The algebraic surgery exact sequence

ANDREW RANICKI (Edinburgh)
http://www.maths.ed.ac.uk/~aar

- The algebraic surgery exact sequence is defined for any space $X$

$$
\cdots \to H_n(X; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \\
\to \mathcal{S}_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to \cdots
$$

with $A$ the $L$-theory assembly map. The functor $X \mapsto \mathcal{S}_*(X)$ is homotopy invariant.

- The 2-stage obstructions of the Browder-Novikov-Sullivan-Wall surgery theory for the existence and uniqueness of topological manifold structures in a homotopy type are replaced by single obstructions in the relative groups $\mathcal{S}_*(X)$ of the assembly map $A$. 

1
Local and global modules

- The assembly map \( A : H_\star(X; \mathbb{L}_\bullet) \rightarrow L_\star(\mathbb{Z}[\pi_1(X)]) \) is induced by a forgetful functor

\[
A : \{(\mathbb{Z}, X)\text{-modules}\} \rightarrow \{\mathbb{Z}[\pi_1(X)]\text{-modules}\}
\]

where the domain depends on the local topology of \( X \) and the target depends only on the fundamental group \( \pi_1(X) \) of \( X \), which is global.

- In terms of sheaf theory \( A = q_! p^! \) with \( p : \widetilde{X} \rightarrow X \) the universal covering projection and \( q : \widetilde{X} \rightarrow \{\text{pt.}\} \).

- The geometric model for the \( L \)-theory assembly \( A \) is the forgetful functor

\[
\{\text{geometric Poincaré complexes}\} \rightarrow \{\text{topological manifolds}\}.
\]

In fact, in dimensions \( n \geq 5 \) this functor has the same fibre as \( A \).
Local and global quadratic Poincaré complexes

• (Global) The $L$-group $L_n(\mathbb{Z}[\pi_1(X)])$ is the cobordism group of $n$-dimensional quadratic Poincaré complexes $(C, \psi)$ over $\mathbb{Z}[\pi_1(X)]$.

• (Local) The generalized homology group $H_n(X; \mathbb{L}_\bullet)$ is the cobordism group of $n$-dimensional quadratic Poincaré complexes $(C, \psi)$ over $(\mathbb{Z}, X)$. As in sheaf theory $C$ has stalks, which are $\mathbb{Z}$-module chain complexes $C(x)$ ($x \in X$).

• (Local-Global) $S_n(X)$ is the cobordism group of $(n - 1)$-dimensional quadratic Poincaré complexes $(C, \psi)$ over $(\mathbb{Z}, X)$ such that the $\mathbb{Z}[\pi_1(X)]$-module chain complex assembly $A(C)$ is acyclic.
The total surgery obstruction

- The total surgery obstruction of an $n$-dimensional geometric Poincaré complex $X$ is the cobordism class

$$s(X) = (C, \psi) \in \mathbb{S}_n(X)$$

of a $\mathbb{Z}[\pi_1(X)]$-acyclic $(n - 1)$-dimensional quadratic Poincaré complex $(C, \psi)$ over $(\mathbb{Z}, X)$. The stalks $C(x) \ (x \in X)$ measure the failure of $X$ to have local Poincaré duality

$$\cdots \to H_r(C(x)) \to H^{n-r}(\{x\}) \to H_r(X, X\setminus\{x\})$$

$$\to H_{r-1}(C(x)) \to H^{n-r+1}(\{x\}) \to \cdots$$

$X$ is an $n$-dimensional homology manifold if and only if $H_\ast(C(x)) = 0$. In particular, this is the case if $X$ is a topological manifold.

- Total Surgery Obstruction Theorem

$s(X) = 0 \in \mathbb{S}_n(X)$ if (and for $n \geq 5$ only if) $X$ is homotopy equivalent to an $n$-dimensional topological manifold.
The proof of the Total Surgery
Obstruction Theorem

The proof is a translation into algebra of the two-stage Browder-Novikov-Sullivan-Wall obstruction for the existence of a topological manifold in the homotopy type of a Poincaré complex $X$:

- The image $t(X) \in H_{n-1}(X; \mathbb{L})$ of $s(X)$ is such that $t(X) = 0$ if and only if the Spivak normal fibration $\nu_X : X \to BG$ has a topological reduction $\tilde{\nu}_X : X \to BTOP$.

- If $t(X) = 0$ then $s(X) \in S_n(X)$ is the image of the surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of the normal map $(f : M \to X, b : \nu_M \to \tilde{\nu}_X)$ determined by a choice of lift $\tilde{\nu}_X : X \to BTOP$.

- $s(X) = 0$ if and only if there exists a reduction $\tilde{\nu}_X : X \to BTOP$ for which $\sigma_*(f, b) = 0$. 
The structure invariant

• The structure invariant of a homotopy equivalence \( h : N \to M \) of \( n \)-dimensional topological manifolds is the cobordism class

\[
s(h) = (C, \psi) \in S_{n+1}(M)
\]

of a globally acyclic \( n \)-dimensional quadratic Poincaré complex \((C, \psi)\). The stalks \( C(x) \) \((x \in M)\) measure the failure of \( h \) to have acyclic point inverses, with

\[
H_\ast(C(x)) = H_\ast(h^{-1}(x) \to \{x\}) = \tilde{H}_{\ast+1}(h^{-1}(x)) \quad (x \in M).
\]

• \( h \) has acyclic point inverses if and only if \( H_\ast(C(x)) = 0 \). In particular, this is the case if \( h \) is a homeomorphism.

• **Structure Invariant Theorem**

\( s(h) = 0 \in S_{n+1}(M) \) if (and for \( n \geq 5 \) only if) \( h \) is homotopic to a homeomorphism.
The proof of the Structure Invariant Theorem (I)

The proof is a translation into algebra of the two-stage Browder-Novikov-Sullivan-Wall obstruction for the uniqueness of topological manifold structures in a homotopy type:

- the image $t(h) \in H_n(M; \mathbb{L}_\ast)$ of $s(h)$ is such that $t(h) = 0$ if and only if the normal invariant can be trivialized

$$(h^{-1})^*v_N - v_M \simeq \{\ast\} : M \rightarrow \mathbb{L}_0 \simeq G/TOP$$

if and only if $1 \cup h : M \cup N \rightarrow M \cup M$ extends to a normal bordism

$$(f, b) : (W; M, N) \rightarrow M \times ([0, 1]; \{0\}, \{1\})$$

- if $t(h) = 0$ then $s(h) \in S_{n+1}(M)$ is the image of the surgery obstruction

$$\sigma_*(f, b) \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$$
The proof of the Structure Invariant Theorem (II)

• $s(h) = 0$ if and only if there exists a normal bordism $(f, b)$ which is a simple homotopy equivalence.

• Have to work with simple $L$-groups here, to take advantage of the $s$-cobordism theorem.

• Alternative proof. The mapping cylinder of $h : N \to M$

$$P = M \cup_h N \times [0, 1]$$

defines an $(n + 1)$-dimensional geometric Poincaré pair $(P, M \cup N)$ with manifold boundary, such that $P$ is homotopy equivalent to $M$. The structure invariant is the rel $\partial$ total surgery obstruction

$$s(h) = s_\partial(P) \in \mathcal{S}_{n+1}(P) = \mathcal{S}_{n+1}(M).$$
The simply-connected case

• For $\pi_1(X) = \{1\}$ the algebraic surgery exact sequence breaks up
  
  $$0 \to S_n(X) \to H_{n-1}(X; \mathbb{L}_\bullet) \to L_{n-1}(\mathbb{Z}) \to 0$$

• The total surgery obstruction $s(X) \in S_n(X)$ maps injectively to the $TOP$ reducibility obstruction $t(X) \in H_{n-1}(X; \mathbb{L}_\bullet)$ of the Spivak normal fibration $\nu_X$. Thus for $n \geq 5$ a simply-connected $n$-dimensional geometric Poincaré complex $X$ is homotopy equivalent to an $n$-dimensional topological manifold if and only if $\nu_X : X \to BG$ admits a $TOP$ reduction $\tilde{\nu}_X : X \to BTOP$.

• The structure invariant $s(h) \in S_{n+1}(M)$ maps injectively to the normal invariant $t(h) \in H_n(M; \mathbb{L}_\bullet) = [M, G/TOP]$. Thus for $n \geq 5$ $h$ is homotopic to a homeomorphism if and only if $t(h) \simeq \{\ast\} : M \to G/TOP$. 
The geometric surgery exact sequence

- The **structure set** $\mathcal{S}^{\text{TOP}}(M)$ of a topological manifold $M$ is the set of equivalence classes of homotopy equivalences $h : N \to M$ from topological manifolds $N$, with $h \sim h'$ if there exist a homeomorphism $g : N' \to N$ and a homotopy $hg \simeq h' : N' \to M$.

- **Theorem** (Quinn, R.) The geometric surgery exact sequence for $n = \dim(M) \geq 5$
  \[ \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}^{\text{TOP}}(M) \]
  \[ \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)]) \]
  is isomorphic to the relevant portion of the algebraic surgery exact sequence
  \[ \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}_{n+1}(M) \]
  \[ \to H_n(M; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)]) \]
with
\[ \mathcal{S}^{\text{TOP}}(\partial M \times D^k, M \times S^{k-1}) = \mathcal{S}_{n+k+1}(M) \).

**Example** $\mathcal{S}^{\text{TOP}}(S^n) = \mathcal{S}_{n+1}(S^n) = 0$. 

10
**The image of the assembly map**

- **Theorem** For any finitely presented group $\pi$ the image of the assembly map

  $$A : H_n(K(\pi, 1); \mathbb{L}) \to L_n(\mathbb{Z}[\pi])$$

  is the subgroup of $L_n(\mathbb{Z}[\pi])$ consisting of the surgery obstructions $\sigma_*(f, b)$ of normal maps $(f, b) : N \to M$ of closed $n$-dimensional manifolds with $\pi_1(M) = \pi$.

- There are many calculations of the image of $A$ for finite $\pi$, notably the Oozing Conjecture proved by Hambleton-Milgram-Taylor-Williams.
**Statement of the Novikov conjecture**

- The $\mathcal{L}$-genus of an $n$-dimensional manifold $M$ is a collection of cohomology classes $\mathcal{L}(M) \in H^{4*}(M; \mathbb{Q})$ which are determined by the Pontrjagin classes of $\nu_M : M \to BTOP$. In general, $\mathcal{L}(M)$ is not a homotopy invariant.

- The Hirzebruch signature theorem for a $4k$-dimensional manifold $M$

  $$\text{signature}(H^{2k}(M), \cup) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}$$

  shows that part of the $\mathcal{L}$-genus is homotopy invariant.

- The **Novikov conjecture** for a discrete group $\pi$ is that the higher signatures for any manifold $M$ with $\pi_1(M) = \pi$

  $$\sigma_x(M) = \langle x \cup \mathcal{L}(M), [M] \rangle \in \mathbb{Q} \ (x \in H^*(K(\pi, 1); \mathbb{Q}))$$

  are homotopy invariant.
Algebraic formulation of the Novikov conjecture

- **Theorem** The Novikov conjecture holds for a group $\pi$ if and only if the rational assembly maps

$$A : H_n(K(\pi, 1); \mathbb{L} \cdot) \otimes \mathbb{Q} = H_{n-4*}(K(\pi, 1); \mathbb{Q})$$

$$\rightarrow L_n(\mathbb{Z}[\pi_1(M)]) \otimes \mathbb{Q}$$

are injective.

- **Trivially** true for finite $\pi$.

- Verified for many infinite groups $\pi$ using algebra, geometric group theory, $C^*$-algebras, etc. See Proceedings of 1993 Oberwolfach conference (LMS Lecture Notes 226, 227) for state of the art in 1995, not substantially out of date.
Statement of the Borel conjecture

- An $n$-dimensional Poincaré duality group $\pi$ is a discrete group such that the classifying space $K(\pi, 1)$ is an $n$-dimensional Poincaré complex.

- $\pi$ must be infinite and torsion-free.

- The Borel conjecture is that for every $n$-dimensional Poincaré duality group $\pi$ there exists an aspherical $n$-dimensional manifold $M \simeq K(\pi, 1)$ with
  \[
  S^{TOP}(M) = 0.
  \]
  This is topological rigidity: every homotopy equivalence $h : N \to M$ is (conjectured) to be homotopic to a homeomorphism. The conjecture also predicts higher rigidity
  \[
  S^{TOP}_{\partial}(M \times D^k, M \times S^{k-1}) = 0 \ (k \geq 0).
  \]
Algebraic formulation of the Borel conjecture

• **Theorem** For \( n \geq 5 \) the Borel conjecture holds for an \( n \)-dimensional Poincaré group \( \pi \) if and only if \( s(K(\pi, 1)) = 0 \in S_n(K(\pi, 1)) \) and the assembly map

\[
A : H_{n+k}(K(\pi, 1); \mathbb{L}) \to L_{n+k}(\mathbb{Z}[\pi_1(M)])
\]

is injective for \( k = 0 \) and an isomorphism for \( k \geq 1 \).

• Verified for many Poincaré duality groups \( \pi \), with \( S_n(K(\pi, 1)) = L_0(\mathbb{Z}) = \mathbb{Z} \).

• True in the classical case \( \pi = \mathbb{Z}^n, K(\pi, 1) = T^n \), which was crucial in the extension due to Kirby-Siebenmann (ca. 1970) of the 1960’s Browder-Novikov-Sullivan-Wall surgery theory from the differentiable and \( PL \) categories to the topological category.
The 4-periodic algebraic surgery exact sequence

- The 4-periodic algebraic surgery exact sequence is defined for any space $X$

\[ \cdots \rightarrow H_n(X; \overline{L}_\bullet) \xrightarrow{\overline{A}} L_n(\mathbb{Z}[\pi_1(X)]) \]
\[ \rightarrow \overline{S}_n(X) \rightarrow H_{n-1}(X; L_\bullet) \rightarrow \cdots \]

with $\overline{L}_0 = L_0(\mathbb{Z}) \times G/\text{TOP}$ and $\overline{A}$ the $L$-theory assembly map.

- Exact sequence

\[ \cdots \rightarrow H_n(X; L_0(\mathbb{Z})) \rightarrow \mathbb{S}_n(X) \rightarrow \overline{S}_n(X) \rightarrow \cdots \]

- The 4-periodic total surgery obstruction $\overline{s}(X) \in \overline{S}_n(X)$ of an $n$-dimensional geometric Poincaré complex $X$ is the image of $s(X) \in \mathbb{S}_n(X)$.
Homology manifolds (I)

- Every $n$-dimensional compact ANR homology manifold $M$ is homotopy equivalent to a finite $n$-dimensional geometric Poincaré complex (West)

- The total surgery obstruction $s(M) \in S_n(M)$ of an $n$-dimensional compact ANR homology manifold $M$ is the image of the Quinn resolution obstruction $i(M) \in H_n(M; L_0(\mathbb{Z}))$. The 4-periodic total surgery obstruction is $\overline{s}(M) = 0 \in \overline{S}_n(M)$.

- The homology manifold structure set $S^H(M)$ of a compact ANR homology manifold $M$ is the set of equivalence classes of simple homotopy equivalences $h : N \to M$ from topological manifolds $N$, with $h \sim h'$ if there exist an $s$-cobordism $(W; N, N')$ and an extension of $h \cup h'$ to a simple homotopy equivalence $(W; N, N') \to N \times ([0, 1]; \{0\}, \{1\})$. 
Homology manifolds (II)

- **Theorem** (Bryant-Ferry-Mio-Weinberger)
  (i) The 4-periodic total surgery obstruction of an $n$-dimensional geometric Poincaré complex $X$ is $\overline{s}(X) = 0 \in \overline{S}_n(X)$ if (and for $n \geq 6$ only if) $X$ is homotopy equivalent to a compact ANR homology manifold.
  (ii) For an $n$-dimensional compact ANR homology manifold $M$ with $n \geq 6$ the 4-periodic rel $\partial$ total surgery obstruction defines a bijection

$$\mathcal{S}^H(M) \to \overline{S}_{n+1}(M) ; \ (h : N \to M) \mapsto \overline{s}(h)$$

- $\overline{S}_{n+1}(S^n) = \mathcal{S}^H(S^n) = L_0(\mathbb{Z})$, i.e. there exists a non-resolvable compact ANR homology manifold $\Sigma^n$ homotopy equivalent to $S^n$, with arbitrary resolution obstruction $i(\Sigma^n) \in L_0(\mathbb{Z})$. 

18
The simply-connected surgery spectrum $\mathbb{L}_\bullet$

- What is $\mathbb{L}_\bullet$? Required properties
  \[ \pi_n(\mathbb{L}_\bullet) = L_n(\mathbb{Z}) \, , \, \mathbb{L}_0 \simeq G/TOP \, . \]

- What are the generalized homology groups $H_\ast(X; \mathbb{L}_\bullet)$? Will construct them as cobordism groups of combinatorial sheaves over $X$ of quadratic Poincaré complexes over $\mathbb{Z}$.

- Confession: so far, have only worked out everything for a (locally finite) simplicial complex $X$, using simplicial homology. In principle, could use singular homology for any space $X$, but this would be even harder. In any case, could use nerves of covers to get Čech theory.
The \((\mathbb{Z}, X)\)-module category

- \(X =\) simplicial complex.

- A \((\mathbb{Z}, X)\)-module is a based f.g. free \(\mathbb{Z}\)-module \(B\) with direct sum decomposition
  \[ B = \sum_{\sigma \in X} B(\sigma) . \]

- A \((\mathbb{Z}, X)\)-module morphism \(f : B \to C\) is a \(\mathbb{Z}\)-module morphism such that
  \[ f(B(\sigma)) \subseteq \sum_{\tau \geq \sigma} C(\tau) . \]

- Proposition (Ranicki-Weiss) A \((\mathbb{Z}, X)\)-module chain map \(f : D \to E\) is a chain equivalence if and only if the \(\mathbb{Z}\)-module chain maps
  \[ f(\sigma, \sigma) : D(\sigma) \to E(\sigma) \quad (\sigma \in X) \]
  are chain equivalences. (This illustrates the \(X\)-local nature of the \((\mathbb{Z}, X)\)-category).
Assembly for \((\mathbb{Z}, X)\)-modules

- Use the universal covering \( p : \tilde{X} \to X \) to define the assembly functor

  \[
  A : \{(\mathbb{Z}, X)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(X)]\text{-modules}\} ;
  \]
  
  \[
  B \mapsto B(\tilde{X}) = \sum_{\tilde{\sigma} \in \tilde{X}} B(p(\tilde{\sigma})).
  \]

- In order to extend \( A \) to \( L \)-theory need involution on the \((\mathbb{Z}, X)\)-category. Unfortunately, it does not have one! The naive dual of a \((\mathbb{Z}, X)\)-module morphism \( f : B \to C \) is not a \((\mathbb{Z}, X)\)-module morphism \( f^* : C^* \to B^* \).

- Instead, have to work with a chain duality, in which the dual of a \((\mathbb{Z}, X)\)-module \( B \) is a \((\mathbb{Z}, X)\)-module chain complex \( T(B) \). Analogue of Verdier duality in sheaf theory.
Dual cells

- The **barycentric subdivision** $X'$ of $X$ is the simplicial complex with one $n$-simplex $\hat{\sigma}_0 \hat{\sigma}_1 \ldots \hat{\sigma}_n$ for each sequence of simplexes in $X$

  $$\sigma_0 < \sigma_1 < \cdots < \sigma_n .$$

- The **dual cell** of a simplex $\sigma \in X$ is the contractible subcomplex

  $$D(\sigma, X) = \{ \hat{\sigma}_0 \hat{\sigma}_1 \ldots \hat{\sigma}_n | \sigma \leq \sigma_0 \} \subseteq X' ,$$

  with boundary

  $$\partial D(\sigma, X) = \{ \hat{\sigma}_0 \hat{\sigma}_1 \ldots \hat{\sigma}_n | \sigma < \sigma_0 \} \subseteq D(\sigma, X) .$$

- Introduced by Poincaré to prove duality.

- A simplicial map $f : M \rightarrow X'$ has acyclic point inverses if and only if

  $$(f|)_* : H_*(f^{-1}D(\sigma, X)) \cong H_*(D(\sigma, X)) (\sigma \in X) .$$

  22
Where do $(\mathbb{Z}, X)$-module chain complexes come from?

- For any simplicial map $f : M \to X'$ the simplicial chain complex $\Delta(M)$ is a $(\mathbb{Z}, X)$-module chain complex:

$$\Delta(M)(\sigma) = \Delta(f^{-1}D(\sigma, X), f^{-1}\partial D(\sigma, X))$$

- The simplicial cochain complex $\Delta(X)^{-*}$ is a $(\mathbb{Z}, X)$-module chain complex with:

$$\Delta(X)^{-*}(\sigma)_r = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$
The \((\mathbb{Z}, X)\)-module chain duality

- The additive category \(\mathbb{A}(\mathbb{Z}, X)\) of \((\mathbb{Z}, X)\)-modules has a chain duality with dualizing complex \(\Delta(X)^{-*}\)

\[
T(B) = \text{Hom}_\mathbb{Z}(\text{Hom}_{\mathbb{Z}, X}(\Delta(X)^{-*}, B), \mathbb{Z})
\]

- \(T(B)_r(\sigma) = \begin{cases} 
\sum_{\tau \geq \sigma} \text{Hom}_\mathbb{Z}(B(\tau), \mathbb{Z}) & \text{if } r = -|\sigma| \\
0 & \text{if } r \neq -|\sigma| \end{cases}\)

- the dual of a \((\mathbb{Z}, X)\)-module chain complex \(C\) is a \((\mathbb{Z}, X)\)-module chain complex \(T(C')\) with

\[
T(C) \simeq_\mathbb{Z} \text{Hom}_{\mathbb{Z}, X}(C, \Delta(X'))^{-*} \simeq_\mathbb{Z} \text{Hom}_\mathbb{Z}(C, \mathbb{Z})^{-*}
\]

- \(T(\Delta(X')) \simeq_{(\mathbb{Z}, X)} \Delta(X)^{-*}\).
The construction of the algebraic surgery exact sequence

- The generalized $\mathbb{L}_\bullet$- homology groups are the cobordism groups of adjusted $n$-dimensional quadratic Poincaré complexes over $(\mathbb{Z}, X)$

$$H_n(X; \mathbb{L}_\bullet) = L_n(\mathbb{Z}, X).$$

Require adjustments to get $\mathbb{L}_0 \simeq G/TOP$. Unadjusted $L$-theory is the 4-periodic $H_n(X; \mathbb{L}_\bullet)$ with $\mathbb{L}_0 \simeq L_0(\mathbb{Z}) \times G/TOP$. Adjust to kill $L_0(\mathbb{Z})$.

- The assembly map $A$ from $(\mathbb{Z}, X)$-modules to $\mathbb{Z}[\pi_1(X)]$-modules induces

$$A : L_n(\mathbb{Z}, X) \rightarrow L_n(\mathbb{Z}[\pi_1(X)]).$$

- The relative groups $S_n(X) = \pi_n(A)$ are the cobordism groups of $(n-1)$-dimensional quadratic Poincaré complexes $(C, \psi)$ over $(\mathbb{Z}, X)$ with assembly $C(\overline{X})$ an acyclic $\mathbb{Z}[\pi_1(X)]$-module chain complex.
Reference

- *Algebraic $L$-theory and topological manifolds*
  Mathematical Tracts 102, Cambridge (1992)