Andrew Ranicki

GEOMETRIC PERIODICITY AND THE INVARIANTS OF MANIFOLDS

Dennis Sullivan

There is a surprisingly rich structure in the theory of manifolds within a given homotopy type.

We begin with a geometric procedure for constructing an isomorphism between two manifolds - either smooth, combinatorial, or topological - in the homotopy class of a given homotopy equivalence between them.
This method which combines transversality and surgery inside these manifolds leads to three natural obstruction theories for the three situations. The obstructions up to codimension two are stable in a certain sense and much of our discussion concerns them. The unstable codimension two obstruction has not been analyzed.

The interesting structure in these theories arises on the one hand from the natural invariants which can be associated to a homotopy equivalence $M \xrightarrow{f} L$. These enable us to obtain a-priori information about the obstructions encountered in constructing an isomorphism between M and L .

By transversality the "varieties in L" can be pulled back to "varieties in M" . This correspondence is a rather deep geometric invariant of f . By a <u>characteristic invariant</u> of f we mean any invariant defined using the induced diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & L \\ \uparrow & & \uparrow \\ f^{-1}V & \xrightarrow{\quad f^V \quad} & V \end{array} \qquad\qquad V \to L \ \text{ is a variety in } L .$$

For example when V is a manifold f^V has a well defined surgery obstruction like

$$\sigma(f, V) = \begin{cases} {}^1/8 \ (\text{signature } f^{-1}V - \text{signature } V) \in Z \text{ or } Z/n \ , \dim V = 4i \\ \\ \text{Arf invariant } (\text{kernel } f_*^V \ , Z/2) \in Z/2, \ \dim V = 4i + 2 \ . \end{cases}$$

Consider the homotopy equivalence $M \xrightarrow{f} L$ where M and L are simply connected and have dimension at least five. Then one of the main objectives is the

Characteristic Variety Theorem:[1] In the topological or combinatorial context we can construct an isomorphism between M and L in the homotopy class of f if and only if a certain[2] finite collection of characteristic invariants $\{\sigma(f, v)\}$ of the signature and Arf invariant type vanish.

1) The fundamental group hypotheses can be replaced by a restriction on $(n - 2)$-handles.

2) Slightly different for the two contexts.

We outline some of the geometrical aspects of the proof. These arise naturally from the "picture" of the obstruction theory. The most interesting step concerns the indeterminancy in the obstructions.

To describe the "total indeterminancy subgroups" we introduce a class of geometric cycles, manifolds with singularities which have a stratified structure like that of a finite join of closed manifolds.

A "periodicity operation" in the theory of cycles with these special singularities (k-varieties)

$$(V \longrightarrow L) \xrightarrow{\times \mathbb{C}P^2} (\mathbb{C}P^2 \times V \longrightarrow L)$$

combines with the "periodicity relation" among the characteristic invariants

$$\sigma(\mathbb{C}P^2 \times V, f) = \sigma(V, f)$$

to pattern the indeterminancy. What finally emerges is the statement that the value of an obstruction on a homology class x is "determinant" (or meaningful for the problem at hand) iff x is represented by a k-variety in L.

The periodicity relation is a special case of a product formula for the invariants

$$\sigma(\mathbb{C} \times V, f) = i(\mathbb{C}) \cdot \sigma(V, f)$$

where $i(\mathbb{C})$ is computed from the signature or Euler characteristic of $\mathbb{C}$ or the difference between the real and mod 2 Euler semi-characteristics of $\mathbb{C}$ or that of a submanifold of $\mathbb{C}$ representing the first Stiefel-Whitney class.

This generalized periodicity relation and the cobordism relation generate all relations among these determining characteristic invariants for all possible M and f (given L).

The "periodicity relation" enters again to simplify the generating set for the invariants.

Thus we need only consider the invariants for "manifolds in L" in the characteristic variety theorem (even though the obstructions are computed on k-varieties) because any k-variety in L is k-homologous to a manifold after applying the "periodicity operation" enough times.

We discuss the difference between the pl and topological invariants and the light shed upon the situation by certain theories of generalized manifolds and the triangulation theory of Kirby and Siebenmann (which allows the topological case to be included).

Turning to the theory for constructing diffeomorphisms we refer briefly to the global or normal invariant viewpoint to the obstructions. There are two real K-theory invariants defined by f and its "homotopy theoretical derivative"

$$\mathcal{T}_M \xrightarrow{df} \mathcal{T}_L \, .$$

One is $\mathcal{T}_f = \mathcal{T}_M - f^*\mathcal{T}_L$, the difference of stable tangent bundles.

The other is Θ_f a unit of K-ring measuring the deviation of f from "degree oneness" in KO-theory.

Finally, there is an invariant α_f in the units of the stable cohomotopy ring of L . The definition of α_f depends on the choice of a "generating element" α of the galois group G of $\overline{Q}$ over Q , $\overline{Q}$ the algebraic closure of the rationals. α_f measures the extent to which df comes from the action of the galois group in the homotopy theory of the finite grassmannians.

Finally we have the

<u>Theorem</u>. The stable obstructions for constructing a diffeomorphism between M and L in the class of f "vanish" iff

$$\mathcal{T}_f = 0 \quad \text{in} \quad \widetilde{KO}(M) \, .$$

$$\Theta_f = 1 \quad \text{in} \quad KO^*(M)$$

$$\alpha_f = 1 \quad \text{in} \quad \pi^*(M) \, .$$

The relations between $\mathcal{T}_f$, Θ_f , and the odd part of the characteristic invariants are the natural ones. α_f (for all possible f) varies freely in a natural subgroup

$$\mathbb{C}(M) \subset \pi_* M \, .$$

In fact the stable part of the smooth obstruction theory factors completely into two theories - one infinite and one finite. The infinite theory K is isomorphic over Q and at every prime to real K-theory. The finite theory is the natural subtheory of stable cohomotopy, $\mathbb{C} \subset \pi^*$. The natural quotients $\pi^*/\mathbb{C}$ form a theory which may be identified with "the rational K-theory".

We end by discussing certain structure and speculation in the topological theory evolving from the "form of the invariants".

When the characteristic invariants are defined in terms of absolute invariants of M and L

$$\mu_2 \cap \{ \sigma(v, f)\} = f_* \mu_M,$$

then the invariants μ_M and μ_L naturally reside in the real K-theory of M and L. Their connection to the Laplacian when M and L are Riemannian suggests these invariants may figure in some natural thermodynamic discussion on topological manfiolds.

The galois symmetry in the invariants and thus in a formal K-theory model of manifold theory (which ignores π_1 and the prime 2) suggests there is a natural profinite form of geometric topology which exhibits this symmetry geometrically.

Finally I happily dedicate this paper to its precursors

Frank Adams
Raoul Bott
William Browder
Michel Kervaire
John Milnor
Sergei Novikov
Daniel Quillen
Stephen Smale
Rene Thom

The Obstruction Theory

Let $M \xrightarrow{f} L$ be a map between two compact[1] manifolds (preserving the boundaries). Our beginning and inductive assumption is that f induces an isomorphisms between certain regions interior to M and L and a homotopy equivalence between their complements.

Denote the isomorphic regions in M and L by Q' and Q . We will assume that L can be obtained by adding handles to ∂Q and so on[2]. The idea of the obstruction theory is to enlarge the "isomorphic region" of f by pulling back the handle structure of L mod Q to a handle structure for M mod Q' .

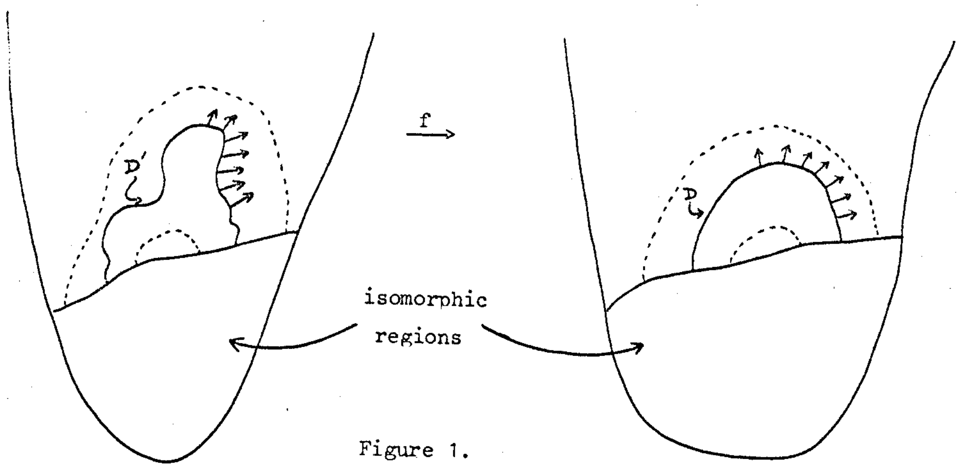

Figure 1.

Deform f slightly on the complement so that it is transversal to the core disks of the first layer of handles not included in the region of isomorphism for f .

Consider one of these (say D in figure 1) and the inverse image D' . Then we have

i) $\partial D'$ is a sphere

ii) D' is normally framed in M

iii) the inclusion $D' \hookrightarrow M$ is naturally null-homotopic.

So from D' we can construct a closed manifold whose stable tangent bundle is parallelized in the complement of a point. Denote the group of cobordism classes of such manifolds by T_i , i = dimension D' .

---

1) There is nothing to prevent one from making progress in the non compact case by considering proper maps.

2) This is only a restriction in the topological case in dimension four or five. We will in fact suppress the low dimensional topological difficulties when they can be avoided by some manoeuvre, e.g. four dimensional topological transversality.

From the above we see that this transversal approximation to f determines a cochain on L relative to Q with values in the group T_i .

It is a pleasant task to show using transversality, that this cochain is a well-defined cocyle.

We obtain an obstruction class

$$O_f \in H^i(L, Q; T_i) .$$

Theorem A. If $i + 2 <$ dimension L , then O_f vanishes iff the region of isomorphism for f may be enlarged to include the i-handles without changing f on the $(i - 2)$ and lower handles.

Sketch of proof: The idea of the proof is simple. Suppose in fact that D' in figure 1 determines the zero element in T_i . Then from surgery theory we know D' is framed cobordant to the disk in a very special way. The cobordism can be obtained by adding to $D' \times I$ handles of dimension no larger than $^i/2 + 1$. We embed this cobordism in $M \times I$ and construct a homotopy of f to another transversal approximation so that $D' \to D$ is an isomorphism.

In our codimension at least three situations, f will be a homotopy equivalence between the complements (respecting the boundary).

The normal framing then insures that we have enlarged the region of isomorphism to include this handle.

More generally if O_f is only cohomologous to zero, we do some preliminary deformation of f on the $(i - 1)$-handles to get into the situation above.

From theorem A we know we can define a sequence of obstructions for deforming f to an isomorphism up to but not including the codimension 2 handles of L mod Q . This is the stable obstruction theory.

The Coefficient Groups and the Codimension Two Obstruction

Theorem A leaves out the cases when the handles have dimension $n - 2$, $n - 1$, or n where $n =$ dimension L .

Some further progress can be made if we assume that n is greater than four (or greater than five if $\partial L \neq \emptyset$) .

If $\pi_1(L - Q) = \{0\}$, the obstruction defined above is adequate for the $(n - 2)$ handles.

The cases $(n - 1)$ and n are treated below and we have a complete obstruction theory in the simply connected case.

In general, the $n-2$ obstruction presents a real difficulty. An adequate class

$$0_{n-2} \in H^{n-2}(L, Q; \tilde{T})$$

has not been defined. The nature of $\tilde{T}$ - whether twisted or not even locally constant - has not been analyzed. To be sure we have to measure the position and knotting of D' in M.

$(n-2)$ is the only difficult dimension however. For if we assume that the region of isomorphism contains the $(n-2)$ handles then in the closed case we are done.

On the complements f is a homotopy equivalence between regular neighborhoods of 1 - complexes which is an isomorphism on the boundary. Once can now show using codimension one surgery techniques that f may be deformed to an isomorphism on the neighborhoods of $(n-1)$ handles.

In the topological or combinatorial case the n-handles present no difficulty.

In the smooth case we have a final obstruction in

$$H^n(L, Q; \theta_n)$$

where θ_n is the group of exotic n-sheres.

In the non-closed case we have to deal with the part of the Whitehead Torsion of f which has not been absorbed by the $(n-2)$ obstruction. This can be analyzed.

A lot is known about the lower or "stable" coefficient groups. In the topological or combinatorial theories the coefficient groups are denoted $\{P_i\}$, the periodic sequence (the Arf-Kervaire invariant and the signature)

$$0 \; Z/_2 \; 0 \; Z \; 0 \; Z/_2 \; 0 \; Z \; 0 \; Z/_2 \; 0 \; Z \; \ldots \; .$$

However, the natural map between the topological and pl coefficients is an isomorphism except in dimension four where we have multiplication by two.

In the C^∞ obstruction theory the groups are denoted $\{A_i\}$. For $i \leq 19$ say, they are

i	1	2	3	4	5	6	7	8	9	10
A_i	0	$Z/_2$	0	Z	0	$Z/_2$	0	$Z \oplus Z/_2$	$(Z/_2)^2$	$Z/_6$

11	12	13	14	15	16
0	Z	$Z/_3$	$(Z/_2)^2$	$Z/_2$	$Z \oplus Z/_2$

17	18	19
$(Z/_2)^3$	$Z/_8 \oplus Z/_2$	$Z/_2$ ... .

Note that up to this point $\{A_i\}$ is the sum of the coefficients for oriented real K-theory and those of a finite theory

$$\{A_i\} = \begin{cases} 0 \;\; Z/_2 \;\; 0 \;\; Z \;\; 0 \;\; 0 \;\; 0 \;\; Z \;\; Z/_2 \;\; Z/_2 \;\; 0 \;\; Z \;\; 0 \;\; 0 \;\; 0 \;\; Z \;\; Z/_2 \;\; Z/_2 \;\; 0 \;\; \ldots \\ \oplus \\ 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; Z/_2 \;\; 0 \;\; Z/_2 \;\; Z/_2 \;\; Z/_3 \;\; 0 \;\; 0 \;\; Z/_3 \;\; (Z/_2)^2 \;\; Z/_2 \;\; Z/_2 \;\; (Z/_2)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Z/_8 \;\; Z/_2 \;\; \ldots \end{cases}.$$

In summary then for the general obstruction theory in closed n-manifolds we have the table of coefficients

topological or piecewise linear theory	1	2	...	i	...	n − 2	n − 1	n
	0	$Z/_2$	...	P_i	...	$\tilde{T}_1 = ?$	0	0

smooth theory	1	2	...	i	...	n − 2	n − 1	n
	0	$Z/_2$	...	A_i	...	$\tilde{T}_2 = ?$	0	Θ_n

The Indeterminancy Subgroups

Trying to deform $M \xrightarrow{f} L$ to an isomorphism with only an obstruction theory such as the above is rather like being in a complicated labyrinth with only a weak torch. The obstructions per se only tell if our progress can be extended a little farther while allowing only small corrections.

We want to understand when we can get completely through the maze and obtain an isomorphism between the two manifolds M and L. In particular we want to know when a given obstruction is "indeterminate" - it can be changed by deforming f on the lower handles to a new partial isomorphism.

To pursue this question we define the "total indeterminancy subgroup"

$$I_k \subseteq H^k(L, T_k)^{1)} .$$

I_k is the subgroup of all cohomology classes occuring as obstructions classes for some deformation of the identity map $L \to L$ to a partial isomorphism.

For a general homotopy equivalence $M \xrightarrow{f} L$ the set of all possible k-dimensional obstructions will either be vacuous or a coset of this subgroup.

---

1) Suppose Q is vacuous for simplicity.

In fact the sequence of possibilities will look like

$$I_1,\ I_2,\ \ldots,\ I_{k-1},\ \begin{matrix}\text{some}\\ \text{non-trivial}\\ \text{coset}\\ \text{of } I_k\end{matrix},\ \emptyset,\ \emptyset,\ \emptyset,\ \ldots$$

if no paths through the obstruction labyrinth go beyond level $k - 1$.

We will analyze these indeterminancy groups in the combinatorial and topological obstructions where they are naturally isomorphic. By looking at the geometry of f near subvarieties of L and M we will show I_k is an odd torsion group. Thus the Arf invariant obstructions are completely determinant. In fact we will see how to give an a-priori calculation of these.

Only the signature obstructions in $H^{4k}(L, Z)$ can by indeterminant. We will see that the indeterminancy subgroup in this case is determined by Pontryagin duality in terms of the classes in

$$\lim_{\substack{\rightarrow \\ j}} H_{4k}(L, Z/_j)$$

which have a nice geometrical representation.

Geometric Computation of the Obstructions

We will concentrate on the piecewise linear and topological theory where the signatures and Arf invariants play a decisive role.

Consider the "signature obstructions"

$$O_f \in H^{4i}(L, Z)\ ^{1)}\ .$$

Recall that an integral cohomology class in L is determined by its evaluations on all mod n homology classes

$$\langle O_f,\ x\rangle \in Z/_n\ ,\ x \in H_{4i}(L,\ Z/_n) \qquad n = 0,\ 1,\ \ldots\ .$$

Suppose that x is represented by a "$Z/_n$-manifold" $V \subseteq L$. That is, a mod n cycle obtained by identifying n isomorphic collections of boundary components of some compact manifold (everything oriented compatibly).

Suppose that all of V outside a $4i$-disk lies in the region of isomorphism for f . Assume the $4i$-disk is the core disk of one of the handles used to compute O_f .

---

1) We take $Q = \emptyset$ for simplicity.

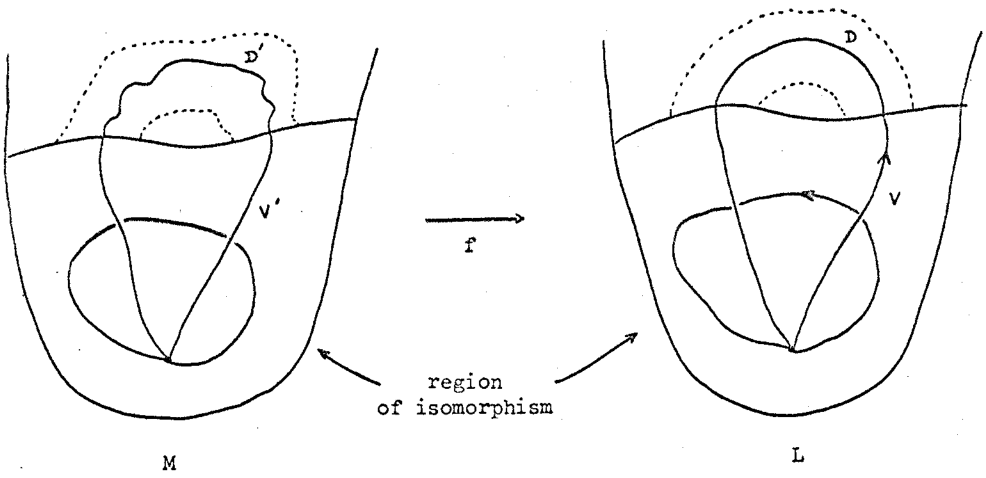

Figure 2.

Let V' be the inverse image of V under the transversal approximation to f used to compute the obstruction.
An examination of figure 2 shows that (calculating modulo n)

$$\begin{aligned} \langle O_f, x\rangle &= \text{value of } D' \text{ in } P_{4i} \\ &= \tfrac{1}{8}\ (\text{signature } D') \\ &= \tfrac{1}{8}\ (\text{signature } V' - \text{signature } V)^{1)}, \end{aligned}$$

where the signature of a $Z/_n$-manifold is computed by intersecting even dimensional cycles in the interior to obtain an integral signature and then reducing mod n .

Now

$$\sigma(f, V) = \tfrac{1}{8}\ (\text{signature } V' - \text{signature } V) \in Z/_n$$

is a characteristic invariant of f . This is clear when n is odd. For then 8 is a unit and only the quantity signature V - signature V' need be determined. But this difference is a cobordism invariant defined for any map $M \xrightarrow{f} L$.
When n is even the definition of $\sigma(f, V)$ uses the fact that f is a homotopy equivalence. For example, one can cobord the map $V' \xrightarrow{f^V} V$ so that the homological situation of figure 2 is realized. Then define

$$\sigma(f, V) = \tfrac{1}{8} \text{ signature } (\ker f_*^V ; Q) .$$

---

1) Actually $\frac{1}{8}$ is replaced by $\frac{1}{16}$ in the pl case for i = 1 .

If V is any $(4i + 2)$ manifold in L representing $x \in H_{4i+2}(L, Z/_2)$, a similar argument shows the "Arf invariant obstruction"

$$O_f \in H^{4i+2}(L, Z/_2)$$

satisfies

$$\langle O_f, x\rangle = \text{Arf invariant } (V' \xrightarrow{f^V} V)$$
$$= \text{Arf invariant } (\ker f^V_* \, ; Z/_2) .$$

But again the right hand side is a characteristic invariant of f,

$$\sigma(f, V) \in Z/_2 \quad , \quad \dim V = 4i + 2$$

defined for any homotopy equivalence $M \xrightarrow{f} L$.

A slight modification of the above argument (using graphs) shows that the condition that V is embedded in L is unnecessary.

From the work of Thom we know that any homology class in $H_i(L, Z/_{2^r})$ is represented by some $Z/_{2^r}$-manifold mapping into L.

Thus we obtain a calculation of the "Arf invariant obstructions" and a partial calculation of the "signature obstructions" in terms of the a-priori geometrical behaviour of f near manifolds in L.

Example of an indeterminant Obstruction

Let L be a manifold with boundary of large dimension having four handles in dimensions 0, 3, 7, and 8.
Suppose the integral homology of L is

i	0	1	2	3	4	5	6	7
H_i	Z	0	0	Z	0	0	0	$Z/_3$

and the seven handle is attached along a generator of $\pi_6(S^3) \sim Z/_{12}$.

Now consider the obstruction theory for some homotopy equivalence $M \xrightarrow{f} L$. There is only one possible obstruction

$$O_f \in H^8(L, Z) \approx Z/_3 .$$

Because the seven handle is attached so vigorously there is no $Z/_3$-manifold in L representing a non-zero element of

$$H_8(L, Z/_3) \approx Z/_3 .$$

Au contraire, it is possible to find a "singular $Z/_3$-manifold" embedded in L representing a generating 8 dimensional class. The "singularity stratum" [1] of this $V \subset L$ lies in the interior of V and has a neighborhood isomorphic to

$$S^3 \times \text{cone } \mathbb{C}P^2 ,$$

the singular points generating the third homology of L.

Now we can construct a homotopy of the identity map of L to a new partial isomorphism on the 0, 3, and 7 handles of L so that the transversal inverse image of "singularity V" $= S^3$ by the homotopy is a cobordism from S^3 to S^3 with signature prime to 3 (for example 16).

The transversal inverse image of V by the new map $L \xrightarrow{I'} L$ now has a new signature in its interior - the signature appearing during the S^3 deformation is multiplied by $\mathbb{C}P^2$ and appears in the interior of V'. (Figure 3).

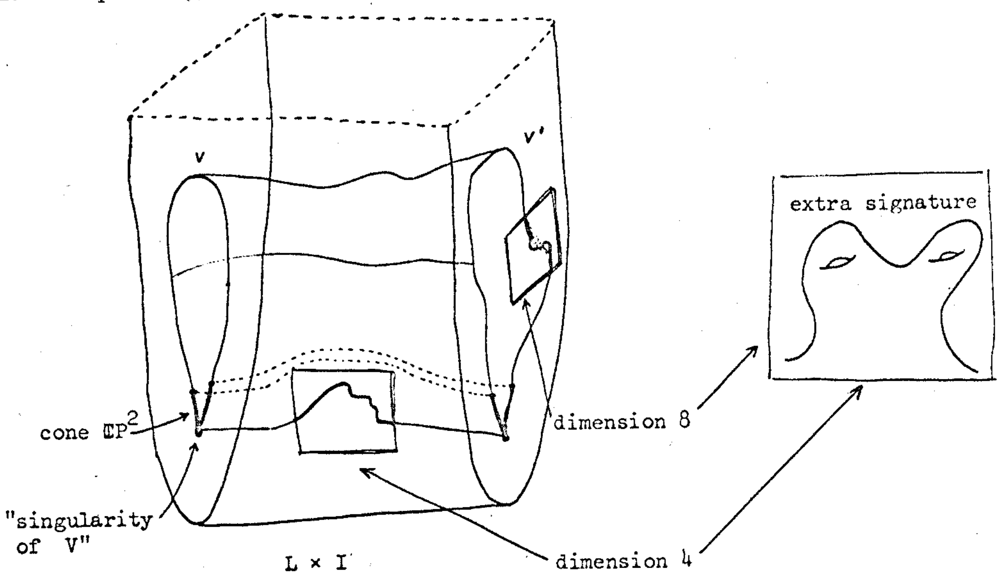

Figure 3.

We then find a non zero obstruction on the eight handle created by a deformation near the three handle. These considerations show that all the obstructions for L are indeterminant, and

$$I_8 \simeq H^8(L, Z) .$$

1) We are not including the Bockstein of V, where the sheets come together, in the "singularity stratum". The Bockstein is there for homological reasons and is not a serious singularity in its untwisted form.

Thus any homotopy equivalence $M \to L$ is homotopic to a homeomorphism.

Balancing the Indeterminancy and Determinancy - k - varieties

So far we have seen that a certain part of our "stable obstructions admit an a-priori calculation in terms of f and a certain part can be created or destroyed by global deformations of f. If we could push our understanding of these two phenomena far enough we might exhaust all possibilities and have a complete analysis.

We will proceed on this course by studying the singularities in geometric cycles representing the various homology classes in L. If the singularities are "signature free" an a-priori calculation of the obstruction is possible. If not, the bad singularity can be used to create an indeterminancy in the value of the obstruction.

Consider stratified spaces (in the sense of Thom) whose stratification schema is isomorphic to that of a finite join[1] of C^∞ manifolds from a given list $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots$.
We can form cycles and homologies from this geometric material and construct geometric homology theories.

There are two sequences of interest for studying these obstructions.
First, if the sequence of closed manifolds gives an irredundant set of generators for the oriented cobordism ring of Thom, then we obtain a generating set of cycles and homologies for ordinary homology theory (H_*, H^*) .[2]

Second if the sequence of manifolds gives an irredundant sequence of generators for the ideal of cobordism classes with vanishing signature, we obtain the cycles and homologies of a generalized homology theory (k_*, k^*) . The non-zero groups of a point are infinite cyclic in every fourth non-negative dimension.
One set of generators is given by the cartesian powers of the complex projective plane.

Let us refer to these manifolds with singularities as H-varieties and k-varieties respectively.

There is a natural exact sequence relating these two theories (for any space)

$$\cdots \longrightarrow k_i \xrightarrow{\text{" } \times\, \mathbb{C}P^2 \text{"}} k_{i+4} \xrightarrow[\text{map}]{\text{natural}} H_{i+4} \xrightarrow[\text{singularity"}]{\text{"}\mathbb{C}P^2} k_{i-1} \longrightarrow \cdots$$

---

[1] The join of two spaces is the space of all segments joining their points.

[2] We calculate modulo finite 2-groups in this section.

The natural map is obtained by adding $\mathbb{C}P^2$ to the list of singularities defining k to obtain a list of singularities for H_* .

" $\times \mathbb{C}P^2$" is just the "periodicity operation" in k_* induced by replacing a k-variety by its cartesian product with the complex projective plane.

The "$\mathbb{C}P^2$ - singularity" map is defined by looking at those points of an H-variety whose link has the form $\mathbb{C}P^2 * L$ ($*$ = "join") for some L . This set has the structure of a k-variety.

The exactness is easy to prove geometrically. In fact such a sequence relating any pair of such theories differing by one singularity is the only proposition needed to prove the assertions above about H_* and k_* .

We can define k-varieties (mod n) , and state the theorem for which they were defined. Recall the homotopy equivalence $L \xrightarrow{f} M$ and the obstruction $\{0_f\}$ to constructing a homeomorphism or pl homeomorphism in the homotopy class of f .

Theorem B. The value of an obstruction 0_f on a homology class $x \in H_{4i}(L, Z/_j)$ is "determinant" iff x is represented by a k-variety in L , $V \to L$. More precisely, there is a characteristic invariant $\sigma(f, V)$ so that

$$\sigma(f, V) = \langle 0_f, x \rangle \quad ,$$

and the "total indeterminancy subgroup"

$$I_{4i} \in H^{4i}(L, Z)$$

is dual under Pontryagin duality to the quotient of

$$\lim_{\substack{\to \\ j \text{ odd}}} H_{4i}(L, Z/_j)$$

by the subgroup of classes represented by k-varieties.

Corollary: f may be deformed to a homeomorphism on some region containing the 1, 2, ..., k handles ($k < n - 2$ or $\pi_1 = 0$) iff all characteristic invariants of the signature and Arf invariant type vanish for k-varieties in L up to dimension k .

Proof: From theorem B an obstruction 0_f lies in the indeterminancy subgroup iff 0_f vanishes on any homology class represented by a k-variety. But these values are given by characteristic invariants. So if enough invariants $\sigma(f, V)$ vanish we can work our way through the maze and construct an isomorphism between the desired regions.

We note that there are certain interesting problems associated to the task of giving a fairly elementary geometrical discussion and proof of theorem B .

First, one would like to define

$$\sigma(f, V) = \frac{1}{8}\,(\text{signature } V' - \text{signature } V) \in Z/_j$$

as we did above when V was a $Z/_j$-manifold in L . It seems reasonable to conjecture that <u>there is a good signature for varieties of this singularity type</u> - which is a cobordism invariant (for these singularities) and which is calculated intrinsically in V by intersecting cycles in the interior, say. This is all that is required for the simple geometrical proof above that

$$\langle O_f, x\rangle = \sigma(f, V) .$$

In fact, a k-variety has a natural formal signature[1)] which agrees with the intrinsic signature in the non-singular case.

The naive argument goes through using this formal signature if L and M are smooth. In general, however V' will have nice join-like singularities but the strata will not be smooth so the naive argument breaks down.

A second problem to a purely geometrical discussion is encountered in establishing the Pontryagin duality between determinancy in the obstructions and the representation of homology by k-varieties. The natural argument here is to use intersection theory for k-varieties to show that there is an intersection duality between k-homology in complementary dimensions for a regular neighborhood of L in euclidean space (at least after making the periodicity operation into an isomorphism). To develop this intersection theory one has to analyze the theory where iterated joins of the same manifold appear in the singularities. (the cartesian product of two k-varieties has this property.)

If the intrinsic signature and the intersection theory are worked out for k-varieties then a pleasant time can be had using the exact sequence above to bash about in the obstruction labyrinth and see how it works.

There is a more formal discussion using the interpretation of k-varieties below, (See S1, S2).

<u>Resolving the Singularities of k-varieties and the Characteristic Variety Theorem</u>

It is possible to formulate all the characteristic invariants in acessible geometric termsby using the periodicity operation on k-varieties.

---

1) A k-variety is k-homologous to $\sigma(\mathbb{C}P^2)^\lambda$. The coefficient σ is the formal signature.

<u>Proposition</u>. For any k-variety in L , $V \xrightarrow{i} L$ there is an integer λ so that

$$(\mathbb{C}P^2)^{\lambda} \times V \longrightarrow L$$

is k-homologous to a non-singular manifold in L .

Now $\mathbb{C}P^2 \times V \longrightarrow L$ has the same characteristic invariant as $V \to L$. Thus we can compute the characteristic invariant on k-varieties by computing them on the associated manifolds after desingularization.

Further, the characteristic invariants for manifolds V in L

$$\sigma(V, f) = \begin{cases} \frac{1}{8}\,(\text{signature } V' - \text{signature } V) \\ \\ \text{Arf invariant } (\ker f_*^V,\ Z/_2) \end{cases}$$

only depend on the cobordism class of V and satisfy the product formula

$$\sigma(V \times \mathbb{C},\ f) = (\text{signature } \mathbb{C})\ \sigma(V,\ f)\ .$$

Thus it is clear that all characteristic invariants of f are known if they are known for a certain finite collection of manifolds - we only have to choose generators for the module of bordism classes of $Z/_n$ manifolds[1)] in L modulo the "generalized periodicity relation"

$$(\mathbb{C} \times V \longrightarrow L) \sim ((\text{signature } \mathbb{C})\ .\ V \longrightarrow L)\ .$$

Assume we can avoid the (n - 2) obstruction - eg. $\pi_1 = 0$ or there are no (n - 2) handles. Then we have the

<u>Characteristic Variety Theorem</u>. In the topological or pl context, we can construct an isomorphism between M and L in the homotopy class of a homotopy equivalence $M \xrightarrow{f} L$ iff a certain finite collection of characteristic invariants $\{\sigma(V,\ f)\}$ for manifolds in L vanish.
The invariants are computed from the signatures and Arf invariants of quadratic forms defined by the transversality structure of f . All relations between the invariants for all possible M and f arise from the cobordism and the product formula

$$\sigma(\mathbb{C} \times V, f) = i(\mathbb{C}) \cdot \sigma(V, f)\ .$$

---

1) For n less than a certain bound depending on L .

Interpretation of k-varieties and the Periodicity

The following interpretation of k-varieties is useful for bringing algebraic methods to bear on our problems.

The homology theory k_*X determined by k-varieties in X has an associated dual cohomology theory k^*X defined so that Alexander duality holds,

$$\text{if } X \subset S^{N+1},\ \tilde{k}^i(X) \equiv \tilde{k}_{N-i}(S^{N+1} - X)\ .$$

The cohomology theory $k^*(X)$ is a connective version of real K-theory,

$$k^0(X) \simeq_{\frac{1}{2}} KO(X)$$

$$k^i(X) \simeq \{0\} \qquad i > \text{dimension } X\ .$$

($\simeq_{\frac{1}{2}}$ means isomorphic modulo finite 2 groups).

The periodicity operation on k-varieties

$$V \longrightarrow \mathbb{C}P^2 \times V$$

corresponds to the square root of the Bott periodicity in real K-theory.

In the normal invariant (see next section for definition) viewpoint we can think of the stable part of the obstruction theory as being that obstruction theory for deforming an associated map $L \xrightarrow[w_f]{} \text{(universal space)}$ to the point map.

There are universal spaces for each of the three situations

$$G/_O\ ,\ G/_{PL}\ ,\ \text{and}\ G/_{Top}\ .$$

Any calculation or decomposition of these universal spaces gives the corresponding information or structure to the related stable obstruction theory.

For example, from what we have seen above for the characteristic invariants, we can deduce the table

stable obstruction theory	at the prime 2 is built from	at the prime p odd is built from
topological theory	ordinary cohomology theory	oriented real K-theory
piecewise-linear theory	the two stage of oriented real K-theory and ordinary cohomology	oriented real K-theory

Comparison of the Piecewise Linear and Topological Invariants

The characteristic invariants and their completeness in the combinatorial situations were developed to study the Hauptvermutung. It was found that the characteristic invariants of the signature and Arf invariant type were sufficient to determine the pl-structure within a homotopy type when the $(n-2)$-dimensional difficulty could be avoided (e.g. $\pi_1 = 0$) or no $(n-2)$ handles).

The motivation for a-priori invariants should now be clear - once one begins homotoping a homeomorphism $M \xrightarrow{f} L$ to a pl-homeomorphism the hypothesis on f is immediately lost. Thus we must know beforehand that all obstructions can be avoided by a careful deformation.

However, there was some delicacy concerning the characteristic invariants for four dimensional submanifolds $V \subseteq L$.

First, a more precise invariant needed to be defined

$$\tilde{\sigma}(f, V^4) = \frac{1}{8}\,(\text{signature } V' - \text{signature } V) \in Z/_{2n}$$

if V^4 is a $Z/_n$-manifold in L and

$V' \to V$ is an isomorphism between the Bocksteins.

Second, a special relation existed among the invariants

$\underline{\tilde{\sigma}(f, V^4)} \in \underline{Z/_{2n}}$ $\underline{\text{is}}$ $\underline{\text{even}}$ "spin relation"

whenever $V^4 \to L$ is a "spin component" - the composition

$$H^2(L, Z/_2) \xrightarrow{S^2_q} H^4(L, Z/_2) \to H^4(V, Z/_2)$$

is zero. (see S1).

Now Novikov had developed a toroidal method for showing $\sigma(f, V) = 0$. It applied whenever f was a proper homotopy equivalence between <u>certain regions</u> $f^{-1}\eta \xrightarrow{f^\eta} \eta$ of M and L and V had dimension greater than five.

The product formula

$$\sigma(f, V \times \mathbb{C}) = (\text{signature } \mathbb{C})\ \sigma(f, V)$$

for the characteristic invariants was developed for the simple purpose of raising the dimension of V to the Novikov range.

From this one could conclude

<u>Theorem C</u>. IF $M \xrightarrow{f} L$ is a proper map between pl-manifolds with Čech contractible point inverses, then the characteristic invariants satisfy

$$\sigma(f, V) = 0 \qquad \dim V \neq 4,$$
$$2\tilde{\sigma}(f, V) = 0 \qquad \dim V = 4 .$$

For example, two simply connected pl-manifolds are isomorphic if they are homeomorphic or related by a "contractible" mapping whenever the cohomology group $H^4(\ , Z)$ has no 2 torsion.

We also have a uniqueness of pl-structure result when there are no $(n - 2)$ handles and the Whitehead torsion works out.

But what about the tantalizing question of a possible non-zero four dimensional obstruction of order 2?

Let us see what was deduced from the form of the characteristic invariants and then compare this to the work of Kirby and Siebenmann on triangulation.

Perhaps this will lend some credence to the speculations given below which are also deduced from the "form" of the invariants.

Our attempt at understanding this order 2 obstruction centered around certain related theories of manifolds where a single obstruction differentiated the theory in question from pl-theory.

Let us work for a moment in the category of polyhedra with the local homology or local homotopy properties of manifolds. The maps (isomorphisms) are proper pl-maps with acyclic or contractible point inverses. A little theory proceeds by studying the problem of resolving the singularities in such spaces or in the mapping cylinders of such maps.

By studying the possibility of inductively replacing the dual cones by acyclic or contractible manifolds with boundary one obtained the following table,

theory of	relation to the category of pl-manifolds	
generalized manifolds	obstructions to resolving singularities in a space	obstructions to resolving singularities in a map
pl homology manifolds and pl maps with acyclic point inverses	one obstruction in $H^4(M, \nu_3)$	one obstruction $\nu_f \in H^3$ (range f, ν_3)
pl homotopy manifolds and pl maps with contractible point inverses	one obstruction in $H^4(M, \Theta_3)$	one obstruction $\Theta_f \in H^3$ (range f, Θ_3)

Here ν_3 is the group of H-cobordism classes of homology three spheres, and Θ_3 is the group of h-cobordism classes of homotopy three spheres.
There is a sequence $\Theta_3 \to \nu_3 \xrightarrow{r} Z/_2$ (exact?) where r is the Kervaire Milnor

Rochlin invariant.

Many wonderful theorems can be formulated in terms of undecided statements about this sequence.

Now consider a pl map of pl-manifolds $M \xrightarrow{f} L$ with acyclic (or contractible) point inverses. It is easy prove the characteristic invariants of f also satisfy the relations

$$\sigma(f, V) = 0 \qquad \dim V \neq 4,$$
$$2\tilde{\sigma}(f, v) = 0 \qquad \dim V = 4 .$$

For $M \xrightarrow{f} L$, being pl, will be transversal to a generic $V \subset L$ and

$$V' \xrightarrow{f^V} V ,$$

having acyclic point inverses, will be a homology isomorphism. Thus the characteristic invariants vanish as stated and for deforming f by a homotopy to a pl isomorphism there is one possible "non-zero" obstruction 0_f in the 2-torsion of $H^4(L, Z)$.

The obstruction mentioned above ν_f , concerning the more precise construction of resolving the singularities in the mapping cylinder of f , is related to 0_f by the composition

$$\begin{array}{ccccc} H^3(L, \nu_3) & \xrightarrow[r_*]{\text{coefficient homomorphism}} & H^3(L, Z/_2) & \xrightarrow{\text{integral Bockstein}} & H^4(L, Z), \\ \cup\!\!\!\mid & & & & \cup\!\!\!\mid \\ \{\nu_f\} & & \longrightarrow & & \{0_f\} \end{array} .$$

Now the existence of the dodecahedral space implies that $\nu_3 \xrightarrow{r} Z/_2$ is onto and for some "pl-acyclic" maps $M \xrightarrow{f} L$,0_f is non-zero.

If a particularly bad counterexample to the three dimensional Poincaré conjecture existed $(\Theta_3 \xrightarrow{\neq 0} Z/_2)$ then for some pl f with contractible point inverses we would have $0_f \neq 0$,

From all this we can conclude that as for as our invariants (and the techniques of transversality, surgery, and h-cobordism) are concerned

i) homeomorphisms are equivalent to maps with Čech contractible point inverses - and the four dimensional obstruction may be non-zero for these - it is for the related pl-acyclic maps, and it can be for pl-Čech contractible maps if the Poincaré conjecture goes awry.

ii) we must abandon trying to prove the four dimensional obstruction vanishes for homeomorphism unless a entirely new idea about homeomorphism arises. (1967).

Now after the work of Kirby and Siebenmann the skies have cleared (assuming the dimension is greater than four), and we can say the following

i) the topological category is a "one obstruction" extension of the pl-category like the generalized manifold categories above

category	existence of pl structure	isotopy uniqueness of pl structure
topological manifolds and homeomorphisms	one obstruction in $H^4(M, Z/_2)$	one obstruction in $w_f \in H^3(M, Z/_2)$

(Kirby-Siebenmann - (KS))

ii) for a homeomorphism $M \xrightarrow{f} L$ of pl-manifolds the one possible non-zero obstruction O_f in $H^4(L, Z)$ is the image of w_f

$$\begin{array}{ccc} H^3(L, Z/_2) & \xrightarrow{\text{integral Bockstein}} & H^4(L, Z) \\ \Psi & & \Psi \\ w_f & \longrightarrow & O_f \end{array}$$

(and can be non-zero - Siebenmann).

iii) homeomorphisms and Cech contractible maps are equivalent for homotopy theoretical or approximation purposes (Siebenmann -(S)).

iv) the characteristic invariants which were shown to be topologically invariant

$$\{\sigma(f, V) \ , \ \dim V > 4\}$$

turn out to be a complete set of invariants for the topological manifolds within a simply connected homotopy type. The relations among invariants are just cobordism, the product formula, and compactibility with respect to reduction mod n .

v) the triangulable manifolds among these are precisely those whose characteristic invariants are compatible with the "spin relation" above.

The n - 2 Obstruction and Characteristic Invariants

In the topological or pl case we have a complete analysis of the obstruction theory for constructing homeomorphisms and pl isomorphisms whenever the (n - 2) obstruction is not encountered. Even after the precise form of the (n - 2) obstruction is determined we will have the problem of indeterminancy.

For this question we make the following points. First the above theory does apply to classify all possible deformations of $M \xrightarrow{f} L$ to partical isomorphisms below the $(n - 2)$-handles. The characteristic invariants classifying these deformations are defined by the transversality properties of f so their effect on the $(n - 2)$ obstruction may be determined by geometrical reasoning.

Second, for the purpose of studying the rigidity of the $(n - 2)$ obstruction by a-priori characteristic invariants it is refreshing to recall that each integral $(n - 2)$ homology class is represented by an embedded submanifold. On the other hand the situation is certainly more complicated for mod q homology in dimension $n - 2$. The singularities required for embedded cycles representing these classes are unknown.

Hopefully, some form of the above indeterminancy analysis can be made for the $(n - 2)$ obstruction by studying surgery on subvarieties in codimension 2.

The Obstruction Theory for Constructing Diffeomorphisms

We cannot make a detailed analysis of the obstructions in the smooth theory. For one thing the coefficient groups are unknown. However, there are interesting global invariants of the homotopy equivalence $M \xrightarrow{f} L$ which lead to a certain understanding of the theory.

For example, the stable part of the smooth theory $(< n - 2)$ can be completely factored as a product of two more familiar obstruction theories - one infinite and one finite. The infinite theory is isomorphic at each prime and over the rationals to oriented real K-theory. The finite theory is an identifiable sub-theory of stable cohomotopy theory.

The Invariants

Besides the characteristic invariants $\{\sigma(f, V)\}$ defined by the transversality structure of f we have certain global invariants -

$\mathcal{T}_f$, the tangent bundle of f, defined by

$$\mathcal{T}_f = \mathcal{T}_M \oplus f^*\nu_L$$

where ν_L the stable normal bundle of L, and $\mathcal{T}_M$ is the stable tangent bundle of M.

df, a natural fibre homotopy equivalence $\mathcal{T}_L \xrightarrow{df} \mathcal{T}_M$ covering f, df determines a fibre homotopy trivialization

$$\mathcal{T}_f \xrightarrow{df} 0$$

called the <u>normal</u> <u>invariant</u> of f.

Θ_f, a unit of the real K ring associated to M. Θ_f is defined by comparing the two natural K-theory Thom isomorphisms in $\mathcal{T}_f$ determined by df.

df determines a spin structure on $\mathcal{T}_f$ and a homotopy equivalence of the Thom space of $\mathcal{T}_f$ to the suspension of M^+.
The "Dirac Thom class" and the suspension isomorphism determine the two Thom isomorphisms whose comparison defines the K-unit Θ_f.

The projection of the smooth theory onto oriented real K-theory is effected by

i) at the prime 2 observing that Θ_f is actually an "oriented unit", the first Stiefel-Whitney class of Θ_f is zero. So we simply take Θ_f.

ii) at the odd prime p by combining Θ_f and $\mathcal{T}_f$ appropriately. The K-ring at p has a canonical splitting in terms of the eigenspaces of completed Adams operations, $K \sim K_1 \oplus K_\xi$. We take

$$(\Theta_f)_1 \oplus (\mathcal{T}_f)_\xi \quad .$$

Alternatively, we could take

$$\{\sigma(f, V)\}_1 \oplus (\mathcal{T}_f)_\xi$$

after building a K-theory unit at p from the characteristic invariants. (see S2 chapter 6.).

The meaning of all this is the following - at each prime the relation between the various K-theoretical invariants above $\Theta_f, \mathcal{T}_f, \{\sigma(f, V)\}_p$ can be completely described in terms of operations in K-theory.

The elements constructed above then are realized independently at the given prime. For example, at the prime 2 there is an exponential isomorphism $\dot{\tilde{K}} \xrightarrow{\rho} K^*$ associated to an Adams operation ψ (in the oriented theory). Then the relation

$$\rho\mathcal{T}_f = \psi\Theta_{f/\Theta_f}$$

holds. Since ρ is an isomorphism Θ_f determines $\mathcal{T}_f$. We take Θ_f then as the invariant which we prove satisfies no further relations at the prime 2.

The Realization of the Invariants

When the invariants

$$\Theta_f, \mathcal{T}_f, \{\sigma(V, f)\}$$

are suitably combined into one KO-invariant K_f we are confronted with the realization question. From the viewpoint of normal invariants we are trying to construct fibre homotopy equivalences between vector bundles over M, say, with given invariants.

In effect we show K_f satisfies no relations at each prime by considering the universal situations of the Grassmannians.

The universal sphere bundle over the union of Grassmannians of n-planes in k-space, $k \to \infty$, is homotopically approximated by the union of Grassmannians of $(n-1)$-planes in k-space, $k \to \infty$.

These finite Grassmannians are nice algebraic varieties whose homotopy types can be (profinitely) approximately by the Čech-like nerves of algebraic coverings of these varieties. The natural algebraic symmetry in these constructions (that of the Galois group of the field of all algebraic numbers over Q, $G = \mathrm{Gal}\,(\tilde{Q}/Q)$.) yield many self-homotopy equivalences of these homotopy types. (See S2).

We use certain of these homotopy automorphisms to construct enough fibre homotopy equivalences to see that K_f satisfies no relations - i.e. the invariants $\theta_f, \mathcal{T}_f\ \{\sigma(f, V)\}_p$ satisfy only the natural K-theory relations.

For example, let α be an element of $G = \mathrm{Gal}\,(\tilde{Q}/Q)$ which generates the p-component of the Abelianization of G for each p. Then we use the above work to analyze the normal invariant

$$\mathcal{T}_f \xrightarrow{df} 0 .$$

The considerations of Galois symmetry in the Grassmannians show that α determines an isomorphism

$$\mathcal{T}_f \overset{b}{\cong} \eta^\alpha - \eta$$

where we work now in the profinite completion of K-theory[1] on which G acts. η is canonical and has a natural fibre homotopy trivialization

$$\eta^\alpha - \eta \xrightarrow{\alpha} 0 .$$

The composition

$$\alpha_f : 0 \overset{df}{\sim} \mathcal{T}_f \simeq \eta^\alpha - \eta \overset{\alpha}{\sim} 0$$

is a new invariant α_f, a unit in the stable cohomotopy ring associated to M. α_f measures the extent to which the normal invariant df comes from Galois symmetry in the Grassmannians.

The invariant α_f always lies in a certain subgroup of the cohomotopy units

$$\mathbb{C} \subseteq \pi^* .$$

1) $\hat{K} = \varprojlim_n K \otimes Z/_n$ for finite complexes.

$\mathbb{C}$ is classified by the "group" of the theory of sphere fibrations with a Galois equivariant KO-theory Thom isomorphism.

The quotient $\pi^*/_{\mathbb{C}}$ can be thought of as the "the rational points of K-theory". This theory K_Q^*, is constructed from the action of the Galois group on K-theory, the image

$$K_Q^* \longrightarrow KO^*$$

is precisely the fixed points of the Galois group.

If we look at these theories over the spheres we find $(n \leq 19)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\mathbb{C}(S^n)$	0	0	0	0	0	$Z/_2$	0	$Z/_2$	$Z/_2$	$Z/_3$	0	0	$Z/_3$	$Z/_2 \oplus Z/_2$	$Z/_2$	$Z/_2$	$Z/_2 \oplus Z/_2$	$Z/_8$	$Z/_2$ ...
$K_Q^*(S^n)$	$Z/_2$	$Z/_2$	$Z/_{24}$	0	0	0	$Z/_{240}$	$Z/_2$	$Z/_2 \oplus Z/_2$	$Z/_2$	$Z/_{504}$	0	0	0	$Z/_{480}$	$Z/_2$	$Z/_2 \oplus Z/_2$	$Z/_2$	$Z/_{264}$ ...

The second line consists of blocks of lenght eight of the form

$$\ldots \quad Z/_n \quad Z/_2 \quad \begin{matrix} Z/_2 \\ \oplus \\ Z/_2 \end{matrix} \quad Z/_2 \quad Z/_m \quad 0 \quad 0 \quad 0 \quad \ldots$$

This rational K-theory satisfies an "adic periodicity theorem" - the Z-indexing can be extended to a $\hat{Z} = \varprojlim Z/_n$ indexing.

We have the table

theory	building blocks
smooth obstruction theory	real K-theory at each prime and the $\mathbb{C}$ part of stable cohomotopy at each prime
stable cohomotopy theory (degree 0)	$\mathbb{C}$ and the "rational K-theory", K_Q.

Now we can state a structure theorem. Recall that α is a chosen element in the Galois group, Gal $(\overline{Q}/_Q)$.

<u>Theorem D</u>. The stable obstructions for constructing a diffeomorphism in the homotopy class of $M \xrightarrow{f} L$ "vanish" iff the invariants

$\mathcal{T}_f$, Θ_f , and α_f

do .

The relations between the invariants $\mathcal{T}_f$, Θ_f, α_f, $\{\sigma(f, V)\}_{p \text{ odd}}$ for all possible f can be described completely. For example α_f varies independently in $\mathbb{C}(M)$, and the other invariants can be combined to give one KO-theory invariant which varies independently.

The stable part of the smooth obstruction theory is isomorphic to a product of theories

$$\mathcal{K} \times \mathbb{C}$$

where $\mathbb{C}$ is the finite theory above and $\mathcal{K}$ is isomorphic over Q and at each prime to real K-theory.

One point left out by Theorem D is the relation between the characteristic invariants at the prime 2 and the invariants Θ_f and α_f .

It would be interesting to have a more geometric hold on these invariants. The most exciting possibility is the method of generic singularities. The geometric structure in the stratification of a generic approximation to

$$M \xrightarrow{f} L$$

should yield geometrical invariants of α_f - the most abstruse among the quantities above.

For example, it is possible to give a criterion on the first order singularities of f equivalent to the triviality of $\mathcal{T}_f$ (unstably). The condition is a cobordism condition on the first Schubert variety regarded as a member of the class of all "abstract Schubert varieties in M ."

Further Implications from the "Form" of the Invariants

Now we wish to discuss what further we can or might deduce from the "form" of the characteristic invariants. We begin with theorems and pass through heartfelt conjecture to speculation.

First we might note that there are natural characteristic classes associated to $M \xrightarrow{f} L$ for computing the characteristic invariants. In fact, for V a closed manifold in L ,

$$\sigma(V, f) = \begin{cases} \langle v^2(V) \cdot k_f, V\rangle \in Z/_2 & \dim V = 4i + 2 \\ \langle \mathcal{L}(V) \cdot \ell_f, V\rangle \in Z & \dim V = 4i \end{cases}$$

where $v(V)$ is the mod 2 Wu class of V (see RS)

$\mathcal{L}(V)$ is the rational Hirzebruch class of V

k_f is a total class in $H^{4*+2}(L, Z/_2)$ defined by f

ℓ_f is a total class in $H^{4*}(L, Q)$ defined by f .

These are formulae of the Riemann-Roch type occuring for example in the Atiyah-Singer formula for computing the index of an elliptic operator.

One can ask if there is a deeper connection between these two situations. We will establish such a connection for the signature case below. A Riemann-Roch type explanation for the Arf invariant formula is unknown.

We remark in passing that a formula of the above type for $\sigma(V, f)$ where V is a $Z/_{2^r}$ manifold in L leads to canonical lifting of ℓ_f and the Hirzebruch class $\mathcal{L}(M)$ to classes with coefficients in the subring of Q consisting of rationals with odd denominators (see MS). The reduction mod 2 of this canonical $\mathcal{L}(M)$ is the square of the Wu class. Thus the two formulae above have related multipliers.

From now on we will mainly consider the characteristic invariants $\sigma(V, f)$ where V^{4i} is a $Z/_n$-manifold in L , for n odd. In that case recall that $\sigma(V, f) \in Z/_n$ is determined by the difference of signatures

$$\text{signature } f^{-1}V - \text{signature } V .$$

This suggests that these characteristic invariants can be determined by comparing intrinsic invariants of the manifolds M and L by some correspondence induced by f .

This is true. From the periodicity phenomenon studied above expressed in terms of real K-theory tensor the dyadic rationals one finds that an oriented topological manifold possesses an intrinsic Poincaré duality in this dyadic K-theory. The isomorphism for a manifold M is given by capping with a fundamental K-homology class

$$\mu_M \in K_m(M) \qquad m = \dim M .$$

Even more is true. If one suppresses the fundamental group and the prime 2 , then the category of compact manifolds and homeomorphisms is equivalent to the category of (CW complexes endowed with the extra structure of a K-duality) and (homotopy equivalences preserving the duality isomorphism).

One can think of the K-orientation μ_M as a prescription for endowing the underlying homotopy type with a non-singular topology (or pl-structure). A homotopy equivalence $M \xrightarrow{f} L$ preserves the canonical duality (and the "topology") precisely when $f_*\mu_M = \mu_L$.

In general the deviation from "homeomorphism" is measured by the unique unit, Δ_f of the K-ring of L satisfying

$$\mu_L \cap \Delta_f = f_* \mu_M .$$

Then the characteristic invariants are calculated by the formula

$$\sigma(V, f) = \langle \Delta_f, V \rangle \in Z/_n , \text{ n odd.}$$

The Laplacian

The K-formula above is also valid when V is closed and we calculate a signature in Z . If the calculation is put in terms of rational cohomology, the formula becomes the one above,

$$\sigma(V, f) = \langle \mathcal{L}(V) . l_f, V \rangle$$

where ℓ_f can now be interpreted as the character of Δ_f .

But the expression

$$\langle \mathcal{L}(V) . \text{ch } \Delta_f, V \rangle$$

is just the Atiyah-Singer Formula for the index of a certain elliptic operator over the smooth manifold V associated to Δ_f . We represent $2^k \Delta_f$ by a vector bundle over V for some k and consider differential forms on V with values in this bundle. If V has a Riemannian metric then will be a Laplacian operator between two spaces of such forms whose index is $2^k \sigma(V, f)$.

Analytical Speculation

There is a more substantial connection with the Laplacian in case our manifold M is smooth. Let M be smoothly embedded in Euclidean space with tubular neighborhood N . Now the K-orientation μ_M determines by Alexander duality a canonical K-class with compact support on N (or $N \times R^n$ for any n) . Denote this dual class by Δ_M . Also, Singer has constructed an elliptic boundary value problem on N which is built from the Laplacian and whose symbol in real K-theory is just Δ_M .

How can we interpret or understand the fact that the symbol of this Laplacian boundary value problem only depends on the underlying pl or topological structure?

Well, in the pl case one might well suppose that there is a theory of difference operators and combinatorial (solid) geometry - a discrete companion to the theories of differential operators and Riemannian geometry in the case of smooth manifolds.

One expecially interesting problem in this connection is to give a solid angle-incidence formula for the rational Pontryagin classes - a discrete analogue of the Chern-Weil formula in Riemannian geometry.

In the topological category we might speculate even further. Firstly, one might ask a-priori - what are the fine structures on topological manifolds which deserve study? For example on a smooth manifold we have the intricate theory of dynamical systems.

One attractive topic which bears some relation to our considerations would be the theory of Brownian motions on the topological manifold M.

Given certain geometrical data on M it should be possible to construct measures $\{P_x : x \in M\}$ on the space of continuous paths in M. The measures P_x should have associated distributions which reflect the lack of memory of a random path and the geometrical bias induced by the given geometrical data.

For example in R^n with the usual metric (or in a Riemannian manifold) Wiener constructed the measures $\{P_x\}$ (1923). The path increments $x(t + s) - x(s)$ in R^n have a symmetric normal distribution with probability density

$p(t, y) = \frac{1}{\sqrt{2\pi t}} e^{-|y|^2/2t}$, for example the probality that a random path starting at a point x in R^n lies in a domain E after time t is

$$P_x\{x(t) \in E\} = \int_{E-x} p(t, y)dy . \quad \text{(See DY).}$$

More recently it was found that the Laplacian operator and harmonic functions have a natural interpretation in terms of Brownian motion. One can use the probabilities to "diffuse" the functions on M

$$f(x) \rightsquigarrow f(t, x) = \int_M f(x)dP_x\{x(t) \in E\} .$$

The infinitesimal generators of this flow of functions is just the Laplacian operator on functions (under good hypotheses).

One might hope then that

i) the nature of the geometrical data required to construct a Brownian process could be understood.

ii) there is an interesting class of Brownian motions on a given topological manifold.

iii) from the connection between Brownian motion and potential theory a topological Laplacian or at least its symbol can be constructed.

iv) one could obtain for each such process a thermodynamic definition of μ_M and Δ_M or at least the real Pontryagin classes of a topological manifold M.

Galois symmetry

Let us consider an algebraic implication of the "form" of the invariant μ_M, determining the K-duality in a manifold.

If we pass to a profinite context - generalizing replacing an integer by a compatible system of residues modulo n for every n - we find a high degree of symmetry in our K-theory model of manifold theory (simply connected, away from 2). The Galois group Gal $(\bar{Q}/Q)$ acts on K-theory, the set of all K-duality isomorphisms and thus on the topological manifold structures on a given homotopy type.

This symmetry also exists in the theories of profinite topological bundles in each dimension and is compatible with the action of the Galois group on the profinite Grassmannians discussed in the smooth theory.

Further we saw above the important role played by this Galois group in analyzing the smooth theory.

What should one make of this structure?

Algebraic Speculation

For one thing it would be very interesting to see the symmetry in manifold theory geometrically. One is especially provoked in this regard by the fact that much of the topological information lost by passing to homotopy theory is explained by this Galois symmetry. (see S2).

To explain the matter one might very well conjecture that there is a natural <u>profinite form</u> of geometric topology which

a) is defined geometrically

b) possesses a high degree of Galois symmetry - again defined simply and geometrically

c) has all the techniques of geometric topology and the additional structure of geometric localization, the Galois symmetry, and a closer connection to some of the problems and techniques of algebraic geometry.

REFERENCES

MS Morgan, J. and Sullivan (to appear).

RS Rourke and Sullivan - On the Kervaire Obstruction (to appear in the Annals of Mathematics).

DY Dynkin and Yushkevich - Markov Processes (Theorems and Problems) Plenum Press New York.

B Browder, W. - Simply Connected Surgery (book - to appear).

W Wall, C.T.C. - Surgery on Compact Manifolds (book - to appear).

KS Kirby, R. and L. Siebenmann - Triangulation

S Siebenmann, L. - The Closure of the Homeomorphism Group.

S1 Sullivan - Geometric Topology Notes, Princeton 1967.

S2 Sullivan - Geometric Topology, Part I, Localization, Periodicity and Galois Symmetry. Preprints available from MIT. Cambridge, Mass.

Massachusetts Institute
of Technology
CAMBRIDGE / Mass.