_n^{X,Y}(f,\varepsilon) \longrightarrow L_{n-1}^X(A,\varepsilon) \longrightarrow \dots .$$

§2. Localization

In setting up the localization exact sequence for quadratic L-theory we follow the pattern established for the localization exact sequence of algebraic K-theory

$$K_1(A) \longrightarrow K_1(S^{-1}A) \longrightarrow K_1(A,S) \longrightarrow K_0(A) \longrightarrow K_0(S^{-1}A)$$

in Chapter IX of Bass [1]. (The extension of the sequence to the lower K-groups K_i $(i \leqslant -1)$ of Bass and the higher K-groups K_i $(i \geqslant 2)$ of Quillen need not concern us here). There are three stages :

I) For any ring morphism $f:A \longrightarrow B$ there is defined a relative K-group $K_1(f)$ to fit into an exact sequence

$$K_1(A) \xrightarrow{f} K_1(B) \longrightarrow K_1(f) \longrightarrow K_0(A) \xrightarrow{f} K_0(B) \quad .$$

Specifically, $K_1(f)$ is a Grothendieck group of triples (P,Q,g) consisting of f.g. projective A-modules P,Q and a B-module isomorphism $g:B\otimes_A P \longrightarrow B\otimes_A Q$.

II) For a localization map $f:A \longrightarrow S^{-1}A$ it is possible to express g as $\frac{h}{s}$ for some $h \in \mathrm{Hom}_A(P,Q)$, $s \in S$ such that h induces an isomorphism over $S^{-1}A$. Thus $K_1(A \longrightarrow S^{-1}A)$ can be expressed as a Grothendieck group of triples such as (P,Q,h).

III) Define $K_1(A,S) = K_0$(exact category of h.d. 1 S-torsion A-modules) and observe that there is a natural isomorphism of abelian groups

$$K_1(A \longrightarrow S^{-1}A) \longrightarrow K_1(A,S) \ ; \ (P,Q,h) \longmapsto [\mathrm{coker}(h:P \longrightarrow Q)] \quad .$$

We have already developed the L-theoretic analogue of I) in §1 above.

(As in the algebraic K-theory of Bass [1] we shall only consider localizations $A \longrightarrow S^{-1}A$ inverting subsets $S \subset A$ of central elements. There is some interest in the L-theory of eccentric localizations, inverting non-central elements. The work of Smith [1] considers localizations of the type $A \longrightarrow S^{-1}A$ with $S = f^{-1}(1) \subset A$ for some ring morphism $f:A \longrightarrow B$ such that a morphism $g \in \mathrm{Hom}_A(P,Q)$ of f.g. projective A-modules P,Q becomes an isomorphism $1 \otimes g \in \mathrm{Hom}_B(B\otimes_A P, B\otimes_A Q)$ if and only if $\ker(g) = 0$ and $\mathrm{coker}(g)$ is an S-torsion A-module. In principle, our methods permit a generalization to quadratic L-theory of any K-theoretic eccentric localization sequence).

Let A be a ring with involution (as in §1).

A <u>multiplicative subset</u> $S \subset A$ is a subset of A such that

i) $st \in S$ for all $s,t \in S$

ii) $\bar{s} \in S$ for all $s \in S$

iii) if $sa = 0$ for some $a \in A, s \in S$ then $a = 0$

iv) $as = sa \in A$ for all $a \in A, s \in S$

v) $1 \in S$.

The <u>localization of A away from S</u> $S^{-1}A$ is the ring with involution defined by the equivalence classes of pairs $(a,s) \in A \times S$ under the relation

$$(a,s) \sim (a',s') \text{ if } s'a = sa' \in A \ ,$$

with addition, multiplication and involution by

$$(a,s) + (b,t) = (at+bs,st) \ , \ (a,s)(b,t) = (ab,st) \ , \ \overline{(a,s)} = (\bar{a},\bar{s}) \ .$$

As usual, the class of (a,s) is denoted by $\frac{a}{s} \in S^{-1}A$. The inclusion

$$A \longrightarrow S^{-1}A \ ; \ a \longmapsto \frac{a}{1}$$

is a morphism of rings with involution. An A-module M induces an $S^{-1}A$-module

$$S^{-1}M = S^{-1}A \otimes_A M$$

which can be identified with the $S^{-1}A$-module of equivalence classes of pairs $(x,s) \in M \times S$ under the relation

$$(x,s) \sim (x',s') \text{ if } s'x = sx' \in M \ .$$

Again, the class of (x,s) is denoted by $\frac{x}{s} \in S^{-1}M$. Given A-modules M,N regard $\mathrm{Hom}_A(M,N)$ as an A-module by

$$A \times \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N) \ ; \ (a,f) \longmapsto (x \longmapsto f(x)\bar{a}) \ ,$$

and use the natural $S^{-1}A$-module isomorphism

$$S^{-1}\mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N) \ ; \ \frac{f}{s} \longmapsto \left(\frac{x}{t} \longmapsto \frac{f(x)}{t\bar{s}}\right)$$

as an identification. In particular, for N = A we have the identification

$$(S^{-1}M)^* = S^{-1}(M^*) \ .$$

For example, if $A = \mathbb{Z}$, $S = \mathbb{Z} - \{0\}$ then $S^{-1}A = \mathbb{Q}$.

Let $L_n^S(S^{-1}A,\varepsilon)$ $(n(\bmod 4))$ be the intermediate ε-quadratic L-groups of $S^{-1}A$ associated to the $*$-invariant subgroup $S=\mathrm{im}(\tilde{K}_0(A)\to\tilde{K}_0(S^{-1}A))\subseteq\tilde{K}_0(S^{-1}A)$ of the projective classes of f.g. projective $S^{-1}A$-modules induced from f.g. projective A-modules. Let $L_n^S(A\longrightarrow S^{-1}A,\varepsilon)$ $(n(\bmod 4))$ be the relative L-groups appearing in the exact sequence

$$\ldots\longrightarrow L_n(A,\varepsilon)\longrightarrow L_n^S(S^{-1}A,\varepsilon)\longrightarrow L_n^S(A\longrightarrow S^{-1}A,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\to\ldots\ .$$

In the first instance we shall express $L_n^S(A\longrightarrow S^{-1}A,\varepsilon)$ for $n=\begin{cases}2i\\2i+1\end{cases}$ as the cobordism group of non-singular $\begin{cases}\text{split } (-)^{i-1}\varepsilon\text{-quadratic formations}\\(-)^i\varepsilon\text{-quadratic forms}\end{cases}$ over A which become $\begin{cases}\text{stably isomorphic to } 0\\\text{hyperbolic}\end{cases}$ over $S^{-1}A$, corresponding to stage II) of the above programme. We shall then use this expression to identify

$$L_n^S(A\longrightarrow S^{-1}A,\varepsilon)=L_n(A,S,\varepsilon)\qquad(n(\bmod 4))$$

with $\begin{cases}L_{2i}(A,S,\varepsilon)\\L_{2i+1}(A,S,\varepsilon)\end{cases}$ $(i(\bmod 2))$ the Witt group of non-singular $(-)^i\varepsilon$-quadratic linking $\begin{cases}\text{forms}\\\text{formations}\end{cases}$ defined using h.d. 1 S-torsion A-modules, corresponding to stage III).

An A-module morphism $f\in\mathrm{Hom}_A(M,N)$ is an <u>S-isomorphism</u> if the induced $S^{-1}A$-module morphism

$$S^{-1}f : S^{-1}M\longrightarrow S^{-1}N\ ;\ \frac{x}{s}\longmapsto\frac{f(x)}{s}$$

is an isomorphism.

An <u>S-isomorphism</u> of ε-quadratic forms over A

$$f : (M,\psi)\longrightarrow(N,\varphi)$$

is a morphism of ε-quadratic forms such that $f\in\mathrm{Hom}_A(M,N)$ is an S-isomorphism. There is induced an isomorphism of ε-quadratic forms over $S^{-1}A$

$$S^{-1}f:S^{-1}(M,\psi)\longrightarrow S^{-1}(N,\varphi)\quad .$$

An ε-quadratic form over A (M,ψ) is non-degenerate if $\psi+\varepsilon\psi^* \in \mathrm{Hom}_A(M,M^*)$ is an S-isomorphism.

An S-lagrangian of a non-degenerate ε-quadratic form over A (M,ψ) is a f.g. projective submodule L of M such that the inclusion $j \in \mathrm{Hom}_A(L,M)$ defines a morphism of forms over A

$$j : (L,0) \longrightarrow (M,\psi)$$

which becomes the inclusion of a lagrangian over $S^{-1}A$. The inclusion j extends to an S-isomorphism of non-degenerate ε-quadratic forms over A

$$(j \;\; k) : (L\oplus L^*, \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}) \longrightarrow (M,\psi)$$

for some $k \in \mathrm{Hom}_A(L^*,M)$, $s \in S$.

A non-degenerate ε-quadratic formation over A $(M,\psi;F,G)$ is a non-singular ε-quadratic form over A (M,ψ) together with a lagrangian F and an S-lagrangian G.

A non-degenerate ε-quadratic $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ over A $\begin{Bmatrix} (M,\psi) \\ (M,\psi;F,G) \end{Bmatrix}$ induces a non-singular ε-quadratic $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ over $S^{-1}A$ $\begin{Bmatrix} S^{-1}(M,\psi) \\ S^{-1}(M,\psi;F,G) \end{Bmatrix}$, representing an element of $\begin{Bmatrix} L_0^S(S^{-1}A,\varepsilon) \\ L_1^S(S^{-1}A,\varepsilon) \end{Bmatrix}$. Conversely, every element of $\begin{Bmatrix} L_0^S(S^{-1}A,\varepsilon) \\ L_1^S(S^{-1}A,\varepsilon) \end{Bmatrix}$ is represented by a $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ of this type.

(We could achieve a more systematic terminology by calling non-degenerate objects over A 'S-non-singular'. We prefer to bow to the tradition of calling forms over $\mathbb{Z}$ which become non-singular over $\mathbb{Q}$ 'non-degenerate').

An <u>ε-quadratic S-form over A</u> $(M,\psi;L)$ is a non-degenerate ε-quadratic form over A (M,ψ) together with an S-lagrangian L. The S-form is <u>non-singular</u> if the form (M,ψ) is non-singular, in which case there is defined an <u>associated</u> relative ε-quadratic formation over $A \longrightarrow S^{-1}A$

$$((M,\psi),S^{-1}L,(j\ \tfrac{k}{s})^{-1} : S^{-1}(M,\psi) \longrightarrow H_\varepsilon(S^{-1}L))$$

with $j \in \mathrm{Hom}_A(L,M)$, $k \in \mathrm{Hom}_A(L^*,M)$, $s \in S$ as above.

An <u>isomorphism</u> of ε-quadratic S-forms over A

$$f : (M,\psi;L) \longrightarrow (M',\psi';L')$$

is an isomorphism of forms

$$f : (M,\psi) \longrightarrow (M',\psi')$$

such that

$$f(L) = L' .$$

A <u>stable isomorphism</u> of ε-quadratic S-forms over A

$$[f] : (M,\psi;L) \longrightarrow (M',\psi';L')$$

is an isomorphism of the type

$$f : (M,\psi;L)\oplus(H_\varepsilon(P);P) \longrightarrow (M',\psi';L')\oplus(H_\varepsilon(P');P')$$

for some f.g. projective A-modules P,P'.

An <u>ε-quadratic S-formation over A</u> $(M,\psi;F,G)$ is a non-degenerate ε-quadratic formation over A such that the A-module morphism

$$G \longrightarrow M/F \ ;\ x \longmapsto [x]$$

is an S-isomorphism. The S-formation is <u>non-singular</u> if G is a lagrangian of (M,ψ).

An <u>isomorphism</u> of ε-quadratic S-formations over A

$$f : (M,\psi;F,G) \longrightarrow (M',\psi';F',G')$$

is an isomorphism of ε-quadratic forms over A

$$f : (M,\psi) \longrightarrow (M',\psi')$$

such that

$$f(F) = F' ,\ f(G) = G' .$$

A <u>stable isomorphism</u> of ε-quadratic S-formations over A

$$[f] : (M,\psi;F,G) \longrightarrow (M',\psi';F',G')$$

is an isomorphism of the type

$$f : (M,\psi;F,G)\oplus(H_\varepsilon(P);P,P^*) \longrightarrow (M',\psi';F',G')\oplus(H_\varepsilon(P');P',P'^*)$$

for some f.g. projective A-modules P,P'.

A <u>split ε-quadratic S-formation over A</u> $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is an ε-quadratic S-formation over A $(H_\varepsilon(F);F,G)$, where $\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G\longrightarrow F\oplus F^*$ is the inclusion, together with a <u>hessian</u> $(-\varepsilon)$-quadratic form over A $(G,\theta\in Q_{-\varepsilon}(G))$ such that

$$\gamma^*\mu = \theta - \varepsilon\theta^* : G \longrightarrow G^* .$$

Such a split S-formation will normally be written as (F,G), denoting $(F,(\begin{pmatrix}-\gamma\\ \mu\end{pmatrix},-\theta)G)$ by $-(F,G)$. Note that $\mu\in \mathrm{Hom}_A(G,F^*)$ is an S-isomorphism.

A split ε-quadratic S-formation (F,G) is <u>non-singular</u> if G is a lagrangian of $H_\varepsilon(F)$, that is if the sequence of A-modules

$$0 \longrightarrow G \xrightarrow{\begin{pmatrix}\gamma\\ \mu\end{pmatrix}} F\oplus F^* \xrightarrow{(\varepsilon\mu^*\ \gamma^*)} G^* \longrightarrow 0$$

is exact. For non-singular (F,G) define the <u>associated</u> relative $(-\varepsilon)$-quadratic form over $A\longrightarrow S^{-1}A$ $((F,G),0,0)$.

An <u>isomorphism</u> of split ε-quadratic S-formations over A

$$(\alpha,\beta,\psi) : (F,G) \longrightarrow (F',G')$$

is defined by A-module isomorphisms $\alpha\in\mathrm{Hom}_A(F,F')$, $\beta\in\mathrm{Hom}_A(G,G')$ together with a $(-\varepsilon)$-quadratic form $(F^*,\psi\in Q_{-\varepsilon}(F^*))$ such that

i) $\alpha\gamma + (\psi - \varepsilon\psi^*)^*\mu = \gamma'\beta : G \longrightarrow F'$

ii) $\alpha^{*-1}\mu = \mu'\beta : G \longrightarrow F'^*$

iii) $\theta + \mu^*\psi\mu - \beta^*\theta'\beta \in \ker(S^{-1}:Q_{-\varepsilon}(G)\longrightarrow Q_{-\varepsilon}(S^{-1}G))$.

A <u>stable isomorphism</u> of split ε-quadratic S-formations over A

$$[\alpha,\beta,\psi] : (F,G) \longrightarrow (F',G')$$

is an isomorphism of the type

$$(\alpha,\beta,\psi) : (F,G)\oplus(P,P^*) \longrightarrow (F',G')\oplus(P',P'^*)$$

for some f.g. projective A-modules P,P'.

The boundary of a non-degenerate ε-quadratic $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over A $\begin{cases}(M,\psi)\\ (M,\psi;F,G)\end{cases}$ is the non-singular $\begin{cases}\text{split } (-\varepsilon)\text{-quadratic } S\text{-formation}\\ \varepsilon\text{-quadratic } S\text{-form}\end{cases}$ over A

$$\begin{cases}\partial(M,\psi) = (M,(\begin{pmatrix}1\\ \psi+\varepsilon\psi^*\end{pmatrix},\psi)M)\\ \partial(M,\psi;F,G) = (M,\psi;G)\end{cases} .$$

Non-singular $\begin{cases}\varepsilon\text{-quadratic } S\text{-forms}\\ \text{split } \varepsilon\text{-quadratic } S\text{-formations}\end{cases}$ over A $\begin{cases}(M,\psi;L),(M',\psi';L')\\ (F,G),(F',G')\end{cases}$ are cobordant if there exists a stable isomorphism

$$\begin{cases}[f] : (M,\psi;L)\oplus(M',-\psi';L')\longrightarrow\partial(N,\varphi;H,K)\\ [\alpha,\beta;\psi] : (F,G)\oplus-(F',G')\longrightarrow\partial(N,\varphi)\end{cases}$$

for some non-degenerate $\begin{cases}\varepsilon\text{-}\\ (-\varepsilon)\text{-}\end{cases}$ quadratic $\begin{cases}\text{formation}\\ \text{form}\end{cases}$ over A $\begin{cases}(N,\varphi;H,K)\\ (N,\varphi)\end{cases}$ such that

$$\begin{cases}S^{-1}(N,\varphi;H,K) = 0\in L_1^S(S^{-1}A,\varepsilon)\\ S^{-1}(N,\varphi) = 0\in L_0^S(S^{-1}A,-\varepsilon)\end{cases} .$$

Proposition 2.1 Cobordism is an equivalence relation on the set of non-singular $\begin{cases}\varepsilon\text{-quadratic } S\text{-forms}\\ \text{split } \varepsilon\text{-quadratic } S\text{-formations}\end{cases}$ over A, such that the equivalence classes define an abelian group with respect to the direct sum ⊕.

The cobordism group of non-singular $\begin{cases}(-)^i\varepsilon\text{-quadratic } S\text{-forms}\\ \text{split } (-)^i\varepsilon\text{-quadratic } S\text{-formations}\end{cases}$ over A is naturally isomorphic (via the associated relative $\begin{cases}\text{formation}\\ \text{form}\end{cases}$ construction) to the relative L-group $L_n^S(A\longrightarrow S^{-1}A,\varepsilon)$ for $n = \begin{cases}2i+1\\ 2i+2\end{cases}$.

The morphisms of the exact sequence

$$\cdots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \longrightarrow L_n^S(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \cdots$$

involving $L_n^S(A \longrightarrow S^{-1}A,\varepsilon)$ are given in terms of S-forms and S-formations by

$$\begin{cases} L_{2i}^S(S^{-1}A,\varepsilon) \longrightarrow L_{2i}^S(A \longrightarrow S^{-1}A,\varepsilon) \ ; \ S^{-1}(M,\psi) \longmapsto \partial(M,\psi) \\ L_{2i+1}^S(S^{-1}A,\varepsilon) \longrightarrow L_{2i+1}^S(A \longrightarrow S^{-1}A,\varepsilon) \ ; \ S^{-1}(M,\psi;F,G) \longmapsto \partial(M,\psi;F,G) \end{cases}$$

$$\begin{cases} L_{2i}^S(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L_{2i-1}(A,\varepsilon) \ ; \ (F,G) \longmapsto (H_{(-)^{i-1}\varepsilon}(F);F,G) \\ L_{2i+1}^S(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L_{2i}(A,\varepsilon) \ ; \ (M,\psi;L) \longmapsto (M,\psi) \ . \end{cases}$$

[]

An A-module M is <u>S-torsion</u> if

$$S^{-1}M = 0 \ ,$$

or equivalently if for every $x \in M$ there exists $s \in S$ such that $sx = 0 \in M$.

An A-module M is <u>h.d. 1</u> (= homological dimension 1) if it admits a f.g. projective A-module resolution of length 1

$$0 \longrightarrow P_1 \xrightarrow{d} P_0 \longrightarrow M \longrightarrow 0 \ .$$

An h.d. 1 S-torsion A-module is thus an A-module which admits a f.g. projective A-module resolution of length 1 with $d \in \mathrm{Hom}_A(P_1,P_0)$ an S-isomorphism.

Regard the abelian group $S^{-1}A/A$ as an A-module by

$$A \times S^{-1}A/A \longrightarrow S^{-1}A/A \ ; \ (a,\tfrac{b}{s}) \longmapsto \frac{ab}{s} \ .$$

The <u>S-dual</u> of an A-module M is the A-module

$$M^\wedge = \mathrm{Hom}_A(M,S^{-1}A/A)$$

with A acting by

$$A \times M^\wedge \longrightarrow M^\wedge \ ; \ (a,f) \longmapsto (x \longmapsto f(x)\bar{a}) \ .$$

The S-dual of an A-module morphism $f \in \mathrm{Hom}_A(M,N)$ is the A-module morphism

$$f^\wedge : N^\wedge \longrightarrow M^\wedge \ ; \ g \longmapsto (x \longmapsto g(f(x))) \ .$$

The S-dual of an h.d. 1 S-torsion A-module $M = \mathrm{coker}(d:P_1 \longrightarrow P_0)$ is an h.d. 1 S-torsion A-module $M^\wedge$, with resolution

$$0 \longrightarrow P_0^* \xrightarrow{d^*} P_1^* \longrightarrow M^\wedge \longrightarrow 0$$

where

$$P_1^* \longrightarrow M^\wedge \ ; \ f \longmapsto ([x] \longmapsto \frac{f(y)}{s}) \quad (x \in P_0,\ y \in P_1,\ s \in S\ ,\ sx = dy \in P_0) \ .$$

The natural A-module morphism

$$M \longrightarrow M^{\wedge\wedge} \; ; \; x \longmapsto (f \longmapsto \overline{f(x)})$$

is an isomorphism if M is an h.d. 1 S-torsion A-module, in which case we shall use it as an identification, and to define the <u>ε-duality</u> involution

$$T_\varepsilon : \mathrm{Hom}_A(M,M^\wedge) \longrightarrow \mathrm{Hom}_A(M,M^\wedge) \; ; \; \varphi \longmapsto (\varepsilon\varphi^\wedge : x \longmapsto (y \longmapsto \varepsilon\overline{\varphi(y)(x)})) \; .$$

An <u>ε-symmetric linking form over (A,S)</u> (M,λ) is an h.d. 1 S-torsion A-module M together with an element $\lambda \in \ker(1-T_\varepsilon : \mathrm{Hom}_A(M,M^\wedge) \longrightarrow \mathrm{Hom}_A(M,M^\wedge))$. Equivalently, λ is given by a pairing

$$\lambda : M \times M \longrightarrow S^{-1}A/A \; ; \; (x,y) \longmapsto \lambda(x)(y)$$

satisfying

$$\begin{aligned} &\text{i) } \lambda(x,ay) = a\lambda(x,y) \in S^{-1}A/A \\ &\text{ii) } \lambda(x,y+y') = \lambda(x,y) + \lambda(x,y') \in S^{-1}A/A \\ &\text{iii) } \lambda(y,x) = \varepsilon\overline{\lambda(x,y)} \in S^{-1}A/A \qquad (x,y,y' \in M) \; . \end{aligned}$$

Define the abelian groups

$$\begin{aligned} Q_\varepsilon(A,S) &= S^{-1}A/\{a + \varepsilon\bar{a} \,|\, a \in A\} \\ Q_\varepsilon(S^{-1}A/A) &= (S^{-1}A/A)/\{b - \varepsilon\bar{b} \,|\, b \in A\} \end{aligned}$$

and the abelian group morphism

$$1+T_\varepsilon : Q_\varepsilon(S^{-1}A/A) \longrightarrow Q_\varepsilon(A,S) \; ; \; c \longmapsto c + \varepsilon\bar{c} \; .$$

An <u>ε-quadratic linking form over (A,S)</u> (M,λ,μ) is an ε-symmetric linking form over (A,S) (M,λ) together with a function

$$\mu : M \longrightarrow Q_\varepsilon(A,S)$$

such that

$$\begin{aligned} &\text{i) } \mu(ax) = a\mu(x)\bar{a} \in Q_\varepsilon(A,S) \\ &\text{ii) } \mu(x+y) - \mu(x) - \mu(y) = \lambda(x,y) + \varepsilon\overline{\lambda(x,y)} \in Q_\varepsilon(A,S) \\ &\text{iii) } [\mu(x)] = \lambda(x)(x) \in S^{-1}A/A \qquad (x,y,y' \in M,\; a \in A). \end{aligned}$$

The linking forms appearing in the work of Wall [2], Passman and Petrie [1], Connolly [1] and Pardon [1],[2] on odd-dimensional surgery obstructions are just the ε-quadratic linking forms over $(\mathbb{Z}[\pi], \mathbb{Z} - \{0\})$, with $\varepsilon = \pm 1$ and π a finite group.

A <u>split ε-quadratic linking form over (A,S)</u> (M,λ,ν) is an ε-symmetric linking form over (A,S) (M,λ) together with a function

$$\nu : M \longrightarrow Q_\varepsilon(S^{-1}A/A)$$

such that

$$\begin{aligned} &\text{i)}\ \ \nu(ax) = a\nu(x)\bar{a} \in Q_\varepsilon(S^{-1}A/A) \\ &\text{ii)}\ \ \nu(x+y) - \nu(x) - \nu(y) = [\lambda(x)(y)] \in Q_\varepsilon(S^{-1}A/A) \\ &\text{iii)}\ \ \nu(x) + \varepsilon\overline{\nu(x)} = \lambda(x)(x) \in S^{-1}A/A \qquad (x,y \in M,\ a \in A). \end{aligned}$$

Split ε-quadratic linking forms were introduced by Karoubi [1].

A <u>morphism</u> (resp. <u>isomorphism</u>) of $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking forms over (A,S)

$$\left\{\begin{array}{l} f : (M,\lambda) \longrightarrow (M',\lambda') \\ f : (M,\lambda,\mu) \longrightarrow (M',\lambda',\mu') \\ f : (M,\lambda,\nu) \longrightarrow (M',\lambda',\nu') \end{array}\right.$$

is a morphism (resp. isomorphism) $f \in \mathrm{Hom}_A(M,M')$ such that

$$f^\wedge\lambda' f = \lambda \in \mathrm{Hom}_A(M,M^\wedge)$$

and also

$$\left\{\begin{array}{l} \mu : M \xrightarrow{f} M' \xrightarrow{\mu'} Q_\varepsilon(A,S) \\ \nu : M \xrightarrow{f} M' \xrightarrow{\nu'} Q_\varepsilon(S^{-1}A/A)\ . \end{array}\right.$$

It can be shown that the forgetful functor

(split ε-quadratic linking forms over (A,S))

$$\longrightarrow (\varepsilon\text{-quadratic linking forms over } (A,S))\ ;$$

$$(M,\lambda,\nu) \longmapsto (M,\lambda,\mu = (1+T_\varepsilon)\nu : M \xrightarrow{\nu} Q_\varepsilon(S^{-1}A/A) \xrightarrow{1+T_\varepsilon} Q_\varepsilon(A,S))$$

defines a surjection of isomorphism classes, which is a bijection if $\frac{1}{2} \in S^{-1}A$, e.g. if $A = \mathbb{Z}[\pi]$, $S = \mathbb{Z} - \{0\}$, $S^{-1}A = \mathbb{Q}[\pi]$. (This may be deduced from Proposition 2.2 below). In §6 we shall give examples of triples (A,S,ε) for which there is a perceptible difference between split ε-quadratic and ε-quadratic linking forms over (A,S).

An $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ is

<u>non-singular</u> if $\lambda \in \mathrm{Hom}_A(M,M^\wedge)$ is an isomorphism.

As in §1 we shall concentrate on the ε-quadratic L-theory, leaving the ε-symmetric L-theory of linking forms to the relevant part of Ranicki [2].

There is a close connection between linking forms over (A,S) and S-formations over A, which was first observed by Wall [1] in the case $A = \mathbb{Z}$, $S = \mathbb{Z} - \{0\}$, $S^{-1}A = \mathbb{Q}$.

<u>Proposition 2.2</u> The isomorphism classes of (non-singular) $\begin{cases} \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking forms over (A,S) $\begin{cases} (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ are in a natural one-one correspondence with the stable isomorphism classes of (non-singular) $\begin{cases} (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic} \end{cases}$ S-formations over A $\begin{cases} (N,\Psi;F,G) \\ (F,G) \end{cases}$. The linking form $\begin{cases} (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ corresponding to the S-formation $\begin{cases} (N,\Psi;F,G) \\ (F,(\binom{\gamma}{\mu},\theta)G) \end{cases}$ is defined by

$$\begin{cases} \begin{cases} M = N/(F+G)\ ,\ \lambda : M \longrightarrow M^\wedge\ ;\ x \longmapsto (y \longmapsto \frac{1}{s}(\Psi-\varepsilon\Psi^*)(x)(g)) \\ \mu : M \longrightarrow Q_\varepsilon(A,S)\ ;\ y \longmapsto \frac{1}{s}(\Psi-\varepsilon\Psi^*)(y)(g) - \Psi(y)(y) \\ \qquad\qquad (x,y\in N,\ s\in S,\ g\in G,\ sy - g \in F) \end{cases} \\ \begin{cases} M = \mathrm{coker}(\mu:G \longrightarrow F^*)\ ,\ \lambda : M \longrightarrow M^\wedge\ ;\ x \longmapsto (y \longmapsto \frac{1}{s}\gamma^*(x)(g)) \\ \nu : M \longrightarrow Q_\varepsilon(S^{-1}A/A)\ ;\ y \longmapsto \frac{\theta(g)(g)}{s\bar{s}} \\ \qquad\qquad (x,y\in F^*,\ s\in S,\ g\in G,\ sy = \mu g \in F^*). \end{cases} \end{cases}$$

[]

A <u>sublagrangian</u> of a non-singular split ε-quadratic linking form over (A,S) (M,λ,ν) is a submodule L of M such that

i) L, M/L are h.d. 1 S-torsion A-modules

ii) the inclusion $j \in \mathrm{Hom}_A(L,M)$ defines a morphism of linking forms

$$j : (L,0,0) \longrightarrow (M,\lambda,\nu)$$

iii) the A-module morphism

$$[\lambda] : M/L \longrightarrow L^\wedge \ ; \ [x] \longmapsto (y \longmapsto \lambda(x)(y)) \quad (x\in M, y\in L)$$

is onto.

The <u>annihilator</u> of a sublagrangian L in (M,λ,ν) is the submodule $L^\perp$ of M defined by

$$L^\perp = \ker(j^\wedge\lambda : M \longrightarrow L^\wedge) ,$$

which is such that $L \subseteq L^\perp$.

A <u>lagrangian</u> of (M,λ,ν) is a sublagrangian L such that

$$L^\perp = L .$$

A non-singular split ε-quadratic linking form which admits a lagrangian is <u>hyperbolic</u>. For example, if L is a sublagrangian of (M,λ,ν) then there is defined a non-singular split ε-quadratic linking form $(L^\perp/L, \lambda^\perp/\lambda, \nu^\perp/\nu)$ such that $(M,\lambda,\nu)\oplus(L^\perp/L,-\lambda^\perp/\lambda,-\nu^\perp/\nu)$ is hyperbolic, with lagrangian

$$L' = \{(x,[x]) \in M\oplus L^\perp/L \mid x \in L^\perp\} .$$

Given an h.d. 1 S-torsion A-module P define the standard hyperbolic split ε-quadratic linking form over (A,S)

$$H_\varepsilon(P) = (P\oplus P^\wedge, \lambda : P\oplus P^\wedge \longrightarrow (P\oplus P^\wedge)^\wedge ; (x,f) \longmapsto ((y,g) \longmapsto f(y) + \varepsilon\overline{g(x)}) ,$$
$$\nu : P\oplus P^\wedge \longrightarrow Q_\varepsilon(S^{-1}A/A) \ ; \ (x,f) \longmapsto f(x) \ ) .$$

A <u>split ε-quadratic linking formation over (A,S)</u> $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is defined by a sublagrangian G in a standard hyperbolic split ε-quadratic linking form over (A,S) $H_\varepsilon(F)$, together with a <u>hessian</u> $(-\varepsilon)$-quadratic linking form over (A,S)

$$(G, \gamma^\wedge\mu \in \mathrm{Hom}_A(G,G^\wedge), \theta : G \longrightarrow Q_{-\varepsilon}(A,S))$$

where $\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : G \longrightarrow F\oplus F^\wedge$ is the inclusion. Such objects first appeared in the work of Pardon [1], and similar structures have been studied by Karoubi [1].

We shall normally write $(F,(\binom{\gamma}{\mu},\theta)G)$ as (F,G), denoting $(F,(\binom{-\gamma}{\mu},-\theta)G)$ by $-(F,G)$.

An <u>isomorphism</u> of split ε-quadratic linking formations over (A,S)

$$f : (F,G) \longrightarrow (F',G')$$

is an isomorphism of the hyperbolic split ε-quadratic linking forms

$$f : H_\varepsilon(F) \longrightarrow H_\varepsilon(F')$$

such that

$$f(F) = F' \ , \ f(G) = G'$$

and also

$$\theta : G \xrightarrow{f|} G' \xrightarrow{\theta'} Q_{-\varepsilon}(A,S) \ .$$

A <u>sublagrangian</u> of a split ε-quadratic linking formation over (A,S) (F,G) is a sublagrangian L of $H_\varepsilon(F)$ such that

i) $L \subseteq G$, with G/L an h.d. 1 S-torsion A-module

ii) $F \cap L = \{0\}$, $F \oplus F^\wedge = F + L^\perp$.

Such a sublagrangian determines an <u>elementary equivalence</u> of split ε-quadratic linking formations over (A,S), the transformation

$$(F,G) \longrightarrow (F',G') \quad ,$$

with (F',G') defined by

$$F' = F \cap L^\perp \quad , \quad G' = G/L$$

$$\gamma' : G' \longrightarrow F' \ ; \ [x] \longmapsto \gamma(x)$$

$$\mu' : G' \longrightarrow F'^\wedge \ ; \ [x] \longmapsto (y \longmapsto \mu(x)(y))$$

$$\theta' : G' \longrightarrow Q_{-\varepsilon}(A,S) \ ; \ [x] \longmapsto \theta(x) \qquad (x \in G, y \in F \ ).$$

Elementary equivalences and isomorphisms generate an equivalence relation on the set of split ε-quadratic linking formations over (A,S), which we shall call <u>stable equivalence</u>.

A split ε-quadratic linking formation over (A,S) (F,G) is <u>non-singular</u> if G is a lagrangian of $H_\varepsilon(F)$, or equivalently if the sequence

$$0 \longrightarrow G \xrightarrow{\binom{\gamma}{\mu}} F \oplus F^\wedge \xrightarrow{(\varepsilon\mu^\wedge\ \gamma^\wedge)} G^\wedge \longrightarrow 0$$

is exact. Any linking formation stably equivalent to a non-singular one is itself non-singular.

There is a close connection between linking formations over (A,S) and S-forms over A.

Proposition 2.3 The stable equivalence classes of (non-singular) split ε-quadratic linking formations over (A,S) (F,G) are in a natural one-one correspondence with the stable isomorphism classes of (non-singular) ε-quadratic S-forms over A $(M,\Psi;L)$. The linking formation (F,G) corresponding to the S-form $(M,\Psi;L)$ is defined as follows: extend the inclusion $j \in \mathrm{Hom}_A(L,M)$ to an S-isomorphism of ε-quadratic forms over A

$$(j \quad k) : (L \oplus L^*, \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}) \longrightarrow (M,\Psi)$$

for some $k \in \mathrm{Hom}_A(L^*,M)$, $s \in S$, set

$$F = \mathrm{coker}(\bar{s}:L \longrightarrow L) \ , \ G = \mathrm{coker}((j\ k):L\oplus L^* \longrightarrow M) \ ,$$

define $\begin{pmatrix} \gamma \\ \mu \end{pmatrix}:G \longrightarrow F \oplus F^{\wedge}$ via the resolution

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L^* & \xrightarrow{(j\ k)} & M & \longrightarrow & G & \longrightarrow & 0 \\
 & & \Big\downarrow \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} k^*(\Psi+\varepsilon\Psi^*) \\ j^*(\Psi+\varepsilon\Psi^*) \end{pmatrix} & & \Big\downarrow \begin{pmatrix} \gamma \\ \mu \end{pmatrix} & & \\
0 & \longrightarrow & L\oplus L^* & \xrightarrow{\begin{pmatrix} \bar{s} & 0 \\ 0 & s \end{pmatrix}} & L\oplus L^* & \longrightarrow & F\oplus F^{\wedge} & \longrightarrow & 0 \ ,
\end{array}$$

and let $(G, \gamma^{\wedge}\mu \in \mathrm{Hom}_A(G,G^{\wedge}), \theta:G \longrightarrow Q_{-\varepsilon}(A,S))$ be the $(-\varepsilon)$-quadratic linking form over (A,S) corresponding to the ε-quadratic S-formation over A

$$(H_{\varepsilon}(M^*);M^*,\mathrm{im}(\begin{pmatrix} -\bar{\varepsilon}\Psi j & \Psi^* k \\ j & k \end{pmatrix}:L\oplus L^* \longrightarrow M^*\oplus M)) \ .$$

[]

The boundary of a split ε-quadratic linking $\begin{Bmatrix} \text{form} \\ \text{formation} \end{Bmatrix}$ over (A,S) $\begin{Bmatrix} (M,\lambda,\nu) \\ (F,G) \end{Bmatrix}$ is the non-singular split $\begin{Bmatrix} (-\varepsilon)\text{-} \\ \varepsilon\text{-} \end{Bmatrix}$ quadratic linking $\begin{Bmatrix} \text{formation} \\ \text{form} \end{Bmatrix}$ over (A,S)

$$\begin{cases} \partial(M,\lambda,\nu) = (M,(\begin{pmatrix} 1 \\ \lambda \end{pmatrix}, (1+T_{\varepsilon})\nu)M) \\ \partial(F,G) = (G^{\perp}/G, \lambda^{\perp}/\lambda, \nu^{\perp}/\nu) \ , \text{ where } H_{\varepsilon}(F) = (F\oplus F^{\wedge},\lambda,\nu) \ . \end{cases}$$

A split ε-quadratic linking $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ is non-singular if and only if its boundary linking $\begin{cases}\text{formation}\\ \text{form}\end{cases}$ is $\begin{cases}\text{stably equivalent to } 0\\ 0\end{cases}$.

Non-singular split ε-quadratic linking $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over (A,S) $\begin{cases}(M,\lambda,\nu),(M',\lambda',\nu')\\ (F,G),(F',G')\end{cases}$ are <u>cobordant</u> if there exists $\begin{cases}\text{an isomorphism}\\ \text{a stable equivalence}\end{cases}$

$$\begin{cases}(M,\lambda,\nu)\oplus(M',-\lambda',-\nu')\longrightarrow\partial(K,L)\\ (F,G)\oplus-(F',G')\longrightarrow\partial(N,\varphi,\psi)\end{cases}$$

for some split $\begin{cases}\varepsilon\text{-}\\ (-\varepsilon)\text{-}\end{cases}$ quadratic linking $\begin{cases}\text{formation}\\ \text{form}\end{cases}$ over (A,S) $\begin{cases}(K,L)\\ (N,\varphi,\psi)\end{cases}$.

<u>Proposition 2.4</u> Cobordism is an equivalence relation on the set of non-singular split ε-quadratic linking $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over (A,S), such that the equivalence classes define an abelian group $\begin{cases}L_0(A,S,\varepsilon)\\ L_1(A,S,\varepsilon)\end{cases}$ with respect to the direct sum $\oplus$. The L-groups defined for $n \pmod 4$ by

$$L_n(A,S,\varepsilon) = \begin{cases}L_0(A,S,(-)^i\varepsilon)\\ L_1(A,S,(-)^i\varepsilon)\end{cases} \text{if } n = \begin{cases}2i\\ 2i+1\end{cases}$$

fit into the localization exact sequence

$$\ldots\longrightarrow L_n(A,\varepsilon)\longrightarrow L_n^S(S^{-1}A,\varepsilon)\longrightarrow L_n(A,S,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\longrightarrow \ldots .$$

The fit is achieved by natural isomorphisms

$$L_n(A,S,\varepsilon)\longrightarrow L_n^S(A\longrightarrow S^{-1}A,\varepsilon) \quad (n \pmod 4),$$

defined by sending a non-singular linking $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over (A,S) to the corresponding non-singular $\begin{cases}S\text{-formation}\\ S\text{-form}\end{cases}$ over A (given by Proposition $\begin{cases}2.2\\ 2.3\end{cases}$).

[]

Note that $L_0(A,S,\varepsilon)$ can also be viewed as the abelian group of equivalence classes of non-singular split ε-quadratic linking forms over (A,S) under the relation

$(M,\lambda,\nu)\sim(M',\lambda',\nu')$ if there exists an isomorphism

$$f : (M,\lambda,\nu)\oplus(N,\varphi,\psi) \longrightarrow (M',\lambda',\nu')\oplus(N',\varphi',\psi')$$

for some hyperbolic split ε-quadratic linking forms $(N,\varphi,\psi),(N',\varphi',\psi')$.

The localization exact sequence of Proposition 2.4 was first obtained by Pardon [1] in the case $A=\mathbb{Z}[\pi]$ (π finite), $S=\mathbb{Z}-\{0\}$ following on from the earlier work of Wall [1],[2], Passman and Petrie [1], Connolly [1] and his own work on rational surgery (Pardon [2]). These authors only work with f.g. free A-modules - we shall discuss the effect of this restriction in §7 below.

Karoubi [1] obtained a localization exact sequence in the context of hermitian K-theory. However, the methods of that paper are not sufficient for a localization sequence in the surgery obstruction groups, since it is frequently assumed that $1/2\in A$, the formula for the quadratic function Q on p.366 of Part I is not well-defined in general, and the quadratic linking formations do not include the hessian θ appearing in the definition of (F,G) (introduced by Pardon [1]) which carries delicate quadratic information such as the Arf invariant.

The localization exact sequence is natural, in the following sense.

Let $f:A\longrightarrow B$ be a morphism of rings with involution such that $f(S)\subseteq T$ for some multiplicative subsets $S\subset A$, $T\subset B$. Given an h.d. 1 S-torsion A-module M with a f.g. projective A-module resolution

$$0\longrightarrow P_1 \xrightarrow{\ d\ } P_0 \longrightarrow M \longrightarrow 0$$

we have that $d\in\mathrm{Hom}_A(P_1,P_0)$ is an S-isomorphism, and hence that $1\otimes d\in\mathrm{Hom}_B(B\otimes_A P_1,B\otimes_A P_0)$ is a T-isomorphism. Also, the functor

$$B\otimes_A - : (A\text{-modules})\longrightarrow(B\text{-modules})\ ;\ P\longmapsto B\otimes_A P$$

is right exact, so that we have a f.g. projective B-module resolution

$$0 \longrightarrow B\otimes_A P_1 \xrightarrow{1\otimes d} B\otimes_A P_0 \longrightarrow B\otimes_A M \longrightarrow 0$$

and $B\otimes_A M$ is an h.d. 1 T-torsion B-module. Thus f induces a functor

$$B\otimes_A - : (\text{h.d. 1 S-torsion A-modules}) \longrightarrow (\text{h.d. 1 T-torsion B-modules}) ;$$
$$M \longmapsto B\otimes_A M$$

and there are defined abelian group morphisms

$$f : L_n(A,S,\varepsilon) \longrightarrow L_n(B,T,\varepsilon) ; x \longmapsto B\otimes_A x \qquad (n(\text{mod } 4)) .$$

Proposition 2.5 A morphism of rings with involution $f:A \longrightarrow B$ such that $f(S) \subseteq T$ for some multiplicative subsets $S \subset A$, $T \subset B$ induces a morphism of exact sequences of abelian groups

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & L_n(A,\varepsilon) & \longrightarrow & L_n^S(S^{-1}A,\varepsilon) & \longrightarrow & L_n(A,S,\varepsilon) & \longrightarrow & L_{n-1}(A,\varepsilon) & \longrightarrow \ldots \\
& \downarrow f & & \downarrow S^{-1}f & & \downarrow f & & \downarrow f & \\
\ldots \longrightarrow & L_n(B,\varepsilon) & \longrightarrow & L_n^T(T^{-1}B,\varepsilon) & \longrightarrow & L_n(B,T,\varepsilon) & \longrightarrow & L_{n-1}(B,\varepsilon) & \longrightarrow \ldots .
\end{array}$$

[]

Were it necessary we could define relative L-groups $L_n(f,S,\varepsilon)$ for n(mod 4) (as cobordism groups of relative linking forms and formations) to fit into exact sequences

$$\ldots \longrightarrow L_n(A,S,\varepsilon) \xrightarrow{f} L_n(B,T,\varepsilon) \longrightarrow L_n(f,S,\varepsilon) \longrightarrow L_{n-1}(A,S,\varepsilon) \longrightarrow \ldots$$
$$\ldots \longrightarrow L_n(f,\varepsilon) \longrightarrow L_n^S(S^{-1}f,\varepsilon) \longrightarrow L_n(f,S,\varepsilon) \longrightarrow L_{n-1}(f,\varepsilon) \longrightarrow \ldots .$$

§3. Cartesian squares

We shall now investigate the conditions under which a morphism of rings with involution and multiplicative subsets

$$f : (A,S) \longrightarrow (B,T)$$

induces excision isomorphisms

$$f : L_n(A,S,\varepsilon) \longrightarrow L_n(B,T,\varepsilon) \qquad (n(\mathrm{mod}\ 4))$$

and a Mayer-Vietoris exact sequence

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon)\oplus L_n(B,\varepsilon) \longrightarrow L_n^T(T^{-1}B,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots .$$

Define a partial ordering on S by

$$s \leqslant s' \text{ if there exists } t\in S \text{ such that } s' = st \in S .$$

Define also a direct system of abelian groups $\{A/sA \mid s\in S\}$ with structure maps

$$A/sA \longrightarrow A/stA \ ; \ x \longmapsto tx \quad .$$

The abelian group morphisms

$$A/sA \longrightarrow S^{-1}A/A \ ; \ a \longmapsto \frac{a}{s}$$

allow the identification

$$\varinjlim_{s\in S} A/sA = S^{-1}A/A \ .$$

The involution

$$- : S^{-1}A/A \longrightarrow S^{-1}A/A \ ; \ \frac{a}{s} \longmapsto \frac{\bar{a}}{\bar{s}}$$

is identified with the involution

$$- : \varinjlim_{s\in S} A/sA \longrightarrow \varinjlim_{s\in S} A/sA \ ; \ \{a_s\in A/sA \mid s\in S\} \longmapsto \{\overline{a_{\bar{s}}} \in A/sA \mid s\in S\} \ .$$

A morphism of rings with involution and multiplicative subsets

$$f : (A,S) \longrightarrow (B,T)$$

is <u>cartesian</u> if $f(S) = T$ and if for every $s\in S$ the map

$$f : A/sA \longrightarrow B/tB \ ; \ x \longmapsto f(x) \quad (t = f(s)\in T)$$

is an isomorphism of abelian groups. It follows that there is induced an isomorphism of abelian groups with involution

$$f : \varinjlim_{s\in S} A/sA = S^{-1}A/A \longrightarrow \varinjlim_{t\in T} B/tB = T^{-1}B/B \ ; \ x \longmapsto f(x) \ ,$$

and hence that the commutative square of rings with involution

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ f\downarrow & & \downarrow f \\ B & \longrightarrow & T^{-1}B \end{array}$$

is cartesian, in the sense that there is defined an exact sequence of abelian groups with involution

$$0 \longrightarrow A \longrightarrow S^{-1}A \oplus B \longrightarrow T^{-1}B \longrightarrow 0 \quad .$$

Cartesian morphisms were introduced by Karoubi [1] (Appendix 5 of Part I), who proved that a cartesian morphism $f:(A,S) \longrightarrow (B,T)$ induces an isomorphism of exact categories

$$f : (\text{h.d. } 1 \text{ } S\text{-torsion } A\text{-modules}) \longrightarrow (\text{h.d. } 1 \text{ } T\text{-torsion } B\text{-modules}) \; ;$$

$$M \longmapsto B \otimes_A M \;(= M \text{ as an } A\text{-module}) \; .$$

As an immediate consequence of this and of the localization exact sequence of Proposition 2.4 we have:

<u>Proposition 3.1</u> A cartesian morphism $f:(A,S) \longrightarrow (B,T)$ induces excision isomorphisms of relative L-groups

$$f : L_n(A,S,\varepsilon) \longrightarrow L_n(B,T,\varepsilon) \qquad (n(\text{mod } 4)) \; ,$$

and there is defined a Mayer-Vietoris exact sequence of absolute L-groups

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \oplus L_n(B,\varepsilon) \longrightarrow L_n^T(T^{-1}B,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots\ldots$$

[]

A Mayer-Vietoris exact sequence of the above type was first obtained by Wall [6] for a cartesian square of arithmetic type (cf. Proposition 3.2 below), by a direct proof which avoided relative L-theory at the expense of invoking the strong approximation theorem. In fact, it is possible to obtain both the Mayer-Vietoris sequence and the excision isomorphisms avoiding the localization sequence, by directly constructing appropriate morphisms

$$\hat{\delta} : L_n^T(B \longrightarrow T^{-1}B,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \qquad (n(\text{mod } 4))$$

(generalizing the method of Wall [6]), using the characterization of the relative L-groups in terms of relative forms and formations of §1. The idea of combining a localization exact sequence with the above isomorphism of categories is due to Karoubi [1], who obtained excision isomorphisms and a

Mayer-Vietoris sequence in hermitian K-theory (with the qualifications regarding the L-groups expressed at the end of §2). Bak [2] has obtained similar results in the context of the KU-theory of Bass [2].

In §7 below we shall generalize the excision isomorphisms and the Mayer-Vietoris sequence of Proposition 3.1 to the intermediate L-groups.

Given a multiplicative subset $S \subset A$ of a ring with involution A define the <u>S-adic completion of A</u> to be the inverse limit

$$\hat{A} = \varprojlim_{s \in S} A/sA$$

of the inverse system of rings $\{A/sA \mid s \in S\}$ with structure maps the natural projections

$$A/stA \longrightarrow A/sA \quad (s,t \in S) \ .$$

Then $\hat{A}$ is a ring, with involution by

$$\bar{\ } : \hat{A} \longrightarrow \hat{A} \ ; \ \{a_s \in A/sA \mid s \in S\} \longmapsto \{\overline{a_s} \in A/sA \mid s \in S\} \ .$$

The inclusion

$$f : A \longrightarrow \hat{A} \ ; \ a \longmapsto \{a \in A/sA \mid s \in S\}$$

is a morphism of rings with involution, such that the image of S is a multiplicative subset $\hat{S} = f(S) \subset A$.

<u>Proposition 3.2</u> The inclusion $f:(A,S) \longrightarrow (\hat{A},\hat{S})$ is a cartesian morphism, so that there are induced excision isomorphisms

$$f : L_n(A,S,\varepsilon) \longrightarrow L_n(\hat{A},\hat{S},\varepsilon) \quad (n(\mathrm{mod}\ 4))$$

and there is defined a Mayer-Vietoris exact sequence

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \oplus L_n(\hat{A},\varepsilon) \longrightarrow L_n^{\hat{S}}(\hat{S}^{-1}\hat{A},\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots \ .$$

[]

In particular, we have a cartesian morphism $f:(\mathbb{Z},\mathbb{Z}-\{0\}) \longrightarrow (\hat{\mathbb{Z}},\mathbb{Z}-\{0\})$, with $\hat{\mathbb{Z}} = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$ the profinite completion of $\mathbb{Z}$. The associated cartesian square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}} & \longrightarrow & \hat{\mathbb{Q}} \end{array}$$

is the 'arithmetic square', with $\hat{\mathbb{Q}}$ the finite adéle ring of $\mathbb{Q}$. In Wall [6] there was obtained an L-theoretic Mayer-Vietoris exact sequence for the cartesian square

$$\begin{array}{ccc} \mathbb{Z}\otimes_{\mathbb{Z}}A & \longrightarrow & \mathbb{Q}\otimes_{\mathbb{Z}}A \\ (=A) & & \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}\otimes_{\mathbb{Z}}A & \longrightarrow & \hat{\mathbb{Q}}\otimes_{\mathbb{Z}}A \end{array}$$

for any ring with involution A such that the additive group of A is finitely generated. For torsion-free A (e.g. $A = \mathbb{Z}[\pi]$, with π a finite group) this is just the cartesian square considered in Proposition 3.2 with $S = \mathbb{Z}-\{0\}\subset A$.

Given a ring with involution A we shall say that multiplicative subsets $S,T\subset A$ are <u>coprime</u> if for any $s\in S$, $t\in T$ the ideals $sA, tA \triangleleft A$ are coprime, that is if there exist $a,b\in A$ such that

$$as + bt = 1 \in A \ .$$

The multiplicative subsets

$$ST = \{st \mid s\in S, t\in T\}\subset A \quad , \quad T^{-1}S = \{\tfrac{s}{1} \mid s\in S\}\subset T^{-1}A$$

are such that there is a natural identification

$$(T^{-1}S)^{-1}(T^{-1}A) = (ST)^{-1}A \ .$$

<u>Proposition 3.3</u> If $S,T\subset A$ are coprime multiplicative subsets then the inclusion $f:(A,S)\longrightarrow(T^{-1}A,T^{-1}S)$ is a cartesian morphism, inducing excision isomorphisms

$$f : L_n(A,S,\varepsilon)\longrightarrow L_n(T^{-1}A,T^{-1}S,\varepsilon) \quad (n(\mathrm{mod}\ 4)) \ ,$$

and there is defined a Mayer-Vietoris exact sequence

$$\ldots\longrightarrow L_n(A,\varepsilon)\longrightarrow L_n^S(S^{-1}A,\varepsilon)\oplus L_n^T(T^{-1}A,\varepsilon)\longrightarrow L_n^{ST}((ST)^{-1}A,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\longrightarrow\ldots \ .$$

[]

For example, if $S = \{p_1^{k_1}p_2^{k_2}\ldots p_r^{k_r} \mid k_1,k_2,\ldots,k_r \geqslant 0\}$ and $T = \{q_1^{j_1}\ldots q_s^{j_s} \mid j_1\ldots j_s \geqslant 0\}$ for some disjoint collections of primes $P = \{p_1,p_2,\ldots\}$, $Q = \{q_1,q_2,\ldots\}$ such that $P\cup Q = \{\text{all primes in } \mathbb{Z}\}$ then $S^{-1}\mathbb{Z} = \mathbb{Z}[\tfrac{1}{P}] = \mathbb{Z}_{(Q)}$ (= localization away from P = localization at Q) and $S,T\subset\mathbb{Z}$ are coprime multiplicative subsets with $(ST)^{-1}\mathbb{Z} = \mathbb{Q}$.

§4. Products

We shall now show that the localization sequence

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \longrightarrow L_n(A,S,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots$$

is an exact sequence of $L^0(R)$-modules if A is an R-module for some ring with involution R. As in §1 $L^0(R)$ denotes the symmetric Witt group of R. We shall use this $L^0(R)$-action to prove that the natural maps

$$L_n(A,\varepsilon) \longrightarrow L_n^S(\mathbb{Q}\otimes_{\mathbb{Z}}A,\varepsilon) \qquad (n(\mathrm{mod}\ 4)\ ,\ S = \mathbb{Z}-\{0\}\subset A)$$

are isomorphisms modulo 8-torsion for any torsion-free ring with involution A, along with other results of this nature.

A ring with involution A is an <u>R-module</u> for some ring with involution R if there is given a morphism of rings with involution

$$R\otimes_{\mathbb{Z}}A \longrightarrow A\ ;\ r\otimes a \longmapsto ra\ ,$$

with the involution on $R\otimes_{\mathbb{Z}}A$ defined by

$$\bar{\ }: R\otimes_{\mathbb{Z}}A \longrightarrow R\otimes_{\mathbb{Z}}A\ ;\ r\otimes a \longmapsto \bar{r}\otimes\bar{a}\ .$$

Note that each $r1_A\in A$ $(r\in R)$ is central in A, so that given an R-module M and an A-module N there is defined an A-module

$$M\otimes_R N = M\otimes_{\mathbb{Z}}N/\{rx\otimes y - x\otimes(r1_A)y \mid x\in M, y\in N, r\in R\}$$

with A acting by

$$A\times M\otimes_R N \longrightarrow M\otimes_R N\ ;\ (a,x\otimes y)\longmapsto x\otimes ay\ .$$

In particular, we have a pairing

$$(\text{f.g. projective } R\text{-modules})\times(\text{f.g. projective } A\text{-modules})$$
$$\longrightarrow (\text{f.g. projective } A\text{-modules})\ ;\ (M,N)\longmapsto M\otimes_R N\ ,$$

with natural identifications

$$(M\otimes_R N)^* = M^*\otimes_R N^*\ .$$

Given a multiplicative subset $S\subset A$ we have that $S^{-1}A$ is an R-module by

$$R\otimes_{\mathbb{Z}}S^{-1}A \longrightarrow S^{-1}A\ ;\ r\otimes\frac{a}{s}\longmapsto\frac{ra}{s}\ ,$$

and that there is defined a pairing

$$(\text{f.g. projective } R\text{-modules})\times(\text{h.d. } 1\ S\text{-torsion } A\text{-modules})$$
$$\longrightarrow (\text{h.d. } 1\ S\text{-torsion } A\text{-modules})\ ;\ (M,N)\longmapsto M\otimes_R N\ ,$$

with natural identifications

$$(M\otimes_R N)^\wedge = M^*\otimes_R N^\wedge \ .$$

Define $L^0(R)$-actions on quadratic L-theory by

$$L^0(R)\otimes_{\mathbb{Z}} L_n(A,\varepsilon) \longrightarrow L_n(A,\varepsilon) \ ;$$

$$\begin{cases} (M,\varphi)\otimes(N,\psi) \longmapsto (M\otimes_R N,\varphi\otimes\psi) \\ (M,\varphi)\otimes(N,\psi;F,G) \longmapsto (M\otimes_R N,\varphi\otimes\psi;M\otimes_R F,M\otimes_R G) \end{cases} \quad \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases} .$$

(In terms of the products defined in Part I of Ranicki [2] these are just the composites

$$L^0(R)\otimes_{\mathbb{Z}} L_n(A,\varepsilon) \xrightarrow{\ \otimes\ } L_n(R\otimes_{\mathbb{Z}} A,1\otimes\varepsilon) \longrightarrow L_n(A,\varepsilon) \qquad (n(\text{mod } 4))).$$

Define also $L^0(R)$-actions

$$L^0(R)\otimes_{\mathbb{Z}} L_n^S(S^{-1}A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \ ;$$

$$\begin{cases} (M,\varphi)\otimes S^{-1}(N,\psi) \longmapsto S^{-1}(M\otimes_R N,\varphi\otimes\psi) \\ (M,\varphi)\otimes S^{-1}(N,\psi;F,G) \longmapsto S^{-1}(M\otimes_R N,\varphi\otimes\psi;M\otimes_R F,M\otimes_R G) \end{cases} \quad \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases} ,$$

$$L^0(R)\otimes_{\mathbb{Z}} L_n(A,S,\varepsilon) \longrightarrow L_n(A,S,\varepsilon) \ ;$$

$$\begin{cases} (M,\varphi)\otimes(N,\lambda,\nu) \longmapsto (M\otimes_R N,\varphi\otimes\lambda,\varphi\otimes\nu : x\otimes y \longmapsto \varphi(x)(x)\nu(y)) \\ (M,\varphi)\otimes(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G) \longmapsto (M\otimes_R F,(\begin{pmatrix}1\otimes\gamma\\ \varphi\otimes\mu\end{pmatrix},\varphi\otimes\theta)M\otimes_R G) \end{cases} \quad \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases} .$$

In each case the element

$$(R,1:R\longrightarrow R^*;q\longmapsto(r\longmapsto r\bar{q}))\in L^0(R)$$

acts by the identity. (In general R is not itself an R-module. However, if R is commutative then it is an R-module in the usual fashion, and the symmetric Witt group $L^0(R)$ is a commutative ring with 1).

<u>Proposition 4.1</u> Let A,R be rings with involution such that A is an R-module, and let $S\subset A$ be a multiplicative subset. The localization sequence

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \longrightarrow L_n(A,S,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots$$

is an exact sequence of $L^0(R)$-modules.

[]

(More generally, if $f:A\longrightarrow B$ is a morphism of rings with involution which is a morphism of R-modules then the symmetric Witt group $L^0(R)$ acts on the exact sequence of Proposition 1.3

$$\ldots \longrightarrow L_n(A,\varepsilon) \xrightarrow{\ f\ } L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots).$$

In our applications of Proposition 4.1 we shall need to know the symmetric Witt groups $L^0(\mathbb{Z}_m)$ of the finite cyclic rings $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.

Let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the factorization of m into prime powers, so that

$$\mathbb{Z}_m = \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i^{k_i}} \quad , \quad L^0(\mathbb{Z}_m) = \bigoplus_{i=1}^{m} L^0(\mathbb{Z}_{p_i^{k_i}}) \quad .$$

Lemma 5 of Wall [4] and Theorem 3.3 of Bak [1] on reduction modulo a complete ideal (alias Hensel's lemma) apply to show that the projections

$$\begin{cases} \mathbb{Z}_{2^k} \longrightarrow \mathbb{Z}_8 \ , \ k \geqslant 3 \\ \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_p \ , \ p \text{ odd} \ , \ k \geqslant 1 \end{cases}$$

induce isomorphisms

$$\begin{cases} L^0(\mathbb{Z}_{2^k}) \longrightarrow L^0(\mathbb{Z}_8) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \\ L^0(\mathbb{Z}_{p^k}) \longrightarrow L^0(\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 1 (\mathrm{mod}\ 4) \\ \mathbb{Z}_4 & \text{if } p \equiv 3 (\mathrm{mod}\ 4) \ . \end{cases} \end{cases}$$

Moreover,

$$L^0(\mathbb{Z}_4) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \quad , \quad L^0(\mathbb{Z}_2) = \mathbb{Z}_2 \quad .$$

For each integer $m \geqslant 2$ define the number

$$\Psi(m) = \text{exponent of } L^0(\mathbb{Z}_m) = \begin{cases} 2 & \text{if } m = d \text{ or } 2d \\ 4 & \text{if } m = 4d, e, 2e \text{ or } 4e \\ 8 & \text{otherwise} \end{cases} \quad ,$$

with

d = a product of odd primes $p \equiv 1 (\mathrm{mod}\ 4)$

e = a product of odd primes, including at least one $p \equiv 3 (\mathrm{mod}\ 4)$.

A ring with involution A is of <u>characteristic</u> m if m is the least integer $\geqslant 2$ such that $m1 = 0 \in A$, in which case $ma = 0$ for all $a \in A$ and A is a $\mathbb{Z}_m$-module.

<u>Proposition 4.2</u> If the ring with involution A is of characteristic m then the localization sequence

$$\cdots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \longrightarrow L_n(A,S,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \cdots$$

is an exact sequence of $L^0(\mathbb{Z}_m)$-modules, so that all the L-groups involved are of exponent $\Psi(m)$.

[]

The symmetric Witt groups $L^0(\hat{\mathbb{Z}}_m)$ of the rings of m-adic integers $\hat{\mathbb{Z}}_m = \varprojlim_k \mathbb{Z}/m^k\mathbb{Z}$ are computed as follows. Again, let $m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ so that

$$\hat{\mathbb{Z}}_m = \bigoplus_{i=1}^{r} \hat{\mathbb{Z}}_{p_i} \quad , \quad L^0(\hat{\mathbb{Z}}_m) = \bigoplus_{i=1}^{r} L^0(\hat{\mathbb{Z}}_{p_i})$$

and

$$L^0(\hat{\mathbb{Z}}_p) = \begin{cases} L^0(\mathbb{Z}_8) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ L^0(\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 1 (\mathrm{mod}\ 4) \\ \mathbb{Z}_4 & \text{if } p \equiv 3 (\mathrm{mod}\ 4) \ . \end{cases} \end{cases}$$

For each integer $m \geq 2$ define the number

$$\hat{\Psi}(m) = \text{exponent of } L^0(\hat{\mathbb{Z}}_m) = \begin{cases} 2 \text{ if } m \text{ is a product of odd primes } p \equiv 1 (\mathrm{mod}\ 4) \\ 4 \text{ if } m \text{ is a product of odd primes at least one} \\ \qquad \text{of which is } p \equiv 3 (\mathrm{mod}\ 4) \\ 8 \text{ if } m \text{ is even} \ . \end{cases}$$

The method of Wall [5] applies to show that the symmetric Witt group of the profinite completion $\hat{\mathbb{Z}} = \varprojlim_m \mathbb{Z}/m\mathbb{Z} = \prod_p \hat{\mathbb{Z}}_p$ is the infinite product

$$L^0(\hat{\mathbb{Z}}) = \prod_p L^0(\hat{\mathbb{Z}}_p) \quad .$$

A ring with involution A is <u>m-torsion-free</u> if $S = \{m^k | k \geq 0\} \subset A$ is a multiplicative subset, so that the localization away from m $S^{-1}A = A[\frac{1}{m}]$ is defined. The m-adic completion $\hat{A} = \varprojlim_k A/m^kA$ is a $\hat{\mathbb{Z}}_m$-module.

A ring with involution A is <u>torsion-free</u> if $S = \mathbb{Z}-\{0\} \subset A$ is a multiplicative subset, so that the localization $S^{-1}A = \mathbb{Q}\otimes_{\mathbb{Z}}A$ is defined. The profinite completion $\hat{A} = \varprojlim_m A/mA$ is a $\hat{\mathbb{Z}}$-module .

<u>Proposition 4.3</u> Let A be a ring with involution which is m-torsion-free (resp. torsion-free) and let $S = \{m^k | k \geq 0\} \subset A$ (resp. $S = \mathbb{Z} - \{0\} \subset A$). The localization sequence of the S-adic completion $\hat{A} = \varprojlim_{s\in S} A/sA$

$$\ldots \longrightarrow L_n(\hat{A},\varepsilon) \longrightarrow L_n^{\hat{S}}(\hat{S}^{-1}\hat{A},\varepsilon) \longrightarrow L_n(\hat{A},\hat{S},\varepsilon) \longrightarrow L_{n-1}(\hat{A},\varepsilon) \longrightarrow \ldots$$

is an exact sequence of $L^0(\hat{\mathbb{Z}}_m)$ (resp. $L^0(\hat{\mathbb{Z}})$)- modules, so that all the L-groups are of exponent $\hat{\Psi}(m)$ (resp. 8). Thus the L-groups $L_n(A,S,\varepsilon) = L_n(\hat{A},\hat{S},\varepsilon)$ are of exponent $\hat{\Psi}(m)$ (resp. 8) and the natural maps

$$L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \qquad (n (\mathrm{mod}\ 4))$$

are isomorphisms modulo $\hat{\Psi}(m)$ (resp. 8)-torsion.

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The integral group ring $\mathbb{Z}[\pi]$ of a group π is torsion-free, with localization $S^{-1}\mathbb{Z}[\pi] = \mathbb{Q}[\pi]$ ($S = \mathbb{Z}-\{0\}$) the rational group ring, so that as a particular case of Proposition 4.3 we have:

Proposition 4.4 The natural maps

$$L_n(\mathbb{Z}[\pi]) \longrightarrow L_n^S(\mathbb{Q}[\pi]) \quad (n(\text{mod } 4))$$

are isomorphisms modulo 8-torsion, for any group π.

[]

Results of this type were first obtained for finite groups π. If we take for granted the result that the natural maps $L_{2i}(\mathbb{Q}[\pi]) \longrightarrow L_{2i}(\mathbb{R}[\pi])$ are isomorphisms modulo 2-primary torsion (π finite, $i(\text{mod } 2)$) then Theorems 13A.3, 13A.4 i) of Wall [3] can be interpreted as stating that the natural maps $L_{2i}(\mathbb{Z}[\pi]) \longrightarrow L_{2i}^S(\mathbb{Q}[\pi])$ are isomorphisms modulo 2-primary torsion. The results of Passman and Petrie [1] and Connolly [1] can be interpreted as stating that the natural maps $L_{2i+1}(\mathbb{Z}[\pi]) \longrightarrow L_{2i+1}^S(\mathbb{Q}[\pi])$ are isomorphisms modulo 8-torsion (π finite, $i(\text{mod } 2)$).

Results similar to those of Propositions 4.3, 4.4 were first obtained by Karoubi [1], for hermitian K-theory.

§5. Dedekind algebra

We shall now investigate the general properties of the L-groups $L_n(A,S,\varepsilon)$ (n(mod 4)) in the case when the ring with involution A is an algebra over a Dedekind ring R and $S = R-\{0\}$. An S-torsion A-module has a canonical direct sum decomposition as a direct sum of $\mathcal{P}$-primary S-torsion A-modules, with $\mathcal{P}$ ranging over all the (non-zero) prime ideals of R, and there is a corresponding decomposition for $L_n(A,S,\varepsilon)$.

Given a multiplicative subset $S \subset A$ in a ring with involution A we shall say that the pair (A,S) is a Dedekind algebra if $R = S \cup \{0\}$ is a Dedekind ring with respect to the ring operations inherited from A. The localization $S^{-1}A = F \otimes_R A$ is the induced algebra over the quotient field $F = S^{-1}R$. For example, a torsion-free ring with involution A is the same as a Dedekind algebra $(A,\mathbb{Z}-\{0\})$. A Dedekind ring with involution R is the same as a Dedekind algebra $(R,R-\{0\})$. In dealing with Dedekind algebras (A,S) and the prime ideals $\mathcal{P}$ of R we shall always exclude the case $\mathcal{P} = \{0\}$.

Let (A,S) be a Dedekind algebra.

The annihilator of an S-torsion A-module M is the ideal of R defined by

$$\mathrm{ann}(M) = \{s \in R \mid sM = 0\} \lhd R \ .$$

Like all ideals of R this has a unique expression as a product of powers of distinct prime ideals $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$

$$\mathrm{ann}(M) = \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_r^{k_r} \quad (k_i \geqslant 1) \ .$$

If M is such that the natural map $M \longrightarrow M^{\wedge\wedge}$ is an isomorphism (e.g. if M is h.d. 1) then

$$\mathrm{ann}(M^{\wedge}) = \overline{\mathrm{ann}(M)} \lhd R \ .$$

An S-torsion A-module M is $\mathcal{P}$-primary for some prime ideal $\mathcal{P}$ of R if

$$\mathrm{ann}(M) = \mathcal{P}^k \ .$$

for some $k \geqslant 1$.

Define the <u>localization of A at $\mathcal{P}$</u> for some prime ideal $\mathcal{P}$ of R to be the ring

$$A_{\mathcal{P}} = (R-\mathcal{P})^{-1}A \ .$$

If $\overline{\mathcal{P}} = \mathcal{P}$ there is defined an involution

$$\bar{\ } : A_{\mathcal{P}} \longrightarrow A_{\mathcal{P}} \ ; \ \frac{a}{r} \longmapsto \frac{\bar{a}}{\bar{r}} \qquad (a \in A, \ \ r \in R-\mathcal{P}) \ .$$

(If $\overline{\mathcal{P}} \neq \mathcal{P}$ there is defined an involution $\bar{\ } : A_{\mathcal{P}} \times A_{\overline{\mathcal{P}}} \longrightarrow A_{\mathcal{P}} \times A_{\overline{\mathcal{P}}} \ ; \ (x,y) \longmapsto (\bar{y},\bar{x})$).

Given an h.d. 1 S-torsion A-module M define an h.d. 1 $\mathcal{P}$-primary S-torsion A-module

$$M_{\mathcal{P}} = A_{\mathcal{P}} \otimes_A M \ .$$

If $\mathrm{ann}(M) = \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_r^{k_r}$ it is possible to identify

$$M_{\mathcal{P}} = \begin{cases} \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_{i-1}^{k_{i-1}}\mathcal{P}_{i+1}^{k_{i+1}}\ldots\mathcal{P}_r^{k_r}M & \text{if } \mathcal{P} = \mathcal{P}_i \text{ for some } i, \ 1 \leqslant i \leqslant r \\ 0 & \text{if } \mathcal{P} \notin \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r\} \end{cases}$$

so that

$$M = \bigoplus_{i=1}^{r} M_{\mathcal{P}_i} \quad , \quad (M^{\wedge})_{\mathcal{P}} = (M_{\overline{\mathcal{P}}})^{\wedge} \quad , \quad \mathrm{Hom}_A(M,M') = \bigoplus_{\mathcal{P}} \mathrm{Hom}_A(M_{\mathcal{P}},M'_{\mathcal{P}}) \ .$$

We thus have a canonical identification of exact categories

$$(\text{h.d. 1 S-torsion A-modules}) = \bigoplus_{\mathcal{P}} (\text{h.d. 1 } \mathcal{P}\text{-primary S-torsion A-modules}) \ ,$$

with $\mathcal{P}$ ranging over all the prime ideals of R. The S-duality functor $M \longmapsto M^{\wedge}$ sends the $\mathcal{P}$-primary component to the $\overline{\mathcal{P}}$-primary component.

Express the spectrum of prime ideals of R as a disjoint union

$$\mathrm{spec}(R) = \{\mathcal{P}\} \cup \{Q\} \cup \{\overline{Q}\}$$

with $\mathcal{P}$ ranging over all the prime ideals such that $\overline{\mathcal{P}} = \mathcal{P}$.

A non-singular split ε-quadratic linking $\begin{cases} \text{form} \\ \text{formation} \end{cases}$ over (A,S) $\begin{cases} (M,\lambda,\nu) \\ (F,G) \end{cases}$ has a canonical direct sum decomposition

$$\begin{cases} (M,\lambda,\nu) = \bigoplus_{\mathcal{P}} (M_{\mathcal{P}},\lambda_{\mathcal{P}},\nu_{\mathcal{P}}) \oplus \bigoplus_{Q} (M_Q \oplus M_{\overline{Q}},\lambda_Q,\nu_Q) \\ (F,G) = \bigoplus_{\mathcal{P}} (F_{\mathcal{P}},G_{\mathcal{P}}) \oplus \bigoplus_{Q} (F_Q \oplus F_{\overline{Q}},G_Q \oplus G_{\overline{Q}}) \end{cases} ,$$

such that for each Q

$$\begin{cases} (M_Q \oplus M_{\bar{Q}}, \lambda_Q, \nu_Q) = 0 \in L_0(A,S,\varepsilon) \\ (F_Q \oplus F_{\bar{Q}}, G_Q \oplus G_{\bar{Q}}) = 0 \in L_1(A,S,\varepsilon) \end{cases} .$$

For each prime ideal $\mathcal{P}$ of R such that $\bar{\mathcal{P}} = \mathcal{P}$ define the L-groups $L_n(A,\mathcal{P}^\infty,\varepsilon)$ (n(mod 4)) in the same way as $L_n(A,S,\varepsilon)$ but using only $\mathcal{P}$-primary h.d. 1 S-torsion A-modules. There is a natural identification

(h.d. 1 $\mathcal{P}$-primary S-torsion A-modules)

= (h.d. 1 $S_{\mathcal{P}}$-torsion $A_{\mathcal{P}}$-modules)

where $S_{\mathcal{P}} = \{\frac{s}{1} \in A_{\mathcal{P}} \mid s \in S\} \subset A_{\mathcal{P}}$, so that we can also identify

$$L_n(A,\mathcal{P}^\infty,\varepsilon) = L_n(A_{\mathcal{P}},S_{\mathcal{P}},\varepsilon) \quad (n(\text{mod } 4)) .$$

If $\mathcal{P} = \pi R$ is a prime ideal of R which is principal, with generator $\pi \in \mathcal{P}$, then $\bar{\pi} = \pi u \in \mathcal{P}$ for some unit $u \in R$ such that $u\bar{u} = 1 \in R$ and there is defined a multiplicative subset $S_\pi = \{\pi^j u^k \mid j \geqslant 0, k \in \mathbb{Z}\} \subset A$ such that

(h.d. 1 $\mathcal{P}$-primary S-torsion A-modules)

= (h.d. 1 S_π-torsion A-modules)

$$L_n(A,\mathcal{P}^\infty,\varepsilon) = L_n(A,S_\pi,\varepsilon) \quad (n(\text{mod } 4)) .$$

Proposition 5.1 The L-groups of a Dedekind algebra (A,S) have a canonical direct sum decomposition

$$L_n(A,S,\varepsilon) = \bigoplus_{\mathcal{P}} L_n(A,\mathcal{P}^\infty,\varepsilon) \quad (n(\text{mod } 4))$$

with $\mathcal{P}$ ranging over all the prime ideals of R such that $\bar{\mathcal{P}} = \mathcal{P}$.
The localization exact sequence of (A,S) can thus be expressed as

$$\ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \longrightarrow \bigoplus_{\mathcal{P}} L_n(A,\mathcal{P}^\infty,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots .$$

[]

The localization sequence in the case (A,S) = (R,R-{0})

$$\ldots \longrightarrow L_n(R,\varepsilon) \longrightarrow L_n(F,\varepsilon) \longrightarrow \bigoplus_{\mathcal{P}} L_n(R,\mathcal{P}^\infty,\varepsilon) \longrightarrow L_{n-1}(R,\varepsilon) \longrightarrow \ldots$$

is closely related to the original localization exact sequence of Milnor (Corollary IV.3.3 of Milnor and Husemoller [1]) for the symmetric Witt group of a Dedekind ring R

$$0 \longrightarrow L^0(R) \longrightarrow L^0(F) \longrightarrow \bigoplus_{\mathcal{P}} L^0(R/\mathcal{P}) .$$

(In the part of Ranicki [2] devoted to localization we shall extend this to an exact sequence

$$0 \longrightarrow L^0(R,\varepsilon) \longrightarrow L^0(F,\varepsilon) \longrightarrow \bigoplus_{\mathcal{P}} L^0(R/\mathcal{P},\varepsilon) \longrightarrow L^1(R,-\varepsilon) \longrightarrow 0$$

with $L^1(R,\varepsilon)$ the cobordism group of non-singular ε-symmetric formations over R). Now $L_1(F,\varepsilon) = 0$, so that the above sequence of quadratic L-groups breaks up into two sequences of the type

$$0 \longrightarrow \bigoplus_{\mathcal{P}} L_1(R,\mathcal{P}^\infty,\varepsilon) \longrightarrow L_0(R,\varepsilon) \longrightarrow L_0(F,\varepsilon) \longrightarrow \bigoplus_{\mathcal{P}} L_0(R,\mathcal{P}^\infty,\varepsilon) \longrightarrow L_1(R,-\varepsilon) \longrightarrow 0 \ .$$

A standard devissage argument shows that the forgetful functors

(f.d. vector spaces over the residue class field $R/\mathcal{P}$)

$\longrightarrow$ (h.d. 1 $\mathcal{P}$-primary S-torsion R-modules) ; $V \longmapsto V$

induce isomorphisms in algebraic K-theory and symmetric L-theory. There are induced morphisms in quadratic L-theory

$$L_n(R/\mathcal{P},\varepsilon) \longrightarrow L_n(R,\mathcal{P}^\infty,\varepsilon) \qquad (n(\text{mod } 4),\ \overline{\mathcal{P}} = \mathcal{P})$$

but these may not be isomorphisms (particularly if $R/\mathcal{P}$ is a field of characteristic 2, cf. Appendix 1 of Part II of Karoubi [1]). For example, neither of the morphisms

$$L_0(\mathbb{Z}_2,1) = \mathbb{Z}_2 \longrightarrow L_0(\mathbb{Z},(2\mathbb{Z})^\infty,1) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \ ;\ 1 \longmapsto (0,1)$$

$$L_1(\mathbb{Z}_2,-1) = 0 \longrightarrow L_1(\mathbb{Z},(2\mathbb{Z})^\infty,-1) = \mathbb{Z}_2$$

is an isomorphism.

Next, we shall describe the Mayer-Vietoris exact sequence of the L-groups of a localization-completion square of a Dedekind algebra (A,S)

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{S}^{-1}\hat{A} \end{array}$$

(Proposition 3.2) in terms of the prime ideal structure of the Dedekind ring $R = S \cup \{0\}$. We shall confine the discussion to the case when $\overline{\mathcal{P}} = \mathcal{P}$ for every prime ideal $\mathcal{P}$ of R, leaving the general case for the reader.

The <u>$\mathcal{P}$-adic completion of A</u> for some prime ideal $\mathcal{P}$ of R is the ring

$$\hat{A}_{\mathcal{P}} = \varprojlim_k A/\mathcal{P}^k A \ ,$$

with involution

$$\bar{\ } : \hat{A}_{\mathcal{P}} \longrightarrow \hat{A}_{\mathcal{P}} \ ;\ \{a_k \in A/\mathcal{P}^k A \mid k \geqslant 1\} \longrightarrow \{\bar{a}_k \in A/\mathcal{P}^k A \mid k \geqslant 1\} \ .$$

The $\mathcal{P}$-adic completion $\hat{A}_{\mathcal{P}}$ of A can be identified with the $S_{\mathcal{P}}$-adic completion of the localization $A_{\mathcal{P}}$ of A at $\mathcal{P}$

$$\hat{A}_{\mathcal{P}} = \varprojlim_{s\in S_{\mathcal{P}}} A_{\mathcal{P}}/sA_{\mathcal{P}} \; (= \varprojlim_{k} A/\pi^k A \text{ if } \mathcal{P} = \pi R,\ \pi\in\mathcal{P}).$$

Given $s\in S = R-\{0\}$ let $sR = \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_r^{k_r} \triangleleft R$, so that

$$A/sA = A/\mathcal{P}_1^{k_1}A\oplus A/\mathcal{P}_2^{k_2}A\oplus\ldots\oplus A/\mathcal{P}_r^{k_r}A$$

$$\frac{1}{s}\in\hat{A}_{\mathcal{P}}\subset\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}} \text{ if } \mathcal{P}\notin\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_r\}\ .$$

It is thus possible to define morphisms of rings with involution

$$\hat{A} = \varprojlim_{s\in S} A/sA \longrightarrow \prod_{\mathcal{P}}\hat{A}_{\mathcal{P}}$$

$$\hat{S}^{-1}\hat{A} \longrightarrow \coprod_{\mathcal{P}}(\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}},\hat{A}_{\mathcal{P}})$$

and hence also abelian group morphisms

$$L_n(\hat{A},\varepsilon) \longrightarrow \prod_{\mathcal{P}} L_n(\hat{A}_{\mathcal{P}},\varepsilon)$$

$$L_n^{\hat{S}}(\hat{S}^{-1}\hat{A},\varepsilon) \longrightarrow \coprod_{\mathcal{P}}(L_n^{\hat{S}_{\mathcal{P}}}(\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}},\varepsilon),L_n(\hat{A}_{\mathcal{P}},\varepsilon)) \quad (n(\mathrm{mod}\ 4))\ .$$

(The restricted product $\coprod_{\mathcal{P}}(G_{\mathcal{P}},H_{\mathcal{P}})$ of a collection of pairs of objects $(G_{\mathcal{P}},H_{\mathcal{P}})$ indexed by $\{\mathcal{P}\}$ and equipped with morphisms $H_{\mathcal{P}}\longrightarrow G_{\mathcal{P}}$ is defined to be the direct limit

$$\coprod_{\mathcal{P}}(G_{\mathcal{P}},H_{\mathcal{P}}) = \varinjlim_{I}(\prod_{\mathcal{P}\in I} G_{\mathcal{P}}\times\prod_{\mathcal{P}\notin I} H_{\mathcal{P}})$$

taken over all the finite subsets I of $\{\mathcal{P}\}$). Wall [5] and Bak [2] have studied some of the circumstances under which the above morphisms are isomorphisms, roughly speaking when A is finitely generated as an R-module and $S^{-1}A = F\otimes_R A$ is a semi-simple F-algebra (e.g. if $(A,S) = (\mathbb{Z}[\pi],\mathbb{Z}-\{0\})$ for a finite group π, with $R = \mathbb{Z}$). At any rate, it is possible to obtain a Mayer-Vietoris exact sequence relating the L-groups of $A,S^{-1}A$ to those of all the $\mathcal{P}$-adic completions $\hat{A}_{\mathcal{P}}$, $\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}}$. Propositions 3.2, 5.1 give morphisms of exact sequences

$$\begin{array}{ccccccccc}
\cdots \to L_n(A,\varepsilon) & \longrightarrow & L_n^S(S^{-1}A,\varepsilon) & \longrightarrow & \bigoplus_{\mathcal{P}} L_n(A,\mathcal{P}^\infty,\varepsilon) & \longrightarrow & L_{n-1}(A,\varepsilon) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to L_n(\hat{A},\varepsilon) & \longrightarrow & L_n^{\hat{S}}(\hat{S}^{-1}\hat{A},\varepsilon) & \longrightarrow & \bigoplus_{\mathcal{P}} L_n(\hat{A}_{\mathcal{P}},\hat{S}_{\mathcal{P}},\varepsilon) & \longrightarrow & L_{n-1}(\hat{A},\varepsilon) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow 1 & & \downarrow & \\
\cdots \to \prod_{\mathcal{P}} L_n(\hat{A}_{\mathcal{P}},\varepsilon) & \longrightarrow & \prod_{\mathcal{P}}(L_n^{\hat{S}_{\mathcal{P}}}(\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}},\varepsilon),L_n(\hat{A}_{\mathcal{P}},\varepsilon)) & \longrightarrow & \bigoplus_{\mathcal{P}} L_n(\hat{A}_{\mathcal{P}},\hat{S}_{\mathcal{P}},\varepsilon) & \longrightarrow & \prod_{\mathcal{P}} L_{n-1}(\hat{A}_{\mathcal{P}},\varepsilon) & \to \cdots
\end{array}$$

involving the isomorphisms

$$L_n(A,\mathcal{P}^\infty,\varepsilon) = L_n(A_{\mathcal{P}},S_{\mathcal{P}},\varepsilon) \longrightarrow L_n(\hat{A}_{\mathcal{P}},\hat{S}_{\mathcal{P}},\varepsilon) \quad (n(\text{mod } 4)) .$$

We deduce the following exact sequence, which is valid even in the case when the Dedekind ring R has prime ideals $\mathcal{P}$ such that $\overline{\mathcal{P}} \neq \mathcal{P}$.

<u>Proposition 5.2</u> Given a Dedekind algebra (A,S) there is defined a Mayer-Vietoris exact sequence

$$\cdots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n^S(S^{-1}A,\varepsilon) \oplus \prod_{\mathcal{P}} L_n(\hat{A}_{\mathcal{P}},\varepsilon) \longrightarrow \prod_{\mathcal{P}}(L_n^{\hat{S}_{\mathcal{P}}}(\hat{S}_{\mathcal{P}}^{-1}\hat{A}_{\mathcal{P}},\varepsilon),L_n(\hat{A}_{\mathcal{P}},\varepsilon)) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \cdots ,$$

with $\mathcal{P}$ ranging over all the prime ideals of $R = S \cup \{0\}$ such that $\overline{\mathcal{P}} = \mathcal{P}$.

[]

§6. Polynomial extensions

Given a central indeterminate x over a ring A there is defined a multiplicative subset $S = \{x^k \mid k \geqslant 0\} \subset A[x]$ with localization $S^{-1}A[x] = A[x,x^{-1}]$. An h.d. 1 S-torsion $A[x]$-module M is the same as a f.g. projective A-module M together with a nilpotent endomorphism $e: M \longrightarrow M; y \longmapsto xy$, and there is in fact a canonical identification of exact categories

(h.d. 1 S-torsion $A[x]$-modules M)

$=$ (f.g. projective A-modules M with a nilpotent endomorphism $e: M \longrightarrow M$).

As in Chapter XII of Bass [1] it is possible to combine this identification with the localization exact sequence of algebraic K-theory

$$K_1(A[x]) \longrightarrow K_1(A[x,x^{-1}]) \longrightarrow K_1(A[x],S) \longrightarrow K_0(A[x]) \longrightarrow K_0(A[x,x^{-1}])$$

to obtain split exact sequences

$$0 \longrightarrow K_1(A[x]) \longrightarrow K_1(A[x,x^{-1}]) \longrightarrow K_1(A[x],S) \longrightarrow 0$$

$$0 \longrightarrow K_1(A) \longrightarrow K_1(A[x]) \oplus K_1(A[x^{-1}]) \longrightarrow K_1(A[x,x^{-1}]) \longrightarrow K_0(A) \longrightarrow 0 \ ,$$

i.e. the 'fundamental theorem of algebraic K-theory'.

It is likewise possible to use an L-theoretic localization exact sequence to describe the L-groups of the polynomial extensions $A[x], A[x,x^{-1}]$ of a ring with involution A, where $\bar{x} = x$. Indeed, such was the approach taken by Karoubi [1]. On the other hand, we have already shown in Part IV of Ranicki [1] that there are defined split exact sequences

$$0 \longrightarrow L_n(A[x]) \longrightarrow L_n^S(A[x,x^{-1}]) \longrightarrow L_n^K(A[x^{-1}]) \longrightarrow 0$$

$$0 \longrightarrow L_n(A) \longrightarrow L_n^K(A[x]) \oplus L_n^K(A[x^{-1}]) \longrightarrow L_n^S(A[x,x^{-1}]) \longrightarrow L_n(A) \longrightarrow 0$$

$$(n(\text{mod } 4),\ K = \text{im}(\check{K}_0(A) \longrightarrow \widetilde{K}_0(A[x^{\pm 1}]))) \ ,$$

by a modification of Part II of Ranicki [1] (which concerned the L-theory of the Laurent extension $A[z,z^{-1}]$ of A, with $\bar{z} = z^{-1}$). We shall now explicitly identify

$$L_n(A[x],S,\varepsilon) = L_n^K(A[x^{-1}],\varepsilon) \quad (n(\text{mod } 4)) \ .$$

The Witt class of a non-singular split ε-quadratic linking form over $(A[x],S)$ corresponds to the Witt class of a non-singular ε-quadratic form over $A[x^{-1}]$, whereas ε-quadratic linking forms over $(A[x],S)$ correspond to even

ε-symmetric forms over $A[x^{-1}]$, so that the extra structure of split ε-quadratic linking forms over $(A[x],S)$ is seen to carry delicate quadratic information such as the Arf invariant.

The <u>polynomial extensions</u> of a ring with involution A are the rings $A[x], A[x^{-1}], A[x,x^{-1}]$ with involution by

$$\bar{x} = x \ .$$

Then $S = \{x^k \mid k \geqslant 0\} \subset A[x]$ is a multiplicative subset in the sense of §2, such that

$$S^{-1}A[x] = A[x,x^{-1}] \ , \ S^{-1}A[x]/A[x] = x^{-1}A[x^{-1}] = \sum_{j=-\infty}^{-1} x^j A \ .$$

Given an h.d. 1 S-torsion $A[x]$-module M we have a f.g. projective A-module together with a nilpotent endomorphism

$$e : M \longrightarrow M \ ; \ y \longmapsto xy \ ,$$

in which case the dual $M^* = \mathrm{Hom}_A(M,A)$ is a f.g. projective A-module with a nilpotent endomorphism

$$e^* : M^* \longrightarrow M^* \ ; \ f \longmapsto (y \longmapsto f(ey))$$

and there is defined a natural $A[x]$-module isomorphism

$$M^* \longrightarrow M^\wedge = \mathrm{Hom}_{A[x]}(M, S^{-1}A[x]/A[x]) \ ; \ f \longmapsto (y \longmapsto \sum_{j=-\infty}^{-1} x^j f(e^{-j-1}y)) \ .$$

Given h.d. 1 S-torsion $A[x]$-modules M, M' there is a natural identification

$$\mathrm{Hom}_{A[x]}(M,M') = \{ f \in \mathrm{Hom}_A(M,M') \mid fe = e'f \} \ .$$

An ε-symmetric linking form over $(A[x],S)$ (M,λ) is the same as a pair (M,e) (as above) together with an element $\varphi \in Q^{\varepsilon}(M)$ such that

$$\varphi e = e^*\varphi \in Q^{\varepsilon}(M) = \ker(1-T_{\varepsilon}:\mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*))$$
$$\lambda : M \times M \longrightarrow S^{-1}A[x]/A[x] \ ; \ (y,z) \longmapsto \sum_{j=-\infty}^{-1} x^j \varphi(y,e^{-j-1}z) \ .$$

An ε-quadratic linking form over $(A[x],S)$ (M,λ,μ) is the same as a triple (M,e,φ) (as above) such that both (M,φ) and $(M,\varphi e)$ are even ε-symmetric forms over A, that is

$$\varphi \ , \ \varphi e \in Q\langle v_0 \rangle^{\varepsilon}(M) \equiv \mathrm{im}(1+T_{\varepsilon}:Q_{\varepsilon}(M) \longrightarrow Q^{\varepsilon}(M)) \ ,$$

in which case

$$\mu : M \longrightarrow Q_{\varepsilon}(A[x],S) = S^{-1}A[x]/\{ b + \varepsilon\bar{b} \mid b \in A[x] \} \ ;$$
$$y \longmapsto \sum_{j=-\infty}^{-1} x^j \varphi(y,e^{-j-1}y) \ .$$

A split ε-quadratic linking form over $(A[x],S)$ (M,λ,ν) is the same as a triple (M,e,φ) (as above) together with elements $\psi_0,\psi_1 \in Q_\varepsilon(M)$ such that

$$\varphi = \psi_0 + \varepsilon\psi_0^* \quad , \quad \varphi e = \psi_1 + \varepsilon\psi_1^* \in Q\langle v_0\rangle^\varepsilon(M) \ ,$$

in which case

$$\nu : M \longrightarrow Q_\varepsilon(S^{-1}A[x]/A[x]) = \sum_{j=-\infty}^{-1} x^j Q_\varepsilon(A) \ ;$$

$$y \longmapsto \sum_{k=-\infty}^{-1} (x^{2k+1}\psi_0(y)(e^{-2k-2}y) + x^{2k}\psi_1(y)(e^{-2k-2}y)).$$

Define an abelian group morphism

$$L_0(A[x],S,\varepsilon) \longrightarrow L_0^K(A[x^{-1}],\varepsilon) \ ; \ (M,\lambda,\nu) \longmapsto (M[x^{-1}],\psi_0 + x^{-1}\psi_1) \ ,$$

where $M[x^{-1}] = A[x^{-1}]\otimes_A M$, $K = \mathrm{im}(\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(A[x^{-1}]))$.

A split ε-quadratic linking formation over $(A[x],S)$ $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is the same as an ε-quadratic formation over A $(H_\varepsilon(F);F,\mathrm{im}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G \longrightarrow F\oplus F^*))$ toogether with nilpotent endomorphisms $f \in \mathrm{Hom}_A(F,F)$, $g \in \mathrm{Hom}_A(G,G)$ such that

$$\gamma g = f\gamma \in \mathrm{Hom}_A(G,F) \ , \ \mu g = f^*\mu \in \mathrm{Hom}_A(G,F^*) \ , \ \gamma^*\mu g \in Q\langle v_0\rangle^{-\varepsilon}(G) \ ,$$

in which case

$$\theta : G \longrightarrow Q_{-\varepsilon}(A[x],S) \ ; \ y \longmapsto \sum_{j=-\infty}^{-1} x^j(\gamma^*\mu g^{-j-1}y) \ .$$

Define an abelian group morphism

$$L_1(A[x],S,\varepsilon) \longrightarrow L_1^K(A[x^{-1}],\varepsilon) \ ;$$

$$(F,G) \longmapsto (H_\varepsilon(F[x^{-1}]);F[x^{-1}],\mathrm{im}(\begin{pmatrix}\gamma\\ \mu(1+x^{-1}g)\end{pmatrix}:G[x^{-1}] \longrightarrow F[x^{-1}]\oplus F[x^{-1}]^*)).$$

In this way there are defined abelian group morphisms

$$L_n(A[x],S,\varepsilon) \longrightarrow L_n^K(A[x^{-1}],\varepsilon) \quad (n(\mathrm{mod}\ 4))$$

which fit into a morphism of exact sequences

$$\begin{array}{ccccccccc}
L_{n+1}(A[x],S,\varepsilon) & \longrightarrow & L_n(A[x],\varepsilon) & \longrightarrow & L_n^S(A[x,x^{-1}],\varepsilon) & \longrightarrow & L_n(A[x],S,\varepsilon) & \rightarrow & L_{n-1}(A[x],\varepsilon) \\
\downarrow & & \downarrow 1 & & \downarrow 1 & & \downarrow & & \downarrow \\
0 & \longrightarrow & L_n(A[x],\varepsilon) & \longrightarrow & L_n^S(A[x,x^{-1}],\varepsilon) & \longrightarrow & L_n^K(A[x^{-1}],\varepsilon) & \longrightarrow & 0
\end{array}$$

The top sequence is the localization sequence given by Proposition 2.4, while the bottom sequence is one of the split exact sequences obtained in the proof of Theorem 4.1 of Part IV of Ranicki [1] (- only the case $\varepsilon = \pm 1 \in A$ was considered there, but the proof generalizes to arbitrary $\varepsilon \in A$). We deduce:

Proposition 6.1 The abelian group morphisms

$$L_n(A[x],S,\varepsilon) \longrightarrow L_n^K(A[x^{-1}],\varepsilon) \qquad (n(\mathrm{mod}\ 4))$$

are isomorphisms.

[]

Define non-singular split (-1)-quadratic linking forms over $(\mathbb{Z}[x],S=\{x^k|k\geqslant 0\})$ (M,λ,ν), (M,λ,ν') by

$$M = \mathbb{Z}\oplus\mathbb{Z} \quad , \quad xM = 0$$

$$\lambda : M\times M \longrightarrow \mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x] \ ; \ ((y,z),(y',z'))\longmapsto x^{-1}(yz'-y'z)$$

$$\nu : M \longrightarrow Q_{-1}(\mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x]) = \mathbb{Z}[x,x^{-1}]/(\mathbb{Z}[x]+2\mathbb{Z}[x,x^{-1}]) \ ;$$
$$(y,z)\longmapsto x^{-1}(y^2+yz+z^2) \ ,$$

$$\nu' : M \longrightarrow Q_{-1}(\mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x]) \ ; \ (y,z)\longmapsto x^{-1}yz$$

with the same associated (-1)-quadratic linking form over $(\mathbb{Z}[x],S)$ (M,λ,μ)

$$\mu : M \longrightarrow Q_{-1}(\mathbb{Z}[x,x^{-1}],S) = \mathbb{Z}[x,x^{-1}] \ ; \ (y,z)\longmapsto 0 \ .$$

The isomorphism given by Proposition 6.1

$$L_0(\mathbb{Z}[x],S,-1) \longrightarrow L_0^K(\mathbb{Z}[x^{-1}],-1) = \mathbb{Z}_2\oplus ?$$

sends (M,λ,ν) to the element $(1,0)$ (= the image of the Arf invariant element $\left(\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\in Q_{-1}(\mathbb{Z}\oplus\mathbb{Z})\right)\in L_0(\mathbb{Z},-1)$ under the map induced by the natural inclusion $\mathbb{Z}\longrightarrow\mathbb{Z}[x^{-1}]$), while (M,λ,ν') is sent to 0. Thus split ε-quadratic linking forms carry more information than ε-quadratic linking forms, in general.

§7. Change of K-theory

We shall now describe the localization exact sequence for quadratic L-theory in the case when all the algebraic K-theory around is restricted to a prescribed $*$-invariant subgroup $X \subseteq \widetilde{K}_m(A)$ ($m = 0$ or 1).

Let A,S,ε be as in §2.

An h.d. 1 S-torsion A-module M has a projective class

$$[M] = [P_0] - [P_1] \in \widetilde{K}_0(A)$$

with P_0, P_1 the f.g. projective A-modules appearing in a resolution

$$0 \longrightarrow P_1 \xrightarrow{\ d\ } P_0 \longrightarrow M \longrightarrow 0 \ .$$

As $d \in \mathrm{Hom}_A(P_1,P_0)$ is an S-isomorphism $[M] \in \ker(\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(S^{-1}A))$.
Given a short exact sequence of h.d. 1 S-torsion A-modules

$$\mathcal{E} : 0 \longrightarrow M \xrightarrow{\ i\ } M' \xrightarrow{\ j\ } M'' \longrightarrow 0$$

there are defined f.g. projective A-module resolutions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\ d\ } & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_0} & & \big\downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & P'_1 & \xrightarrow{\ d'\ } & P'_0 & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle j_1} & & \big\downarrow{\scriptstyle j_0} & & \big\downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & P''_1 & \xrightarrow{\ d''\ } & P''_0 & \longrightarrow & M'' & \longrightarrow & 0 \ ,
\end{array}$$

and there exists a chain homotopy $k \in \mathrm{Hom}_A(P_0,P''_1)$ such that

$$j_0 i_0 = d''k \in \mathrm{Hom}_A(P_0,P''_0) \ , \ j_1 i_1 = kd \in \mathrm{Hom}_A(P_1,P''_1) \ .$$

Thus there is defined an acyclic f.g. projective A-module chain complex

$$C(\mathcal{E}) : 0 \longrightarrow P_1 \xrightarrow{\begin{pmatrix} d \\ -i_1 \end{pmatrix}} P_0 \oplus P'_1 \xrightarrow{\begin{pmatrix} i_0 & d' \\ k & j_1 \end{pmatrix}} P'_0 \oplus P''_1 \xrightarrow{(-j_0 \ \ d'')} P''_0 \longrightarrow 0 \ ,$$

giving the sum formula

$$[M] - [M'] + [M''] = 0 \in \widetilde{K}_0(A) \ .$$

The S-dual $M\hat{} = \mathrm{Hom}_A(M,S^{-1}A/A)$ of an h.d. 1 S-torsion A-module M has projective class

$$[M\hat{}] = [P_1^*] - [P_0^*] = -[M]^* \in \widetilde{K}_0(A) \ .$$

The <u>projective class</u> of a split ε-quadratic linking $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over (A,S) $\begin{cases}(M,\lambda,\nu)\\ (F,G)\end{cases}$ is defined to be

$$\begin{cases}[(M,\lambda,\nu)] = [M] \in \widetilde{K}_0(A)\\ [(F,G)] = [G] - [F^\wedge] \in \widetilde{K}_0(A)\ .\end{cases}$$

If $\begin{cases}(M,\lambda,\nu)\\ (F,G)\end{cases}$ is non-singular then

$$\begin{cases}[(M,\lambda,\nu)]^* = -[(M,\lambda,\nu)] \in \widetilde{K}_0(A)\\ [(F,G)]^* = [(F,G)] \in \widetilde{K}_0(A)\ .\end{cases}$$

Given a $*$-invariant subgroup $X \subseteq \widetilde{K}_0(A)$ let $L_n^X(A,S,\varepsilon)$ $(n(\mathrm{mod}\ 4))$ be the Witt groups of non-singular split $\pm\varepsilon$-quadratic linking forms and formations over (A,S) defined exactly as $L_n(A,S,\varepsilon)$, but using only h.d. 1 S-torsion A-modules with projective class in $X \subseteq \widetilde{K}_0(A)$. In particular,

$$L_n^{\widetilde{K}_0(A)}(A,S,\varepsilon) = L_n(A,S,\varepsilon) \quad (n(\mathrm{mod}\ 4))\ .$$

Define $*$-invariant subgroups

$$X^S = X \cap \ker(S^{-1}:\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(S^{-1}A)) \subseteq \widetilde{K}_0(A)$$
$$S^{-1}X = \{[S^{-1}P] \mid [P] \in X\} \subseteq \widetilde{K}_0(S^{-1}A)\ ,$$

so that there is defined a short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \longrightarrow X^S \longrightarrow X \longrightarrow S^{-1}X \longrightarrow 0$$

inducing a long exact sequence of Tate $\mathbb{Z}_2$-cohomology groups

$$\ldots \longrightarrow \hat{H}^n(\mathbb{Z}_2;X^S) \longrightarrow \hat{H}^n(\mathbb{Z}_2;X) \longrightarrow \hat{H}^n(\mathbb{Z}_2;S^{-1}X) \longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2;X^S) \longrightarrow \ldots\ .$$

The exact sequences of Propositions 1.2, 2.4 can be generalized to the intermediate projective L-groups, as follows.

<u>Proposition 7.1</u> Given $*$-invariant subgroups $X \subseteq Y \subseteq \widetilde{K}_0(A)$ there is defined a commutative diagram of abelian groups with exact rows and columns

$$\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & L_n^X(A,\varepsilon) & \longrightarrow & L_n^{S^{-1}X}(S^{-1}A,\varepsilon) & \longrightarrow & L_n^X(A,S,\varepsilon) & \longrightarrow & L_{n-1}^X(A,\varepsilon) & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & L_n^Y(A,\varepsilon) & \longrightarrow & L_n^{S^{-1}Y}(S^{-1}A,\varepsilon) & \longrightarrow & L_n^Y(A,S,\varepsilon) & \longrightarrow & L_{n-1}^Y(A,\varepsilon) & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;Y/X) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;S^{-1}Y/S^{-1}X) & \longrightarrow & \hat{H}^{n-1}(\mathbb{Z}_2;Y^S/X^S) & \longrightarrow & \hat{H}^{n-1}(\mathbb{Z}_2;Y/X) & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & L_{n-1}^X(A,\varepsilon) & \longrightarrow & L_{n-1}^{S^{-1}X}(S^{-1}A,\varepsilon) & \longrightarrow & L_{n-1}^X(A,S,\varepsilon) & \longrightarrow & L_{n-2}^X(A,\varepsilon) & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array} .$$

[]

In dealing with based A-modules we shall assume (as in §1) that f.g. free A-modules have a well-defined rank, and that $\tau(\varepsilon : A \longrightarrow A) = 0 \in \widetilde{K}_1(A)$.

An h.d. 1 S-torsion A-module M is <u>based</u> if there is given a f.g. free A-module resolution

$$0 \longrightarrow P_1 \xrightarrow{d} P_0 \longrightarrow M \longrightarrow 0$$

such that P_0 and P_1 are based, in which case there is defined a torsion

$$\tau_S(M) = \tau(S^{-1}d : S^{-1}P_1 \longrightarrow S^{-1}P_0) \in \widetilde{K}_1(S^{-1}A) \quad .$$

The S-dual $M^\wedge$ is also based, with torsion

$$\tau_S(M^\wedge) = \tau_S(M)^* \in \widetilde{K}_1(S^{-1}A) \ .$$

A short exact sequence of based h.d. 1 S-torsion A-modules

$$\mathcal{E} : 0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

has a torsion

$$\tau(\mathcal{E}) \equiv \tau(C(\mathcal{E})) \in \widetilde{K}_1(A)$$

such that

$$S^{-1}\tau(\mathcal{E}) = \tau_S(M) - \tau_S(M') + \tau_S(M'') \in \widetilde{K}_1(S^{-1}A) \ .$$

The torsion of a non-singular split ε-quadratic linking $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over (A,S) $\begin{cases}(M,\lambda,\nu)\\ (F,G)\end{cases}$ with $\begin{cases}M\\ F,G\end{cases}$ based is defined by

$$\begin{cases}\tau(M,\lambda,\nu) = (\tau(\mathcal{E}:0\longrightarrow M\xrightarrow{\lambda}M^\wedge\longrightarrow 0\longrightarrow 0),\tau_S(M))\\ \quad\in \ker\left(\begin{pmatrix}1+T & 0\\ -S^{-1} & 1-T\end{pmatrix}: \tilde{K}_1(A)\oplus\tilde{K}_1(S^{-1}A)\longrightarrow\tilde{K}_1(A)\oplus\tilde{K}_1(S^{-1}A)\right)\\ \tau(F,G) = (\tau(\mathcal{E}:0\longrightarrow G\xrightarrow{\binom{\gamma}{\mu}}F\oplus F^\wedge\xrightarrow{(\varepsilon\mu^\wedge\ \gamma^\wedge)}G^\wedge\longrightarrow 0),\tau_S(G)-\tau_S(F^\wedge))\\ \quad\in \ker\left(\begin{pmatrix}1-T & 0\\ -S^{-1} & 1+T\end{pmatrix}: \tilde{K}_1(A)\oplus\tilde{K}_1(S^{-1}A)\longrightarrow\tilde{K}_1(A)\oplus\tilde{K}_1(S^{-1}A)\right)\ ,\end{cases}$$

with $T:x\longmapsto x^*$ the duality involution.

Given $*$-invariant subgroups $X\subseteq\tilde{K}_1(A)$, $Y\subseteq\tilde{K}_1(S^{-1}A)$ such that

$$S^{-1}X \equiv \{\tau(S^{-1}f)\in\tilde{K}_1(S^{-1}A)\,|\,\tau(f)\in X\}\subseteq Y$$

let $L_n^{X,Y}(A,S,\varepsilon)$ $(n(\text{mod } 4))$ be the Witt groups of non-singular split $\pm\varepsilon$-quadratic linking forms and formations over (A,S) defined exactly as $L_n(A,S,\varepsilon)$, but using only based h.d. 1 S-torsion A-modules and requiring the torsions to lie in

$$\{(x,y)\in X\oplus Y\,|\,x^* = (-)^{n-1}x\ ,\ S^{-1}x = y+(-)^{n-1}y^*\}\subseteq\tilde{K}_1(A)\oplus\tilde{K}_1(S^{-1}A)$$

In particular,

$$L_n^{\tilde{K}_1(A),\tilde{K}_1(S^{-1}A)}(A,S,\varepsilon) = L_n^{\{0\}\subseteq\tilde{K}_0(A)}(A,S,\varepsilon)\quad (n(\text{mod } 4))\ .$$

Given a morphism of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$f : G\longrightarrow H$$

define relative Tate $\mathbb{Z}_2$-cohomology groups

$$\hat{H}^n(\mathbb{Z}_2;f:G\longrightarrow H) = \frac{\{(x,y)\in G\oplus H\,|\,x^*=(-)^{n-1}x\,,fx=y+(-)^{n-1}y^*\}}{\{(u+(-)^{n-1}u^*,fu+v+(-)^n v^*)\,|\,(u,v)\in G\oplus H\}}\qquad (n(\text{mod } 2))$$

to fit into a long exact sequence

$$\ldots\longrightarrow\hat{H}^n(\mathbb{Z}_2;G)\xrightarrow{f}\hat{H}^n(\mathbb{Z}_2;H)\longrightarrow\hat{H}^n(\mathbb{Z}_2;f)\longrightarrow\hat{H}^{n-1}(\mathbb{Z}_2;G)\longrightarrow\ldots\ .$$

The exact sequences of Propositions 1.2, 2.4, 7.1 can be generalized to the intermediate torsion L-groups, as follows.

Proposition 7.2 Given $*$-invariant subgroups $X \subseteq X' \subseteq \widetilde{K}_1(A)$, $Y \subseteq Y' \subseteq \widetilde{K}_1(S^{-1}A)$ such that $S^{-1}X \subseteq Y$, $S^{-1}X' \subseteq Y'$ there is defined a commutative diagram of abelian groups with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & & L_n^X(A,\varepsilon) & \longrightarrow & L_n^Y(S^{-1}A,\varepsilon) & \longrightarrow & L_n^{X,Y}(A,S,\varepsilon) & \longrightarrow & L_{n-1}^X(A,\varepsilon) & \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & & L_n^{X'}(A,\varepsilon) & \longrightarrow & L_n^{Y'}(S^{-1}A,\varepsilon) & \longrightarrow & L_n^{X',Y'}(A,S,\varepsilon) & \longrightarrow & L_{n-1}^{X'}(A,\varepsilon) & \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & & \hat{H}^n(\mathbb{Z}_2;X'/X) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;Y'/Y) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;X'/X \to Y'/Y) & \longrightarrow & \hat{H}^{n-1}(\mathbb{Z}_2;X'/X) & \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & & L_{n-1}^X(A,\varepsilon) & \longrightarrow & L_{n-1}^Y(S^{-1}A,\varepsilon) & \longrightarrow & L_{n-1}^{X,Y}(A,S,\varepsilon) & \longrightarrow & L_{n-2}^X(A,\varepsilon) & \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & .
\end{array}
$$

[]

Let

$$V_n(A,\varepsilon) = L_n^{\{0\} \subseteq \widetilde{K}_0(A)}(A,\varepsilon) = L_n^{\widetilde{K}_1(A)}(A,\varepsilon) \quad (n(\bmod 4))$$

be the L-groups defined using only f.g. free A-modules, and let

$$V_n(A,S,\varepsilon) = L_n^{\{0\} \subseteq \widetilde{K}_0(A)}(A,S,\varepsilon) = L_n^{\widetilde{K}_1(A),\widetilde{K}_1(S^{-1}A)}(A,S,\varepsilon) \ (n(\bmod 4))$$

be the L-groups defined using only h.d. 1 S-torsion A-modules which admit a f.g. free A-module resolution of length 1. As a special case of either of the localization sequences of Propositions 7.1, 7.2 we have an exact sequence of V-groups

$$\dots \longrightarrow V_n(A,\varepsilon) \longrightarrow V_n(S^{-1}A,\varepsilon) \longrightarrow V_n(A,S,\varepsilon) \longrightarrow V_{n-1}(A,\varepsilon) \longrightarrow \dots .$$

For example, the localization exact sequence of Pardon [1] is of this type.

The excision isomorphisms and the Mayer-Vietoris exact sequence for the L-theory of the cartesian square

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ f\downarrow & & \downarrow f \\ B & \longrightarrow & T^{-1}B \end{array}$$

associated to a cartesian morphism $f:(A,S)\longrightarrow(B,T)$ (Proposition 3.1) can be generalized as follows.

Proposition 7.3 Let $f:(A,S)\longrightarrow(B,T)$ be a cartesian morphism of rings with involution and multiplicative subsets, and let $X\subseteq\widetilde{K}_m(A)$, $Y\subseteq\widetilde{K}_m(S^{-1}A)$, $Z\subseteq\widetilde{K}_m(B)$, $W\subseteq\widetilde{K}_m(T^{-1}B)$ ($m = 0$ or 1) be $*$-invariant subgroups such that

$$S^{-1}X\subseteq Y\ ,\ B\otimes_A X\subseteq Z\ ,\ T^{-1}Z\subseteq W\ ,\ T^{-1}B\otimes_{S^{-1}A}Y\subseteq W,\ \ker(\widetilde{K}_m(A)\to\widetilde{K}_m(S^{-1}A)\oplus\widetilde{K}_m(B))\subseteq X$$

and such that the sequence

$$0\longrightarrow X/\ker(\widetilde{K}_m(A)\longrightarrow\widetilde{K}_m(S^{-1}A)\oplus\widetilde{K}_m(B))\longrightarrow Y\oplus Z\longrightarrow W\longrightarrow 0$$

is exact. Then there are defined excision isomorphisms

$$f\ :\ L_n^{X,Y}(A,S,\varepsilon)\longrightarrow L_n^{Z,W}(B,T,\varepsilon)\quad (n(\mathrm{mod}\ 4))$$

and a Mayer-Vietoris exact sequence

$$\ldots\longrightarrow L_n^X(A,\varepsilon)\longrightarrow L_n^Y(S^{-1}A,\varepsilon)\oplus L_n^Z(B,\varepsilon)\longrightarrow L_n^W(T^{-1}B,\varepsilon)\longrightarrow L_{n-1}^X(A,\varepsilon)\longrightarrow\ldots\ .$$

[]

(In the case $m = 0$ the groups $L_n^{X,Y}(A,S,\varepsilon)$ are to be interpreted as the relative groups $L_n^{X,Y}(A\longrightarrow S^{-1}A,\varepsilon)$ appearing in the exact sequence

$$\ldots\longrightarrow L_n^X(A,\varepsilon)\longrightarrow L_n^Y(S^{-1}A,\varepsilon)\longrightarrow L_n^{X,Y}(A\longrightarrow S^{-1}A,\varepsilon)\longrightarrow L_{n-1}^X(A,\varepsilon)\longrightarrow\ldots\ .$$

For $Y = S^{-1}X$ these are the groups defined previously

$$L_n^{X,S^{-1}X}(A,S,\varepsilon) = L_n^X(A,S,\varepsilon)\quad (n(\mathrm{mod}\ 4))\ ,$$

but for general X,Y it is not possible to express these relative L-groups in terms of linking forms and formations over (A,S)).

For example, the Mayer-Vietoris sequence of Theorem 6.6 of Wall [6] is a special case of the sequence of Proposition 7.3, with $(B,T) = (\hat{A},\hat{S})$ and

$$X = \ker(\widetilde{K}_1(A)\longrightarrow\widetilde{K}_1(\hat{S}^{-1}\hat{A}))\ ,\ Y = \ker(\widetilde{K}_1(S^{-1}A)\longrightarrow\widetilde{K}_1(\hat{S}^{-1}\hat{A}))$$

$$Z = \ker(\widetilde{K}_1(\hat{A})\longrightarrow\widetilde{K}_1(\hat{S}^{-1}\hat{A}))\ ,\ W = \{0\}\subseteq\widetilde{K}_1(\hat{S}^{-1}\hat{A})\ .$$

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