Links

\[ L = S^n \cup \ldots \cup S^n \rightarrow S^{n+2} \]

E.G. Hopf Link

\[ S^3 \]

Relations: 1) Isotopy

2) Cobordism = concordance
**Boundary Links**

**Defn:** \( L : \bigcup_{i=1}^{n} S_i \hookrightarrow S^{n+2} \)

is a boundary link if

\[ L = \emptyset \left( \bigcup_{i=1}^{n} F_i \hookrightarrow S^{n+2} \right) \]

\( F_i \) is a Seifert surface for \( S_i \).

**E.g.**

[Diagrams of boundary links with examples of accepted and rejected links]
Seifert surface:

\[ \bigcirc \quad (x^2) \]

**Proposition:** (Gutierrez, Smythe)

A link \( L \) is a boundary link \( \iff \exists \) a group homomorphism 

\[ \Theta : \pi_1(S^{n+2} - L) \to \mathbb{E}_n \]

\[ m_i \rightarrow \mathbb{Z}_i \]

\( \{ z_1, \ldots, z_n \} \) a basis for \( \mathbb{E}_n \)

\( \{ m_1, \ldots, m_n \} \) a choice of meridian
Definition: (Cappell & Shaneson)

A pair \((L, \Theta)\) as above is called an \textbf{Eu-link}.

- \(C(n, \mu) := \left\{ \mu\text{-links} \right\}/\text{cone.}\)

- \(B(\eta, \mu) := \left\{ \mu\text{-boundary links} \right\}/\text{boundary-cone.}\)

- \(C_n(F_m) := \left\{ \text{Eu-links} \right\}/\text{boundary-cone.}\)

\[ C_n(F_m) \rightarrow B(\eta, \mu) \rightarrow C(n, \mu) \]

\(C_n(F_i) = B(n, 1) = C(n, 1) = \text{knot concordance gp.}\)
History:

- Fox & Milnor (1966)
  Defined: $CC(1,1)$, $1CC(1,1) | = \infty$

- Kervaire (1965)
  $C(2q,1) = 0$

- Levine (1969)
  $C(2q+1,1) \cong \mathbb{Z}^{\circ} \oplus \mathbb{Z}_{4}^{\circ} \oplus \mathbb{Z}_{2}^{\circ}$ ($q \geq 3$).

Seifert surface $\Rightarrow$ Seifert form.

$\Rightarrow C_{2q+1}(F_\mu)$ looks tractable.
History II

• Cappell + Shaneson:
  \[ C(2q, \mu) = 0 \]

• \[ \prod_{\mu} C(2q+1) \rightarrow B(2q+1, \mu) \ (1980) \]
  "Not all boundary links are conc. to split links."

• \[ C_{2q+1}(F_{\mu}) \cong \Gamma \left( \frac{\mathbb{Z}_2 \to \mathbb{Z}_2}{\mathbb{Z}_2 \to 1} \right) \]

Here for any unital map of rings with involution \( R \rightarrow S \)
\[ \Rightarrow \Gamma(R \rightarrow S). \]
History

- Cochran + Orr (1993)
  \[ B(2q+1, \mu) \xrightarrow{I} C(2q+1, \mu) \]
  "Not all links are conc. to boundary links."

- Still open:
  A) What is \( \mathbb{C}_{2q+1}(E_n) \)?
  B) Does \( I \) have a kernel?

Problem 9 of Levine-Orr (1996)

Des begins Ph.D. E. A. Ranicki.
Des' Thesis

Theorem A:

\[ \mu \geq 2. \]

\[ C_{2q-1}(F_4) \cong \mathbb{Z}^8 \oplus \mathbb{Z}^{10} \oplus \mathbb{Z}^{18} \]

- 30 year old problem
- Completely solved
- "Could only be solved by a student" A.R.
- "Regarded as intractable" K.O.
Open Problem:

Injectivity of $I$:

\[ \mathbb{C}^{2q+1}(\mathbb{F}_m) \xrightarrow{I'} \mathbb{B}^{2q+1}(\mathbb{F}_m) \xrightarrow{I} \mathbb{C}(2q+1, \mu) \]

$\text{Ker}(I') \subset \text{Ker}

\text{Vogel}

\text{Let } f : \mathbb{F}_m \to \mathbb{F}_m \text{ s.t. } H_i(f) \text{ an } \mathbb{F}_m.

\Sigma(f) : \Gamma(\mathbb{Z} \mathbb{F}_m \to \mathbb{Z}) \to \Gamma(\mathbb{Z} \mathbb{F}_m \to \mathbb{Z})$
Poet as Mathematician

Having perceived the connections, he sees the proof, the clean revelation in its simplest form, never doubting that somewhere waiting in the chaos is the unique elegance, the precise, airy structure defined, swift-lined, and indestructible.

Remember: you have been touched by Des’ presence.

Ivan Sheiham.