are given by

$$\eta(f_*(P,g)) \ = \ \eta(Q,h) \ = \ t(h,d) \ ,$$
$$(f^{-1})^*\eta(P,g) \ = \ (f^{-1})^*\eta(g,c) \ = \ t(f',b') \in \mathcal{T}^{TOP}(M) \ = \ H_n(M;\mathbb{L}_\bullet) \ ,$$

differing by

$$t(h,d) - t(f',b') \ = \ t(f,b)t(f',b') \in \mathcal{T}^{TOP}(M) \ = \ H_n(M;\mathbb{L}_\bullet) \ .$$

Thus

$$\begin{aligned} \eta(N,f) + \eta(f_*(P,g)) &= \ t(f,b) + t(h,d) \\ &= \ t(f,b) + t(f',b') + t(f,b)t(f',b') \\ &= \ \eta(N,f) \oplus \eta(N',f') \\ &= \ \eta(N,f) \oplus (f^{-1})^*\eta(P,g) \in \mathcal{T}^{TOP}(M) \ = \ H_n(M;\mathbb{L}_\bullet) \ . \end{aligned}$$

□

We conclude with a specific example, $M = S^p \times S^q$, one of the two cases for which the manifold structure composition formula $s(fg) = s(f) + f_*s(g)$ of Theorem 2.3 is used by Kreck and Lück [8].

Example 3.6. (i) Let $M = S^p \times S^q$ for $p,q \geqslant 2$, so that $\pi_1(M) = \{1\}$. The assembly map in quadratic L-theory is given by

$$A \ : \ H_{p+q}(M;\mathbb{L}_\bullet) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \to L_{p+q}(\mathbb{Z}) \ ; \ (x,y,z) \mapsto z$$

and

$$\mathcal{S}_{p+q+1}(M) \ = \ \ker(A) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \ .$$

The addition and intersection pairing in $H_{p+q}(M;\mathbb{L}_\bullet)$ are given by

$$(x,y,z) + (x',y',z') \ = \ (x+x', y+y', z+z') \ ,$$
$$(x,y,z)(x',y',z') \ = \ (0,0,xy'+x'y) \in H_{p+q}(M;\mathbb{L}_\bullet) \ .$$

The product $L_p(\mathbb{Z}) \otimes L_q(\mathbb{Z}) \to L_{p+q}(\mathbb{Z})$ factors through $L_p(\mathbb{Z}) \otimes L^q(\mathbb{Z})$ and the quadratic and symmetric L-groups of $\mathbb{Z}$ are given by

$$L_n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}_2 \\ 0 \end{cases} , \ L^n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ 0 \\ 0 \end{cases} \text{for } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

so the intersection pairing is non-zero only in the case $p \equiv q \equiv 0 (\mathrm{mod}\, 4)$. Given a topological normal map $(f,b) : N \to M$ make f transverse regular at $S^p \times \{*\}$, $\{*\} \times S^q \subset M$ to obtain topological normal maps

$$(f_p,b_p) \ = \ (f,b)| \ : \ N_p \ = \ f^{-1}(S^p \times \{*\}) \to S^p \ ,$$
$$(f_q,b_q) \ = \ (f,b)| \ : \ N_q \ = \ f^{-1}(\{*\} \times S^q) \to S^q \ .$$

and write the surgery obstructions as

$$(\sigma_*(f_p,b_p), \sigma_*(f_q,b_q), \sigma_*(f,b)) \ = \ (x_f,y_f,z_f) \in L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ .$$

The algebraic normal invariant defines a bijection

$$\mathcal{T}^{TOP}(M) \to L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ ; \ \eta(f,b) \mapsto t(f,b) = (x_f,y_f,z_f) \ .$$

The Whitney sum addition in $\mathcal{T}^{TOP}(M)$ corresponds to the addition

$$(x,y,z) \oplus (x',y',z') \ = \ (x+x', y+y', xy'+x'y+z+z') \in L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ .$$

Given a homotopy equivalence $f : N \to M$ let $(f,b) : N \to M$ be the corresponding topological normal map. The function

$$\mathcal{S}^{TOP}(M) \to L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \ ; \ s(f) \mapsto (x_f,y_f)$$

is a bijection, and

$$t(f,b) \ = \ (x_f,y_f,0) \in \mathcal{T}^{TOP}(M) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ .$$

Given also a homotopy equivalence $g : P \to N$ with corresponding topological normal map $(g,c) : P \to N$ let

$$f_*s(g) \ = \ (x_g,y_g) \in \mathcal{S}^{TOP}(M) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \ ,$$

so that

$$f_*t(g,c) \ = \ (x_g,y_g,0) \in \mathcal{T}^{TOP}(M) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ .$$

As in the proof of Corollary 3.5 let $(f',b') : N' \to M$ be a topological normal map with topological normal invariant

$$\eta(f',b') \ = \ (f^{-1})^*\eta(g,c) \in \mathcal{T}^{TOP}(M) \ ,$$

let $h : Q \to N$ be a homotopy equivalence with

$$s(h) \ = \ f_*s(g) \ = \ (x_g,y_g) \in \mathcal{S}^{TOP}(M) \ = \ \mathcal{S}_{p+q+1}(M) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \ ,$$

and let $(h,d) : Q \to N$ be the corresponding topological normal map. Then

$$\eta(f,b) \oplus \eta(f',b') \ = \ \eta(fg,bc) \in \mathcal{T}^{TOP}(M) \ ,$$

$$t(f',b') \ = \ (x_g, y_g, -x_f y_g - x_g y_f) \ , \ t(h,d) \ = \ (x_g,y_g,0) \ ,$$

$$\begin{aligned} t(fg,bc) \ &= \ t(f,b) + f_* t(g,c) \\ &= \ t(f,b) \oplus t(f',b') \ = \ t(f,b) + t(f',b') + t(f,b)t(f',b') \\ &= \ (x_f + x_g, y_f + y_g, 0) \in H_{p+q}(M;\mathbb{L}_\bullet) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \ , \end{aligned}$$

$$s(fg) \ = \ s(f) + f_* s(g) \ = \ (x_f + x_g, y_f + y_g) \in \mathcal{S}^{TOP}(M) \ = \ L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \ .$$

(ii) For the simplest example of the non-additivity of the surgery obstruction function

$$\theta \ : \ \mathcal{T}^{TOP}(M) \to L_n(\mathbb{Z}[\pi_1(M)])$$

with respect to $\oplus$ set $p = q = 4$ in (i), and let $f : N \to M = S^4 \times S^4$ be a homotopy equivalence with

$$\begin{aligned} s(f) \ &= \ (x,y) \in \mathcal{S}^{TOP}(M) \ = \ L_4(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \ , \\ t(f,b) \ &= \ (x,y,0) \in \mathcal{T}^{TOP}(M) \ = \ L_4(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \oplus L_8(\mathbb{Z}) \ , \\ \theta(\eta(f,b)) \ &= \ \sigma_*(f,b) \ = \ A(t(f,b)) \ = \ 0 \in L_8(\mathbb{Z}) \end{aligned}$$

for any $x,y \in \mathbb{Z}\backslash\{0\}$. By (i) a homotopy inverse $g = f^{-1} : P = M \to N$ is then such that

$$\begin{aligned} f_* s(g) \ &= \ -s(f) \ = \ (-x,-y) \in \mathcal{S}^{TOP}(M) \ = \ L_4(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \ , \\ f_* t(g,c) \ &= \ -t(f,b) \ = \ (-x,-y,0) \in \mathcal{T}^{TOP}(M) \ = \ L_4(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \oplus L_8(\mathbb{Z}) \end{aligned}$$

and a topological normal map $(f',b') : N' \to M$ with topological normal invariant $\eta(f',b') = (f^{-1})^*\eta(g,c)$ has

$$\begin{aligned} t(f',b') \ &= \ (-x,-y,2xy) \in \mathcal{T}^{TOP}(M) \ = \ L_4(\mathbb{Z}) \oplus L_4(\mathbb{Z}) \oplus L_8(\mathbb{Z}) \ , \\ \theta(\eta(f',b')) \ &= \ \sigma_*(f',b') \ = \ A(t(f',b')) \ = \ 2xy \neq 0 \in L_8(\mathbb{Z}) \ = \ \mathbb{Z} \ . \end{aligned}$$

The Whitney sum

$$\eta(f,b) \oplus \eta(f',b') \ = \ \eta(fg,bc) \ = \ \eta(1 : M \to M) \ = \ 0 \in \mathcal{T}^{TOP}(M)$$

has surgery obstruction

$$\sigma_*(fg,bc) \ = \ \sigma_*(f,b) + \sigma_*(f',b') + A(t(f,b)t(f',b')) \ = \ 0 + 2xy - 2xy \ = \ 0 \in L_8(\mathbb{Z}) \ ,$$

so

$$\theta(\eta(f,b) \oplus \eta(f',b')) = 0 \neq \theta(\eta(f,b)) + \theta(\eta(f',b')) = 2xy \in L_8(\mathbb{Z}) = \mathbb{Z} \ .$$

□

Remark 3.7. See the preprint by Jahren and Kwasik [5] for an application of the composition formula obtained in this paper to the classification of free involutions on $S^1 \times S^n$ for $n \geqslant 3$. □

References

[1] G.Brumfiel, *Homotopy equivalences of almost smooth manifolds*, Comm. Math. Helv. 46, 381–407 (1971)

[2] S.Cappell and S.Weinberger, *A geometric interpretation of Siebenmann's periodicity phenomenon*, Geometry and topology (Athens, Ga., 1985), Lecture Notes in Pure and Appl. Math. 105, 47–52, Dekker (1987)

[3] G.E.Cooke, *The Hauptvermutung according to Casson and Sullivan* (1968), published in The Hauptvermutung Book, 165–187, Kluwer (1996)

[4] M.Freedman and F.Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series 39, Princeton University Press (1990)

[5] B.Jahren and S.Kwasik, *Free involutions on $S^1 \times S^n$*, ArXiv e-print math.GT/08022035 (2008)

[6] L.Jones, *Patch spaces: a geometric represntation for Poincaré spaces*, Ann. of Maths. 97, 306–343 (1973), *Corrections for patch spaces*, ibid. 102, 183–185 (1975)

[7] R.Kirby and L.Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Mathematics Studies 88, Princeton (1977)

[8] M.Kreck and W.Lück, *Topological rigidity for non-aspherical manifolds*, to appear in the Hirzebruch 80th birthday volume of the Quarterly Journal of Pure and Applied Mathematics, Part 2, e-print arXiv:math.GT/0509238

[9] T.Kro, *Geometrically defined group structures in 3-dimensional surgery*, Oslo Cand. Scient. Thesis (2000), http://www.maths.ntnu.no/~toreak/hopp.pdf

[10] I.Madsen, L.Taylor and B.Williams, *Tangential homotopy equivalences*, Comm. Math. Helv. 55, 445-484 (1980)

[11] A.Nicas, *Induction theorems for groups of homotopy manifold structures.* Memoirs AMS 267 (1982)

[12] F.Quinn, *A geometric formulation of surgery*, Topology of manifolds, Proceedings 1969 Georgia Topology Conference, Markham Press, 500–511 (1970)

[13] ———, *$B_{(TOP_n)^\sim}$ and the surgery obstruction.* Bull. A.M.S. 77, 596–600 (1971)

[14] A.A.Ranicki, *The total surgery obstruction*, Proc. 1978 Arhus Topology Conference, Springer Lecture Notes in Mathematics 763, 275–316 (1979)

[15] ———, *The algebraic L-theory of surgery I. Foundations*, Proc. L.M.S. (3) 40, 87–192 (1980)

[16] ———, *The algebraic L-theory of surgery II. Applications to topology*, Proc. L.M.S. (3) 40, 193–283 (1980)

[17] ______, *Algebraic L-theory and topological manifolds*, Tracts in Mathematics 102, Cambridge (1992)
[18] ______, *Singularities, double points, controlled topology and chain duality*, Documenta Math. 4, 1–59 (1999)
[19] ______ and M.Weiss, *Chain complexes and assembly*, Math. Z. 204, 157–185 (1990)
[20] L.Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11, 271–294 (1972)
[21] C.T.C.Wall, *Poincaré complexes*, Ann. Math. 86, 213–245 (1967)
[22] ______, *Surgery on compact manifolds*, Academic Press (1970), 2nd edition, A.M.S. (1999)

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