Knot theory from Vandermonde to Jones

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Abstract. Leibniz wrote in 1679: "I consider that we need yet another kind of analysis, ... which deals directly with position." He called it "geometry of position"(geometria situs). The first convincing example of geometria situs was Euler's solution to the bridges of Königsberg problem (1735). The first mathematical paper which mentions knots was written by A. T. Vandermonde in 1771: "Remarques sur les problemes de situation". We will sketch in this essay [1] the history of knot theory from Vandermonde to Jones stressing the combinatorial aspect of the theory that is so visible in Jones type invariants.

> *"When Alexander reached Gordium, he was seized with a longing to ascend to the acropolis, where the palace of Gordius and his son Midas was situated, and to see Gordius' wagon and the knot of the wagon's yoke.... Over and above this there was a legend about the wagon, that anyone who untied the knot of the yoke would rule Asia. The knot was of cornel bark, and you could not see where it began or ended. Alexander was unable to find how to untie the knot but unwilling to leave it tied, in case this caused a disturbance among the masses; some say that he struck it with his sword, cut the knot, and said it was now untied - but Aristobulus says that he took out the pole-pin, a bolt driven right through the pole, holding the knot together, and so removed the yoke from the pole. I cannot say with confidence what Alexander actually did about this knot, but he and his suite certainly left the wagon with the impression that the oracle about the undoing of the knot had been fulfilled, and in fact that night there was thunder and lightning, a further sign from heaven; so Alexander in thanksgiving offered sacrifice next day to whatever gods had shown the signs and the way to undo the knot."* [Lucius Flavius Arrianus, Anabasis Alexandri, Book II, c.150 A.D.]

[1] This essay is an extended version of talks given at U.C. Riverside (April 18, 1991), at the Mexican National Congress of Mathematicians (November 1991) [93], and at the Minisemester on Knot Theory at Banach Center (July 18, 1995).

The goal is to present the history of ideas which lead up to the development of modern knot theory. We will try to be more detailed when pre-XX century history is reported. With more recent times we are more selective, stressing developments related to Jones type invariants of links. I am not a professional historian so my account is *amateurish* but I hope it can be treated as an invitation to more thorough exposition. [2]

Knots have fascinated people from the dawn of the human history. We can wonder what caused a merchant living about 1700 BC. in Anatolia and exchanging goods with Mesapotamians, to choose braids and knots as his seal sign; Fig.1. We can guess however that knots on seals or cylinders appeared before proper writing was invented about 3500 BC.

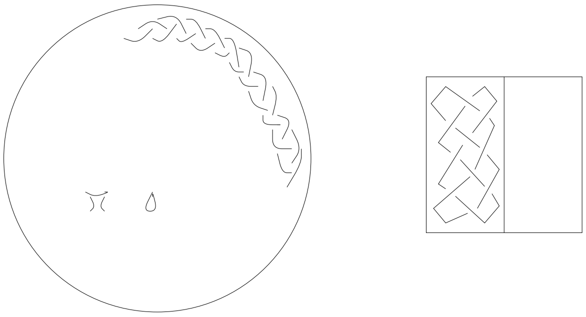


Figure 1. Stamp seal, about 1700 BC (the British Museum).

On the octagonal base [of hammer-handled haematite seal] are patterns surrounding a hieroglyphic inscription (largely erased). Four of the sides are blank and the other four are engraved with elaborate patterns typical of the period (and also popular in Syria) alternating with cult scenes...([22], p.93).

It is tempting to look for the origin of knot theory in Ancient Greek mathematics (if not earlier). There is some justification to do so: a Greek physician named Heraklas, who lived during the first century A.D. and who was likely a pupil or associate of Heliodorus, wrote an essay on surgeon's slings[3]. Heraklas explains, giving step-by-step instructions, eighteen ways to tie orthopedic slings. His work survived because Oribasius of Pergamum (ca. 325-400; physician of the emperor Julian the Apostate) included it toward the end of the fourth century in his "Medical Collections". The oldest extant manuscript of

[2]There are books which treat the history of topics related to knot theory [7, 21, 27, 36]. J.Stillwell's textbook [107] contains very interesting historical digressions.

[3]Heliodorus, who lived at the time of Trajan, also mentions in his work knots and loops [101]

"Medical Collections" was made in the tenth century by the Byzantine physician Nicetas. The Codex of Nicetas was brought to Italy in the fifteenth century by an eminent Greek scholar, J. Lascaris, a refugee from Constantinople. Heraklas' part of the Codex of Nicetas has no illustrations, and around 1500 an anonymous artist depicted Heraklas' knots in one of the Greek manuscripts of Oribasus "Medical Collections" (in Figure 2 we reproduce the drawing of the third Heraklas knot together with its original, Heraklas', description). Vidus Vidius (1500-1569), a Florentine who became physician to Francis I (king of France, 1515-1547) and professor of medicine in the Collège de France, translated the Codex of Nicetas into Latin; it contains also drawings of Heraklas' surgeon's slings by the Italian painter, sculptor and architect Francesco Primaticcio (1504-1570); [28, 97].

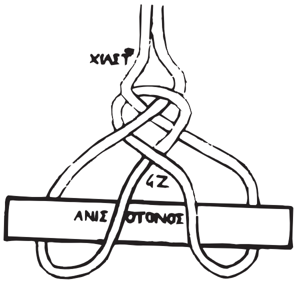


Figure 2. The crossed noose

"For the tying the crossed noose, a cord, folded double, is procured, and the ends of the cord are held in the left hand, and the loop is held in the right hand. Then the loop is twisted so that the slack parts of the cord crossed. Hence the noose is called crossed. After the slack parts of the cord have been crossed, the loop is placed on the crossing, and the lower slack part of the cord is pulled up through the middle of the loop. Thus the knot of the noose is in the middle, with a loop on one side and two ends on the other. This likewise, in function, is a noose of unequal tension"; [28].

Heraklas' essay should be taken seriously as far as knot theory is concerned. It is not knot theory proper but rather its application. The story of the survival of Heraklas' work; and efforts to reconstruct his knots in Renaissance is typical of all science disciplines and efforts to recover lost Greek books provided the important engine for development of modern science. This was true in Mathematics as well: the beginning of modern calculus in XVII century can be traced to efforts of reconstructing lost books of Archimedes and other ancient Greek mathematicians. It was only the work of Newton and Leibniz which went much farther than their Greek predecessors.

Gottfried Wilhelm Leibniz (1646-1716) speculated that aside from calculus and ana-

lytical geometry there should exist a "geometry of position" (geometria situs) which deals with relations depending on position alone (ignoring magnitudes). In a letter to Christian Huygens (1629-1695), written in 1679 [72], he declared:
"I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude".

I do not know whether Leibniz had any convincing example of a problem belonging to the geometry of position. According to [70]:
"As far back as 1679 Leibniz, in his Characteristica Geometrica, tried to formulate basic geometric properties of geometrical figures, to use special symbols to represent them, and to combine these properties under operations so as to produce others. He called this study analysis situs or geometria situs... To the extent that he was at all clear, Leibniz envisioned what we now call combinatorial topology".

The first convincing example of geometria situs was studied by Leonard Euler (1707-1783). This concerns the bridges on the river Pregel at Königsberg (then in East Prussia). Euler solved (and generalized) the bridges of Königsberg problem and on August 26, 1735 presented his solution to the Russian Academy at St. Petersburg (it was published in 1736), [40]. With the Euler paper, graph theory and topology were born. Euler started his paper by remarking:
"The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position" (geometria situs). This branch of geometry deals with relations dependent on position; it does not take magnitudes into considerations, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them".

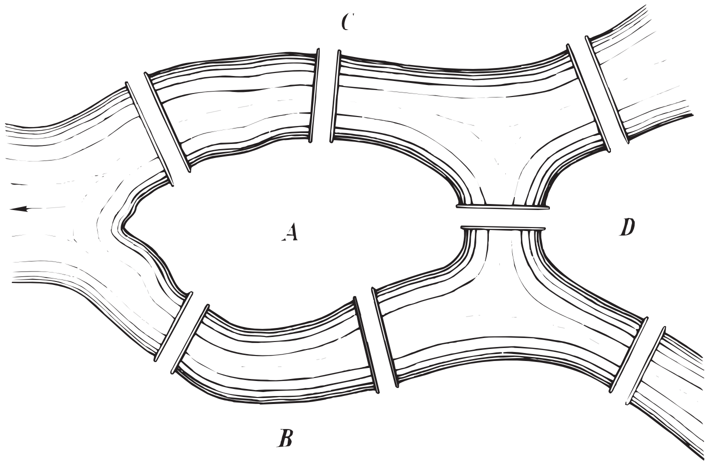


Figure 3. Bridges of Königsberg

For the birth of knot theory one had to wait another 35 years. In 1771 Alexandre-Theophile Vandermonde (1735-1796) wrote the paper: *Remarques sur les problèmes de situation* (Remarks on problems of positions) where he specifically places braids and knots as a subject of the geometry of position [126]. In the first paragraph of the paper Vandermonde wrote:

Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced.

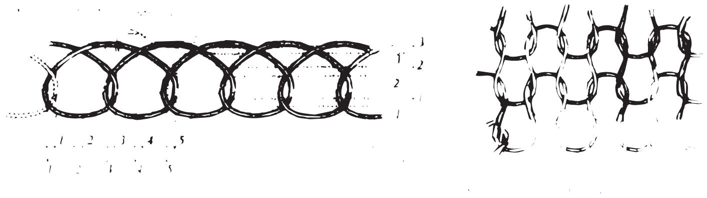


Figure 4. Knots of Vandermonde

In our search for the origin of knot theory, we arrive next at Carl Friedrich Gauss (1777-1855). According to [106, 39] :

"One of the oldest notes by Gauss to be found among his papers is a sheet of paper with the date 1794. It bears the heading "A collection of knots" and contains thirteen neatly sketched views of knots with English names written beside them... With it are two additional pieces of paper with sketches of knots. One is dated 1819; the other is much later, ...". [4]

In July of 1995 I finally visited the old library in Göttingen, I looked at knots from 1794 - in fact not all of them are drawn - some only described (see Fig. 5 for one of the drawings).

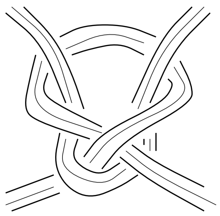
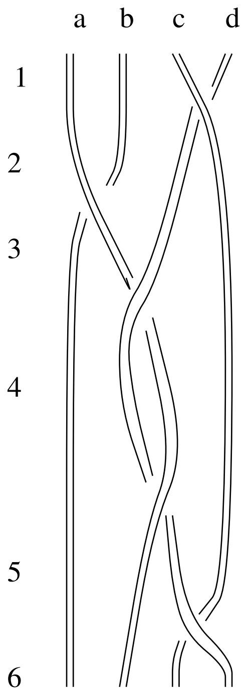


Figure 5. *Meshing knot*, 10'th knot of Gauss from 1794.

There are other fascinating drawings in Gauss' notebooks. For example, the drawing of a braid (with analytic description at each height), Fig.6, and the note that it is a good method of coding a knotting. It is difficult to date the drawing; one can say for sure that it was done between 1814 and 1830, I would guess closer to 1814. [5]

[4]According to [48], the first English sailing book with pictures of knots appeared in 1769 [41].

[5]As a curiosity one can add that of one of the notebooks (Handb. 3) in which Gauss had also drawn some knot diagrams has braids motives on its cover.

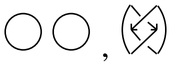


Veraindrungder Coordiniz

a	1	1	2+i	3+i	2+2i	2+2i
b	2	2	1	1	1	1
c	3	4	4	4	4	3
d	4	3+i	3+i	2+2i	3+2i	4+3i

Figure 6.

It is a good method of coding a knotting (from a Gauss' notebook (Handb.7)).

In his note (Jan. 22 1833) Gauss introduces the linking number of two knots[6]. Gauss' note presents the first deep incursion into knot theory; it establishes that the following two links are substantially different: . Gauss' analytical method has been recently revitalized by Witten's approach to knot theory [133].

James Clerk Maxwell (1831-1879), in his fundamental book of 1873 "A treatise on electricity & magnetism" [77] writes [7] : "It was the discovery by Gauss of this very integral, expressing the work done on a magnetic pole while describing a closed curve in presence of a closed electric current, and indicating the geometrical connection between the two closed curves, that led him to lament the small progress made in the Geometry of Position since the time of Leibnitz, Euler and Vandermonde. We have now, however, some progress to report chiefly due to Riemann, Helmholtz and Listing."[8] Maxwell goes on to describe two

[6]His method is analytical - the Gauss integral; in modern language Gauss integral computes analytically the degree of the map from a torus parameterizing a 2 component link to the unit 2-sphere.

[7]The note of Gauss was first published in his collected works in 1867.

[8]Gauss wrote in 1833, in the same note in which he introduced the linking number: "On the geometry

closed curves which cannot be separated but for which the value of the Gauss integral is equal to zero; Fig.7.

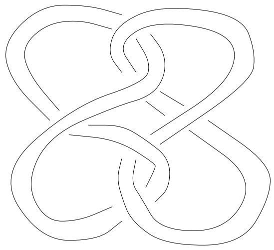
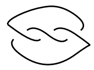


Figure 7. The link of Maxwell.

In 1876, O.Boeddicker observed that, in a certain sense, the linking number is the number of the crossing points of the second curve with a surface bounded by the first curve [13, 14, 15]. Hermann Karl Brunn[9] [19] observed more than hundred years ago that the linking number of a two-component link, considered by Gauss, can be read from a diagram of the link[10]. If the link has components K_1 and K_2, we consider any diagram of the link and count each point at which K_1 crosses under K_2 as $+1$ for  and -1 for . The sum of these, over all crossings of K_1 under K_2, is the Gauss linking number.

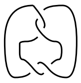
The year of 1847 was very important for the knot theory (graph theory and topology as well). On one hand, Gustav Robert Kirchoff (1824-1887) published his fundamental paper on electrical circuits [67]. It has deep connections with knot theory, however the relations were discovered only about a hundred years later (e.g. the Kirchoff complexity of a circuit corresponds to the determinant of the knot or link determined by the circuit). On the other hand, Johann Benedict Listing (1808-1882), a student of Gauss, published his monograph (Vorstudien zur Topologie, [73]). A considerable part of the monograph is devoted to knots[11]. Listing stated in particular that the right handed trefoil knot () and the left handed trefoil knot () are not equivalent. Later Listing showed that the

of position, which Leibniz initiated and to which only two geometers, Euler and Vandermonde, have given a feeble glance, we know and possess, after a century and a half, very little more than nothing."

[9]Born August 1, 1862 Rome (Italy), died Sept. 20, 1939 (Munchen, Germany)[11].

[10]It is also noted by Tait in 1877 ([111], page 308).

[11]Earlier, on 1 April 1836, Listing wrote a letter from Catania to "Herr Muller", with the heading "Topology", concerning ... (2) winding paths of knots; and (3) paths in a lattice [17, 106].

figure eight knot  and its mirror image  are equivalent (we say that the figure eight knot, also called the Listing knot, is amphicheiral)[12].

As we mentioned before, Maxwell, in his study of electricity and magnetism, had some thoughts on knots and links (in particular motivated by the freshly published Gauss' collected works). The origin of modern knot theory should be associated with four physicists: Hermann Von Helmholtz (1821-1894), William Thomson (Lord Kelvin) (1824-1907), Maxwell and Peter Guthrie Tait (1831-1901). We can quote after Tait's assistant in Edinburgh and later biographer, C.G.Knott [71]:

Tait was greatly impressed with Helmholtz's famous paper on vortex motion [[53]; 1858]... Early in 1867 he devised a simple but effective method of producing vortex smoke rings; and it was when viewing the behaviour of these in Tait's Class Room that Thomson was led to the conception of the vortex atom. In his first paper to the Royal Society of Edinburgh on February 18, 1867 [[119]], Sir William Thomson refers... to the genesis of the conception. In turn Thomson's theory was Tait's motivation to understand the structure of knots. In Tait's words: *I was led to the consideration of the form of knots by Sir W. Thomson's Theory of Vortex Atoms, and consequently the point of view which, at least at first, I adopted was that of classifying knots by the number of their crossings... The enormous number of lines in the spectra of certain elementary substances show that, if Thomson's suggestion be correct, the form of the corresponding vortex atoms cannot be regarded as very simple[[111]].*

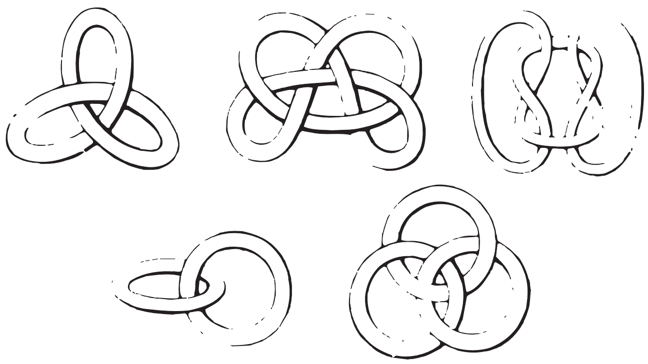


Figure 8. Knots and links of William Thomson (Kelvin) from 1867.

[12]This was observed in the note dated March 18, 1849 [75]

There is an interesting letter from Maxwell to Tait dated Nov. 13, 1867, which shows that Tait was sharing his ideas of knots with his friend [71, 76]. In one of his rhymes Maxwell wrote (clearly referring to Tait):

Clear your coil of kinkings
Into perfect plaiting,
Locking loops and linkings
Interpenetrating.
[71]

Tait describes his work on knots in the following words ([111],1877): *When I commenced my investigations I was altogether unaware that anything had been written (from a scientific point of view) about knots. No one in Section A at the British Association of 1876, when I read a little paper[[110]] on the subject, could give me any reference; and it was not till after I had sent my second paper to this Society that I obtained, in consequence of a hint from Professor Clerk-Maxwell, a copy of the very remarkable Essay by Listing*[13]*, Vorstudien zur Topologie[[73]], of which (so far as it bears upon my present subject) I have given a full abstract in the Proceedings of the Society for Feb. 3, 1877. Here, as was to be expected, I found many of my results anticipated, but I also obtained one or two hints which, though of the briefest, have since been very useful to me. Listing does not enter upon the determination of the number of distinct form of knots with a given number of intersections, in fact he gives only a very few forms as examples, and they are curiously enough confined to three, five and seven crossings only; but he makes several very suggestive remarks about the representation of a particular class of "reduced" knots... This work of Listing's and an acute remark made by Gauss (which with some comments on it by Clerk-Maxwell, will be referred to later), seem to be all of any consequence that has been as yet written on the subject.* Tait's paper was revised May 11, 1877; he finishes the paper as follows: *After the papers, of which the foregoing is a digest, had been read, I*

[13] In 1883 Tait wrote in *Nature* obituary after Listing death [112]: *One of the few remaining links that still continued to connect our time with that in which Gauss had made Göttingen one of the chief intellectual centres of the civilised world has just be broken by the death of Listing... This paper [Vorstudien zur Topologie], which is throughout elementary, deserves careful translation into English....* After more than a hundred years the paper is still not translated (only the Tait summary exists [113]) and one should repeat Tait's appeal again: the paper very much deserves translation. One can add that in 1932 the paper was translated into Russian.

obtained from Professor Listing[14] *and Klein a few references to the literature of the subject of knots. It is very scanty, and has scarcely any bearing upon the main question which I have treated above. Considering that Listing's Essay was published thirty years ago, and that it seems to be pretty well known in Germany, this is a curious fact. From Listing's letter (Proc. R.S.E.. 1877, p.316), it is clear that he has published only a small part of the results of his investigations. Klein himself [69] has made the very singular discovery that in space of four dimensions there cannot be knots.*

The value of Gauss's integral has been discussed at considerable length by Boeddicker ... in an Inaugural Dissertation, with the title Beitrag zur Theorie des Winkels*, Göttingen, 1876.*

An inaugural Dissertation by Weith, Topologische Untersuchung der Kurven-Verschlingung, Zürich, 1876 [129], is professedly based on Listing's Essay. It contains a proof that there is an infinite number of different forms of knots![15] *The author points out what he (erroneously) supposes to be mistakes in Listing's Essay; and, in consequence, gives as something quite new an illustration of the obvious fact that there can be irreducible knots in which the crossing are not alternately over and under* [16]*. The rest of this paper is devoted to the relations of knots to Riemann's surfaces.*

Tait, in collaboration with Reverend Thomas Penyngton Kirkman (1806-1895), and independently Charles Newton Little[17], made a considerable progress on the enumeration problem so that by 1900 there were in existence tables of (prime) knots up to ten crossings [111, 68, 74, 75]. These tables were partially extended in M.G. Haseman's doctoral dissertation of 1916[18], [51]. Knots up to 11 crossings were enumerated by John H. Conway [24] before 1969 [19].

[14] Library of the University of California has a copy of *Vorstudien zur Topologie* which Listing sent to Tait with the dedication.

[15] In fact it was proven only 20-30 years later and depended on the fundamental work of Poincaré on foundation of algebraic topology.

[16] It was proven only in 1930 by Bankwitz [6], using the determinant of a knot

[17] Born Madura India, May 19, 1858. A.B., Nebraska 1879, A.M. 1884; Ph.D, Yale, 1885. Instructor math. and civil eng, Nebraska, 1880-84, assoc.prof.civil eng.84-90, prof, 90-93; visited Göttingen and Berlin, 1898-1899; math, Stanford, 1893-1901; civil eng, Moscow, Idaho, from 1901 , dean, col. eng, from 1911., died August 31, 1923 [2, 134].

[18] Mary Gertrude Haseman was the fifth doctoral student of C.A.Scott at Bryn Mawr College.

[19] K.A.Perko, a student of Fox at Princeton, and later a lawyer in New York City, observed a duplication

Figure 9. Knots of Perko.

Knots up to 13 crossings were enumerated by C .H .Dowker and M. B. Thistlethwaite [38, 114], 1983 [20]. To be able to make tables of knots, Tait introduced three basic principles (called now the Tait conjectures). All of them have been recently solved. The use of the Jones polynomial makes the solution of the first two Tait conjectures astonishingly easy [83, 115, 63] and the solution of the third Tait conjecture also uses essentially Jones type polynomials [79, 80]. We formulate these conjectures below:

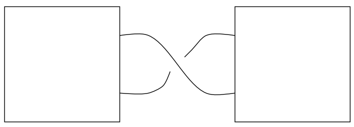
T1. An alternating diagram with no nugatory crossings, of an alternating link realizes the minimal number of crossings among all diagrams representing the link. A nugatory crossing is drawn (defined) in Figure 10(a).

T2. Two alternating diagrams, with no nugatory crossings, of the same oriented link have the same Tait (or writhe) number, i.e. the signed sum of all crossings of the diagram with the convention that is $+1$ and is -1.

T3. Two alternating diagrams, with no nugatory crossings, of the same link are related by a sequence of flypes (see Figure 10(b)).

in the tables: two 10-crossing diagrams represented the same knot, see Figure 9. Perko corrected also the Conway's eleven crossing tables: 4 knots were missed [85, 87].

[20]Jim Hoste has informed me that he and, independently, M.B.Thistlethwaite are working on the extension up to 15 crossings of the existing knots' tables. They found already that there are 19 536 prime alternating, 14 crossing knots and 85 263 prime alternating, 15 crossing knots. Census for all knots (not necessary alternating) is not yet verified but suggests over 40 000 knots of 14 crossings and over 200 000 knots of 15 crossings For prime alternating, 16 crossing knots the preliminary number obtained by Hoste is 379 799. Knots and their mirror images are not counted separately [56].


(a)

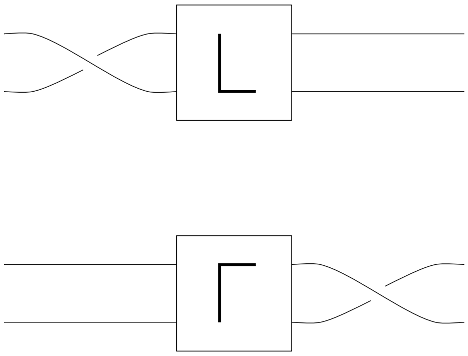

(b)

Figure 10.

A very interesting survey on the developments in knot theory in XIX century can be found in the Dehn and Heegaard article in the Mathematical Encyclopedia [35], 1907[21]. In this context, the papers of Oscar Simony from Vienna [22] are of great interest [105, 121]. Figure 11 describes torus knots of Simony.

[21] According to Dehn's wife, Mrs. Toni Dehn, *Dehn and Heegaard met at the third International Congress of Mathematicians at Heidelberg in 1904 and took to each other immediately. They left Heidelberg on the same train, Dehn going to Hamburg and Heegaard returning to Copenhagen. They decided on the train that an Encyclopedia article on topology would be desirable, that they would propose themselves as authors to the editors, and that Heegaard would take care of literature whereas Dehn would outline a systematic approach which would lay the foundations of the discipline [78].*

[22] Born April 23, 1852 in Vienna, died in 1915 [89].

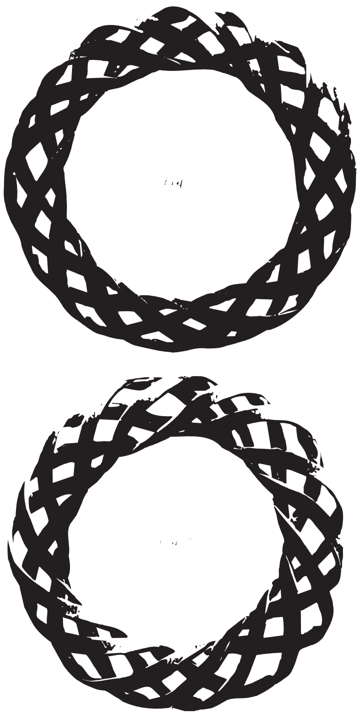


Figure 11. Torus knots of Simony from 1884.

The fundamental problem in knot theory is to be able to distinguish non-equivalent knots. It was not achieved (even in the simple case of the unknot and the trefoil knot) until Jules Henri Poincaré (1854-1912) in his “Analysis Situs” paper ([90] 1895) laid foundations for algebraic topology. Poul Heegaard (1871-1948) in his Copenhagen Dissertation of 1898 ([52]) constructed the 2-fold branch cover of a trefoil knot and showed that it is the lens space, $L(3,1)$, in modern terminology. He also showed that the analogous branch cover of the unknot is S^3. He distinguished $L(3,1)$ from S^3 using the Betti numbers (more precisely he showed that the first homology group is nontrivial and he clearly understood that it is a 3-torsion group). He didn’t state however that it can be used to distinguish the trefoil knot from the unknot; see [107] p.226.[23] Heinrich Tietze (1880-1964) used in 1908 the fundamental group of the exterior of a knot in R^3, called the knot group, to distinguish the unknot from the trefoil knot [120]. The fundamental group was essentially[24] introduced by Poincaré in his 1895 paper [90].

[23] For the English translation of the topological part of the Heegaard thesis see the appendix to [94].

[24] According to [21] Hurwitz’ paper of 1891 [57] “may very well be interpreted as giving the first approach to the idea of a fundamental group of a space of arbitrarily many dimensions.”

Wilhelm Wirtinger (1865-1945) in his lecture delivered at a meeting of the German Mathematical Society in 1905 outlined a method of finding a knot group presentation (it is called now the Wirtinger presentation of a knot group) [132]. Max Dehn (1878-1952), in his 1910 paper [31] refined the notion of the knot group and in 1914 was able to distinguish the right handed trefoil knot ( ) from its mirror image, the left handed trefoil knot ( ); that is Dehn showed that the trefoil knot is not amphicheiral [32] [25].

Tait was the first to notice the relation between knots and planar graphs. He colored the regions of the knot diagram alternately white and black (following Listing) and constructed the graph by placing a vertex inside each white region, and then connecting vertices by edges going through the crossing points of the diagram (see Figure 12)[35].

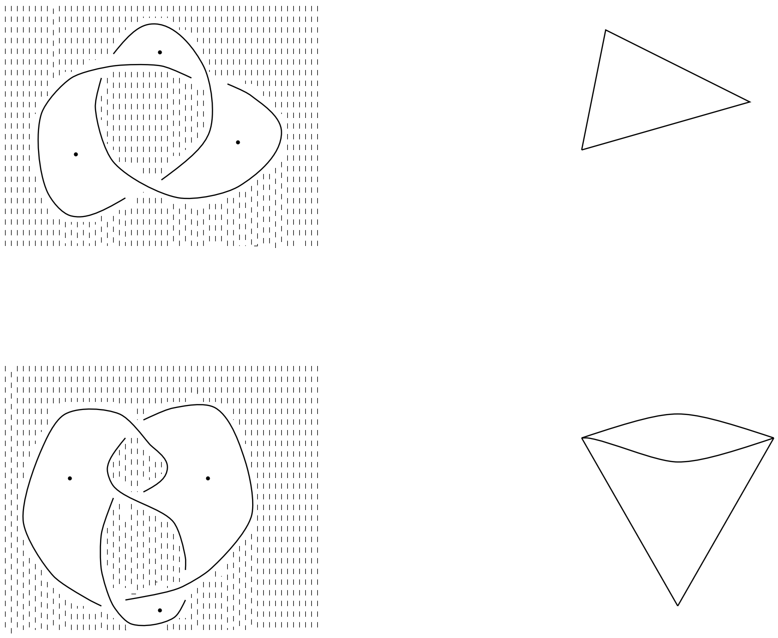


Figure 12.

In 1912, George David Birkhoff (1884-1944) when trying to prove the four-color problem (formulated in 1852 by Francis Guthrie (1831-1899)), introduced the chromatic poly-

[25]In 1978, W.Magnus wrote [78]: *Today, it appears to be a hopeless task to assign priorities for the definition and the use of fundamental groups in the study of knots, particularly since Dehn had announced [30] one of the important results of his 1910 paper (the construction of Poincaré spaces with the help of knots) already in 1907.*

nomial of a graph [8].

The breakthrough, from the point of view which focuses on the Jones type link invariants, was the invention by James Waddell Alexander (1888-1971) of a Laurent polynomial invariant of links ([1], 1928). Alexander was a colleague of Birkhoff and we can conjecture that he knew about the chromatic polynomial.[26] We know for sure that when W. T. Tutte was generalizing the chromatic polynomial in 1947 [125], he was motivated by the knot polynomial of Alexander. The Alexander polynomial can be derived from the group of the knot (or link). This point of view has been extensively developed since Alexander's discovery. More generally, the study of the fundamental group of a knot complement and the knot complement alone was the main topic of research in knot theory for the next fifty years, culminating in 1988 in the proof of Tietze [120] conjecture (that a knot is determined by its complement) by Gordon and Luecke [46]. We can refer to the survey articles by Ralph Hartzler Fox (1913 -1973) [42] and Gordon [45] or books [12, 20, 64, 98, 100] in this respect. However Alexander observed also that if three oriented links, L_+, L_- and L_0, have diagrams which are identical except near one crossing where they look as in Figure 13, then their polynomials are linearly related [1]. An analogous discovery about the chromatic polynomial of graphs was made by Ronald M. Foster in 1932 (see [130]; compare also [9] Formula (10)). In early 1960's, J. Conway rediscovered Alexander's formula and normalized the Alexander polynomial, $\Delta_L(t) \in Z[t^{\pm 1/2}]$, defining it recursively as follows ([24]):

(i) $\Delta_o(t) = 1$, where o denotes a knot isotopic to a simple circle

(ii)

$$\Delta_{L_+} - \Delta_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}$$

[26]Birkhoff writes in [10] : "...Alexander, then [1911] a graduate student [at Princeton], began to be especially interested in the subject [analysis situs]. His well known "duality theorem," his contributions to the theory of knots, and various other results, have made him a particularly important worker in the field". We can also mention that in the fall of 1909 Birkhoff became a member of the faculty at Princeton and left for Harvard in 1912. His 1912 paper [8] ends with "Princeton University, May 4, 1912."

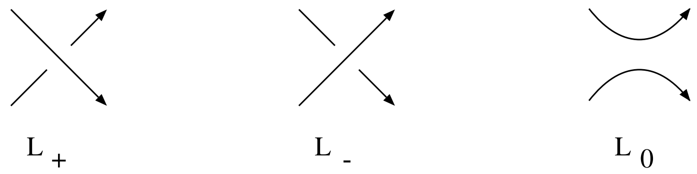


Figure 13

In the spring of 1984, Vaughan Jones discovered his invariant of links, $V_L(t)$ [58, 59, 60][27], and still in May of 1984 he was trying various substitutions to the variable t, in particular $t = -1$. He observed that $V_L(-1)$ is equal to the classical knots invariant – determinant of a knot, but, initially, was unable to prove it. Soon he realized that his polynomial satisfies the local relation analogous to that discovered by Alexander and Conway and established the meaning of $t = -1$.[28] Thus the Jones polynomial is defined recursively as follows:

(i) $V_o = 1$,

(ii) $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$.

In the summer and the fall of 1984, the Alexander and the Jones polynomials were generalized to the skein (named also Conway-Jones, Flypmoth, Homfly, Homflypt, generalized Jones, 2-variable Jones, Jones-Conway, Thomflyp, twisted Alexander) polynomial, $P_L \in Z[a^{\pm 1}, z^{\pm 1}]$, of oriented links [43, 96]. This polynomial is defined recursively as follows [43, 96]:

[27]Jones wrote in [61]: "It was a warm spring morning in May, 1984, when I took the uptown subway to Columbia University to meet with Joan S. Birman, a specialist in the theory of "braids"... In my work on von Neumann algebras, I had been astonished to discover expressions that bore a strong resemblance to the algebraic expression of certain topological relations among braids. I was hoping that the techniques I had been using would prove valuable in knot theory. Maybe I could even deduce some new facts about the Alexander polynomial. I went home somewhat depressed after a long day of discussions with Birman. It did not seem that my ideas were at all relevant to the Alexander polynomial or to anything else in knot theory. But one night the following week I found myself sitting up in bed and running off to do a few calculations. Success came with a much simpler approach than the one that I had been trying. I realized I had generated a polynomial invariant of knots."

[28]The relation was also discovered independently in July 1984 by Lickorish and Millett.

(i) $P_o = 1$;

(ii) $aP_{L_+} + a^{-1}P_{L_-} = zP_{L_0}$.

In particular $\Delta_L(t) = P_L(i, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$, $V_L(t) = P_L(it^{-1}, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$. In August 1985 L. Kauffman found another approach to the Jones polynomial [29]. It starts from an invariant, $< D > \in Z[\mu, A, B]$, of an unoriented link diagram D called the Kauffman bracket [63]. The Kauffman bracket is defined recursively by:

(i)

$$< \underbrace{o \dots o}_{i} > = \mu^{i-1}$$

(ii)

$$< L_+ > = A < L_0 > + B < L_\infty >$$

(iii)

$$< L_- > = B < L_0 > + A < L_\infty >$$

where L_+, L_-, L_0 and L_∞ denote four diagrams that are identical except near one crossing as shown in Figure 14, and $< \underbrace{o \dots o}_{i} >$ denotes a diagram of i trivial components (i simple circles).

If we assign $B = A^{-1}$ and $\mu = -(A^2 + A^{-2})$ then the Kauffman bracket gives a variant of the Jones polynomial for oriented links. Namely, for $A = t^{-\frac{1}{4}}$ and D being an oriented diagram of L we have

$$V_L(t) = (-A^3)^{-w(D)} < D > \tag{1}$$

where $w(D)$ is the *planar writhe* (*twist* or *Tait number*) of D equal to the algebraic sum of signs of crossings.

It should be noted, as observed first by Kauffman, that bracket $< >_{\mu,A,B}$ is an isotopy invariant of alternating links (and their connected sums) under the assumption that the third Tait conjecture (recently proven by Menasco and Thistlethwaite [79, 80]) holds.

[29] First he thought that he produced a new knot polynomial and only analyzing the polynomial he realized that he found a variant of the Jones polynomial.

In the summer of 1985 (two weeks before discovering the "bracket"), L. Kauffman invented another invariant of links [65], $F_L(a,z) \in Z[a^{\pm 1}, z^{\pm 1}]$, generalizing the polynomial discovered at the beginning of 1985 by Brandt, Lickorish, Millett and Ho [16, 55]. To define the Kauffman polynomial we first introduce the polynomial invariant of link diagrams $\Lambda_D(a,z)$. It is defined recursively by:

(i) $\Lambda_o(a,z) = 1$,

(ii) $\Lambda(a,z) = a\Lambda_{|}(a,z)$; $\Lambda(a,z) = a^{-1}\Lambda_{|}(a,z)$,

(iii) $\Lambda_{D_+}(a,z) + \Lambda_{D_-}(a,z) = z(\Lambda_0(a,z) + \Lambda_{D_\infty}(a,z))$.

The Kauffman polynomial of oriented links is defined by

$$F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$$

where D is any diagram of an oriented link L.

Figure 14

Jones type invariants lead to invariants of three-dimensional manifolds [5, 133, 92, 122, 99, 124, 123, 95]. We mentioned already that Jones type invariants of knots have been used to solve Tait conjectures. The Jones discovery however not only introduced a delicate method of analyzing knots in 3-manifolds but related knot theory to other disciplines of mathematics and theoretical physics, for example statistical mechanics, quantum field theory, operator algebra, graph theory and computational complexity. On the other hand the Jones polynomial gives a simple tool to recognize knots and as such is of great use for biologists (e.g. for analysis of DNA) and chemists (see for example [109]).

Biographical Notes

We give below a chronological list of selected people mentioned in the article (compare [7]:

Gottfried Wilhelm Leibniz (1646-1716)
Leonhard Euler (1707-1783)
Alexandre-Theophile Vandermonde (1735-1796)
Carl Friedrich Gauss (1777-1855)
Thomas Penyngton Kirkman (1806-1895)
Johann Benedict Listing (1808-1882)
Hermann Von Helmholtz (1821-1894)
Gustav Robert Kirchoff (1824-1887)
William Thomson (Lord Kelvin) (1824-1907)
Peter Guthrie Tait (1831-1901)
James Clerk Maxwell (1831-1879)
Oscar Simony (1852-1915)
Jules Henri Poincaré (1854-1912)
Charles Newton Little (1858-1923)
Hermann Karl Brunn (1862- 1939)
Wilhelm Wirtinger (1865-1945)
Paul Heegaard (1871-1948)
Max Dehn (1878-1952)
Heinrich Tietze (1880-1964)
George David Birkhoff (1884-1944)
James Waddell Alexander (1888-1971)
Kurt W.F. Reidemeister (1893-1971)
Ralph Hartzler Fox (1913-1973)

References

[1] J. W. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* 30 (1928) 275-306.

[2] American Men of Science, 1921, 3'rd edition, Edited by Catell, J.McKeen Brimhall, Dean R. Garrison, NY, The Science Press.

[3] T.Archibald, Connectivity and Smoke-Rings: Green's Second Identity in Its First Fifty Years, *Math.Magazine*, 62(2) 1989, 219-232.

[4] L. F. Arrian, *Anabasis Alexandri, Book II*, c.150 AD (English translation P.A. Brunt, Harvard University Press, 1976).

[5] M. Atiyah, *The geometry and physics of knots*, Cambridge Univ. Press, 1990, 75 pp.

[6] C. Bankwitz, Über die Torsionszahlen der alternierenden Knoten, *Math. Ann.*, 103, 1930, 145-161.

[7] N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph theory 1736-1936*, Clarendon Press, Oxford, 1986.

[8] G.D.Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. of Math.* (2)14, 1912, 42-46.

[9] G.D.Birkhoff, On the polynomial expressions for the number of ways of coloring a map, *Annali, Pisa, Scoula Normali Superiore*, s.2, v.3 (1934), 85-104. Also in George David Birkhoff Collected Mathematical Papers, Vol.3, AMS (1950), 29-47.

[10] G.D.Birkhoff, Fifty years of American Mathematics, Amer. Math. Soc., *Semicentennial Addresses*, v. 2, 1938, 270-315. Also in George David Birkhoff Collected Mathematical Papers, Vol.3, AMS (1950), 606-651.

[11] W.Blaschke, Obituary: Hermann Brunn. (German) Jber. Deutsch. Math. Verein. 50, (1940). 163–166.

[12] J.S. Birman, *Braids,Links and mapping class groups*, Ann.Math. Studies 82, Princeton N.J.: Princeton Univ. Press, 1974.

[13] O. Boeddicker, Beitrag zur Theorien des Winkels, Dissertation Göttingen 1876.

[14] O. Boeddicker, Erweiterung der Gausschen Theorie der Verschlingungen, Stuttgart, 1876.

[15] M.Bognár, *Foundations of linking theory*, Akadémiai Kiadó, Budapest 1992.

[16] R. D. Brandt, W. B. R.Lickorish, K. C. Millett, A polynomial invariant for unoriented knots and links, *Invent. Math.* 84 (1986), 563-573.

[17] E. Breitenberger, Gauss und Listing: Topologie und Freundschaft. (German) [Gauss and Listing: topology and friendship] Gauss-Ges. Göttingen Mitt. No. 30 (1993), 3–56.

[18] A.Brill, Math. Ann. 18, 1881.

[19] H.Brunn, Topologische Betrachtungen, *Zeitschrift für Mathematik und Physik*, 37 (1892), 106-116.

[20] G. Burde, H. Zieschang, *Knots*, De Gruyter (1985).

[21] B. Chandler, W. Magnus, The history of Combinatorial group theory: A case study in the history of ideas, *Studies in the History of Mathematics and Physical Sciences* 9, Springer-Verlag, 1982.

[22] D. Collon, *Ancient Near Eastern Art*, University of California Press, Berkeley, Los Angeles, 1995.

[23] Comprehensive Dissertation Index, 1861-1972, Volume 5: Mathematics and Statistics, Xerox University Microfilms, Ann Arbor, Michigan, 1973.

[24] J. H. Conway, An enumeration of knots and links, *Computational problems in abstract algebra* (ed. J.Leech), Pergamon Press (1969) 329 - 358.

[25] P.Cromwell, E.Beltrami, M.Rampichini, The Borromean rings, preprint 1997.

[26] W. Crönert, *Sprachliches zu griechischen Ärzten, Archiv für Papyrus-forschungen und Verwandte Gebiete*, II (1902-1903), 474-482.

[27] M.J.Crowe, *A history of vector analysis: The evolution of the idea of a vectorial system*, Dover Publications, 1994 (first publ. 1967).

[28] C. L. Day, *Quipus and Witches' Knots, With a Translation and Analysis of "Oribasius De Laqueis"*, The University of Kansas Press, Lawrence 1967.

[29] , R.Dedekind, H.Weber, Theorie der algebraischen Funktionen einer Veränderlichen, *J. de Crelle*, XCII (1882), 181-290. (Also in R.Dedekind, Ges.Math.Werke,t. I, p. 248-349).

[30] M. Dehn, *Jahresbericht der deutschen Mathematiker–Vereinigung*, Vol.16, 1907, p.573.

[31] M. Dehn, Über die Topologie des dreidimensionalen Raumes, *Math. Annalen.* 69 (1910), 137-168.

[32] M. Dehn, Die beiden Kleeblattschlingen, *Math.Annalen.* 75 (1914), 1-12.

[33] M. Dehn, The mentality of the mathematician. A characterization. *Math. Intelligencer*, 5 (1983), no. 2, 18–26.

[34] M. Dehn, Papers on group theory and topology. (English) Translated from the German and with introductions and an appendix by John Stillwell. With an appendix by Otto Schreier. Springer-Verlag, New York-Berlin, 1987. viii+396 pp

[35] M.Dehn, P.Heegaard, Analysid situs, *Encykl. Math. Wiss.*, vol. III AB3 Leipzig, 1907, 153-220.

[36] J. A. Dieudonné, *A history of algebraic and differential topology 1900-1960*, Birkhäuser, Boston, Basel, 1989.

[37] Dilgendey, *Topologische Studien*, Leipzig, 1890.

[38] C. H. Dowker, M. B. Thistlethwaite, Classification of knot projections, *Topology and its applications 16 (1983)*, 19-31.

[39] G. W. Dunnington, *Carl Friedrich Gauss, Titan of Science*, Hafner Publishing Co., New York, 1955.

[40] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 8 (1736), 128-140.

[41] W.Falconer, *A universal dictionary of the marine*, London 1769.

[42] R.H. Fox, A quick trip through knot theory, *In: Top. 3-manifolds*, Proc. 1961 Top.Inst.Univ. Georgia (ed. M.K.Fort, jr), 120-167. Englewood Cliffs. N.J.: Prentice-Hall, 1962.

[43] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.*, 12 (1985) 239-249.

[44] K.F. Gauss, Zur mathematischen Theorie der electrodynamischen Wirkungen, 1833, *Werke*, Königliche Gesellschaft der Wissenschaften zu Gottingen, 5 (1877), 602-629.

[45] C.McA. Gordon, Some aspects of classical knot theory, *In: Knot theory*, L.N.M. 685 (1978) 1-60.

[46] C.McA. Gordon, J. Luecke, Knots are Determined by their Complements, *Bulletin of the A.M.S*, 20 (1989), 83-87.

[47] P. van de Griend, The history of knot theory, preprint 18, Aarhus University, April 1992.

[48] P. van de Griend, Notes on Knots, Århus, March 1993.

[49] B.Grünbaum, G.C.Shephard, Interlace patterns in Islamic and Moorish art, ”The Visual Mind - Art and Mathematics”. Edited by Michele Emmer, MIT Press (1993); Chapter 22), 147-155. (Articles from “Leonardo”).

[50] P. de la Harpe, M.Kervaire, C.Weber, On the Jones polynomial, *L'enseignement Mathématique*, 32 (1986), 271-335.

[51] M. G. Haseman, On knots, with a census of the amphicheirals with twelve crossings, *Trans. Roy. Soc. Edinburgh*, 52 (1918), 235-255 (also Ph.D thesis, Bryn Mawr College, 1918 [23]).

[52] P.Heegaard, Forstudier til en Topologisk Teori for de algebraiske Fladers Sammenhæng, København, 1898, Filosofiske Doktorgrad; French translation: Sur l'Analysis situs, *Soc. Math. France Bull.*, 44 (1916), 161-242.

[53] H.Helmholtz, Über Integrale der hydrodynamischen Gleichungen, Welche den Wirbelbewegungen entsprechen, *Crelle's Journal für Mathematik*, 55, 1858, 25-55; English translation by P.G.Tait: *Philosophical Magazine*, June 1867, 485-511.

[54] Heraklas, first century A.D., included in Oribasius ”Medical collections”, fourth century A.D.

[55] C. F. Ho, A new polynomial for knots and links; preliminary report. *Abstracts AMS* 6(4) (1985), p 300.

[56] J. Hoste, e-mails, 2 and 7 Sept. 1993, and 9 June 1994.

[57] A.Hurwitz, Über Riemannsche Flächen mit gegebenen Verzweigungspunkten, *Math. Annalen*, 39, (1891), 1-61.

[58] V. Jones, Letter to J.Birman, May 1984.

[59] V. Jones, A polynomial invariant for knots via Von Neumann algebras, *Bull. Amer. Math. Soc* 12 (1985) 103-111.

[60] V. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (1987) 335-388.

[61] V. Jones, Knot theory and statistical mechanics, *Scientific American*, Vol. 262(11), 1990, 98-103.

[62] L.H. Kauffman, *Formal knot theory*, Mathematical Notes 30, Princeton University Press, 1983.

[63] L.H. Kauffman, State models and the Jones polynomial, *Topology 26* (1987) 395-407.

[64] L.H. Kauffman, *On Knots*, Princeton University Press, 1987.

[65] L.H. Kauffman, An invariant of regular isotopy, *Trans. Amer. Math. Soc.*, 318(2), 1990, 417–471.

[66] A.E.Kennelly, The equivalence of triangles and three-pointed stars in conducting networks, *Electrical World and Engineer*, 34, 1899, 413-414.

[67] G. R. Kirchoff, Über die Auflosung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer strome geführt wird, *Annalen d. Physik und Chemie* 72 (1847) 497-508.

[68] T.P.Kirkman, The enumeration description and construction of knots with fewer than 10 crossing, Trans. R. Soc. Edinburgh, 32, 1885, 281-309.

[69] F. Klein, *Mathematische Annalen*, IX, 478.

[70] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.

[71] C.G. Knott, *Life and Scientific work of Peter Guthrie Tait*, Supplementing the two volumes of scientific papers published in 1898 and 1900, Cambridge University Press, 1911.

[72] G. W. Leibniz, *Math. Schriften,* 1 Abt., Vol 2, (1850), 19-20.

[73] J. B. Listing, *Vorstudien zur Topologie*, Göttinger Studien (Abtheilung 1) 1 (1847), 811-875.

[74] C.N. Little, On knots, with a census for order 10, *Trans. Connecticut Acad. Sci.*, 18 (1885), 374-378 (also Ph.D. thesis, 1885, Yale University [23]).

[75] C.N. Little, Non-alternate $\pm$ knots, *Trans. Royal. Soc. Edinburgh*, XXXIX, Part III (No. 30), for the session 1898-89 (read July 3, 1899), 771-778.

[76] S.L.Lomonaco, The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, Proceedings of Symposia in Applied Mathematics, Vol. 51, 1996, 145-166.

[77] J.C.Maxwell, *A treatise on electricity and magnetism*, Oxford 2 (1873).

[78] W. Magnus, Max Dehn. *Math. Intelligencer* 1 (1978), 132–143. (Also in: Wilhelm Magnus Collected Paper, Edited by G.Baumslag and B.Chandler, Springer-Verlag, 1984).

[79] W. M. Menasco, M.B.Thistlethwaite, The Tait Flyping Conjecture, *Bull. Amer. Math. Soc.* 25 (2) (1991) 403-412.

[80] W. M. Menasco, M.B.Thistlethwaite, The classification of alternating links, *Annals of Math.*, 138 (1993), 113-171.

[81] Fr. Meyer, Dissertation, München 1878. Also Edinb. Proc., 13, 1886.

[82] L. G. Miller, The earliest(?) description of a string figure, *The American Anthropologist*, XLVII (1945), 461-462.

[83] K. Murasugi, Jones' polynomials and classical conjectures in knot theory, *Topology*, 26 (1987), 187-194.

[84] L.Olesko (ed.), Science in Germany, Osiris (2nd series) 5 (1989).

[85] Perko, Kenneth A., Jr. On the classification of knots, *Proc. Amer. Math. Soc.*, 45 (1974), 262–266.

[86] K.A.Perko, NAMS 25(6), 1978.

[87] K.A.Perko, Primality of certain knots, *Topology Proceedings*, vol. 7, 1982, 109-118.

[88] Plutarch of Chareonera, The age of Alexander, translated by I. Scott-Kilvert, Penguin Books, 1982.

[89] Poggendorff's Biographisch Literarisches Handwörterbuch, Vol. 3, 1858-83 (A-Z), 1898.

[90] H. Poincaré, Analysis Situs (&12), *Journal d'Ecole Polytechnique Normale*, 1 (1895), 1-121.

[91] H. Poincaré, First Complément à l'analysis situs, 1899; in Œuvres, vol VI, Gauthier-Villars, Paris, 1953.

[92] J.H. Przytycki, Skein modules of 3-manifolds, *Bull. Polish Acad. Science*, 39(1-2), 1991, 91-100.

[93] J. H. Przytycki, History of the knot theory from Vandermonde to Jones, *Aportaciones Matemáticas Comunicaciones*, 11 (1992), 173-185. (XXIV Congreso Nacional de la Sociedad Matematica Mexicana celebrado en Oaxtepec, Morelos del 10 al 16 de noviembre de 1991.)

[94] J.H.Przytycki, Knot theory from Vandermonde to Jones,
with the translation of the topological part of Poul Heegaard Dissertation: Forstudier til en topologisk teori for de algebraiske fladers sammenh[ae]ng, København 1898, Filosofisk Doktorgrad (Preliminary studies towards the topological theory of connectivity of algebraic surfaces), by A.H. Przybyszewska, Preprint 43, Odense University 1993.

[95] J.H. Przytycki, Algebraic topology based on knots: an introduction, Proceedings of *Knots 96*, edited by Shin'ichi Suzuki, @1997 World Scientific Publishing Co., 279-297, to appear.

[96] J. H. Przytycki, P. Traczyk, Invariants of links of Conway type, *Kobe J. Math.* 4 (1987) 115-139.

[97] J. Raeder, *Oribassi Collectionum Medicarum Reliquiae*, 4 volumes, Leipzig and Berlin, 1928-1933.

[98] K.Reidemeister, *Knotentheorie.* Ergebn. Math. Grenzgeb., Bd.1; Berlin: Springer-Verlag (1932) (English translation: Knot theory, BSC Associates, Moscow, Idaho, USA, 1983).

[99] N.Y.Reshetikhin, V.Turaev, Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.* 103 (1991) 547-597.

[100] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[101] G.Sarton, Ancient Science through the Golden Age of Greece, Dover Pub., 1993 (first edition: Harvard Univ. Press, 1952).

[102] H.I. Sharlin, T.Sharlin, *Lord Kelvin, the dynamic Victorian*, Penn. State University Press, 1979.

[103] R.B.Sher, Max Dehn and Black Mountain College, *Math. Intelligencer* 16 (1994), no. 1, 54–55.

[104] R.H.Silliman, William Thomson: smoke rings and nineteenth-century atomism, *Isis*, 54, December 1963, 472-.

[105] O.Simony, Über eine Reihe neuer Thatsachen aus dem Gebiete der Topologie, *Math. Annalen*, 19 (1882), 110-120, and 24 (1884), 253-280.

[106] P. Stäckel, Gauss als Geometer, in the X'th Volume of Gauss' Collected Works.

[107] J.Stillwell, Classical Topology and Combinatorial Group Theory, Grad. Texts in Math. 72, Springer-Verlag 1980.

[108] D. J. Struik, *A concise history of mathematics*, Dover Publications, Inc. New York, 1967.

[109] D.W.Sumners (Editor), N.R.Cozzarelli, L.H.Kauffman, J.Simon, D.W.Sumners, J.H.White, S.G.Whittington, *New Scientific Applications of Geometry and Topology*, Proceedings of Symposia in Applied Mathematics, Vol. 45, AMS 1992.

[110] P.G.Tait, Some elementary properties of closed plane curves, *Messenger of Mathematics*, New Series, No.69, 1877, (communicated at the 1876 Meeting of the British Association). Reprinted in Scientific Papers of P.G.Tait, 270-272.

[111] P.G.Tait, On knots I, II, III, Scientific Papers, Cambridge University Press, 1898-1900. Including: *Trans. R. Soc. Edin.*, 28, 1877, 35-79. Reprinted by Amphion Press, Washington D.C., 1993.

[112] P.G.Tait, Johann Benedict Listing, *Nature*, February 1, 1883. Reprinted in Scientific Papers of P.G.Tait.

[113] P.G.Tait, Listing's Topologie (Introductory address to the Edinburgh Mathematical Society, November 9, 1883), *Philosophical Magazine*, January, 1884. Reprinted in Scientific Papers of P.G.Tait.

[114] M. B. Thistlethwaite, Knot tabulations and related topics, *Aspects of topology, LMS Lecture Notes Series*, 93 (1985), 1-76.

[115] M.B.Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology*, 26 (1987), 297-309.

[116] S.P. Thompson, The life of William Thomson: Baron Kelvin of Largs, 2 vols., London: Macmillan, 1910.

[117] D'A.W. Thompson, On Growth and Form, Dover Publications, 1992 (republication of "A New Edition" by Cambridge University Press, 1942; "I wrote this book in wartime, and its revision has employed me during another war".)

[118] J.J. Thomson, A treatise on the motion of vortex rings, The Adams prize in 1882, (published in 1883).

[119] W.H. Thomson, On vortex motion, *Trans. R. Soc. Edin.*, 25 (1867), 217-260. See also *Proceedings of the Royal Society of Edinburgh*, February 1867.

[120] H. Tietze, Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten, *Monatshefte für Mathematik and Physik* 19 (1908), 1-118.

[121] H. Tietze, Über Simony-Knoten und Simony-Ketten mit vorgeschriebenen singulären Primzahlen für die Figur und für ihr Spiegelbild, *Math. Zeitschrift*, 49, 1943, 351-369.

[122] V.G. Turaev, The Conway and Kauffman modules of the solid torus, *Zap. Nauchn. Sem. Lomi* 167 (1988), 79-89. English translation: *J. Soviet Math.*.

[123] V.G. Turaev, H.Wenzl, Quantum invariants of 3-manifolds associated with classical simple Lie algebras, *Internat. J. Math.*, 4(2), 1993,323-358.

[124] V.G. Turaev, O.Y.Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, *Topology*, 31 (1992), 865-902.

[125] W.T.Tutte. A ring in graph theory. *Proc. Cambridge Phil. Soc.* 43 (1947) 26-40.

[126] A. T. Vandermonde, Remarques sur les problèmes de situation, *Mémoires de l'Académie Royale des Sciences* (Paris) (1771), 566-574.

[127] A.Weil, Riemann, Betti and the birth of topology, *Arch. for Hist. Exact Sci.*, 16 (1979), 91-96.

[128] C.Weber, A stroll through knot theory, *New. J. Chem.*, 17 (1993), 627-644.

[129] H.Weith, *Topologische Untersuchung der Kurven-Verschlingung*, Inaugural Dissertation, Zürich 1876.

[130] H. Whitney, The coloring of graphs, *Ann. of Math.*, 33 (1932), 688-718.

[131] W. Whitten, Knot complements and groups, *Topology*, 26 (1987), 41-44.

[132] W.Wirtinger, Über die Verzweigungen bei Funktionen von zwei Veränderlichen, *Jahresbericht d. Deutschen Mathematiker Vereinigung*, 14 (1905), 517. (The title of the talk supposedly given at September 26 1905 at the annual meeting of the German Mathematical Society in Meran).

[133] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.*, 121 (1989), 351-399.

[134] Yale University, Obituary Record of graduates deceased during the year ending July 1, 1924, New Haven (Published by the University), 1924, p.1211.

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Appendix

Paul Heegaard (1871-1948) is well known to topologists because of his theorem about decomposition of any 3-manifold into 2 handlebodies. His achievements in knot theory are much less known.[30] One of the reasons for this was the fact that his 1898 Doctoral Dissertation [52] was written in Danish. It was translated into French only 18 years later. I learned of the Heegaard work on knots only from the book by J. C. Stillwell [107]. We will present here the translation from Danish of the Preface and the second (topological) part of Heegaard's Dissertation. One can hope that the whole Dissertation will be translated soon. As to importance of Heegaard results we can best quote from the Stillwell book:

"Heegaard's results lay dormant (although noted by Tietze 1908) until the publication of the French translation of his thesis in 1916. The translation was checked for mathematical soundness by J.W.Alexander, fresh from his work on homology groups, and we may surmise that the collision of these ideas led to the fruitful discoveries which were to follow. Alexander must also have read Tietze 1908 at this time, because in short order he disposed of two of the most important of Tietze's conjectures: Alexander 1919a[31] shows that there are nonhomeomorphic lens spaces with the same group, while Alexander 1919b[32] proves that any orientable 3-manifold is a branched cover of S^3. Later in 1920 he finally took the cue from Heegaard's example and looked for torsion in cyclic covers of S^3 branched over various knots."

Heegaard's bibliography:

[Heegaard] P.Heegaard, Forstudier til en Topologisk Teori for de algebraiske Fladers Sammenhæng, København, 1898, Filosofiske Doktorgrad; French translation: Sur l'Analysis situs, *Soc. Math. France Bull.*, 44 (1916), 161-242.

[D-H] M.Dehn, P.Heegaard, Analysid situs, *Encykl. Math. Wiss.*, vol. III AB3 Leipzig, 1907, 153-220.

[H-1] P.Heegaard, Der Mathematikunterricht in Dänemark, *Internationale Mathematische Unterrichtskommission I.M.U.K*, København 1912. (Teaching of Mathematics in Denmark).

[H-2] P.Heegaard, Bidrag til Grafernes Theori, Tredie Skandinaviske Matematikerkongres i

[30]The only, known to me, research papers on topology by Heegaard are [35, 52]. For other Heegaard's work see his bibliography (Brjan Toft found the box donated to Odense Math. Dept. by Niels Erik Nørlund (1885-1981) with Heegaard papers (as in the Heegaard's bibliography except [35] and only French version of [52]).

[31]Note on two three-dimensional manifolds with the same group, *Trans. Amer. Math. Soc.*, 20 (1919), 339-342.

[32]Note on Riemann spaces, *Bull. Amer. Math. Soc.*, 26 (1919), 370-372.

Kristiania, 1913, 1-6.

[H-3] P.Heegaard, Omriss av kronologiens historie i Europa (med sœrlig henblikk pá den norske almanakk av 1644, ????? 51-96.

[H-4] P.Heegaard, Hamiltons Dodekaederspil, *Nyt Tidsskrift Matematik*, 29(4), 1919, 81-88.

[H-5] P.Heegaard, Mindetale, oer prof..dr.H.G.Zeuthen, Særtrykk av Videnskapsselskapets Forhandlinger 1920, 1-3.

[H-6] P.Heegaard, Hovedlinjer i H.G.Zeuthens videnskabelige production, Foredrag i Norsk Matematisk Forening, *Særtrykk av Norsk Matematisk Tidsskrift*, 1920, 1-5.

[H-7] P.Heegaard, Eudoksos' Hippopede. Et foredrag holdt i Norsk Matematisk Forening, *Særtrykk av Norsk Matematisk Tidsskrift*, 1921, 1-10.

[H-8] P.Heegaard, Felix Klein, Et foredrag holdt i Norsk Matematisk Forening 25. mars 1926, *Særtrykk av Norsk Matematisk Tidsskrift*, 1926, 1-7.

[H-9] P.Heegaard, Om en generalisasjon til rummet av Sophus Lies fremstilling av imaginære elementer i planet, Den syvende Skandinaviske Matematikerkongress, I Oslo 19-22 August 1929, Oslo 1930, 92-95.

[H-10] P.Heegaard, Scandinavie. Les modifications essentielles de l'enseignement mathématique dans les principaux pays depuis 1910, *L'Enseignement mathématique*, XXIX(4-6), 1930, 307-314.

[H-11] P.Heegaard, Appendix P.Osl 6. The date of the Horoscope of Philoe (in "Papyri Osloenses" by S.Eitrem, L.Amundsen), Oslo, on commission by Jacob Dybwad, 1931, 146-151.

[H-12] P.Heegaard, Über die Heawoodschen Kongruenzen, *Særtrykk av Norsk Matematisk, Forenings Skrifter*,Serie II, Nr. 1-12, 1933, 47-54.

[H-13] P.Heegaard, Norvège. La préparation théorique et pratique des professeurs de Mathématiques de l'enseignement secondaire, *De L'Enseignement mathématique*, 5-6, 1933, 360-364.

[E-H] F.Engel, P.Heegaard, Sophus Lies Samlede Avhandlinger. Blind I, *Særtrykk av Norsk Matematisk Tidsskrift*, 1934, 1-9.

[H-14] P.Heegaard, Et brev fra Abel til Degen, *Særtrykk av Norsk Matematisk Tidsskrift*, 2. hefte 1935, 1-6.

[H-15] P.Heegaard, Bemerkungen zum Vierfarbenproblem, *Mat. Sb.* (New Ser.), 1(43):5, (1936), 685-693.

[H-16] P.Heegaard, La représentation des points imaginaires de Sophus Lie at sa valeur didactique, Atti del Congresso Internazionale dei Matematici, Bologna, 3-10 settembre 1938 - VI, 421-423.

[H-17] P.Heegaard, Die Topologie und die Theorie der algebraischen Funktionen mit zwei komplexen Variabeln, Congrès des Mathématiques à Helsingfors 1938, 1-8.