TRANSVERSALITY IN GENERALIZED MANIFOLDS

J. L. BRYANT AND W. MIO

Abstract. Suppose that $X$ is a generalized $n$-manifold, $n \geq 5$, satisfying the disjoint disks property, and $M$ and $Q$ are topological $m$- and $q$-manifolds, respectively, 1-LCC embedded in $X$, with $n - m \geq 3$ and $n - q \geq 3$. We define what it means for $M$ to be stably transverse to $Q$ in $X$. In the metastable range, $3m \leq 2(n - 1)$ and $3(m + q) < 4(n - 1)$, we show that there is an arbitrarily small homotopy of $M$ to a 1-LCC embedding that is stably transverse to $Q$.

1. Introduction

In this paper we introduce a notion of transversality for submanifolds of a generalized $n$-manifold. One of the major difficulties in arriving at suitable criteria for transversality is that a (generalized) submanifold $M$ of a generalized manifold $X$ may not have a stable euclidean normal (micro)bundle neighborhood in $X$. This situation occurs, for example, when $M$ is a topological manifold, which has Quinn index $[22] \iota(M) = 1$, and $X$ is a generalized manifold with $\iota(X) \neq 1$. Examples of generalized manifolds $X$ with $\iota(X) \neq 1$ were constructed in [4]. An embryonic form of transversality was established in [5] for codimension three topological submanifolds $M$ and $Q$ of a generalized manifold $X$ having complementary dimensions in $X$. Specifically, it was shown that if $m \leq q \leq n - 3$, $m + q = n \geq 6$, and $M$ and $Q$ are orientable topological manifolds of dimensions $m$ and $q$, respectively, tamely embedded in an orientable generalized $n$-manifold $X$ with the disjoint disks property, then there is an arbitrarily small homotopy of $M$ to a tame embedding $f : M \to X$ such that $f(M) \cap Q$ is a finite set and the intersection number of $f(M) \cap Q$ at each point of intersection is $\pm 1$. Assuming the metastable codimension restriction $3m \leq 2(n - 1)$, $3(m + q) < 4(n - 1)$, we find a small homotopy of $M$ to a tame embedding $f : M \to X$ such that $f(M)$ and $Q$ are stably transverse, in an sense to be described. In fact, we need only assume that $Q$ is a generalized $q$-manifold with the disjoint disks property. In particular, $f(M) \cap Q$ will be a tame topological submanifold of $f(M)$ and $Q$ of the expected dimension, $m + q - n$. The proof makes use of the transversality theorems of Kirby-Siebenmann [15] and Marin [16], the Main Construction of [5], and a splitting theorem of [7]. Map transversality, which can be obtained from submanifold transversality, has been studied by Johnston [14] in the special case where the homology submanifold has a bundle neighborhood.

2. Definitions

A generalized $n$-manifold ($n$-gm) without boundary is a locally compact euclidean neighborhood retract (ENR) $X$ such that for each $x \in X$,
Following Mitchell [19] we say that an ENR $X$ is an $n$-gm with boundary if the condition $H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ is replaced by $H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ or 0, and if $\partial X = \{x \in X : H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong 0\}$ is an $(n-1)$-gm embedded in $X$ as a Z-set. (In [19] Mitchell shows that $\partial X$ is a homology $(n-1)$-manifold.) Recall that $Y$ is a Z-set in $X$ if, for each open set $U$ in $X$, the inclusion $U \setminus Y \to U$ is a homotopy equivalence. A $n$-gm $X$, $n \geq 5$, has the disjoint disks property (DDP) if every pair of maps of the 2-cell $B^2$ into $X$ can be approximated arbitrarily closely by maps that have disjoint images. A subset $A$ of $X$ is 1-LCC in $X$ if for each $x \in A$ and neighborhood $U$ of $x$ in $X$, there is a neighborhood $V$ of $x$ in $X$ lying in $U$ such that the inclusion induced homomorphism $\pi_1(V \setminus A) \to \pi_1(U \setminus A)$ is trivial. An ENR $A$ in $X$ of codimension at least three will be called tame in $X$ if it is 1-LCC in $X$.

Given an $n$-gm $X$, a manifold approximate fibration with fiber $F(MAF)$ over $X$ is an approximate fibration $p: N \to X$, where $N$ is a topological manifold and the homotopy fiber of $p$ is homotopy equivalent to $F$. (Equivalently, each $p^{-1}(x)$ has the shape of the space $F$.) (See [8], [13].) If $Q$ is a (topological or generalized) manifold in $X$ and $p: N \to X$ is a MAF, then $p$ is said to be split over $Q$ if $p[p^{-1}(Q)]: p^{-1}(Q) \to Q$ is also a MAF.

Suppose that $M_p$ is the mapping cylinder of a MAF $p: N \to X$ with fiber a sphere and mapping cylinder projection $\pi: M_p \to X$. If $M_p$ is a topological manifold, then we will call $\pi: M_p \to X$ (or, sometimes, just $M_p$) a manifold stabilization of $X$. As the following proposition shows, this last condition is almost always satisfied.

**Proposition 2.1.** Suppose that $N$ is a topological $n$-manifold, $X$ is a generalized manifold, and $M_p$ is the mapping cylinder of a MAF $p: N \to X$ with fiber a $k$-sphere and mapping cylinder projection $\pi: M_p \to X$. If $n \geq 5$, then $M_p$ is a topological manifold. If, in addition, $k \geq 2$, then $X$ is 1-LCC embedded in $M_p$.

**Proof.** That $M_p$ is a homology manifold follows easily from results of Gottlieb [11] and Quinn [20]. Since $M_p$ has manifold points, $M_p$ has a resolution [22], and, hence, by a theorem of Edwards (see [9]), it suffices to observe that $M_p$ has the DDP. We consider three cases.

**Case 1.** $k \geq 2$. In this case it enough to show that $X$ is 1-LCC in $M_p$, since we can then use ordinary general position in $M_p \setminus X$. Suppose then that $f: B^2 \to M_p$ and $T$ is a fine triangulation of $B^2$. By Alexander duality, $X$ is 0-LCC in $M_p$; hence, we may assume that, if $T^{(1)}$ denotes the 1-skeleton of $T$, then $f(T^{(1)}) \cap X = \emptyset$. Let $\Delta$ be a 2-simplex of $T$ with boundary $\Sigma$, such that $f(\Delta) \cap X \neq \emptyset$. By a small homotopy of $f[\Sigma]$ in $M_p \setminus X$, we can assume that $f(\Sigma)$ lies in some $t$-level $N_t$ of the mapping cylinder near $X$. Since $\pi|\Sigma$ is null-homotopic in $X$, we can use the approximate lifting property of $p$ to assume that $f(\Sigma)$ lies near a fiber of $p$ in $N_t$. Since the fibers have the shape of $S^k$, $k \geq 2$, we can homotope $f[\Sigma]$ to a constant in a neighborhood of a fiber in $N_t$. Thus there is a small homotopy of $f|\Delta$ to a map of $\Delta$ into $M_p \setminus X$.

**Case 2.** $k = 1$. Since $X$ is 0-LCC in $M_p$, we can begin as in Case 1. Given $f: B^2 \to M_p$, we can assume that $f(T^{(1)}) \cap X = \emptyset$, where $T$ is a fine triangulation of $B^2$. If $f(\Delta) \cap X \neq \emptyset$, for some 2-simplex $\Delta$ of $T$ with boundary $\Sigma$, then we
may assume that $f(\Sigma)$ lies near a fiber of $p$ in some $t$-level $N_t$ of $M_p$, as above. Thus, there is a small homotopy of $f|_{\Delta}$ to $f': \Delta \to M_p$ such that $f'(\Delta) \cap X$ is a single point. This process gives a small homotopy of $f$ to $f': B^2 \to M_p$ such that $f'(B^2) \cap X$ is a finite set. Given another mapping $g: B^2 \to X$, we can get a small homotopy of $g$ to $g'$ such that $g(B^2) \cap X$ is a finite set disjoint from $f'(B^2) \cap X$. We can then use general position in $M_p \setminus X$ to get $f'(B^2)$ and $g'(B^2)$ disjoint.

Case 3. $k = 0$. In this case $X$ locally separates $M_p$, and the approximate lifting property of $p$ implies that $X$ is 1-LCC in $M_p$. If $f: B^2 \to M_p$, and $T$ is a fine triangulation of $B^2$, then it is easy to get a small homotopy of $f$ to $f'$ such that $\dim f'(B^2) \cap X \leq 1$. Since $\dim X \geq 4$, $f'(B^2) \cap X$ is 0-LCC in $X$. Thus, if $g: B^2 \to M_p$ is another mapping, then there is a small homotopy of $g$ to $g'$ such that $g'(B^2) \cap (f'(B^2) \cap X) = \emptyset$. We can then use general position in $M_p \setminus X$ to get $f'(B^2)$ and $g'(B^2)$ disjoint as before. \qed

Suppose $M, Q \subseteq N$ are topological manifolds without boundary of dimensions $m, q,$ and $n$, respectively. Let $p = m + q - n$. Then $M$ and $Q$ are locally transverse if, for each $x \in M \cap Q$, there is a neighborhood $W$ of $x$ in $N$, with $W \cap M = U$ and $W \cap Q = V$, such that

$$(W, U, V, U \cap V) \cong (\mathbb{R}^m, \mathbb{R}^{m-p} \times \mathbb{R}^p \times 0, 0 \times \mathbb{R}^p \times \mathbb{R}^{q-p}, 0 \times \mathbb{R}^q \times 0).$$

This implies, in particular, that $P = M \cap Q$ is a $p$-dimensional submanifold of both $M$ and $Q$. If $M$ (or $Q$) has boundary, and $x \in \partial M$ (or $x \in \partial Q$), then local transversality at $x$ can be described by replacing $\mathbb{R}^m$ by $\mathbb{R}^{m-1} \times \mathbb{R}^+$, (or $\mathbb{R}^q$ by $\mathbb{R}^{q-1} \times \mathbb{R}^+$), and so on. Following [15], we say that $M$ is stably microbundle transverse to $Q$ in $N$ if $M$ and $Q$ are locally transverse and, for some integer $s \geq 0$, there exists a normal microbundle $\xi$ to $Q \times 0$ in $N \times \mathbb{R}^s$ so that $M \times \mathbb{R}^s$ is embedded microbundle transverse to $\xi$ in $N \times \mathbb{R}^{s}$. That is, $M \cap Q$ has a normal microbundle $\nu$ in $M$ each of whose fibers lies in a fiber of $\xi$. Marin shows that this relation is symmetric [16] and, with help from Scharlemann [23] when $p = 4$, that local transversality implies stable microbundle transversality, provided $n - m \leq 3$ and $n - q \leq 3$. With these ideas in mind, we make the following definition.

**Definition 2.2.** Given a topological manifold $M$ and generalized manifold $Q$ in a generalized manifold $X$, $Q$ is stably locally transverse to $M$ if there is a manifold stabilization $\pi: M_p \to X$ of $X$, split over $Q$, such that $\pi^{-1}(Q)$ and $M$ are locally transverse in $M_p$.

3. **Transversality in the Metastable Range**

**Theorem 3.1.** Suppose that $X$ is an $n$-gm with the DDP, $n \geq 5$, $M$ is a topological $m$-manifold embedded in $X$ (with or without boundary), and $Q$ is either a topological $q$-manifold or a $q$-gm with the DDP if $q \geq 5$, 1-LCC embedded in $X$, such that $n - q \geq 3, 3m \leq 2(n - 1)$, and $3(m + q) < 4(n - 1)$. Then for every $\epsilon > 0$ there is an $\epsilon$-homotopy of the inclusion of $M$ in $X$ to a 1-LCC embedding $f: M \to X$ such that $Q$ is stably locally transverse to $f(M)$ in $X$.

The following corollary is a consequence of Theorem 3.1 and Corollary 1.3 of [5].

**Corollary 3.2.** Suppose that $M$ and $Q$ are topological $m$- and $q$-manifolds, respectively, in an $n$-gm $X$, $n \geq 5$, with the DDP, such that $3m \leq 2(n - 1), 3q \leq 2(n - 1), 3(m + q) < 4n - 4$. Then there are arbitrarily small homotopies of the inclusions.
to 1-LCC embeddings \( f : M \to X \) and \( g : Q \to X \) such that \( f(M) \) is stably locally transverse to \( g(Q) \) in \( X \).

The proof of Theorem 3.1 ultimately depends upon a transversality theorem of Kirby-Siebenmann [15] and Marin [16]. One of the main ingredients of the proof is the following splitting theorem proved in [7].

**Theorem 3.3 ([7]).** Suppose that \( X \) is an \( n \)-gm without boundary, \( n \geq 5 \), and \( Q \subseteq X \) is an \( q \)-gm (with or without boundary), \( n - q \geq 3 \), 1-LCC in \( X \). Assume \( Q \) is a topological manifold if \( q \leq 4 \). Then there is a manifold stabilization \( \pi : M_p \to X \) of \( X \) of dimension \( \geq n + 3 \) that is split over \( Q \).

The manifold stabilization \( X \) of Theorem 3.3 is obtained in [7] by first taking a mapping cylinder neighborhood \( M_p \) of \( X \) is some euclidean space \([18, 25]\), where \( p' : N \to X \) is a MAF with homotopy fiber a sphere, and then homotoping \( p' \) to a MAF \( p : N \to X \) such that \( p^{-1}(M) \) is a topological manifold. A similar argument can be found in [6], wherein \( X \) is a topological manifold.

Another important ingredient is the Main Construction of [5]. It can be summarized in the following theorem.

**Theorem 3.4 ([5]).** Suppose that \( M \) is a topological \( m \)-manifold and \( X \) is an \( n \)-gm with the DDP, \( n \geq 5 \), \( 3m \leq 2(n - 1) \). Then for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( f : M \to X \) is a \((\delta, 2m - n + 1)\)-connected map, then \( f \) is \( \epsilon \)-homotopic to a 1-LCC embedding. Moreover, the homotopy is supported in a neighborhood of a 1-LCC subset of \( X \) of dimension \( \leq 2m - n + 2 \).

A map \( f : M \to X \) is \((\epsilon, k)\)-connected if the pair \((M, X)\) is \((\epsilon, i)\)-connected for \( 0 \leq i \leq k \). If \( M \), in 3.3 or 3.4, is not compact, then \( f \) should be a proper map and \( \epsilon \) and \( \delta \) should be interpreted as positive, continuous functions on \( M \). The “moreover” part of Theorem 3.4 has the following consequence, which will be important for us here.

**Addendum.** If \( P \) is a (closed) ANR in \( M \), with \( \dim P < m \), such that \( f|f^{-1}f(P) \) is a 1-LCC embedding, then we can arrange to have the homotopy \( f_t, t \in [0, 1] \), of \( f \) to an embedding satisfy \( f_1|P = f|P \) and \( f_t^{-1}f_t(P) = P \) for all \( t \in [0, 1] \).

**Proof of Theorem 3.1.** Suppose that \( X \), \( M \), and \( Q \) are given as in the hypothesis of Theorem 3.1. By Theorem 3.3, there is a manifold stabilization \( \pi : M_p \to X \) of \( X \) of dimension \( n + k \), with \( k \geq 3 \), that is split over \( Q \). Let \( W = \pi^{-1}(Q) \). Choose \( k \) large enough so that, by 2.1, \( W \) is a topological \((q + k)\)-manifold. Since \( Q \) is 1-LCC in \( X \), \( W \) is 1-LCC in \( M_p \), hence, locally flat [3]. Thus, by [15], [16], and [23], there is an arbitrarily small ambient isotopy of the inclusion of \( M \) in \( M_p \) to a locally flat embedding \( h : M \to M_p \) such that \( h(M) \) and \( W \) are locally transverse. Let \( P = h(M) \cap W \). Then \( P \) is a manifold of dimension \( p = m + q - n \), locally flatly embedded in \( h(M) \) and in \( W \). The next step is to push \( h(M) \) down into \( X \), sending \( P \) into \( Q \) and \( h(M) \to X - Q \), to a 1-LCC embedding close to \( M \). Observe that \( \pi|h(M) \) has all but the last of these properties.

The first step is to observe that the inequalities \( 3m \leq 2(n - 1) \), \( 3(m + q) < 4(n - 1) \) imply \( 2p + 1 \leq q \). General position then implies that \( \pi|P : P \to Q \) can be approximated by a 1-LCC embedding. (If \( Q \) is a manifold, this is immediate. If \( Q \) is a \( q \)-gm with the DDP, then the general position results of [2] and [24] may be applied.) Since \( k \geq 3 \), there is a small ambient isotopy of \( W \) taking \( P \) to this
embedding [1], which can be extended to $M_j$ by [12]. After composing with $\pi$, we get a map $h' : (M, M \setminus h^{-1}(P)) \to (X, X \setminus Q)$ such that $h'$ approximates the inclusion of $M$ into $X$ and $h'|P$ is a 1-LCC embedding into $Q$. Finally, as long as $\pi \circ h'$ is a sufficiently close approximation to the inclusion of $M$ in $X$, it will have the desired connectivity properties to apply Theorem 3.4. Thus we can get a small homotopy of $h'$ rel $P$ to a 1-LCC embedding in $X$. According to Theorem 3.4, this homotopy is supported on a 1-LCC set of dimension $2m - n + 2$, and our dimension restrictions imply that $(2m - n + 2) + q < n$. By the general position results of [2] and [24], we can assume that these supports can be made to miss $Q$. Thus, the homotopy of $h'$ to a 1-LCC embedding can be constructed so as not to introduce any new intersections of $M$ with $Q$ as guaranteed by the Addendum to Theorem 3.4. This final adjustment provides the map $f : M \to X$ promised in the theorem.

\begin{thebibliography}{99}

\end{thebibliography}

Department of Mathematics, Florida State University, Tallahassee, FL 32306
E-mail address, J. L. Bryant: bryant@math.fsu.edu
E-mail address, W. Mio: mio@math.fsu.edu