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GEOMETRIC PERIODICITY AND THE INVARIANTS OF MANIFOLDS

Dennis Sullivan

There is a surprisingly rich structure in the theory of manifolds within a given homotopy type.

We begin with a geometric procedure for constructing an isomorphism between two manifolds - either smooth, combinatorial, or topological - in the homotopy class of a given homotopy equivalence between them.
This method which combines transversality and surgery inside these manifolds leads to three natural obstruction theories for the three situations. The obstructions up to codimension two are stable in a certain sense and much of our discussion concerns them. The unstable codimension two obstruction has not been analyzed.

The interesting structure in these theories arises on the one hand from the natural invariants which can be associated to a homotopy equivalence $M \xrightarrow{f} L$. These enable us to obtain a-priori information about the obstructions encountered in constructing an isomorphism between M and L.

By transversality the "varieties in L" can be pulled back to "varieties in M". This correspondence is a rather deep geometric invariant of f. By a <u>characteristic invariant</u> of f we mean any invariant defined using the induced diagram

$$\begin{tikzcd}
M \arrow[r, "f"] & L \\
f^{-1}V \arrow[u] \arrow[r, "f^V"] & V \arrow[u]
\end{tikzcd}$$

$V \to L$ is a variety in L.

For example when V is a manifold f^V has a well defined surgery obstruction like

$$\sigma(f, V) = \begin{cases} {}^1/8 \;(\text{signature } f^{-1}V - \text{signature } V) \in Z \text{ or } Z/n, \; \dim V = 4i \\ \\ \text{Arf invariant } (\text{kernel } f_*^V, \; Z/2) \in Z/2, \; \dim V = 4i + 2 . \end{cases}$$

Consider the homotopy equivalence $M \xrightarrow{f} L$ where M and L are simply connected and have dimension at least five. Then one of the main objectives is the

Characteristic Variety Theorem:[1] In the topological or combinatorial context we can construct an isomorphism between M and L in the homotopy class of f if and only if a certain[2] finite collection of characteristic invariants $\{\sigma(f, v)\}$ of the signature and Arf invariant type vanish.

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[1] The fundamental group hypotheses can be replaced by a restriction on $(n - 2)$-handles.

[2] Slightly different for the two contexts.

We outline some of the geometrical aspects of the proof. These arise naturally from the "picture" of the obstruction theory. The most interesting step concerns the indeterminancy in the obstructions.

To describe the "total indeterminancy subgroups" we introduce a class of geometric cycles, manifolds with singularities which have a stratified structure like that of a finite join of closed manifolds.

A "periodicity operation" in the theory of cycles with these special singularities (k-varieties)

$$(V \longrightarrow L) \xrightarrow{\times\, \mathbb{C}P^2} (\mathbb{C}P^2 \times V \longrightarrow L)$$

combines with the "periodicity relation" among the characteristic invariants

$$\sigma(\mathbb{C}P^2 \times V,\ f) = \sigma(V,\ f)$$

to pattern the indeterminancy. What finally emerges is the statement that the value of an obstruction on a homology class x is "determinant" (or meaningful for the problem at hand) iff x is represented by a k-variety in L.

The periodicity relation is a special case of a product formula for the invariants

$$\sigma(\mathbb{C} \times V, f) = i(\mathbb{C}) \cdot \sigma(V, f)$$

where $i(\mathbb{C})$ is computed from the signature or Euler characteristic of $\mathbb{C}$ or the difference between the real and mod 2 Euler semi-characteristics of $\mathbb{C}$ or that of a submanifold of $\mathbb{C}$ representing the first Stiefel-Whitney class.

This generalized periodicity relation and the cobordism relation generate all relations among these determining characteristic invariants for all possible M and f (given L).

The "periodicity relation" enters again to simplify the generating set for the invariants.

Thus we need only consider the invariants for "manifolds in L" in the characteristic variety theorem (even though the obstructions are computed on k-varieties) because any k-variety in L is k-homologous to a manifold after applying the "periodicity operation" enough times.

We discuss the difference between the pl and topological invariants and the light shed upon the situation by certain theories of generalized manifolds and the triangulation theory of Kirby and Siebenmann (which allows the topological case to be included).

Turning to the theory for constructing diffeomorphisms we refer briefly to the global or normal invariant viewpoint to the obstructions. There are two real K-theory invariants defined by f and its "homotopy theoretical derivative"

$$\mathcal{T}_M \xrightarrow{df} \mathcal{T}_L \,.$$

One is $\mathcal{T}_f = \mathcal{T}_M - f^*\mathcal{T}_L$, the difference of stable tangent bundles.
The other is Θ_f a unit of K-ring measuring the deviation of f from "degree oneness" in KO-theory.

Finally, there is an invariant α_f in the units of the stable cohomotopy ring of L . The definition of α_f depends on the choice of a "generating element" α of the galois group G of $\overline{Q}$ over Q , $\overline{Q}$ the algebraic closure of the rationals. α_f measures the extent to which df comes from the action of the galois group in the homotopy theory of the finite grassmannians.

Finally we have the

Theorem. The stable obstructions for constructing a diffeomorphism between M and L in the class of f "vanish" iff

$$\mathcal{T}_f = 0 \quad \text{in} \quad \widetilde{KO}(M) \,.$$
$$\Theta_f = 1 \quad \text{in} \quad KO^*(M)$$
$$\alpha_f = 1 \quad \text{in} \quad \pi^*(M) \,.$$

The relations between $\mathcal{T}_f$, Θ_f , and the odd part of the characteristic invariants are the natural ones. α_f (for all possible f) varies freely in a natural subgroup

$$\mathbb{C}(M) \subset \pi_* M \,.$$

In fact the stable part of the smooth obstruction theory factors completely into two theories - one infinite and one finite. The infinite theory K is isomorphic over Q and at every prime to real K-theory. The finite theory is the natural subtheory of stable cohomotopy, $\mathbb{C} \subset \pi^*$. The natural quotients $\pi^*/\mathbb{C}$ form a theory which may be identified with "the rational K-theory".

We end by discussing certain structure and speculation in the topological theory evolving from the "form of the invariants".

When the characteristic invariants are defined in terms of absolute invariants of M and L

$$\mu_2 \cap \{ \sigma(v, f)\} = f_* \mu_M,$$

then the invariants μ_M and μ_L naturally reside in the real K-theory of M and L. Their connection to the Laplacian when M and L are Riemannian suggests these invariants may figure in some natural thermodynamic discussion on topological manfiolds.

The galois symmetry in the invariants and thus in a formal K-theory model of manifold theory (which ignores π_1 and the prime 2) suggests there is a natural profinite form of geometric topology which exhibits this symmetry geometrically.

Finally I happily dedicate this paper to its precursors

Frank Adams
Raoul Bott
William Browder
Michel Kervaire
John Milnor
Sergei Novikov
Daniel Quillen
Stephen Smale
Rene Thom

The Obstruction Theory

Let $M \xrightarrow{f} L$ be a map between two compact[1] manifolds (preserving the boundaries). Our beginning and inductive assumption is that f induces an isomorphisms between certain regions interior to M and L and a homotopy equivalence between their complements.

Denote the isomorphic regions in M and L by Q' and Q. We will assume that L can be obtained by adding handles to ∂Q and so on[2]. The idea of the obstruction theory is to enlarge the "isomorphic region" of f by pulling back the handle structure of L mod Q to a handle structure for M mod Q'.

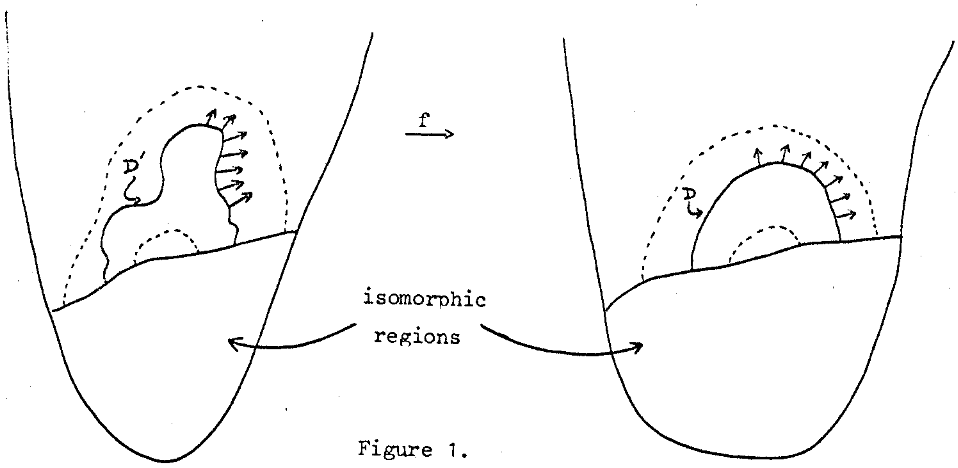

Figure 1.

Deform f slightly on the complement so that it is transversal to the core disks of the first layer of handles not included in the region of isomorphism for f.

Consider one of these (say D in figure 1) and the inverse image D'. Then we have

i) $\partial D'$ is a sphere

ii) D' is normally framed in M

iii) the inclusion $D' \hookrightarrow M$ is naturally null-homotopic.

So from D' we can construct a closed manifold whose stable tangent bundle is parallelized in the complement of a point. Denote the group of cobordism classes of such manifolds by T_i, i = dimension D'.

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[1] There is nothing to prevent one from making progress in the non compact case by considering proper maps.

[2] This is only a restriction in the topological case in dimension four or five. We will in fact suppress the low dimensional topological difficulties when they can be avoided by some manoeuvre, e.g. four dimensional topological transversality.

From the above we see that this transversal approximation to f determines a cochain on L relative to Q with values in the group T_i.

It is a pleasant task to show using transversality, that this cochain is a well-defined cocyle.

We obtain an obstruction class

$$O_f \in H^i(L, Q; T_i) .$$

Theorem A. If $i + 2 <$ dimension L, then O_f vanishes iff the region of isomorphism for f may be enlarged to include the i-handles without changing f on the $(i - 2)$ and lower handles.

Sketch of proof: The idea of the proof is simple. Suppose in fact that D' in figure 1 determines the zero element in T_i. Then from surgery theory we know D' is framed cobordant to the disk in a very special way. The cobordism can be obtained by adding to $D' \times I$ handles of dimension no larger than $i/2 + 1$. We embed this cobordism in $M \times I$ and construct a homotopy of f to another transversal approximation so that $D' \to D$ is an isomorphism.

In our codimension at least three situations, f will be a homotopy equivalence between the complements (respecting the boundary).

The normal framing then insures that we have enlarged the region of isomorphism to include this handle.

More generally if O_f is only cohomologous to zero, we do some preliminary deformation of f on the $(i - 1)$-handles to get into the situation above.

From theorem A we know we can define a sequence of obstructions for deforming f to an isomorphism up to but not including the codimension 2 handles of L mod Q. This is the stable obstruction theory.

The Coefficient Groups and the Codimension Two Obstruction

Theorem A leaves out the cases when the handles have dimension $n - 2$, $n - 1$, or n where $n =$ dimension L.

Some further progress can be made if we assume that n is greater than four (or greater than five if $\partial L \neq \emptyset$).

If $\pi_1(L - Q) = \{0\}$, the obstruction defined above is adequate for the $(n - 2)$ handles.

The cases $(n - 1)$ and n are treated below and we have a complete obstruction theory in the simply connected case.

In general, the $n - 2$ obstruction presents a real difficulty. An adequate class

$$0_{n-2} \in H^{n-2}(L, Q; \tilde{T})$$

has not been defined. The nature of $\tilde{T}$ - whether twisted or not even locally constant - has not been analyzed. To be sure we have to measure the position and knotting of D' in M.

$(n - 2)$ is the only difficult dimension however. For if we assume that the region of isomorphism contains the $(n - 2)$ handles then in the closed case we are done.

On the complements f is a homotopy equivalence between regular neighborhoods of 1 - complexes which is an isomorphism on the boundary. Once can now show using codimension one surgery techniques that f may be deformed to an isomorphism on the neighborhoods of $(n - 1)$ handles.

In the topological or combinatorial case the n-handles present no difficulty.

In the smooth case we have a final obstruction in

$$H^n(L, Q; \Theta_n)$$

where Θ_n is the group of exotic n-sheres.

In the non-closed case we have to deal with the part of the Whitehead Torsion of f which has not been absorbed by the $(n - 2)$ obstruction. This can be analyzed.

A lot is known about the lower or "stable" coefficient groups. In the topological or combinatorial theories the coefficient groups are denoted $\{P_i\}$, the periodic sequence (the Arf-Kervaire invariant and the signature)

$$0 \ Z/_2 \ 0 \ Z \ 0 \ Z/_2 \ 0 \ Z \ 0 \ Z/_2 \ 0 \ Z \ \ldots \ .$$

However, the natural map between the topological and pl coefficients is an isomorphism except in dimension four where we have multiplication by two.

In the C^∞ obstruction theory the groups are denoted $\{A_i\}$. For $i \leq 19$ say, they are

i	A_i
1	0
2	$Z/_2$
3	0
4	Z
5	0
6	$Z/_2$
7	0
8	$Z \oplus Z/_2$
9	$(Z/_2)^2$
10	$Z/_6$
11	0
12	Z
13	$Z/_3$
14	$(Z/_2)^2$
15	$Z/_2$
16	$Z \oplus Z/_2$
17	$(Z/_2)^3$
18	$Z/_8 \oplus Z/_2$
19	$Z/_2$ ... .

Note that up to this point $\{A_i\}$ is the sum of the coefficients for oriented real K-theory and those of a finite theory

$$\{A_i\} = \begin{cases} 0 \;\; Z/_2 \;\; 0 \;\; Z \;\; 0 \;\; 0 \;\; 0 \;\; Z \;\; Z/_2 \;\; Z/_2 \;\; 0 \;\; Z \;\; 0 \;\; 0 \;\; 0 \;\; Z \;\; Z/_2 \;\; Z/_2 \;\; 0 \;\; \ldots \\ \qquad\qquad\qquad\qquad\qquad \oplus \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; Z/_2 \;\; 0 \;\; Z/_2 \;\; Z/_2 \;\; Z/_3 \;\; 0 \;\; 0 \;\; Z/_3 \;\; (Z/_2)^2 \;\; Z/_2 \;\; Z/_2 \;\; (Z/_2)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Z/_8 \;\; Z/_2 \;\; \ldots \end{cases} .$$

In summary then for the general obstruction theory in closed n-manifolds we have the table of coefficients

topological or piecewise linear theory	1	2	...	i	...	n - 2	n - 1	n
	0	$Z/_2$	...	P_i	...	$\tilde{T}_1 = ?$	0	0

smooth theory	1	2	...	i	...	n - 2	n - 1	n
	0	$Z/_2$	...	A_i	...	$\tilde{T}_2 = ?$	0	Θ_n

The Indeterminancy Subgroups

Trying to deform $M \xrightarrow{f} L$ to an isomorphism with only an obstruction theory such as the above is rather like being in a complicated labyrinth with only a weak torch. The obstructions per se only tell if our progress can be extended a little farther while allowing only small corrections.

We want to understand when we can get completely through the maze and obtain an isomorphism between the two manifolds M and L . In particular we want to know when a given obstruction is "indeterminate" - it can be changed by deforming f on the lower handles to a new partial isomorphism.

To pursue this question we define the "total indeterminancy subgroup"

$$I_k \subseteq H^k(L, T_k)^{1)} .$$

I_k is the subgroup of all cohomology classes occuring as obstructions classes for some deformation of the identity map $L \to L$ to a partial isomorphism.

For a general homotopy equivalence $M \xrightarrow{f} L$ the set of all possible k-dimensional obstructions will either be vacuous or a coset of this subgroup.

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1) Suppose Q is vacuous for simplicity.

In fact the sequence of possibilities will look like

$$I_1,\ I_2,\ \dots,\ I_{k-1},\ \begin{array}{c}\text{some}\\ \text{non-trivial}\\ \text{coset}\\ \text{of } I_k\end{array},\ \emptyset,\ \emptyset,\ \emptyset,\ \dots$$

if no paths through the obstruction labyrinth go beyond level $k - 1$.

We will analyze these indeterminancy groups in the combinatorial and topological obstructions where they are naturally isomorphic. By looking at the geometry of f near subvarieties of L and M we will show I_k is an odd torsion group. Thus the Arf invariant obstructions are completely determinant. In fact we will see how to give an a-priori calculation of these.

Only the signature obstructions in $H^{4k}(L, Z)$ can by indeterminant. We will see that the indeterminancy subgroup in this case is determined by Pontryagin duality in terms of the classes in

$$\varinjlim_{j} H_{4k}(L, Z/_j)$$

which have a nice geometrical representation.

Geometric Computation of the Obstructions

We will concentrate on the piecewise linear and topological theory where the signatures and Arf invariants play a decisive role.

Consider the "signature obstructions"

$$O_f \in H^{4i}(L, Z)\ ^{1)}\ .$$

Recall that an integral cohomology class in L is determined by its evaluations on all mod n homology classes

$$\langle O_f, x\rangle \in Z/_n\ ,\ x \in H_{4i}(L, Z/_n)\qquad n = 0, 1, \dots\ .$$

Suppose that x is represented by a "$Z/_n$-manifold" $V \subseteq L$. That is, a mod n cycle obtained by identifying n isomorphic collections of boundary components of some compact manifold (everything oriented compatibly).
Suppose that all of V outside a $4i$-disk lies in the region of isomorphism for f . Assume the $4i$-disk is the core disk of one of the handles used to compute O_f .

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1) We take $Q = \emptyset$ for simplicity.

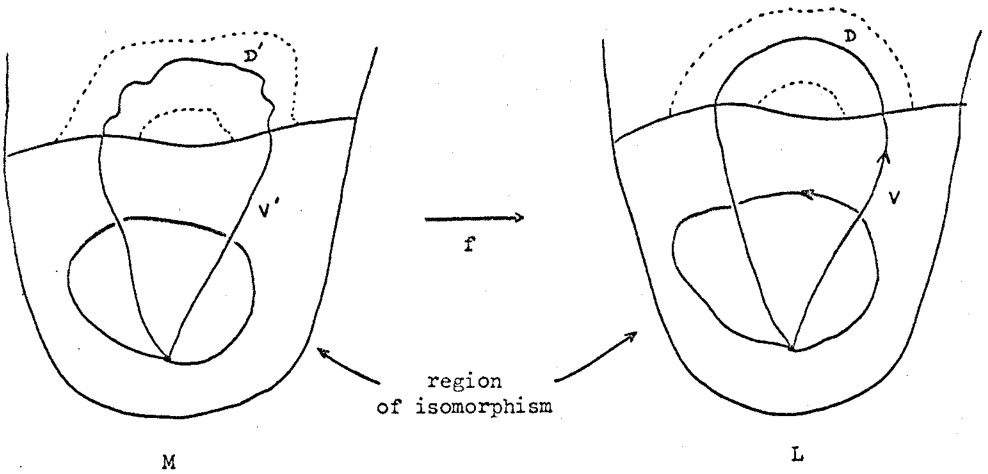

Figure 2.

Let V' be the inverse image of V under the transversal approximation to f used to compute the obstruction.
An examination of figure 2 shows that (calculating modulo n)

$$\begin{aligned}\langle O_f, x\rangle &= \text{value of } D' \text{ in } P_{4i}\\ &= \tfrac{1}{8}\,(\text{signature } D')\\ &= \tfrac{1}{8}\,(\text{signature } V' - \text{signature } V)^{1)},\end{aligned}$$

where the signature of a $Z/_n$-manifold is computed by intersecting even dimensional cycles in the interior to obtain an integral signature and then reducing mod n.

Now

$$\sigma(f, V) = \tfrac{1}{8}\,(\text{signature } V' - \text{signature } V) \in Z/_n$$

<u>is a characteristic invariant of f</u>. This is clear when n is odd. For then 8 is a unit and only the quantity signature V - signature V' need be determined. But this difference is a cobordism invariant defined for any map $M \xrightarrow{f} L$.
When n is even the definition of $\sigma(f, V)$ uses the fact that f is a homotopy equivalence. For example, one can cobord the map $V' \xrightarrow{f^V} V$ so that the homological situation of figure 2 is realized. Then define

$$\sigma(f, V) = \tfrac{1}{8}\ \text{signature}\ (\ker f^V_* \,;\, Q)\,.$$

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1) Actually $\frac{1}{8}$ is replaced by $\frac{1}{16}$ in the pl case for $i = 1$.

If V is any $(4i + 2)$ manifold in L representing $x \in H_{4i+2}(L, Z/_2)$, a similar argument shows the "Arf invariant obstruction"

$$O_f \in H^{4i+2}(L, Z/_2)$$

satisfies

$$\begin{aligned} \langle O_f, x\rangle &= \text{Arf invariant } (V' \xrightarrow{f^V} V) \\ &= \text{Arf invariant } (\ker f^V_* \,;\, Z/_2) \,. \end{aligned}$$

But again the right hand side is a characteristic invariant of f,

$$\sigma(f, V) \in Z/_2 \quad , \quad \dim V = 4i + 2$$

defined for any homotopy equivalence $M \xrightarrow{f} L$.

A slight modification of the above argument (using graphs) shows that the condition that V is embedded in L is unnecessary.

From the work of Thom we know that any homology class in $H_i(L, Z/_{2^r})$ is represented by some $Z/_{2^r}$-manifold mapping into L.

Thus we obtain a calculation of the "Arf invariant obstructions" and a partial calculation of the "signature obstructions" in terms of the a-priori geometrical behaviour of f near manifolds in L.

Example of an indeterminant Obstruction

Let L be a manifold with boundary of large dimension having four handles in dimensions 0, 3, 7, and 8.
Suppose the integral homology of L is

i	0	1	2	3	4	5	6	7
H_i	Z	0	0	Z	0	0	0	$Z/_3$

and the seven handle is attached along a generator of $\pi_6(S^3) \sim Z/_{12}$.

Now consider the obstruction theory for some homotopy equivalence $M \xrightarrow{f} L$. There is only one possible obstruction

$$O_f \in H^8(L, Z) \cong Z/_3 \,.$$

Because the seven handle is attached so vigorously there is no $Z/_3$-manifold in L representing a non-zero element of

$$H_8(L, Z/_3) \cong Z/_3 \,.$$

Au contraire, it is possible to find a "singular $Z/_3$-manifold" embedded in L representing a generating 8 dimensional class. The "singularity stratum" [1] of this $V \subset L$ lies in the interior of V and has a neighborhood isomorphic to

$$S^3 \times \text{cone } \mathbb{C}P^2 ,$$

the singular points generating the third homology of L .

Now we can construct a homotopy of the identity map of L to a new partial isomorphism on the 0, 3, and 7 handles of L so that the transversal inverse image of "singularity $V" = S^3$ by the homotopy is a cobordism from S^3 to S^3 with signature prime to 3 (for example 16).

The transversal inverse image of V by the new map $L \xrightarrow{I'} L$ now has a new signature in its interior - the signature appearing during the S^3 deformation is multiplied by $\mathbb{C}P^2$ and appears in the interior of V' . (Figure 3).

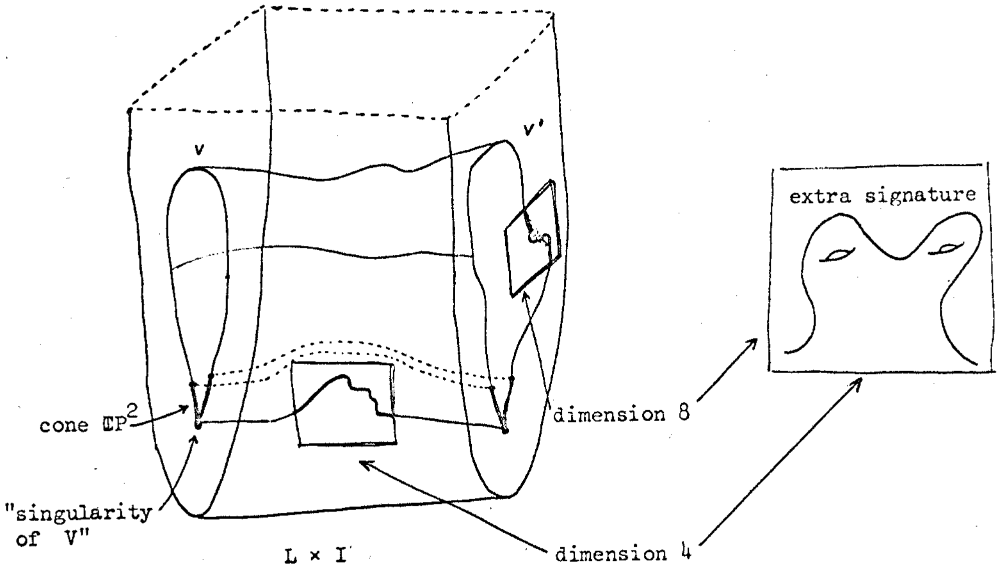

Figure 3.

We then find a non zero obstruction on the eight handle created by a deformation near the three handle. These considerations show that all the obstructions for L are indeterminant, and

$$I_8 \simeq H^8(L, Z) .$$

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[1] We are not including the Bockstein of V , where the sheets come together, in the "singularity stratum". The Bockstein is there for homological reasons and is not a serious singularity in its untwisted form.

Thus any homotopy equivalence $M \to L$ is homotopic to a homeomorphism.

Balancing the Indeterminancy and Determinancy - k - varieties

So far we have seen that a certain part of our "stable obstructions admit an a-priori calculation in terms of f and a certain part can be created or destroyed by global deformations of f. If we could push our understanding of these two phenomena far enough we might exhaust all possibilities and have a complete analysis.

We will proceed on this course by studying the singularities in geometric cycles representing the various homology classes in L. If the singularities are "signature free" an a-priori calculation of the obstruction is possible. If not, the bad singularity can be used to create an indeterminancy in the value of the obstruction.

Consider stratified spaces (in the sense of Thom) whose stratification schema is isomorphic to that of a finite join[1] of C^∞ manifolds from a given list $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots$.
We can form cycles and homologies from this geometric material and construct geometric homology theories.

There are two sequences of interest for studying these obstructions.
First, if the sequence of closed manifolds gives an irredundant set of generators for the oriented cobordism ring of Thom, then we obtain a generating set of cycles and homologies for ordinary homology theory (H_*, H^*) .[2]

Second if the sequence of manifolds gives an irredundant sequence of generators for the ideal of cobordism classes with vanishing signature, we obtain the cycles and homologies of a generalized homology theory (k_*, k^*) . The non-zero groups of a point are infinite cyclic in every fourth non-negative dimension.
One set of generators is given by the cartesian powers of the complex projective plane.

Let us refer to these manifolds with singularities as H-varieties and k-varieties respectively.

There is a natural exact sequence relating these two theories (for any space)

$$\ldots \longrightarrow k_i \xrightarrow{\text{"}\times \mathbb{C}P^2\text{"}} k_{i+4} \xrightarrow[\text{map}]{\text{natural}} H_{i+4} \xrightarrow[\text{singularity"}]{\text{"}\mathbb{C}P^2} k_{i-1} \longrightarrow \ldots$$

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[1] The join of two spaces is the space of all segments joining their points.

[2] We calculate modulo finite 2-groups in this section.

The natural map is obtained by adding $\mathbb{C}P^2$ to the list of singularities defining k to obtain a list of singularities for H_*.

" $\times \mathbb{C}P^2$" is just the "periodicity operation" in k_* induced by replacing a k-variety by its cartesian product with the complex projective plane.

The "$\mathbb{C}P^2$ - singularity" map is defined by looking at those points of an H-variety whose link has the form $\mathbb{C}P^2 * L$ ($*$ = "join") for some L. This set has the structure of a k-variety.

The exactness is easy to prove geometrically. In fact such a sequence relating any pair of such theories differing by one singularity is the only proposition needed to prove the assertions above about H_* and k_*.

We can define k-varieties (mod n), and state the theorem for which they were defined. Recall the homotopy equivalence $L \xrightarrow{f} M$ and the obstruction $\{0_f\}$ to constructing a homeomorphism or pl homeomorphism in the homotopy class of f.

Theorem B. The value of an obstruction 0_f on a homology class $x \in H_{4i}(L, Z/_j)$ is "determinant" iff x is represented by a k-variety in L, $V \to L$. More precisely, there is a characteristic invariant $\sigma(f, V)$ so that

$$\sigma(f, V) = \langle 0_f, x\rangle \quad ,$$

and the "total indeterminancy subgroup"

$$I_{4i} \in H^{4i}(L, Z)$$

is dual under Pontryagin duality to the quotient of

$$\lim_{\substack{\rightarrow \\ j \text{ odd}}} H_{4i}(L, Z/_j)$$

by the subgroup of classes represented by k-varieties.

Corollary: f may be deformed to a homeomorphism on some region containing the 1, 2, ..., k handles ($k < n - 2$ or $\pi_1 = 0$) iff all characteristic invariants of the signature and Arf invariant type vanish for k-varieties in L up to dimension k.

Proof: From theorem B an obstruction 0_f lies in the indeterminancy subgroup iff 0_f vanishes on any homology class represented by a k-variety. But these values are given by characteristic invariants. So if enough invariants $\sigma(f, V)$ vanish we can work our way through the maze and construct an isomorphism between the desired regions.