HYPERBOLIC MONOPOLES AND RATIONAL NORMAL CURVES

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A NOTE ON THE TANGENTS OF A TWISTED CUBIC

By M. F. ATIYAH

Communicated by J. A. Todd

Received 8 May 1951

1. Consider a rational normal cubic C_3. In the Klein representation of the lines of S_3 by points of a quadric Ω in S_5, the tangents of C_3 are represented by the points of a rational normal quartic C_4. It is the object of this note to examine some of the consequences of this correspondence, in terms of the geometry associated with the two curves.

2. C_4 lies on a Veronese surface V, which represents the congruence of chords of C_3 (1). Also C_4 determines a 4-space Σ meeting Ω in Ω_1, say; and since the surface of

M F Atiyah, *A note on the tangents to a twisted cubic*, Proc. Camb. Phil. Soc. **48** (1952) 204–205

"... The tangents at four points of a twisted cubic have a unique transversal if and only if the four points are equianharmonic".

RATIONAL NORMAL CURVES

- $\mathbf{P}^1 \subset \mathbf{P}^n$ of degree n
- ... not contained in any hyperplane
- $=$ image by a projective transformation of $z \mapsto [1, z, z^2, \ldots, z^n]$

- Symmetric product $S^n(\mathbf{P}^1) = \mathbf{P}^n$

- Diagonal $\Delta \subset S^n(\mathbf{P}^1) = \{(x, x, \ldots, x) : x \in \mathbf{P}^1\}$

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- Diagonal $\Delta \subset S^n(\mathbf{P}^1) = \{(x, x, \ldots, x) : x \in \mathbf{P}^1\}$

- V = 2-dim symplectic vector space, S^nV symmetric tensor product

- $S^n(\mathbf{P}(V)) = \mathbf{P}(S^nV)$, $\Delta = \{[v \otimes v \otimes \ldots \otimes v] : v \in V\}$

- rational normal curve $C \subset \mathbf{P}(W)$ defines an isomorphism

  $W^* \cong H^0(C, \mathcal{O}(n) = S^n H^0(C, \mathcal{O}(1)) = S^n V^*$

- $S^n V$ has a symplectic/ orthogonal (n odd/even) structure

EXAMPLES

- conic in $\mathbf{P}^2$

- twisted cubic in $\mathbf{P}^3$

EXAMPLES

- conic in $\mathbf{P}^2$

- twisted cubic in $\mathbf{P}^3$

- tangents to a twisted cubic $\subset \mathbf{Q}^4 \subset \mathbf{P}^5$ ...

- ... lies in $\mathbf{P}^4 \cap \mathbf{Q}^4$

- (S^3V symplectic, $\mathbf{P}(S^3V)$ contact, twisted cubic Legendrian)

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VECTOR BUNDLES ON THE PROJECTIVE PLANE

By R. L. E. SCHWARZENBERGER

[Received 13 October 1960]

LET k be an algebraically closed field, and P_n the n-dimensional projective space defined over k. We consider algebraic vector bundles with fibre k^r, group $GL(r, k)$, and base P_n, and then speak of k^r-bundles, or, when $r = 1$, of line bundles. The equivalence classes of line bundles on an algebraic variety have been classified (**10**): they are in one-one correspondence with the divisor classes. In particular, on P_n there is one equivalence class of line bundles for each (positive or negative) integer. If H is the line bundle

The construction of k^n-bundles on P_n in § 1 is based on unpublished work of Hodge and Atiyah for the case $n = 2$.

1. SCHWARZENBERGER BUNDLES

2. RESTRICTION TO RATIONAL NORMAL CURVES

3. HYPERBOLIC MONOPOLES AND RATIONAL MAPS

SCHWARZENBERGER BUNDLES

FIRST DEFINITION

- $S^rV \longrightarrow S^{r-n}V \otimes S^nV$

- $\mathbf{P}^n = \mathbf{P}(S^nV)$

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- $S^r V \longrightarrow S^{r-n} V \otimes S^n V$

- $\mathbf{P}^n = \mathbf{P}(S^n V)$

- $S^r V \longrightarrow S^{r-n} V \otimes H^0(\mathbf{P}^n, \mathcal{O}(1))$

- $0 \to E_n^{r*} \to S^r V \to \mathcal{O}(1) \otimes S^{r-n} V \to 0$

SECOND DEFINITION

- $f : Y = \mathbf{P}(S^{n-1}V) \times \mathbf{P}(V) \to \mathbf{P}(S^n V)$

- or $S^n V$ = degree n homogeneous polynomials $p(z_0, z_1)$ and

- $Y = \{([p(z)], [w]) : p(w) = 0\}$

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- or S^nV = degree n homogeneous polynomials $p(z_0, z_1)$ and

- $Y = \{([p(z)], [w]) : p(w) = 0\}$

- $f : \mathbf{P}(S^{n-1}V) \times \mathbf{P}(V) \to \mathbf{P}(S^nV)$ n-fold branched covering

- $E_n^r = f_*\mathcal{O}(0, r)$

PROPERTIES

- $c(E_n^r) = (1-h)^{n-r-1}$

- $T\mathbf{P}^n = E_n^n(1)$

PROPERTIES

- $c(E_n^r) = (1-h)^{n-r-1}$
- $T\mathbf{P}^n = E_n^n(1)$
- E_n^r is stable
- The unstable hyperplanes ($H^0(\mathbf{P}^{n-1}, (E_n^r)^*) \neq 0$) are defined by the dual curve of Δ
- ($S^n V \cong S^n V^*$ so $\mathbf{P}(S^n V) \cong \mathbf{P}(S^n V)^\vee$)

RESTRICTION TO RATIONAL CURVES

Birkhoff-Grothendieck: Any holomorphic vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles.

RESTRICTION TO RATIONAL NORMAL CURVES

- $C \subset \mathbf{P}^n$ rational normal curve: degree n
- $C \cong \mathbf{P}^1$, $\mathcal{O}_{\mathbf{P}^n}(1)|_C \cong \mathcal{O}_{\mathbf{P}^1}(n)$

RESTRICTION TO RATIONAL NORMAL CURVES

- $C \subset \mathbf{P}^n$ rational normal curve: degree n
- $C \cong \mathbf{P}^1$, $\mathcal{O}_{\mathbf{P}^n}(1)|_C \cong \mathcal{O}_{\mathbf{P}^1}(n)$
- $c_1(E_n^r) = (r+1-n)h$, degree $(r+1-n)n$ on C
- generic splitting type $\mathbf{C}^n \otimes \mathcal{O}(r+1-n)$

WHEN DOES $E^r_n|_C$ CONTAIN $\mathcal{O}(m)$ FOR $m \geq r$?

- $0 \longrightarrow \mathcal{O}(-1) \otimes S^{r-n}V \longrightarrow S^rV \longrightarrow E^r_n \longrightarrow 0$
- $\ldots \longrightarrow H^1(\mathbf{P}^1, \mathcal{O}(-n-r)) \otimes S^{r-n}V \xrightarrow{\alpha} H^1(\mathbf{P}^1, \mathcal{O}(-r)) \otimes S^rV \longrightarrow \ldots$
- $H^0(\mathbf{P}^1, E^r_n(-r)) = \ker \alpha$

- $\alpha : \mathbf{C}^{n+r-1} \otimes \mathbf{C}^{r-n+1} \to \mathbf{C}^{r-1} \otimes \mathbf{C}^{r+1}$

- matrices $A : \mathbf{C}^m \to \mathbf{C}^n$ of non-maximal rank are codimension $(n - m + 1)$

- $\alpha : \mathbf{C}^{n+r-1} \otimes \mathbf{C}^{r-n+1} \to \mathbf{C}^{r-1} \otimes \mathbf{C}^{r+1}$

- matrices $A : \mathbf{C}^m \to \mathbf{C}^n$ of non-maximal rank are codimension $(n-m+1)$

- $(r-1)(r+1)-(r+n-1)(r-n+1)+1 = (n-1)^2$ constraints

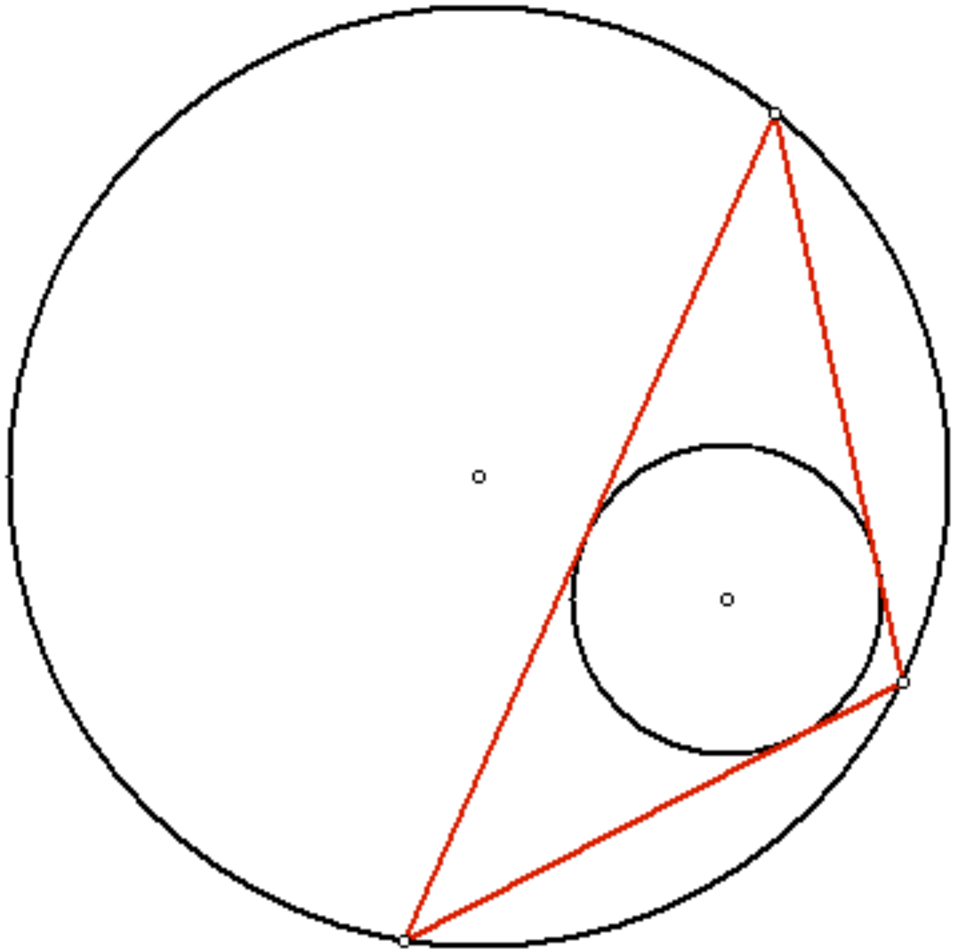
THE CASE $n = 2$

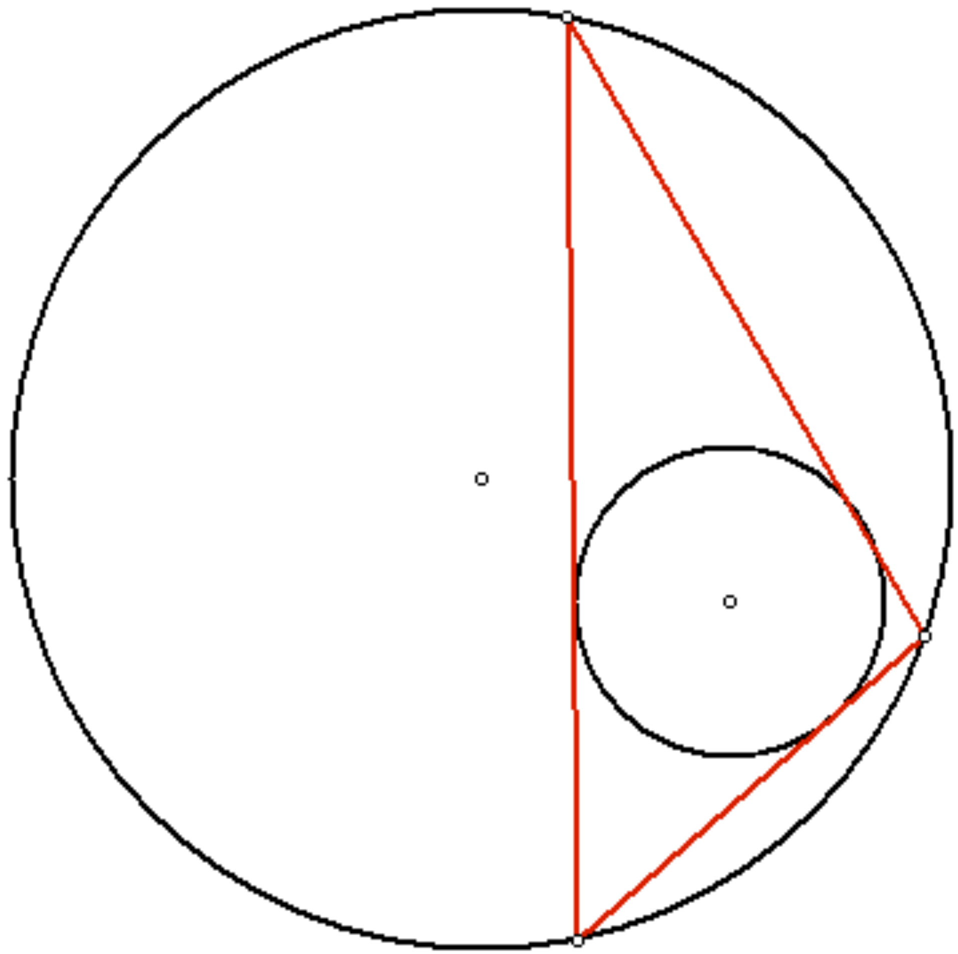
- $\Delta, C \subset \mathbf{P}^2$ conics

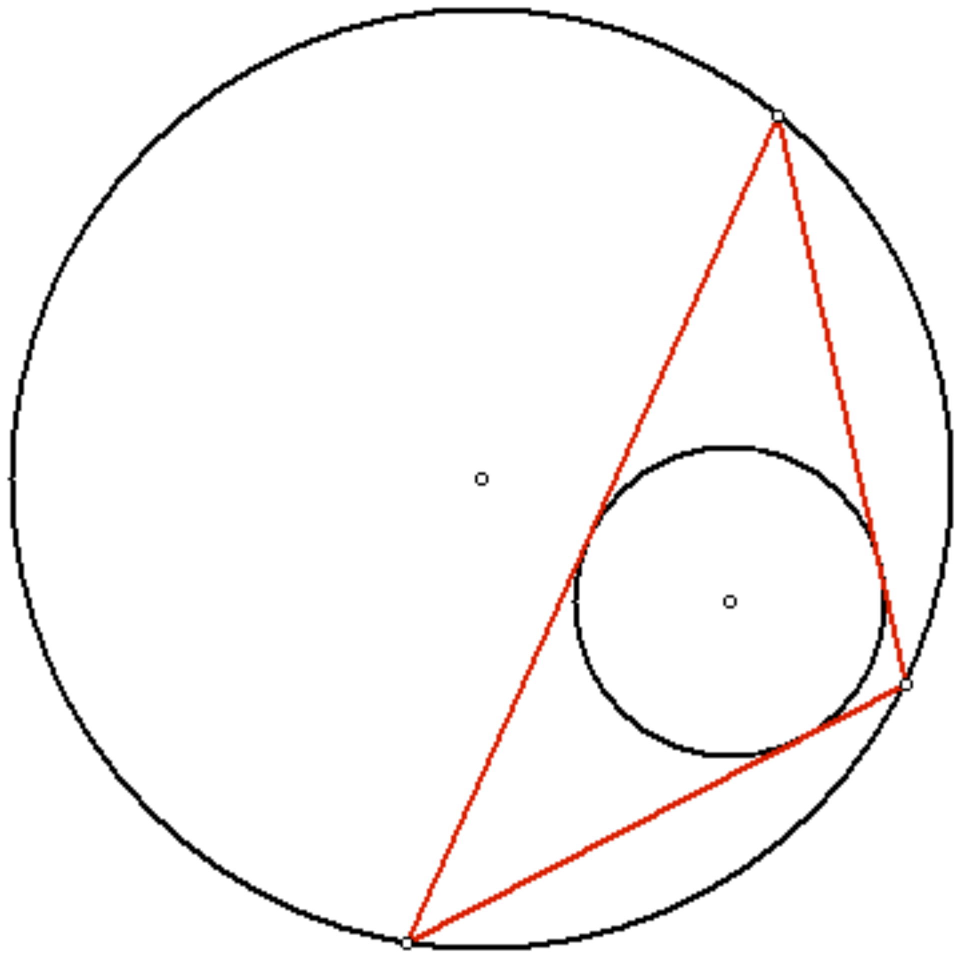
- $(n-1)^2 = 1$ constraint

- jumping conics – four parameter family.

“in-and-circumscribed polygon”





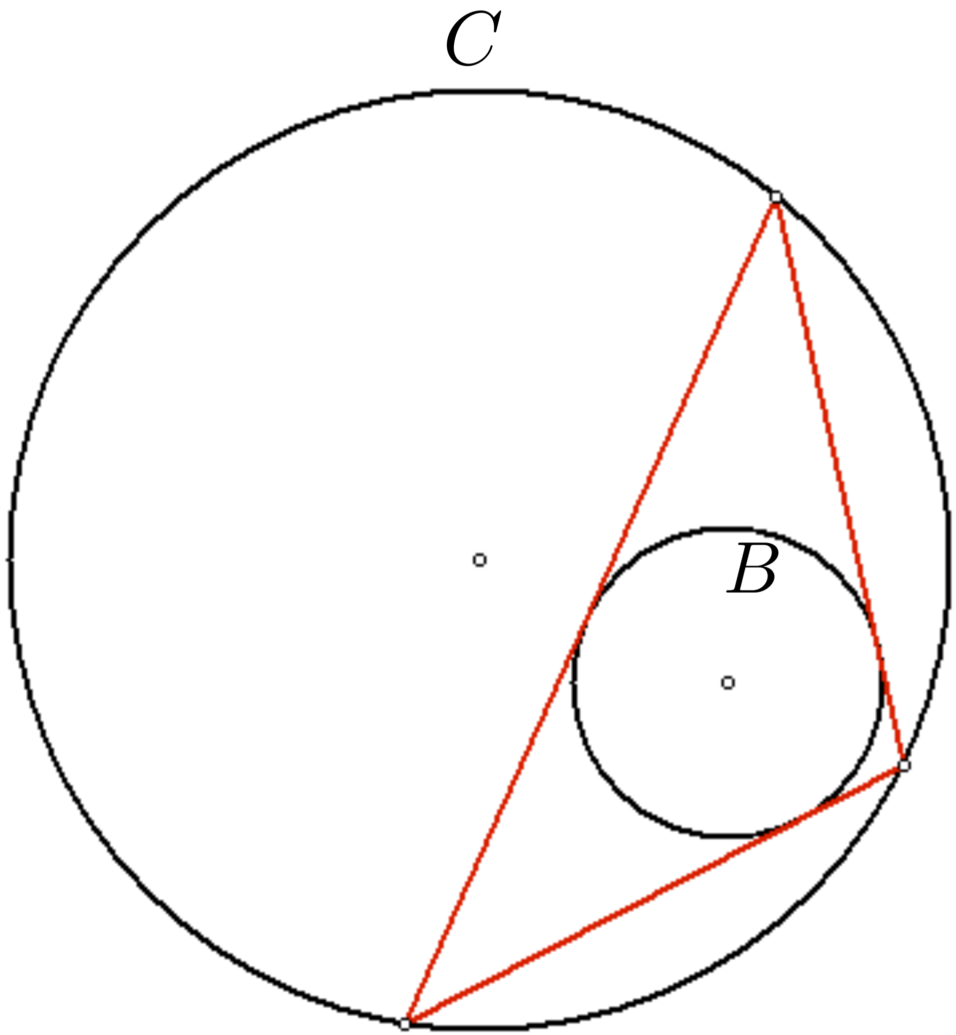


DUALITY

- rational normal curve B defines a vector bundle $E^r(B)$

- take another rational normal curve C

- Define $C < B$ if $H^0(C, E^r(B)(-r)) \neq 0$



C
B

Theorem: $C < B$ if and only if $B^{\vee} < C^{\vee}$

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- $B = \phi(\Delta), C = \psi(\Delta)$

- $B^\vee = (\phi^T)^{-1}(\Delta)$

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- $B = \phi(\Delta), C = \psi(\Delta)$
- $B^\vee = (\phi^T)^{-1}(\Delta)$
- $C < B \Leftrightarrow \phi^{-1}\psi(\Delta) < \Delta$
- $B^\vee < C^\vee \Leftrightarrow \psi^T(\phi^T)^{-1}(\Delta) < \Delta$
- $\Leftrightarrow (\phi^{-1}\psi)^T(\Delta) < \Delta$

Theorem: $C < B$ if and only if $B^\vee < C^\vee$

- $B = \phi(\Delta), C = \psi(\Delta)$
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- $B^\vee < C^\vee \Leftrightarrow \psi^T(\phi^T)^{-1}(\Delta) < \Delta$
- $\Leftrightarrow (\phi^{-1}\psi)^T(\Delta) < \Delta$
- RTP: $\psi(\Delta) < \Delta$ if and only if $\psi^T(\Delta) < \Delta$

$$\mathbf{P}^{n-1} \times \mathbf{P}^1 \to \mathbf{P}^n$$

$$\cup$$

$$C$$

$$
\begin{array}{ccc}
\mathbf{P}^{n-1} \times \mathbf{P}^1 & \rightarrow & \mathbf{P}^n \\
\cup & & \cup \\
S & \longrightarrow & C
\end{array}
$$

- n-fold covering
- $S \hookrightarrow C \times \mathbf{P}^1$

$$\begin{array}{ccc} \mathbf{P}^{n-1} \times \mathbf{P}^1 & \to & \mathbf{P}^n \\ \cup & & \cup \\ S & \longrightarrow & C \end{array}$$

- n-fold covering

- $S \hookrightarrow C \times \mathbf{P}^1$

- choose an identification $C \cong \mathbf{P}^1 = \Delta$ (condition invariant under Aut(Δ))

$$S: \sum_{i,j=0}^{n} \phi_{ij} z^i (-w)^{n-j} = 0$$

$$\begin{array}{ccc} \mathbf{P}^{n-1} \times \mathbf{P}^1 & \to & \mathbf{P}^n \\ \cup & & \cup \\ S & \longrightarrow & C \end{array}$$

- $E_n^r = f_* \mathcal{O}(0, r)$

- $E_n^r(-r)|_C = f_* \mathcal{O}_{C \times \mathbf{P}^1}(-r, r)|_S$

$$
\begin{array}{ccc}
\mathbf{P}^{n-1} \times \mathbf{P}^1 & \rightarrow & \mathbf{P}^n \\
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\end{array}
$$

- $E_n^r = f_* \mathcal{O}(0, r)$

- $E_n^r(-r)|_C = f_* \mathcal{O}_{C \times \mathbf{P}^1}(-r, r)|_S$

- $H^0(C, E_n^r(-r)) \cong H^0(S, \mathcal{O}(-r, r))$

- $H^0(C, E_n^r(-r)) \neq 0$ if and only if $\mathcal{O}(-r, r)$ is trivial on S

- $\phi \mapsto \phi^T \Leftrightarrow (w, z) \mapsto (z, w)$

- $\mathcal{O}(-r, r)$ is trivial on S if and only if its inverse $\mathcal{O}(r, -r)$ is trivial.

HYPERBOLIC MONOPOLES

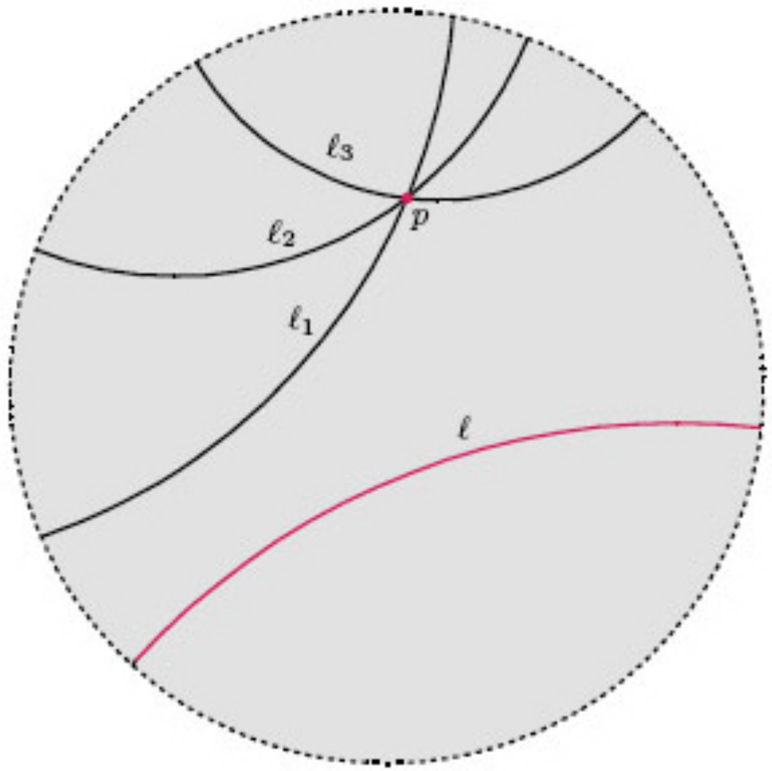
- H^3 hyperbolic three-space of curvature -1

- Bogomolny equations $F_A = *d_A\phi$ for $SU(2)$ connection A

- boundary conditions:

  mass $= |\phi| \to p$ *charge* $= n = \deg \phi : S^2_R \to S^2$

- M F Atiyah, *Magnetic monopoles in hyperbolic spaces* in "Vector bundles on algebraic varieties (Bombay, 1984)" 1–33, Tata Inst. Fund. Res. Stud. Math., 11, Bombay, 1987.



- space of geodesics $S^2 \times S^2 \setminus \Delta$

SPECTRAL CURVE

- geodesics: $\mathbf{P}^1 \times \mathbf{P}^1 \setminus \{w = \bar{z}\}$

- spectral curve S: divisor of a section of $\mathcal{O}(n, n)$

- constraint: $\mathcal{O}(r, -r)$ is trivial on S where $r = 2p + n$ ($p =$ mass, $n =$ charge)

Theorem: $C < B$ if and only if $B^\vee < C^\vee$

$$\Leftrightarrow$$

Fact: A monopole (A, ϕ) transforms to a monopole (with opposite orientation) under a hyperbolic reflection in a point.

MONOPOLE MODULI SPACES

The Geometry and Dynamics of Magnetic Monopoles

MICHAEL ATIYAH AND NIGEL HITCHIN

M. B. PORTER LECTURES

RICE UNIVERSITY

MONOPOLES ON $\mathbf{R}^3$

- moduli space M^{4n} is hyperkähler

- twistor space complex manifold Z^{2n+1}

- holomorphic fibration $p : Z \to \mathbf{P}^1$

- complex symplectic fibres

- M = a space of sections

- each fibre of $p \cong$ based degree n rational maps

- $S(z) = p(z)/q(z)$, zeros of q: $z_1, \ldots, z_n$

- symplectic form:

$$\sum_i dz_i \wedge d \log p(z_i)$$

L Faybusovich & M Gekhtman, *Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices,* Phys. Lett. A 272 (2000), 236–244

K L Vaninsky, *The Atiyah-Hitchin bracket and the open Toda lattice,* J. Geom. Phys. 46 (2003) 283–307

K L Vaninsky, *The Atiyah-Hitchin bracket and the cubic nonlinear Schrödinger equation,* IMRP (2006), 17683, 1–60.

- fix p: Lagrangian submanifold
- fix q: Lagrangian submanifold
- Define $f_x(S) = p(x)$, $g_x(S) = q(x)$
- Poisson bracket:

$$\{f_x, g_y\} = \frac{p(x)q(y) - q(x)p(y)}{x - y}$$

(Bezoutian)

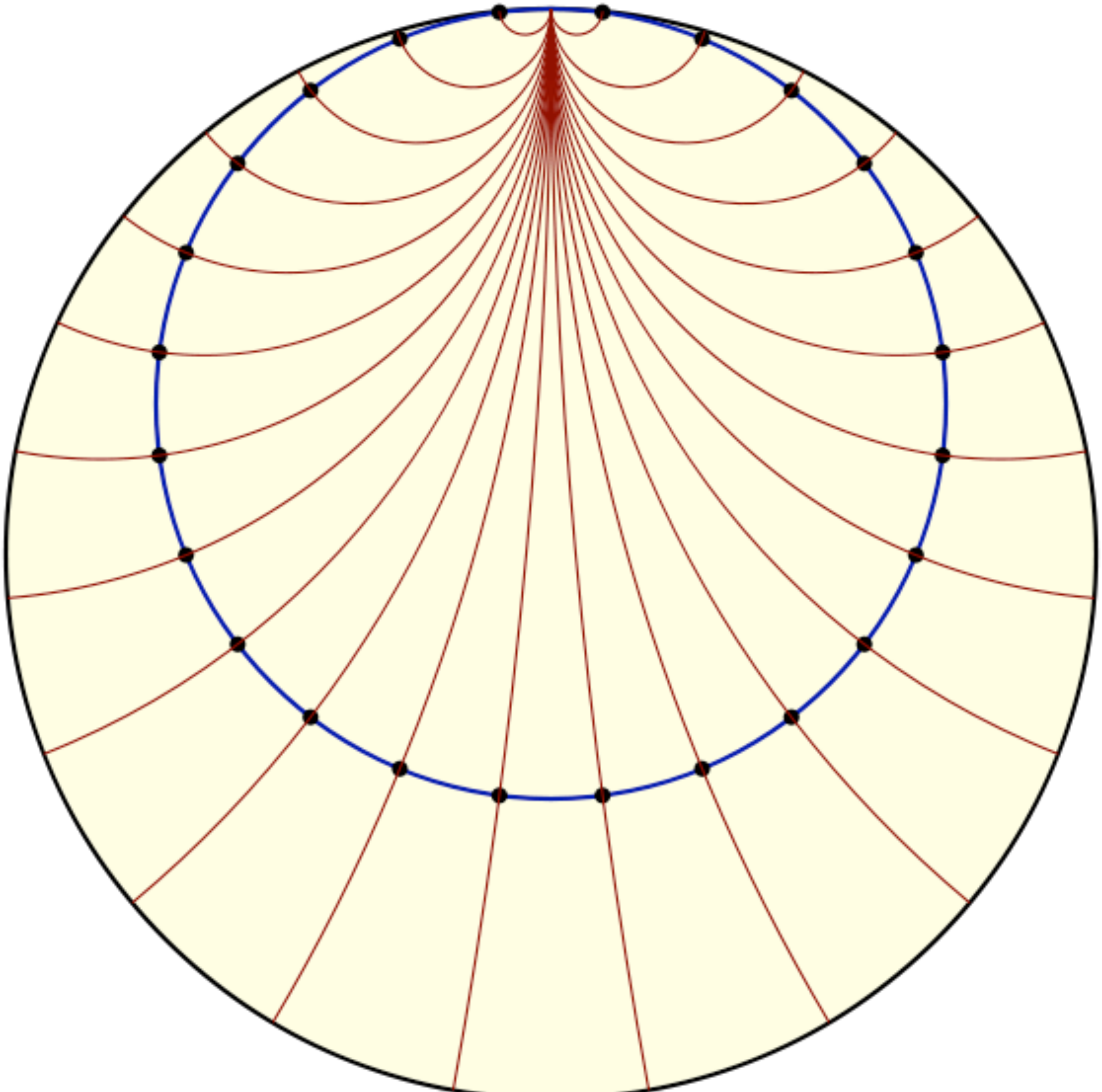
MONOPOLES ON H^3

For each point on $S^2 = \partial H^3$ the moduli space is isomorphic to the space of based rational maps.

M F Atiyah, *Instantons in two and four dimensions*, Commun. Math. Phys. **93** (1984), 437–451

P J Braam & D M Austin, *Boundary values of hyperbolic monopoles* Nonlinearity **3** (1990), 809–823

M K Murray, P Norbury & M A Singer, *Hyperbolic monopoles and holomorphic spheres*, Ann. Global Anal. Geom. **23** (2003) 101–128



SYMPLECTIC STRUCTURE

O Nash, *A new approach to monopole moduli spaces*, Nonlinearity **20** (2007) 1645-1675

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- spectral curve $S \subset \mathbf{P}^1 \times \mathbf{P}^1$

- constraint lifts S to $\mathcal{O}(-r, r)$

- deformation theory of a curve in a three-manifold

SCHWARZENBERGER BUNDLES AND RATIONAL MAPS

- $0 \to S^{r-n}V(-1) \to S^rV \to E_n^r \to 0$

- S^kV = homogeneous polynomials $q(z_0, z_1)$ of degree k

- fibre of E_n^r over $[q] \in \mathbf{P}(S^nV) \cong$ polynomials p of degree r modulo q

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- fibre of E_n^r over $[q] \in \mathbf{P}(S^nV) \cong$ polynomials p of degree r modulo q

- common factor?

- $f : Y \to X$

- evaluation map $\mathrm{ev} : f^*f_*L \to L$

- $f : Y \to X$

- evaluation map ev : $f^* f_* L \to L$

- $\Rightarrow$ section α of Hom$(f^* E_n^r, \mathcal{O}(0, r))$ on $\mathbf{P}^{n-1} \times \mathbf{P}^1$

- kernel of α = rank $(n-1)$ bundle over $\mathbf{P}^{n-1} \times \mathbf{P}^1 = p, q$ with common factor

- $(E_n^r)_0$ = complement

- choose $[a_0, a_1] \in \mathbf{P}^1$, restrict to q with $q(a_0, a_1) \neq 0$

- $[a_0, a_1] = [0, 1]$, q = degree n polynomial in $z = z_1/z_0$

- $p = aq + b$, $\deg b < n$

- based rational map $b(z)/q(z)$

TWISTOR SPACES

- spectral curve S defines a rational normal curve $C \subset \mathbf{P}(S^n V)$

$$w \mapsto \sum_{i,j=0}^{n} \phi_{ij} z^i (-w)^{n-j}$$

- constraint $H^0(C, E_n^r(-r)) \neq 0$ lifts C to $E_n^r(-r)_0$

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- (Note: $\mathcal{O}(-r) = \mathcal{O}_{\mathbf{P}^n}(-r/n)|_C$)

- $C \Rightarrow S$ requires an isomorphism $C \cong \mathbf{P}^1$

$$\mathbf{P}(\bar{V}) \times \mathbf{P}(S^n V) \setminus \{(w, q) : q(\bar{w}) = 0\}$$

$$Z^{2n+1} = \begin{array}{c} E_n^r(r, -r/n)_0 \\ \downarrow \\ \mathbf{P}(\bar{V}) \times \mathbf{P}(S^n V) \setminus \{(w,q) : q(\bar{w}) = 0\} \end{array}$$

MONOPOLES ON H^3

- complex manifold Z^{2n+1}
- holomorphic fibration $p : Z \to \mathbf{P}^1$
- complex symplectic fibres
- M = a space of sections

PROBLEMS

- no real structure
- symplectic forms along fibres do not vary holomorphically

CHARGE 2 MONOPOLES

CENTRES

- $V \cong \bar{V} \Rightarrow$ Hermitian form $=$ point in H^3
- spectral curve equation $\in S^nV \otimes S^n\bar{V}$
- $S^nV \otimes S^nV = 1 + S^2V + \ldots + S^{2n}V$
- centred monopole: S^2V component vanishes

- $V \cong \bar{V} \Rightarrow$

- real structure on Schwarzenberger bundle

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- real structure on Schwarzenberger bundle

- if $C = \phi(\Delta)$, $\bar{C} = \phi^T(\Delta)$

- charge 2 centred: $1 + S^4 V$ symmetric

The projective Schwarzenberger bundle $\mathbf{P}((E_2^r)_0)$ is the twistor space for a 4-dimensional self-dual Einstein manifold.

N J Hitchin *A new family of Einstein metrics,* in "Manifolds and geometry (Pisa, 1993)", 190–222, Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge, 1996

EXAMPLE: CHARGE 2

$$g = f dr^2 + T_1 \sigma_1^2 + T_2 \sigma_2^2 + T_3 \sigma_3^2$$

$$T_1 = \frac{(1 - r^2)^2}{(1 + r + r^2)(r + 2)(2r + 1)}$$

$$T_2 = \frac{1 + r + r^2}{(r + 2)(2r + 1)^2}$$

$$T_3 = \frac{r(1 + r + r^2)}{(r + 2)^2(2r + 1)}$$

$$f = \frac{1 + r + r^2}{r(r + 2)^2(2r + 1)^2}$$

- $M^4 = S^4 \setminus \mathbf{R}P^2$

- (irreducible 5-dimensional rep of $SO(3)$)

- orbifold singularity around $\mathbf{R}P^2$, $(r-2)$-fold quotient.

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- ... $SO(3)$ bundle H_r over M^4 – smooth, Einstein (3-Sasakian)

- $r = 3$, $M^4 = S^4$

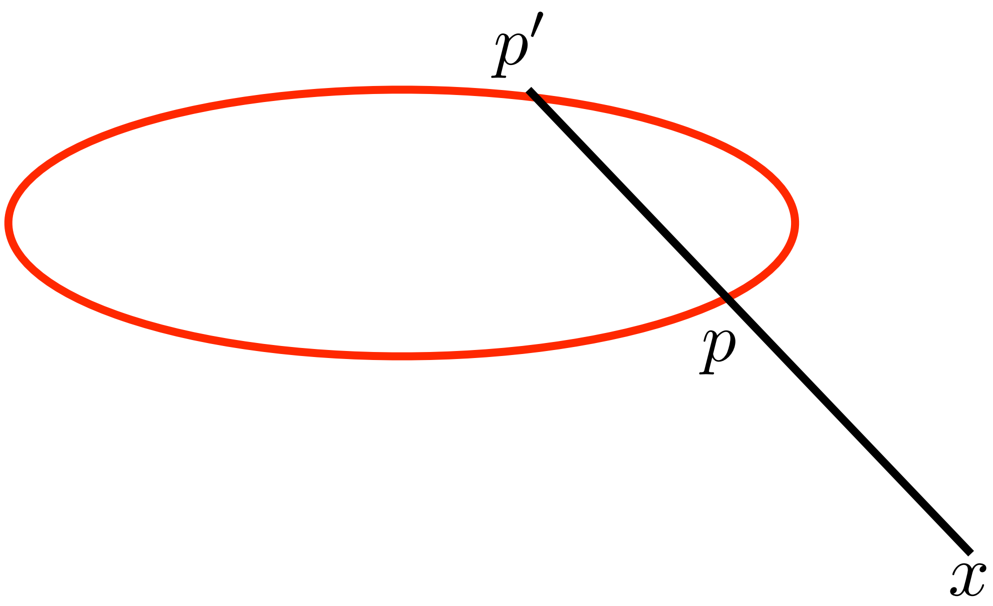
- twistor space $\mathbf{P}^3 = \mathbf{P}(S^3 V)$

- What's the link with $\mathbf{P}(E_2^3)$?

THE TWISTED CUBIC

- $C \subset \mathbf{P}^3$ rational normal curve

- $x \neq C \Rightarrow$ unique secant through x



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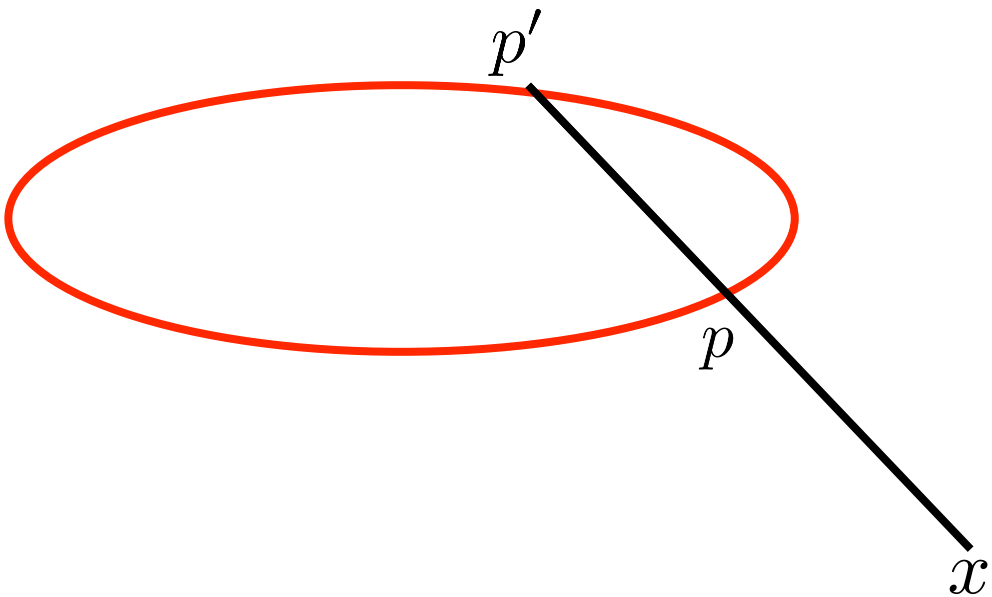

- $f : \mathbf{P}^3 \setminus C \to S^2C = \mathbf{P}^2$ $\qquad f(x) = (p, p')$

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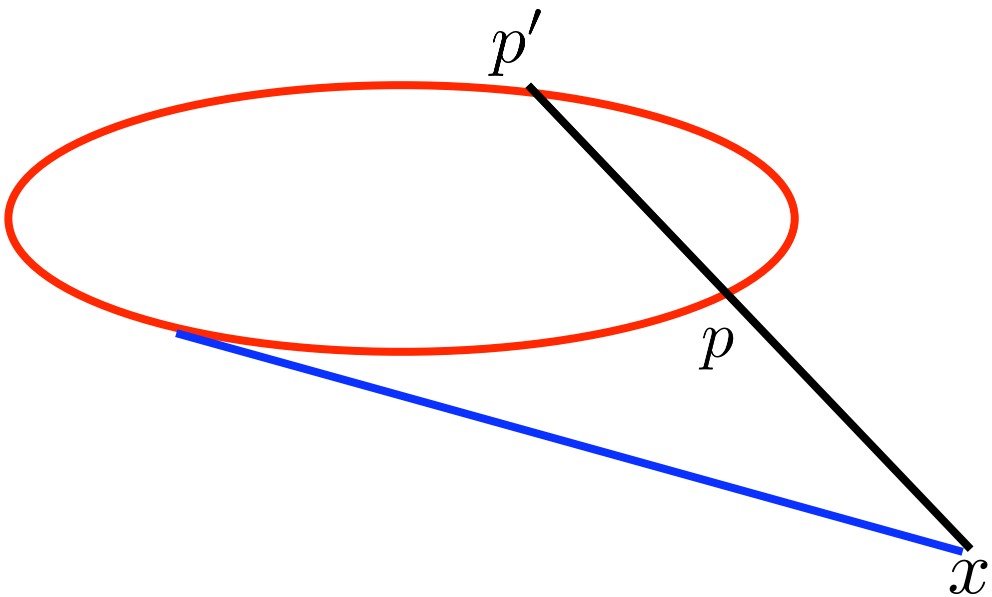

- $f : \mathbf{P}^3 \setminus C \to S^2C = \mathbf{P}^2$ $\qquad f(x) = (p, p')$

- Blow up C: $\mathbf{P}^1$ fibration $= \mathbf{P}(E_2^3)$

- lines in $\mathbf{P}^3 \sim$ sections of $\mathbf{P}(E_2^3)$ ...

- ... constrained conics in $\mathbf{P}^2$

M F Atiyah, *A note on the tangents to a twisted cubic,* Proc. Camb. Phil. Soc. **48** (1952) 204–205

".. The tangents at four points of a twisted cubic have a unique transversal if and only if the four points are equianharmonic".

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".. The tangents at four points of a twisted cubic have a unique transversal if and only if the four points are equianharmonic".

$$\Leftrightarrow$$

There is a unique constrained conic passing through four points of Δ if and only if the four points are equianharmonic.

- $\Delta : z_0^2 + z_1^2 + z_2^2 = 0$

- $C : (x_1 + x_2)z_0^2 + (x_2 + x_0)z_1^2 + (x_0 + x_1)z_2^2 = 0$

- cross-ratio of intersection points: $(x_1 - x_0)/(x_2 - x_0)$

- constraint: $\sigma_2 = x_1 x_2 + x_2 x_0 + x_0 x_1 = 0$

- in pencil: $x_i \mapsto x_i + t$,

  $\sigma_2 + 2\sigma_1 t + 3t^2 = 0$

- constraint: $\sigma_2 = x_1 x_2 + x_2 x_0 + x_0 x_1 = 0$

- in pencil: $x_i \mapsto x_i + t$,

  $\sigma_2 + 2\sigma_1 t + 3t^2 = 0$

- one root: $\sigma_1^2 = 3\sigma_2$

- constraint: $\sigma_2 = x_1 x_2 + x_2 x_0 + x_0 x_1 = 0$

- in pencil: $x_i \mapsto x_i + t$,

  $\sigma_2 + 2\sigma_1 t + 3t^2 = 0$

- one root: $\sigma_1^2 = 3\sigma_2$

- $x_1^2 - x_1(x_0 + x_2) + x_0^2 + x_2^2 - x_0 x_2 = 0$

- constraint: $\sigma_2 = x_1x_2 + x_2x_0 + x_0x_1 = 0$

- in pencil: $x_i \mapsto x_i + t$,

  $\sigma_2 + 2\sigma_1 t + 3t^2 = 0$

- one root: $\sigma_1^2 = 3\sigma_2$

- $x_1^2 - x_1(x_0 + x_2) + x_0^2 + x_2^2 - x_0x_2 = 0$

$$x_1 = \frac{x_0 + x_2 \pm i\sqrt{3}(x_0 - x_2)}{2}$$

cross-ratio: $\dfrac{1 \pm i\sqrt{3}}{2}$

- HAPPY BIRTHDAY, SIR MICHAEL!