



THE UNIVERSITY *of* EDINBURGH

PROJECT IN MATHEMATICS (GROUP)

Celtic Knot Theory

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Abstract

The purpose of this project is to explore the subject of knot theory. We consider knot invariants, in particular the Alexander polynomial which we show to be well defined and invariant and also the Alexander-Conway polynomial. We discuss both alternating and non-alternating knots in relation to such invariants. The main part of the paper concerns Celtic knots; their construction and the proofs that Celtic knots are alternating, and that alternating knots are Celtic. Finally we investigate another example of mathematical knots in art, the Brunnian links.

Summary

This project report is submitted in partial fulfilment of the requirements for the degree of *BSc Mathematics*.

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Chapter 1

Introduction

In this paper we give an introduction to knot theory, the study of mathematical knots. While inspired by knots that we see in real life, in ropes, laces and wires, a mathematical knot differs in that the two loose ends of a strand are joined together. This forms a continuous loop which cannot be undone by manipulation. In mathematical terminology we say that a **knot**¹ is an embedding of S^1 (a circle) in \mathbb{R}^3 that does not intersect itself.

Knot theory may seem to stand alone as a field of study, but it has strong connections to many other mathematical fields, in particular topology and graph theory. Outside of mathematics, the study of knots has major applications in other disciplines such as physics, biology and chemistry. We discuss how knot theory has developed historically and the implications it has had to date. We also show that it is a very current area of study with many problems left unsolved.

The great unsolved problem in knot theory is how to tell when two knots are **equivalent** by using **knot invariants**. Currently, no invariant exists that can successfully distinguish between all non-equivalent knots. In this paper we examine two of the milestones in addressing this problem, the discovery of the Alexander and Alexander-Conway polynomial invariants. We investigate methods of calculating these invariants and provide a thorough proof that the Alexan-

¹words that are in bold in the body of the text appear in the glossary.

der Polynomial is both well defined and invariant.

The paper also looks at properties and examples of alternating and non-alternating knots in relation to these invariants. This study provides some background for the main section of the paper, Celtic knots.

We include a novel method for constructing Celtic knots and this brings together all of the other aspects studied in the paper. The link between alternating knots and Celtic knots is investigated, with a proof demonstrating that knots are alternating if, and only if, they are Celtic. A connection to the previously mentioned invariants is also found, whilst new invariants are considered that measure properties specific to Celtic knots.

The final section of the paper is on the Brunnian links. We are able to bring together two of the most popular representations of mathematical knots in art by constructing a Celtic representation of the well known Borromean rings. This shows a fully worked example of the devised algorithm and concludes our study.

This paper focuses on the aspects of knot theory that we found most interesting, rather than briefly discussing a large range of areas. Whilst considering the background of these areas, we have proven certain properties in order to make the basis of the field accessible and justified to others. We hope that this in turn will inspire others to study this field further, as, in writing this paper we became truly aware how much there is left to be discovered.

Chapter 2

Knot Theory

2.1 History of Knots

Although **knots** have a long history in sailing, stamps, seal signs, Celtic art, etc. the first reference from a mathematical perspective did not come until 1771, when Vandermonde wrote the paper ‘*Remarques sur les Problemes de Situation*’ [24]. In his paper, Vandermonde specifically places knots as a subject of the geometry of position, an observation which provided a basis for further mathematical study.

The work of Gauss is seen by most as the birth of knot theory, as he was able to apply mathematical concepts to the study of knots. Consequently and significantly, he derived the ‘Gauss **linking number**’ in 1833, the earliest discovered **knot invariant** [23, p. 363].

Knot theory achieved prominence and wider interest when, in 1867, Sir William Thomson (Lord Kelvin) released his paper ‘*On Vortex Atoms*’. Inspired by Helmholtz’s work on vortex motion, Thomson theorised that matter was composed of ‘vortex atoms’ or 3-dimensional knotted tubes of ether [25, p. 162]. Atoms could thus be classified by the knots that they resembled, with different crossing and twisting formations corresponding to elements with unique chemical properties.

Thomson's ideas gained the attention of his friend, the physicist James Clerk Maxwell. Although Maxwell was primarily interested in how the study of knots could be applied to his own work on electromagnetism, he made an important advancement to the field. Maxwell defined three moves on **link diagrams** which would become famous as the **Reidemeister moves** some sixty years later [19, p. 213].

Thomson's theory of vortex atoms required a system of classification of knots and it was this that motivated Peter Guthrie Tait to tabulate them. Tait's extensive study led him to explore the structure of knots in an attempt to understand when two knots were 'different'. Tait began to make the first table of knots in 1876 [25, p. 163] and went on to compile further tables alongside Thomas Kirkman and Charles Newton Little which have only been slightly modified to this day.

On 16 October 1876, Tait delivered a sealed envelope to the Royal Society of Edinburgh containing two conjectures. Although these conjectures appeared implicitly in his later work, they were never stated as explicitly as in the envelope, which remained sealed for 111 years [8, p. 356]. Eventually, Tait's conjectures for knot diagrams were established (and proven, following the discovery of the Jones Polynomial) and provided greater understanding of link diagram manipulation.

The study of knot theory suffered a blow in history in 1871 when Mendeleev published the Periodic Table of Elements. This was widely accepted by the scientific community, immediately discrediting Thomson's theory and leading knot theory to become practically forgotten.

Around 1900, after Poincaré formalised the modern theory of topology, knot theory was reconsidered by mathematicians. Knot theorists aimed to further address the question of 'sameness'. In 1926, Kurt Reidemeister proved how Maxwell's three moves are the only moves required to illustrate **equivalence** [22]. As a result, the influential moves bear Reidemeister's name.

Progressing the idea of equivalence, mathematicians wanted to find more sensitive knot invariants, so that one could identify two equivalent (or far more easily, two different) knots given different link diagrams. The first polynomial knot invariant was discovered in 1928 by James Waddell Alexander and successfully differentiates between many non-equivalent knots (see Section 3.1). However, it was by no means a complete invariant. John Conway increased the understanding of the Alexander Polynomial by creating his own polynomial invariant in the 1960s, which, whilst only slightly modified, was very differently defined [6].

A further improved and more perceptive polynomial invariant was discovered in 1984, by Vaughan Jones, using the theory of ‘operator algebras’. The Jones Polynomial was the first polynomial invariant to distinguish many knots from their mirror images [13] and was, therefore, a major advancement in knot theory. It also enabled the first rigorous proof of the Tait Conjectures as mentioned previously.

The Jones Polynomial has since been generalised, examined from different approaches and derived using several different methods, and has led to applications of knot theory in a wider context.

2.2 Knots in Context

Knot theory is a relatively young subject within mathematics, with some of the greatest discoveries occurring in the last forty years. Such discoveries have opened up applications in other disciplines, leading to the requirement of greater understanding of the field. As a result of this demand, alongside mathematical interest, knot theory is constantly growing.

Principles of topology, in particular knot theory, have led to concrete applications in the study of DNA, allowing cell biologists to measure properties concerning the life and replication of cells.

For instance, DNA strands can become knotted, with the formation of the molecule influencing its function within the cell. Additionally, in the early 1970s it was discovered that the process of replication of DNA, from ‘splicing’ to ‘recombination’, happened as a result of a single enzyme, topoisomerase [22, p. 268]. This is the enzyme that causes the DNA to become knotted. The name comes from the manipulations performed on topological isomers¹ to change the linking number, and thus the study of knot theory is necessary in understanding this process and in gaining a greater insight into the structure and function of the molecule.

Knot theory has also been used to understand and determine the chirality of molecules. This has important implications in the pharmaceutical industry, as well as aiding general understanding in chemistry and biology. Pharmaceutical companies have had to study the chirality of drug molecules to ensure that the detrimental side effects of one enantiomer (two molecules which are mirror images of each other) are minimal or non-existent when compared to the positive effects of the other [9, p. 15].

In the 1950s and 60s there was a particularly disastrous demonstration of the consequences that chiral molecules can have. The drug Thalidomide was prescribed to pregnant women as a treatment for morning sickness. Although the (*R*)-thalidomide enantiomer was successful in alleviating morning sickness, the (*S*)-thalidomide enantiomer (the mirror image) was unknowingly responsible for causing many birth defects. Studying knots has since provided scientists with understanding of chirality to avoid such disasters.

Another relatively new application of knot theory is within statistical mechanics. Following the discovery of the Jones polynomial, a connection was made between statistical mechanical concepts and the properties of knots. Statistical models can provide an alternative approach to the calculation of knot invariants

¹molecules with the same sequence of base pairs but topologically different structures, e.g. different linking numbers.

[26, pp. 108-112] and this has consequently motivated mathematicians to use such models as a means to search for new knot invariants.

Applications can even be seen in the construction of quantum computers and in quantum physics, as knot invariants arise in gravitational physics. It seems there is little doubt that further applications will arise, but the major advancements are still generally being discovered by those interested in knot theory for knot theory's sake.

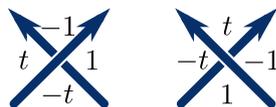
Chapter 3

Polynomial Invariants

3.1 The Alexander Polynomial

Comparing **knot invariants** can be an effective method to distinguish between two **knots**. However, the invariant needs to be sensitive enough to give different values for non-**equivalent** knots. In his efforts to address this, the first polynomial invariant in knot theory was discovered in 1928 by J.W. Alexander. For a knot K , the Alexander polynomial, denoted $\Delta_K(t)$, is a ‘Laurent polynomial’. This means it has positive and negative powers of t and, in this case, has integer coefficients, i.e. $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$.

In Alexander’s original paper, he gives the definition of his polynomial from a combinatorial standpoint. He goes on to define the method of the Alexander matrix to achieve his polynomial [3, pp. 279-280]. To obtain the Alexander matrix, we label the orientated knot diagram for each crossing as below:



We notice that by Euler’s formula, a knot K , with n crossings, creates $(n + 2)$ regions of the diagram¹. The Alexander matrix $M'(t)$ is indexed in rows by crossings and in columns by regions. Thus $M'(t)$ is an $n \times (n + 2)$ matrix. The ij^{th} entry is the appropriate value from $\{1, -1, t, -t\}$ where the i^{th} crossing meets the

¹ $V = n$ vertices, $E = 2n$ edges (since 4-valent) and $V - E + F = 2 \Rightarrow F = n + 2$ faces.

j^{th} region, or takes the value zero if the region does not touch that crossing.

Next, we remove any two columns from the matrix which correspond to adjacent regions² to obtain the reduced Alexander matrix, $M(t)$. From the remaining $n \times n$ matrix, we then find the determinant $\det(M(t))$.

The final stage is to factor out the largest possible power of t and multiply by ± 1 to ensure the term of lowest degree in the determinant is a positive constant, just for convention. An alternative convention is to then divide through by $t^{s/2}$ where s is the highest power of t . This gives a palindromic polynomial. The resulting expressions are Alexander polynomials, $\Delta_K(t)$. Note that if two Alexander polynomials differ by a factor of $\pm t^m$ we consider them to be equivalent. A calculation of the Alexander polynomial is given in Chapter 5.

It is important to note that the determinant of the reduced matrix is dependent on the choice made as to which two columns are removed. This consequently adds some ambiguity to the answer, however it will only change within a factor of $\pm t^m$ for some $m \in \mathbb{Z}$ as shown in Theorem 1.

Despite the advancements that the Alexander polynomial made, it is still not a complete invariant as it cannot successfully distinguish between all knots, in particular the **unknot**, which has Alexander polynomial $\Delta_{0_1}(t) = 1$. This is because nontrivial knots exist with the same Alexander polynomial, such as the Kinoshita-Terasaka knot [15, p. 151] which has **crossing number** 11.

3.1.1 Proof that $\Delta_K(t)$ is well defined

Theorem 1. *The following three choices do not affect the determinant $|M|$ by more than a factor of $\pm t^m$:*

1. *The orientation of the knot.*
2. *The way that the regions and crossings in the knot are indexed.*

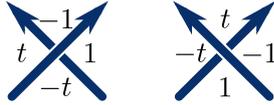
²two regions which share an edge.

3. The two adjacent columns that are removed.

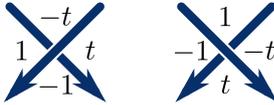
Proof. Let us examine the effect of each case individually.

1. The orientation of the knot.

In the process of calculating the Alexander polynomial, we labelled each crossing of the knot, say K_+ , as follows:



If we were to change the orientation of the knot, naming the new knot K_- , the crossings and their respective labels would become:



From the diagrams, it can be seen that all $\pm 1 \rightarrow \pm t$ and all $\pm t \rightarrow \pm 1$. We instead let $\pm t \rightarrow \pm 1$ and $\pm 1 \rightarrow \pm t^{-1}$. Thus if a_{ij} refers to the label of the j^{th} region where it meets the i^{th} crossing, the new knot will have $\frac{a_{ij}}{t}$ for the same region and crossing.

Now, the reduced matrix for K_+ can be denoted M_0 , and we have $\Delta_{K_+}(t) = \pm t^m \cdot \det(M_0)$ by definition. Thus if the entries of M_0 are a_{ij} , then the reduced matrix for K_- , denoted M^* , has entries of the form $\frac{a_{ij}}{t}$. Now for each row of the matrix we multiply the row by t and divide the determinant by t . Given that the number of crossings is n , we find:

$$\det(M^*(t)) = t^n \cdot \det(M_0(t)) = \pm t^{m'} \Delta_{K_+}(t) \tag{3.1}$$

However, the reduced matrix $M^*(t)$ is not quite what we were initially looking for. We should look at $M^*(t^{-1})$ so that $\pm t \rightarrow \pm 1$ and $\pm 1 \rightarrow \pm t$ as we wanted. We remember $\Delta_{K_+}(t)$ can be written as a palindromic polynomial, which implies that $\Delta_{K_+}(t) = \Delta_{K_+}(t^{-1})$. Therefore:

$$\det(M_0(t)) = \pm t^{m''} \cdot \det(M_0(t^{-1})) = \pm t^{n'} \cdot \det(M^*(t^{-1})) \text{ by equation 3.1.}$$

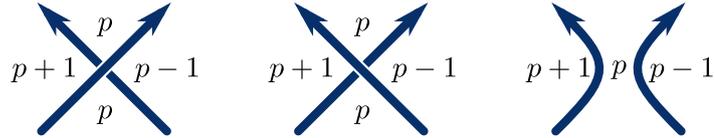
This shows that changing the orientation of our knot only changes the determinant by a factor of $\pm t^m$.

2. Indexing the regions and crossings in the knot

Altering the way that the regions and crossings in the knot are indexed only alters the order of the rows and columns of the matrix $M(t)$. Swapping rows or columns in the matrix only multiplies the determinant by ± 1 and thus the way the regions and crossing in the knot are indexed only affects the determinant by a factor of ± 1 .

3. Removing two adjacent columns [3, pp. 280-281]

First of all, let us look at a new way of indexing the regions of our knot diagram. The infinite region is given subscript 0. The other regions are assigned a subscript whereby travelling along the knot in the orientation specified, the region on the left of the strand has subscript one higher than the subscript of the region on the right. We also index the crossings so that if two regions with index p are incident at it then it also takes index p , as in the following diagrams.



So, in our example in Chapter 5, 8_{19} 's regions 1 – 10 become $1_{-1}, 2_{-2}, 3_{-1}, 4_0, 5_1, 6_0, 7_{-1}, 8_{-1}, 9_{-1}, 10_0$. We note that any two neighbouring regions will necessarily have subscripts differing by 1. And the crossings from the same example become $1_{-1}, 2_{-1}, 3_{-1}, 4_{-1}, 5_{-1}, 6_0, 7_0, 8_0$.

We let R denote the set of all regions and r_p denote a column vector of the Alexander matrix, M , corresponding to a region with index p . We find

$$\sum_R r_p = \underline{0} \tag{3.2}$$

where $\underline{0}$ is the zero-vector and p is the appropriate index for each region $r \in R$. This is because each row represents a crossing and must contain the sum of 1, -1 , t , and $-t$ in some order. We also find

$$\sum_R t^{-p} \cdot r_p = \underline{0} \tag{3.3}$$

with p as above. If we look at a row corresponding to a crossing of index q this is easy to understand. We look at how the Alexander matrix, M , assigns 1, -1 , t , and $-t$ to the regions at the crossing and how the regions are indexed. For a

crossing of index q we sum either $1 \cdot t^{-(q-1)} - 1 \cdot t^{-q} + t \cdot t^{-(q+1)} - t \cdot t^{-q} = 0$ or $1 \cdot t^{-q} - 1 \cdot t^{-(q-1)} + t \cdot t^{-q} - t \cdot t^{-(q+1)} = 0$ if it is a positive or negative crossing respectively. Combining equations 3.2 and 3.3 we get

$$\sum_R (t^{-p} - 1) \cdot r_p = \underline{0}.$$

We see that for $p = 0$ we have $(t^0 - 1) \cdot r_0$ for each $r_0 \in R$ but $t^0 - 1 = 0$ so the regions with index 0 disappear. Therefore, for any r^* with index q we have

$$(t^{-q} - 1) \cdot r^* = - \sum_{R \setminus \{r^*\}} (t^{-p} - 1) \cdot r_p.$$

We now define $|M_{pq}(t)| = |M_{qp}(t)|$ to be the determinant of the reduced Alexander matrix where we have removed a pair of columns with indices p, q for $p \neq q$. For ease, let us rename the columns of M so that $M = (v_0, v_1, \dots, v_i, \dots, v_j, \dots, v_n)$. We let v_0, v_i and v_j have indices 0, q and s respectively. We call M_{0q} the reduced matrix with v_0 and v_i removed and call M_{0s} the reduced matrix with v_0 and v_j removed. We find

$$\begin{aligned} (t^{-s} - 1) \cdot |M_{0q}| &= |v_1, \dots, v_{i-1}, v_{i+1}, \dots, (t^{-s} - 1) \cdot v_j, \dots, v_n| \\ &= |v_1, \dots, v_{i-1}, v_{i+1}, \dots, - \sum_{R \setminus \{v_j, v_0\}} (t^{-p} - 1) \cdot r_p, \dots, v_n|, \end{aligned}$$

and since the sum has multiples of all the other columns, they can be subtracted without changing the determinant, leaving a multiple of v_i :

$$\begin{aligned} &= |v_1, \dots, v_{i-1}, v_{i+1}, \dots, -(t^{-q} - 1) \cdot v_i, \dots, v_n| \\ &= (t^{-q} - 1) \cdot |v_1, \dots, v_{i-1}, v_{i+1}, \dots, -v_i, \dots, v_n| \\ &= \pm (t^{-q} - 1) \cdot |v_1, \dots, v_i, \dots, v_{j-1}, v_{j+1}, \dots, v_n| \\ &= \pm (t^{-q} - 1) \cdot |M_{0s}| \end{aligned}$$

where, in each step, we have either multiplied a column by a constant, added multiples of other columns to a column, or re-ordered the columns, so as not to change the determinant by more than a constant factor. Moreover, since the indices are only determined up to an additive constant, we can replace 0 with p . Then

$$\begin{aligned} (t^{p-s} - 1) \cdot |M_{pq}| &= \pm (t^{p-q} - 1) \cdot |M_{ps}| \\ (t^{s-r} - 1) \cdot |M_{sp}| &= \pm (t^{s-p} - 1) \cdot |M_{sr}| \end{aligned}$$

$$\begin{aligned}
 \Rightarrow |M_{pq}| &= \frac{\pm(t^{p-q} - 1)}{(t^{p-s} - 1)} \cdot |M_{ps}| \\
 |M_{sp}| &= \frac{\pm(t^{s-p} - 1)}{(t^{s-r} - 1)} \cdot |M_{sr}| \\
 \Rightarrow |M_{pq}| &= \frac{\pm(t^{p-q} - 1) \cdot (t^{s-p} - 1)}{(t^{p-s} - 1) \cdot (t^{s-r} - 1)} \cdot |M_{sr}| \\
 &= \frac{\pm t^{s-p} \cdot (t^{p-q} - 1)}{(t^{s-r} - 1)} \cdot |M_{sr}|.
 \end{aligned}$$

In particular, if $q = p + 1$, $r = s + 1$, we find

$$\begin{aligned}
 |M_{p(p+1)}| &= \frac{\pm t^{s-p} \cdot (t^{-1} - 1)}{(t^{-1} - 1)} \cdot |M_{s(s+1)}| \\
 &= \pm t^{s-p} \cdot |M_{s(s+1)}|.
 \end{aligned}$$

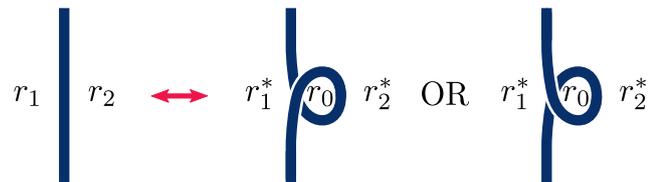
We find that whichever two columns are removed, as long as they have indices differing by 1, (in particular, if they are neighbouring,) we find that the determinant of the matrix is changed by at most a factor of $\pm t^m$. \square

We have checked that the only choices we can make in calculating $\Delta_K(t)$ do not change the polynomial by more than a factor of $\pm t^m$. Thus, the Alexander polynomial is well defined up to sign and multiplication by powers of t .

3.1.2 Proof of Invariance

In order to prove invariance it must be shown that $K \sim K' \Rightarrow \Delta_K(t) = \Delta_{K'}(t)$. Since $K \sim K'$ iff K and K' are related by a sequence of **Reidemeister moves** it is enough to show that the Alexander polynomial remains unchanged by the Reidemeister moves. Since the last few steps of the calculation are just ‘tidying up’, we need only show that the determinants of the matrices are only different by a factor of $\pm t^m$.

Type I. Let M_0 be the reduced matrix of the original knot diagram, K_0 , with the two columns corresponding to the adjacent regions r_1 and r_2 removed.



Carrying out a Type I move on K_0 and then removing the columns corresponding to the adjacent regions r_1^* and r_2^* gives us the matrix M^* . This is the

In M^* , if we now send $r_3^* \rightarrow r_3^* + r_0$ we do not change the determinant and we obtain the original matrix M_0 bordered by the two new rows and columns. This is because $r_3^* + r_0 = \text{original } r_3$ and the other entries are not affected.

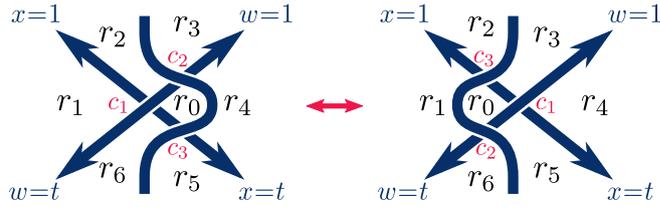
$$\left| \begin{array}{cc|ccc} \pm t^m & \pm t^n & 0 & \dots & 0 \\ \mp t^m & 0 & 0 & \dots & 0 \\ \hline 0 & \xi & & & \\ \vdots & & & & \\ 0 & & & & \end{array} M_0 \right| = \left| \begin{array}{cc|ccc} 0 & \pm t^n & 0 & \dots & 0 \\ \mp t^m & 0 & 0 & \dots & 0 \\ \hline 0 & \xi & & & \\ \vdots & & & & \\ 0 & & & & \end{array} M_0 \right| = \pm t^{m'} \cdot \left| M_0 \right|$$

In the first matrix we let the top row of M^* represent the top crossing of our image and the second row the lower crossing. Also, we let the first column represent r_0 and the second column r_2^* . In r_2^* we write ξ for a potentially non-zero column vector.

Now, in order to show M^* and M_0 are equivalent, we add the second row of M^* to the first as the two non-zero r_0 entries must be the negation of each other. Then we divide the first row by $\pm t^n$ and the second row by $\mp t^m$, whilst multiplying the determinant by both. Next we want to make the entries in ξ equal zero. To do this we add multiples of the first row to all of the other rows. This does not affect the rest of the matrix because all of the other entries in the row are zero. Then we can drop the first and second rows and columns as required, to get $\pm t^{m'} |M_0|$.

As M^* differs from M_0 by at most a factor of $\pm t^m$, the polynomials yielded from M_0 and M^* are equivalent. Thus, $\Delta_K(t)$ is invariant under moves of Type II.

Type III. To investigate the effect that carrying out a Type III move has on the entries of our initial matrix, we must first note that the number of regions does not change as a result of the move, it is only the entries around the crossings that are altered. There are four possible orientations to consider, each of which are addressed in the following diagram:



Claim. If $M = (a_{ij})$ is the Alexander matrix, $N = (|a_{ij}|)$ is equivalent to it.

Proof. We start with M . If we index the regions as with the proof of Theorem 1.3, any two neighbouring regions will have subscripts differing by one, so that the indices will alternate between odd and even. The labels at the crossings, $\{t, -t, 1, -1\}$, also alternate, this time between positive and negative. Therefore, for a given crossing, the entries corresponding to the regions with odd indices will either both be positive or both negative and vice versa for even indices. Let us now multiply all of the columns with odd indices by -1 . Now each crossing has all its entries with the same sign, be they all positive or all negative. So we simply multiply those negative rows by -1 and we are left with N . \square

Now, as the matrices are relatively large, let us only examine part of the matrix (with all positive values as above) with rows corresponding to crossings c_1, c_2, c_3 and with columns corresponding to regions $r_0, r_1, r_2, r_3, r_4, r_5, r_6$ (the part of the matrix representing the above diagram). Before and after the Type III move respectively, this part of the matrix has the form:

$$\begin{pmatrix} x & y & x & 0 & 0 & 0 & y \\ w & 0 & z & z & w & 0 & 0 \\ x & 0 & 0 & 0 & x & y & y \end{pmatrix} \longleftrightarrow \begin{pmatrix} y & 0 & 0 & x & x & y & 0 \\ z & z & 0 & 0 & 0 & w & w \\ y & y & x & x & 0 & 0 & 0 \end{pmatrix}$$

$$\text{where } x \in \{1, t\}, y = \begin{cases} 1 & \text{if } x = t \\ t & \text{otherwise} \end{cases} \quad \text{and } w \in \{1, t\}, z = \begin{cases} 1 & \text{if } w = t \\ t & \text{otherwise} \end{cases}.$$

The matrices above show the general case of the effect of a Type III move, with all four orientations. Now we can use elementary matrix transformations to take one to the other, ensuring that the determinant of the matrix does not change by more than a factor of $\pm t^m$. It is important to consider the entries in the rest of the matrix, as in the general case this is unknown. We do know however, that the rest of the entries in the first three rows are made up of zeros,

since no other regions are incident at these crossings. We also know that below the first three rows, there are zeros in the first column. With this knowledge, we are only able to *a)* multiply the first column or any of the rows by $\pm t^m$; *b)* add multiples (by $\pm t^m$) of the first column or any of the rows to another column or row; and *c)* swap rows or columns.

Let us look at the case that $w = y$. First of all, we swap c_1 and c_3 :

$$\begin{aligned} & \begin{pmatrix} x & 0 & 0 & 0 & x & y & y \\ y & 0 & x & x & y & 0 & 0 \\ x & y & x & 0 & 0 & 0 & y \end{pmatrix} \xrightarrow{c_2 \rightarrow c_1 - \frac{x}{y}c_2} \begin{pmatrix} x & 0 & 0 & 0 & x & y & y \\ 0 & 0 & \frac{-x^2}{y} & \frac{-x^2}{y} & 0 & y & y \\ x & y & x & 0 & 0 & 0 & y \end{pmatrix} \xrightarrow{r_6 \rightarrow r_6 - \frac{y}{x}r_0} \\ & \begin{pmatrix} x & 0 & 0 & 0 & x & y & 0 \\ 0 & 0 & \frac{-x^2}{y} & \frac{-x^2}{y} & 0 & y & y \\ x & y & x & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{c_2 \rightarrow c_2 + \frac{x}{y}c_3} \begin{pmatrix} x & 0 & 0 & 0 & x & y & 0 \\ \frac{x^2}{y} & x & 0 & \frac{-x^2}{y} & 0 & y & y \\ x & y & x & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_3 \rightarrow r_3 + r_0} \\ & \begin{pmatrix} x & 0 & 0 & x & x & y & 0 \\ \frac{x^2}{y} & x & 0 & 0 & 0 & y & y \\ x & y & x & x & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_0 \rightarrow \frac{y}{x}r_0} \begin{pmatrix} y & 0 & 0 & x & x & y & 0 \\ x & x & 0 & 0 & 0 & y & y \\ y & y & x & x & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

For the case that $w = x$ we use similar matrix transformations as above to obtain the required result.

We have found that, for any orientation, we can use elementary matrix transformations to take the initial matrix to the matrix resulting from a Type III move. This ensures that the determinant of the reduced Alexander matrices differ at most by a factor of $\pm t^m$ between the matrix in the initial case and the matrix after the move, thus they are equivalent. That is, $\Delta_K(t)$ is invariant under moves of Type III.

As we have now shown that $\Delta_K(t)$ is invariant under each of the Reidemeister moves, we find that $\Delta_K(t)$ is a polynomial knot invariant.

3.2 The Alexander-Conway Polynomial

We will now discuss the Conway polynomial ∇_K , which is a variation of the Alexander polynomial using a recursive method devised by J.H. Conway in 1969.

Conway himself stated that ‘ ∇_K is just a disguised and normalized form of the Alexander polynomial Δ_K ’ [6, p. 337]. In fact ∇_K can be defined completely by different properties which we will investigate.

Definition. Given an oriented knot or link, K , there is an associated polynomial $\nabla_K(z)$ that can be determined by means of the following two axioms [6, p. 338]:

Axiom 1. If 0_1 denotes the standard unknot, then $\nabla_{0_1}(z) = 1$.

Axiom 2. Suppose that L_+ , L_- and L_0 are three knots or links that have identical diagrams except in the neighbourhood of one crossing point. In this neighbourhood, the diagrams differ in the following manner:



Then the polynomials of the three knots or links are related by the following:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$$

This relation is known as the skein relation and Conway showed that this, alongside a choice of value for the unknot, was enough to determine the polynomial invariant $\nabla_K(z)$. He also showed that it is well defined by relating it to Alexander’s matrix however he failed to find an explanation for the skein relation [6, p. 339].

Conway’s invariant ∇_K is related to the Alexander polynomial Δ_K by the following identity:

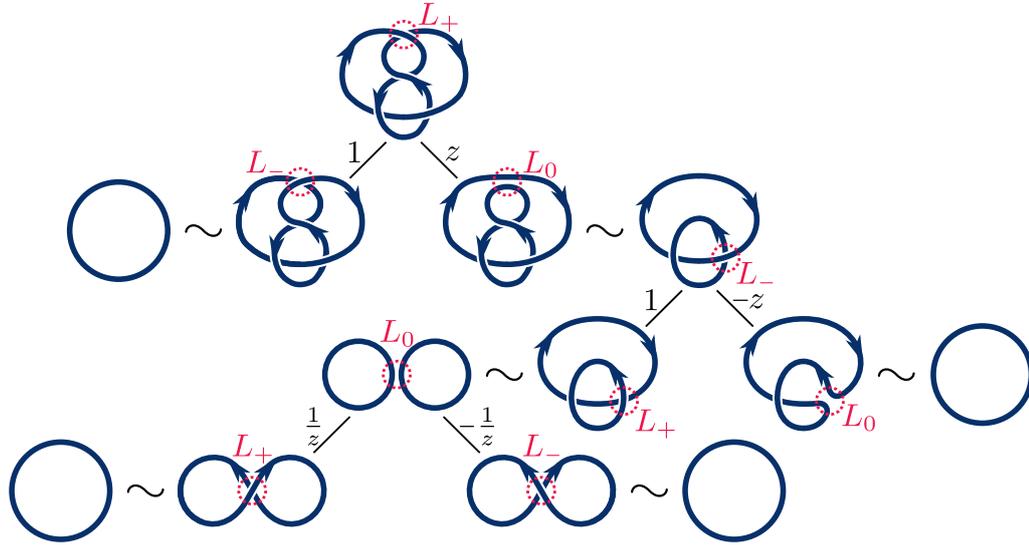
$$\Delta_K(t) = \nabla_K \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right)$$

Thus, replacing z by $\sqrt{t} - \frac{1}{\sqrt{t}}$ in the Conway polynomial yields the Alexander polynomial, which is why Conway’s polynomial is generally known as the Alexander-Conway polynomial. It differs slightly in that after calculating $\nabla_K(z)$ there is no ‘tidying up’ to be done, one always has the same result.

3.2.1 Calculation for Figure Eight Knot

We will now calculate the Conway polynomial of 4_1 using the skein relation:

Note. Sometimes we might write ∇_K instead of $\nabla_K(z)$.



With the ‘skein tree’ above we show that:

$$\begin{aligned}
 \nabla_{4_1}(z) &= 1 \cdot \nabla_{0_1} + z \cdot \nabla_{H_-} \\
 &= 1 \cdot \nabla_{0_1} + z \cdot (1 \cdot \nabla_{0_1^2} - z \cdot \nabla_{0_1}) \\
 &= 1 \cdot \nabla_{0_1} + z \cdot \left(1 \cdot \left(\frac{1}{z} \cdot \nabla_{0_1} - \frac{1}{z} \cdot \nabla_{0_1}\right) - z \cdot \nabla_{0_1}\right) \\
 &= 1 + z(0 - z(1)) = 1 - z^2
 \end{aligned}$$

where H_- is the oriented Hopf Link with **linking number** -1, 0_1^2 is the **unlink** of two components, and 0_1 is the unknot.

3.3 Further Polynomial Invariants

As a result of Conway’s discovery of the skein relation, Vaughan Jones discovered the Jones polynomial in 1984. The Jones polynomial, $V_K(t)$, is a more sensitive polynomial invariant which is calculated using a skein relation [13]. $V_K(t)$ also addresses a limitation of the Alexander-Conway polynomial in that it can sometimes distinguish chirality, e.g. right- and left-handed trefoil knots have Jones polynomials $V_{3R}(t) = t + t^3 - t^4$ and $V_{3L}(t) = t^{-1} + t^{-3} - t^{-4}$ respectively [31].

Open Problem 1. Prove that there are no non-trivial knots with $V_K(t) = 1$.

Although more sensitive, the Jones polynomial is by no means a complete invariant and its discovery led to renewed exploration in this area. A year later, a two-variable polynomial invariant was discovered, the HOMFLY (or HOMFLYPT) polynomial, which generalises both the Alexander and the Jones polynomials. The HOMFLY polynomial is again more sensitive and opens further the field of knot theory, however it is still not complete.

Open Problem 2. *Find a complete knot invariant³.*

³a knot invariant, say κ , such that $\kappa(K) = \kappa(K') \Rightarrow K \sim K'$.

Chapter 4

Alternating Knots

Definition. An alternating knot is a **knot** for which there exists a **knot diagram** in which crossings alternate between under and over as one travels around the knot in a given direction. Note that not all knot diagrams of alternating knots need be alternating [1, p. 7].

4.1 Examples

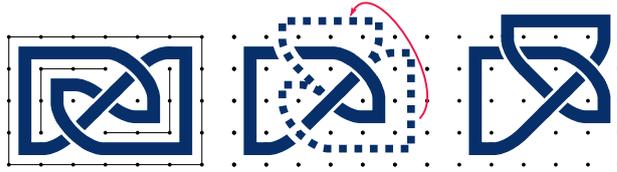
4.1.1 Trefoil Knot, 3_1



The trefoil knot has the lowest **crossing number** of any **prime knot** not **equivalent** to the **unknot**. It is denoted 3_1 since it is the first (and only) prime knot with crossing number 3. The trefoil knot is **chiral** [4, p. 367].

The Conway polynomial of the trefoil knot is $\nabla_{3_1}(z) = 1 + z^2$ [5]. The Alexander polynomial is $\Delta_{3_1}(t) = 1 - t + t^2$. This is unique for knots with up to 10 crossings [31], so if we find a knot with 10 or fewer crossings with the same polynomial then we know it is equivalent to the trefoil knot.

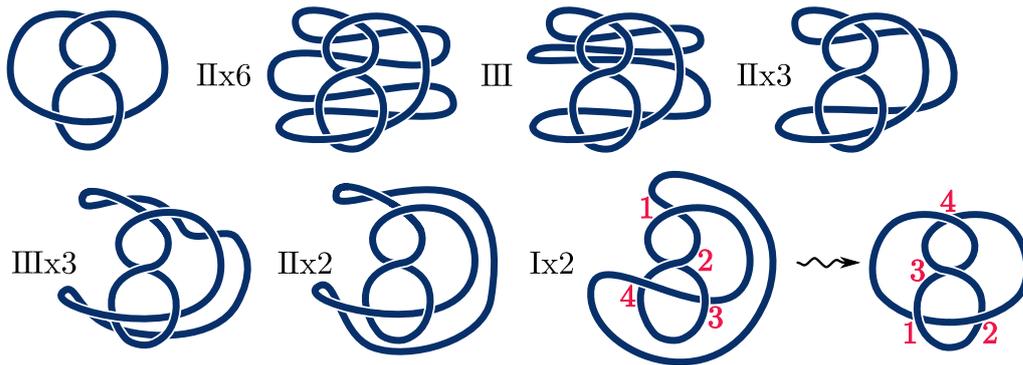
The following is a Celtic knot representation (defined in Chapter 6) of a trefoil knot and a demonstration that it is equivalent:



4.1.2 Figure Eight Knot, 4_1



The figure eight knot is the only prime knot with crossing number 4. Hence, it is denoted 4_1 . We calculated the Conway polynomial in 3.2.1. The Alexander polynomial is $\Delta_{4_1}(t) = 1 - 3t + t^2$ [5]. The figure eight knot is amphichiral as we shall show, listing the type of **Reidemeister move** used in each step (and with labelled crossings for the last step, adjusting the knot without crossing/uncrossing):



We see that with crossing number 3 there are no amphichiral knots but with crossing number 4 there is one. This pattern of zero and non-zero seems to continue, leading Tait to think that there are no prime amphichiral knots with an odd number of crossings. However, Hoste et al. discovered one with crossing number 15 in 1998 [12, p. 37].

Open Problem 3. *Are there prime amphichiral knots for every crossing number ≥ 14 ?¹*

Open Problem 4. *Find a knot invariant that determines whether a knot is amphichiral or not.*

¹there certainly are for crossing numbers 14, 15 and 16.

4.2 Properties

Alternating knots have many special properties. Looking at knots with low crossing numbers, it can seem as if almost all knots are alternating. However, with crossing number 13 there are more non-alternating prime knots than alternating prime knots. And with crossing number 16 there are well over twice as many non-alternating prime knots than alternating prime knots. In fact it has been conjectured that the proportion of alternating knots tends exponentially to zero as the crossing number increases [12, p. 36].

Probably the most striking properties unique to alternating knots, are those conjectured by Tait in the 1870s.

Tait's Conjectures [12, p. 35].

1. ***Reduced** alternating knot diagrams have the minimal number of crossings.*
2. *Any two equivalent reduced alternating knot diagrams have equal **writhe**.*
3. *Any two equivalent reduced alternating knot diagrams are related by a succession of **flypes**.*

It was not until Jones developed his polynomial over 100 years later [13], that any of these conjectures were solved. In 1987, Kauffman, Murasugi and Thistlethwaite proved the first and second conjectures using the Jones polynomial (and Kauffman polynomial) [21], then four years later Menasco and Thistlethwaite proved the third conjecture [18].

There are two other significant properties relating to the Alexander polynomial of alternating knots.

Properties of Alexander polynomial.

1. *If K is an alternating knot, all coefficients of $\Delta_K(t)$ are non-zero.*
2. *If K is an alternating knot, the coefficients of $\Delta_K(t)$ are alternating ².*

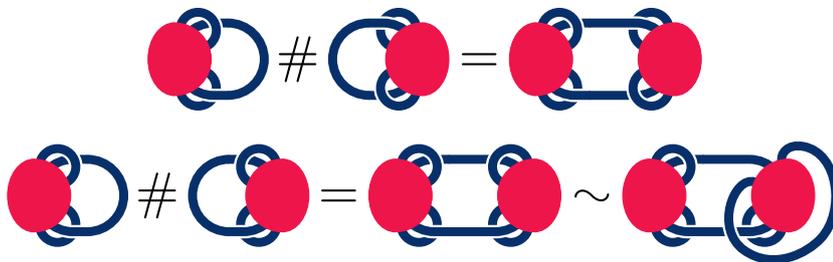
²a polynomial $\sum a_n t^n$ is alternating iff $(-1)^{i+j} a_i a_j \geq 0$

The first theorem was proven by Murasugi in 1958 [20, p. 181] and the second by Crowell in 1959 [7, p. 262].

We can use Tait's first conjecture to prove another property of alternating knots.

Theorem 2. *Alternating knots are adequate³ knots.*

Proof. First we must show that the **knot composition** of two alternating knots is alternating. The easiest way to do this is with knot diagrams, where the pair on the left represent any two reduced alternating knot diagrams. We then give a reduced alternating knot diagram of their composition:



These diagrams, along with their mirror images, cover every combination of alternating knot compositions. In the first case, the composition is clearly reduced and alternating, since both knots on the left hand side are. In the second case, we get a non-alternating knot diagram. By ‘lifting’ one of the strands completely over the rest of the right hand factor knot, we find an alternating knot diagram. We may not have given the set of Reidemeister moves used, but considering our knot in \mathbb{R}^3 we can see we have an equivalent knot, so we have found a reduced alternating knot diagram.

Clearly every crossing which was in the two knot diagrams on the left appears in the knot diagram of the composition, i.e. the number of crossings on each side are equal. Both sides are reduced alternating knot diagrams, and so, by the second conjecture, the numbers of crossings are minimal, i.e. are equal to the crossing numbers. □

Open Problem 5. *Prove that every knot is an adequate knot [16, p. 45].*

³a class of knots whose crossing numbers satisfy: $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

Chapter 5

Non-alternating Knots

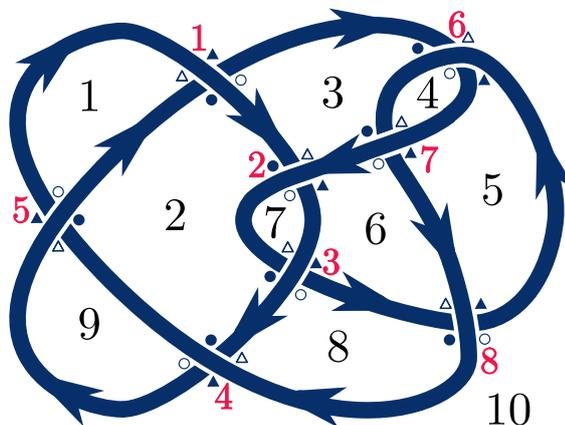
Definition. A non-alternating knot is any **knot** which is not an alternating knot. That is, a non-alternating knot is a knot for which there does not exist a knot diagram in which crossings alternate between under- and over-passes.

There are no prime non-alternating knots with **crossing number** ≤ 7 . With crossing number 8 there are three non-alternating knots: 8_{19} , 8_{20} , and 8_{21} [5].



Theorem 3. 8_{19} is a non-alternating knot.

Proof. Let us compute the Alexander polynomial of 8_{19} . First of all, we give the knot diagram an orientation. Then we number all of the regions (in black) and all of the crossings (in red) and give the appropriate value from $\{1, -1, t, -t\}$ at each crossing/region (where we denote 1 by \bullet , -1 by \circ , t by \blacktriangle and $-t$ by \triangle):



We then create a matrix, where each row corresponds to a crossing, each column to a region, and each entry to the appropriate value in $\{1, -1, t, -t\}$.

$$\begin{pmatrix} -t & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & -t & 0 & 0 & t & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & t & -t & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -t & -1 & t \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -t & t \\ 0 & 0 & 1 & -1 & t & 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 1 & -t & t & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & -t & 0 & 1 & 0 & -1 \end{pmatrix}$$

We then remove any two columns which correspond to adjacent regions. Let us remove the columns corresponding to regions 9 and 10, since they share an edge (between crossings 4 and 5). And we then calculate the determinant:

$$\begin{vmatrix} -t & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 & 0 & t & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & t & -t & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -t \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & -t & t & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & -t & 0 & 1 \end{vmatrix} = t - t^2 + t^4 - t^6 + t^7$$

We now factor out the largest possible power of t , i.e. divide by t^1 and multiply by ± 1 to make sure the constant is positive and we get the Alexander polynomial: $\Delta_{8_{19}}(t) = 1 - t + t^3 - t^5 + t^6$. This polynomial has coefficient zero for t^2 and t^4 . Also, t^1 and t^3 have opposite coefficients. This polynomial fails to meet either of the properties of Alexander polynomials of alternating knots on page 27. Thus 8_{19} is non-alternating. \square

Chapter 6

Celtic Knots

Note. In this chapter we will sometimes use *graph* when we mean *multigraph*¹ or *pseudograph*². Also, we often use *knot* instead of *link*. However, we rarely need to restrict ourselves to one component.

6.1 What is a Celtic knot?

Celtic knots are stylized, interlaced patterns, representing ropes or threads tied in a knot. To be strictly accurate, they are mis-named. They also appear in the art of various other peoples including Romans, Vikings and Saxons. Generally, like our mathematical knots, they have no ends. In this case, they can also be called Gordian knots, due to an ancient Greek myth. As the legend goes, it was said that the person who untied Gordius' knot would rule all of Asia. It remained unsolved for many years, resisting all attempted solutions until 333BC when Alexander the Great cut it through with a sword.

For the purposes of this paper we shall look at certain kinds of Celtic knots that are drawn on a grid of dots. From now on we shall speak of Celtic knots as if they are all of this kind.

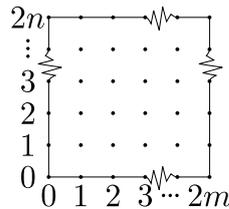
Definition. A Celtic knot is a **link diagram** created by a boundary, barriers (horizontal or vertical lines going from one dot to another, that will 'reflect' the

¹a graph which permits multiple edges between two vertices.

²a multigraph which allows loops (an edge starting and finishing at one vertex).

strands) and tiles, in the following way.

We set a boundary (a rectangular barrier) with dimensions $2m \times 2n$ for $m, n \in \mathbb{N}$. Next we add barriers of even length, starting and finishing (note that we treat an intersection of barriers as a start/end) on dots (x, y) where $x + y$ is even (i.e. either x, y are both even or both odd), where we count rows as follows:



We are now ready to place the tiles. Each tile of the grid must be a rotation of one of the following, which we name, respectively, *corner*, *straight*, *left-bend*, *right-bend* and *diagonal*:



The tiles with two edging barriers can be placed, rotating so each space's and tile's barriers match up.

If we number the tiles by the dot at the bottom-left of the tile, so the bottom left tile is $(0, 0)$ and the top-right tile is $(2m - 1, 2n - 1)$, then the placing of a tile (x, y) is dependent only on its neighbouring barriers and the sum $x + y$. If the space (x, y) has no neighbouring barriers then we place a diagonal tile, NW to SE if $x + y$ is even and SW to NE if $x + y$ is odd.

Note. We see that every diagonal tile has a strand going from (p, q) to (r, s) where both $p + q$ and $r + s$ are odd. Thus crossings will be on dots (x, y) where $x + y$ is odd (the converse is true, that if the sum of the coordinates is odd, then there is a crossing on it, UNLESS there is a boundary or barrier through it), and if the sum of the dot's coordinates are even then the dot has no strands on it. The placing of the bends will ensure this is always the case:

If the space (x, y) has one neighbouring barrier b then we place a:

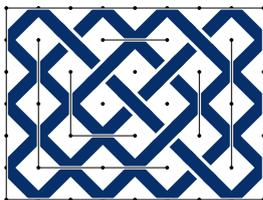
- left-bend if (b horizontal and $x + y$ even) OR (b vertical and $x + y$ odd)

6.3 Celtic Knots vs. Alternating Knots

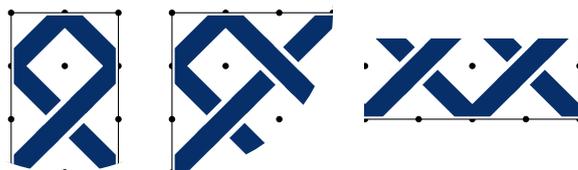
6.3.1 Celtic Knots

Theorem 4. *Every Celtic knot is alternating.*

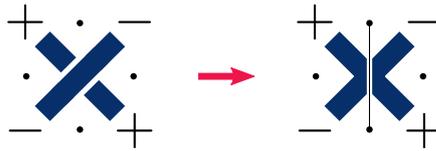
Proof. Let us replace bends, straights and corners with diagonal tiles (ensuring that these would fit our rules if we were to ignore barriers). This will clearly not affect the crossings (since these always come at the corners of tiles anyway). Now, every stretch is either NW to SE or SW to NE. The knot from the construction example in Section 6.2 would become:



Let us take a Celtic knot with no barriers (with all diagonal tiles as above). Let us assume either the width or height is more than two (otherwise we have the unknot). If we follow a stretch away from the boundaries, we will obviously move from odd to even rows or even to odd rows, clearly meeting alternate crossings. At the boundaries the strands still alternate since we can only have a rotation or reflection of the following:

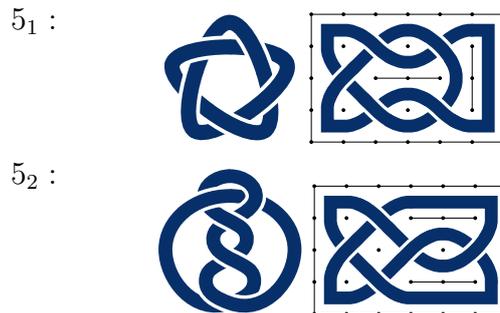


(\dagger) Now we look at adding a barrier of length 2. A barrier goes from some (a, b) to (c, d) where both $a + b$ and $c + d$ are even, and cannot intersect another barrier (otherwise the new barrier will have two stretches of length 1 which is odd and therefore not allowed). Thus the middle of a barrier must be the dot where there was previously a crossing (by the Note on page 32) and the ends of the barrier must be dots where there was previously no strand incident (by the rules of placing diagonals, the end of each diagonal must be at some (x, y) where $x + y$ is odd). Thus we must have a rotation or reflection of the following, where $-$ and $+$ denote the next crossing in this direction being under- and over-crossings respectively:



The link is still alternating after adding a barrier. Therefore we can add barriers repeatedly and we will still have an alternating link. Thus all Celtic knots are alternating. \square

Since we now know that all Celtic knots are alternating, let us examine some alternating knots to see whether they are Celtic:



6.3.2 Alternating Knots

Before meeting the main theorem of this section, we must first define some tools.

Inverse Medial Graphs

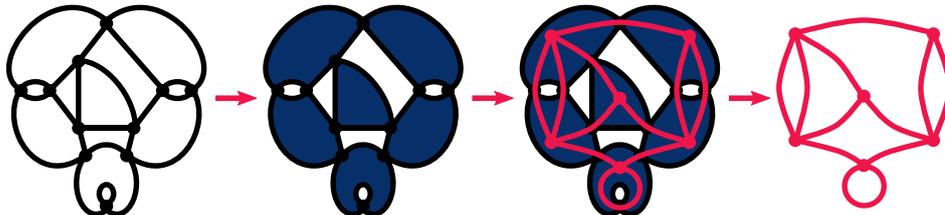
Definition. If we set m to be the function that takes a graph G to its **medial graph**, $m(G)$, then we want to define the *inverse medial graph*, (IMG), to be the image of the map m^{-1} , that is, the map that takes $m(G)$ to G . We construct one as follows [11, p. 89]:

- Colour one of the faces of $m(G)$ blue.
- Colour the opposite faces³ of the blue face in blue.
- Repeat the previous step until there are no neighbouring faces with the same colour, i.e. the faces are 2-coloured.
- Place a vertex in the centre of each blue face.
- Place an edge between two vertices if their corresponding faces share a vertex.

³faces incident at the same vertex but with no edges in common.

- Delete $m(G)$ and the colouring of the faces.

We give an example of constructing an IMG:



Let us check that the construction is well defined. Only the second and third steps make any assumptions so we need just check that these are valid.

First of all, let us check that opposite faces are well defined. Let us take a vertex v of $m(G)$. By definition this corresponds to an edge, say e of G . e has two ends and must be the border between two faces. Thus it has ‘a vertex and face in common’ twice (once for each face) with each end. Therefore v has 4 edges incident at it, i.e. has degree⁴ 4. Since this is true for all vertices of $m(G)$, we say $m(G)$ is *4-valent*. Each vertex of $m(G)$ looks the following image. As we can see, after colouring one face, the opposite face which we are to colour is well defined:



Since each vertex of $m(G)$ has degree 4, the dual⁵ $(m(G))'$ must have edges from each face to the neighbouring faces, i.e. a 4-cycle around v . Since this is true for all vertices of $m(G)$ we see that all cycles in $(m(G))'$ are even. Thus $m(G)$ 4-valent \Rightarrow bipartite dual $(m(G))' \Rightarrow$ the vertices of $(m(G))'$ can be 2-coloured \Rightarrow the faces of $m(G)$ can be 2-coloured. This shows the construction is well defined.

Note. The only criterion to ensure that m^{-1} can work is that $m(G)$ is 4-valent. Indeed, any 4-valent graph G does have an IMG $m^{-1}(G)$, i.e. any 4-valent graph G is a medial graph of some graph $m^{-1}(G)$.

We have set up the construction such that we can return to G .⁶ However, if we reverse the colouring (e.g. if we had first coloured the infinite face in our example) the output is the dual of G , G' . This is unsurprising since $m(G) = m(G')$.

⁴the number of edges incident at a vertex, with loops counted twice.

⁵denoted G' , a graph which has vertices for the faces of G , and edges between vertices v, w if v, w correspond to neighbouring faces of G .

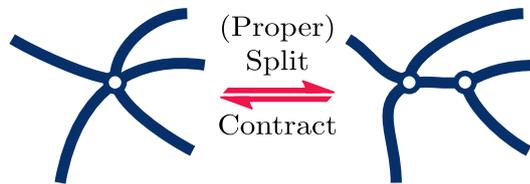
⁶see **medial graph** in the glossary, where we take G to $m(G)$.

Since we started out by 2-colouring the faces of $m(G)$ we cannot get any other possibilities for $m^{-1}(m(G))$.

We find that we have not necessarily found the map that takes $m(G)$ to G , as it may take it to G' , i.e. $m^{-1}(m(G)) = G$ OR G' . However, we have found a map such that $m(m^{-1}(G)) = G$.

Splitting Vertices

Definition. We define *splitting a vertex* to be the inverse operation of edge contraction⁷ [11, p. 91].



When splitting a vertex, we can do this in a number of ways. Importantly, if the degree of a vertex u is ≥ 4 , we can split u such that the two vertices it splits into, say v and w , have degree $\leq \text{degree}(u)$, with no new crossings of edges. Let us call this a *proper split* of vertex u .

This is simple to show, since we can split u so that v takes $\lceil \text{degree}(u)/2 \rceil$ of the edges from u and w takes the other $\lfloor \text{degree}(u)/2 \rfloor$ edges. There is then just the one new edge between v and w , meaning the maximum degree of v and w is $\lceil \text{degree}(u)/2 \rceil + 1 \leq \text{degree}(u)$ for $\text{degree}(u) \geq 4$. We can easily avoid any extra crossings by letting the two new vertices only take edges that were already next to each other, and by placing the last edge carefully.

Orthogonal Grid of Size n

We define an orthogonal grid of size n to be a graph which has vertices at $(1 + an, 1 + bn)$, $\forall a, b \in \mathbb{Z}$, and edges of length n connecting all orthogonal vertices. We define an ‘inner grid’ to be an orthogonal grid of size 2 such that it

⁷removing an edge of a graph, whilst merging the two vertices it previously joined.

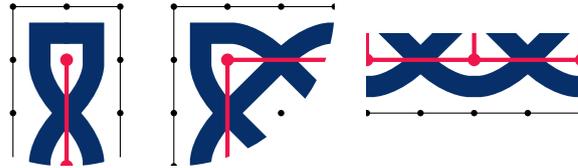
stays within our boundary.

Let us now look at **shadows** of knots. Each crossing becomes a vertex, and since each crossing is of two strands, each vertex has degree 4 (each strand goes into and out of the crossing), i.e. shadows are 4-valent \Rightarrow shadows have IMGs.

Theorem 5. *The IMG (leaving the infinite face white) of a Celtic knot can be constructed as follows [11, p. 90]:*

1. Start with the inner grid H .
2. Delete every edge of H that meets a barrier orthogonally at its mid-point.
3. Contract every edge of H that coincides with a barrier.

Proof. Let us first look at a Celtic knot with no barriers, say C . At the edges and corners we have the following:

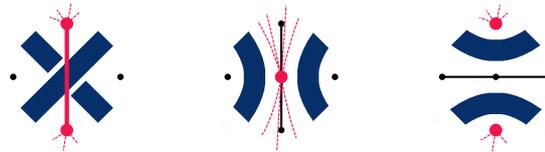


In each case we can see that the IMG (leaving the infinite face white), $m^{-1}(C)$, is the same as the inner grid, H . This is still the case for the third picture since the enclosed face contains the vertex one in from the edge, which means it has one odd coordinate and it is not a crossing so the sum of the coordinates is even (by the Note on page 32) \Rightarrow both coordinates are odd and so it is in the inner grid.

Next we look at the interior of C . By definition, every edge of H starts and finishes on vertices with both odd x - and y -coordinates. Thus the center of each edge is a dot with coordinates (w, z) where $w + z$ is odd, i.e. precisely where there are crossings of C (again, by the Note on page 32). Now we just need to check that the edges go the right direction through the crossings, i.e. that we have chosen the right faces to be coloured. Suppose not, that is, suppose that we have coloured some face containing a dot (u, v) with u, v both even.

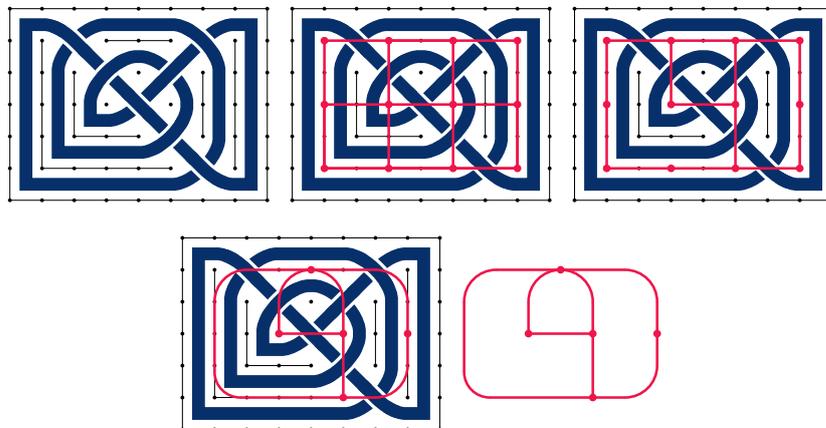
Then as long as we pass through crossings, we keep meeting other coloured faces. In particular, if we move directly Southward we keep passing through crossings to faces containing dots $(u, v - 2), (u, v - 4), \dots$ until we get to $(u, 2)$. But either $(u - 1, 1)$ or $(u + 1, 1)$ are already coloured as in our picture before, and neighbouring $(u, 2)$ so we have a contradiction. We find that there are no faces coloured which contain a dot (u, v) with u, v both even, and so the other faces are coloured in the process of constructing $m^{-1}(C) \Rightarrow H = m^{-1}(C)$.

Now let us look at what happens when we add a barrier of length 2. Note that, by \downarrow in the proof of Theorem 6.3.1 a barrier must be placed in the following way, where the tiles may start as bends (and therefore finish as straights or corners) but only the crossing is important and the graph in red gives the IMG (where the dashed edges represent any number of edges):



We see that, in the second picture, adding a barrier that coincides with an edge of the inner grid contracts the corresponding edge of the IMG and that, in the third picture, adding a barrier that intersects an edge of the inner grid removes the corresponding edge of the IMG. Thus, we can add as many barriers as we want and see that the IMG can be constructed using the theorem. \square

Let us now take an example:



So in the first image we have the Celtic knot, let's call it K , that we found in 6.2. The second image is K with its inner grid overlaid. We then delete the

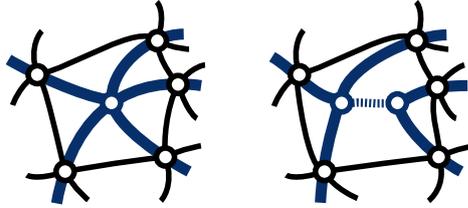
edges of the inner grid which orthogonally intersect barriers of K . This gives us the third image. Next we contract the edges of the inner grid which coincide with barriers of K . Removing the Celtic knot K from the background, we are left with one of the two IMGs of the shadow $s(K)$. We should note that since we are only looking at Celtic knots and therefore alternating knots, $s(K)$ can only be the shadow of 1 or 2 non-equivalent knots. If we replace one vertex with a crossing, all of the other crossings follow alternately. With the first vertex we have a choice of an under- or over-crossing and so there are two possible knots that the shadow defines. However, if K is amphichiral these two knots are equivalent anyway.

We are going to try to create a Celtic knot representation of some knot K by precisely inverting the steps above. We will start by using the IMG of the shadow of the knot, $m^{-1}(s(K))$. Now, $m^{-1}(s(K))$ may have vertices of any degree. However, as we can see in the previous example, if we split a vertex and replace the new edge with a *barrier* (moving from the fourth image to the third), the medial graph remains the same. Intuitively this may seem to work, but we need to be careful with this reuse of the word *barrier*. We have not yet reached a grid environment, and so our definition of a barrier does not hold. Let us therefore redefine it, in the context of an inverse medial graph.

Definition. A barrier b in a graph G is an edge with the following properties:

- Another edge cannot cross (under or over) b .
- b cannot be **subdivided**, i.e. a new vertex cannot be placed on b .
- If edges u and v are incident at either of the vertices that b is incident at, then we say u and v share a vertex.

We draw a barrier as a dashed edge to distinguish it from the other edges. When creating $m(G)$ we can think of b and its end vertices as one ‘long vertex’, since a barrier’s properties make it act like a vertex for this operation. The following image demonstrates a split with a barrier replacing the new edge. It also shows the unchanging medial graph overlaid:



Subdividing an edge but replacing one of the two new edges with a barrier will also leave the medial graph unchanged, since subdividing an edge is the same as splitting one of the end vertices.

Theorem 6. *All alternating knots are Celtic, that is, they can be drawn as Celtic knots.*

Proof. We show this by constructing a Celtic knot equivalent to any given alternating knot, K [11, pp. 91-92]:

1. Construct $G = m^{-1}(s(K))$, leaving the infinite face white.

Note. If K' is the mirror image of K then $s(K) = s(K')$. If K is chiral then by definition it is not equivalent to K' yet has the same shadow. However, we still have uniqueness up to chirality.

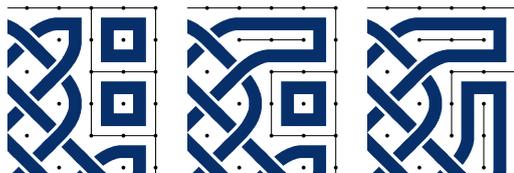
2. If G has vertices of degree ≥ 4 we split those vertices (properly). Each time we split a vertex, we replace the new edge with a barrier. Note that this means the medial graph does not change, as we saw previously.
3. Repeat the previous step until the maximum degree of a vertex is ≤ 4 .
4. **Subdivide** as needed until our graph is a subgraph⁸ of an orthogonal grid of size 2 (possible since if a graph has maximum degree ≤ 4 then it is **homeomorphic** to a subgraph of an orthogonal grid [28, p. 941]). After each subdivision, replace one of the two new edges with a barrier. We call the resulting graph G^* . Here we should note that $m(G^*) = m(G) = s(K)$.
5. Create a boundary, as small as possible around G^* .
6. Next, we place barriers of length 2 within our boundary, orthogonally to, and bisecting, each edge in the inner grid but not in G^* . (See †, ‡ in Section 7.2 for an example.)

⁸ S is a subgraph of G if all the vertices and edges of S appear in G .

7. Remove G^* , leaving behind its barriers and the new barriers.
8. Construct the knot that this set of barriers uniquely defines. (As shown in Section 6.2.)

Once this is done, we have a Celtic knot C . Step 6 ensures the rest of the algorithm constructs our knot by Theorem 5, since we have inverted both the second and third steps of the theorem. Thus $s(C) = m(G^*) = m(G) = s(K)$. That is, up to chirality, C is equivalent to K .

Note. In step 6 there is a chance that the barriers may cordon off sections. When step 8 is followed this would then lead to separate unknots. However, this can be easily fixed. One of the separating barriers can just be replaced with its orthogonal barrier. This can be repeated until all of the unknots become composed with the original knot ($K \# 0_1 = K$). This process is demonstrated below:



□

Theorem 7. *All knots are ‘Celtish’, that is up to crossings they are Celtic.*

Proof. As our proof of Theorem 6 depends only on $s(K)$, the shadow of the alternating knot K , we may also apply the result to non-alternating knots. This is because in the final step of the algorithm, the construction of the knot, it will only differ in the choice of the crossings. □

6.4 Polynomials of Celtic Knots

Let us look at Celtic knots with no barriers, which we call *plaits*. We let C_m^n denote a plait with dimensions $2m \times 2n$. Depending on our choice of m and n , we may create links of 1 or more components. We quickly discover if $\gcd(m, n) = 1$, i.e. m and n are coprime, then we create a link of 1 component (a knot) [17, p. 233].

Theorem 8. *If $\gcd(m, n) = k$ then the plait C_m^n has exactly k components.*

Proof. Let us replace bends and corners with diagonal tiles (ensuring that these would fit our rules if we were to ignore barriers). We must still have the same number of components, but now, walking along any tile's strand we move ± 1 both horizontally and vertically. Then to walk from one tile, say (x, y) , along its entire component and get back to (x, y) , we must make $4am$ horizontal steps (to the end and back a times) and $4bn$ vertical steps (to the top and bottom b times) for some $a, b \in \mathbb{N}$. Since every tile is diagonal, we must take as many vertical as horizontal steps, so $4am = 4bn \Rightarrow am = bn$. We want to know the least number of steps in order to return, that is we want to find the a and b such that $am = bn = \text{lcm}(m, n)$. We note:

$$\text{lcm}(m, n) = \frac{mn}{\gcd(m, n)} \Rightarrow a = \frac{n}{\gcd(m, n)} \text{ and } b = \frac{m}{\gcd(m, n)}$$

Since a is the number of times we crossed to the end and back, it must be the number of times we turned around at, and hit, the vertical axis. This happens at $(0, 1), (0, 3), \dots, (0, 2n - 1)$ i.e. at n locations. (x, y) could have been on any component, so every component must meet the vertical axis exactly a times. Thus, the number of components is $\frac{n}{a} = \gcd(m, n)$. \square

Theorem 9. C_m^2 has Conway polynomial $\frac{(-z + \sqrt{1 + z^2})^m - (-z - \sqrt{1 + z^2})^m}{2\sqrt{1 + z^2}}$.

Proof. We see from Theorem 6.4 that C_m^2 has one or two components (for m odd or even respectively). If this gives two components they are oriented as follows (note $C_3^2 \sim 7_4$):



Looking at the Conway polynomials, we find:

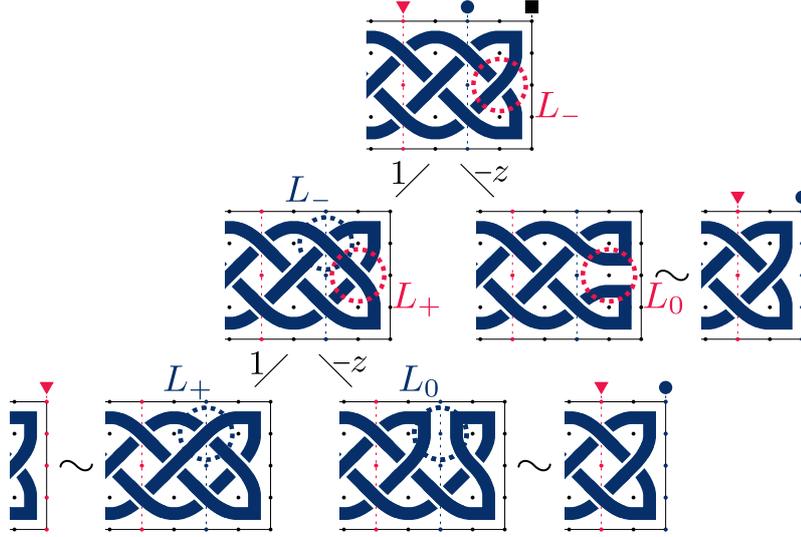
$$\nabla_{C_1^2}(z) = 1 \tag{6.1}$$

$$\nabla_{C_2^2}(z) = -2z \tag{6.2}$$

$$\nabla_{C_m^2}(z) = \nabla_{C_{m-2}^2}(z) - 2z\nabla_{C_{m-1}^2}(z) \quad \forall m \geq 2 \tag{6.3}$$

where 6.1 and 6.2 are the initial conditions since $C_1^2 \sim 0_1$ and $C_2^2 \sim 4_1^2$ where $\nabla_{4_1^2}(z) = 1 \cdot H_- - z \cdot 0_1 = 1(-z) - z(1) = -2z$ ($\Rightarrow \nabla_{C_0^2}(z) = 0$). Equation 6.3

comes from the following skein tree (where $\blacktriangledown = 2(m-2)$, $\bullet = 2(m-1)$, $\blacksquare = 2m$:



Now that we have a recurrence relation and initial conditions we can find the solution for any m . First of all we find the roots of the characteristic equation:

$$\begin{aligned} c(r) &= r^2 + 2zr - 1 \\ (r + z)^2 - z^2 - 1 &= 0 \\ r &= -z \pm \sqrt{1 + z^2}. \end{aligned}$$

Then the solution to the recurrence takes the form:

$$\begin{aligned} \nabla_{C_m^2}(z) &= \alpha_1(-z + \sqrt{1 + z^2})^m + \alpha_2(-z - \sqrt{1 + z^2})^m \\ m = 0 &\Rightarrow \alpha_1 = -\alpha_2 \\ m = 1 &\Rightarrow 1 = \alpha_1(-z + \sqrt{1 + z^2}) - \alpha_1(-z - \sqrt{1 + z^2}) \\ &= 2\alpha_1\sqrt{1 + z^2} \\ \alpha_1 &= \frac{1}{2\sqrt{1 + z^2}}, \quad \alpha_2 = \frac{-1}{2\sqrt{1 + z^2}} \\ \nabla_{C_m^2}(z) &= \frac{(-z + \sqrt{1 + z^2})^m - (-z - \sqrt{1 + z^2})^m}{2\sqrt{1 + z^2}}. \end{aligned}$$

□

$\Rightarrow 7_4 \sim C_3^2$ has Conway polynomial given by:

$$\begin{aligned} \nabla_{7_4}(z) &= \frac{(-z + \sqrt{1 + z^2})^3 - (-z - \sqrt{1 + z^2})^3}{2\sqrt{1 + z^2}} \\ &= \frac{8z^2\sqrt{1 + z^2} + 2\sqrt{1 + z^2}}{2\sqrt{1 + z^2}} \\ &= 1 + 4z^2. \end{aligned}$$

6.5 Celtic Invariants

Now that we have examined how the Conway polynomial invariant relates to Celtic knots, we can investigate other invariants specific to the rules that we have defined for Celtic (and Celtish) knots:

Minimum Area, α . For a knot, K , the minimum area is defined to be the minimum number of tiles that any Celtish knot, K' , can be drawn on, for all $K' \sim K$. Clearly if our boundary dimensions are $2m \times 2n$ then the area is $4mn$.

Minimum Number of Barriers, β . The minimum b , where b is the number of barriers of length 2, that any Celtish knot, K' , can be drawn with, for all $K' \sim K$.

Minimum Dimension, δ . The minimum dimension for a knot, K , is the smallest width (or height) that any Celtish knot, K' , can be drawn with, for all $K' \sim K$. For all non-trivial knots, the minimum dimension ≥ 4 . This is because if we have a knot with boundary dimensions $2m \times 2n$ and either m or $n = 1$, say $n = 1$, then with $n - 1 - b$ twists (i.e. Type I moves) we have the unknot.

Minimum Perimeter, π . Defined as for area and barriers.

The following relations are useful to derive lower bounds for the above invariants:

Theorem 10. *If c is the number of crossings in the Celtish knot we have that $c + b = 2mn - m - n$. Also, $4mn \geq 2b + 2c + 8$ and $4mn \geq 2c + 8$.*

Proof. For every dot inside our boundary with coordinates (x, y) , where $x + y$ is odd, there is a crossing or a barrier of length 2 by the Note on page 32. The Note also shows that barriers and crossings cannot be anywhere else.

Now, for the m odd columns (at $y = 1, \dots, 2m - 1$) we need even rows (so that $x + y$ is odd as above) which are at $x = 2, \dots, 2n - 2$ implying that we have $n - 1$ even rows. For the $m - 1$ even columns (at $y = 2, \dots, 2m - 2$) we need odd rows (so

that $x+y$ is odd) which are at $x = 1, \dots, 2n-1$ implying that we have n odd rows.

Thus the number of crossings and barriers is $c + b = m(n-1) + n(m-1) = 2mn - m - n$.

We find if b is minimised and the boundary dimensions, $2m \times 2n$, are minimised then the minimum area is $4mn = 2b + 2c + 2m + 2n$. Now, since the minimum dimension is ≥ 4 we have $m, n \geq 2 \Rightarrow 4mn \geq 2b + 2c + 8$ and $b \geq 0 \Rightarrow 4mn \geq 2c + 8$. \square

Let us look at some examples and their Celtic invariants:

	0_1	3_1	4_1	5_1	5_2	7_4	6_2^3
α	4	16	24	24	24	24	36
β	0	1	3	2	2	0	6
δ	2	4	4	4	4	4	6
π	8	16	20	20	20	20	24

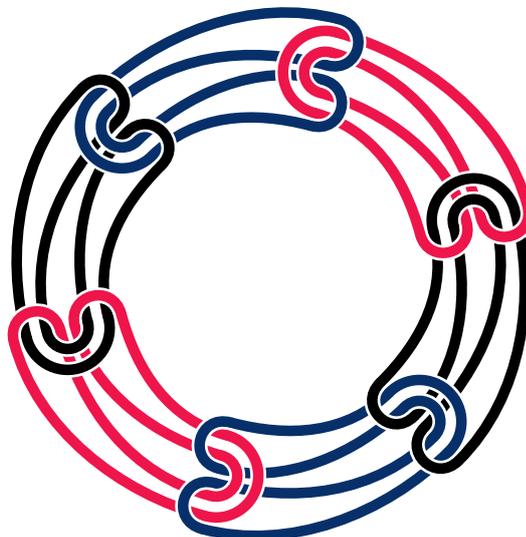
We see that none of these Celtic invariants are particularly sensitive, and even a combination of all four is not enough to tell 5_1 and 5_2 apart. However, if we were to consider barriers within the inner grid as separate from the other barriers (those orthogonal to some edge of the inner grid), we may start to get a more precise invariant. These sorts of invariants are more akin to the ‘unknotting number’ in that it is very easy to find upper bounds for them but difficult to check that they cannot be improved. We can, at least, use the relations we have in order to discern lower bounds.

Chapter 7

Brunnian Links

Definition. A Brunnian link is a set of n linked **unknots** such that the removal of any one component leaves the **unlink** of $n - 1$ components.[1, p. 22]

It is worth noting that there are Brunnian links $\forall n \geq 3$. This is easily seen by example, with the Rubberband Brunnian link. In the image below, this link is drawn with six components but using the same pattern, they could clearly be drawn $\forall n \geq 3$:



Although often drawn as circles (particularly in the case of the Borromean rings), it turns out that they cannot be constructed out of any rigid, circular loops with finite thickness. However, they can be constructed out of ellipses as closely approximating circles as desired as long as they are not exactly circles, [10, p. 81]¹.

¹a much more accessible adaptation of Freedman's proof is found at <http://homepages.math.uic.edu/~agol/circles.pdf>, written by Ian Agol.

7.1 $n = 3$: Borromean Rings, 6_2^3

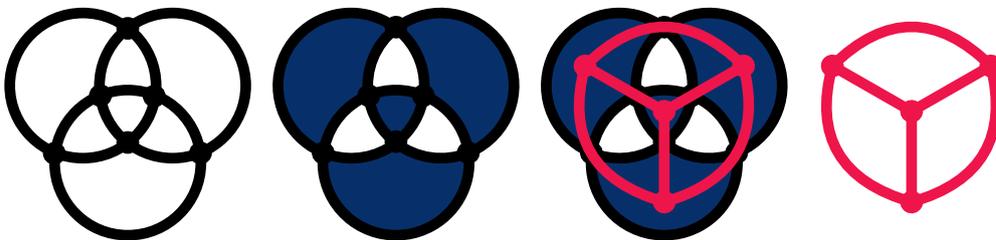
In the case that $n = 3$ we have the most commonly seen Brunnian link, the Borromean rings. The Borromean rings have a rich history, spanning many centuries and various cultures. They are seen throughout the world, as the Brunnian property is seen as a symbol of strength through unity and of interconnectedness.

The rings got their name from the Italian Renaissance family, Borromeo, who used them on their coat of arms. The first time that the Borromean rings were studied from a mathematical point of view was in 1876 by Tait [27, p. 300].



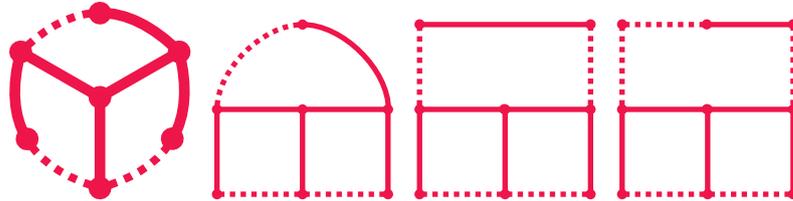
7.2 Celtic Borromean Rings

Let us follow the algorithm from the proof of Theorem 6 to show that the Borromean Rings are equivalent to a Celtic knot. First of all we create $s(K)$, the shadow of the Borromean Rings, and then two-colour it, leaving the infinite face white. Next, we create the inverse medial graph $G = m^{-1}(s(K))$ ($= K_4$, the complete graph with four vertices).

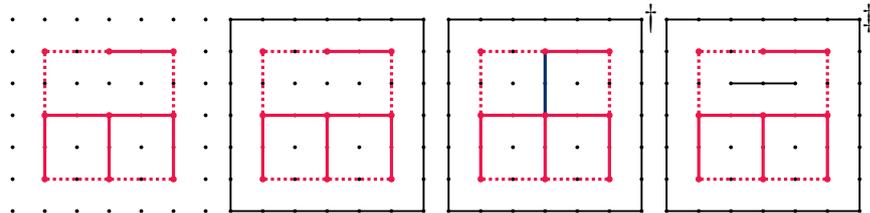


We find that we do not need to do step 2 (or 3) of the algorithm, as we have no vertices of degree ≥ 4 , so we go to step 4 and start subdividing. We are aiming to make our graph into a subgraph of a grid, and although there are computer algorithms to do this, we are just going to use our intuition. First of all, we

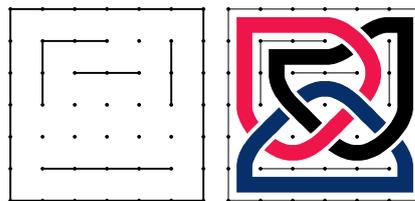
note that cycles in a grid, and therefore cycles in a subgraph of a grid, must have length at least 4. Therefore we subdivide one of the edges in each 3-cycle, each time replacing one of the two new edges with a barrier. However this is not enough, as we see in the second image below. If we subdivide the top-right edge twice more, we get to a subgraph of a grid:



We now make our graph into a subgraph of an orthogonal grid of size 2. This translates to scaling our graph so that every vertex is on some (x, y) where x and y are both odd. Next, we draw a boundary around our graph to get the second picture below. Now, we compare our graph with the inner grid (in †, which contains the extra edges of the inner grid in blue). In the third image we see that there is one edge of the inner grid which does not appear in our graph. In following step 6 of the algorithm, we draw a barrier orthogonally to this edge.



We now remove the graph, leaving behind all of its boundaries and barriers, and construct the link as in Section 6.2.



Chapter 8

Conclusion

In summary, knot theory is a very interesting subject to study. It is tangible and visible which makes it interesting to the non-mathematical outsider - however it can become very complicated, very quickly. Throughout this paper we discovered that knowledge of many different fields of mathematics were required, even for the more simple proofs. In writing this paper we aimed to make such proofs accessible and comprehensive, despite the fact some are quite lengthy.

This paper took its own shape, with study of one area leading into another. Examining invariants and their properties led into the interest of alternating knots which subsequently led to our study of Celtic knots. The investigations into Celtic knots and the Brunnian links bring together two different artistic representations of knot theory. The connection between pure maths and something based entirely upon aesthetics was particularly interesting.

In our studies we found that there is much more to be discovered. Throughout the paper we have highlighted a small selection of the open problems within the field. There is a demand for further understanding of such problems, not just for interest's sake, but in relation to some of the applications in science.

Glossary

Ambient Isotopy: Two knots are ambient isotopic if there exists a deformation of one knot, without breaking it or allowing it to intersect itself, into the other [1, p. 12].

Chirality: A knot is amphichiral if it is equivalent to its mirror image [1, p. 175], e.g. the figure eight knot, see 4.1.2. Otherwise, it is a chiral knot, e.g. the trefoil knot, see 4.1.1.

Crossing Number: n is the crossing number of a knot, K , i.e. $c(K) = n$, if there exists a link diagram of K with n crossings and no link diagrams of K with fewer than n crossings [1, p. 3].

Flype: A flype is a 180° flip of a 2-subtangle (a region of a link diagram such that the link or knot crosses the edge of the region exactly four times), R , of a knot [14, p. 231].



Graph Homeomorphism: Two graphs G, G' , are homeomorphic if there exists a graph that is a subdivision of both [32, p. 62].

Knot: Knots in mathematics are just like those that appear every day, except that the ends are connected. More precisely, a knot is an embedding of S^1 (a circle) in \mathbb{R}^3 that does not intersect itself [29].

Knot Composition: If K_1 and K_2 are (oriented) knots, their composition, $K_1 \# K_2$, or ‘connect-sum’, is obtained by removing an arc from each knot

and splicing the ends to achieve a single component (ensuring the orientation is consistent on the new knot) [1, pp. 7-11]. We then call K_1 and K_2 the ‘factor knots’.



Knot Equivalence: Two knots K, K' are equivalent if one is an ambient isotopy of the other. We write $K \sim K'$. Reidemeister’s Theorem states that two link diagrams (and therefore any two links) are equivalent if and only if they are connected by a sequence of Reidemeister moves [22].

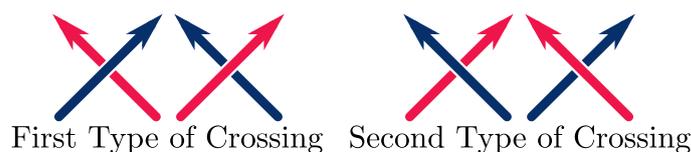
Knot Invariant: A knot invariant is some measure of a knot, K , that is fixed for all knots equivalent to K [1, p. 21].

Link: A link is a set of knots that do not intersect but may be linked together (if not, we say the link is splittable). Thus a knot is a link of one component [1, p. 17].



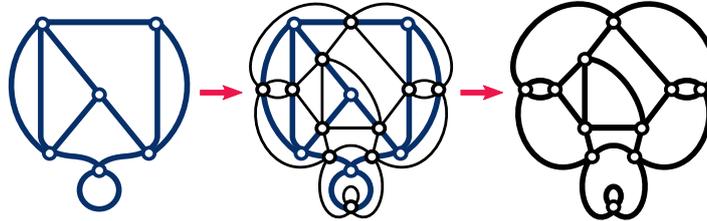
Link Diagram: A link diagram of a link is a projection of the link onto a plane. By selecting an appropriate direction of projection one can arrange it so that the projection is one-to-one except at the transverse (or ‘single point’) intersections of two strands, called crossings. The strands are then identified by creating a break in the under-strand [30]. (See (4.1.1) for an example.)

Linking Number: The linking number is a measure of how ‘linked up’ two components are. After assigning an orientation to both components, crossings (between one component and another - we are not interested in the crossings between a component and itself) are of the following types:



We then define the linking number to be $(+1(\# \text{ first type of crossing}) - 1(\# \text{ second type of crossing}))/2$ [1, p. 19].

Medial Graph: A medial graph $m(G)$ of a graph G is defined by having vertices halfway along the edges of G and having edges joining the vertices of $m(G)$ if the vertices represent edges in G that have a vertex and face in common [11, pp. 88-89].



Orientation: A knot is oriented if it is given a continuous orientation, that is, a direction on which to travel along the knot. The orientation is denoted by placing arrows along the link diagram [1, p. 10].

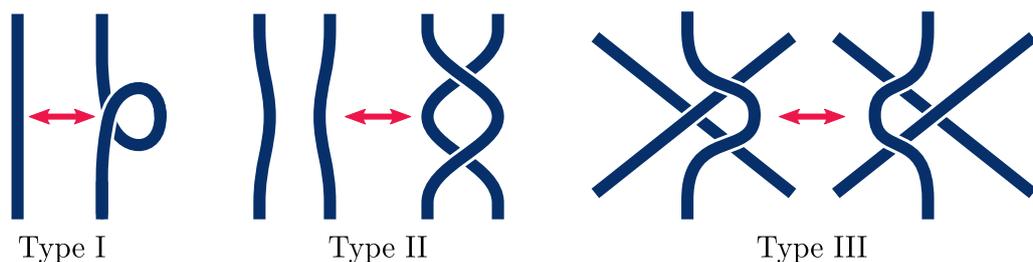


Prime Knot: A knot is prime if it is not the composition of two knots (that are not equivalent to the unknot) [1, p. 9].

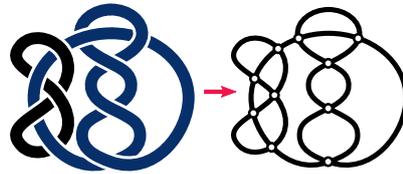
Reduced: The link diagram of a knot is reduced if it does not contain any easily removed crossings [1, p. 68].



Reidemeister Moves: The Reidemeister moves are a set of three moves on link diagrams, that correspond to ambient isotopy (by Reidemeister's Theorem [22]). They are Type I, the twist move, Type II, the poke move, and Type III, the slide move [1, p. 13].



Shadow (or Normal Projection): A shadow $s(K)$ of a link K is obtained by replacing each crossing of the link diagram by vertices of degree 4 to obtain a 4-regular graph [11, p. 89].



Subdivision: A subdivision of a graph G is obtained by subdividing its edges (that is, inserting new vertices of degree 2 into its edges). Below, we subdivide an edge e between two vertices u and v creating a new graph with one new vertex w , and two new edges $\{u, w\}$ and $\{w, v\}$ replacing e [2, p. 205].

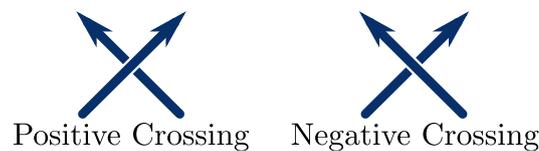


Unknot, 0_1 : Intuitively, the unknot is a loop that is not knotted. That is, an unknot is an embedding of S^1 in \mathbb{R}^3 , such that it is an ambient isotopy of a geometrically round circle [29].



Unlink: The unlink of n components is a splittable set of n unknots, that is n unknots that are equivalent to a link diagram without any crossings [1, p. 18].

Writhe: On an oriented link diagram, L , the writhe, w , is given by $w(L) = +1(\# \text{ positive crossings}) - 1(\# \text{ negative crossings})$ [1, p. 152].



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