

# Generalisations and applications of block bundles

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## Introduction

Rourke and Sanderson [10] introduced the idea of a **block bundle**. They used block bundles in the  $PL$  (piecewise linear) category as a substitute for the normal bundles of differential topology. Their block bundles had fibre  $I^q$  (the unit cube in  $q$ -dimensional space).

We generalise the idea to allow any compact  $PL$  manifold  $F$  as fibre. Chapter I sets up the theory; in particular, it is shown that there is a classifying space  $B\widetilde{PL}_F$  for block bundles with fibre  $F$ .

In Chapter II we compare block bundles with Hurewicz fibrations. Let  $F$  be a compact  $PL$  manifold with boundary  $\partial F$ , and let  $BG_F$  classify Hurewicz fibrations with fibre  $(F, \partial F)$ . We produce a map  $\chi : B\widetilde{PL}_F \rightarrow BG_F$ , arising from a natural transformation of bundle functors.

We wish to obtain information about  $B\widetilde{PL}_F$ ; in fact we can study  $BG_F$  (which is purely homotopy theoretic) and the fibre  $G_F/\widetilde{PL}_F$  of  $\chi$ . In Chapter III we construct a map  $\theta : G_F/\widetilde{PL}_F \rightarrow (G/PL)^F$ , where  $G/PL$  is the space studied in Sullivan's thesis under the name  $F/PL$ , and  $(G/PL)^F$  is the space of all unbased maps from  $F$  to  $G/PL$ . Theorems 5,6 show that, under suitable conditions,  $\theta$  is almost a homotopy equivalence. For these results it is essential to work with block bundles rather than fibre bundles.

Sullivan shows in his thesis (see [13] for a summary) that  $G/PL$  is closely related to the problem of classifying  $PL$  manifolds homotopy equivalent to a given manifold. Therefore it is important to have information about the homotopy type of  $G/PL$ . In Chapter IV we apply Theorem 5 to show that  $\Omega^4(G/PL)$  is homotopy equivalent to  $\Omega^8(G/PL)$ ; it is almost true that  $G/PL$  is homotopy equivalent to  $\Omega^4(G/PL)$ .

In Chapter V (which is almost independent of the earlier chapters) we show that, with certain restrictions on the base-space, a block bundle with fibre  $\mathbb{R}^q$  which is topologically trivial is necessarily piecewise linearly trivial. It follows from this (again using the results of [13]) that the Hauptvermutung is true for closed 1-connected  $PL$  manifolds  $M$  with  $\dim M \geq 5$  and  $H^3(M; \mathbb{Z}_2) = 0$ .

I should like to thank Professor C.T.C.Wall for suggesting the study of generalized block bundles and for much encouragement. I am also very grateful to Dr. D.P.Sullivan for several conversations during the summer of 1966.

## Origins of the ideas

Chapter I is based on §1 of [10]; the definitions and technical details are new, but the general plan is similar. The use of block bundles with arbitrary fibres was suggested to me by Professor Wall.

Chapter II is mainly technical, and new as far as I know.

Chapter III generalizes results in Sullivan's thesis (but the proofs are based on the references given rather than on Sullivan's work).

The result and method of proof in Chapter IV is new, as far as I know.

I believe that Sullivan\* has a stronger result than Theorem 8 of Chapter V, but have not seen his proofs. I proved Theorem 8 before hearing of Sullivan's latest result. My proof is an extension of the idea of [16].

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\* See D.P.Sullivan, **On the Hauptvermutung for manifolds**, Bull. Amer. Math. Soc. 73 (1967) 598-600. The theorem announced there includes ours, but the proof seems somewhat different.

## I. Block Bundles

A **polyhedron** is a topological space together with a maximal family of *PL* related locally finite triangulations. A **cell complex**  $B$  is a collection of cells *PL* embedded in a polyhedron  $X$  such that :

- (1)  $B$  is a locally finite covering of  $X$ ,
- (2) if  $\beta, \gamma \in B$  then  $\partial\beta, \beta \cap \gamma$  are unions of cells of  $B$ ,
- (3) if  $\beta, \gamma$  are distinct cells of  $B$ , then  $\text{Int } \beta \cap \text{Int } \gamma = \emptyset$ .

We write  $|B|$  for  $X$  and do not distinguish between a cell  $\beta$  of  $B$  and the subcomplex it determines. A cell complex  $B'$  is a **subdivision** of  $B$  if  $|B'| = |B|$  and every cell of  $B$  is a union of cells of  $B'$ . A **based polyhedron** is a polyhedron with a preferred base-point; a **based cell complex** is a cell complex with a preferred vertex. All base-points will be denoted by 'bpt'.

Let  $F$  be a polyhedron and let  $B$  be a based cell complex. A **block bundle**  $\xi$  over  $B$  with **fibre**  $F$  consists of a polyhedron  $E(\xi)$  (the **total space** of  $\xi$ ) with a closed sub-polyhedron  $E_\beta(\xi)$  for each  $\beta \in B$  and a *PL* homeomorphism  $b(\xi) : F \longrightarrow E_{\text{bpt}}(\xi)$ , such that :

- (1)  $\{E_\beta(\xi) | \beta \in B\}$  is a locally finite covering of  $E(\xi)$ ,
- (2) if  $\beta, \gamma \in B$  then

$$E_\beta(\xi) \cap E_\gamma(\xi) = \bigcup_{\delta \subset \beta \cap \gamma} E_\delta(\xi) ,$$

- (3) if  $\beta \in B$ , there is a *PL* homeomorphism  $h : F \times \beta \longrightarrow E_\beta(\xi)$  such that

$$h(F \times \gamma) = E_\gamma(\xi) \quad (\gamma \subset \partial\beta) .$$

If  $\xi$  is a block bundle over  $B$  and  $B_0$  is a subcomplex of  $B$ , the **restriction**  $\xi|_{B_0}$  is defined by

$$E(\xi|_{B_0}) = \bigcup_{\beta \in B_0 \cup \{\text{bpt.}\}} E_\beta(\xi) ,$$

$$E_\beta(\xi|_{B_0}) = E_\beta(\xi) , \quad b(\xi|_{B_0}) = b(\xi) .$$

Note that  $\xi|_{B_0}$  is a block bundle over  $B_0 \cup \{\text{bpt.}\}$ , not necessarily over  $B_0$  itself.

If  $\xi, \eta$  are block bundles over  $B$ , an **isomorphism**  $h : \xi \longrightarrow \eta$  is a *PL* homeomorphism  $h : E(\xi) \longrightarrow E(\eta)$  such that

$$hE_\beta(\xi) = E_\beta(\eta) \quad (\beta \in B) , \quad hb(\xi) = b(\eta) .$$

A particular block bundle  $\epsilon$  over  $B$  is obtained by setting

$$\begin{aligned} E(\epsilon) &= F \times B , \quad E_\beta(\epsilon) = F \times \beta , \\ b(\epsilon) &= 1 \times \text{bpt} : F \longrightarrow F \times \text{bpt} . \end{aligned}$$

A trivial block bundle is one isomorphic to  $\epsilon$ ; an isomorphism  $h : \epsilon \longrightarrow \xi$  is a **trivialisation** of  $\xi$ . It follows from condition (3) that  $\xi|\beta$  is trivial for each  $\beta \in B$ .

Let  $B, C$  be based cell complexes and let  $\xi$  be a block bundle over  $B$ . Define a block bundle  $\xi \times C$  over  $B \times C$  by

$$E(\xi \times C) = E(\xi) \times C, \quad E_{\beta \times \gamma}(\xi \times C) = E_{\beta}(\xi) \times \gamma$$

(for cells  $\beta \in B, \gamma \in C$ ) and  $b(\xi \times C) = b(\xi) \times \text{bpt}$ .

**Lemma 1.** *Suppose  $|B| = \beta$ , where  $\beta$  is an  $n$ -cell of  $B$ , and let  $\gamma$  be an  $(n-1)$ -cell over  $B$ . If  $\xi$  and  $\eta$  are block bundles over  $B$ , any isomorphism  $h : \xi|(\partial\beta - \gamma) \longrightarrow \eta|(\partial\beta - \gamma)$  can be extended to an isomorphism  $h : \xi \longrightarrow \eta$ .*

**Proof.** Since  $\xi = \xi|\beta, \eta = \eta|\beta$ ,  $\xi$  and  $\eta$  are both trivial. Let  $k$  and  $l$  be trivialisations of  $\xi, \eta$ , respectively. Then a *PL* homeomorphism

$$l^{-1}hk : F \times (\partial\beta - \gamma) \longrightarrow F \times (\partial\beta - \gamma)$$

is defined.

Choose a *PL* homeomorphism

$$f : (\partial\beta - \gamma) \times I \longrightarrow B$$

such that  $f_0 : (\partial\beta - \gamma) \longrightarrow B$  is the inclusion, and let

$$g = 1 \times f : F \times (\partial\beta - \gamma) \times I \longrightarrow F \times \beta.$$

The required extension of  $f$  is given by

$$h = lg(l^{-1}hk \times I)g^{-1}k^{-1} : E(\xi) \longrightarrow E(\eta).$$

**Lemma 2.** *Let  $B$  be a based cell complex and take  $\text{bpt} \times 0$  as base-point for  $B \times I$ . If  $\xi, \eta$  are block bundles over  $B \times I$ , then any isomorphism*

$$h : \eta|(B \times 0) \cup (\text{bpt} \times I) \longrightarrow \xi|(B \times 0) \cup (\text{bpt} \times I)$$

*can be extended to an isomorphism  $h : \xi \longrightarrow \eta$ .*

**Proof.** Write  $B^r$  for the  $r$ -skeleton of  $B$ , and let  $C^r = (B \times 0) \cup (B^r \times I)$ . Suppose inductively that  $h$  can be extended to an isomorphism  $h : \xi|C^r \longrightarrow \eta|C^r$ ; the induction starts trivially with  $r = 0$ . Let  $\beta$  be an  $(r+1)$ -cell of  $B$ . By Lemma 1,

$$h : \eta|(\beta \times 0) \cup (\partial\beta \times I) \longrightarrow \xi|(\beta \times 0) \cup (\partial\beta \times I)$$

can be extended to an isomorphism  $h : \xi|(\beta \times I) \longrightarrow \eta|(\beta \times I)$ . Thus we have defined an isomorphism

$$h : \xi|C^r \cup (\beta \times I) \longrightarrow \eta|C^r \cup (\beta \times I).$$

Do this for all  $r$ -cells of  $B$  to obtain

$$h : \xi|C^{r+1} \longrightarrow \eta|C^{r+1}$$

extending the given isomorphism. The Lemma now follows by induction.

Let  $\xi$  be a block bundle over  $B$  and let  $B'$  be a subdivision of  $B$ . A block bundle  $\xi'$  over  $B'$  is a subdivision of  $\xi$  if  $E(\xi') = E(\xi)$ ,  $E_{\beta'}(\xi') \subset E_{\beta}(\xi)$  (for all cells  $\beta' \in B'$ ,  $\beta \in B$  with  $\beta' \subset \beta$ ) and  $b(\xi') = b(\xi)$ .

**Theorem 1.** *Let  $B'$  be a subdivision of a cell complex  $B$ . Any block bundle over  $B'$  is a subdivision of some block bundle over  $B$ . Any block bundle  $\xi$  over  $B$  has a subdivision over  $B'$ , and any two subdivisions of  $\xi$  over  $B'$  are isomorphic.*

**Proof.** First we prove the following propositions together by induction on  $n$ .

$P_n$  : If  $|B|$  is homeomorphic to an  $n$ -cell, then any block bundle over  $B$  is trivial.

$Q_n$  : Let  $\dim B \leq n$  and let  $B_0$  be a subcomplex of  $B$ . Let  $B'$  be a subdivision of  $B$ , inducing subdivision  $B'_0$  of  $B_0$ . Let  $\xi$  be a block bundle over  $B$  and let  $\xi'_0$  be a subdivision of  $\xi|_{B_0}$  over  $B'_0$ . Then there is a subdivision  $\xi'$  of  $\xi$  over  $B'$  such that  $\xi'_0 = \xi'|_{B'_0}$ .

Observe that  $P_0$  and  $Q_0$  are both true. We shall prove that  $Q_n \implies P_n$  and  $P_n \& Q_n \implies Q_{n+1}$ .

**Proof that  $Q_n \implies P_n$ .** Suppose  $|B|$  is homeomorphic to an  $n$ -cell, and let  $\xi$  be a block bundle over  $B$ . Since  $|B|$  is collapsible, there is a simplicial subdivision  $B'$  of  $B$  which collapses simplicially to the base point [19]. Assuming  $Q_n$ , there is a subdivision  $\xi'$  of  $\xi$  over  $B'$ . It is enough to prove that  $\xi'$  is trivial.

Let

$$B' = K_k \searrow^s K_{k-1} \searrow^s \dots \searrow^s K_0 = \{\text{bpt.}\}$$

be a sequence of elementary simplicial collapses. Suppose inductively that  $\xi'|_{K_r}$  is trivial; the induction starts with  $r = 0$ . Write

$$K_{r+1} = K_r \cup \Delta, \quad K_r \cap \Delta = \Lambda,$$

where  $\Delta$  is a simplex of  $K_{r+1}$  and  $\Lambda$  is the complement of a principal simplex in  $\partial\Delta$ . Let  $h : F \times K_r \rightarrow E(\xi'|_{K_r})$  be a trivialisaton of  $\xi'|_{K_r}$ . By Lemma 1,  $h|_{F \times \Lambda}$  extends to a trivialisaton of  $\xi|\Delta$ . Thus we obtain a trivialisaton of  $\xi''|_{K_{r+1}}$ . By induction,  $\xi'$  is trivial, as required.

**Proof that  $P_n \& Q_n \implies Q_{n+1}$ .** Suppose  $B, B_0, B', \xi, \xi'_0$  satisfy the hypotheses of  $Q_{n+1}$ . If  $A$  is any subcomplex of  $B$ , we write  $A'$  for the subdivision of  $A$  induced by  $B'$ . Let  $B_1 = B_0 \cup B^n$ , assuming  $Q_n$  there is a subdivision  $\xi'_1$  of  $\xi|_{B_1}$  over  $B'_1$  such that  $\xi'_0|(B_0 \cap B^n)' = \xi'_1|(B_0 \cap B^n)'$ . Let  $\beta$  be an  $(n+1)$ -cell of  $B - B_0$ , and let

$\gamma$  be an  $n$ -cell of  $B$  contained in  $\partial\beta$ . Since  $|\partial\beta - \gamma|$  is homeomorphic to an  $n$ -cell,  $\xi'_1|(\partial\beta - \gamma)'$  is trivial by  $P_n$ . Let  $h$  be a trivialisation of  $\xi'_1|(\partial\beta - \gamma)'$ ; a fortiori,  $h$  is a trivialisation of  $\xi|(\partial\beta - \gamma)$ .

By Lemma 1,  $h$  extends to a trivialisation of  $\xi|\beta$ . Let  $C$  be the cell complex consisting of  $\beta, \gamma$  and the cells of  $(\partial\beta - \gamma)'$ . Define a block bundle  $\eta$  over  $C$  by

$$E_\beta(\eta) = E_\beta(\xi), \quad E_\gamma(\eta) = E_\gamma(\xi)$$

and  $E_{\delta'}(\eta) = E_{\delta'}(\xi'_1)$  for each cell  $\delta'$  of  $(\partial\beta - \gamma)'$ . Then  $k$  is a trivialisation of  $\eta$ , so  $\eta$  satisfies condition (3) in the definition of block bundle.

Let  $\delta'$  be an  $n$ -cell of  $(\partial\beta - \gamma)'$ , so  $|\partial\beta' - \delta'|$  is homeomorphic to an  $n$ -cell. Assuming  $P_n, \xi'_1|(\partial\beta' - \delta')$  is trivial; let  $h'$  be a trivialisation. A fortiori,  $h'$  is a trivialisation of  $\eta|(\partial\beta' - \delta')$ .

By Lemma 1,  $h'$  extends to a trivialisation  $k'$  of  $\eta$ . In fact,  $k'|F \times \partial\beta'$  is a trivialisation of  $\xi'_1|\partial\beta'$ , because  $k'$  extends  $h'$  and  $k'(F \times \delta') = E_{\delta'}(\xi'_1)$ . To extend  $\xi'_1|\partial\beta'$  to a subdivision  $\xi'|\beta'$  of  $\xi|\beta$ , we define  $E_{\alpha'}(\xi') = k'(F \times \alpha')$  for each cell  $\alpha'$  of  $\beta'$ .

Do this for all  $(n+1)$ -cells of  $B - B_0$ , and define  $\xi'|\beta = \xi'_0|\beta$  for each  $(n+1)$ -cell  $\beta$  of  $B_0$ . We obtain a subdivision  $\xi'$  of  $\xi$  over  $B'$  such that  $\xi'_0 = \xi'|B'_0$ , as required.

By induction,  $P_n$  and  $Q_n$  are true for all  $n$ . Let  $B$  be any based cell complex and let  $B'$  be a subdivision of  $B$ .

Let  $\xi'$  be a block bundle over  $B'$ . We define a block bundle  $\xi$  over  $B$  with  $E(\xi) = E(\xi')$  by setting  $E_\beta(\xi) = E(\xi'|\beta')$  (where  $\beta'$  is the subdivision of  $\beta$  induced by  $B'$ ) for each cell  $\beta$  of  $B$ . This clearly satisfies conditions (1),(2) in the definition of block bundle. By  $P_n$ ,  $\xi'|\beta'$  is trivial, so  $\xi$  also satisfies condition (3). Clearly,  $\xi'$  is a subdivision of  $\xi$ .

If  $\xi$  is a block bundle over  $B$ , it follows from  $Q_n$  (by induction on the skeleton of  $B$ ) that  $\xi$  has a subdivision over  $B'$ . Let  $\xi'_0, \xi'_1$  be two such subdivisions. Recall that  $\eta = \xi \times I$  is a block bundle over  $B \times I$ . Define a block bundle  $\eta'_0$  over  $B' \times \partial I$  by  $\xi'_t = \eta'_0|B' \times \{t\}$ , ( $t = 0, 1$ ). Again it follows from  $Q_n$  that  $\eta$  has a subdivision  $\eta'$  over  $B' \times I$  such that  $\eta'_0 = \eta'|B' \times \partial I$ . Observe that  $\eta'|bpt \times I = \xi'_0 \times I|bpt \times I$ . By Lemma 2, the identity isomorphism

$$\eta'|((B \times 0) \cup (bpt \times I)) \longrightarrow \xi'_0 \times I|((B \times 0) \cup (bpt \times I))$$

extends to an isomorphism  $\eta' \longrightarrow \xi'_0 \times I$ ; it follows that  $\xi'_1 \cong \xi'_0$ . This completes the proof of Theorem 1.

Let  $X$  be a polyhedron and let  $B, C$  be cell complexes with  $|B| = |C| = X$ ; suppose all three have the same base-point. Let  $\xi, \eta$  be block bundles over  $B, C$  respectively. We call  $\xi, \eta$  **equivalent** if, for some common subdivision  $D$  of  $B, C$ , the subdivision  $\xi$  over  $D$  is isomorphic to the subdivision of  $\eta$  over  $D$ . This relation is clearly reflexive and symmetric; by Theorem 1 it is also transitive.

Let  $I_F(X)$  be the set of equivalence classes of block bundles over cell complexes  $B$  with  $|B| = X$ . It is easily checked that, if  $|B| = X$ , then each member of  $I_F(X)$  is represented by a unique isomorphism class of block bundles over  $B$ .

Suppose  $X, Y$  are polyhedra and let  $y \in I_F(Y)$ . Let  $B, C$  be cell complexes with  $|B| = X, |C| = Y$ , and let  $\eta$  be a block bundle over  $C$  representing  $y$ . If  $p_2 : X \times Y \rightarrow Y$  is the projection, let  $p_2^*(y) \in I_F(X \times Y)$  be the equivalence class of  $B \times \eta$ .

If  $i : X \rightarrow Y$  is a closed based  $PL$  embedding, let  $C'$  be a subdivision of  $C$  with a subcomplex  $D'$  such that  $|D'| = i(X)$ . Let  $\eta'$  be a subdivision of  $\eta$  over  $C'$ , and let  $\xi' = \eta'|D'$ . It follows from Theorem 1 that the equivalence class  $x' \in I_F(i(X))$  of  $\xi'$  depends only on  $y$ . Let  $i^*(y) \in I_F(X)$  correspond to  $x'$  via the  $PL$  homeomorphism  $i : X \rightarrow i(X)$ . The next lemma will enable us to define  $f^* : I_F(Y) \rightarrow I_F(X)$  for any based  $PL$  map  $f : X \rightarrow Y$ .

**Lemma 3.** *Let  $X, Y, V, W$  be polyhedra and let  $i : X \rightarrow V \times Y, j : X \rightarrow W \times Y$  be closed based  $PL$  embeddings such that  $p_2 i = p_2 j : X \rightarrow Y$ . Then*

$$i^* p_2^* = j^* p_2^* : I_F(Y) \rightarrow I_F(X) .$$

**Proof.** Let  $k : X \rightarrow V \times W \times Y$  be defined by

$$\begin{aligned} p_{13} k &= i : X \rightarrow V \times Y , \\ p_{23} k &= j : X \rightarrow W \times Y . \end{aligned}$$

In the diagram

$$(1) \quad \begin{array}{ccccc} & & I_F(V \times Y) & & \\ & \swarrow i^* & \downarrow p_{13}^* & \nwarrow p_2^* & \\ I_F(X) & \xleftarrow{k^*} & I_F(V \times W \times Y) & \xleftarrow{p_3^*} & I_F(Y) \\ & \searrow j^* & \uparrow p_{23}^* & \swarrow p_2^* & \\ & & I_F(W \times Y) & & \end{array}$$

the right-hand triangles are clearly commutative. We prove that the bottom left-



$$\begin{array}{c}
I_F(X) \\
\uparrow i^* \\
I_F(V \times Y) \\
\uparrow p_2^* \\
I_F(Y) \\
\uparrow p_{23}^* \\
I_F(W \times Z) \\
\uparrow p_2^* \\
I_F(Z)
\end{array}
\quad
\begin{array}{c}
(1 \times j)^* \\
\swarrow \\
I_F(V \times W \times Z) \\
\searrow p_{23}^*
\end{array}$$

This clearly commutes; the right route defines  $f^*g^*$  and the left route defines  $(gf)^*$ . This proves that  $I_F$  is a contravariant functor.

**Theorem 2.** *If  $F$  is compact, then there is a based polyhedron  $\widetilde{BPL}_F$  and an element  $w_I \in I_F(\widetilde{BPL}_F)$  such that  $f \mapsto f^*(w_I)$  defines a natural equivalence  $[\quad, \widetilde{BPL}_F] \longrightarrow I_F$ .*

**Proof.** First we show that  $I_F$  satisfies the following axioms:

- (1) If  $X, Y$  are based polyhedra and  $f_0 \simeq f_1 : X \longrightarrow Y$  by a based  $PL$  homotopy, then  $f_0^* = f_1^* : I_F(Y) \longrightarrow I_F(X)$ .
- (2) If  $X_i$  is a based polyhedron ( $i \in I$ ) and  $u_j : X_j \longrightarrow \bigvee_{i \in I} X_i$  is the inclusion, then  $\prod_{i \in I} u_i^* : I_F(\bigvee_{i \in I} X_i) \longrightarrow \prod_{i \in I} I_F(X_i)$  is an isomorphism.
- (3) Suppose that  $X, X_0, X_1, X_2$  are polyhedra with  $X = X_1 \cup X_2, X_0 = X_1 \cap X_2$ , and that the inclusions  $u_i : X_0 \longrightarrow X_i, v_i : X_i \longrightarrow X$  are based maps. If  $x_i \in I_F(X_i), (i = 1, 2)$  satisfy  $u_1^*(x_1) = u_2^*(x_2)$ , then there exists  $x \in I_F(X)$  with  $x_i = v_i^*(x), (i = 1, 2)$ .
- (4)  $I_F(S^0)$  is a single point and  $I_F(S^n)$  is countable where  $S^n$  denotes the boundary of an  $(n + 1)$ -cell.

**Proof of (1).** This follows from Lemma 2 and a short argument about base-points.

**Proof of (2).** Let  $B_i$  be a cell complex with  $|B_i| = X_i$ . Let  $x \in \prod_{i \in I} X_i$  and let  $\xi_j$  be a block bundle over  $B_j$  representing  $p_j(x)$ . Let  $A = \cup_{i \in I} E(\xi_i), A_0 = \cup_{i \in I} E_{\text{bpt}}(\xi_i), b = \cup_{i \in I} b(\xi)^{-1} : A_0 \longrightarrow F$  and define  $E(\eta) = A \cup_b F$ . If  $\beta$  is a cell of

$\bigvee_{i \in I} B_i$ , then  $\beta$  is a cell of some  $B_j$ , so we can define  $E_\beta(\eta) = E_\beta(\xi_j) \subset E(\eta)$ . Let  $b(\eta) = b(\xi_j) : F \longrightarrow E_{\text{bpt}}(\eta)$ , which is independent of  $j$ . Then  $\eta$  is a block bundle over  $\bigvee_{i \in I} B_i$ ; let  $y \in I_F(\bigvee_{i \in I} X_i)$  be the class of  $\eta$ . Then  $x \mapsto y$  defines an inverse to  $\prod_{i \in I} u_i^*$ , so (2) is proved.

**Proof of (3).** Let  $B$  be a cell complex with  $|B| = X$  and with subcomplexes  $B_0, B_1, B_2$  such that  $|B_i| = X_i$  ( $i = 0, 1, 2$ ). Let  $\xi_i$  be a block bundle over  $B_i$  representing  $x_i$  ( $i = 1, 2$ ). Since  $u_1^*(x_1) = u_2^*(x_2)$ , there is an isomorphism  $h : \xi_1|_{B_0} \longrightarrow \xi_2|_{B_0}$ . Let  $E(\xi) = E(\xi_1) \cup_h E(\xi_2)$ , let  $E_\beta(\xi) = E_\beta(\xi_i)$  if  $\beta \in B_i$  and let  $b(\xi) = b(\xi_1) = b(\xi_2) : F \longrightarrow E_{\text{bpt}}(\xi)$ . Then the class  $x$  of  $\xi$  has the required properties.

**Proof of (4).** Clearly  $I_F(S^0)$  is a single point. Let  $B$  be a cell complex such that  $S^n = |B|$ , and let  $\beta$  be an  $n$ -cell of  $B$ . Any element  $x \in I_F(S^n)$  can be represented by a block bundle  $\xi$  over  $B$ . Let  $k, l$  be trivialisations of  $\xi|_\beta$ ,  $\xi|_{B-\beta}$ , and let  $h = k^{-1}l : F \times \partial\beta \longrightarrow F \times \partial\beta$ .

Since  $F$  is compact, there are finite simplicial complexes  $K, L$  with  $|K| = F \times \beta$ ,  $|L| = F \times (B - \beta)$  and such that  $h$  is simplicial. Clearly the simplicial isomorphism class of the triple  $(K, L, h)$  determines  $x$  completely. But there are only countably many such classes (of triples), so  $I_F(S^n)$  is countable.

Now we can apply Brown's Theorem on representable functors [4] to  $I_F$ . We deduce that there is a countable based  $CW$  complex  $W$  and a natural equivalence  $R : [ \quad, W ] \longrightarrow I_F$ . By a theorem of J. H. C. Whitehead [18], there is a polyhedron  $\widetilde{BPL}_F$  and a homotopy equivalence  $\phi : \widetilde{BPL}_F \longrightarrow W$ . Let  $w_I = R(\phi) \in I_F(\widetilde{BPL}_F)$ ; then the pair  $(\widetilde{BPL}_F, w_I)$  has the required properties.

**Remark.** The compactness of  $F$  was only required to make the classifying space  $\widetilde{BPL}_F$  a polyhedron. If  $F$  were an infinite discrete space (for example), then the space  $W$  constructed above would have uncountable fundamental group.

Our main concern is with block bundles having a compact  $PL$  manifold  $F$  as fibre. If  $\xi$  is such a bundle over a cell complex  $B$ , we can define a block bundle  $\partial\xi$  over  $B$  with fibre  $\partial F$  as follows.

Let  $\beta$  be a cell of  $B$ , let  $k, l$  be trivialisations of  $\xi|_\beta$  and let  $h = k^{-1}l : F \times \beta \longrightarrow F \times \beta$ . Since  $h(F \times \gamma)$  for each  $\gamma \subset \partial\beta$ ,  $h(F \times \partial\beta) = F \times \partial\beta$ . Therefore

$$h(\partial F \times \beta) = \overline{h(\partial(F \times \beta) - F \times \partial\beta)} = \partial F \times \beta ;$$

it follows that  $k(\partial F \times \beta) = l(\partial F \times \beta)$ . Define

$$E_\beta(\partial\xi) = k(\partial F \times \beta) ,$$

where  $k$  is any trivialisation of  $\xi|\beta$ , and define

$$E(\partial\xi) = \bigcup_{\beta \in B} E_\beta(\partial\xi), \quad b(\partial\xi) = b(\xi)|\partial F.$$

Then  $\partial\xi$  is a block bundle over  $B$  with fibre  $\partial F$ .

**Lemma 5.** *Suppose that  $|B|, F$  are compact PL manifolds, that  $\partial\beta$  contains the base-point of  $B$  and let  $\xi$  be a block bundle over  $B$  with fibre  $F$ . Then  $E(\xi)$  is a compact PL manifold and  $\partial E(\xi) = E(\partial\xi) \cup E(\xi|\partial B)$ .*

**Proof.** Let  $B'$  be a simplicial division of  $B$  and  $\xi'$  be a subdivision of  $\xi$  over  $B'$ . Clearly  $\partial\xi'$  is then a subdivision of  $\partial\xi$ . If  $p \in E(\xi)$ , then

$$p \in P, \quad \text{where } P = E(\xi'|St(q, B')) - E(\xi'|Lk(q, B'))$$

for some vertex  $q$  of  $B'$ ; we can choose  $q \in \text{Int}B$  unless  $p \in E(\xi|\partial B)$ . Let

$$Q = St(q, B') - Lk(q, B'),$$

so  $Q$  is an open ball if  $p \notin E(\xi|\partial B)$ , and a half-open ball if  $p \in E(\xi|\partial B)$ .

A trivialisation  $k$  of  $\xi'|St(q, B')$  defines a homeomorphism  $k : F \times Q \rightarrow P$  such that  $k(\partial F \times Q) = P \cap E(\partial\xi)$ . Let  $N$  be an open ball neighbourhood of  $p_1 k^{-1}(p)$  in  $F$  if  $p \notin E(\partial\xi)$ , or a half-open ball neighbourhood if  $p \in E(\partial\xi)$ .

If  $p \notin E(\partial\xi) \cup E(\xi|\partial B)$ , then  $k(N \times Q)$  is an open ball neighbourhood of  $p$  in  $E(\xi)$ . If  $p \in E(\partial\xi) \cup E(\xi|\partial B)$ , then  $k(N \times Q)$  is a half-open ball neighbourhood of  $p$ , and  $p \in k(\partial(N \times Q))$ . This proves that  $E(\xi)$  is a PL manifold (obviously compact) with boundary  $E(\partial\xi) \cup E(\xi|\partial B)$ .

## II. Homotopy Properties of Block Bundles

Let  $\xi$  be a block bundle over  $B$  with fibre  $F$ . A **block fibration** for  $\xi$  is a PL map  $\pi : E(\xi) \rightarrow |B|$  such that  $E_\beta(\xi) = \pi^{-1}(\beta)$  for each  $\beta \in B$ . A **block homotopy** for  $\xi$  is a PL map  $H : E(\xi) \times I \rightarrow |B|$  such that, for all  $t \in I$ ,  $H_t : E(\xi) \rightarrow |B|$  is a block fibration for  $\xi$ .

**Lemma 6.** *Any block bundle  $\xi$  has a block fibration, and any two block fibrations for  $\xi$  are block homotopic.*

**Proof.** Write  $B^r$  for the  $r$ -skeleton of  $B$ . There is a unique block fibration  $\pi : E(\xi|B^0) \rightarrow |B^0|$ . Suppose inductively that  $\pi$  can be extended to a block fibration  $\pi : E(\xi|B^r) \rightarrow |B^r|$ , and let  $\beta$  be an  $(r+1)$ -cell of  $B$ . Then  $\pi : E(\xi|\partial\beta) \rightarrow |\partial\beta|$  can be extended to a PL map  $\pi : E(\xi|\beta) \rightarrow |\beta|$  such that  $\pi^{-1}(|\partial\beta|) = E(\xi|\partial\beta)$ . Do this for all  $(r+1)$ -cells of  $B$  to obtain a block fibration  $\pi : E(\xi|B^{r+1}) \rightarrow |B^{r+1}|$

extending the given block fibration. By induction,  $\xi$  has a block fibration; it is obvious that any two block fibrations for  $\xi$  are block homotopic.

Let  $\xi$  be a block bundle over  $B$  with fibre  $F$ , and let  $\pi$  be a block fibration for  $\xi$ . Let

$$\mathcal{E} = \{(x, \psi) : x \in E, \psi : I \longrightarrow |B| \text{ such that } \pi(x) = \psi(0)\},$$

with the compact open topology. Define  $i : E(\xi) \longrightarrow \mathcal{E}$ ,  $p : \mathcal{E} \longrightarrow |B|$  by  $i(x) = (x, \text{constant})$  and  $p(x, \psi) = \psi(1)$ . Then  $i$  is a homotopy equivalence and  $p$  is a Hurewicz fibre map. Let  $\mathcal{F} = p^{-1}(\text{bpt})$  be the fibre of  $p$ .

**Theorem 3.** *The map  $ib(\xi) : F \longrightarrow \mathcal{F}$  is a homotopy equivalence.*

**Proof.** By [9],  $\mathcal{F}$  has the homotopy type of a  $CW$  complex. Choose a component  $\mathcal{F}_1$  of  $\mathcal{F}$ ;  $\mathcal{F}_1$  lies in some component  $\mathcal{E}_0$  of  $\mathcal{E}$ . Let  $E_0$  be the corresponding component of  $E(\xi)$ , and let  $B_0$  be the component of  $B$  containing the base-point. It is easy to see that  $\pi|_{E_0} \longrightarrow B_0$  must be surjective, so  $F_0 = E_0 \cap E_{\text{bpt}}(\xi)$  is non-empty. Choose a base-point for  $\mathcal{F}_0 = \mathcal{E}_0 \cap \mathcal{F}$ .

If  $n \geq 1$ , there is a commutative diagram

$$\begin{array}{ccc} \pi_n(E_0, F_0) & \xrightarrow{i_*} & \pi_n(\mathcal{E}_0, \mathcal{F}_0) \\ & \searrow \pi_* & \swarrow p_* \\ & \pi_n(|B_0|, \text{bpt}) & \end{array}$$

Since  $p : \mathcal{E}_0 \longrightarrow |B_0|$  is a Hurewicz fibration,  $p_*$  is an isomorphism. Using the fact that  $\pi : E_0 \longrightarrow |B_0|$  is a block fibration, we shall prove that  $\pi_*$  is an isomorphism. It will follow that  $i_*$  is an isomorphism; hence there is a unique component  $F_1$  of  $F_0$  with  $i(F_1) \subset \mathcal{F}_1$ . An application of the Five Lemma will show that  $i_* : \pi_r(F_1) \longrightarrow \pi_r(\mathcal{F}_1)$  is an isomorphism for all  $r \geq 1$ , and the Theorem will follow by the Whitehead theorem.

To prove that  $\pi_*$  is surjective, consider an element  $\alpha \in \pi_n(|B_0|, \text{bpt})$ . By subdividing, we may assume that  $B_0$  is a simplicial complex (note that subdivision does not alter the homotopy class of  $\pi : E_0, F_0 \longrightarrow |B_0|, \text{bpt}$ ). Let  $D^n$  be a standard  $n$ -cell. There is a triangulation of  $D^n$  such that  $D^n \searrow^s \text{bpt} \in S^{n-1}$  and  $\alpha$  is represented by a simplicial map  $f : D^n, S^{n-1} \longrightarrow B_0, \text{bpt}$ . Let

$$D^n = K_k \searrow^s K_{k-1} \searrow^s \dots \searrow^s K_0 = \text{bpt}$$

be a sequence of elementary simplicial collapses.

Suppose inductively that there is a map  $g : K_r \rightarrow E_0$  such that, for all  $x \in |K_r|$ ,  $\pi g(x)$  is in the closed carrier of  $f(x)$  in  $B_0$ . We can write  $K_{r+1} = K_r \cup \Delta$ ,  $K_r \cap \Delta = \Lambda$  for some simplex  $\Delta \in K_{r+1}$ . Let  $\Delta_1 = \overline{\Delta - \Lambda}$ , so  $\Delta_1$  is a principal simplex of  $\partial\Delta$ . Let  $\beta = f(\Delta)$ ,  $\beta_1 = f(\Delta_1)$  be the image simplices in  $B_0$ . Then

$$g : \Lambda, \partial\Lambda \longrightarrow E_\beta(\xi), E_{\beta_1}(\xi)$$

is defined. Since  $E_{\beta_1}(\xi)$  is a deformation retract of  $E_\beta(\xi)$ ,  $g$  can be extended to a map

$$g : \Delta, \Delta_1 \longrightarrow E_\beta(\xi), E_{\beta_1}(\xi) .$$

Thus we obtain an extension of  $g$  to  $g : K_{r+1} \rightarrow E_0$  such that, for all  $x \in |K_{r+1}|$ ,  $\pi g(x)$  is in the closed carrier of  $f(x)$  in  $B_0$ .

Now we have completed our induction and have obtained a map  $g : D^n \rightarrow E_0$  such that for all  $x \in D^n$ ,  $\pi g(x)$  is in the closed carrier of  $f(x)$  in  $B_0$ . In particular,  $g(S^{n-1}) \subset F_0$ , so  $g$  represents an element  $\beta \in \pi_n(E_0, F_0)$ . Clearly  $\pi_*\beta = \alpha$ , so  $\pi_*$  is injective as asserted. A similar argument shows that  $\pi_*$  is injective, and the Theorem is proved.

We now restrict  $F$  to be a compact  $PL$  manifold with boundary  $\partial F$ . Let  $X$  be a based polyhedron; a **Hurewicz fibration** over  $X$  with fibre  $(F, \partial F)$  consists of a pair of topological spaces  $(\mathcal{E}, \partial\mathcal{E})$ , a map  $p : \mathcal{E} \rightarrow X$  and a homotopy equivalence of pairs

$$b : F, \partial F \longrightarrow p^{-1}(\text{bpt}), p^{-1}(\text{bpt}) \cap \partial\mathcal{E}$$

such that;

- (1) For all  $x \in X$ ,  $(p^{-1}(x), p^{-1}(x) \cap \partial\mathcal{E}) \simeq (F, \partial F)$ ,
- (2) Given a pair of topological spaces  $A, \partial A$ , a map  $f : A, \partial A \rightarrow \mathcal{E}, \partial\mathcal{E}$  and a homotopy  $G : A \times I \rightarrow X$  such that  $G_0 = pf$ , then there exists a homotopy  $H : A \times I, \partial A \times I \rightarrow \mathcal{E}, \partial\mathcal{E}$  with  $H_0 = f$ ,  $G = pH$ .

Two Hurewicz fibrations  $(\mathcal{E}, \partial\mathcal{E}, p, b)$ ,  $(\mathcal{E}', \partial\mathcal{E}', p', b')$  are **fibre homotopy equivalent** if there are maps

$$h : \mathcal{E}, \partial\mathcal{E} \longrightarrow \mathcal{E}', \partial\mathcal{E}' \quad , \quad h' : \mathcal{E}', \partial\mathcal{E}' \longrightarrow \mathcal{E}, \partial\mathcal{E}$$

and homotopies  $H : h'h \simeq 1$ ,  $H' : hh' \simeq 1$  such that, for all  $t \in I$ ,

$$pH_t = p \quad , \quad H_t b = b \quad , \quad p' H'_t = p' \quad , \quad H'_t b' = b' .$$

We write  $H_F(X)$  for the set of fibre homotopy equivalence classes of Hurewicz fibrations over  $X$  with fibre  $(F, \partial F)$ . The well-known construction for induced fibrations makes  $H_F$  into a contravariant functor from the category of based polyhedra and based  $PL$  maps to the category of based sets. A proof that  $H_F$  is representable is indicated in [4]; the step which is given without proof can be dealt with by the methods of Theorem 3 above. We summarise the conclusion as follows.

**Proposition.** *If  $F$  is a compact PL manifold, then there is a based polyhedron  $BG_F$  and an element  $w_H \in H_F(BG_F)$  such that  $f \mapsto f^*(w_H)$  defines a natural equivalence  $[ \ , BG_F ] \longrightarrow H_F$ .*

**Lemma 7.** *There is a natural transformation  $S : I_F \longrightarrow H_F$ .*

**Construction of  $S$ .** Let  $X$  be a based polyhedron and let  $x \in I_F(X)$ . Let  $B$  be a cell complex with  $|B| = X$ , and let  $\xi$  be a block bundle over  $B$  representing  $x$ . By Lemma 6,  $\xi$  has a block fibration  $\pi : E(\xi) \longrightarrow X$ . Construct  $p : \mathcal{E} \longrightarrow X$  as above, and let  $\partial\mathcal{E} = \{(x, \psi) \in \mathcal{E} : x \in E(\partial\xi)\}$ . It is easily proved that  $p : \mathcal{E}, \partial\mathcal{E} \longrightarrow X$  satisfies part (2) of the definition of Hurewicz fibration. By Theorem 3, part (1) is also satisfied, and

$$ib(\xi) : F, \partial F \longrightarrow p^{-1}(\text{bpt}), p^{-1}(\text{bpt}) \cap \partial\mathcal{E}$$

is a homotopy equivalence. Therefore  $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$  defines an element  $S(\xi, \pi) \in H_F(X)$ .

Let  $\pi'$  be another block fibration for  $\xi$ . Construct  $(\mathcal{E}', \partial\mathcal{E}', p', i')$  from  $\pi'$  as above. Define  $j' : \mathcal{E}', \partial\mathcal{E}' \longrightarrow \mathcal{E}, \partial\mathcal{E}$  by  $j'(x, \psi) = x$ ; then  $j'$  is a homotopy inverse to  $i'$ . Thus  $ij' : \mathcal{E}', \partial\mathcal{E}' \longrightarrow \mathcal{E}, \partial\mathcal{E}$  is a homotopy equivalence of pairs and  $p' \simeq p \cdot ij'$  via a homotopy  $H$  with  $H_t \cdot i'b(\xi) = ib(\xi)$ . It follows from Theorem 6.1 of [5] (modified to take account of base-points and pairs of fibres) that  $(\mathcal{E}', \partial\mathcal{E}', p', i'b(\xi))$  is fibre homotopy equivalent to  $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$ . Therefore  $S(\xi, \pi)$  depends only on  $\xi$ .

If  $\xi'$  is a subdivision of  $\xi$ , then  $S(\xi, \pi) = S(\xi, \pi') = S(\xi', \pi')$  for any block fibrations  $\pi, \pi'$  of  $\xi, \xi'$ . Therefore  $S(\xi, \pi)$  depends only on the equivalence class  $x$  of  $\xi$ ; we write  $S(x) = S(\xi, \pi)$ .

**Naturality of  $S$ .** It is enough to prove that  $S$  is natural

- (1) with respect to projections  $p_2 : Y \times X \longrightarrow X$ ,
- (2) with respect to closed based PL embeddings  $j : Y \longrightarrow X$ .

**Proof of (1).** Let  $B, C$  be cell complexes with  $|B| = X, |C| = Y$ . Let  $\xi$  be a block bundle over  $B$  representing  $x \in I_F(X)$ , and let  $\pi$  be a block fibration for  $\xi$ . Then  $\pi \times 1 : E(\xi) \times Y \longrightarrow X \times Y$  is a block fibration for  $\xi \times C$  (which represents  $p_2^*(x)$ ). Construct  $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$  representing  $S(x)$ . Let  $Y^I$  be the space of **unbased** maps  $\psi : I \longrightarrow Y$  and define  $e_1 : Y^I \longrightarrow Y$  by  $e_1(\psi) = \psi(1)$ . Then

$$(\mathcal{E} \times Y^I, \partial\mathcal{E} \times Y^I, p \times e_1, (i \times \text{bpt})b(\xi))$$

represents  $S(p_2^*(x))$ . But this fibration is equivalent to

$$(\mathcal{E} \times Y, \partial\mathcal{E} \times Y, p \times 1, (i \times \text{bpt})b(\xi)) ,$$

which represents  $p_2^*(S(x))$ .

**Proof of (2).** Let  $B$  be a cell complex with  $|B| = X$  and with a subcomplex  $C$  such that  $|C| = j(Y)$ . Let  $\xi$  be a block bundle over  $B$  representing  $x \in I_F(X)$ , and let  $\pi$  be a block fibration for  $\xi$ . then  $\pi|E(\xi|C) \rightarrow |C|$  is a block fibration for  $\xi|C$  (which represents  $j^*(x)$ ). Construct  $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$  representing  $S(x)$ . Write  $i'$  for the restriction

$$i| : E(\xi|C), E(\partial\xi|C) \longrightarrow p^{-1}|C|, p^{-1}|C| \cap \partial\mathcal{E} ;$$

clearly  $\pi|E(\xi|C) = pi'$ .

Identify  $F$  with  $b(\xi)F$  and write  $\mathcal{F}$  for  $p^{-1}(\text{bpt})$ . Consider the commutative diagram

$$\begin{array}{ccc} \pi_n(E(\xi|C), F) & \xrightarrow{i'_*} & \pi_n(p^{-1}|C|, \mathcal{F}) \\ & \searrow \pi_* & \swarrow p_* \\ & & \pi_n(|C|, \text{bpt}) \end{array}$$

As in Theorem 3,  $\pi_*$  is an isomorphism; since  $p_*$  is an isomorphism,  $i'_*$  is also an isomorphism. Therefore  $i' : E(\xi|C) \rightarrow p^{-1}|C|$  is a homotopy equivalence. Similarly  $i' : E(\partial\xi|C) \rightarrow p^{-1}|C| \cap \partial\mathcal{E}$  is a homotopy equivalence, so  $i'$  is a homotopy equivalence of pairs. It follows from Theorem 6.1 of [5] that

$$(p^{-1}|C|, p^{-1}|C| \cap \partial\mathcal{E}, p|p^{-1}|C|, ib(\xi))$$

represents  $S(j^*x)$ , so  $j^*S(x) = S(j^*x)$ . This completes the proof of Lemma 7.

Recall that  $w_I \in I_F(\widetilde{BPL}_F)$ ,  $w_H \in H_F(BG_F)$  are the universal elements. There is a based map  $\chi : \widetilde{BPL}_F \rightarrow BG_F$  such that  $S(w_I) = \chi^*(w_H)$ . This defines the based homotopy class of  $\chi$  uniquely.

Consider the topological space  $L = \{(x, \psi)\}$  of pairs with  $x \in \widetilde{BPL}_F$ ,  $\psi : I \rightarrow BG_F$  such that  $\chi(x) = \psi(0)$ ,  $\psi(1) = \text{bpt}$ , with  $(\text{bpt}, \text{constant})$  as base-point. There is a based map  $\chi' : L \rightarrow \widetilde{BPL}_F$  defined by  $\chi'(x, \psi) = x$ . By theorems of Milnor [9] and J. H. C. Whitehead [18], there is a based polyhedron  $G_F/\widetilde{PL}_F$  and a homotopy equivalence  $i : G_F/\widetilde{PL}_F \rightarrow L$ . Define  $\chi_1 = \chi' i : G_F/\widetilde{PL}_F \rightarrow \widetilde{BPL}_F$ .

Let  $B$  be a based cell complex and let  $F$  be a compact  $PL$  manifold. A  $G_F/\widetilde{PL}_F$ -**bundle** over  $B$  consists of a block bundle  $\xi$  over  $B$  with fibre  $F$  and a  $PL$  map

$$t : E(\xi), E(\partial\xi) \longrightarrow F, \partial F$$

such that  $tb(\xi) = 1$ . Two  $G_F/\widetilde{PL}_F$ -bundles  $(\xi, t)$  and  $(\eta, u)$  over  $B$  are **isomorphic** if there is an isomorphism  $h : \xi \rightarrow \eta$  such that  $uh \simeq t$  (rel  $b(\xi)(F)$ ). Define

the **equivalence** of  $G_F/\widetilde{PL}_F$ -bundles over a polyhedron  $X$  as in Chapter I, and let  $J_F(X)$  be the set of equivalence classes.

**Lemma 8.** *Let  $(\xi, t)$  be a  $G_F/\widetilde{PL}_F$ -bundle over  $B$  and let  $\pi : E(\xi) \rightarrow |B|$  be a block fibration for  $\xi$ . Then*

$$t \times \pi : E(\xi), E(\partial\xi) \longrightarrow F \times |B|, \partial F \times |B|$$

*is a homotopy equivalence of pairs.*

**Proof.** Apply Theorem 3 as in the proof of Lemma 7.

We make  $J_F$  into a contravariant functor as follows. Let  $f : X \rightarrow Y$  be a based  $PL$  map, and suppose  $f = p_2 j$ , where  $j : X \rightarrow V \times Y$  is a closed based  $PL$  embedding. Let  $B, C, D$  be cell complexes with

$$|B| = X, \quad |C| = Y, \quad |D| = Z,$$

and let  $(\eta, u)$  be a  $G_F/\widetilde{PL}_F$ -bundle over  $C$  representing  $y \in J_F(Y)$ . Let  $(D \times C)'$  be a subdivision of  $D \times C$  with  $j(B)$  as a subcomplex, and let  $(D \times \eta)'$  be a subdivision of  $D \times \eta$  over  $(D \times C)'$ . Then  $f^*(y)$  is represented by

$$((C \times \eta)'|j(B), \quad up_2|E((C \times \eta)'|j(B))).$$

The proof of Lemma 3 shows that  $f^* : J_F(Y) \rightarrow J_F(X)$  is well-defined, and that  $J_F$  is a contravariant functor.

**Theorem 4.** *If  $F$  is a compact  $PL$  manifold, then there is an element  $w_J \in J_F(G_F/\widetilde{PL}_F)$  such that  $f \mapsto f^*(w_J)$  defines a natural equivalence  $[, G_F/\widetilde{PL}_F] \rightarrow J_F$ .*

**Proof.** Let  $C_I, C_J, C_H$  be cell complexes with

$$|C_I| = B\widetilde{PL}_F, \quad |C_J| = G_F/\widetilde{PL}_F \text{ and } |C_H| = BG_F.$$

Let  $\eta_I$  be a block bundle over  $C_I$  representing  $w_I$ , and let  $\eta_H$  be a Hurewicz fibration over  $|C_H|$  representing  $w_H$ . Let  $\eta_J$  be a block bundle over  $C_J$  representing  $\chi_1^*(w_I)$ , and let  $\pi_I, \pi_J$  be block fibrations for  $\eta_I, \eta_J$ .

Recall that  $S(w_I) = \chi^*(w_H)$ ; let  $h : S(\eta_I, \pi_I) \rightarrow \chi^*(\eta_H)$  be a fibre homotopy equivalence. The proof of naturality of  $S$  (Lemma 7) provides a fibre homotopy equivalence

$$S(\eta_J, \pi_J) \longrightarrow \chi_1^* S(\eta_I, \pi_I).$$

Compose this with

$$\chi_1^* h : \chi_1^* S(\eta_I, \pi_I) \longrightarrow \chi_1^* \chi^*(\eta_H)$$

to obtain a fibre homotopy equivalence

$$h_1 : S(\eta_J, \pi_J) \longrightarrow \chi_1^* \chi^*(\eta_H).$$

Now  $\chi\chi_1 = \chi\chi'i$ , where  $\chi' : L \rightarrow BG_F$  sends  $(x, \psi)$  to  $x$ . There is an obvious null-homotopy  $H : L \times I \rightarrow BG_F$  of  $\chi\chi'$ , so  $H' = H(i \times 1)$  is a null-homotopy of  $\chi\chi_1$ . Let  $h' : \chi_1^*\chi^*(\eta_H) \rightarrow \epsilon$  be the trivialisation defined by  $H'$ . The composite

$$E(\eta_J) \xrightarrow{i} S(\eta_J, \pi_J) \xrightarrow{h_1} \chi_1^*\chi^*(\eta_H) \xrightarrow{h'} \epsilon \xrightarrow{p_1} F$$

defines a map

$$u_J : E(\eta_J), E(\partial\eta_J) \rightarrow F, \partial F$$

such that  $u_J b(\eta_J) = 1$ . Let  $w_J$  be the equivalence class of  $(\eta_J, u_J)$  in  $J_F(G_F/\widetilde{PL}_F)$ .

Clearly  $f \mapsto f^*(w_J)$  defines a natural transformation from  $[ \ , G_F/\widetilde{PL}_F ]$  to  $J_F$ . Let  $B$  be a cell complex and let  $(\xi, t)$  be a  $G_F/\widetilde{PL}_F$ -bundle over  $B$ ; we have to prove that the equivalence class of  $(\xi, t)$  corresponds to a unique element of  $[|B|, G_F/\widetilde{PL}_F]$ . Let  $\pi$  be a block fibration for  $\xi$ .

There is a map  $g : |B| \rightarrow B\widetilde{PL}_F$  such that  $\xi$  represents  $g^*(w_I)$ ;  $g$  is unique up to homotopy. The proof of naturality of  $S$  (Lemma 7) provides a fibre homotopy equivalence  $S(\xi, \pi) \rightarrow g^*S(\eta_I, \pi_I)$ . Compose this with  $g^*h : g^*S(\eta_I, \pi_I) \rightarrow g^*\chi^*(\eta_H)$  to obtain a fibre homotopy equivalence  $k : S(\xi, \pi) \rightarrow g^*\chi^*(\eta_H)$ .

Now  $tk^{-1} : g^*\chi^*(\eta_H) \rightarrow F$  defines a fibre homotopy trivialisation of  $g^*\chi^*(\eta_H)$ , unique up to fibre homotopy. Let  $K : |B| \times I \rightarrow BG_F$  be the corresponding null-homotopy of  $\chi g : |B| \rightarrow BG_F$ . Then  $(g, K)$  defines the unique homotopy class of maps  $f : |B| \rightarrow G_F/\widetilde{PL}_F$  such that  $(\eta, t)$  represents  $f^*(w_J)$ . This completes the proof of Theorem 4.

### III. Tangential Properties of Block Bundles

Let  $I^n$  denote the product of  $n$  copies of the unit interval; we write  $G_n/\widetilde{PL}_n$  for  $G_{I^n}/\widetilde{PL}_{I^n}$ . The obvious natural transformation  $J_{I^n} \rightarrow J_{I^{n+1}}$  (multiply the fibre of each bundle by  $I$ ) defines a homotopy class of maps  $G_n/\widetilde{PL}_n \xrightarrow{i_n} G_{n+1}/\widetilde{PL}_{n+1}$ . Write  $G/PL$  for the direct limit of the sequence

$$\xrightarrow{i_{n-1}} G_n/\widetilde{PL}_n \xrightarrow{i_n} G_{n+1}/\widetilde{PL}_{n+1} \xrightarrow{i_{n+1}} \dots$$

More precisely, for  $n = 1, 2, 3, \dots$  replace  $G_{n+1}/\widetilde{PL}_{n+1}$  by a homotopy equivalent polyhedron in such a way that  $i_n$  is an injection, and identify  $G_n/\widetilde{PL}_n$  with  $i_n(G_n/\widetilde{PL}_n)$ . Now define  $G/PL$  to be the nested union of the  $G_n/\widetilde{PL}_n$ ; it can be shown that the homotopy type of  $G/PL$  is independent of the choices made (see Lemma 1.7 of [3]).

$G/PL$  was studied by Sullivan in his thesis (but he called it  $F/PL$ ). The aim of this chapter is to obtain a map  $\theta : G_F/\widetilde{PL}_F \rightarrow (G/PL)^F$ , where  $(G/PL)^F$

is the space of all **unbased** maps from  $F$  into  $G/PL$  (with the compact open topology). Let  $\mathcal{C}$  be the category of based, compact, stably parallelizable  $PL$  manifolds and based  $PL$  maps. Our first step is to define a natural transformation

$$T : [ \ , G_F/\widetilde{PL}_F ] \longrightarrow [ \ , (G/PL)^F ] ,$$

where the functors are defined on  $\mathcal{C}$ .

Let  $N$  be an object of  $\mathcal{C}$ , with boundary  $\partial N$ , and let  $B$  be a cell complex with  $|B| = N$ . Let  $\beta$  be a principal cell of  $B$  with the base-point as one vertex. Let  $x \in [N, G_F/\widetilde{PL}_F]$  be represented by a  $G_F/\widetilde{PL}_F$ -bundle  $(\xi, t)$  over  $B$ . Extend  $b(\xi)p_1 : F \times \text{bpt} \longrightarrow E(\xi|\text{bpt})$  to a homeomorphism  $b : F \times \beta \longrightarrow E(\xi|\beta)$ . Change  $t$  by a homotopy (rel  $b(\xi)(F)$ ) until  $tb = p_1 : F \times \beta \longrightarrow F$ .

We write  $E = E(\xi)$ , so  $E$  is a  $PL$  manifold with  $\partial E = E(\partial\xi) \cup E(\xi|\partial B)$ . We write  $W$  for  $F \times \beta$  and identify  $W$  with  $b(W)$ . Let  $\pi : E \longrightarrow N$  be a block fibration such that  $\pi|F \times \beta = p_2$ . Let  $Q = F \times N$ , so by Lemma 8,  $t \times \pi : E, \partial E \longrightarrow Q, \partial Q$  is a homotopy equivalence of pairs. Note that  $t \times \pi|W = 1$  and  $(t \times \pi)^{-1}(W) = W$ . Let  $g : Q, \partial Q \longrightarrow E, \partial E$  be a homotopy inverse to  $t \times \pi$  such that  $g|W = 1$  and  $g^{-1}(W) = W$ .

Let  $k$  be large, and choose embeddings

$$e : E, \partial E \longrightarrow D^k, S^{k-1} \quad , \quad q : Q, \partial Q \longrightarrow D^k, S^{k-1}$$

such that  $e|W = q|W$ . By [6], there exists normal bundles  $\nu_Q, \nu_E$  of  $Q, E$  in  $D^k$ . Choose  $\nu_Q, \nu_E$  so that  $\nu_Q|W = \nu_E|W$  (using the uniqueness theorem of [6] and regular neighbourhood theory). Let  $Q^\nu, E^\nu, W^\nu$  be Thom spaces for  $\nu_Q, \nu_E, \nu_Q|W$ , and let

$$\gamma : Q^\nu/\partial Q^\nu \longrightarrow W^\nu/\partial W^\nu \quad , \quad \gamma : E^\nu/\partial E^\nu \longrightarrow W^\nu/\partial W^\nu$$

be the collapsing maps. Let  $\bar{\nu}_Q = g^*(\nu_E)$  have Thom space  $Q^{\bar{\nu}}$  and collapsing map

$$\bar{\gamma} : Q^{\bar{\nu}}/\partial Q^{\bar{\nu}} \longrightarrow W^{\bar{\nu}}/\partial W^{\bar{\nu}} = W^\nu/\partial W^\nu .$$

There is a homotopy equivalence  $\bar{h} : E^\nu/\partial E^\nu \longrightarrow Q^{\bar{\nu}}/\partial Q^{\bar{\nu}}$  covering  $t \times \pi : E \longrightarrow Q$  and such that  $\bar{\gamma}\bar{h} = \gamma$ .

There is a map  $D^k \longrightarrow Q^\nu/\partial Q^\nu$  which collapses

$$S^{k-1} \cup (\text{complement of total space of } \nu_Q)$$

to a point. If we identify  $S^k$  with  $D^k/S^{k-1}$ , we obtain a map  $\phi : S^k \longrightarrow Q^\nu/\partial Q^\nu$ ; let  $\psi : S^k \longrightarrow E^\nu/\partial E^\nu$  be defined similarly. Let  $\bar{\phi} = \bar{h}\psi : S^k \longrightarrow Q^{\bar{\nu}}/\partial Q^{\bar{\nu}}$ ; then  $\bar{\gamma}\bar{\phi} = \gamma\phi = \gamma\psi$ .

By theorems of Atiyah [1] and Wall [15, Th 3.5] there is a fibre homotopy equivalence  $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$  such that  $\bar{f}\bar{\phi} \simeq \phi$ . It follows from Wall's theorem that  $\bar{f}$  is unique up to fibre homotopy. Consider  $\tilde{f} = \bar{f}|(\bar{\nu}_Q|W) \longrightarrow (\nu_Q|W)$ ; this has the

property that

$$\gamma\phi \simeq \gamma\bar{f}\bar{\phi} = \tilde{f}(\bar{\gamma}\bar{\phi}) = \tilde{f}(\gamma\phi) .$$

By the uniqueness clause in Wall's theorem,  $\tilde{f}$  is fibre homotopic to the identity. Therefore we can alter  $\bar{f}$  by a fibre homotopy until it is the identity on  $\bar{\nu}_Q|W$ .

Let  $G$  be defined as in [8] (this agrees with the definition used in [15]), so  $G$  is an  $H$ -space. Since  $W$  is a retract of  $Q$ , the map  $[Q/W, G] \rightarrow [Q, G]$  is injective. It follows that two fibre equivalences  $\bar{f}_0, \bar{f}_1 : \bar{\nu}_Q \rightarrow \nu_Q$  which are the identity on  $\bar{\nu}_Q|W$  are fibre homotopic (rel  $\bar{\nu}_Q|W$ ) if and only if they are fibre homotopic. Therefore the fibre homotopy equivalence  $\bar{f} : \bar{\nu}_Q \rightarrow \nu_Q$  obtained above is unique up to fibre homotopy (rel  $\bar{\nu}_Q|W$ ).

Let  $\tau_Q$  be the tangent bundle on  $Q$ , and choose a fixed trivialisation  $\kappa : \tau_Q \oplus \nu_Q \rightarrow \epsilon$ . Then

$$f = \kappa(1 \oplus \bar{f}) : \tau_Q \oplus \bar{\nu}_Q \rightarrow \epsilon$$

is a fibre homotopy equivalence, which agrees with  $\kappa$  on  $\tau_Q \oplus \bar{\nu}_Q|W$ . The pair  $(\tau_Q \oplus \bar{\nu}_Q, f)$  represents an element

$$T(x) \in [Q/W, G/PL] \cong [N, (G/PL)^F] .$$

Since the normal invariants  $\phi, \psi$  are unique up to homotopy and  $PL$  bundle automorphisms,  $T(x)$  depends only on  $x$ . Thus we have defined a map

$$T : [N, G_F/\widetilde{PL}_F] \rightarrow [N, (G/PL)^F] .$$

**Lemma 9.** *T is a natural transformation (between functors from  $\mathcal{C}$  to the category of based sets).*

**Proof.** Let  $f : M \rightarrow N$  be a based  $PL$  map. Express  $f$  as a composite

$$M \xrightarrow{\times 0} M \times D^r \xrightarrow{u} N \times D^s \xrightarrow{p_1} N ,$$

where  $u$  is a codimension 0 embedding. We prove that  $T$  is natural

- (1) with respect to  $\times 0$  and  $p_1$ ,
- (2) with respect to codimension 0 embeddings.

**Proof of 1.** Consider  $p_1 : N \times D^s \rightarrow N$ ; let  $B$  be a cell complex with  $|B| = N$ . Let  $(\xi, t)$  be a  $G_F/\widetilde{PL}_F$ -bundle over  $B$  representing  $x \in [N, G_F/\widetilde{PL}_F]$ , so that  $(\xi \times D^s, tp_1)$  represents  $p_1^*(x)$ . Let

$$Q , W , \nu_Q , \bar{\nu}_Q , \phi : S^k \rightarrow Q^\nu / \partial Q^\nu , \bar{\phi} : S^k \rightarrow Q^{\bar{\nu}} / \partial Q^{\bar{\nu}}$$

be defined for  $(\xi, t)$  as above. The corresponding objects for  $(\xi \times D^s, tp_1)$  are

$$\begin{aligned} Q_s &= Q \times D^s, \quad W_s = W \times D^s, \\ \nu_{Q_s} &= \nu_Q \times D^s, \quad \bar{\nu}_{Q_s} = \bar{\nu}_Q \times D^s, \\ S^s \phi &: S^{s+k} \longrightarrow Q_s^\nu / \partial Q_s^\nu, \quad S^s \bar{\phi} : S^k \longrightarrow Q_s^{\bar{\nu}} / \partial Q_s^{\bar{\nu}} \end{aligned}$$

(note that  $Q_s^\nu / \partial Q_s^\nu \cong S^s(Q^\nu / \partial Q^\nu)$ ,  $Q_s^{\bar{\nu}} / \partial Q_s^{\bar{\nu}} \cong S^s(Q^{\bar{\nu}} / \partial Q^{\bar{\nu}})$ ).

Let  $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$  be a fibre homotopy equivalence such that  $\bar{f}\bar{\phi} \simeq \phi$  and  $\bar{f}$  is the identity on  $\bar{\nu}_Q|W$ . Then  $\bar{f}_s = \bar{f} \times 1 : \bar{\nu}_{Q_s} \longrightarrow \nu_{Q_s}$  is the identity on  $\bar{\nu}_{Q_s}|W$ , and  $\bar{f}_s(S^s \bar{\phi}) \simeq S^s \phi$ . Therefore  $(\tau_{Q_s} \oplus \bar{\nu}_{Q_s}, 1 \oplus \bar{f}_s)$  represents  $T(p_1^*(x))$ . It follows that  $T(p_1^*(x)) = p_1^*(T(x))$ , as required. Since  $\times 0 : N \longrightarrow N \times D^s$  is a homotopy inverse to  $p_1$ ,  $T$  is also natural with respect to  $\times 0$ .

**Proof of 2.** Let  $u : M \longrightarrow N$  be a codimension 0 embedding. Let  $B$  be a cell complex with  $|B| = N$  and with a subcomplex  $A$  such that  $|A| = u(M)$ . Choose  $\beta$  to be a cell of  $A$  containing the base-point, as above. Let  $(\xi, t)$  be a  $G_F/\widetilde{PL}_F$ -bundle over  $B$  representing  $x \in [N, G_F/\widetilde{PL}_F]$ ; then  $(\xi|A, t|E(\xi|A))$  represents  $u^*(x)$ . Let

$$E = E(\xi), \quad D = D(\xi|A), \quad Q = F \times N, \quad P = F \times M.$$

Identify  $W$  with  $b(W) \subset D \subset E$ , as above. Let

$$g : Q, P, \partial Q, \partial P \longrightarrow E, D, \partial E, \partial D$$

be a homotopy inverse to  $t \times \pi$  such that  $g|W$  is the identity.

Choose embeddings

$$g : Q, \partial Q \longrightarrow D^k, S^{k-1}, \quad e : E, \partial E \longrightarrow D^k, S^{k-1}$$

agreeing on  $W$ , as above. Let  $\nu_Q, \nu_E$  be normal bundles with  $\nu_Q|W = \nu_E|W$ , and let  $\nu_P = \nu_Q|P$ ,  $\nu_D = \nu_E|D$ . We obtain collapsing maps

$$\eta : Q^\nu / \partial Q^\nu \longrightarrow P^\nu / \partial P^\nu, \quad \eta : E^\nu / \partial E^\nu \longrightarrow D^\nu / \partial D^\nu.$$

Let  $\bar{\nu}_Q = g^*(\nu_E)$ , let  $\bar{\nu}_P = \bar{\nu}_Q|P$  and let  $\bar{h} : E^\nu / \partial E^\nu \longrightarrow Q^\nu / \partial Q^\nu$  be a homotopy equivalence covering  $t \times \pi : E \longrightarrow Q$ , such that  $\bar{\gamma}\bar{h} = \gamma$  (where  $\gamma, \bar{\gamma}$  are as above).

If

$$\phi : S^k \longrightarrow Q^\nu / \partial Q^\nu, \quad \psi : S^k \longrightarrow E^\nu / \partial E^\nu$$

are collapsing maps for  $Q, E$ , then  $\eta\phi, \eta\psi$  are collapsing maps for  $P, D$ . Let  $\bar{\phi} = \bar{h}\psi : S^k \longrightarrow Q^\nu / \partial Q^\nu$ ; the corresponding map for  $P$  is  $\bar{h}\eta\psi : S^k \longrightarrow P^\nu / \partial P^\nu$ . Let  $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$  be a fibre homotopy equivalence such that  $\bar{f}$  is the identity on  $\bar{\nu}_Q|W$  and  $\bar{f}\bar{\phi} \simeq \phi$ .

Now  $\tilde{f} = \bar{f}|_{\bar{\nu}_P} \longrightarrow \nu_P$  is a fibre homotopy such that

$$\tilde{f}(\bar{h}\eta\psi) = \tilde{f}(\eta\bar{\phi}) = \eta\tilde{f}\bar{\phi} \simeq \eta\phi$$

and  $\tilde{f}$  is the identity on  $\bar{\nu}_P|W$ . Therefore  $T(x)$ ,  $T(u^*(x))$  are represented by  $(\tau_Q \oplus \bar{\nu}_Q, 1 \oplus \tilde{f})$ ,  $(\tau_P \oplus \bar{\nu}_P, 1 \oplus \tilde{f})$  respectively. It follows that  $T(u^*(x)) = u^*(T(x))$ , as required. This proves the Lemma.

Since  $G_F/\widetilde{PL}_F$  and  $(G/PL)^F$  have the homotopy type of countable  $CW$  complexes, it follows from Lemma 1.7 of [3] that there is a map  $\theta : G_F/\widetilde{PL}_F \rightarrow (G/PL)^F$  such that  $T = \theta_*$ . Unfortunately, the homotopy class of  $\theta$  is not uniquely determined by this condition.

**Theorem 5.** *Let  $F^n$  be a closed 1-connected  $PL$  manifold with  $n \geq 4$ . Let  $F^* = \overline{F - D^n}$ , and let  $\rho : (G/PL)^F \rightarrow (G/PL)^{F^*}$  be the restriction map. Then the composite  $\rho\theta$  induces isomorphisms*

$$(\rho\theta)_* : \pi_r(G_F/\widetilde{PL}_F) \rightarrow \pi_r((G/PL)^{F^*})$$

for  $r \geq 1$ .

**Remark.** For any based space  $X$  let  $X_0$  be the component of  $X$  containing the base-point. Then  $(G_F/\widetilde{PL}_F)_0$  is homotopy equivalent to  $((G/PL)^{F^*})_0$ , but  $(G/PL)^{F^*}$  usually has more components than  $G_F/\widetilde{PL}_F$ .

**Proof.** First we prove that  $(\rho\theta)_*$  is surjective; we defer the case  $n = 4, r = 1$  until after Theorem 7. Let  $B$  be a cell complex with  $|B| = S^r$ , and let  $\beta$  be a principal cell of  $B$ . Let  $f : S^r, \beta \rightarrow (G/PL)^{F^*}$ ,  $\text{bpt}$  represent an element of  $x \in \pi_r((G/PL)^{F^*})$ . Let

$$g : F^* \times S^r, F^* \times \beta \rightarrow (G/PL), \text{bpt}$$

be the adjoint map. Extend  $g$  over  $(F^* \times S^r) \cup (F \times \beta)$  by defining  $g(F \times \beta) = \text{bpt}$ . Let  $Q = F \times S^r$ ,  $W = F \times \beta$  and let  $Q^*$  be obtained from  $Q$  by deleting the interior of an  $(n+r)$ -disc in  $Q - W$ . Then  $Q^*$  deformation retracts onto  $(F^* \times S^r) \cup (F \times \beta)$ , so  $g$  defines a homotopy class of maps  $h : Q^*, W \rightarrow G/PL, \text{bpt}$ .

Let  $k$  be large, identify  $D^k$  with the northern hemisphere of  $S^k$  and identify  $2D^k$  with the closed region to the north of the Antarctic circle. Let  $q : Q \rightarrow S^k$  be an embedding such that  $q^{-1}(D^k) = W$ ,  $q^{-1}(2D^k) = Q^*$ . Let  $\nu_Q$  be a normal bundle of  $Q$  in  $S^k$  such that  $\nu_Q|W, \nu_Q|Q^*$  are normal bundles of  $W, Q^*$  in  $D^k, 2D^k$  respectively. Let  $\phi^* : S^k \rightarrow Q^{*\nu}/\partial Q^{*\nu}$  be the collapsing map.

Choose a piecewise linear bundle  $\bar{\nu}_{Q^*}$  over  $Q^*$  and a fibre homotopy equivalence  $\bar{f} : \bar{\nu}_{Q^*} \rightarrow \nu_{Q^*}$  such that  $\bar{\nu}_{Q^*}|W = \nu_{Q^*}|W$ ,  $\bar{f}$  is the identity on  $\bar{\nu}_{Q^*}|W$  and  $(\tau_{Q^*} \oplus \bar{\nu}_{Q^*}, 1 \oplus \bar{f})$  represents  $h$ . By the theorem of Wall quoted above, there is a map  $\bar{\phi} : S^k \rightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$  such that  $\bar{f}\bar{\phi} \simeq \phi^*$ . Let  $\eta : Q^{*\bar{\nu}} \rightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$  be the collapsing map; if  $k$  is large enough then there is a map  $\psi' : 2D^k, 2S^{k-1} \rightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$  such that  $\eta\psi'$  and  $\bar{\phi}$  represent the same element of  $\pi_k(Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}})$ .

Adjust  $\psi'$  by a homotopy until  $\psi'|D^k = \phi|D^k$ , and  $\psi'$  is transverse regular on  $Q^* \subset Q^{*\bar{\nu}}$ ; let  $E' = \psi'^{-1}(Q^*)$ , so  $W \subset E'$ . We shall modify  $E', \partial E'$  by surgery (keeping  $W$  fixed), attempting to make  $\psi'| : E', \partial E' \longrightarrow Q^*, \partial Q^*$  a homotopy equivalence of pairs.

Since the inclusion induces an isomorphism  $\pi_1(\partial Q^*) \longrightarrow \pi_1(\overline{Q^* - W})$  (in fact both groups are zero) and  $n+r \geq 6$ , we can use Theorem 3.3 of [17] to the manifold  $\overline{E' - W}$ . This has two boundary components, namely  $\partial W$  and  $\partial E'$ ; we wish to do surgery on  $\text{Int}(\overline{E' - W})$  and  $\partial E'$ , but **not** on  $\partial W$ .

We obtain a map  $\psi^* : 2D^k, 2S^{k-1} \longrightarrow Q^{*\bar{\nu}}, \partial Q^{*\bar{\nu}}$ , which is transverse regular on  $Q^*$  and is homotopic to  $\psi'$  (rel  $D^k$ ), with the following property. Let

$$E^* = \psi^{*-1}(Q^*) ;$$

then

$$\psi^*| : \overline{E' - W}, \partial E^* \longrightarrow \overline{Q^* - W}, \partial Q^*$$

is a homotopy equivalence of pairs. It follows that  $\psi^*| : E^*, \partial E^* \longrightarrow Q^*, \partial Q^*$  is a homotopy equivalence of pairs.

Since  $\partial Q^* \cong S^{n+r-1}$  and  $n+r-1 \geq 5$ ,  $\partial E^*$  is homeomorphic to  $S^{n+r-1}$ . Let  $E = E^* \cup_{\partial E^*} D^{n+r}$ , and extend the embedding  $E^* \subset 2D^k$  to an embedding  $E \subset S^k$ . Let  $\nu_E$  be a normal bundle of  $E$  in  $S^k$  such that  $\nu_E|W, \nu_E|E^*$  are normal bundles of  $W, E^*$  in  $D^k, 2D^k$  respectively. Extend  $\psi^* : 2D^k \longrightarrow Q^{*\bar{\nu}}$  to a map  $\psi : S^k \longrightarrow Q^{\bar{\nu}}$ , transverse regular on  $Q \subset Q^{\bar{\nu}}$  and with  $E = \psi^{-1}(Q)$ . Then  $\psi|E \longrightarrow Q$  is a homotopy equivalence, and  $\psi|W$  is the identity.

Recall that  $B$  is a cell complex with  $|B| = S^r$ , and  $\beta$  is a principal cell of  $B$ . Let  $\gamma$  be an  $(r-1)$ -cell of  $B$  contained in  $\partial\beta$ . Choose a *PL* homeomorphism  $k : |\partial\beta - \gamma| \times I \longrightarrow |B - \beta|$  such that  $k_0$  is the inclusion. Recall that  $Q = F \times |B|$ . Since  $n+r \geq 6$ , we can use the relative *h*-cobordism theorem [12] to extend

$$\psi^{-1}(1 \times k)| : F \times |\partial\beta - \gamma| \times 0 \longrightarrow \overline{\partial E - W}$$

to a homeomorphism  $H : F \times |\partial\beta - \gamma| \times I \longrightarrow \overline{E - W}$ .

Define a block bundle  $\xi$  over  $B$  with  $E(\xi) = E$  by  $E_\beta(\xi) = W$  and, for each cell  $\delta$  in  $(B - \beta)$ ,  $E_\delta(\xi) = H(1 \times k^{-1})(F \times \delta)$ . Then  $\xi$  satisfies the local triviality condition in the definition of a block bundle. Let

$$b(\xi) = 1 \times \text{bpt} : F \longrightarrow F \times \text{bpt} = E_{\text{bpt}}(\xi) .$$

Let  $t = p_1\psi : E \longrightarrow F$ ; then  $(\xi, t)$  is a  $G_F/\widetilde{PL}_F$ -bundle over  $S^r$ , representing an element  $y \in \pi_r(G_F/\widetilde{PL}_F)$ . It is easily checked that  $p_*(T(y)) \in \pi_r((G/PL)^{F^*})$  is represented by  $(\tau_{Q^*} \oplus \bar{\nu}_{Q^*}, 1 \oplus \bar{f})$  so  $p_*(T(y)) = x$ . Therefore  $(\rho\theta)_*(y) = x$ , so  $(\rho\theta)_*$  is surjective, as required (provided  $n+r \geq 6$ ).

Similar arguments prove that  $(\rho\theta)_*$  is injective; we have to consider  $G_F/\widetilde{PL}_F$ -bundles  $(\xi_0, t_0), (\xi_1, t_1)$  over  $S^r \times 0, S^r \times 1$ . We prove that they are isomorphic by extending them to a  $G_F/\widetilde{PL}_F$ -bundle  $(\xi, t)$  over  $S^r \times I$ . Since  $n + \dim(S^r \times I) \geq 6$ , we can always carry out surgery and use the  $h$ -cobordism theorem. Thus the Theorem is established, except for surjectivity of  $(\rho\theta)_*$  when  $n = 4, r = 1$ .

**Theorem 6.** *Let  $F^n$  be a compact  $PL$  manifold with  $\pi_1(\partial F)$  isomorphic to  $\pi_1(F)$  by inclusion and  $n \geq 6$ . Then  $\theta$  induces isomorphisms*

$$\theta_* : \pi_r(G_F/\widetilde{PL}_F) \longrightarrow \pi_r((G/PL)^F)$$

for  $r \geq 1$ .

**Proof.** Since the proof is essentially the same as the proof of Theorem 5, we shall not give the details. To prove that  $\theta_*$  is surjective, let  $B, \beta, \xi, Q, W$  be as above. Since  $Q$  has a boundary  $\partial Q$  such that  $\pi_1(\partial Q) \longrightarrow \pi_1(\overline{Q - W})$  is an isomorphism, it is unnecessary to cut out a disc from  $Q$ . We can use Theorem 3.3 of [17] to construct a manifold  $E \supset W$  with boundary  $\partial E$  and a **simple** homotopy equivalence  $\psi : E, \partial E \longrightarrow Q, \partial Q$  with  $\psi|_W$  equal to the identity.

In the construction of the block bundle  $\xi$  above, we used the  $h$ -cobordism theorem to construct a homeomorphism  $F \times |B - \beta| \longrightarrow \overline{E - W}$ . Here we can use the  $s$ -cobordism theorem [7] twice (first for  $\partial F \times |B - \beta|$ , then for  $F \times |B - \beta|$ ), since  $\psi$  is a simple homotopy equivalence and  $\dim(\partial F \times |B - \beta|) \geq 6$ . The rest of the proof proceeds as above.

#### IV. Periodicity of $G/PL$

In his thesis, Sullivan interpreted  $[M, G/PL]$  in terms of  $PL$  structures on manifolds homotopy equivalent to  $M$ . Thus it is useful to have information about  $G/PL$  which facilitates computation of  $[M, G/PL]$ . It has been known for some time that  $\pi_r(G/PL) \cong \mathbb{Z}, 0, \mathbb{Z}_2, 0$  according as  $r \equiv 0, 1, 2, 3 \pmod{4}$ ; in particular,  $\pi_r(G/PL) \cong \pi_{r+4}(G/PL)$ .

**Theorem 7.** *There is a map  $\lambda : G/PL \longrightarrow \Omega^4(G/PL)$  such that  $\lambda_* : \pi_r(G/PL) \longrightarrow \pi_{r+4}(G/PL)$  is an isomorphism if  $r \neq 0, 4$  and a monomorphism onto a subgroup of index 2 if  $r = 4$ .*

**Proof.** Let  $F^n$  be a closed 1-connected  $PL$  manifold with  $n \geq 4$ . If  $X$  is a based space we write  $X_0$  for the component of  $X$  containing the base-point. Consider the diagram

$$(1) \quad \begin{array}{ccccc} & & (G_F/\widetilde{PL}_F)_0 & & \\ & \theta \swarrow & & \searrow \rho\theta & \\ \Omega^n(G/PL)_0 & \xrightarrow{\alpha} & (G/PL)_0^F & \xrightarrow{\rho} & (G/PL)_0^{F*} \end{array}$$

where  $\alpha$  is induced by a map  $F \rightarrow S^n$  of degree 1.

Suppose first that  $n \geq 5$ . Then  $\rho\theta$  is a homotopy equivalence by Theorem 5 (we are not using the unproved case!). Let  $\gamma'$  be a homotopy inverse to  $\rho\theta$ . Let  $\gamma = \theta\gamma'$ , so  $\rho\gamma \simeq 1 : (G/PL)_0^{F*} \rightarrow (G/PL)_0^F$ .

The Whitney sum construction gives a multiplication map  $\mu : G/PL \times G/PL \rightarrow G/PL$ . If  $K$  is a finite  $CW$  complex,  $\mu$  defines Abelian group structures on

$$[K, \Omega^n(G/PL)_0] \quad , \quad [K, (G/PL)_0^F] \quad , \quad [K, (G/PL)_0^{F*}]$$

such that  $\alpha_*$ ,  $\rho_*$  are homeomorphisms. Let  $x \in [K, (G/PL)_0^F]$  and let  $y = (1 - \gamma_*\rho_*)(x)$ , so  $\rho_*(y) = 0$ . Therefore  $y = \alpha_*(z)$  for some  $z \in [K, \Omega^n(G/PL)_0]$ . Since  $\rho$  has a right homotopy inverse,  $\alpha_*$  is injective and  $z$  is unique. Define a natural transformation

$$S : [ \quad , (G/PL)_0^F ] \longrightarrow [ \quad , \Omega^n(G/PL)_0 ]$$

on finite  $CW$  complexes by  $S(x) = z$ . By Lemma 1.7 of [3], there is a map

$$\sigma : (G/PL)_0^F \longrightarrow \Omega^n(G/PL)_0$$

with  $S = \sigma_*$ . Observe that

$$\alpha_*\sigma_*\alpha_* = \alpha_* - \gamma_*\rho_*\alpha_* = \alpha_* \quad ,$$

since  $\rho\alpha \simeq \text{bpt}$ . Since  $\alpha_*$  is injective,  $\sigma_*\alpha_* = 1$ .

Let  $r \geq 1$  and consider the homomorphism

$$\sigma : \pi_r((G/PL)^F) \longrightarrow \pi_{n+r}(G/PL) \quad .$$

This is an epimorphism (with right inverse  $\alpha_*$ ). Let  $x \in \pi_r((G/PL)^F)$  be represented by

$$g : F \times S^r, F \times \beta \longrightarrow G/PL, \text{ bpt}$$

(where  $\beta$  is a cell of  $S^r$  containing the base-point). Let  $Q = F \times S^r$ ,  $W = F \times \beta$ , as above.

Let  $k$  be large and identify  $D^k$  with the northern hemisphere of  $S^k$ . Let  $q : Q \rightarrow S^k$  be an embedding such that  $q^{-1}(D^k) = W$ . Let  $\nu_Q$  be a normal bundle

of  $Q$  in  $S^k$  such that  $\nu_Q|W$  is a normal bundle of  $W$  in  $D^k$ . Let  $\phi : S^k \longrightarrow Q^\nu$  be the collapsing map.

Choose a piecewise linear bundle  $\bar{\nu}_Q$  over  $Q$  and a fibre homotopy equivalence  $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$  such that  $\bar{\nu}_Q|W = \nu_Q|W$ ,  $\bar{f}$  is the identity on  $\bar{\nu}_Q|W$  and  $(\tau_Q \oplus \bar{\nu}_Q, 1 \oplus \bar{f})$  represents  $g$ . As in Chapter III, there is a map  $\psi' : S^k \longrightarrow Q^\nu$  such that  $\bar{f}\psi' \simeq \phi$ . Adjust  $\psi'$  by a homotopy until  $\psi'|D^k = \phi|D^k$  and  $\psi'$  is transverse regular on  $Q \subset Q^\nu$ ; let  $E' = \psi'^{-1}(Q)$ , so  $W \subset E'$ . We attempt to modify  $E'$  by surgery (keeping  $W$  fixed), to make  $\psi'|E' \longrightarrow Q$  a homotopy equivalence.

We seek a map  $\psi : S^k \longrightarrow Q^\nu$  which is transverse regular on  $Q$  and is homotopic to  $\psi'$  (rel  $D^k$ ), and with the following property. Let  $E = \psi^{-1}(Q)$ ; then  $\psi|E \longrightarrow Q$  is a homotopy equivalence. Let  $P_r = \mathbb{Z}, 0, \mathbb{Z}_2, 0$  according as  $r \equiv 0, 1, 2, 3 \pmod{4}$  (as in [8]). By [14, §4], since  $Q$  is 1-connected and  $\dim Q \geq 5$ , there is an obstruction  $\bar{\sigma}(x) \in P_{n+r}$  to performing the surgery. Note that  $\bar{\sigma}(x)$  depends only on  $x$ .

Using the homotopy group addition in  $\pi_r((G/PL)^F)$  (**not** the  $H$ -structure on  $G/PL$ ) and the interpretation of  $\bar{\sigma}(x)$  as a signature or Arf invariant, we see that  $\bar{\sigma} : \pi_r((G/PL)^F) \longrightarrow P_{n+r}$  is a homomorphism. Consider the homomorphism  $\bar{\sigma}\alpha_* : \pi_{n+r}(G/PL) \longrightarrow P_{n+r}$ . This coincides with the canonical homomorphism obtained in [13], and is therefore an isomorphism. It follows that  $\bar{\sigma}$  is an epimorphism.

If  $x \in \pi_r((G/PL)^F)$ , then  $\gamma_*\rho_*(x)$  is represented by  $\bar{g} : Q, W \longrightarrow G/PL$ , bpt, where  $\bar{g}$  agrees with  $g$  on  $Q^* = Q - D^{n+r}$ , and  $\bar{g}|D^{n+r}$  is chosen so that  $\bar{\sigma}(\bar{g}) = 0$  (because the surgery problem for  $\gamma_*\rho_*(x) \in \text{im}(\theta_*)$  is clearly soluble). Since  $\bar{\sigma}\alpha_*$  is a monomorphism, these conditions characterise the homotopy class of  $\bar{g}$ . Therefore  $x = \gamma_*\rho_*(x)$  if and only if  $\bar{\sigma}(x) = 0$ , so  $\ker \sigma_* = \ker \bar{\sigma}$ . If we identify  $\pi_{n+r}(G/PL)$  with  $P_{n+r}$  via the canonical isomorphism, we see that  $\bar{\sigma}(x) = \sigma_*(x)$ .

Let  $\epsilon : G/PL \longrightarrow (G/PL)^F$  be induced by the map  $F \longrightarrow \text{point}$ , and let  $\lambda^F$  denote the composite

$$G/PL \xrightarrow{\epsilon} (G/PL)^F \xrightarrow{\sigma} \Omega^n(G/PL) .$$

If  $\dim F = n = 4$ , this construction fails as

$$(\rho\theta)_* : \pi_1(G_F/\widetilde{PL}_F) \longrightarrow \pi_1((G/PL)^{F*})$$

is not yet known to be surjective. However, we can construct a map  $\bar{\lambda}^F : \Omega^4(G/PL)_0 \longrightarrow \Omega^{n+4}(G/PL)_0$ ; simply apply the functor  $\Omega^4$  to diagram (1) and argue as above.

Let  $x \in \pi_r(G/PL)$  be represented by  $g : S^r \longrightarrow G/PL$ . Then  $\epsilon_*(x)$  is represented by  $gp_2 : F \times S^r \longrightarrow G/PL$ . Note that  $x = \bar{\sigma}(x)$  and

$$\bar{\sigma}(\epsilon_*(x)) = \sigma_*\epsilon_*(x) = \lambda_*^F(x) .$$

Now  $\bar{\sigma}(x)$  is the obstruction to making a certain map  $\psi'| : V' \longrightarrow S^r$  a homotopy equivalence by surgery (where  $V'$  is a certain framed  $r$ -manifold). Similarly  $\bar{\sigma}(\epsilon_*(x))$  is the obstruction to making  $1 \times \psi'| : F \times V' \longrightarrow F \times S^r$  a homotopy equivalence.

Take  $F = \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$ . Suppose  $r \equiv 0 \pmod{4}$ ; then by [14],  $\bar{\sigma}(x) = \frac{1}{8}(\text{signature of } V')$  if  $r \geq 8$ ; but  $\bar{\sigma}(x) = \frac{1}{16}(\text{signature of } V')$  if  $r = 4$ . Similarly,

$$\begin{aligned} \bar{\sigma}(\epsilon_*(x)) &= \frac{1}{8}(\text{signature of } F \times V' - \text{signature of } F \times S^r) \\ &= \frac{1}{8}(\text{signature of } V') \text{ for all } r. \end{aligned}$$

Thus  $\bar{\sigma}(\epsilon_*(x)) = \bar{\sigma}(x)$  unless  $r = 4$ , when  $\bar{\sigma}(\epsilon_*(x)) = 2\bar{\sigma}(x)$ .

If  $r \equiv 2 \pmod{4}$ , then it follows from Theorem 9.9 of [17] that  $\bar{\sigma}(x) = \bar{\sigma}(\epsilon_*(x))$ . (The theorem is stated for  $r \geq 5$ , but the argument seems to work when  $r = 2$ .) Since  $\pi_r(G/PL) = \pi_{r+8}(G/PL) = 0$  if  $r$  is odd, we have proved that  $\lambda_*^F : \pi_r(G/PL) \longrightarrow \pi_{r+8}(G/PL)$  is an isomorphism if  $r \neq 0, 4$ , and a monomorphism onto a subgroup of index 2 if  $r = 4$ .

Similar arguments show that, if  $F = \mathbb{C}\mathbb{P}^2$  and  $r \geq 1$ , then  $\bar{\lambda}_*^F : \pi_{r+4}(G/PL) \longrightarrow \pi_{r+8}(G/PL)$  is an isomorphism. Therefore  $\bar{\lambda}^F : \Omega^4(G/PL)_0 \longrightarrow \Omega^8(G/PL)_0$  is a homotopy equivalence. Let  $\lambda : G/PL \longrightarrow \Omega^4(G/PL)$  be the composite of  $\lambda^{\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2}$  with a homotopy inverse to  $\bar{\lambda}^{\mathbb{C}\mathbb{P}^2}$ ; then  $\lambda$  has the desired properties.

Now we can complete the proof of Theorem 5 by showing that, if  $\dim F = 4$ , then

$$(\rho\theta)_* : \pi_1(G_F/\widetilde{PL}_F) \longrightarrow \pi_1((G/PL)^{F*})$$

is surjective. Consider the following diagram :

$$\begin{array}{ccccc} \pi_1((G/PL)^F) & \xrightarrow{\rho_*} & \pi_1((G/PL)^{F*}) & \xrightarrow{\partial} & \pi_0(\Omega^4(G/PL)) \\ \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* \\ \pi_1((\Omega^4(G/PL))^F) & \xrightarrow{\rho_*} & \pi_1((\Omega^4(G/PL))^{F*}) & \xrightarrow{\partial} & \pi_0(\Omega^8(G/PL)) . \end{array}$$

The rows are taken from the homotopy exact sequences of the Hurewicz fibrations

$$(G/PL)^F \longrightarrow (G/PL)^{F*} \quad , \quad (\Omega^4(G/PL))^F \longrightarrow (\Omega^4(G/PL))^{F*} .$$

The proof of Theorem 7 shows that, in the bottom row,  $\rho_*$  is surjective so  $\partial = 0$ . But

$$\lambda_* : \pi_0(\Omega^4(G/PL)) \longrightarrow \pi_0(\Omega^8(G/PL))$$

is injective, so  $\partial = 0$  in the top row. Therefore

$$\rho_* : \pi_1((G/PL)^F) \longrightarrow \pi_1((G/PL)^{F^*})$$

is surjective.

Let  $x \in \pi_1((G/PL)^{F^*})$ , and choose an element

$$\bar{x} \in \pi_1((G/PL)^F)$$

such that  $\rho_*(\bar{x}) = x$ . Let  $\beta$  be an interval in  $S^1$  containing the base-point, let  $Q = F \times S^1$ ,  $W = F \times \beta$ . Let  $\nu_Q, \psi'$  be as in the proof of Theorem 7. Since  $\bar{\sigma}(x) \in P_5 = 0$ , we can do surgery to find a map  $\psi : S^k \longrightarrow Q^{\bar{\nu}}$  which is transverse regular to  $\psi'$  (rel  $D^k$ ), with the following property. Let  $E = \psi^{-1}(Q)$ ; then  $\psi| : E \longrightarrow Q$  is a homotopy equivalence.

Let  $b_0, b_1$  be the end-points of  $\beta$ , and let  $B$  be the cell complex  $\{b_0, b_1, \beta, \overline{S^1 - \beta}\}$ . Then  $\overline{E - W}$  is an  $h$ -cobordism between  $F \times b_0$  and  $F \times b_1$ , and the  $PL$  homeomorphism  $1 \times b_1 : F \times b_0 \longrightarrow F \times b_1$  is in the preferred homotopy class. By Barden's  $h$ -cobordism theorem for 5-manifolds [2], there is a  $PL$  homeomorphism  $H : F \times |B - \beta| \longrightarrow \overline{E - W}$  with  $H(F \times b_i) = F \times b_i$ . Now we can define a block bundle  $\xi$  over  $B$  with  $E(\xi) = E$ , and a map  $t : E \longrightarrow F$ , as in the proof of Theorem 5. We obtain a  $G_F/\widetilde{PL}_F$ -bundle  $(\xi, t)$  over  $B$ , representing an element  $y \in \pi_1(G_F/\widetilde{PL}_F)$  such that  $\theta_*(y) = \bar{x}$ . Therefore  $x = (\rho\theta)_*(y)$ , so

$$(\rho\theta)_* : \pi_1(G_F/\widetilde{PL}_F) \longrightarrow \pi_1((G/PL)^{F^*})$$

is surjective. This completes the proof of Theorem 5.

## V. Topologically Trivial Block Bundles

Let  $\xi$  be a block bundle over  $B$  with fibre  $F$ . A **proper trivialisation** of  $\xi$  is a proper map

$$h : E(\xi) \longrightarrow F \times |B|$$

such that

$$h(E_\beta(\xi)) \subset F \times \beta \quad \text{for each } \beta \in B$$

(base-points will be irrelevant in this chapter). Two proper trivialisations  $h_0, h_1$  of  $\xi$  are **properly homotopic** if there is a proper map

$$H : E(\xi) \times I \longrightarrow F \times |B|$$

such that

$$H(E_\beta(\xi) \times I) \subset F \times \beta$$

for each  $\beta \in B$  and  $H_t = h_t$  ( $t = 0, 1$ ). A **topological trivialisation** of  $\xi$  is a proper trivialisation which is a topological homeomorphism; a  **$PL$  trivialisation** is defined similarly.

**Theorem 8.** *Let  $\xi$  be a block bundle over  $B$  with fibre  $\mathbb{R}^q$  ( $q \geq 3$ ). Let  $h : E(\xi) \rightarrow \mathbb{R}^q \times |B|$  be a topological trivialisation of  $\xi$ . Then there is an obstruction  $w \in H^3(B; \mathbb{Z}_2)$  which vanishes if and only if  $h$  is properly homotopic to a  $PL$  trivialisation of  $\xi$ .*

**Proof.** Let  $V, W$  be  $PL$  manifolds and let  $N$  be a compact submanifold of  $W$  with  $\partial N = N \cap \partial W$ . A map  $\phi : V \rightarrow W$  is  **$h$ -regular** on  $N$  if it is transverse regular on  $N$  and  $\phi| : \phi^{-1}(N) \rightarrow N$  is a homotopy equivalence. Let  $Q$  denote  $\mathbb{C} \mathbb{P}^2 \times \mathbb{C} \mathbb{P}^2$ . Our first objective is to construct the following:

- (1) A proper map  $f : E(\xi) \times Q \rightarrow \mathbb{R}^q \times |B| \times Q$  such that, for each  $\beta \in B$ ,

$$f| : E_\beta(\xi) \times Q \rightarrow \mathbb{R}^q \times \beta \times Q$$

is  $h$ -regular on  $0 \times \beta \times Q$ .

- (2) A proper homotopy  $F$  from  $h \times 1$  to  $f$  such that, for each  $\beta \in B$ ,

$$F(E_\beta(\xi) \times Q \times I) \subset \mathbb{R}^q \times \beta \times Q .$$

We shall eventually use  $f$  and  $F$  to construct a  $PL$  trivialisation of  $\xi$ . The factor  $Q$  is introduced to avoid difficulties with low-dimensional manifolds.

Let  $T = \partial \Delta^2$  and write  $T^r$  for the product of  $r$  copies of  $T$ . Note that the universal covering space  $\tilde{T}^r$  of  $T$  is  $PL$  homeomorphic to  $\mathbb{R}^r$ . Choose a  $PL$  embedding  $\mathbb{R} \times T^{q-1} \subset \mathbb{R}^q$  and a  $PL$  homeomorphism  $\mathbb{R}^q \rightarrow \mathbb{R} \times \tilde{T}^{q-1}$  such that the composite

$$e : \mathbb{R}^q \rightarrow \mathbb{R} \times \tilde{T}^{q-1} \rightarrow \mathbb{R} \times T^{q-1} \subset \mathbb{R}^q$$

is the identity on a neighbourhood of the origin.

Let  $A$  be a subcomplex of  $B$ . Let  $W_{A,r}$  denote  $\mathbb{R}^r \times T^{q-r} \times |A| \times Q$  and let  $N_{A,r} = 0 \times T^{q-r} \times A \times Q \subset W_{A,r}$ . We have an embedding  $W_{A,1} \subset \mathbb{R}^q \times A \times Q$  and there is a covering map  $p : W_{A,r} \rightarrow W_{A,r-1}$ . Define  $V_{A,1} = (h \times 1)^{-1}(W_{A,1})$  and let  $g_{A,1} = h \times 1| : V_{A,1} \rightarrow W_{A,1}$ . Define  $V_{B,r}, g_{B,r}$  ( $r \geq 2$ ) inductively as follows. Let  $p : V_{B,r} \rightarrow V_{B,r-1}$  be the covering map induced from  $p : W_{B,r} \rightarrow W_{B,r-1}$  by the homeomorphism  $g_{B,r-1} : V_{B,r-1} \rightarrow W_{B,r-1}$ . Let  $g_{B,r} : V_{B,r} \rightarrow W_{B,r}$  be a homeomorphism such that  $pg_{B,r} = g_{B,r-1}p$ . Finally let  $V_{A,r} = p^{-1}(V_{A,r-1})$  and let  $g_{A,r} = g_{B,r}|_{V_{A,r}}$ . We write  $W_r^n, N_r^n, V_r^n, g_r^n$  for  $W_{B^n,r}, N_{B^n,r}, V_{B^n,r}, g_{B^n,r}$  respectively, and abbreviate  $W_{B,r}, N_{B,r}, V_{B,r}, g_{B,r}$  to  $W_r, N_r, V_r, g_r$ .

Suppose inductively that we have constructed the following, for some integer  $n$ :

- (1) A proper map  $f_1^{n-1} : V_1^{n-1} \rightarrow W_1^{n-1}$  such that, for each  $\beta \in B^{n-1}$ ,  $f_1^{n-1}|_{V_{\beta,1}} \rightarrow W_{\beta,1}$  is  $h$ -regular on  $N_{\beta,1}$ .
- (2) A proper homotopy  $F_1^{n-1}$  from  $g_1^{n-1}$  to  $f_1^{n-1}$  such that, for each  $\beta \in B^{n-1}$ ,  $F_1^{n-1}(V_{\beta,1} \times I) \subset W_{\beta,1}$ .

Suppose also that  $f_1^{n-1}, F_1^{n-1}$  are extensions of  $f_1^{n-2}, F_1^{n-2}$ .

Now let  $\beta \in B^n - B^{n-1}$ . Let  $f_{\partial\beta,1} = f_1^{n-1}|_{V_{\partial\beta,1}}$  and let  $F_{\partial\beta,1} = F_1^{n-1}|_{V_{\partial\beta,1} \times I}$ . The inductive hypothesis ensures that  $f_{\partial\beta,1}$  is transverse regular on  $N_{\partial\beta,1}$ . Thus  $M_{\partial\beta,1} = f_{\partial\beta,1}^{-1}(N_{\partial\beta,1})$  is a submanifold of  $V_{\partial\beta,1}$  of codimension 1.

**Lemma 10.**  $f_{\partial\beta,1}$  is  $h$ -regular on  $N_{\partial\beta,1}$ .

**Proof.** Let  $B$  be a cell complex. A **blocked space**  $E$  over  $B$  consists of a topological space  $E$  and, for each  $\beta \in B$ , a subspace  $E_\beta$  of  $E$  such that the following conditions are satisfied:

- (1)  $\{E_\beta : \beta \in B\}$  is a locally finite covering of  $E$ .
- (2) If  $\beta, \gamma \in B$ , then  $E_\beta \cap E_\gamma = \bigcup_{\delta \subset \beta \cap \gamma} E_\delta$ .
- (3) If  $\beta$  is a face of  $\gamma \in B$ , then the inclusion  $E_\beta \subset E_\gamma$  is a homotopy equivalence.
- (4) If  $\beta \in B$  and  $E_{\partial\beta} = \bigcup_{\gamma \subset \partial\beta} E_\gamma$ , then the pair  $(E_\beta, E_{\partial\beta})$  has the absolute extension property.

If  $E^{(1)}, E^{(2)}$  are blocked spaces over  $B$ , a **blocked equivalence**  $\phi : E^{(1)} \longrightarrow E^{(2)}$  is a continuous map such that  $\phi(E_\beta^{(1)}) \subset E_\beta^{(2)}$  and  $\phi| : E_\beta^{(1)} \longrightarrow E_\beta^{(2)}$  is a homotopy equivalence for each  $\beta \in B$ . Observe that  $M_{\partial\beta,1}$  and  $N_{\partial\beta,1}$  are blocked spaces over  $\partial\beta$ , and  $f_{\partial\beta,1}| : M_{\partial\beta,1} \longrightarrow N_{\partial\beta,1}$  is a blocked equivalence.

Suppose inductively that, if  $E^{(1)}, E^{(2)}$  are blocked spaces over  $B^{s-1}$ , then any blocked equivalence  $\phi : E^{(1)} \longrightarrow E^{(2)}$  is a homotopy equivalence. Now let  $\phi : E^{(1)} \longrightarrow E^{(2)}$  be a blocked equivalence over  $B^s$ .

Let  $C^{(i)} = \bigcup_{\beta \in B^{s-1}} E_\beta^{(i)}$  and let  $D^{(i)}, \partial D^{(i)}$  be the disjoint unions of

$$\{E_\beta^{(i)} : \beta \in B^s - B^{s-1}\}, \quad \{E_{\partial\beta}^{(i)} : \beta \in B^s - B^{s-1}\}.$$

Then  $\partial D^{(i)} \subset D^{(i)}$  and there are maps  $\lambda^{(i)} : \partial D^{(i)} \longrightarrow C^{(i)}$  such that  $E^{(i)} = C^{(i)} \cup_{\lambda^{(i)}} D^{(i)}$ . By induction,  $\phi : C^{(1)} \longrightarrow C^{(2)}$  is a homotopy equivalence.

Now  $\phi$  defines a homotopy equivalence  $\psi : D^{(1)} \longrightarrow D^{(2)}$  such that  $\phi\lambda^{(1)} = \lambda^{(2)}\psi|_{\partial D^{(1)}}$ . By induction,  $\psi|_{\partial D^{(1)}} \longrightarrow \partial D^{(2)}$  is a homotopy equivalence. The pairs  $(D^{(i)}, \partial D^{(i)})$  satisfy the absolute extension condition; using a result in homotopy theory we deduce that  $\phi : E^{(1)} \longrightarrow E^{(2)}$  is a homotopy equivalence. By induction, any blocked equivalence over a finite-dimensional complex is a homotopy equivalence, and the Lemma follows.

Now the  $PL$  manifold  $V_{\beta,1}$  has two tame ends (for definition see [11]) with free Abelian fundamental groups. Since  $M_{\partial\beta,1} \subset V_{\partial\beta,1}$  is a homotopy equivalence (by Lemma 10),  $M_{\partial\beta,1}$  bounds collars of the ends of  $V_{\partial\beta,1}$ . Since  $\dim V_{\beta,1} \geq 8$ , we can apply Siebenmann's theorem [11,§5] to construct a compact submanifold  $M_{\beta,1}$  of

$V_{\beta,1}$  with boundary  $M_{\partial\beta,1}$  and such that  $M_{\beta,1} \subset V_{\beta,1}$  is a homotopy equivalence. As in [16], we can extend  $f_{\partial\beta,1}$  to  $f_{\beta,1} : V_{\beta,1} \longrightarrow W_{\beta,1}$ , transverse regular on  $N_{\beta,1}$  and with  $M_{\beta,1} = f_{\beta,1}^{-1}(N_{\beta,1})$ . We can also extend  $F_{\partial\beta,1}$  to a proper homotopy  $F_{\beta,1}$  from  $g_{\beta,1}$  to  $f_{\beta,1}$ .

Do this for all  $n$ -cells  $\beta$  of  $B$  to obtain extensions  $f_1^n, F_1^n$  of  $f_1^{n-1}, F_1^{n-1}$  satisfying the inductive hypotheses. This completes our induction on  $n$ ; we have defined the following:

- (1) A proper map  $f_1 : V_1 \longrightarrow W_1$  such that for each  $\beta \in B$ ,  $f_1| : V_{\beta,1} \longrightarrow W_{\beta,1}$  is  $h$ -regular on  $N_{\beta,1}$ .
- (2) A proper homotopy  $F_1$  from  $g_1$  to  $f_1$  such that, for each  $\beta \in B$ ,  $F_1(V_{\beta,1} \times I) \subset W_{\beta,1}$ .

Suppose inductively that we have defined the following, for some integer  $r \geq 1$ :

- (1) A proper map  $f_r : V_r \longrightarrow W_r$  such that for each  $\beta \in B$ ,  $f_r| : V_{\beta,r} \longrightarrow W_{\beta,r}$  is  $h$ -regular on  $N_{\beta,r}$ .
- (2) A proper homotopy  $F_r$  from  $g_r$  to  $f_r$  such that, for each  $\beta \in B$ ,  $F_r(V_{\beta,r} \times I) \subset W_{\beta,r}$ .

Let

$$\tilde{N}_r = 0 \times \mathbb{R} \times T^{q-r-1} \times |B| \times Q \subset W_{r+1} .$$

If  $p : W_{r+1} \longrightarrow W_r$  is the covering map then  $\tilde{N}_r = p^{-1}(N_r)$ . Lift  $F_r$  to a proper homotopy  $\tilde{F}_r$  from  $g_{r+1}$  to a map  $\tilde{f}_r : V_{r+1} \longrightarrow W_{r+1}$ . Let  $\tilde{M}_r = \tilde{f}_r^{-1}(\tilde{N}_r)$  and let  $M_r = f_r^{-1}(N_r)$ . Since  $p| : \tilde{M}_r \longrightarrow M_r$  is a covering map and  $f_r : M_r \longrightarrow N_r$  is a homotopy equivalence,  $\tilde{f}_r| : \tilde{M}_r \longrightarrow \tilde{N}_r$  is a proper homotopy equivalence. Let  $A$  be a subcomplex of  $B$ . Let  $\tilde{W}_{A,r} = p^{-1}(N_{A,r})$ ,  $\tilde{f}_{A,r} = \tilde{f}_r|_{V_{A,r+1}}$ ,  $\tilde{F}_{A,r} = \tilde{F}_r|_{V_{A,r+1} \times I}$ ,  $\tilde{M}_{A,r} = \tilde{M}_r \cap V_{A,r+1}$  and  $M_{A,r} = M_r \cap V_{A,r}$ .

We construct the following:

- (1) A proper map  $\phi_r : \tilde{M}_r \longrightarrow \tilde{N}_r$  such that for each  $\beta \in B$ ,  $\phi_r| : \tilde{M}_{\beta,r} \longrightarrow \tilde{N}_{\beta,r}$  is  $h$ -regular on  $N_{\beta,r+1}$ .
- (2) A proper homotopy  $\Phi_r$  from  $\tilde{f}_r|_{\tilde{M}_r}$  to  $\phi_r$  such that, for each  $\beta \in B$ ,  $\Phi_r(\tilde{M}_{\beta,r} \times I) \subset \tilde{N}_{\beta,r}$ .

The construction is exactly the same as the one given above for  $f_1$  and  $F_1$ . We apply Siebenmann's theorem to  $\tilde{M}_{\beta,r}$  instead of  $V_{\beta,1}$ ; the details will be omitted.

Using the product structure on a neighbourhood of  $\tilde{M}_r$  in  $V_{r+1}$ , we can construct the following:

- (1) A proper map  $f_{r+1} : V_{r+1} \longrightarrow W_{r+1}$  such that for each  $\beta \in B$ ,  $f_{r+1}| : V_{\beta,r+1} \longrightarrow W_{\beta,r+1}$  is  $h$ -regular on  $N_{\beta,r+1}$ .

- (2) A proper homotopy  $F_{r+1}$  from  $g_{r+1}$  to  $f_{r+1}$  such that, for each  $\beta \in B$ ,  $F_{r+1}(V_{\beta,r+1} \times I) \subset W_{\beta,r+1}$ .

This completes the induction on  $r$ . When  $r = q$  we obtain a proper map  $f_q : V_q \longrightarrow W_q = \mathbb{R}^q \times |B| \times Q$  and a proper homotopy  $F_q$  from  $g_q$  to  $f_q$ , satisfying the inductive hypotheses.

Consider the commutative diagram :

$$\begin{array}{ccccc} V_q & \xrightarrow{g_q} & W_q & \xlongequal{\quad} & \mathbb{R}^q \times |B| \times Q \\ \downarrow \epsilon & & \downarrow \epsilon & & \downarrow e \times 1 \\ E(\xi) \times Q & \xrightarrow{h \times 1} & \mathbb{R}^q \times |B| \times Q & \xlongequal{\quad} & \mathbb{R}^q \times |B| \times Q \end{array}$$

where  $\epsilon$  denotes a covering map followed by an inclusion. Recall that  $e : \mathbb{R}^q \longrightarrow \mathbb{R}^q$  is the identity on an open disc neighbourhood  $U$  of the origin.

Let  $A$  be a subcomplex of  $B$ , let  $X_A$  denote

$$h^{-1}(U \times |A|) \times Q - h^{-1}(0 \times |A|) \times Q \subset E(\xi) \times Q ,$$

and let  $X = X_B$ ,  $X^n = X_{B^n}$ . Suppose inductively that we have constructed the following, for some integer  $n$ .

- (1) A subset  $Y^{n-1}$  of  $X^{n-1}$  such that, for each  $\beta \in B^{n-1}$ ,  $Y_\beta = Y^{n-1} \cap X_\beta$  is a compact submanifold of  $X_\beta$  of codimension one and  $Y_\beta \subset X_\beta$  is a homotopy equivalence. Then  $E(\xi|B^{n-1}) \times Q - Y^{n-1}$  has two components; let  $Z^{n-1}$  be the closure of the bounded component. Let  $(Z')^{n-1}$  be the component of  $\epsilon^{-1}(Z^{n-1})$  which lies in  $g_q^{-1}(U \times |B^{n-1}| \times Q)$ , and let  $(Y')^{n-1} = (Z')^{n-1} \cap \epsilon^{n-1}(Y^{n-1})$ .
- (2) *PL* homeomorphisms

$$\begin{aligned} \gamma^{n-1} : Y^{n-1} \times [0, \infty) &\longrightarrow \overline{E(\xi|B^{n-1}) \times Q - Z^{n-1}} , \\ (\gamma')^{n-1} : (Y')^{n-1} \times [0, \infty) &\longrightarrow \overline{V_q^{n-1} - (Z')^{n-1}} \end{aligned}$$

such that  $\gamma_0^{n-1}, (\gamma')_0^{n-1}$  are the inclusions.

Suppose further that  $\gamma^{n-1}, (\gamma')^{n-1}$  are extensions of  $\gamma^{n-2}, (\gamma')^{n-2}$ .

Now let  $\beta \in B^n - B^{n-1}$ . Let

$$\begin{aligned} Y_{\partial\beta} &= Y^{n-1} \cap X_{\partial\beta} , \\ \gamma_{\partial\beta} &= \gamma^{n-1}|Y_{\partial\beta} \times [0, \infty) , \\ \gamma'_{\partial\beta} &= (\gamma')^{n-1}|Y'_{\partial\beta} \times [0, \infty) . \end{aligned}$$

Then  $Y_{\partial\beta}$  bounds a collar of the end of  $E(\xi|\partial\beta) \times Q$ . It follows that  $Y_{\partial\beta} \subset X_{\partial\beta}$  is a homotopy equivalence; since  $\dim X_{\partial\beta} \geq 8$ ,  $Y_{\partial\beta}$  bounds a collar of the ends of

$X_{\partial\beta}$ .

Since the ends of  $X_\beta$  are tame and have trivial fundamental groups, Siebenmann's theorem shows that there is a compact submanifold  $Y_\beta$  of  $X_\beta$  with boundary  $Y_{\partial\beta}$  and such that  $Y_\beta \subset X_\beta$  is a homotopy equivalence. It follows that  $Y_\beta$  bounds a collar of the end of  $E(\xi|\beta) \times Q$ . Let

$$\gamma_\beta : Y_\beta \times [0, \infty) \longrightarrow \overline{E(\xi|\beta) \times Q - Z_\beta}$$

be a  $PL$  homeomorphism such that  $(\gamma_\beta)_0$  is the inclusion and  $\gamma_\beta|_{Y_{\partial\beta} \times [0, \infty)} = \gamma_{\partial\beta}$ . Do this for all  $n$ -cells  $\beta$  of  $B$  to obtain  $Y^n$ ,  $\gamma^n$  satisfying the inductive hypotheses.

Define  $(Z')^n$ ,  $(Y')^n$  as in (1) above, and note that  $\epsilon : (Z')^n \longrightarrow Z^n$  is a  $PL$  homeomorphism. Then, for each  $\beta \in B^n - B^{n-1}$ ,  $Y'_\beta \subset \overline{V_{\beta,q} - Z'_\beta}$  is a homotopy equivalence, so  $Y'_\beta$  bounds a collar of the end of  $V_{\beta,q}$ . Let

$$\gamma'_\beta : Y'_\beta \times [0, \infty) \longrightarrow \overline{V_{\beta,q} - Z'_\beta}$$

be a  $PL$  homeomorphism such that  $(\gamma'_\beta)_0$  is the inclusion and  $\gamma'_\beta|_{Y'_{\partial\beta} \times [0, \infty)} = \gamma'_{\partial\beta}$ . Then the  $\gamma'_\beta$  fit together to define an extension  $(\gamma')^n$  of  $(\gamma')^{n-1}$  satisfying the inductive hypotheses. This completes the induction on  $n$ .

Let

$$Y = \bigcup_{n=1}^{\infty} Y^n, \quad Z = \bigcup_{n=1}^{\infty} Z^n, \quad \gamma = \bigcup_{n=1}^{\infty} \gamma^n, \quad \gamma' = \bigcup_{n=1}^{\infty} (\gamma')^n.$$

Define a  $PL$  homeomorphism  $\psi : E(\xi) \times Q \longrightarrow V_q$  by  $\psi = \epsilon^{-1}$  on  $Z$  and  $\psi = \gamma^{-1}(\epsilon^{-1} \times 1)\gamma$  elsewhere. Define a proper homotopy  $\Psi$  from  $g_q\psi$  to  $h \times 1$  as follows. If  $x \in \mathbb{R}^q$ ,  $y \in |B|$ ,  $z \in Q$  and  $t \in [0, 1)$ , let  $g_q\psi(h^{-1}(tx, y), z) = (x', y', z')$ , and define

$$\Psi(h^{-1}(x, y), z, t) = (t^{-1}x', y', z').$$

Define

$$\Psi(h^{-1}(x, y), z, 0) = (x, y, z);$$

this makes  $\Psi$  continuous since  $(x', y', z') = (tx, y, z)$  provided  $t$  is sufficiently small.

Now we can define the proper map  $f : E(\xi) \times Q \longrightarrow \mathbb{R} \times |B| \times Q$  and proper homotopy  $F$  from  $h \times 1$  to  $f$ , as promised at the beginning of the proof. Let  $f = f_q\psi$  and let  $F = \Psi * (F_q\psi)$  be defined by

$$F(x, t) = \begin{cases} \Psi(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ F_q\psi(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then  $f$  and  $F$  have the required properties (1) and (2).

Suppose inductively that we have constructed the following, for some integer

$n$ .

- (1) A proper trivialisation  $j^{n-1} : E(\xi|B^{n-1}) \longrightarrow \mathbb{R}^q \times |B^{n-1}$ .
- (2) A proper homotopy  $J^{n-1}$  from  $h^{n-1}$  to  $j^{n-1}$ .
- (3) A proper homotopy  $L^{n-1}$  from  $f^{n-1}$  to  $j^{n-1} \times 1$  such that, for each  $\beta \in B^{n-1}$ ,  $L^{n-1}|E(\xi|\beta) \times Q$  is  $h$ -regular on  $0 \times \beta \times Q$ .
- (4) A proper homotopy  $\mathcal{L}^{n-1}$  from  $\bar{F}^{n-1} * (J^{n-1} \times 1)$  to  $L^{n-1}$  (rel  $\mathbb{R}^q \times |B^{n-1}| \times Q \times \partial I$ ).

Suppose further that  $j^{n-1}$ ,  $J^{n-1}$ ,  $L^{n-1}$ ,  $\mathcal{L}^{n-1}$  are extensions of  $j^{n-2}$ ,  $J^{n-2}$ ,  $L^{n-2}$ ,  $\mathcal{L}^{n-2}$  respectively.

Let  $\beta \in B^n - B^{n-1}$ . If  $A$  is a subcomplex of  $B^{n-1}$ , then  $j_A$ ,  $J_A$ ,  $L_A$ ,  $\mathcal{L}_A$  will have the usual meanings. As in Lemma 10 we see that

$$L_{\partial\beta} : E(\xi|\partial\beta) \times Q \times I \longrightarrow \mathbb{R}^q \times \partial\beta \times Q$$

is  $h$ -regular on  $0 \times \partial\beta \times Q$ . Note that  $L_{\partial\beta}$  is a proper homotopy from  $f_{\partial\beta}$  to  $j_{\partial\beta} \times 1$ . Extend  $L_{\partial\beta}$  to a proper homotopy  $K_\beta$  from  $f_\beta$  to a proper map  $k_\beta : E_\beta(\xi) \times Q \longrightarrow \mathbb{R}^q \times \beta \times Q$ . We can arrange for  $K_\beta$  to be  $h$ -regular on  $0 \times \beta \times Q$ .

Now  $J_{\partial\beta}$  is a proper homotopy from  $h_{\partial\beta}$  to  $j_{\partial\beta}$ . Extend  $J_{\partial\beta}$  to a proper map  $I_\beta$  from  $h_\beta$  to a proper map  $i_\beta : E_\beta(\xi) \longrightarrow \mathbb{R} \times \beta$ . Using the homotopies  $(I_\beta \times 1) * F_\beta * K_\beta$  and  $\mathcal{L}_{\partial\beta}$ , we see that  $i_\beta \times 1$  is properly homotopic (rel  $\mathbb{R}^q \times \partial\beta \times Q$ ) to  $k_\beta$ .

The obstruction to deforming  $i_\beta$  properly (rel  $E(\xi|\partial\beta)$ ) to a  $PL$  homeomorphism  $j'_\beta : E_\beta(\xi) \longrightarrow \mathbb{R}^q \times \beta$  is an element  $x \in \pi_n(G/PL)$ . Let  $\lambda_* : \pi_n(G/PL) \longrightarrow \pi_{n+8}(G/PL)$  be the periodicity homomorphism discussed in Chapter IV. Then  $\lambda_*(x)$  is the obstruction to deforming  $i_\beta \times 1$  properly (rel  $E(\xi|\partial\beta) \times Q$ ) to a map  $k'_\beta$  which is  $h$ -regular on  $0 \times \beta \times Q$ . The previous paragraph shows that  $\lambda_*(x) = 0$ ; since  $\lambda_*$  is a monomorphism,  $x = 0$ . Choose a  $PL$  homeomorphism  $j'_\beta : E_\beta(\xi) \longrightarrow \mathbb{R} \times \beta$  and a proper homotopy  $J'_\beta$  from  $h_\beta$  to  $j'_\beta$  extending  $J_{\partial\beta}$ .

Now  $\mathcal{L}_{\partial\beta}$  is a proper homotopy from  $\bar{F}_{\partial\beta} * (J_{\partial\beta} \times 1)$  to  $L_{\partial\beta}$ . Extend  $\mathcal{L}_{\partial\beta}$  to a proper homotopy  $\mathcal{G}_\beta$  (rel  $E_\beta(\xi) \times Q \times \partial I$ ) from  $\bar{F}_\beta * (J'_\beta \times 1)$  to a proper homotopy  $G_\beta$  between  $f_\beta$  and  $j'_\beta \times 1$ . Let  $y \in \pi_{n+9}(G/PL)$  be the obstruction to deforming  $G_\beta$  properly (rel  $\partial(\mathbb{R}^q \times \beta \times Q \times I)$ ) to a homotopy  $G'_\beta$  which is  $h$ -regular on  $0 \times \beta \times Q$ .

If we vary  $(j'_\beta, J'_\beta)$  by an element  $z \in \pi_{n+1}(G/PL)$ , we replace  $y$  by  $y + \lambda_*(z)$ . If  $n \neq 3$  then  $\lambda_*$  is surjective, so we can choose  $z$  so that  $y + \lambda_*(z) = 0$ . In other words, we can replace  $(j'_\beta, J'_\beta)$  by a pair  $(j_\beta, J_\beta)$  for which  $y$  vanishes. Then there is a proper homotopy  $L_\beta$  from  $f_\beta$  to  $j_\beta \times 1$  which is  $h$ -regular on  $0 \times \beta \times Q$ , and a proper homotopy  $\mathcal{L}_\beta$  (rel  $\mathbb{R}^q \times \beta \times Q \times \partial I$ ) from  $\bar{F}_\beta * (J_\beta \times 1)$  to  $L_\beta$ ;  $L_\beta$  and  $\mathcal{L}_\beta$  are extensions of  $L_{\partial\beta}$  and  $\mathcal{L}_{\partial\beta}$  respectively.

Do this for all  $n$ -cells  $\beta$  of  $B$  to obtain  $j^n, J^n, L^n, \mathcal{L}^n$  satisfying conditions (1)–(4). This completes the induction provided  $n \neq 3$ . In case  $\beta$  is a 3-cell of  $B$ , let  $c(\beta) \in \mathbb{Z}_2$  be the mod 2 reduction of  $y \in \pi_{12}(G/PL) = \mathbb{Z}$ . This defines a cochain  $c \in C^3(B; \mathbb{Z}_2)$ . The above argument enables us to construct  $j^3, J^3, L^3, \mathcal{L}^3$  provided  $c = 0$ .

We consider the effect of varying  $L^2$ . Suppose  $j^1, J^1, L^1, \mathcal{L}^1, j^2, J^2, L^2, \mathcal{L}^2$  are constructed, and let  $\beta$  be a 3-cell in  $B$ . Observe that, if the cells  $\alpha \subset \partial\beta$  are oriented suitably, then  $\partial\beta = \sum_{\alpha \subset \partial\beta} \alpha \in C_2(B; \mathbb{Z})$ . If we vary  $L_\alpha$  by an element  $u_\alpha \in \pi_{12}(G/PL) = \mathbb{Z}$ , it can be seen that  $c(\beta)$  is replaced by  $c(\beta) + (\sum_{\alpha \subset \partial\beta} u_\alpha)_2$ . Let  $u \in C^2(B; \mathbb{Z}_2)$  be the cochain defined by  $u(\alpha) = u_\alpha$ ; then we have replaced  $c$  by  $c + \delta u$ .

Now let  $\gamma$  be a 4-cell of  $B$ , so  $\partial\gamma = \sum_{\beta \subset \partial\gamma} \beta \in C_3(B; \mathbb{Z})$ . For each 3-cell  $\beta \subset \partial\gamma$ , define  $j'_\beta, J'_\beta$  as above, and define

$$J'_{\partial\gamma} : E(\xi|\partial\gamma) \times I \longrightarrow \mathbb{R}^q \times \partial\gamma$$

by  $J'_{\partial\gamma}|E(\xi|\beta) \times I = J'_\beta$ . It is easy to adjust  $(j'_\beta, J'_\beta)$  on one cell  $\beta \subset \partial\gamma$  until  $J'_{\partial\gamma}$  extends to a proper homotopy  $J'_\gamma$  from  $h_\gamma$  to a  $PL$  homeomorphism  $j'_\gamma : E(\xi|\partial\gamma) \longrightarrow \mathbb{R}^q \times \partial\gamma$ .

Define

$$G_{\partial\gamma} : E(\xi|\partial\gamma) \times Q \times I \longrightarrow \mathbb{R}^q \times \partial\gamma \times Q$$

by  $G_{\partial\gamma}|E(\xi|\beta) \times Q \times I = G_\beta$ . Let  $v_\gamma \in \pi_{12}(G/PL) = \mathbb{Z}$  be the obstruction to deforming  $G_{\partial\gamma}$  properly (rel  $E(\xi|\partial\gamma) \times Q \times \partial I$ ) to a proper homotopy  $G'$  which is  $h$ -regular on  $0 \times \partial\gamma \times Q$ . Then it can be seen that  $v_\gamma = \sum_{\beta \subset \partial\gamma} y_\beta$ , so  $(\delta c)(\gamma) = c(\partial\gamma)$  is equal to the mod 2 reduction of  $v_\gamma$ .

On the other hand,  $v_\gamma$  is the obstruction to deforming  $\bar{F}_{\partial\gamma} * (J'_{\partial\gamma} \times 1)$  properly (rel  $E(\xi|\partial\gamma) \times Q \times \partial I$ ) to a proper homotopy  $G'_{\partial\gamma}$  which is  $h$ -regular on  $0 \times \partial\gamma \times Q$ . But  $\bar{F}_{\partial\gamma} * (J'_{\partial\gamma} \times 1)$  extends to a proper homotopy  $\bar{F}_\gamma * (J'_\gamma \times 1)$  from  $f_\gamma$  to  $j'_\gamma \times 1$ , both of which are  $h$ -regular on  $0 \times \gamma \times Q$ . Now it follows from Wall's surgery theorem [14] that  $v_\gamma = 0$ . Therefore  $(\delta c)(\gamma) = 0$ , so  $c$  is a cocycle.

Let  $w \in H^3(B; \mathbb{Z}_2)$  be the cohomology class of  $c$ ; we have shown that our construction can be carried out if  $w = 0$ . Assume now that  $w = 0$ , and let  $j = \bigcup_{n=1}^{\infty} j^n, J = \bigcup_{n=1}^{\infty} J^n$ . Then  $j$  is a  $PL$  trivialisation of  $\xi$  and  $J$  is a proper homotopy from  $h$  to  $j$ , as required. It is not hard to see that  $w = 0$  whenever  $h$  is properly homotopic to a  $PL$  trivialisation, so Theorem 8 is proved.

Theorem 8 implies a result on the Hauptvermutung by fairly well-known arguments, given in Sullivan's thesis.

**Corollary.** *Let  $M^n, N^n$  be closed, 1-connected PL manifolds with  $n \geq 5$  and let  $h : M \rightarrow N$  be a topological homeomorphism. If  $H^3(M; \mathbb{Z}_2) = 0$ , then  $h$  is homotopic to a PL homeomorphism.*

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