

SMSTC Geometry and Topology 2011–2012

Lecture 7

The classification of surfaces

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Manifolds

- ▶ An n -**dimensional manifold** M is a topological space such that each $x \in M$ has an open neighbourhood $U \subset M$ homeomorphic to n -dimensional Euclidean space \mathbb{R}^n

$$U \cong \mathbb{R}^n .$$

- ▶ Strictly speaking, need to include the condition that M be Hausdorff and paracompact = every open cover has a locally finite refinement.
- ▶ Called n -**manifold** for short.
- ▶ Manifolds are the topological spaces of greatest interest, e.g. $M = \mathbb{R}^n$.
- ▶ Study of manifolds initiated by Riemann (1854).
- ▶ A **surface** is a 2-dimensional manifold.
- ▶ Will be mainly concerned with manifolds which are compact = every open cover has a finite refinement.

Why are manifolds interesting?

- ▶ Topology.
- ▶ Differential equations.
- ▶ Differential geometry.
- ▶ Hyperbolic geometry.
- ▶ Algebraic geometry. Uniformization theorem.
- ▶ Complex analysis. Riemann surfaces.
- ▶ Dynamical systems,
- ▶ Mathematical physics.
- ▶ Combinatorics.
- ▶ Topological quantum field theory.
- ▶ Computational topology.
- ▶ Pattern recognition: body and brain scans.
- ▶ ...

Examples of n -manifolds

- ▶ The n -dimensional Euclidean space \mathbb{R}^n
- ▶ The n -sphere S^n .
- ▶ The n -dimensional projective space

$$\mathbb{R}P^n = S^n / \{z \sim -z\} .$$

- ▶ **Rank theorem in linear algebra.** If $J : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ is a linear map of rank k (i.e. onto) then $J^{-1}(0) = \ker(J) \subseteq \mathbb{R}^{n+k}$ is an n -dimensional vector subspace.
- ▶ **Implicit function theorem.** The solutions of differential equations are generically manifolds. If $f : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ is a differentiable function such that for every $x \in f^{-1}(0)$ the Jacobian $k \times (n+k)$ matrix $J = (\partial f_i / \partial x_j)$ has rank k , then

$$M = f^{-1}(0) \subseteq \mathbb{R}^{n+k}$$

is an n -manifold.

- ▶ In fact, every n -manifold M admits an embedding $M \subseteq \mathbb{R}^{n+k}$ for some large k .

Manifolds with boundary

- ▶ An n -**dimensional manifold with boundary** $(M, \partial M \subset M)$ is a pair of topological spaces such that
 - (1) $M \setminus \partial M$ is an n -manifold called the **interior**,
 - (2) ∂M is an $(n - 1)$ -manifold called the **boundary**,
 - (3) Each $x \in \partial M$ has an open neighbourhood $U \subset M$ such that

$$(U, \partial M \cap U) \cong \mathbb{R}^{n-1} \times ([0, \infty), \{0\}) .$$

- ▶ A manifold M is **closed** if $\partial M = \emptyset$.
- ▶ The boundary ∂M of a manifold with boundary $(M, \partial M)$ is closed, $\partial \partial M = \emptyset$.
- ▶ **Example** (D^n, S^{n-1}) is an n -manifold with boundary.
- ▶ **Example** The product of an m -manifold with boundary $(M, \partial M)$ and an n -manifold with boundary $(N, \partial N)$ is an $(m + n)$ -manifold with boundary

$$(M \times N, M \times \partial N \cup_{\partial M \times \partial N} \partial M \times N) .$$

The classification of n -manifolds I.

- ▶ Will only consider compact manifolds from now on.
- ▶ A function

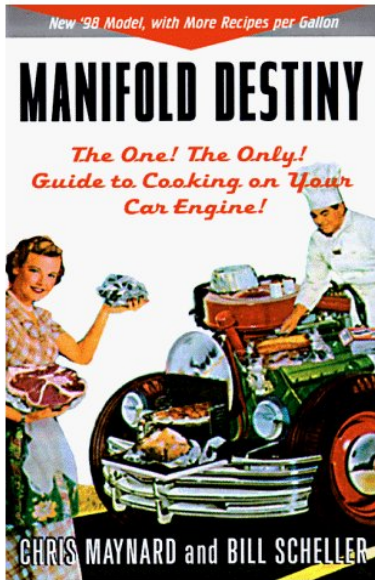
i : a class of manifolds \rightarrow a set ; $M \mapsto i(M)$

is a **topological invariant** if $i(M) = i(M')$ for homeomorphic M, M' . Want the set to be finite, or at least countable.

- ▶ **Example 1** The dimension $n \geq 0$ of an n -manifold M is a topological invariant (Brouwer, 1910).
- ▶ **Example 2** The number of components $\pi_0(M)$ of a manifold M is a topological invariant.
- ▶ **Example 3** The orientability $w(M) \in \{-1, +1\}$ of a connected manifold M is a topological invariant.
- ▶ **Example 4** The Euler characteristic $\chi(M) \in \mathbb{Z}$ of a manifold M is a topological invariant.
- ▶ A **classification** of n -manifolds is a topological invariant i such that $i(M) = i(M')$ if and only if M, M' are homeomorphic.

The classification of n -manifolds II. $n = 0, 1, 2, \dots$

- ▶ **Classification of 0-manifolds** A 0-manifold M is a finite set of points. Classified by $\pi_0(M) = \text{no. of points} \geq 1$.
- ▶ **Classification of 1-manifolds** A 1-manifold M is a finite set of circles S^1 . Classified by $\pi_0(M) = \text{no. of circles} \geq 1$.
- ▶ **Classification of 2-manifolds** Classified by $\pi_0(M)$, and for connected M by the fundamental group $\pi_1(M)$. Details to follow!
- ▶ For $n \leq 2$ homeomorphism \iff homotopy equivalence.
- ▶ It is theoretically possible to classify 3-manifolds, especially after the Perelman solution of the Poincaré conjecture.
- ▶ It is not possible to classify n -manifolds for $n \geq 4$. Every finitely presented group is realized as $\pi_1(M) = \langle S | R \rangle$ for a 4-manifold M . The word problem is undecidable, so cannot classify $\pi_1(M)$, let alone M .



How does one classify surfaces?

- ▶ (1) Every surface M can be **triangulated**, i.e. is homeomorphic to a finite 2-dimensional cell complex

$$M \cong \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 .$$

- ▶ (2) Every connected triangulated M is homeomorphic to a normal form

$M(g)$ orientable, genus $g \geq 0$,

$N(g)$ nonorientable, genus $g \geq 1$

- ▶ (3) No two normal forms are homeomorphic.
- ▶ Similarly for surfaces with boundary, with normal forms $M(g, h)$, $N(g, h)$ with genus g , and h boundary circles.
- ▶ History: (2)+(3) already in 1860-1920 (Möbius, Clifford, van Dyck, Dehn and Heegaard, Brahana). (1) only in the 1920's (Rado, Kerékjártó). Today will only do (3), by computing π_1 of normal forms.

A page from Dehn and Heegaard's *Analysis Situs* (1907)

B. NEXUS II. 4. Anwendungen der Normalform.

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*Jede geschlossene Fläche kann stets mit drei Elementarflächenstücken bedeckt werden. Jede nicht geschlossene Fläche und jede Kugel-
fläche kann mit zwei Elementarflächen bedeckt werden^{95a)}.*

d) Normalformen für geschlossene Flächen⁹⁶⁾.

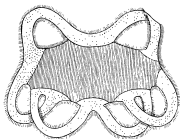


Fig. 8.



Fig. 9.

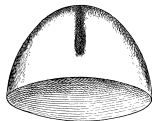


Fig. 10.



Fig. 11.

a) *Zweiseitige Flächen.* Eine Fläche, deren Restfläche p Doppelbänder hat, ist homöomorph mit einer Kugel mit p „Henkeln“ (Fig 9);

⁹⁵⁾ Möbius, Leipzig Ber. 15 (1863) — Werke 2, p. 450.

⁹⁶⁾ Diese Formen für geschlossene Flächen sind, soweit zweiseitige Flächen in Betracht kommen, als betrachtet worden von Riemann (cf. Klein, Über Riemanns Theorie . . . (1882), p. IV), Möbius, a. a. O. § 16, Tonelli (Rom Line. Atti (2) 2 (1876), p. 594, vgl. Rom Line. Rend. (5) 4¹ (1895), p. 309; W. K. Clifford London Proc. Math. Soc. 8 (1877), p. 292). Normalformen für einseitige Flächen sind von Dyck a. a. O. aufgestellt.

The connected sum I.

- ▶ Given an n -manifold with boundary $(M, \partial M)$ with M connected use any embedding $D^n \subset M \setminus \partial M$ to define the **punctured n -manifold with boundary**

$$(M_0, \partial M_0) = (\text{cl.}(M \setminus D^n), \partial M \cup S^{n-1}).$$

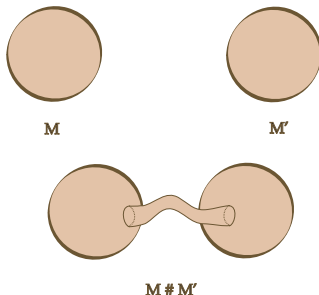
- ▶ The **connected sum** of connected n -manifolds with boundary $(M, \partial M)$, $(M', \partial M')$ is the connected n -manifold with boundary

$$(M \# M', \partial(M \# M')) = (M_0 \cup_{S^{n-1}} M'_0, \partial M \cup \partial M').$$

Independent of choices of $D^n \subset M \setminus \partial M$, $D^n \subset M' \setminus \partial M'$.

- ▶ If M and M' are closed then so is $M \# M'$.

The connected sum II.



- ▶ The connected sum $\#$ has a neutral element, is commutative and associative:

- (i) $M \# S^n \cong M$,
- (ii) $M \# M' \cong M' \# M$,
- (iii) $(M \# M') \# M'' \cong M \# (M' \# M'')$.

The fundamental group of a connected sum

- ▶ If $(M, \partial M)$ is an n -manifold with boundary and M is connected then M_0 is also connected. Can apply the Seifert-van Kampen Theorem to

$$M = M_0 \cup_{S^{n-1}} D^n$$

to obtain

$$\pi_1(M) = \pi_1(M_0) *_{\pi_1(S^{n-1})} \{1\} = \begin{cases} \pi_1(M_0) & \text{for } n \geq 3 \\ \pi_1(M_0) / \langle \partial \rangle & \text{for } n = 2 \end{cases}$$

with $\langle \partial \rangle \triangleleft \pi_1(M_0)$ the normal subgroup generated by the boundary circle $\partial : S^1 \subset M_0$.

- ▶ Another application of the Seifert-van Kampen Theorem gives

$$\begin{aligned} \pi_1(M \# M') &= \pi_1(M_0) *_{\pi_1(S^{n-1})} \pi_1(M'_0) \\ &= \begin{cases} \pi_1(M) * \pi_1(M') & \text{for } n \geq 3 \\ \pi_1(M_0) *_{\mathbb{Z}} \pi_1(M'_0) & \text{for } n = 2. \end{cases} \end{aligned}$$

Orientability for surfaces

- ▶ Let M be a connected surface, and let $\alpha : S^1 \rightarrow M$ be an injective loop.
 - ▶ α is **orientable** if the complement is not connected, in which case it has 2 components.
 - ▶ α is **nonorientable** if the complement $M \setminus \alpha(S^1)$ is connected.
- ▶ **Definition** M is **orientable** if every $\alpha : S^1 \rightarrow M$ is orientable.
- ▶ **Jordan Curve Theorem** \mathbb{R}^2 is orientable.
- ▶ **Example** The 2-sphere S^2 and the torus $S^1 \times S^1$ are orientable.
- ▶ **Definition** M is **nonorientable** if there exists a nonorientable $\alpha : S^1 \rightarrow M$, or equivalently if Möbius band $\subset M$.
- ▶ **Example** The Möbius band, the projective plane $\mathbb{R}P^2$ and the Klein bottle K are nonorientable.
- ▶ **Remark** Can similarly define orientability for connected n -manifolds M , using $\alpha : S^{n-1} \rightarrow M$, $\pi_0(M \setminus \alpha(S^{n-1}))$.

The orientable closed surfaces $M(g)$ I.

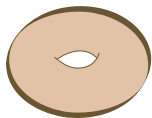
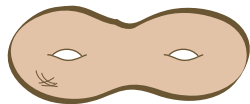
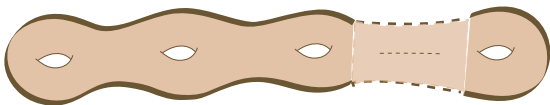
- ▶ **Definition** Let $g \geq 0$. The **orientable connected surface with genus g** is the connected sum of g copies of $S^1 \times S^1$

$$M(g) = \#_g(S^1 \times S^1)$$

- ▶ **Example** $M(0) = S^2$, the 2-sphere.
- ▶ **Example** $M(1) = S^1 \times S^1$, the torus.
- ▶ **Example** $M(2) =$ the 2-holed torus, by Henry Moore.



The orientable closed surfaces $M(g)$ II.

 $M(0)$  $M(1)$  $M(2)$  $M(g)$

The nonorientable surfaces $N(g)$ I.

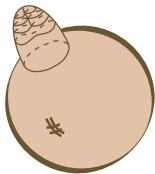
- ▶ Let $g \geq 1$. The **nonorientable connected surface with genus g** is the connected sum of g copies of $\mathbb{R}P^2$

$$N(g) = \#_g \mathbb{R}P^2$$

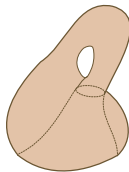
- ▶ **Example** $N(1) = \mathbb{R}P^2$, the projective plane.
- ▶ **Boy's immersion of $\mathbb{R}P^2$ in \mathbb{R}^3** (in Oberwolfach)



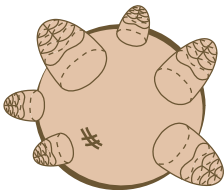
The nonorientable closed surfaces $N(g)$ II.



Projective plane = $N(1)$



Klein bottle = $N(2)$



$N(g)$

The Klein bottle

- ▶ **Example** $N(2) = K$ is the Klein bottle.
- ▶ The Klein bottle company



The classification theorem for closed surfaces

- ▶ **Theorem** Every connected closed surface M is homeomorphic to exactly one of

$$M(0), M(1), \dots, M(g) = \#_g S^1 \times S^1, \dots \text{ (orientable)}$$

$$N(1), N(2), \dots, N(g) = \#_g \mathbb{RP}^2, \dots \text{ (nonorientable)}$$

- ▶ Connected surfaces are classified by the genus g and orientability.
- ▶ Connected surfaces are classified by the fundamental group :

$$\pi_1(M(g)) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] \rangle$$

$$\pi_1(N(g)) = \langle c_1, c_2, \dots, c_g \mid (c_1)^2 (c_2)^2 \dots (c_g)^2 \rangle$$

- ▶ Connected surfaces are classified by the Euler characteristic and orientability

$$\chi(M(g)) = 2 - 2g, \quad \chi(N(g)) = 2 - g.$$

The punctured torus I.

- ▶ The computation of $\pi_1(M(g))$ for $g \geq 0$ will be by induction, using the connected sum

$$M(g+1) = M(g) \# M(1)$$

- ▶ So need to understand the fundamental group of the torus $M(1) = T = S^1 \times S^1$ and the puncture torus (T_0, S^1) .
- ▶ Clear from $T = S^1 \times S^1$ that $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$.
- ▶ Can also get this by applying the Seifert-van Kampen theorem to $M(1) = M(1) \# M(0)$, i.e. $T = T_0 \cup_{S^1} D^2$.
- ▶ The punctured torus

$$(T_0, \partial T_0) = (\text{cl.}(S^1 \times S^1 \setminus D^2), S^1)$$

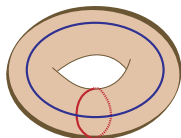
is such that $S^1 \vee S^1 \subset T_0$ is a homotopy equivalence.

The punctured torus II.

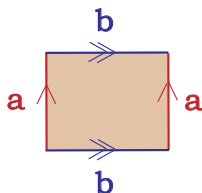
- ▶ The inclusion $\partial T_0 = S^1 \subset T_0$ induces

$$\pi_1(S^1) = \mathbb{Z} \rightarrow \pi_1(T_0) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle ;$$

$$1 \mapsto [a, b] = aba^{-1}b^{-1} .$$



Torus

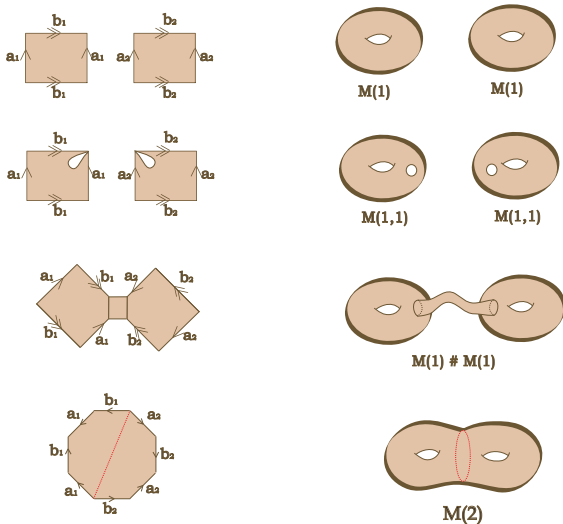


- ▶ The Seifert-van Kampen Theorem gives

$$\pi_1(T) = \pi_1(T_0) *_{\mathbb{Z}} \{1\} = \langle a, b \mid [a, b] \rangle = \mathbb{Z} \oplus \mathbb{Z} .$$

The calculation of $\pi_1(M(g))$ I.

- The initial case $g = 2$, using $M(2) = M(1) \# M(1)$



The calculation of $\pi_1(M(g))$ II. General case

- ▶ Assume inductively that

- ▶ $\pi_1(M(g)) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] \rangle$,
- ▶ the punctured surface

$$(M(g)_0, \partial M(g)_0) = (\text{cl.}(M(g) \setminus D^2), S^1)$$

is such that $\bigvee_{2g} S^1 \subset M(g)_0$ is a homotopy equivalence,

- ▶ the inclusion $\partial M(g)_0 = S^1 \subset M(g)_0$ induces

$$\begin{aligned} \pi_1(S^1) = \mathbb{Z} \rightarrow \pi_1(M(g)_0) &= \underset{2g}{*} \mathbb{Z} = \langle a_1, b_1, \dots, a_g, b_g \rangle ; \\ 1 &\mapsto [a_1, b_1][a_2, b_2] \dots [a_g, b_g] . \end{aligned}$$

- ▶ Apply the Seifert-van Kampen Theorem to

$$M(g+1) = M(g) \# M(1)$$

to obtain

$$\begin{aligned} \pi_1(M(g+1)) &= \pi_1(M(g)_0) *_{\mathbb{Z}} \pi_1(M(1)_0) \\ &= \langle a_1, b_1, \dots, a_{g+1}, b_{g+1} \mid [a_1, b_1] \dots [a_{g+1}, b_{g+1}] \rangle \end{aligned}$$

Cross-cap

- ▶ If M is a surface the connected sum

$$M' = M \# \mathbb{R}P^2$$

is the surface obtained from M by forming a **crosscap** (*Kreuzhaube* in German).

- ▶ M' is homeomorphic to the identification space obtained from the punctured surface (M_0, S^1) by identifying $z \sim -z$ for $z \in S^1$

$$M' = M_0 / \{z \sim -z\} .$$

- ▶ Equivalently, M' is obtained from M by punching out $D^2 \subset M$ and replacing it by a Möbius band.
- ▶ M' is nonorientable.
- ▶ **Example** If $M = S^2$ then $M' = \mathbb{R}P^2$.

The punctured projective plane I.

- ▶ The computation of $\pi_1(N(g))$ for $g \geq 1$ will be by induction, using the connected sum

$$N(g+1) = N(g) \# N(1)$$

with $N(1) = \mathbb{RP}^2$. Abbreviate $\mathbb{RP}^2 = P$.

- ▶ Need to understand the fundamental group of P and the punctured projective plane (P_0, S^1) , i.e. the Möbius band.
- ▶ Clear from the universal double cover $p: S^2 \rightarrow P$ that

$$\pi_1(P) = \text{Homeo}_p(P) = \mathbb{Z}_2 .$$

- ▶ Can also get this by applying the Seifert-van Kampen Theorem to $N(1) = N(1) \# M(0)$, i.e. $P = P_0 \cup_{S^1} D^2$.

The punctured projective plane II.

- ▶ The punctured projective plane

$$(P_0, \partial P_0) = (\text{cl.}(P \setminus D^2), S^1)$$

is a Möbius band, such that $S^1 \subset P_0 \setminus \partial P_0$ is a homotopy equivalence.

- ▶ The inclusion $\partial P_0 = S^1 \subset P_0$ induces

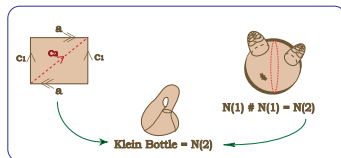
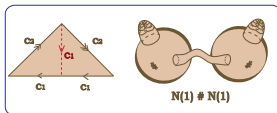
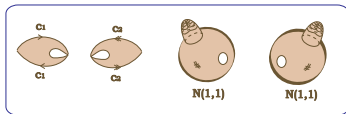
$$\pi_1(S^1) = \mathbb{Z} \rightarrow \pi_1(P_0) = \pi_1(S^1) = \mathbb{Z} ; 1 \mapsto 2 .$$

- ▶ The Seifert-van Kampen Theorem gives

$$\pi_1(P) = \pi_1(P_0) *_{\mathbb{Z}} \{1\} = \langle c \mid c^2 \rangle = \mathbb{Z}_2 .$$

The calculation of $\pi_1(N(g))$ I.

- The initial case $g = 2$, using $N(2) = N(1) \# N(1)$ and $(N(1)_0, S^1) = (\text{Möbius band}, \text{boundary circle})$.



- By the Seifert-van Kampen Theorem, with $c_2 = (c_1')^{-1}$,
- $$\begin{aligned} \pi_1(N(2)) &= \pi_1(N(1) \# N(1)) \\ &= \langle c_1, c_1' \mid (c_1)^2 = (c_1')^2 \rangle = \langle c_1, c_2 \mid (c_1)^2 (c_2)^2 \rangle . \end{aligned}$$

The calculation of $\pi_1(N(g))$ II.

- ▶ Assume inductively that

- ▶ $\pi_1(N(g)) = \langle c_1, c_2, \dots, c_g \mid (c_1)^2(c_2)^2 \dots (c_g)^2 \rangle$,
- ▶ the punctured surface

$$(N(g)_0, \partial N(g)_0) = (\text{cl.}(N(g) \setminus D^2), S^1)$$

is such that $\bigvee_g S^1 \subset N(g)_0$ is a homotopy equivalence,

- ▶ the inclusion $\partial N(g)_0 = S^1 \subset N(g)_0$ induces

$$\begin{aligned} \pi_1(S^1) = \mathbb{Z} \rightarrow \pi_1(N(g)_0) &= *_{\mathbb{Z}} \mathbb{Z} = \langle c_1, c_2, \dots, c_g \rangle ; \\ 1 &\mapsto (c_1)^2 \dots (c_g)^2 . \end{aligned}$$

- ▶ Apply the Seifert-van Kampen Theorem to

$$N(g+1) = N(g) \# N(1)$$

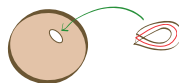
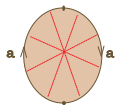
to obtain

$$\begin{aligned} \pi_1(N(g+1)) &= \pi_1(N(g)_0) *_{\mathbb{Z}} \pi_1(N(1)_0) \\ &= \langle c_1, \dots, c_{g+1} \mid (c_1)^2 \dots (c_{g+1})^2 \rangle . \end{aligned}$$

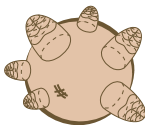
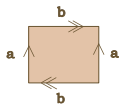
The calculation of $\pi_1(N(g))$ III.



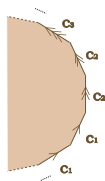
N(1)



N(2)



N(g)



The Euler characteristic

- **Definition** The **Euler characteristic** of a finite cell complex

$$X = \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 \cup \dots \cup \bigcup_{c_n} D^n$$

with c_k k -cells is

$$\chi(X) = \sum_{k=0}^n (-1)^k c_k \in \mathbb{Z}.$$

- $\chi(D^n) = 1$, $\chi(S^n) = 1 + (-1)^n$
- If X is homotopy equivalent to Y then $\chi(X) = \chi(Y)$
- $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \in \mathbb{Z}$.
- A punctured n -manifold has $\chi(M_0) = \chi(M) + (-1)^n$
- A connected sum of n -manifolds has

$$\chi(M \# M') = \chi(M) + \chi(M') - \chi(S^n)$$

- If $F \rightarrow \tilde{X} \rightarrow X$ is a regular cover with finite fibre F then $\chi(\tilde{X}) = \chi(F)\chi(X)$, with $\chi(F) = |F|$.

The Euler characteristic of $M(g)$

- ▶ The fundamental group of $M(g)$ determines the genus g .
- ▶ The first homology group of $M(g)$ is the free abelian group of rank $2g$

$$H_1(M(g)) = \pi_1(M(g))^{ab} = \bigoplus_{2g} \mathbb{Z}$$

- ▶ $M(g)$ is homotopy equivalent to the 2-dimensional cell complex

$$\left(\bigvee_{2g} S^1 \right) \cup_{[a_1, b_1] \dots [a_g, b_g]} D^2 = D^0 \cup \bigcup_{2g} D^1 \cup_{[a_1, b_1] \dots [a_g, b_g]} D^2 .$$

- ▶ The Euler characteristic of $M(g)$ is

$$\chi(M(g)) = 2 - 2g .$$

- ▶ A closed surface M is homeomorphic to S^2 if and only if $\chi(M) = 2$.

The Euler characteristic of $N(g)$

- ▶ The fundamental group determines the genus g .
- ▶ The first homology group of $N(g)$ is direct sum of the free abelian group of rank $g - 1$ and the cyclic group of order 2

$$H_1(N(g)) = \pi_1(N(g))^{ab} = \left(\bigoplus_g \mathbb{Z} \right) / (2, 2, \dots, 2) = \left(\bigoplus_{g-1} \mathbb{Z} \right) \oplus \mathbb{Z}_2$$

- ▶ $N(g)$ is homotopy equivalent to the 2-dimensional cell complex

$$\left(\bigvee_g S^1 \right) \cup_{(c_1)^2(c_2)^2 \dots (c_g)^2} D^2 = D^0 \cup \bigcup_g D^1 \cup_{(c_1)^2 \dots (c_g)^2} D^2 .$$

- ▶ $N(g)$ has Euler characteristic

$$\chi(N(g)) = 2 - g .$$

The orientable surfaces with boundary $M(g, h)$

- ▶ Let $g \geq 0, h \geq 1$.
- ▶ **Definition** The **orientable surface of genus g and h boundary components** is

$$(M(g, h), \partial) = (\text{cl.}(M(g) \setminus \bigcup_h D^2), \bigcup_h S^1).$$

- ▶ Cell structure $M(g, h) \simeq \bigvee_{2g+h-1} S^1 = D^0 \cup \bigcup_{2g+h-1} D^1$
- ▶ Fundamental group $\pi_1(M(g, h)) = \ast_{2g+h-1} \mathbb{Z}$
- ▶ Euler characteristic $\chi(M(g, h)) = 2 - 2g - h$
- ▶ **Classification Theorem** Every connected orientable surface with non-empty boundary is homeomorphic to exactly one of $(M(g, h), \partial M(g, h))$.
- ▶ Set $M(g, 0) = M(g)$.

Examples of orientable surfaces with boundary

- ▶ $(M(0, 1), \partial) = (D^2, S^1)$, 2-disk
- ▶ $(M(0, 2), \partial) = (S^1 \times [0, 1], S^1 \times \{0, 1\})$, cylinder
- ▶ $(M(1, 1), \partial) = ((S^1 \times S^1)_0, S^1)$, punctured torus.
- ▶ $(M(0, 3), \partial) = (\text{pair of pants}, S^1 \cup S^1 \cup S^1)$.
- ▶ The pair of pants is an essential feature of topological quantum field theory, and so appeared in Ida's birthday cake for the [80th birthday of Michael Atiyah](#) (29 April, 2009)



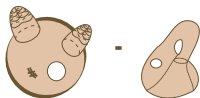
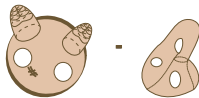
The nonorientable surfaces with boundary $N(g, h)$ I.

- ▶ Let $g \geq 1, h \geq 1$.
- ▶ **Definition** The **nonorientable surface with boundary with genus g with h boundary components** is

$$(N(g, h), \partial N(g, h)) = (\text{cl.}(N(g) \setminus \bigcup_h D^2), \bigcup_h S^1).$$

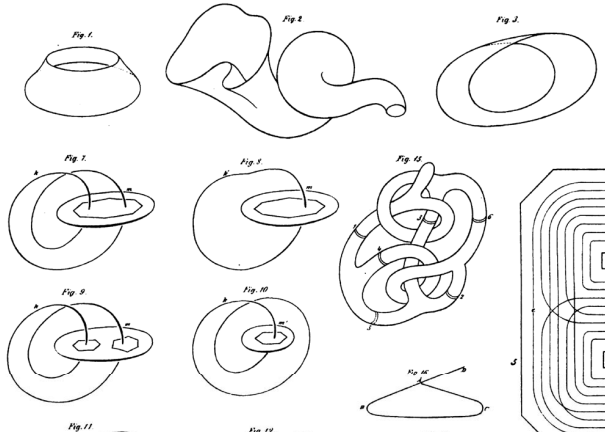
- ▶ Cell structure $N(g, h) \simeq \bigvee_{g+h-1} S^1 = D^0 \cup \bigcup_{g+h-1} D^1$.
- ▶ Fundamental group $\pi_1(N(g, h)) = \ast_{g+h-1} \mathbb{Z}$
- ▶ Euler characteristic $\chi(N(g, h)) = 2 - g - h$
- ▶ **Classification Theorem** Every connected nonorientable surface with non-empty boundary is homeomorphic to exactly one of $(N(g, h), \partial N(g, h))$.
- ▶ Set $N(g, 0) = N(g)$.

The nonorientable surfaces with boundary $N(g, h)$ II.

 $N(1,1)$  $N(1,2)$  $N(1, h)$  $N(2,1)$  $N(2,1)$  $N(2, h)$  $N(g, h)$

The Möbius band

- ▶ The Möbius band $(N(1, 1), \partial N(1, 1)) = ((\mathbb{RP}^2)_0, S^1)$.
- ▶ The first drawing of a Möbius band, from Listing' s 1862 *Census der Räumlichen Complexe*



The orientation double cover

- ▶ A **double cover** of a space N is a regular cover $\tilde{N} \rightarrow N$ with fibre $F = \{0, 1\}$. Connected double covers of connected N are classified by index 2 subgroups $\pi_1(\tilde{N}) \triangleleft \pi_1(N)$.
- ▶ A surface N has an **orientation double cover** $p : \tilde{N} \rightarrow N$, with \tilde{N} an orientable surface. For connected N classified by the kernel of the orientation character group morphism

$$w : \pi_1(N) \rightarrow \mathbb{Z}_2 = \{+1, -1\}$$

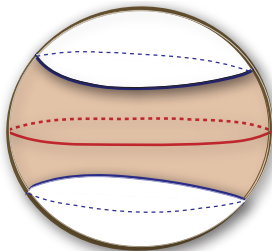
sending orientable (resp. nonorientable) α to $+1$ (resp. -1).

- ▶ If N is orientable $\tilde{N} = N \cup N$ is the trivial double cover of N .
- ▶ If N is nonorientable w is onto, $\pi_1(\tilde{N}) = \ker w$. Pullback along nonorientable $\alpha : S^1 \rightarrow N$ is the nontrivial double cover

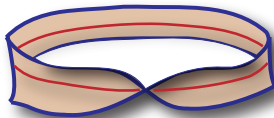
$$q = \alpha^* p : S^1 \rightarrow S^1 ; z \mapsto z^2$$

$$\begin{array}{ccc} S^1 & \xrightarrow{\tilde{\alpha}} & \tilde{N} \\ q \downarrow & & \downarrow p \\ S^1 & \xrightarrow{\alpha} & N \end{array}$$

The orientation double cover
of a Möbius band is a cylinder



$$S^1 \times I = M(0,2)$$



$$M = N(1,1)$$

$M(g-1, 2h)$ is the orientation double cover of $N(g, h)$

- ▶ **Proposition** The orientation double cover of $N(g, h)$ is

$$\widetilde{N(g, h)} = M(g-1, 2h) \quad (g \geq 1, h \geq 0)$$

- ▶ **Proof** Let N be a connected nonorientable surface with orientation double cover \widetilde{N} . The boundary circle of $N_0 = \text{cl.}(N \setminus D^2)$ is orientable. The orientation double cover of N_0 is the twice-punctured \widetilde{N} , $\widetilde{N}_{00} = \text{cl.}(\widetilde{N} \setminus D^2 \cup D^2)$. The orientation double cover of $N' = N \# \mathbb{R}P^2$ is

$$\widetilde{N'} = \widetilde{N}_{00} \cup_{S^1 \cup S^1} S^1 \times I.$$

with $\chi(\widetilde{N'}) = \chi(\widetilde{N}_{00}) = \chi(\widetilde{N}) - 2$. This gives the inductive step in checking that $\widetilde{N(g, h)} = M(g-1, 2h)$.

- ▶ **Example** For $h = 0$, $g \geq 1$ have $\widetilde{N(g)} = M(g-1)$. Simply-connected for $g = 1$. For $g \geq 2$ universal cover \mathbb{R}^2 .

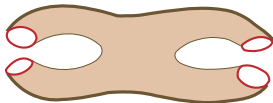
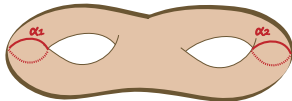
The genus measures connectivity

I. The orientable case

- ▶ The genus g of an orientable surface M is the maximum number of disjoint loops $\alpha_1, \alpha_2, \dots, \alpha_g : S^1 \rightarrow M$ such that the complement $M \setminus \bigcup_{i=1}^g \alpha_i(S^1)$ is connected. The complement is homeomorphic to $M(0, 2g) \setminus \partial M(0, 2g)$.
- ▶ **Example** For $M = M(2)$ let $\alpha_1, \alpha_2 : S^1 \rightarrow M$ be disjoint loops which go round as in the diagram. The complement

$$M \setminus (\alpha_1(S^1) \cup \alpha_2(S^1)) = M(0, 4) \setminus \partial M(0, 4)$$

is the sphere $M(0) = S^2$ with 4 holes punched out.



The genus measures connectivity

II. The nonorientable case

- ▶ The genus g of a nonorientable surface N is the maximum number of disjoint nonorientable loops

$\beta_1, \beta_2, \dots, \beta_g : S^1 \rightarrow N$ such that the complement

$N \setminus \bigcup_{i=1}^g \beta_i(S^1)$ is connected.

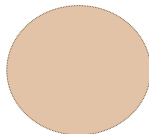
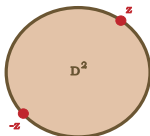
The complement is homeomorphic to $M(0, g) \setminus \partial M(0, g)$.

- ▶ **Example** Let $N = \mathbb{RP}^2 = D^2 / \{z \sim -z \mid z \in S^1\}$ and

$$\beta : S^1 = \mathbb{RP}^1 \rightarrow \mathbb{RP}^2 ; z \mapsto [\sqrt{z}] .$$

The complement is

$$\mathbb{RP}^2 \setminus \beta(S^1) = M(0, 1) \setminus \partial M(0, 1) = D^2 \setminus S^1 = \mathbb{R}^2 .$$



Morse theory

- ▶ For an orientable surface $M \subset \mathbb{R}^3$ in general position the height function

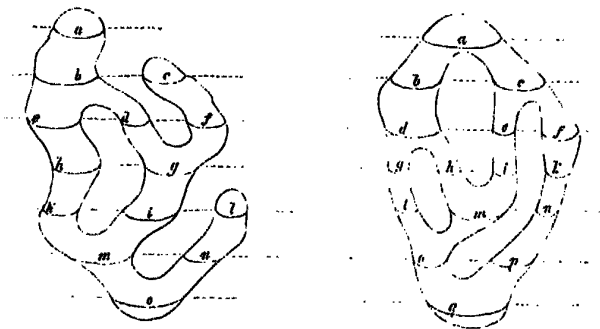
$$f : M \rightarrow \mathbb{R} ; (x, y, z) \mapsto z$$

has the property that the inverse image $f^{-1}(c) \subset M$ is a 1-dimensional submanifold for all except a finite number $c \in \mathbb{R}$ called the critical values of f .

- ▶ Can recover the genus g of M by looking at the jumps in the number of circles in $f^{-1}(a)$ and $f^{-1}(b)$ for $a < b < c$.
- ▶ Morse theory developed (since 1926) is the key tool for studying n -manifolds for all $n \geq 0$.

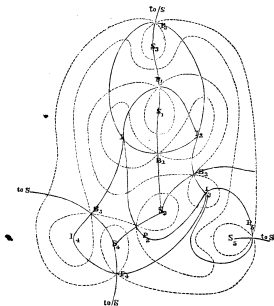
An early exponent of Morse theory on a surface

- ▶ August Ferdinand Möbius
Theorie der elementaren Verwandtschaften (1863)
- ▶ Fill a surface shaped bathtub with water, and recover the genus of the surface from a film of the cross-sections.



Another early exponent of Morse theory on a surface

- ▶ James Clerk Maxwell (1870) [On hills and dales](#)
- ▶ Reconstruct surface of the earth ($= S^2$) from contour lines.



- ▶ Mountaineer's equation for surface of Earth

$$\text{no. of peaks} - \text{no. of pits} + \text{no. of passes} = \chi(S^2) = 2.$$

Modern account in Chapter 8 of *Surfaces* (CUP, 1976) by H.B.Griffiths

Complex algebraic curves

- ▶ The **complex projective space** $\mathbb{C}\mathbb{P}^2$ is the space of 1-dimensional complex linear subspaces $L \subset \mathbb{C}^3$. A closed 4-manifold. Homogeneous coordinates $[x, y, z] \in \mathbb{C}\mathbb{P}^2$.
- ▶ For a degree d homogeneous complex polynomial $P(x, y, z)$ let

$$M(P) = \{[x, y, z] \in \mathbb{C}\mathbb{P}^2 \mid P(x, y, z) = 0\}$$

- ▶ **Theorem** (Special case of the Riemann-Hurwitz formula)
If $(\partial P/\partial x, \partial P/\partial y, \partial P/\partial z) \neq (0, 0, 0)$ for all $(x, y, z) \in M(P)$ then $M(P)$ is a closed orientable surface with genus

$$g = (d-1)(d-2)/2$$

0 or one of the triangular numbers

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \dots$$

- ▶ **Complex algebraic curves** by Frances Kirwan (CUP, 1992)

Further reading

- ▶ [Google for "Classification of Surfaces"](#) (147,000 hits)
- ▶ [An Introduction to Topology. The classification theorem for surfaces](#) by E.C. Zeeman (1966)
- ▶ [A Guide to the Classification Theorem for Compact Surfaces](#) by Jean Gallier and Dianna Xu (2011)
- ▶ [Home Page for the Classification of Surfaces and the Jordan Curve Theorem](#) Online resources, including many of the original papers.