Triangulating and Smoothing Homotopy Equivalences and Homeomorphisms. Geometric Topology Seminar Notes

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Introduction

We will study the smooth and piecewise linear manifolds within a given homotopy equivalence class. In the first part we find an obstruction theory for deforming a homotopy equivalence between manifolds to a diffeomorphism or a piecewise linear homeomorphism. In the second part we analyze the piecewise linear case and characterize the obstructions in terms of a geometric property of the homotopy equivalence. In the third part we apply this analysis to the Hauptvermutung and complex projective space.

I. Triangulating and Smoothing Homotopy Equivalences

Definition 1. Let A_i denote the Abelian group of almost framed[1] cobordism classes of almost framed smooth i-manifolds.
Let P_i denote the Abelian group of almost framed cobordism classes of almost framed piecewise linear i-manifolds.

Theorem 1. (**The obstruction theories**) *Let $f:(L,\partial L)\longrightarrow(M,\partial M)$ be a homotopy equivalence between connected piecewise linear n-manifolds. Let Q be an $(n-1)$-dimensional submanifold of ∂L such that $f(\partial L - Q) \subseteq \partial M - f(Q)$. Suppose that $n \geq 6$ and that $\pi_1(L) = \pi_1(\text{each component of } \partial L - Q) = 0$.*

(a) *If $f|Q$ is a PL-homeomorphism, then f may be deformed (mod Q) to a PL-homeomorphism on all of L iff a sequence of obstructions in $H^i(L,Q;P_i)$ $0<i<n$ vanish.*

(b) *If L and M are smooth, $f|Q$ is a diffeomorphism, and $\partial L \neq Q$ then f may be deformed (mod Q) to a diffeomorphism on all of L iff a sequence of obstructions in $H^i(L,Q;A_i)$ $0<i<n$ vanish.*

Remark. From the work of Kervaire and Milnor [KM] we can say the following about the above coefficient groups:

[1] almost framed means framed over some $(i-1)$-skeleton

(i) If θ_i denotes the finite Abelian group of oriented equivalence classes of differentiable structures on S^i, then there is a natural exact sequence

$$\ldots \longrightarrow P_{i+1} \xrightarrow{\partial} \theta_i \xrightarrow{i} A_i \xrightarrow{j} P_i \longrightarrow \theta_{i-1} \longrightarrow \ldots$$

with image$(\partial) = \theta_i\partial\pi = \{\pi\text{-boundaries}\} \subseteq \theta_i$.

(ii) $P_* = P_1, P_2, P_3, \ldots, P_i, \ldots$ is just the (period four) sequence

i	1	2	3	4	5	6	7	8
P_i	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$

9	10	11	12	13	14	15	16
0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$

(iii) For $i \leq 19$, A_i may be calculated

i	1	2	3	4	5	6	7
A_i	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0

8	9	10	11	12	13
$\mathbb{Z} \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_6$	0	$\mathbb{Z}$	$\mathbb{Z}_3$

14	15	16	17	18	19
$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\mathbb{Z}_2$

Note that Theorem 1 is analogous to a fundamental theorem in smoothing theory. In that case f is a PL-homeomorphism, $f|Q$ is a diffeomorphism, and f may be deformed by a weak-isotopy (mod Q) to a diffeomorphism iff a sequence of obstructions in $H^i(L, Q; \theta_i)$ vanish.

These three obstruction theories are related by the exact sequence of coefficients above.

Proof of Theorem 1: There are several approaches to Theorem 1. The most direct method seeks to alter f by a homotopy so that it becomes a diffeomorphism or a PL-homeomorphism on a larger and larger region containing Q. Suppose for example that M is obtained from $f(Q)$ by attaching one i-handle with core disk D_i.

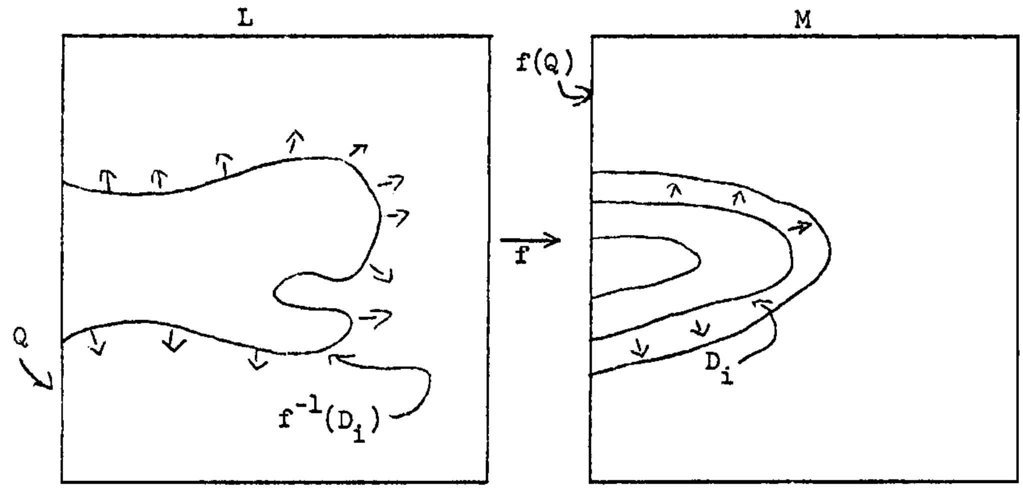

Figure 1

Then f is deformed (mod Q) so that it is transverse regular to the framed manifold D_i. The framed manifold $f^{-1}(D_i)$ has a (smooth or PL) sphere boundary and determines an element in A_i or P_i. If this element is zero, then surgery techniques may be employed to deform f so that it is a diffeomorphism or PL-homeomorphism on a neighborhood of $f^{-1}D_i$.

Theorem 1 asserts that the cochain with values in A_i or P_i determined by the $f^{-1}D_i$'s has the properties of an obstruction cochain.

A complete description is given in [S1]. See also [W1].

The obstruction theories of Theorem 1 have the usual complications of an Eilenberg-Whitney obstruction theory. The k^{th} obstruction in $H^k(L, Q; P_k)$ or $H^k(L, Q; A_k)$ is defined only when the lower obstructions are zero; and its value depends on the nature of the deformation of f to a PL-homeomorphism or diffeomorphism on a thickened region of L containing the $(k-2)$-skeleton of $L - Q$.

Thus applications of a theory in this form usually treat only the first obstruction or the case when the appropriate cohomology groups are zero.

For more vigorous applications of the theory one needs to know more precisely how the obstructions depend on the homotopy equivalence f — for example, is it possible to describe the higher obstructions and their indeterminacies in terms of a priori information about f?[1]

[1] I am indebted to Professor Steenrod for suggesting this problem at my Thesis Defense, January 1966.

We will concentrate on the PL obstruction theory – where a complete analysis can be made. We will replace the sequence of conditions in Theorem 1 by one condition which depends only on the geometrical invariants of f (Theorem 2). These geometrical invariants are the classical surgery obstructions in P_* obtained by studying the behavior of f on the inverse image of certain characteristic (singular) submanifolds of M.

It would be very interesting if a similar analysis can be made of the smooth theories.

The PL Theory

Definition 2. Let M be an oriented PL m-manifold whose oriented boundary is the disjoint union of n copies of the closed oriented $(m-1)$-manifold L. We call the polyhedron V obtained from M by identifying the copies of L to one another a $\mathbb{Z}_n$-**manifold**. We denote the subcomplex $L \subseteq V$ by δV, the **Bockstein** of V.

A finite disjoint union of $\mathbb{Z}_n$-manifolds for various n's and of various dimensions is called a **variety**.

If X is a polyhedron, a **singular variety in** X is a piecewise linear map $f : V \longrightarrow X$ of a variety V into X.

Remark. (i) Note that if V is a $\mathbb{Z}_n$-manifold of dimension m then V is locally Euclidean except along points of $\delta V = L$. A neighborhood of L in V is PL-homeomorphic to $L \times \text{cone}\,(n \text{ points})$.
(ii) A $\mathbb{Z}_n$-manifold carries a well-defined fundamental class in $H_m(V;\mathbb{Z}_n)$. It is the nicest geometric model of a $\mathbb{Z}_n$-homology class.
(iii) A closed oriented manifold is a ($\mathbb{Z}_0$ or $\mathbb{Z}$)-manifold.

We return to the homotopy equivalence $f : (L, \partial L) \longrightarrow (M, \partial M)$. Let $g : V \longrightarrow M$ be a connected singular $\mathbb{Z}_n$-manifold in the interior of M, of dimension v. The graph of g defines V as a $\mathbb{Z}_n$-submanifold of the $\mathbb{Z}_n$-manifold $M \times V$.[1] Consider $\overline{f} = f \times (\text{identity on } V)$ mapping $(L, \partial L) \times V$ to $(M, \partial M) \times V$. If $\pi_1(M) = \pi_1(V) = \pi_1(\delta V) = 0$, $v = 2s$, and $\dim(M) \geq 3$, then we may deform $\overline{f}$ so that it has the following properties:

(i) $\overline{f}$ is transverse regular to $(V, \delta V) \subseteq M \times (V, \delta V)$ with $(U, \delta U) \subseteq L \times (V, \delta V)$ where $U = f^{-1}V$.
(∗) (ii) $\overline{f} : \delta U \longrightarrow \delta V$ is a homotopy equivalence.[2]
(iii) $\overline{f} : U \longrightarrow V$ is s-connected where $v = \dim(V) = 2s$. See [S1] and [W1].

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[1] Using the graph of g is unnecessary if g is an embedding. Note that this construction is the Gysin homomorphism for bordism.

[2] We assume further that $\overline{f} : \delta U \longrightarrow \delta V$ is a PL-homeomorphism in case $\dim(\delta U) = 3$. This is possible.

Let $K_s = \ker \overline{f}_* \subseteq H_s(U;\mathbb{Z})$. If s is even K_s admits a symmetric quadratic form (the intersection pairing) which is even ($\langle x, x\rangle$ is even) and non-singular. Thus K_s has an index which is divisible by 8. If s is odd, then $K_s \otimes \mathbb{Z}_2$ admits a symmetric quadratic form which has an Arf-Kervaire invariant in $\mathbb{Z}_2$.

We define the **splitting obstruction** of $f : (L, \partial L) \longrightarrow (M, \partial M)$ **along** V by

$$\mathcal{O}_f(V) \;=\; \begin{cases} \text{Arf-Kervaire } (K_s) & \in \mathbb{Z}_2 \text{ if } s = 2k+1 \\ \frac{1}{8}\text{Index}(K_s) \text{ (modulo } n) & \in \mathbb{Z}_n \text{ if } s = 2k > 2 \\ \frac{1}{8}\text{Index}(K_s) \text{ (modulo } 2n) & \in \mathbb{Z}_{2n} \text{ if } s = 2. \end{cases}$$

We claim that $\mathcal{O}_f(V)$ only depends on the homotopy class of f. Also for $s \neq 2$, $\mathcal{O}_f(V) = 0$ iff f may be deformed to a map split along V, i.e. $f^{-1}(V, \delta V)$ is homotopy equivalent to $(V, \delta V)$.

More generally we make the following:

Definition 3. Let $f : (L, \partial L) \longrightarrow (M, \partial M)$ be a homotopy equivalence and let $g : V \longrightarrow M$ be a singular variety in M. **The splitting invariant of** f **along the variety** V is the function which assigns to each component of V the splitting obstruction of f along that component.

Now we replace the Eilenberg obstruction theory of Theorem 1 by a first-order theory. We assume for simplicity that Q is $\emptyset$.

Theorem 2. (The Characteristic Variety Theorem)
Let $f : (L, \partial L) \longrightarrow (M, \partial M)$ be a homotopy equivalence as in Theorem 1. Then there is a (characteristic) singular variety in M, $V \longrightarrow M$, with the property that f is homotopic to a piecewise linear homeomorphism iff the splitting invariant of f along V is identically zero.

For example:

(i) (characteristic variety of $\mathbb{QP}^n$) $= (\mathbb{QP}^1 \cup \mathbb{QP}^2 \cup \ldots \cup \mathbb{QP}^{n-1} \xrightarrow{\text{inclusion}} \mathbb{QP}^n)$

(ii) (characteristic variety of $S^p \times S^q \times S^r$)

$=$ even dimensional components of

$(S^p \cup S^q \cup S^r \cup S^p \times S^q \cup S^p \times S^r \cup S^q \times S^r \xrightarrow{\text{inclusion}} S^p \times S^q \times S^r)$

(iii) (characteristic variety of $\mathbb{CP}^n$) $= (\mathbb{CP}^2 \cup \mathbb{CP}^3 \cup \ldots \cup \mathbb{CP}^{n-1} \xrightarrow{\text{inclusion}} \mathbb{CP}^n)$

(iv) (characteristic variety of a regular neighborhood M of $S^{4k-1} \cup_r e^{4k}$)

$$= (V^{4k} \xrightarrow{\text{degree } 1} M)$$

where V^{4k} is the $\mathbb{Z}_r$-manifold obtained from S^{4k} by removing the r open disks and identifying the boundaries.

We remark that there is not in general a canonical characteristic variety for M. We will discuss below conditions that insure that a variety in M is characteristic and what choices are available.

First we consider the natural question raised by Theorem 2 – what are the relations on the set of all splitting invariants of homotopy equivalences $f : (L, \partial L) \longrightarrow (M, \partial M)$?

One relation may be seen by example – if $f : (L, \partial L) \longrightarrow (M, \partial M)$ is a homotopy equivalence and $S^4 \subseteq M$, then Index $f^{-1}(S^4) \equiv 0 \pmod{16}$ by a theorem of Rochlin. Thus the splitting obstruction of f along S^4 is always even.

More generally, if V is a singular variety in M, then a **four** dimensional component N of V is called a **spin component of V in M** if:

(i) N is a ($\mathbb{Z}$ or $\mathbb{Z}_{2^r}$)-manifold,

(ii) $\langle x \cup x, [N]_2 \rangle = 0$ for all $x \in H^2(M; \mathbb{Z}_2)$ where $[N]_2$ is the orientation class of N taken mod 2.

Then we can state the following generalization of Theorem 2.

Theorem 2′. *Let $(M, \partial M)$ be a simply connected piecewise linear manifold pair with* $\dim(M) \geq 6$. *Then there is a* **characteristic singular variety V in M** *with the following properties:*

(i) *Let $g_i : (L_i, \partial L_i) \longrightarrow (M, \partial M)$ be homotopy equivalences $i = 0, 1$. Then there is a piecewise linear homeomorphism $c : L_0 \longrightarrow L_1$ such that*

$$\begin{array}{ccc} (L_0, \partial L_0) & & \\ \cong \downarrow c & \overset{g_0}{\searrow} & (M, \partial M) \\ & \underset{g_1}{\nearrow} & \\ (L_1, \partial L_1) & & \end{array}$$

is homotopy commutative iff

(*Splitting invariant of g_0 along V*) = (*Splitting invariant of g_1 along V*) .

(ii) *A function on the components of V with the proper range is the splitting invariant of a homotopy equivalence iff its values on the four dimensional spin components are even.*

Note that Theorem 2 follows from Theorem 2′ (i) by taking $g_0 = f$ and $g_1 =$ identity map of $(M, \partial M)$

$$\begin{array}{ccc} (L,\partial L) & \xrightarrow{\ f\ } & \\ \cong\big\downarrow c & & (M,\partial M) \\ (M,\partial M) & \xrightarrow[\text{identity}]{} & \end{array}$$

Proof of Theorem 2:

The Kervaire Obstruction in $H^{4*+2}(M; \mathbb{Z}_2)$

There is a very nice geometrical argument proving one half of the characteristic variety theorem. Namely, assume the homotopy equivalence $f : (L, \partial L) \longrightarrow (M, \partial M)$ can be deformed to a PL-homeomorphism on some neighborhood Q of the $(k-1)$-skeleton of L and $f(L - Q) \subseteq M - f(Q)$:

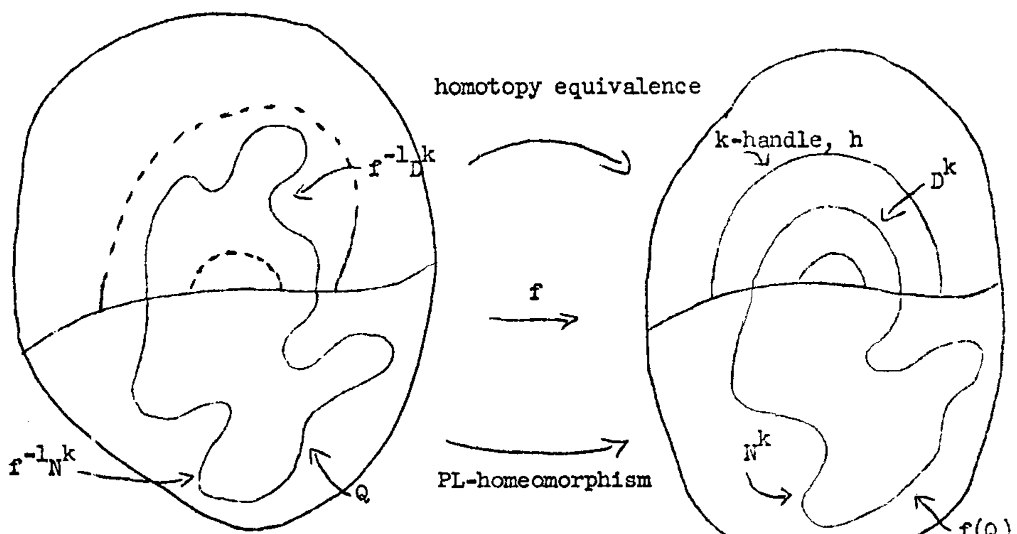

Figure 2

Suppose that $k = 4s + 2$ and recall that the obstruction class in

$$H^{4s+2}(M; \mathbb{Z}_2)$$

is represented by a cochain c calculated by looking at various $(f^{-1}D^k)$'s – where the

D^k's are the core disks of handles attached along $\partial f(Q)$. In fact, for a particular handle h, $c(D^k)$ is the Kervaire invariant of the framed manifold $f^{-1}(D^k)$ (which equals the class of $f^{-1}(D^k)$ in $P_{4s+2} = \mathbb{Z}_2$). See Figure 2.

NOW assume that there is a $\mathbb{Z}_2$-manifold N^k embedded in $f(Q)$ union the k-handle h which intersects the k-handle in precisely D^k. Then $f^{-1}N^k$ consists of two pieces – one is PL homeomorphic to N^k intersect $f(Q)$ and one is just $f^{-1}D^k$. Thus it is clear that the obstruction to deforming f on all of M so that $f^{-1}N^k$ is homotopy equivalent to N^k is precisely $c(D^k) =$ Kervaire Invariant of $f^{-1}D^k$.

This means that $c(D^k)$ is *determined by the splitting obstruction of f along N^k – it does not depend on the deformation of f to a PL-homeomorphism on Q.*

Roughly speaking, the part of N^k in $f(Q)$ binds all possible deformations of f together.

From cobordism theory [CF1] we know that any homology class in $H_k(M;\mathbb{Z}_2)$ is represented by a possibly singular $\mathbb{Z}_2$-manifold N^k in M. So for part of the characteristic variety[1] we choose a collection of singular $\mathbb{Z}_2$-manifolds in M of dimension $4s+2$, $2 \le 4s+2 \le \dim M$. We suppose that these represent a basis of $H_{4s+2}(M;\mathbb{Z}_2)$, $2 \le 4s+2 \le \dim M$.

The splitting obstructions for f along these $\mathbb{Z}_2$-manifolds in M determine homomorphisms $H_{4s+2}(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ which in turn determine cohomology classes in $H^{4s+2}(M;\mathbb{Z}_2)$. The argument above (generalized slightly)[2] shows that the lowest dimensional non-zero class among these is the first non-vanishing Eilenberg obstruction in dimension $4s+2$, (if it exists).

This would complete the proof of Theorem 2 if we did not have to cope with the obstructions in $H^{4i}(M;\mathbb{Z})$. So now the fun begins.

The Infinite (Index) Obstructions in $H^{4*}(M;\mathbb{Z})$

Of course we can try to apply the argument of Figure 2 to characterize the Eilenberg obstructions in dimensions $4s$.

The attempt succeeds in *characterizing the Eilenberg obstructions in $H^{4s}(M;\mathbb{Z})$ modulo odd torsion elements.*

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[1] We shall see below that some of the two dimensional components are not needed and others are replaced by certain four-dimensional (non-spin) components, e.g. $M = \mathbb{C}\mathbb{P}^n$.

[2] The fact that the submanifolds are singular presents no difficulty – for we may look at graphs or cross the problem with a high dimensional disk.

Let D^k, $k = 4s$ be the core disk of a handle h attached along the boundary of $f(Q)$ which determines an element of infinite order in $H^{4s}(M;\mathbb{Z})$. Then by obstruction theory in MSO there is an oriented submanifold N^k of $f(Q) \cup h$ which intersects the handle in a certain positive number of oriented copies of D^k. We can then calculate $[\frac{1}{8}\text{ Index of } f^{-1}D^k] = [\text{class of } f^{-1}D^k \text{ in } P_{4s} = \mathbb{Z}]$[1] in terms of Index $f^{-1}N^k$ − Index N^k. The latter integer is determined by the splitting obstruction of f along N^k. This characterizes the Eilenberg obstructions modulo torsion elements in $H^{4s}(M;\mathbb{Z})$.

The 2-Torsion (Index) Obstructions in $H^{4*}(M;\mathbb{Z})$

Now suppose D^k represents a generator of order n in $H^{4s}(M;\mathbb{Z})$ $(k = 4s)$ and there is a singular $\mathbb{Z}_n$-manifold N^k in $f(Q) \cup h$ which intersects the handle h in D^k. Then the argument of Figure 2 again shows that the value of an Eilenberg obstruction cochain on $[D^k]$ taken mod n is just the splitting invariant of f along N^k.

From cobordism theory we can show that such an N^k exists if n is a power of 2.

So now we can characterize the Eilenberg obstructions modulo odd torsion elements. We add to our characteristic variety the manifolds considered in the previous two paragraphs – namely:

(i) an appropriate (as above) closed oriented manifold of dimension $4s$ for each element of a basis of $H^{4s}(M;\mathbb{Z})/\text{Torsion}$.[2]

(ii) an appropriate $\mathbb{Z}_{2^r}$-manifold of dimension $4s$ for each $\mathbb{Z}_{2^r}$-summand in $H^{4s}(M;\mathbb{Z})$.

All this for $4 \le 4s < \dim M$.

The above applications of cobordism theory are based on the fact that the Thom spectrum for the special orthogonal group, MSO, has only *finite* k-invariants of *odd* order. (See [CF1]).

We also use the fact that the homotopy theoretical bordism homology with $\mathbb{Z}_n$-coefficients is just the geometric bordism homology theory defined by $\mathbb{Z}_n$-manifolds.

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[1] Except when $4s = k = 4$, in which case we calculate $\frac{1}{16}$ Index $f^{-1}D^k$.

[2] We will later impose an additional restriction on these manifolds so that Theorem 2′ (ii) will hold.

Now the proof of Theorem 2 would be complete if $H^{4*}(M;\mathbb{Z})$ had no odd torsion.

The Odd Torsion Obstructions in $H^{4*}(M;\mathbb{Z})$, Manifolds With Singularities, and k-Homology.

We have reduced our analysis to the case when the Eilenberg obstructions are concentrated in the odd torsion subgroup of $H^{4*}(M;\mathbb{Z})$.

However, we are stopped at this point by the crucial fact that $\mathbb{Z}_n$-manifolds are not general enough to represent $\mathbb{Z}_n$-homology when n is odd. (For example the generator of $H_8(K(\mathbb{Z},3);\mathbb{Z}_3)$ is not representable.) Thus the crucial geometrical ingredient of the "Figure 2" proof is missing.

In the n odd case we can change the format of the proof slightly. Let D^k be the core disk of a k-handle h representing a generator of *odd* order n in $H^{4s}(M;\mathbb{Z})$. Let N^k be a $\mathbb{Z}_n$-manifold in $f(Q)\cup h$ situated as usual. *Then we claim that the class of* $f^{-1}D^k$ *in* $P_k\otimes\mathbb{Z}_n$ *is determined by* Index $f^{-1}N^k$ – Index $N^k\in\mathbb{Z}_n$.[1] (We can recover the $\frac{1}{8}$ factor since n is odd.) But the index of $f^{-1}N^k$ only depends on the homotopy class of f because of transversality and the cobordism invariance of the mod n index. ([N2])

Thus we see that the rigidity of the odd torsion Eilenberg obstruction "follows" from the existence of a **geometrical** $\mathbb{Z}_n$-manifold object which:
(i) is general enough to represent $\mathbb{Z}_n$-homology,
(ii) is nice enough to apply transversality,
(iii) has an additive index $\in\mathbb{Z}_n$ which is a cobordism invariant and which generalizes the usual index.

Finding a reasonable solution of (i) is itself an interesting problem.*

We proceed as follows. Let $\mathbb{C}_1,\mathbb{C}_2,\ldots$ denote a set of ring generators for smooth bordism modulo torsion, $\Omega_*/\text{Torsion}$ ($\dim\mathbb{C}_i=4i$). We say that a polyhedron is "like" $S^n*\mathbb{C}_1$ ($=S^n$ join $\mathbb{C}_1$) if it is of the form $W\cup L\times\text{cone}\,\mathbb{C}_1$, W a PL-manifold, $\partial W=L\times\mathbb{C}_1$, i.e. has a singularity structure like $S^n*\mathbb{C}_1$. More generally we say that a polyhedron is "like" $Q=S^n*\mathbb{C}_{i_1}*\mathbb{C}_{i_2}*\ldots*\mathbb{C}_{i_r}$[2] if it admits a global decomposition "like" Q.

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[1] The index of a $\mathbb{Z}_n$-manifold $(N^{4k},\delta N)$ is the index of $N/\delta N$ taken modulo n. The index of $N/\delta N$ is the signature of the (possibly degenerate) cup product pairing on $H^{2k}(N/\delta N;\mathbb{Q})$.

* A solution is given by Rourke, Bull. L.M.S. **5** (1973) 257–262

[2] We require $i_1<i_2<\ldots<i_r$.

For example, $S^n * \mathbb{C}_1 * \mathbb{C}_2$ may be decomposed as in Figure 3.

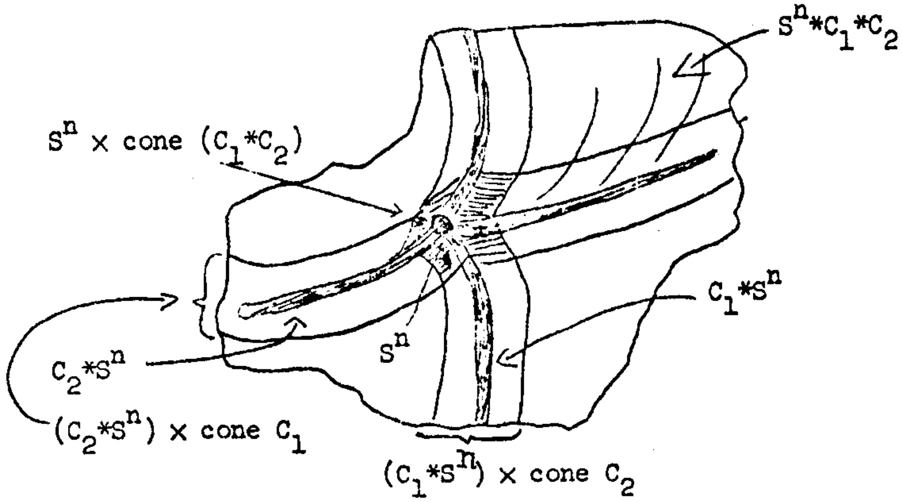


Figure 3

These polyhedra play the role of closed "*manifolds*" in our theory. Part of their structure is the join-like decomposition of a neighborhood of the singularity set together with a compatible linear structure on the stable tangent bundle of the complement of the singular set.

"Manifolds with boundary" and "$\mathbb{Z}_n$-manifolds" are easy generalizations.

(i) *The bordism homology theory defined by these manifolds with singularities is usual integral homology theory.*
(ii) Also transversality and other geometrical constructions are fairly easy with these varieties.
(iii) They do *not however have a good index.* Our mistake came when we introduced the cone on $\mathbb{C}_1$ = cone on $\mathbb{C}\mathbb{P}^2$, say.

If we make the analogous construction using only $\mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \ldots$ where Index $\mathbb{C}_i = 0$ $i = 2, 3, 4, \ldots$ then *we can define a proper index.*

However, *we no longer have ordinary homology theory but a theory V_* such that V_*(pt.) is a polynomial* algebra on one 4-dimensional generator $[\mathbb{C}\mathbb{P}^2]$. V_* is in fact a geometric representation of connective k-homology and the natural transformation $\Omega_* \longrightarrow V_*$[1] is closely related to the transformation $I : \Omega_* \longrightarrow K_*$ constructed below.[2]

[1] This is obtained by regarding a non-singular manifold as a variety.
[2] We are working modulo 2-torsion in this paragraph.

Thus we see that the Eilenberg obstructions in dimension $4k$ are not well-defined. Their values may be varied on those $\mathbb{Z}_n$-classes whose geometric representative requires a $\mathbb{C}\mathbb{P}^2$-singularity (i.e. does not come from V_*). This may be seen quite clearly in dimension 8. In fact, from the homotopy theory below we see that the $H^{4k}(M;\mathbb{Z}_n)$ modulo the indeterminacy of the Eilenberg obstructions (reduced mod n) is precisely dual to the subgroup of V_* representable elements in $H_{4k}(M;\mathbb{Z}_n)$.

This duality may also be seen geometrically but it is more complicated.

II. The Characteristic Bundle of a Homotopy Equivalence

The proof of Theorem 2 (the Characteristic Variety Theorem) can be completed by studying the obstruction theory of Theorem 1 from the homotopy theoretical point of view.

Definition. (F/PL-bundle, F/O-bundle). An $F/PL)_n$**-bundle** over a finite complex X is a (proper) homotopy equivalence $\theta : E \longrightarrow X \times \mathbb{R}^n$ where $\pi : E \longrightarrow X$ is a piecewise linear $\mathbb{R}^n$-bundle and

$$\begin{array}{ccc} E & \xrightarrow{\ \theta\ } & X \times \mathbb{R}^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{\text{identity}} & X \end{array}$$

is homotopy commutative.

Two $F/PL)_n$-bundles θ_0 and θ_1 are **equivalent** iff there is a piecewise linear bundle equivalence $b : E_0 \longrightarrow E_1$ so that

$$\begin{array}{ccc} E_0 & & \\ & \searrow^{\theta_0} & \\ {\scriptstyle b}\downarrow & & X \times \mathbb{R}^n \\ & \nearrow_{\theta_1} & \\ E_1 & & \end{array}$$

is properly homotopy commutative.

An $F/O)_n$-bundle is the corresponding linear notion.[1]

[1] These bundle theories are classified [B1] by the homotopy classes of maps into certain CW complexes $F/PL)_n$ and $F/O)_n$. The correspondence $\theta \longrightarrow \theta \times \text{identity}_{\mathbb{R}}$ defines stabilization maps $F/PL)_n \longrightarrow F/PL)_{n+1}$, $F/O)_n \longrightarrow F/O)_{n+1}$. The stable limits are denoted by F/PL and F/O respectively. Using a "Whitney sum" operation F/O and F/PL become homotopy associative, homotopy commutative H-spaces.

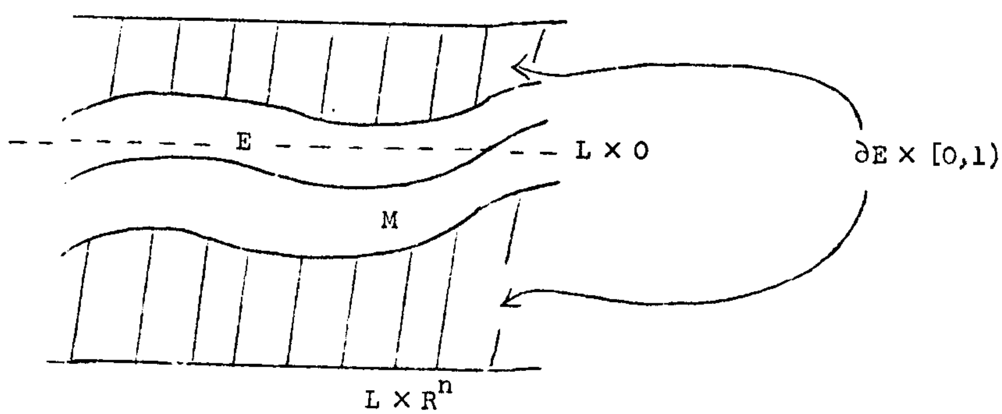
Let $g : (L, \partial L) \longrightarrow (M, \partial M)$ be a homotopy equivalence of compact piecewise linear manifolds with homotopy inverse $\overline{g} : (M, \partial M) \longrightarrow (L, \partial L)$.

A **characteristic** $F/PL)_n$**-bundle of** g is any composition θ_g, given by

$$E \xrightarrow[\cong]{c\,=PL\text{-homeomorphism}} L \times \mathbb{R}^n \xrightarrow{g\times\text{identity}} M \times \mathbb{R}^n \ ,$$

where E is the normal bundle of an embedding $M \subset L \times \mathbb{R}^n$ ($n \gg \dim L$) approximating $\overline{g} \times 0$. c is any identification (homotopic to $\overline{g}$) of the total space of E with all of $L \times \mathbb{R}^n$. (c may be constructed for example á lá Mazur using the "half-open" h-cobordism theorem.)

Notice that θ_g is transverse regular to $M \times 0$ with inverse image PL-homeomorphic to L



The characteristic bundle of the homotopy equivalence $g : (L, \partial L) \longrightarrow (M, \partial M)$ is the stable equivalence class of θ_g considered as a homotopy class of maps

$$\theta_g \ : \ M \longrightarrow F/PL \ .$$

If L and M are smooth, E is a vector bundle, c will be a diffeomorphism, and we obtain the characteristic F/O-bundle of a homotopy equivalence between smooth manifolds

$$\eta_g \ : \ M \longrightarrow F/O \ .$$

The Classification of h-Triangulations and h-Smoothings

To state the homotopy theoretical analogue of Theorem 1 we consider a homotopy equivalence of a PL-manifold pair with $(M, \partial M)$ as defining a "**homotopy-triangulation**" of $(M, \partial M)$. Two h**-triangulations** $g_0 : (L_0, \partial L_0) \longrightarrow (M, \partial M)$ and $g_1 : (L_1, \partial L_1) \longrightarrow (M, \partial M)$ are "concordant" iff there is a PL-homeomorphism

$c : (L_0, \partial L_0) \longrightarrow (L_1, \partial L_1)$ so that

$$\begin{array}{ccc} (L_0, \partial L_0) & & \\ & \searrow^{g_0} & \\ \cong \big\downarrow c & & (M, \partial M) \\ & \nearrow_{g_1} & \\ (L_1, \partial L_1) & & \end{array}$$

is homotopy commutative. We denote the set of concordance classes of h-triangulations of M by $hT(M)$.

Note that the characteristic variety theorem asserts that the concordance class of an h-triangulation $g : (L, \partial L) \longrightarrow (M, \partial M)$ is *completely* determined by the splitting invariant of g.

In a similar fashion we obtain the set of concordance classes of **h-smoothings** of M, $hS(M)$.

The zero element in $hT(M)$ or $hS(M)$ is the class of id. $: M \longrightarrow M$. The characteristic bundle construction for a homotopy equivalence defines transformations

$$\theta \ : \ hT(M) \longrightarrow (M, F/PL)$$
$$\eta \ : \ hS(M) \longrightarrow (M, F/O)$$

where (X, Y) means the set of homotopy classes of maps from X to Y.

Assume $\pi_1(M) = \pi_1(\partial M) = 0$, $n = \dim M \geq 6$.

Theorem 3. *If $\partial M \neq \emptyset$, then*

$$\theta \ : \ hT(M) \ = \ \left\{ \begin{array}{c} \textit{concordance classes of} \\ h\textit{-triangulations of } M \end{array} \right\} \longrightarrow (M, F/PL)$$

and

$$\eta \ : \ hS(M) \ = \ \left\{ \begin{array}{c} \textit{concordance classes of} \\ h\textit{-smoothings of } M \end{array} \right\} \longrightarrow (M, F/O)$$

are isomorphisms.
If $\partial M = \emptyset$, we have the exact sequences (of based sets)

$$\text{(i)} \ \ 0 \longrightarrow hT(M) \xrightarrow{\ \theta\ } (M, F/PL) \xrightarrow{\ \mathcal{S}\ } P_n$$

$$\text{(ii)} \ \ \theta_n \partial\pi \xrightarrow{\ \#\ } hS(M) \xrightarrow{\ \eta\ } (M, F/O) \xrightarrow{\ \mathcal{S}\ } P_n \ .$$

Proof. See [S1].

Here $\mathcal{S}$ is the surgery obstruction for an F/PL or F/O bundle over a closed (even dimensional) manifold; and $\#$ is obtained from the action of θ_n on $hS(M)$,

$(g : L \longrightarrow M) \mapsto (g : L \# \Sigma \longrightarrow M)$.

$\mathcal{S}$ will be discussed in more detail below. We remark that the exactness of (ii) at $hS(M)$ is stronger, namely

$$\{\text{orbits of } \theta_n \partial \pi\} \;\cong\; \text{image}\, \eta \ .$$

Also (i) may be used to show that $\partial M = \emptyset$ implies the:

Corollary. *If M is closed then $\theta : hT(M) \cong (M - \text{pt.}, F/PL)$.*

Easy transversality arguments show that

$$\pi_i(F/PL) \;=\; P_i \quad , \quad \pi_i(F/O) \;=\; A_i \ .$$

Thus the Theorem 1 obstructions in

$$H^i(M; A_i) \ \text{ or } \ H^i(M; P_i)$$

for deforming $g : (L, \partial L) \longrightarrow (M, \partial M)$ to a diffeomorphism or a PL-homeomorphism become the homotopy theoretical obstructions in

$$H^i(M; \pi_i(F/O)) \ \text{ or } \ H^i(M; \pi_i(F/PL))$$

for deforming η_g or θ_g to the point map.

In fact using naturality properties of θ and η we can precisely recover the obstruction theory of Theorem 1 from statements about the "kernels" of θ and η given in Theorem 3.

We obtain *new* information from the statements about the "cokernels" of θ and η given in Theorem 3. For example, if we consider the map

$$\mathbb{CP}^4 - \text{pt.} \;\cong\; \mathbb{CP}^3 \xrightarrow{\ \deg 1\ } S^6 \xrightarrow{\ \text{gen } \pi_6\ } F/PL$$

we obtain an interesting h-triangulation of $\mathbb{CP}^4$, $M^8 \longrightarrow \mathbb{CP}^4$.

Now we may study the obstruction theory of Theorem 1 by studying the homotopy theory of F/O and F/PL.

For example using the fact that F/O and F/PL are H-spaces (Whitney sum) (and thus have trivial k-invariants over the rationals) one sees immediately that the triangulating and smoothing obstructions for a homotopy equivalence f are torsion cohomology classes iff f is a correspondence of rational Pontrjagin classes.

To describe the obstructions completely we must again restrict to the piecewise linear case.

The Homotopy Theory of F/PL

We have already seen that the homotopy groups of F/PL are very nice:

i	1	2	3	4	5	6	7	8	...
$\pi_i(F/PL)$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	...

This regularity is also found in the global homotopy structure of the space.

For describing this structure we will localize F/PL *at* the prime 2 and then *away* from the prime 2.

If X is a homotopy associative homotopy commutative H-space, then

"X localized at 2" $\equiv X_{(2)}$ is the H-space which represents the functor

$$(\ \ , X) \otimes \mathbb{Z}_{(2)}$$

where $\mathbb{Z}_{(2)} \ = \ \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{p_i}, \ldots]$, p_i the i^{th} odd prime.

"X localized away from 2" $\equiv X_{(\text{odd})}$ is the H-space which represents the functor

$$(\ \ , X) \otimes \mathbb{Z}_{(\text{odd})}$$

where $\mathbb{Z}_{(\text{odd})} \ = \ \mathbb{Z}[\frac{1}{2}]$.

Note that there are natural projections $p_{(2)}$ and $p_{(\text{odd})}$

$$\begin{array}{ccccc} & & & \overset{p_{(2)}}{\nearrow} & X_{(2)} \\ Y \ = \ CW\text{-complex} & \xrightarrow{\ f\ } & X & & \\ & & & \underset{p_{(\text{odd})}}{\searrow} & X_{(\text{odd})} \end{array}$$

Also $f : Y \longrightarrow X$ is homotopic to zero iff $p_{(2)} \circ f$ and $p_{(\text{odd})} \circ f$ are homotopic to zero. Thus it suffices to study $X_{(2)}$ and $X_{(\text{odd})}$.

Let BO denote the classifying space for stable equivalence classes of vector bundles over finite complexes.

Let $K(\pi, n)$ denote the Eilenberg-MacLane space, having one non-zero homotopy group π in dimension n.

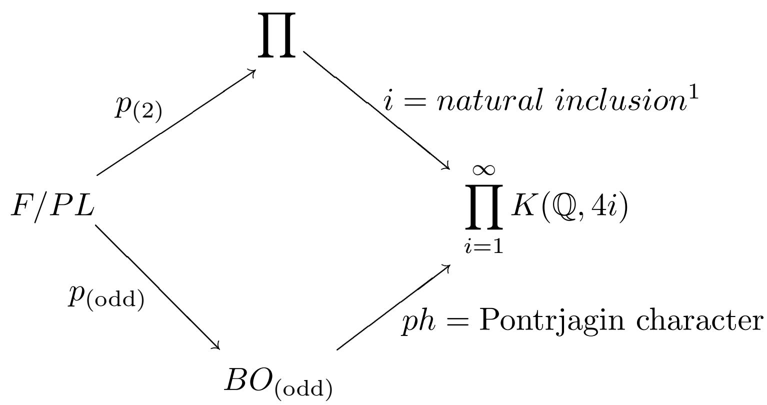
Let δSq^2 denote the unique element of order 2 in $H^5(K(\mathbb{Z}_2,2);\mathbb{Z}_{(2)})$, and $K(\mathbb{Z}_2,2)\times_{\delta Sq^2} K(\mathbb{Z}_{(2)},4)$ the total space of the principal fibration over $K(\mathbb{Z}_2,2)$ with $K(\mathbb{Z}_{(2)},4)$ as fibre and principal obstruction (k-invariant) δSq^2.

Then we have the following:

Theorem 4. (i) *$F/PL_{(2)}$ is homotopy equivalent to*

$$\prod \;=\; K(\mathbb{Z}_2,2)\times_{\delta Sq^2} K(\mathbb{Z}_{(2)},4)\times \prod_{i=1}^{\infty} K(\mathbb{Z}_2,4i+2)\times K(\mathbb{Z}_{(2)},4i+4)\ .$$

(ii) *$F/PL_{(\mathrm{odd})}$ is homotopy equivalent to $BO_{(\mathrm{odd})}$.*
(iii) *F/PL is homotopy equivalent to the fibre product of i and ph in the diagram*



Corollary. *$H^*(F/PL;\mathbb{Z})$ has no odd torsion and the 2-torsion may be calculated.*

Corollary. *$\mathbb{Z}\times F/PL$ is an infinite loop space. In fact it is homotopy equivalent to the 0th space in the Ω-spectrum of a multiplicative cohomology theory.*

Corollary. *If $\mathcal{O}=\Omega(F/PL)=$ loop space of F/PL, then $\mathcal{O}$ satisfies a Bott periodicity of length four, namely*

$$\Omega^4\mathcal{O}\ \cong\ \mathcal{O}\ ,\quad \text{as } H\text{-spaces}\ .$$

We use the notation $\mathcal{O}$ because $\mathcal{O}_{(\mathrm{odd})}\cong O_{(\mathrm{odd})}$ where O is the infinite orthogonal group.

Bordism, Homology Theory, and K-Theory.

In order to prove and apply Theorem 4 we need to study the relationship between smooth bordism and ordinary homology on the one hand (for the prime 2) and smooth bordism and K-theory on the other hand (for odd primes). Recall that $\Omega_*(X)$ is a module over $\Omega_*=\Omega_*(\mathrm{pt})$ by the operation

$$(f:M\longrightarrow X,N)\longrightarrow(fp_2:N\times M\longrightarrow X)\ .$$

[1] On π_4 i_* is twice the natural embedding $\mathbb{Z}_{(2)}\longrightarrow\mathbb{Q}$.

$\mathbb{Z}$ is a module over Ω_* by Index : $\Omega_* \longrightarrow \mathbb{Z}$.
Then we can form $\Omega_*(X) \otimes_{\Omega_*} \mathbb{Z}$, and obtain a $\mathbb{Z}_4$-graded *functor*.
Let $K_*(X)$ denote the $\mathbb{Z}_4$-graded homology theory defined by $KO_*(X) \otimes \mathbb{Z}_{(\mathrm{odd})}$.
KO_* is the homology theory dual to real K-theory, $KO^*(X)$.

Theorem 5. *There are natural equivalences of functors:*

(i) $\Omega_*(X) \otimes \mathbb{Z}_{(2)} \;\cong\; H_*(X; \Omega_* \otimes \mathbb{Z}_{(2)})$ *as $\mathbb{Z}$-graded Ω_*-modules,*

(ii) $\Omega_*(X) \otimes_{\Omega_*} \mathbb{Z}_{(\mathrm{odd})} \;\cong\; KO_*(X) \otimes \mathbb{Z}_{(\mathrm{odd})} \;\equiv\; K_*(X)$ *as $\mathbb{Z}_4$-graded $\mathbb{Z}$-modules* .

Proof. (i) can be found in [CF1]. (ii) is analogous to [CF2] but different (and simpler) because the module structure on the left is different. ([CF2] uses the $\widehat{A}$ genus and the first cobordism Pontrjagin class.)

Proof of (ii) : We construct a (multiplicative) transformation

$$I \;:\; \Omega_*(X) \longrightarrow K_*(X)$$

which on the point is essentially the index.

I was first constructed by introducing the singularities described above into cobordism theory and then taking a direct limit.

It can also be constructed by first producing an element in $K^0(BSO)$ whose Pontrjagin character is

$$\frac{\widehat{A}}{L} = \frac{\widehat{A}\text{-genus}}{\text{Hirzebruch } L\text{-genus}}$$

(this is a calculation), and then applying the usual Thom isomorphism to obtain the correct element in $K^0(MSO)$.

I induces a transformation

$$I \;:\; \Omega^0(X) \longrightarrow K^0(X)$$

which is in turn induced by a map of universal spaces

$$I \;:\; \Omega^\infty MSO_{(\mathrm{odd})} \longrightarrow \mathbb{Z} \times BO_{(\mathrm{odd})} \ .$$

I is onto in homotopy (since there are manifolds of index 1 in each dim $4k$), thus the fibre of I only has homotopy in dimensions $4k$. Obstruction theory implies that I has a cross section. Therefore the transformation induced by I

$$I \;:\; \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z}_{(\mathrm{odd})} \longrightarrow K^*(X)$$

is onto for dimension zero cohomology.

When $X = MSO$, I is easily seen to be an isomorphism. This fact together with the cross section above implies that I is injective for dimension zero. (ii) now follows by Alexander duality and the suspension isomorphism.

The point of Theorem 5 is the following – the theories on the left have a nice geometrical significance for our problems while those on the right are nice algebraically.

Of course $H_*(X;\Omega_*)$ is constructed from a chain complex. We thus have the classical duality theorems (universal coefficient theorems) relating H_* and H^*.

These results are also true for K_* and K^*. For example, the multiplicative structure (see [A]) in K^* defines homomorphisms

$$\begin{aligned} e\ &: K^0(X) \longrightarrow \mathrm{Hom}(K_0(X), \mathbb{Z}_{(\mathrm{odd})}) \\ &= \\ e_n\ &: K^0(X) \longrightarrow \mathrm{Hom}(K_0(X;\mathbb{Z}_n), \mathbb{Z}_n)\ ,\ n \text{ odd}\ . \end{aligned}$$

Theorem 6. *If X is finite*

(i) *e is onto,*

(ii) *if $\sigma \in K^0(X)$, then $\sigma = 0$ iff $e_n(\sigma) = 0$ for all odd integers n,*

(iii) *any compatible (w.r.t. $\mathbb{Z}_n \longrightarrow \mathbb{Z}_{n'}$) set of homomorphisms (f, f_n) determines an element σ in $K^0(X)$ such that $e_n(\sigma) = f_n$, $e(\sigma) = f$.*

(i) and (ii) were first proved by the author using intersection theory and the geometrical interpretation of I. (The hard part was to construct e and e_n.)

However, using the multiplication in $K^0(X)$ coming from the tensor product of vector bundles (plus the extension to $\mathbb{Z}_n$-coefficients in [AT]) (i), (ii), and (iii) follow immediately from Bott periodicity and general nonsense.

The duality theorems for K^* and K_* were first proved by Anderson [A1]. We denote by $\Omega_*(X;\mathbb{Z}_n)$ the homology theory defined by bordism of $\mathbb{Z}_n$-manifolds. We make $P_* = 0, \mathbb{Z}_2, 0, \mathbb{Z}, \ldots$ into a Ω_**-module* by Index : $\Omega_* \longrightarrow P_*$. Then we have:

Theorem 7. **(The splitting obstruction of an F/PL bundle)**
There are onto Ω_*-module homomorphisms $\mathcal{S}$ *and $\mathcal{S}_n$ so that the following square commutes*

$$\begin{array}{ccc} \Omega_*(F/PL) & \xrightarrow{\quad \mathcal{S} \quad} & P_* \\ \text{natural inclusion} \big\downarrow & & \big\downarrow \text{reduction mod } n \\ \Omega_*(F/PL;\mathbb{Z}_n) & \xrightarrow{\quad \mathcal{S}_n \quad} & P_* \otimes \mathbb{Z}_n \end{array}$$

The composition

$$\pi_*(F/PL) \xrightarrow{\text{Hurewicz}} \Omega_*(F/PL) \xrightarrow{\ \mathcal{S}\ } P_*$$

is an isomorphism if $* \neq 4$, and multiplication by 2 if $* = 4$.

Using Theorem 5 (i) and $\mathcal{S}_2 : \Omega_{4*+2}(F/PL;\mathbb{Z}_2)\longrightarrow\mathbb{Z}_2$ which is more generally a $\Omega_*(\mathrm{pt};\mathbb{Z}_2)$-module homomorphism[1] we obtain:

Corollary 1. (**A Formula for the Kervaire Invariant of an F/PL-bundle over a $\mathbb{Z}_2$-manifold**). *There is a unique class*

$$\mathcal{K} \;=\; k_2 + k_6 + k_{10} + \ldots \in H^{4*+2}(F/PL;\mathbb{Z}_2)$$

such that for any $\mathbb{Z}_2$-manifold in F/PL

$$f \;:\; M^{4k+2} \longrightarrow F/PL$$

we have

$$\mathcal{S}_2(M^{4k+2}, f) \;=\; W(M)\cdot f^*\mathcal{K}[M] \; .$$

Using Theorem 5 (i) and $\mathcal{S} : \Omega_{4*}(F/PL)\longrightarrow\mathbb{Z}$ and working modulo torsion we obtain:

Corollary 2. *There is a class*

$$\mathcal{L} \;=\; \ell_4 + \ell_8 + \ell_{12} + \ldots \in H^{4*}(F/PL;\mathbb{Z}_{(2)})$$

which is unique modulo torsion, such that

$$\mathcal{S}(M^{4k}, f) \;=\; L(M)\cdot f^*\mathcal{L}[M] \; .$$

Here $W(M)$ and $L(M)$ are respectively the total Stiefel Whitney class and total Hirzebruch class.

The point of Corollary 2 is that $\mathcal{L}$ is a class with $\mathbb{Z}_{(2)}$-coefficients. $\mathcal{L}$ regarded as a class with rational coefficients is familiar, namely

$$\mathcal{L} \;=\; \ell_4 + \ell_8 + \ldots \;=\; \frac{1}{8} j^*(L_1 + L_2 + \ldots)$$

where $j : F/PL\longrightarrow B_{PL}$ is the natural map and L_i is the universal Hirzebruch class in $H^{4i}(B_{PL};\mathbb{Q})$.

Now use

$$\begin{array}{ccc} & \Omega_*(F/PL) & \\ {\scriptstyle\text{natural projection}}\swarrow & & \searrow{\scriptstyle\mathcal{S}} \\ K_0(F/PL) & \xrightarrow{\;\mathcal{S}\otimes_{\Omega_*}\mathbb{Z}_{(\mathrm{odd})}\;} & \mathbb{Z}_{(\mathrm{odd})} \end{array}$$

(with $K_0(F/PL) \cong \Omega_{4*}(F/PL)\otimes_{\Omega_*}\mathbb{Z}_{(\mathrm{odd})}$) and Theorem 6 (i) to obtain:

[1] Via the mod 2 Euler characteristic $\Omega_*(\mathrm{pt},\mathbb{Z}_2)\longrightarrow\mathbb{Z}_2$.

Corollary 3. *There is a unique element $\sigma \in \widetilde{K}^0(F/PL)$ such that the Pontrjagin character of $\sigma = \mathcal{L}$ in $H^{4*}(F/PL;\mathbb{Q})$.*

Uniqueness follows from existence – which implies that $\widetilde{K}^0(F/PL)$ is a free $\mathbb{Z}_{(\mathrm{odd})}$-module.

Proof of Theorem 7. If $\theta : E \longrightarrow M \times \mathbb{R}^n$ is an F/PL-bundle over a $\mathbb{Z}_n$-manifold M^{2s}, then suppose $\theta^{-1}(M, \delta M)$ has the properties of $(U, \delta U)$ in $(*)$ (in the construction of the splitting obstruction in Section I) and define $\mathcal{S}(M, f)$ by

$$\mathcal{S}(M,f) \;=\; \begin{cases} \frac{1}{8}\big\{\mathrm{Index}(U,\delta U) - \mathrm{Index}(M,\delta M)\big\} \pmod n \\ \qquad s \text{ even}^1 \\ \text{Kervaire invariant of } \theta : (U,\delta U) \longrightarrow (M,\delta M) \\ \qquad s \text{ odd}\ . \end{cases}$$

Remark. The fact that $\mathcal{S} : \Omega_4(F/PL) \longrightarrow \mathbb{Z}$ is onto[2] while $\pi_4(F/PL) \longrightarrow \Omega_4(F/PL) \xrightarrow{\mathcal{S}} \mathbb{Z}$ is multiplication by 2 implies the first k-invariant of F/PL in $H^5(K(\mathbb{Z}_2, 2); \mathbb{Z}) = \mathbb{Z}_4$ is *non-zero.*

Since F/PL is an H-space the reduction of the k-invariant to $\mathbb{Z}_2$-coefficients must be primitive and have Sq^1 zero. It is therefore zero (by an easy calculation – pointed out to me by Milnor). This singles out δSq^2 = "integral Bockstein of square two" as the first k-invariant of F/PL (= first k-invariant of BSO, F/O, $BSPL$ etc.)

Proof of Theorem 4:

I. The $\sigma \in \widetilde{K}_0(F/PL)$ of Corollary 3 determines

$$p_{(\mathrm{odd})} \;:\; F/PL \longrightarrow BO_{(\mathrm{odd})}\ .$$

II. Using $\mathcal{K}$ and $\mathcal{L}$ of Corollaries 1 and 2 and the remark above we construct[3]

$$p_{(2)} \;:\; F/PL \longrightarrow \prod\ ,$$

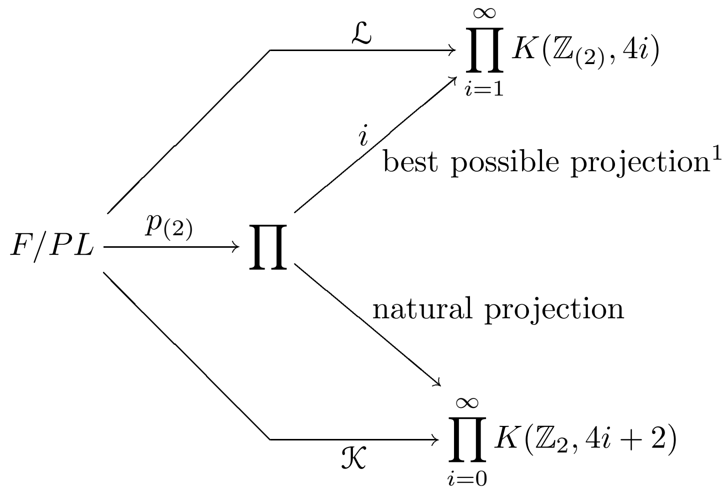
---

[1] If $\dim M = 4$ $\mathcal{S}(M, f)$ is well-defined (modulo $2n$) if cobordisms of δM are restricted to spin manifolds.

[2] 24 times the canonical complex line bundle over $\mathbb{C}\mathbb{P}^2$ is fibre homotopically trivial.

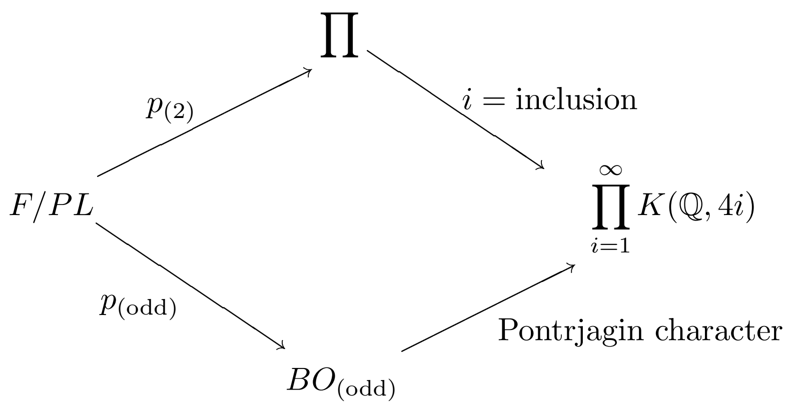
[3] i is onto in homotopy except in dimension 4 where it has index 2

so that



is homotopy commutative.

A calculation shows that $p_{(\mathrm{odd})}$ and $p_{(2)}$ are correct in homotopy. It is clear from the construction that



is homotopy commutative, so (iii) is proven.

Remark. The only part of the construction of the localizing projections $p_{(2)}$ and $p_{(\mathrm{odd})}$ which is not completely canonical is the construction of $\mathcal{L}$ used in the definition of $p_{(2)}$. $\mathcal{L}$ was only determined modulo torsion (this can be improved to "modulo torsion elements divisible by 2"). This difficulty arises from the lack of a nice geometrical description of the product of two $\mathbb{Z}_n$-manifolds (as a $\mathbb{Z}_n$-manifold).

We can make this aspect of F/PL-homotopy theory more intrinsic by formulating the results in terms of a characteristic variety.

If X is a finite complex and $g : V \longrightarrow X$ is a singular variety in X, then for each F/PL-bundle over X we can by restriction to V associate a splitting obstruction on each component of V. We use the splitting invariant defined in the proof of Theorem 7 (with the refinement in dimension 4). We obtain a "function on V" for each F/PL-bundle over X.

Notice that no fundamental group hypothesis need be made to define the "splitting invariant along V" because every element in $\Omega_{2*}(F/PL; \mathbb{Z}_n)$ is represented by a simply connected pair $(M, \delta M)$.

Theorem 4′. (**The characteristic variety theorem for** F/PL).
Let X be a finite complex. Then there is a characteristic variety in X, $g : V \longrightarrow X$ with the property that:

(i) *two F/PL-bundles over X are equivalent iff their "splitting invariants along V" are equal.*

(ii) *a "function on V" is the splitting invariant of a bundle iff its values on the 4-dimensional* spin components[1] *of V are even.*

Remark. It is easy to see that if $h : (M, \partial M) \longrightarrow (L, \partial L)$ is a homotopy equivalence then the splitting invariant of h along a singular variety V in M is the same as the splitting invariant of θ_h along V. Thus Theorems 3 and 4′ imply Theorem 2′.

Proof of Theorem 4′. We first describe a suitable characteristic variety.

(i) Choose a collection of $(4i + 2)$-dimensional $\mathbb{Z}_2$-manifolds in X, $f : \bigcup_i K_i \longrightarrow X$, so that $\{f_*(\text{fundamental class } K_i)\}$ is a basis of $A \bigoplus_{i>0} H_{4i+2}(X; \mathbb{Z}_2)$ where $A \subseteq H_2(X; \mathbb{Z}_2)$ is a subgroup dual to $\ker(Sq^2 : H^2(X; \mathbb{Z}_2) \longrightarrow H^4(X; \mathbb{Z}_2))$.

(ii) Choose a collection of $4i$-dimensional $\mathbb{Z}_{2^r}$-manifolds $f : \bigcup_j N_j \longrightarrow X$ such that $\{f_*(\delta N_j)\}$ is a basis of [2-torsion $\bigoplus_i H_{4i-1}(M; \mathbb{Z})$].

(iii) Choose a collection $\{V_\alpha\}$ of $4N$-dimensional $\mathbb{Z}_{p^r}$-manifolds for each odd prime p, $\{f_\alpha : V_\alpha \longrightarrow X\}$ such that $\{f_\alpha : \delta V_\alpha \longrightarrow X\}$ form a basis of the odd torsion subgroup of

$$\Omega_{4*-1}(X) \otimes_{\Omega_*} \mathbb{Z}_{(\text{odd})} \;\cong\; K_{-1}(X)\ .$$

(iv) Choose a collection C of singular closed oriented $4i$-dimensional manifolds $\{g_s : M_s \longrightarrow X\}$ such that in

[1] For definition see remarks before Theorem 2′.

$$\begin{array}{ccc}
\Omega_{4*}(X) & \xrightarrow{S_* = \text{fund. class}} & \bigoplus_{i>0} H_{4i}(X;\mathbb{Z}_{(2)})/\text{Torsion} \\
\Big\downarrow {\scriptstyle I_* = \text{natural projection}} & & \Big\downarrow {\scriptstyle \text{inclusion}} \\
\Omega_{4*}(X) \otimes_{\Omega_*} \mathbb{Z}_{(\text{odd})}/\text{Torsion} & \xrightarrow{ph_*} & \bigoplus_{i>0} H_{4i}(X;\mathbb{Q})
\end{array}$$

S_*C and I_*C form bases.

I. If we assign "splitting obstructions" in $\mathbb{Z}_{p^r}$ or $\mathbb{Z}$ to each manifold in group (iii) or (iv) we define a collection of homomorphisms

$$\phi_n \ : \ K_0(X;\mathbb{Z}_n) \longrightarrow \mathbb{Z}_n \ , \ n \text{ odd or zero} \ .$$

The collection $\{\phi_n\}$ defines a unique element in $\sigma \in K^0(X)$, $\sigma : X \longrightarrow BO_{(\text{odd})}$.

The commutativity[1] of

$$\begin{array}{ccccc}
K_0(BO_{(\text{odd})};\mathbb{Z}_n) & & \xrightarrow[\text{eval. of id.}]{e_n(\text{id})} & & \mathbb{Z}_n \\
& \nwarrow {\scriptstyle \sigma_*} & & \nearrow {\scriptstyle \phi_n} & \\
{\scriptstyle p_{(\text{odd})}} \Big\downarrow \cong & & K_0(X;\mathbb{Z}_n) & & \Big\uparrow {\scriptstyle \mathcal{S}\otimes} \\
K_0(F/PL;\mathbb{Z}_n) & & \xrightarrow{\cong} & & \Omega_{4*}(F/PL;\mathbb{Z}_n) \otimes_{\Omega_*} \mathbb{Z}_{(\text{odd})}
\end{array}$$

implies any lifting of σ

$$\begin{array}{ccc}
& & F/PL \\
& \overset{\overline{\sigma}}{\nearrow} & \Big\downarrow {\scriptstyle p_{(\text{odd})}} \\
X & \xrightarrow[\sigma]{} & BO_{(\text{odd})}
\end{array}$$

will have the desired splitting obstructions on these components.

II. If we give splitting obstructions for the 2 and 4 dimensional components, we can construct a homomorphism $H_2(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ using the given values on A and the values (reduced mod 2) on the non-spin components of dimension 4 to obtain a cohomology class $u \in H^2(M;\mathbb{Z}_2)$ such that $\delta Sq^2 u = 0$.

[1] The outer commutativity is clear since for $Y = F/PL_{(\text{odd})}$ or $BO_{(\text{odd})}$ we have $K_0(Y;\mathbb{Z}_n) = K_0(Y) \otimes \mathbb{Z}_n$.

u defines a map $X \longrightarrow K(\mathbb{Z}_2, 2)$ which may be lifted to $K(\mathbb{Z}_2, 2) \times_{\delta Sq^2} K(\mathbb{Z}_{(2)}, 4)$. We alter this lifting by a map $f : X \longrightarrow K(\mathbb{Z}_{(2)}, 4)$ to obtain the desired splitting invariants in dimension 2 and 4.

The splitting obstructions in the other dimensions may be obtained by mapping into the appropriate $K(\mathbb{Z}_{(2)}, 4i)$ or $K(\mathbb{Z}_2, 4i+2)$ independently. We obtain

$$\beta : X \longrightarrow \prod .$$

σ and β determine a unique map $f : X \longrightarrow F/PL$ with the desired splitting obstructions.

Discussion of the characteristic variety

We can replace any component $g : N \longrightarrow X$ of the characteristic variety constructed for Theorem 4′ by $gp_2 : \mathbb{C} \times N \longrightarrow X$ if $\operatorname{Index} \mathbb{C} = \pm 1$ and N is not a $\mathbb{Z}$ or $\mathbb{Z}_{2^r}$ manifold of dimension 4. The new variety is still characteristic for X.

For determining whether two F/PL-bundles are the same we may further replace the four dimensional $\mathbb{Z}$-components N by $\mathbb{C} \times N$. (The realization property is then disturbed however).

We cannot replace the 4-dimensional $\mathbb{Z}_{2^r}$-components by higher dimensional components because we thereby lose the delicate property that the splitting invariant is well defined modulo 2^{r+1} on these components.

Thus in either case the characteristic variety has two “parts” – one of dimension four and one of infinite (or stable) dimension.

The ability to stabilize is the real reason why only $\mathbb{Z}_n$-*manifolds* appear in Theorem 4′ and not varieties of the more complicated type discussed earlier (for the study of odd torsion). Such a variety $\times \mathbb{C}\mathbb{P}^n$ is cobordant to a non-singular manifold.

The $\mathbb{Z}_n$-manifolds with singularities can be used to describe F/PL-bundles over X together with filtrations (the highest skeleton over which the bundle is trivial).

III. The Hauptvermutung

We can apply the *first* part of the *characteristic variety theorem* for F/PL to study homeomorphisms.

Theorem H. *Let $h : (L, \partial L) \longrightarrow (M, \partial M)$ be a homeomorphism and $\theta_h : M \longrightarrow F/PL$ be the characteristic F/PL-bundle for h. Then there is only one possible non-zero obstruction to the triviality of θ_h, an element of order 2 in $H^4(M; \mathbb{Z})$.*

Then if $H_3(M;\mathbb{Z})$ has no 2-torsion we have:

Corollary 1. *If $\pi_1(M) = \pi_1(\text{each component of } \partial M) = 0$ and $\dim M \geq 6$, then $h : (L, \partial L) \longrightarrow (M, \partial M)$ is homotopic to a PL-homeomorphism.*

Corollary 2. *In the non-simply-connected case, any dimension, we have that*

$$h \times \mathrm{id}_{\mathbb{R}^N} \;:\; (L, \partial L) \times \mathbb{R}^N \longrightarrow (M, \partial M) \times \mathbb{R}^N$$

is properly homotopic to a PL-homeomorphism. We may take $N = 3$.

Corollary 3. *The localizing projections for the natural map $H : TOP/PL \longrightarrow F/PL$ satisfy*

$$\begin{array}{ccc}
(*) & \xrightarrow{\;Pt\;} & (F/PL)_{(\mathrm{odd})} \\
\uparrow{\scriptstyle Pt} & & \uparrow{\scriptstyle p_{\mathrm{odd}}} \\
TOP/PL & \xrightarrow{\;H\;} & F/PL \\
\downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle p_{(2)}} \\
K(\mathbb{Z}_{(2)}, 4) & \xrightarrow[\text{inclusion}]{} & (F/PL)_{(2)}
\end{array}$$

where Θ is an h-map and has order 2.
($\mathbb{Z}_{(2)} = \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{p_i}, \ldots]$, p_i the i^{th} odd prime).[1]

Corollary 4. *Let M be as in Corollary 1. The subgroup of $hT(M)$ generated by homeomorphisms $h : (L, \partial L) \longrightarrow (M, \partial M)$ is a $\mathbb{Z}_2$-module of dimension not exceeding the dimension of $[\text{2-torsion}\, H_3(M;\mathbb{Z})] \otimes \mathbb{Z}_2$.*

Proof of Theorem H.

Let V be a characteristic variety for M. Replace each component N of V by $\mathbb{CP}^4 \times N$. If M is simply connected, we may use the splitting theorem of Novikov [N1] to see that the splitting invariant of h along $\mathbb{CP}^4 \times V$ equals zero.[2] This proves Theorem H in the simply connected case.

If M is a general manifold we use a strengthened version of Novikov's splitting theorem (originally proved to treat the manifolds with singularities described

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[1] We are considering the spaces of Corollary 3 as being defined by functors on the category of finite CW complexes.

[2] V may be chosen in this case so that $\pi_1(V) = \pi_1(\delta V) = 0$.

above).

Definition. A **manifold complex** is a polyhedron constructed inductively by attaching an n-dimensional PL-manifold to the previously constructed $(n-1)$-dimensional polyhedron along the boundary of the manifold which is embedded. The components of the n-manifold are the n**-cells** of the manifold complex.

Splitting Lemma. *Let K be a manifold complex whose "cells" have dimension ≥ 5 and free abelian fundamental group. Let $t : E \longrightarrow K \times \mathbb{R}^n$ be a topological trivialization of the piecewise linear $\mathbb{R}^n$-bundle E over K. Then t is properly homotopic to a map which is transverse regular to $K \times 0 \subset K \times \mathbb{R}^n$ and such that*

$$t| \ : \ t^{-1}(K \times 0) \longrightarrow K \times 0$$

is a cell-wise homotopy equivalence of manifold complexes.

Proof of Splitting Lemma. Assume first that K has one cell. Consider

$$T^{n-1} \ = \ S^1 \times \ldots \times S^1 \ (n-1 \text{ factors}) \subset \mathbb{R}^n$$

and let

$$W \ = \ t^{-1}(K \times T^{n-1} \times \mathbb{R}) \ .$$

Then $t_0 : W \longrightarrow K \times T^{n-1} \times \mathbb{R}$ is a *proper homotopy equivalence.*[1] We may apply Siebenmann's Thesis [S] to split t_0, namely we find a PL-homeomorphism $W \cong W_1 \times \mathbb{R}$ and a map $t_1 : W \longrightarrow K \times T^{n-1}$ so that

$$\begin{array}{ccc} W & \xrightarrow{\ t_0\ } & K \times T^{n-1} \times \mathbb{R} \\ \cong \big\downarrow & \nearrow_{t_1 \times \mathrm{id}_{\mathbb{R}}} & \\ W_1 \times \mathbb{R} & & \end{array}$$

is properly homotopy commutative.

We then apply Farrell's fibring theorem [F] to deform

$$W_1 \xrightarrow{t_1} K \times T^{n-1} \xrightarrow{\text{last factor}} S^1$$

[1] This is the only place "homeomorphism" is used in the proof of the Hauptvermutung.

to a fibration and thus split t_1. Then we have a diagram

$$\begin{array}{ccc} W_2 & \xrightarrow{\ t_2\ } & K \times T^{n-2} \\ \subset\downarrow & & \downarrow\subset \\ W_1 & \xrightarrow{\ t_1'\ } & K \times T^{n-1} \\ & \searrow \qquad \swarrow & \\ & S^1 & \end{array}$$

where t_1' is transverse regular to $K \times T^{n-2}$ and t_2 is a homotopy equivalence.

We similarly split t_2 and find a t_3, etc. Finally, after n steps we obtain the desired splitting of t.

Now each of the above steps is relative (Siebenmann's $M \times \mathbb{R}$ theorem and Farrell's Fibring Theorem).

The desired splitting over a manifold complex may then be constructed inductively over the "cells". The only (and very crucial) requirement is that each manifold encountered has dimension ≥ 5 and free abelian π_1.

Proof of Theorem H (contd.)

We may assume that $\theta_h : E \longrightarrow M \times \mathbb{R}^n$ is a topological bundle map (by increasing n if necessary).

Now notice that (any $\mathbb{Z}_n$-manifold)$\times \mathbb{C}\mathbb{P}^4$ has the structure of a manifold complex satisfying the hypotheses of the *Splitting Lemma*. Thus the splitting invariant of θ_h along (characteristic variety)$\times \mathbb{C}\mathbb{P}^4$ is zero.

Proof of Corollaries.

Corollary 1 follows from Theorem 3.

Corollary 2 follows from the definition of θ_h.

Corollary 3 follows from Theorem 4.

Corollary 4 follows from Theorems 5, 3 and 4.

Lashof and Rothenberg [LR] have proved the Hauptvermutung for 3–connected manifolds by deforming the 3–connective covering of

$$H \ : \ TOP/PL \longrightarrow F/PL$$

to zero. The argument is somewhat like that of the splitting lemma.

Application to complex projective space

We will apply the general theory to the special case of complex projective space. We choose this example because (a) the results have immediate applications to the theory of free S^1-actions on homotopy spheres, (b) $\mathbb{CP}^n$ is interesting enough to illustrate certain complications in the theory, and finally (c) certain simplifications occur to make the theory especially effective in this case.

We illustrate the last point first. We assume $n > 2$ throughout.

Theorem 8. (i) *Any self-homotopy equivalence of* $\mathbb{CP}^n$ *is homotopic to the identity or the conjugation.*
(ii) *Any self-piecewise linear homeomorphism of* $\mathbb{CP}^n$ *is weakly isotopic to the identity or the conjugation.*

Definition 5. If M is homotopy equivalent to $\mathbb{CP}^n$, we call a generator of $H^2(M;\mathbb{Z})$ **a** c**-orientation of** M.

Corollary 1. *The group of concordance classes of h-triangulations of* $\mathbb{CP}^n$ *is canonically isomorphic to the set of* PL*-homeomorphism classes of piecewise linear homotopy* $\mathbb{CP}^n$*'s.*

Corollary 2. *The set of concordance classes of h-smoothings of* $\mathbb{CP}^n$ *is canonically isomorphic to the set of c-oriented diffeomorphism classes of smooth homotopy* $\mathbb{CP}^n$*'s.*

Corollary 3. *The group of concordance classes of smoothings of* $\mathbb{CP}^n$ *is canonically isomorphic to the set of c-oriented diffeomorphism classes of smooth manifolds homeomorphic (or* PL*-homeomorphic) to* $\mathbb{CP}^n$*.*

Proof of Theorem 8.

(i) Theorem 8 (i) follows from the fact that $\mathbb{CP}^n$ is the $(2n+1)$-skeleton of $K(\mathbb{Z},2) = \mathbb{CP}^\infty$.

(ii) Any PL-homeomorphism $P : \mathbb{CP}^n \longrightarrow \mathbb{CP}^n$ is homotopic to the identity or the conjugation by (i). Choose (mod $\mathbb{CP}^n \times I$) such a homotopy H and try to deform it to a weak isotopy (mod $\mathbb{CP}^n \times \partial I$)

$$H \ : \ \mathbb{CP}^n \times I \longrightarrow \mathbb{CP}^n \times I \ .$$

Such a deformation is obstructed (according to Theorem 1) by cohomology classes in

$$H^*(\mathbb{CP}^n \times (I, \partial I); P_*)\ .$$

But these groups are all zero. This proves Theorem 8 (ii).

Proof of Corollaries.

Corollaries 2 and 3 follow immediately from Theorem 8 (i) and (ii) and the definitions.

Corollary 1 follows from the additional fact that

$$\begin{array}{ccc} \mathbb{CP}^n & \overset{f}{\searrow} & \\ {\scriptstyle\text{conjugation}}\downarrow & & F/PL \\ \mathbb{CP}^n & \underset{f}{\nearrow} & \end{array}$$

is homotopy commutative for any f. The corollaries show that the three groups

$$(\mathbb{CP}^n, PL/O)\ \ ,\ \ (\mathbb{CP}^n, F/O)\ \ ,\ \ \text{and}\ \ (\mathbb{CP}^n, F/PL)$$

solve the correct problems.

Theorem 9. *The characteristic variety of* $\mathbb{CP}^n$ *may be taken to be*

$$V\ =\ \mathbb{CP}^2 \cup \mathbb{CP}^3 \cup \ldots \cup \mathbb{CP}^{n-1} \longrightarrow \mathbb{CP}^n\ .$$

Thus any PL-manifold M homotopy equivalent to $\mathbb{CP}^n$ is determined uniquely by choosing any homotopy equivalence $g : M \longrightarrow \mathbb{CP}^n$ and calculating the splitting invariant of g along V. Furthermore *all such invariants are realizable.*

The set of PL-homeomorphism classes of such M is therefore canonically isomorphic to

$$\begin{array}{lll} \mathbb{Z} & \text{for} & \mathbb{CP}^3 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for} & \mathbb{CP}^4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & \text{for} & \mathbb{CP}^5 \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for} & \mathbb{CP}^6 \\ \vdots & \vdots & \vdots \end{array}$$

etc.

Remark. Any $\mathbb{CP}^n$ admits a c-orientation reversing PL-homeomorphism, i.e. a piecewise linear conjugation. This follows from Corollary 1.

Remark. The characteristic variety V does not contain $\mathbb{CP}^1$ because $A = \ker Sq^2 \subseteq H^2(\mathbb{CP}^n;\mathbb{Z}_2) = 0$. In fact, the splitting obstruction of g along $\mathbb{CP}^1$ (i.e. Kervaire invariant of $g^{-1}(\mathbb{CP}^1)$) is just the splitting obstruction of g along $\mathbb{CP}^2$ taken modulo 2 (i.e. $\frac{1}{8}$(Index $(g^{-1}\mathbb{CP}^2) - 1$)(modulo 2)).

Remark. $hT(\mathbb{CP}^n)$ has another group structure coming from the isomorphism

$$hT(\mathbb{CP}^n) \;\cong\; (\mathbb{CP}^n - \{\mathrm{pt}\}, F/PL) \;\cong\; (\mathbb{CP}^{n-1}, F/PL)\ .$$

If we denote the F/PL-structure by $\otimes$ and the characteristic variety group structure by $+$ then the operation $a \circ b = (a \otimes b) - (a + b)$ is a multiplication.

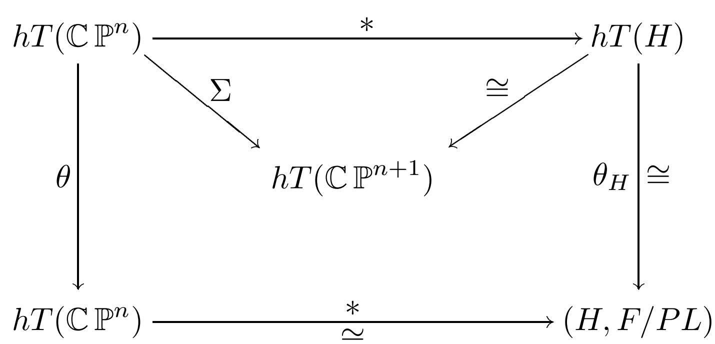
The operations $\circ$ and $+$ make $hT(\mathbb{CP}^n)$ into a commutative associative ring.

The ring $hT(\mathbb{CP}^n) \otimes \mathbb{Z}_{(\mathrm{odd})}$ has *one* generator η obtained by *suspending* the additive generator of $hT(\mathbb{CP}^3) = \mathbb{Z}$. The elements $\eta, \eta^2, \ldots, \eta^{[\frac{n-1}{2}]}$ span $hT(\mathbb{CP}^n) \otimes \mathbb{Z}_{\mathrm{odd}}$ additively.

Remark. A suspension map $\Sigma : hT(\mathbb{CP}^n) \longrightarrow hT(\mathbb{CP}^{n+1})$ is defined by

$$(g : M \longrightarrow \mathbb{CP}^n) \;\longmapsto$$
$$(\overline{g} : g^* \text{ (line bundle)} \cup \text{cone on boundary} \longrightarrow \mathbb{CP}^{n+1})\ .$$

If H denotes the total space of the canonical D^2-bundle over $\mathbb{CP}^n$ then we have the diagram



with $*$ given by the induced bundle.[1]

Thus the image of Σ is isomorphic to

$$\mathrm{image}(\theta) \;=\; \ker(\mathcal{S} : [\mathbb{CP}^n, F/PL] \longrightarrow P_n)$$

by Theorem 3.

Corollary. *An element in $hT(\mathbb{CP}^{n+1})$ is a suspension iff its top splitting invariant is zero.*

---

[1] θ_H is an isomorphism because $\pi_1(H) = \pi_1(\partial H) = 0$.

(Note: When we suspend elements of $hT(\mathbb{C}\,\mathbb{P}^n)$ we merely add zeroes to the string of splitting invariants.)

Smoothing elements of $hT(\mathbb{C}\,\mathbb{P}^n)$

One interesting problem is to determine which elements of $hT(\mathbb{C}\,\mathbb{P}^n)$ are determined by smoothable manifolds.

For example,

$$(0,1) \in hT(\mathbb{C}\,\mathbb{P}^4)\ ,$$
$$(0,0,0,0,0,1) \in hT(\mathbb{C}\,\mathbb{P}^8)\ ,\ \text{and}$$
$$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1) \in hT(\mathbb{C}\,\mathbb{P}^{16})\ \text{are.}$$

In fact, these manifolds are stably PL-homeomorphic to the corresponding projective spaces.

A non-smoothable example is provided by

$$(0,0,0,0,1) \in hT(\mathbb{C}\,\mathbb{P}^6)\ .$$

In fact any element of $hT(\mathbb{C}\,\mathbb{P}^n)$ *with fifth invariant non-zero is non-smoothable.*[1]

Also any suspension of a non-smoothable homotopy $\mathbb{C}\,\mathbb{P}^n$ is likewise.

Understanding which $(4K+2)$-invariants are realizable by smooth manifolds is quite hard in general.

The corresponding problem for the $4K$-invariants is theoretically possible because of Adams' work on $J(\mathbb{C}\,\mathbb{P}^n)$.

This problem is further complicated by the fact that the set $hS(\mathbb{C}\,\mathbb{P}^n)$ has no natural group structure when n is even. Theorem 3 asserts there are exact sequences

$$0 \longrightarrow hS(\mathbb{C}\,\mathbb{P}^{2n+1}) \longrightarrow (\mathbb{C}\,\mathbb{P}^{2n+1}, F/O) \xrightarrow{\mathcal{S}_1} \mathbb{Z}_2$$
$$0 \longrightarrow hS(\mathbb{C}\,\mathbb{P}^{2n}) \longrightarrow (\mathbb{C}\,\mathbb{P}^{2n}, F/O) \xrightarrow{\mathcal{S}_2} \mathbb{Z}\ .$$

$\mathcal{S}_1$ is a homomorphism, but $\mathcal{S}_2$ is not.

If we consider homotopy *almost smoothings* of $\mathbb{C}\,\mathbb{P}^n$ we do get a group

$$h^+S(\mathbb{C}\,\mathbb{P}^n)\ \cong\ (\mathbb{C}\,\mathbb{P}^{n-1}, F/O)\ ,$$

[1] This follows from the fact that the 10-dimensional Kervaire manifold is not a PL-boundary (mod 2).

and

$$h^+S(\mathbb{C}\,\mathbb{P}^n) \xrightarrow{\text{natural}} hT(\mathbb{C}\,\mathbb{P}^n)$$
$$hS(\mathbb{C}\,\mathbb{P}^n) \xrightarrow{\text{natural}} hT(\mathbb{C}\,\mathbb{P}^n)$$

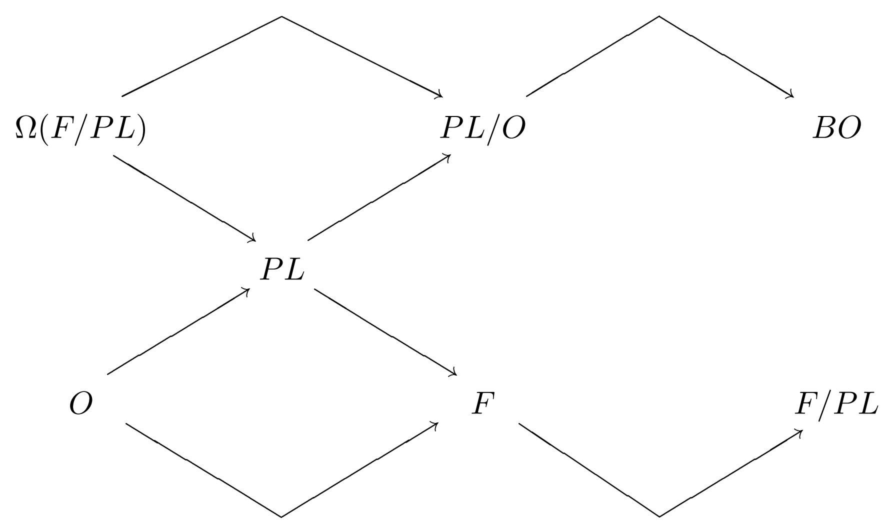
are homomorphisms with the $\otimes$ structure on $hT(\mathbb{C}\,\mathbb{P}^n)$.

It would be interesting to describe all these group structures geometrically for $\mathbb{C}\,\mathbb{P}^n$.

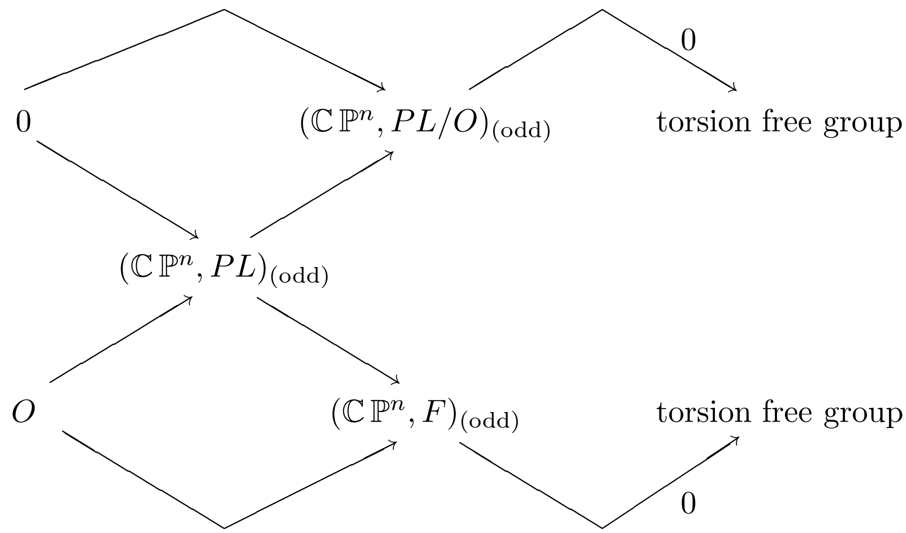
Corollary 3 asserts that the set of c-oriented equivalence classes of differentiable structures on $\mathbb{C}\,\mathbb{P}^n$ is isomorphic to $[\mathbb{C}\,\mathbb{P}^n, PL/O]$, a finite group.

We can calculate this group in another way if we ignore 2-torsion.

Proposition. $[\mathbb{C}\,\mathbb{P}^n, PL/O]$ *is isomorphic to the zeroth stable cohomotopy group of* $\mathbb{C}\,\mathbb{P}^n$ *modulo 2-torsion.*
Proof. We apply $(\mathbb{C}\,\mathbb{P}^n, \;\;) \otimes \mathbb{Z}[\frac{1}{2}]$ to the diagram



and obtain



Thus

$$\begin{aligned}(\mathbb{C}\mathbb{P}^n, PL/O)_{(\mathrm{odd})} &\cong (\mathbb{C}\mathbb{P}^n, PL)_{(\mathrm{odd})} \\ &\cong (\mathbb{C}\mathbb{P}^n, F)_{(\mathrm{odd})} \equiv \pi_s^0(\mathbb{C}\mathbb{P}^n)_{(\mathrm{odd})} \,.\end{aligned}$$

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