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On crystal bases and Enright's completions

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Abstract

We investigate the interplay of crystal bases and completions in the sense of Enright on certain nonintegrable representations of quantum groups. We define completions of crystal bases, show that this notion of completion is compatible with Enright's completion of modules, prove that every module in our category has a crystal basis which can be completed and that a completion of the crystal lattice is unique. Furthermore, we give two constructions of the completion of a crystal lattice.

Keywords: Enright's completion; Crystal basis; Crystal lattice; Nonintegrable representations; Quantum groups

Introduction

In this paper we are concerned with bringing together two distinct notions of the representation theory of quantized enveloping algebras—those of crystal bases and completions.

The theory of crystal and canonical bases is one of the most remarkable developments in representation theory. It was introduced independently by M. Kashiwara [7,8] and G. Lusztig [11] in a combinatorial and a geometric way, respectively. In this paper we follow Kashiwara's approach. Roughly speaking, a crystal basis of an integrable representation for the quantized enveloping algebra $U_q(\mathfrak{g})$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ (or more generally, of a symmetrizable Kac–Moody Lie algebra) is a pair consisting of a lattice of the module, called a crystal lattice, and a vector space basis of a quotient of the crystal lattice, called a crystal. It is actually a certain parametrization of bases of the module with a number

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of desirable properties and it encodes the intrinsic structure of the module in a combinatorial way. One of the most important combinatorial realizations of crystals is Littelmann's path model [9,10].

T. Enright [3] introduced a notion of completion with respect to a simple root of $\mathfrak{g}$ on the category $\mathcal{I}(\mathfrak{g})$ of weight modules of $\mathfrak{g}$ that are $U^-(\mathfrak{g})$-torsion free and $U^+(\mathfrak{g})$-finite. Completion is an effective process of obtaining new representations from a given one, containing the original one as a subrepresentation. Enright used the completion functor in order to algebraically construct the fundamental series representations. Soon after, V. Deodhar [2] realized the completion functor via Ore localization giving a concrete model of completion and used it to prove Enright's uniqueness conjecture arising from considering successive completions (stressing the "$\mathfrak{sl}_2$-nature" of completions). A. Joseph [5] then generalized this functor to the Bernstein–Gelfand–Gelfand category $\mathcal{O}(\mathfrak{g})$ and gave a refinement of the Jantzen conjecture by studying the lifting of a contravariant form under the action of the completion functor. Later, Y.M. Zou [15] extended some of the results obtained by Enright and Deodhar to the quantum groups setting.

The starting motivation of this work was to study both crystal bases and completions of modules belonging to the category $\mathcal{I}(U_q(\mathfrak{g}))$ (thus nonintegrable ones) and examine how the two concepts relate to each other with the aim of introducing a notion of completion of crystal bases which would be compatible with Enright's completion of modules. It is natural to expect that such a program may eventually lead to some potentially interesting interplay between the theory of crystal bases and that of fundamental series representations. Furthermore, since all the Verma modules in a lattice of inclusions of Verma modules may be obtained from the corresponding irreducible one by means of completions, one may expect that a successful crystal base theory related to completions could produce a combinatorial tool relevant for studying Jantzen filtrations and some other Kazhdan–Lusztig Theory related topics.

Beside defining crystal bases of integrable representations, Kashiwara [8] also defined the crystal basis of the quantization $U_q^-(\mathfrak{g})$ of the universal enveloping algebra of the nilpotent part of $\mathfrak{g}$ by considering $U_q^-(\mathfrak{g})$ as a module for Kashiwara's algebra $B_q(\mathfrak{g})$. Although the former was the main goal of [8], the latter provided a way to simultaneously consider the bases of all integrable representations, and the interplay of the two notions of crystal bases played a central role in the paper. Since [8] there have been only a few papers concerned with crystal bases or crystals of representations which are not necessarily integrable, including [6,13,14]. Also, in a joint work with V. Chari and A. Moura [1] we considered the problem of tensor product decompositions into indecomposables for nonintegrable modules in the BGG category $\mathcal{O}$ by introducing the combinatorial objects called branched crystals which satisfy a relaxed axiom for formal invertibility of Kashiwara's operators.

In this paper, we follow Kashiwara's definition of crystal bases, thus the U_q^--torsion free modules in question are naturally viewed as $\mathcal{B}_q(\mathfrak{g})$-modules. On the other hand, in regard to completions they are thought of as $U_q(\mathfrak{g})$-modules. This aspect makes the situation more interesting and a synchronization of the two structures becomes essential. As the $U_q(\mathfrak{sl}_2)$-case is already quite intricate, we restrict ourselves to that case in this paper leaving the consideration on how to extend this theory to the higher rank case to a future work.

We introduce a category $\widetilde{\mathcal{I}}$ consisting of U_q-modules in $\mathcal{I} = \mathcal{I}(U_q(\mathfrak{sl}_2))$ with a compatible $\mathcal{B}_q$-structure and obtain a decomposition of every module in $\widetilde{\mathcal{I}}$ into a direct sum of indecomposable U_q-submodules which are also $\mathcal{B}_q$-invariant (cf. Theorem 2.4). Due to this decomposition, we are able to introduce weight spaces into our consideration of crystal bases of modules in $\widetilde{\mathcal{I}}$ even though these crystal bases arise from the $\mathcal{B}_q$-structures. Therefore, we get a setting resembling the one of integrable U_q-modules. We define a notion of a complete crystal basis and a completion of a crystal basis. In the process, we take advantage of a symmetric action of the Kashiwara operators $\widetilde{e}$ and $\widetilde{f}$ on crystal bases. However, unlike the case of modules where the completion of a module contains the module, one cannot expect a completion of a crystal lattice to contain the crystal lattice; this being indeed clear from the case of Verma modules. Our definition of completion of bases involves a very natural connection of crystal bases arising from $\mathcal{B}_q$-structures with the ones arising from U_q-structures. We prove that a crystal basis is complete if and only if its corresponding module is. We also show that every module in $\widetilde{\mathcal{I}}$ has a crystal basis which can be completed and moreover a completion of the crystal lattice is unique (cf. Theorems 3.5 and 3.6).

The paper is arranged as follows. In Section 1, we set the notation and review relevant results on completions and crystal bases. We think it instructive to have this section done for $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is any simple Lie algebra, as it is then more transparent how crucial a step the $U_q(\mathfrak{sl}_2)$-case is in solving the problem. However, the remaining sections treat the $U_q(\mathfrak{sl}_2)$-case, except where stated otherwise. In Section 2, we consider some natural $\mathcal{B}_q$-structures on the indecomposable U_q-modules in $\mathcal{I}$, collect the desirable properties that a $\mathcal{B}_q$-structure should have with respect to a U_q-structure in order to define the category $\widetilde{\mathcal{I}}$, and prove the simultaneous decomposition theorem for modules in $\widetilde{\mathcal{I}}$. In Section 3, we look into the crystal bases with which modules in $\widetilde{\mathcal{I}}$ are naturally endowed via their $\mathcal{B}_q$-structures, define complete crystal bases, show they correspond to complete modules, define a completion of a crystal basis, and prove the aforementioned Theorems 3.5 and 3.6. In Section 4 we give two constructions of the completion of a crystal lattice—one obtained by modifying Deodhar's model of completion of modules and the other by applying an operator used by Kashiwara in [7] to construct the operators $\widetilde{e}$ and $\widetilde{f}$.

1. Preliminaries

1.1. Let $(a_{ij})_{1\leqslant i,j\leqslant l}$ be the Cartan matrix of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and d_i unique positive integers such that gcd $(d_1,\dots,d_l)=1$ and the matrix $(d_i a_{ij})_{1\leqslant i,j\leqslant l}$ is symmetric. Let q be an indeterminate and $\mathbb{Q}(q)$ the field of rational functions of q with coefficients in $\mathbb{Q}$. Set $q_i=q^{d_i}$. The quantized enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra with generators e_i, f_i, t_i, t_i^{-1}, $1\leqslant i\leqslant l$, and defining relations

$$t_i t_i^{-1}=1=t_i^{-1}t_i, \qquad t_i t_j=t_j t_i,$$
$$t_i e_j t_i^{-1}=q_i^{a_{ij}}e_j, \qquad t_i f_j t_i^{-1}=q_i^{-a_{ij}}f_j,$$
$$e_i f_j-f_j e_i=\delta_{ij}\frac{t_i-t_i^{-1}}{q_i-q_i^{-1}},$$
$$\sum_{s=0}^{1-a_{ij}}(-1)^s e_i^{(s)}e_j e_i^{(1-a_{ij}-s)}=0, \qquad \sum_{s=0}^{1-a_{ij}}(-1)^s f_i^{(s)}f_j f_i^{(1-a_{ij}-s)}=0 \quad (i\neq j)$$

where

$$e_i^{(n)}=\frac{e_i^n}{[n]_i!}, \qquad f_i^{(n)}=\frac{f_i^n}{[n]_i!}, \quad [n]_i!=[1]_i[2]_i\dots[n]_i \ \left(n\in\mathbb{Z}^+\right), \quad \text{and}$$
$$[n]_i=\frac{q_i^n-q_i^{-n}}{q_i-q_i^{-1}} \quad (n\in\mathbb{Z}).$$

Denote by $U_q^+(\mathfrak{g})$ (respectively $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by e_i (respectively f_i), $1 \leqslant i \leqslant l$. Let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by t_i and t_i^{-1}, $1 \leqslant i \leqslant l$. Multiplication defines an isomorphism of $\mathbb{Q}(q)$-vector spaces $U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) \to U_q(\mathfrak{g})$.

Denote by $U_q(\mathfrak{g})_i$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i and t_i^{-1}. Then $U_q(\mathfrak{g})_i \cong U_{q_i}(\mathfrak{sl}_2)$.

1.2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\{\alpha_i\}_{1 \leqslant i \leqslant l} \subset \mathfrak{h}^*$ be a set of simple roots of $\mathfrak{g}$ and $\{h_i\}_{1 \leqslant i \leqslant l} \subset \mathfrak{h}$ the corresponding set of coroots so that $\alpha_j(h_i) = a_{ij}$. Let $Q = \sum_{i=1}^l \mathbb{Z}\alpha_i$ be the root lattice and $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z},\ 1 \leqslant i \leqslant l\}$ the weight lattice for $\mathfrak{g}$. Also, set $Q^+ = \sum_{i=1}^l \mathbb{Z}^+\alpha_i$ and $P^+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}^+,\ 1 \leqslant i \leqslant l\}$. Denote by W be the Weyl group of Φ and by s_i the reflection with respect to the simple root α_i $(1 \leqslant i \leqslant l)$.

For a $U_q(\mathfrak{g})$-module M and $\lambda \in P$, we define the λ-weight space of M as $M_\lambda = \{m \in M \mid t_i m = q_i^{\lambda(h_i)} m,\ 1 \leqslant i \leqslant l\}$. M is a weight module if it is a direct sum of its weight spaces. If in addition there exist $\lambda \in P$ and a nonzero vector $m \in M_\lambda$ such that $e_i \cdot m = 0$ for all i and $M = U_q(\mathfrak{g}) \cdot m$, then M is a highest weight module with highest weight λ and highest weight vector m.

The Verma module $M(\lambda)$ is the $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})/J(\lambda)$ where $J(\lambda)$ is the left ideal of $U_q(\mathfrak{g})$ generated by $\{t_i - q_i^{\lambda(h_i)}, e_i \mid 1 \leqslant i \leqslant l\}$. $M(\lambda)$ is the universal highest weight module of weight λ. The unique irreducible quotient $V(\lambda)$ of $M(\lambda)$ is finite-dimensional iff $\lambda \in P^+$.

1.3. We recall the definitions and some properties of category $\mathcal{I}$, completions, and T modules (cf. [2–4,15]).

Let $\mathcal{I}(U_q(\mathfrak{g}))$ be the category of $U_q(\mathfrak{g})$-modules M satisfying: (i) M is a weight module, (ii) $U_q^-(\mathfrak{g})$-action on M is torsion free, and (iii) M is $U_q^+(\mathfrak{g})$-finite, i.e., e_i acts locally nilpotently on M for all i.

Fix $i \in \{1, \ldots, l\}$. Set $M_n = \{m \in M \mid t_i m = q_i^n m\}$ for $n \in \mathbb{Z}$, $M^{e_i} = \{m \in M \mid e_i m = 0\}$, and $M_n^{e_i} = M_n \cap M^{e_i}$. A module M in $\mathcal{I}(U_q(\mathfrak{g}))$ is said to be complete with respect to i if $f_i^{n+1} : M_n^{e_i} \to M_{-n-2}^{e_i}$ is bijective for all $n \in \mathbb{Z}^+$. A module N in $\mathcal{I}(U_q(\mathfrak{g}))$ is a completion of M with respect to i provided: (i) N is complete with respect to i, (ii) M is embedded in N, and (iii) N/M is f_i-finite.

Theorem. *(Cf. [2,3,15].)*

(i) *Every module M in $\mathcal{I}(U_q(\mathfrak{g}))$ has a completion $C_i(M)$ with respect to i $(1 \leqslant i \leqslant l)$ and any two such completions are naturally isomorphic.*
(ii) *Let $w \in W$ and $M \in \mathcal{I}(U_q(\mathfrak{g}))$. For any two reduced expressions $w = s_{i_1} \ldots s_{i_k} = s_{j_1} \ldots s_{j_k}$, there exists an isomorphism $F : C_{i_1}(C_{i_2} \ldots (C_{i_k}(M)) \ldots) \to C_{j_1}(C_{j_2} \ldots (C_{j_k}(M)) \ldots)$ such that $F|_M$ is the identity.*

Thus the process of completion depends essentially on the $U_q(\mathfrak{sl}_2)$-representation theory and so we pay special attention to the case $\mathfrak{g} = \mathfrak{sl}_2$.

1.4. For brevity, write $U_q = U_q(\mathfrak{sl}_2)$ and $\mathcal{I} = \mathcal{I}(U_q(\mathfrak{sl}_2))$. Denote the generators of $\mathfrak{sl}_2$ by e, f, $t^{\pm 1}$.

Let $n \in \mathbb{Z}$. The quantum Casimir element $C = \dfrac{qt + q^{-1}t^{-1}}{(q - q^{-1})^2} + fe$ acts on the Verma module $M(n)$ as multiplication by scalar $c_n = \dfrac{q^{n+1} + q^{-n-1}}{(q - q^{-1})^2}$. Consider the left ideal $I(n)$ of U_q defined by $I(n) = U_q\{t - q^{-n-2}, e^{n+2}, (C - c_n)^2\}$. The T module $T(n)$ is defined as $T(n) = U_q/I(n)$. It is an indecomposable module belonging to the category $\mathcal{I}$ and $0 \to M(n) \to T(n) \to M(-n-2) \to 0$ is exact.

Theorem. *(Cf. [3,15].)*

(i) *The $M(n)$ ($n \in \mathbb{Z}$) and the $T(n)$ ($n \in \mathbb{Z}^+$) are precisely all the indecomposable objects of the category $\mathcal{I}$. Among these, the $M(n)$ for $n \geqslant -1$ and $T(n)$ for $n \in \mathbb{Z}^+$ are the complete ones. The completion of $M(-n-2)$ is $M(n)$ for $n \geqslant -1$.*

(ii) *Every module in $\mathcal{I}$ is a direct sum (not necessarily finite) of indecomposable ones.*

1.5. In the subsections that follow, we recall the main results on Kashiwara's crystal bases (cf. [8]).

Let M be a finite-dimensional $U_q(\mathfrak{g})$-module. Fix an index i $(1 \leqslant i \leqslant l)$. By the representation theory of $U_q(\mathfrak{sl}_2)$, $M = \bigoplus_{\lambda \in P, 0 \leqslant k \leqslant \lambda(h_i)} f_i^{(k)}(\operatorname{Ker} e_i \cap M_\lambda)$. Hence, every element $u \in M_\lambda$ can be uniquely written as $u = \sum_{k \geqslant 0} f_i^{(k)} u_k$ where $u_k \in \operatorname{Ker} e_i \cap M_{\lambda + k\alpha_i}$. The Kashiwara operators $\widetilde{e_i}$ and $\widetilde{f_i}$ are defined by $\widetilde{e_i}u = \sum_{k \geqslant 1} f_i^{(k-1)} u_k$ and $\widetilde{f_i}u = \sum_{k \geqslant 0} f_i^{(k+1)} u_k$. It follows that $\widetilde{e_i} M_\lambda \subseteq M_{\lambda + \alpha_i}$ and $\widetilde{f_i} M_\lambda \subseteq M_{\lambda - \alpha_i}$. Also, if $u \in \operatorname{Ker} e_i$ and $n > 0$, then $\widetilde{f_i}^n u = f_i^{(n)} u$.

Let $\mathbb{A}$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$.

Definition. (See [8].) A free $\mathbb{A}$-submodule L of M is called a crystal lattice if: (a) $M \cong \mathbb{Q}(q) \otimes_{\mathbb{A}} L$, (b) $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap M_\lambda$, and (c) $\widetilde{e_i} L \subseteq L$ and $\widetilde{f_i} L \subseteq L$, $1 \leqslant i \leqslant l$. A crystal basis of M is a pair (L, B) satisfying the following conditions: (i) L is a crystal lattice of M, (ii) B is a $\mathbb{Q}$-basis of L/qL, (iii) $B = \bigsqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$, (iv) $\widetilde{e_i} B \subseteq B \cup \{0\}$ and $\widetilde{f_i} B \subseteq B \cup \{0\}$, $1 \leqslant i \leqslant l$, and (v) for $b, b' \in B$, $\widetilde{f_i} b = b'$ if and only if $b = \widetilde{e_i} b'$.

For $\lambda \in P^+$, consider the finite-dimensional irreducible $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight λ and highest weight vector u_λ. Let $L(\lambda)$ be the smallest $\mathbb{A}$-submodule of $V(\lambda)$ containing u_λ which is stable under $\widetilde{f_i}$'s, i.e., $L(\lambda)$ is the $\mathbb{A}$-span of the vectors of the form $\widetilde{f_{i_1}} \ldots \widetilde{f_{i_r}} u_\lambda$, where $1 \leqslant i_j \leqslant l$ and $r \in \mathbb{Z}^+$. Set $B(\lambda) = \{b \in L(\lambda)/qL(\lambda) \mid b = \widetilde{f_{i_1}} \ldots \widetilde{f_{i_r}} u_\lambda \bmod qL(\lambda)\} \setminus \{0\}$. Then $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$ (cf. [8, Theorem 2]).

Since crystal bases are stable under direct sums, every finite-dimensional $U_q(\mathfrak{g})$-module M has a crystal basis.

1.6. Isomorphism of crystal bases is defined as follows:

Definition. Let (L_1, B_1) and (L_2, B_2) be crystal bases of finite-dimensional $U_q(\mathfrak{g})$-modules M_1 and M_2, respectively. We say that $(L_1, B_1) \cong (L_2, B_2)$ if there exists a $U_q(\mathfrak{g})$-isomorphism $\varphi : M_1 \to M_2$ which induces an isomorphism of $\mathbb{A}$-lattices $\varphi : L_1 \to L_2$ such that $\overline{\varphi}(B_1) = B_2$ where $\overline{\varphi} : L_1/qL_1 \to L_2/qL_2$ is the induced isomorphism of $\mathbb{Q}$-vector spaces.

If (L, B) is a crystal basis of a finite-dimensional $U_q(\mathfrak{g})$-module M, then there exists an isomorphism $M \cong \bigoplus_j V(\lambda_j)$ by which (L, B) is isomorphic to $\bigoplus_j (L(\lambda_j), B(\lambda_j))$ (cf. [8, Theorem 3]).

1.7. Kashiwara's algebra $\mathcal{B}_q(\mathfrak{g})$ (cf. [8,12]) is the $\mathbb{Q}(q)$-algebra generated by e'_i and f_i, $1 \leqslant i \leqslant l$, subject to relations

$$e'_i f_j = q_i^{-a_{ij}} f_j e'_i + \delta_{ij}, \tag{1.1}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \big(e'_i\big)^{(s)} \big(e'_j\big) \big(e'_i\big)^{(1-a_{ij}-s)} = 0, \qquad \sum_{s=0}^{1-a_{ij}} (-1)^s f_i^{(s)} f_j f_i^{(1-a_{ij}-s)} = 0 \quad (i \neq j). \tag{1.2}$$

For the case $\mathfrak{g} = \mathfrak{sl}_2$, write $\mathcal{B}_q = \mathcal{B}_q(\mathfrak{sl}_2)$. Then the generators e' and f of $\mathcal{B}_q$ satisfy

$$e' f = q^{-2} f e' + 1. \tag{1.3}$$

For each $P \in U_q^-(\mathfrak{g})$, there exist unique $R, Q \in U_q^-(\mathfrak{g})$ such that $[e_i, P] = \dfrac{t_i Q - t_i^{-1} R}{q_i - q_i^{-1}}$. Define $e'_i \in \operatorname{End}(U_q^-(\mathfrak{g}))$ by $e'_i(P) = R$. Then, $U_q^-(\mathfrak{g})$ is a left $\mathcal{B}_q(\mathfrak{g})$-module where e'_i acts as described and f_i acts by the left multiplication. Moreover, $U_q^-(\mathfrak{g})$ is an irreducible $\mathcal{B}_q(\mathfrak{g})$-module and $U_q^-(\mathfrak{g}) \in \mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$, where $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$ is the category of $\mathcal{B}_q(\mathfrak{g})$-modules M satisfying that for all $u \in M$, there exists $r \geqslant 0$ such that $e'_{i_1} e'_{i_2} \ldots e'_{i_r} u = 0$ for any $1 \leqslant i_1, \ldots, i_r \leqslant l$. Furthermore, $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$ is semisimple and $U_q^-(\mathfrak{g})$ is the unique irreducible object of $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$, up to isomorphism.

Remark. For each i, $e'_i(1) = 0$ and $e'_i(f_i^p) = q_i^{-(p-1)} [p]_i f_i^{p-1}$ for $p \geqslant 1$.

1.8. Let M belong to the category $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$. Fix an index i $(1 \leqslant i \leqslant l)$. Then

$$M = \bigoplus_{k \geqslant 0} f_i^{(k)} \operatorname{Ker} e'_i. \tag{1.4}$$

Define Kashiwara's operators $\widetilde{e_i}, \widetilde{f_i} \in \operatorname{End}(M)$ by

$$\widetilde{e_i}\big(f_i^{(k)} u\big) = \begin{cases} f_i^{(k-1)} u, & k \geqslant 1, \\ 0, & k = 0, \end{cases} \qquad \widetilde{f_i}\big(f_i^{(k)} u\big) = f_i^{(k+1)} u, \tag{1.5}$$

for $u \in \operatorname{Ker} e'_i$ and extend linearly.

Then $\widetilde{e_i}\widetilde{f_i} = 1$. Also, $\widetilde{f_i}\widetilde{e_i}$ is the projection onto $f_i M$ with respect to $M = f_i M \oplus \operatorname{Ker} e'_i$.

Definition. (See [8].) A free $\mathbb{A}$-submodule L of M is called a crystal lattice if: (a) $M \cong \mathbb{Q}(q) \otimes_{\mathbb{A}} L$, and (b) $\widetilde{e_i} L \subseteq L$ and $\widetilde{f_i} L \subseteq L$, $1 \leqslant i \leqslant l$. A crystal basis of M is a pair (L, B) satisfying the following conditions: (i) L is a crystal lattice of M, (ii) B is a $\mathbb{Q}$-basis of L/qL, (iii) $\widetilde{e_i} B \subseteq B \cup \{0\}$ and $\widetilde{f_i} B \subseteq B$, $1 \leqslant i \leqslant l$, and (iv) if $b \in B$ such that $\widetilde{e_i} b \in B$, then $b = \widetilde{f_i}\widetilde{e_i} b$.

Isomorphism of crystal bases of modules in $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$ is defined analogously to isomorphism of crystal bases of finite-dimensional modules replacing $U_q(\mathfrak{g})$ by $\mathcal{B}_q(\mathfrak{g})$ in Definition 1.6.

It will be clear from the context if we mean crystal basis in the sense of Definition 1.5 or in the sense of Definition 1.8. Otherwise, we will make the distinction.

Let $L(\infty)$ be the smallest $\mathbb{A}$-submodule of $U_q^-(\mathfrak{g})$ containing 1 that is stable under $\widetilde{f}_i$'s, i.e., $L(\infty) = \mathbb{A}$-span of $\{\widetilde{f}_{i_1} \dots \widetilde{f}_{i_r} \cdot 1 \mid 1 \leqslant i_j \leqslant l;\ r \in \mathbb{Z}^+\}$. Set $B(\infty) = \{b \in L(\infty)/qL(\infty) \mid b = \widetilde{f}_{i_1} \dots \widetilde{f}_{i_r} \cdot 1 \bmod qL(\infty)\}$. Then $(L(\infty), B(\infty))$ is a crystal basis of $U_q^-(\mathfrak{g})$ [8, Theorem 4]. Moreover, any crystal basis of $U_q^-(\mathfrak{g})$ coincides with $(L(\infty), B(\infty))$ up to a constant multiple.

The relation of $(L(\infty), B(\infty))$ to $(L(\lambda), B(\lambda))$ is given by [8, Theorem 5].

2. U_q-modules with $\mathcal{B}_q$-module structure

In this section we are concerned with U_q-modules endowed with B_q-module structures which allow decompositions that are simultaneously U_q and B_q-invariant. In 2.1 we drop the assumption $\mathfrak{g} = \mathfrak{sl}_2$, but we uphold it otherwise.

2.1. For $\lambda \in P$, we consider $M(\lambda)$, the Verma module with highest weight λ. If m_λ is a highest weight vector of $M(\lambda)$, then $M(\lambda) = U_q(\mathfrak{g})m_\lambda = U_q^-(\mathfrak{g})m_\lambda$. Since $U_q^-(\mathfrak{g})$ is a $\mathcal{B}_q(\mathfrak{g})$-module, $M(\lambda)$ has a natural $\mathcal{B}_q(\mathfrak{g})$-structure via composition of $\mathbb{Q}(q)$-algebra homomorphisms

$$\mathcal{B}_q(\mathfrak{g}) \to \operatorname{End}\big(U_q^-(\mathfrak{g})\big) \xrightarrow{\Theta} \operatorname{End}\big(M(\lambda)\big).$$

Namely, if $\varphi_\lambda : U_q^-(\mathfrak{g}) \to M(\lambda)$ is the $U_q^-(\mathfrak{g})$-module isomorphism sending 1 to m_λ, then $\Theta(g) = \varphi_\lambda g \varphi_\lambda^{-1}$ for $g \in \operatorname{End}(U_q^-(\mathfrak{g}))$.

Hence, if $f_i, e_i' \in \operatorname{End}(U_q^-(\mathfrak{g}))$, $1 \leqslant i \leqslant l$, are defined as in 1.7, then $f_i, e_i' \in \operatorname{End}(M(\lambda))$ are given as follows: $f_i \cdot um_\lambda = \Theta(f_i)(um_\lambda) = \varphi_\lambda f_i(u) = f_i um_\lambda$ and $e_i' \cdot um_\lambda = \Theta(e_i')(um_\lambda) = \varphi_\lambda e_i'(u) = e_i'(u)m_\lambda$ for $u \in U_q^-(\mathfrak{g})$. Using the same symbols for f_i and e_i' in both cases should create no confusion.

Remark. The following conclusions are evident.

(1) $e_i' \cdot m_\lambda = 0$ for each i.
(2) $f_i, e_i' \in \operatorname{End}(M(\lambda))$ do not depend on the choice of a highest weight vector.
(3) $\varphi_\lambda : U_q^-(\mathfrak{g}) \to M(\lambda)$ is also a $\mathcal{B}_q(\mathfrak{g})$-module isomorphism.
(4) $M(\lambda) \in \mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$.
(5) $M(\lambda)$ is irreducible as a $\mathcal{B}_q(\mathfrak{g})$-module.

We emphasize Remark 2.1(5) at this point. For example, in $\mathfrak{sl}_2$-case, $M(-n-2)$ is a submodule of $M(n)$ ($n \in \mathbb{Z}^+$) with respect to the U_q-structure, but as $\mathcal{B}_q$-modules $M(-n-2) \cong M(n)$.

2.2. Next, we study the $\mathcal{B}_q$-structure on T modules and therefore we consider $\mathfrak{g} = \mathfrak{sl}_2$. Let $n \in \mathbb{Z}^+$. We aim to extend the $\mathcal{B}_q$-action on $M(n)$ embedded in $T(n)$ to the whole of $T(n)$.

Let z be a U_q-generator of $T(n)$ of weight $-n-2$ and let v be a highest weight vector of the Verma submodule of $T(n)$ with highest weight n. Consider the U_q^--decomposition

$$T(n) = U_q^- \cdot v \oplus U_q^- \cdot z. \tag{2.1}$$

We define $f, e' \in \mathrm{End}(T(n))$ using (2.1) and $f, e' \in \mathrm{End}(U_q^-)$ as follows. For $u \in U_q^-$, set

$$f \cdot uv = fuv, \qquad f \cdot uz = fuz, \qquad e' \cdot uv = e'(u)v, \qquad e' \cdot uz = e'(u)z. \tag{2.2}$$

The next assertion is easily seen.

Lemma. *Equations* (2.2) *define a $\mathcal{B}_q$-module structure on $T(n)$.*

Remark.

(1) $\operatorname{Ker} e' = \mathbb{Q}(q)v \oplus \mathbb{Q}(q)z$ (cf. Remark 1.7).
(2) $T(n) \in \mathcal{O}(\mathcal{B}_q)$.

We call the above *standard $\mathcal{B}_q$-structures* on Verma and T modules.

2.3. We now consider the category $\mathcal{I}$ and collect together desirable properties that *any* $\mathcal{B}_q$-structure must have with respect to the U_q-structure so that a synchronization of the two would be plausible.

Let M be in $\mathcal{I}$ and denote by M^c the generalized c-eigenspace of the quantum Casimir element C on M. Since M is U_q^+-finite and C is central, then $M = \bigoplus_{r\in\mathbb{Z}} M^{c_r}$ where $c_r = \dfrac{q^{r+1} + q^{-r-1}}{(q-q^{-1})^2}$. Clearly, $c_r = c_{-r-2}$.

Definition. Let $\widetilde{\mathcal{I}}$ be the category with objects all finitely generated U_q-modules M in the category $\mathcal{I}$ which are also equipped with a $\mathcal{B}_q$-module structure satisfying the following conditions:

(a) $f \in \mathcal{B}_q$ acts the same as $f \in U_q$.
(b) If $e \cdot m = 0$ for a weight vector $m \in M \setminus f \cdot M$ of weight $n \in \mathbb{Z}$, then $e' \cdot m = 0$.
(c) If $e \cdot m \neq 0$ for a weight vector $m \in M^{c_n} \setminus f \cdot M$ of weight $-n-2$ for $n \in \mathbb{Z}^+$, then there exist a $\mathbb{Q}(q)$-subspace T of M such that
   (1) T is both U_q and $\mathcal{B}_q$-submodule of M,
   (2) $T \cong T(n)$ both as U_q and $\mathcal{B}_q$-module where $T(n)$ is endowed with standard $\mathcal{B}_q$-structure,
   (3) $e \cdot m = e \cdot \widetilde{m}$ for some U_q-generator $\widetilde{m}$ of T of weight $-n-2$.

Morphisms are defined to be $\mathbb{Q}(q)$-linear maps that are both U_q and $\mathcal{B}_q$-morphisms.

The following lemma and proposition are immediate.

Lemma. *Let M be in $\widetilde{\mathcal{I}}$, $m \in M \setminus f \cdot M$, and $p \geqslant 1$. Then*

$$e' f^p m = q^{-2p} f^p e' m + \frac{1-q^{-2p}}{1-q^{-2}} f^{p-1} m.$$

By the above lemma, the action of e' on M is determined by its action on $M \setminus f \cdot M$.

Proposition.

(i) *The modules $M(r)$, $r \in \mathbb{Z}$, and $T(n)$, $n \in \mathbb{Z}^+$, with the standard $\mathcal{B}_q$-structures belong to $\widetilde{\mathcal{I}}$.*
(ii) *Finite direct sums of modules in $\widetilde{\mathcal{I}}$ are also in $\widetilde{\mathcal{I}}$.*

2.4. We next aim to show:

Theorem. *Let M be in $\widetilde{\mathcal{I}}$. Then $M = M^{(1)} \oplus \cdots \oplus M^{(s)}$ for some $s > 0$, where*:

- (i) *$M^{(j)}$ is a U_q and $\mathcal{B}_q$-submodule of M*;
- (ii) *$M^{(j)}$ is U_q-isomorphic either to $M(r)$, $r \in \mathbb{Z}$, or $T(n)$, $n \in \mathbb{Z}^+$*;
- (iii) *$M^{(j)}$ is $\mathcal{B}_q$-isomorphic either to $M(r)$, $r \in \mathbb{Z}$, or $T(n)$, $n \in \mathbb{Z}^+$, with standard $\mathcal{B}_q$-structures.*

Proof. If follows from Theorem 1.4 and finite generation of M that $M = N^{(1)} \oplus \cdots \oplus N^{(p)} \oplus \cdots \oplus N^{(s)}$ where, as U_q-modules, $N^{(j)} \cong M(r_j)$ for some $r_j \in \mathbb{Z}$ for $1 \leqslant j \leqslant p$, and $N^{(j)} \cong T(n_j)$ for some $n_j \in \mathbb{Z}^+$ for $p+1 \leqslant j \leqslant s$.

For $1 \leqslant j \leqslant p$, let m_j be a highest weight vector of $N^{(j)}$. It is clear that $m_j \notin f \cdot M$ and $e \cdot m_j = 0$. So by Definition 2.3(b), $e' \cdot m_j = 0$. It then follows from Lemma 2.3 that $N^{(j)}$ is stable under e', i.e., $N^{(j)}$ is a $\mathcal{B}_q$-submodule as well, and moreover it is equipped with the standard $\mathcal{B}_q$-structure.

Now consider $N^{(p+1)}$ which is U_q-isomorphic to $T(n_{p+1})$ and for simplicity let $n = n_{p+1}$ and $N = N^{(p+1)}$. Let z be a U_q-generator of N of weight $-n-2$ and let v be a highest weight vector of the Verma submodule of N with highest weight n. Note that $z \notin f \cdot M$, $e \cdot z \neq 0$, and $z \in M^{c_n}$. So by Definition 2.3(c), there exist a $\mathbb{Q}(q)$-subspace T of M which is both U_q and $\mathcal{B}_q$-isomorphic to $T(n)$ with standard $\mathcal{B}_q$-action and a generator $\widetilde{z}$ of T such that $e \cdot z = e \cdot \widetilde{z}$. Set $w = z - \widetilde{z}$. Then $e \cdot w = 0$ and w is of weight $-n-2$.

Let $R = \bigoplus_{j=1,\, j \neq p+1}^{s} N^{(j)}$. Then $M = R \oplus N$. Hence $w = w_R + w_N$ for some $w_R \in R$, $w_N \in N$, and $e \cdot w_R = 0 = e \cdot w_N$. Since w_N is of weight $-n-2$, it is a linear combination of z and $f^{n+1}v$ and since $e \cdot z \neq 0$, then $w_N = \alpha f^{n+1} v$ for some $\alpha \in \mathbb{Q}(q)$. Consider $z_0 = z - w_N = z - \alpha f^{n+1} v$. Then, z_0 also generates N and $z_0 = w_R + \widetilde{z}$. Therefore $N \subseteq R + T$ and so $M = R + T$. It is easily seen by weight consideration that the U_q-submodule $R \cap T$ of T cannot be isomorphic neither to T nor to Verma submodules of T since the corresponding weight spaces of T and N have the same dimension and $R + T = R \oplus N$. Hence $R \cap T = \{0\}$, and so $M = R \oplus T$ as U_q-modules.

Therefore, we can replace $N^{(p+1)}$ in the direct sum we started with by T. Denote T by $T^{(p+1)}$. Now we replace in the same way one by one the U_q-submodules $N^{(p+2)}, \ldots, N^{(s)}$ by U_q-submodules $T^{(p+2)}, \ldots, T^{(s)}$ with standard $\mathcal{B}_q$-module structure such that $M = N^{(1)} \oplus \cdots \oplus N^{(p)} \oplus T^{(p+1)} \oplus \cdots \oplus T^{(s)}$. This proves the theorem. □

Corollary. *Let M be in $\widetilde{\mathcal{I}}$. Then*:

- (i) $\operatorname{Ker} e' = \bigoplus_{r \in \mathbb{Z}} (\operatorname{Ker} e' \cap M_r)$;
- (ii) $\operatorname{Ker} e$ *has a $\mathbb{Q}(q)$-basis consisting of weight vectors of the form $f^k u$ where $k \in \mathbb{Z}^+$ and $u \in \operatorname{Ker} e'$.*

3. Completions of crystal bases of modules in the category $\widetilde{\mathcal{I}}$

Now we consider crystal bases in the sense of Definition 1.8 for modules in $\widetilde{\mathcal{I}}$ with any $\mathcal{B}_q$-structure. Since every module in $\widetilde{\mathcal{I}}$ has completion belonging to $\widetilde{\mathcal{I}}$, as well as a crystal basis, it is natural to examine the interplay between these two concepts. We aim to define a notion of completion of crystal bases of modules in $\widetilde{\mathcal{I}}$ that will be compatible with the notion of completion of modules.

3.1. The first two subsections are written for any simple Lie algebra $\mathfrak{g}$. Let $m, n \in \mathbb{Z}^+$. Recall (cf. [8]),

$$f_i^{(n)} f_i^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_i f_i^{(n+m)}, \tag{3.1}$$

$$[n]_i \in q_i^{-n+1}(1 + q\mathbb{A}) \quad \text{for } n \neq 0, \tag{3.2}$$

$$[n]_i! \in q_i^{-\frac{n(n-1)}{2}}(1 + q\mathbb{A}), \tag{3.3}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_i \in q_i^{-n(m-n)}(1 + q\mathbb{A}) \tag{3.4}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i[m-1]_i \dots [m-n+1]_i}{[1]_i[2]_i \dots [n]_i} \quad \text{for } n > 0 \quad \text{and} \quad \begin{bmatrix} m \\ 0 \end{bmatrix}_i = 1.$$

Denote by $\mathbb{A}^\times$ the units in $\mathbb{A}$. The next lemma is immediate and it holds for crystal lattices both in the sense of Definition 1.8 and in the sense of Definition 1.5.

Lemma. *Let L be a crystal lattice of a module M and let $\widetilde{L} = q^r aL$ for some $r \in \mathbb{Z}$ and $a \in \mathbb{A}^\times$. Then $\widetilde{L}$ is also a crystal lattice of M. If $r > 0$ then $\widetilde{L} \subsetneqq L$, while if $r < 0$ then $\widetilde{L} \supsetneqq L$. If $r = 0$ then $\widetilde{L} = L$.*

3.2. The following proposition is an analogue of a statement for crystal bases of finite-dimensional modules, i.e., in the sense of Definition 1.5. We give a proof for reader's convenience.

Proposition. *Let M and M' be in $\mathcal{O}(\mathcal{B}_q(\mathfrak{g}))$ and let $\varphi : M \to M'$ be a $B_q(\mathfrak{g})$-module isomorphism. If (L, B) is a crystal basis of M, then $(\varphi(L), \overline{\varphi}(B))$ is a crystal basis of M', where $\overline{\varphi}$ is the induced map $L/qL \to \varphi(L)/q\varphi(L)$.*

Proof. Consider the decompositions of both M and M' as in (1.4) and $\widetilde{e_i}$ and $\widetilde{f_i}$ defined as in (1.5). It is plain that $\operatorname{Ker} e_i'|_{M'} = \varphi(\operatorname{Ker} e_i'|_M)$ and both $\widetilde{e_i}$ and $\widetilde{f_i}$ commute with φ. Thus, $\varphi(L)$ is a crystal lattice of M'. The restriction $\varphi|_L : L \to \varphi(L)$ is an $\mathbb{A}$-module isomorphism, and so the induced map $\overline{\varphi} : L/qL \to \varphi(L)/q\varphi(L)$ is a $\mathbb{Q}$-vector space isomorphism. Hence $\overline{\varphi}(B)$ is a $\mathbb{Q}$-basis of $\varphi(L)/q\varphi(L)$. Clearly both $\widetilde{e_i}$ and $\widetilde{f_i}$ commute with $\overline{\varphi}$. □

For $\lambda \in P$, let m_λ be a highest weight vector of the Verma module $M(\lambda)$ and let $\varphi_\lambda : U_q^-(\mathfrak{g}) \to M(\lambda)$ be a $\mathcal{B}_q(\mathfrak{g})$-module isomorphism such that $\varphi_\lambda(u) = um_\lambda$. Let $L^{(\lambda)}$ be the $\mathbb{A}$-span of $\{\widetilde{f_{i_1}} \dots \widetilde{f_{i_p}} \cdot m_\lambda \mid 1 \leqslant i_j \leqslant l;\ p \in \mathbb{Z}^+\}$ and $B^{(\lambda)} = \{\widetilde{f_{i_1}} \dots \widetilde{f_{i_p}} \cdot \overline{m_\lambda} \mid 1 \leqslant i_j \leqslant l;\ p \in \mathbb{Z}^+\}$, where $\overline{m_\lambda} = m_\lambda + qL^{(\lambda)}$. Since $(L(\infty), B(\infty))$ is a crystal basis of $U_q^-(\mathfrak{g})$, it immediately follows:

Corollary. *$(L^{(\lambda)}, B^{(\lambda)})$ is a crystal basis of the Verma module $M(\lambda)$. Every crystal basis of $M(\lambda)$ is a nonzero scalar multiple of $(L^{(\lambda)}, B^{(\lambda)})$.*

Example. If $\mathfrak{g} = \mathfrak{sl}_2$ and $r \in \mathbb{Z}$, then any crystal basis of the Verma module $M(r)$ is of the form $(L^{(r)}, B^{(r)})$ where $L^{(r)} = \bigoplus_{k\in\mathbb{Z}^+} \mathbb{A} f^{(k)} m_r$ and $B^{(r)} = \{f^{(k)}\overline{m_r} \mid k \in \mathbb{Z}^+\}$ for a highest weight vector m_r.

3.3. From this point on, we focus exclusively on the $U_q(\mathfrak{sl}_2)$-case.

We consider T modules. Recall that $(L(\infty)^{\oplus 2}, B(\infty)^{\oplus 2})$ is a crystal basis of $U_q^- \oplus U_q^-$, where $L(\infty)^{\oplus 2} = L(\infty) \oplus L(\infty)$, $B(\infty)^{\oplus 2} = (B(\infty) \times 0) \cup (0 \times B(\infty)) \subset L(\infty)^{\oplus 2}/qL(\infty)^{\oplus 2}$, $L(\infty) = \bigoplus_{k\in\mathbb{Z}^+} \mathbb{A} f^{(k)} \cdot 1$ and $B(\infty) = \{f^{(k)} \cdot \overline{1} \mid k \in \mathbb{Z}^+\}$ where $\overline{1} = 1 + qL(\infty)$. The next corollary follows immediately from considering the map $\psi_n : U_q^- \oplus U_q^- \to T(n)$ given by $(u_1, u_2) \mapsto u_1 v + u_2 z$ which is a $\mathcal{B}_q$-module isomorphism where $U_q^- \oplus U_q^-$ is equipped with the obvious $\mathcal{B}_q$-structure.

Corollary. *Let z be a U_q-generator of $T(n)$ of weight $-n-2$ and let v be a highest weight vector of Verma submodule of $T(n)$ with highest weight n. Set*

$$L^{\{n\}} = \left(\bigoplus_{k\in\mathbb{Z}^+} \mathbb{A} f^{(k)} v\right) \oplus \left(\bigoplus_{k\in\mathbb{Z}^+} \mathbb{A} f^{(k)} z\right) \quad \textit{and}$$

$$B^{\{n\}} = \left\{ f^{(k)} v \bmod qL^{\{n\}} \;\middle|\; k \in \mathbb{Z}^+ \right\} \cup \left\{ f^{(k)} z \bmod qL^{\{n\}} \;\middle|\; k \in \mathbb{Z}^+ \right\}.$$

Then $(L^{\{n\}}, B^{\{n\}})$ is a crystal basis of $T(n)$. Moreover, if (L, B) is any crystal basis of $T(n)$, then there exist z and v as above such that (L, B) is of the form $(L^{\{n\}}, B^{\{n\}})$.

The following theorem now easily follows from Theorem 2.4, Corollaries 3.2 and 3.3.

Theorem. *Let M be a module in the category $\widetilde{\mathcal{I}}$. Then M belongs to the category $\mathcal{O}(\mathcal{B}_q)$ and there exist unique integers $r_1, \ldots, r_m$ and unique nonnegative integers $n_1, \ldots, n_t$ such that M has a crystal basis (L, B) which is a direct sum of crystal bases of the form $(L^{(r_i)}, B^{(r_i)})$, for $i = 1, \ldots, m$, and crystal bases of the form $(L^{\{n_j\}}, B^{\{n_j\}})$, for $j = 1, \ldots, t$. Furthermore, any crystal basis of M is isomorphic to (L, B).*

Definition. We call the crystal basis (L, B) from the previous theorem a standard crystal basis of M.

Evidently, the integers $r_1, \ldots, r_m$ and $n_1, \ldots, n_t$ depend only on M and not on (L, B). Every crystal basis of an indecomposable module in $\widetilde{\mathcal{I}}$ is obviously a standard crystal basis.

3.4. For M in $\widetilde{\mathcal{I}}$, recall the decomposition $M = \bigoplus_{k\geqslant 0} f^{(k)} \operatorname{Ker} e'$ (see 1.8). By Corollary 2.4, $\operatorname{Ker} e' = \bigoplus_{n\in\mathbb{Z}} (\operatorname{Ker} e' \cap M_n)$. Thus we bring the weight spaces of M in the picture and have:

$$M = \bigoplus_{\substack{k\geqslant 0\\ n\in\mathbb{Z}}} f^{(k)} (\operatorname{Ker} e' \cap M_n). \tag{3.5}$$

For a crystal lattice L of M, set $L_n = L \cap M_n$. Then $L = \bigoplus_{n\in\mathbb{Z}} L_n$. Also, set $L^e = \{m \in L \mid em = 0\} = L \cap M^e$ and $L_n^e = L^e \cap M_n = L \cap M_n^e$.

Lemma. *Let M be in $\widetilde{\mathcal{I}}$ and let L be a crystal lattice of M. Then $\tilde{f}^{n+1}(M^e_n) \subseteq M^e_{-n-2}$ and $\tilde{f}^{n+1}(L^e_n) \subseteq L^e_{-n-2}$ for $n \in \mathbb{Z}^+$.*

Proof. Let $m \in M^e_n$. By Corollary 2.4, $m = \sum_{k \geqslant 0} f^{(k)} u_k$ where $f^{(k)} u_k \in M^e$ and $u_k \in \operatorname{Ker} e' \cap M_{n+2k}$. Thus $\tilde{f}^{n+1} m = \sum_{k \geqslant 0} \tilde{f}^{n+1} f^{(k)} u_k = \sum_{k \geqslant 0} f^{(n+1+k)} u_k \in M_{-n-2}$. Furthermore, utilizing (3.1) and commutation relations,

$$e \tilde{f}^{n+1} f^{(k)} u_k = \frac{1}{\left[\begin{smallmatrix} n+k+1 \\ n+1 \end{smallmatrix}\right]} e f^{(n+1)} f^{(k)} u_k = \frac{1}{\left[\begin{smallmatrix} n+k+1 \\ n+1 \end{smallmatrix}\right]} \left(f^{(n+1)} e + f^{(n)} \frac{q^{-n} t - q^n t^{-1}}{q - q^{-1}} \right) f^{(k)} u_k = 0$$

for all $k \geqslant 0$. Therefore $\tilde{f}^{n+1} m \in M^e$, and so $\tilde{f}^{n+1}(M^e_n) \subseteq M^e_{-n-2}$. In addition, since $\tilde{f}(L) \subseteq L$, then $\tilde{f}^{n+1}(L^e_n) \subseteq L^e_{-n-2}$. □

Definition. Let L be a crystal lattice of a module M in $\widetilde{\mathcal{I}}$. We say L is complete if $\tilde{f}^{n+1} : L^e_n \to L^e_{-n-2}$ is bijective for all $n \in \mathbb{Z}^+$. A crystal lattice $\widetilde{L}$ of the completion $C(M)$ of M is said to be a completion of a crystal lattice L of M provided

- (i) $L^e \subseteq \widetilde{L}$ and $\widetilde{L} \not\subset L$;
- (ii) $\widetilde{L} \cap M = \widetilde{L} \cap L$;
- (iii) $\widetilde{L}/(\widetilde{L} \cap L)$ is a crystal lattice of $C(M)/M$ (to be precise, its image under $\widetilde{L}/(\widetilde{L} \cap L) \hookrightarrow C(M)/M$).

Remark.

- (i) Unlike the case of modules where $C(M)$ contains M, one cannot expect $\widetilde{L}$ to contain L. If we consider crystal lattices of a reducible Verma module and its irreducible Verma U_q-submodule, denoted by L_1 and L_2 respectively, although L_1 is isomorphic to L_2, when talking about completions we are interested in their exact relationship and indeed $L_2 \not\subseteq L_1$. Moreover, we observe that there is no $p \in \mathbb{Z}$ such that $q^p L_2 \subseteq L_1$. In fact, the actions of the Kashiwara operators $\tilde{e}$ and $\tilde{f}$ on L_1 and L_2 are different, i.e., $\tilde{e}_M \neq (\tilde{e}_{C(M)})|_M$ and $\tilde{f}_M \neq (\tilde{f}_{C(M)})|_M$.
- (ii) Condition (iii) in Definition 3.4 is a very natural connection to expect between crystal lattices and completions. Also, we note that $C(M)/M$ is a finite-dimensional U_q-module, thus condition (iii) gives a connection between crystal lattices arising from the $\mathcal{B}_q$-structures, i.e., in the sense of Definition 1.8, with crystal lattices arising from the U_q-structures, i.e., in the sense of Definition 1.5.
- (iii) The condition $\widetilde{L} \not\subset L$, i.e., $\widetilde{L}$ is not properly contained in L, follows from condition (iii) in the case that L is a crystal lattice of a module M that is not complete.

Proposition. *Let L be a crystal lattice of a module M in $\widetilde{\mathcal{I}}$ and let $\widetilde{L}$ be a completion of L. Then $\widetilde{L}^e_{-n-2} = L^e_{-n-2}$ for all $n \in \mathbb{Z}^+$.*

Proof. Let $n \in \mathbb{Z}^+$. We note that since $C(M)/M$ is a finite-dimensional U_q-module, then $(C(M)/M)^e_{-n-2} = 0$, and so $C(M)^e_{-n-2} = M^e_{-n-2}$. Now, by Definition 3.4(ii), $\widetilde{L}^e_{-n-2} = \widetilde{L} \cap C(M)^e_{-n-2} = \widetilde{L} \cap M^e_{-n-2} = (\widetilde{L} \cap M) \cap M^e_{-n-2} = (\widetilde{L} \cap L) \cap M^e_{-n-2} = \widetilde{L}^e_{-n-2} \cap L^e_{-n-2}$. Hence, $\widetilde{L}^e_{-n-2} \subseteq L^e_{-n-2}$. The claim now follows from Definition 3.4(i). □

3.5. Let (L, B) be a crystal basis of M. Set $B_n = L_n/qL_n \cap B$ where $L_n/qL_n = \{m \bmod qL \mid m \in L_n\}$. Then $B = \bigsqcup_{n\in\mathbb{Z}} B_n$. Also set $B_n^e = (L_n/qL_n)^e \cap B$ where $(L_n/qL_n)^e = \{m \bmod qL \mid m \in L_n^e\}$.

Lemma. *Let M be in $\widetilde{\mathcal{I}}$ and let (L, B) be a crystal basis of M. Then*:

(i) $\widetilde{f}^{n+1}(B_n^e) \subseteq B_{-n-2}^e$ *for* $n \in \mathbb{Z}^+$;
(ii) *If L is complete, then $\widetilde{f}^{n+1} : B_n^e \to B_{-n-2}^e$ is bijective for all $n \in \mathbb{Z}^+$.*

Proof. Part (i) of the lemma follows from Lemma 3.4 and from B being invariant under $\widetilde{f}$. It is easily seen from Corollary 2.4 that B_n^e is a basis of $(L_n/qL_n)^e$, thus implying part (ii). □

Let $\overline{\varphi} : \widetilde{L}/q\widetilde{L} \to \dfrac{\widetilde{L}/(L \cap \widetilde{L})}{q(\widetilde{L}/(L \cap \widetilde{L}))}$ be the induced $\mathbb{Q}$-epimorphism from the canonical $\mathbb{A}$-epimorphism $\varphi : \widetilde{L} \to \widetilde{L}/(L \cap \widetilde{L})$.

Definition. A crystal basis (L, B) of M in $\widetilde{\mathcal{I}}$ is said to be complete if the crystal lattice L is complete.

A crystal basis $(\widetilde{L}, \widetilde{B})$ of $C(M)$ is a completion of a crystal basis (L, B) if

(1) $\widetilde{L}$ is a completion of L,
(2) $(\widetilde{L}/(L \cap \widetilde{L}),\ \overline{\varphi}(\widetilde{B}))$ is a crystal basis of $C(M)/M$.

The next theorem verifies that the above definition of completeness for crystal bases is compatible with the notion of completeness for modules.

Theorem. *Let (L, B) be a crystal basis of a module M in $\widetilde{\mathcal{I}}$. Then (L, B) is complete if and only if M is complete.*

Proof. Let M be complete and let $n \in \mathbb{Z}^+$. Then $f^{n+1} : M_n^e \to M_{-n-2}^e$ is bijective. Since M is in $\widetilde{\mathcal{I}}$, its weight spaces are finite-dimensional. Hence,

$$\dim_{\mathbb{Q}(q)} M_n^e = \dim_{\mathbb{Q}(q)} M_{-n-2}^e. \tag{3.6}$$

L is a crystal lattice of M, so there is an $\mathbb{A}$-basis of L that is also a $\mathbb{Q}(q)$-basis of M. This basis is made up of weight vectors of the form $f^{(k)}u$ where $k \geqslant 0$ and $u \in \operatorname{Ker} e'$. If a nonzero sum of such vectors is annihilated by e, then also each of them is (see Theorem 2.4 and Corollary 2.4). Thus, a $\mathbb{Q}(q)$-basis of M_n^e is an $\mathbb{A}$-basis of L_n^e. By (3.6), $\operatorname{rank}_{\mathbb{A}} L_n^e = \operatorname{rank}_{\mathbb{A}} L_{-n-2}^e$. Now, since $\mathbb{A}$ has the invariant dimension property and since $\widetilde{e}\widetilde{f} = 1$ on M, then $\widetilde{f}^{n+1} : L_n^e \to L_{-n-2}^e$ is bijective. The converse is true since f is injective on M, and the above steps can be reversed for a fixed n. □

Example. The modules $T(n)$ for $n \in \mathbb{Z}^+$ and $M(n)$ for $n \geqslant -1$ are complete, so their crystal bases defined in 3.3 and 3.2, respectively, are complete by the theorem. This can also be seen directly from the Definitions 3.4 and 3.5.

Corollary. *A complete crystal basis (L, B) of a module M in $\widetilde{\mathcal{I}}$ is a completion of itself. Furthermore, L is the only completion of itself.*

Proof. By the previous theorem, (L, B) being complete implies $M = C(M)$. It is obvious that a crystal basis of a complete module satisfies Definitions 3.4 and 3.5 for being a completion of itself. Now assume $\widetilde{L}$ is another completion of L. Then $\widetilde{L} = \widetilde{L} \cap C(M) = \widetilde{L} \cap M = \widetilde{L} \cap L$, thus $\widetilde{L} \subseteq L$. The desired conclusion follows from the condition $\widetilde{L} \not\subset L$. □

3.6. The rest of the section is devoted to the proof of the following theorem.

Theorem. *A standard crystal basis (L, B) of any module M in $\widetilde{\mathcal{I}}$ has a completion. Moreover, there exists a unique standard crystal lattice which is a completion of L.*

By Theorem 3.3, every module M in category $\widetilde{\mathcal{I}}$ has a standard crystal basis, thus it has a crystal basis that can be completed as in the above theorem.

We first prove:

Proposition. *A crystal basis (L, B) of any indecomposable module M in $\widetilde{\mathcal{I}}$ has a completion. Furthermore, L has a unique completion.*

Proof. By Theorem 3.5 and Corollary 3.5, in order to show existence, we need to consider only Verma modules in $\widetilde{\mathcal{I}}$ which are not complete. Let M be a Verma module with highest weight $-n-2$ for some $n \in \mathbb{Z}^+$ and let L be its crystal lattice. Then $L = \bigoplus_{k \geqslant 0} \mathbb{A} f^{(k)} m_0$ for some highest weight vector m_0 of M and L is clearly not complete. Since M has a completion $C(M)$ and $f^{n+1} : C(M)^e_n \to C(M)^e_{-n-2} = M^e_{-n-2}$ is bijective, there exists $\widetilde{m}$ in $C(M)^e_n$ such that $f^{(n+1)}\widetilde{m} = m_0$. Let $\widetilde{L} = \bigoplus_{k \geqslant 0} \mathbb{A} f^{(k)} \widetilde{m}$ and $\widetilde{B} = \{f^{(k)}\widetilde{m} + q\widetilde{L} \mid k \in \mathbb{Z}^+\}$. Then $(\widetilde{L}, \widetilde{B})$ is a crystal basis of $C(M)$ and we claim it is a completion of (L, B).

It follows from Section 3.1 that

$$L = \bigoplus_{k \geqslant 0} \mathbb{A} f^{(k)} f^{(n+1)} \widetilde{m} = \bigoplus_{k \geqslant 0} \mathbb{A} \begin{bmatrix} n+1+k \\ k \end{bmatrix} f^{(n+1+k)} \widetilde{m} = \bigoplus_{k \geqslant 0} \mathbb{A} q^{-k(n+1)} f^{(n+1+k)} \widetilde{m}.$$

The last equality is a consequence of the elements of $1 + q\mathbb{A}$ being units in $\mathbb{A}$.

We have $L^e = \mathbb{A} m_0 \subseteq \widetilde{L}$, $\widetilde{L} \cap L = \bigoplus_{k \geqslant 0} \mathbb{A} f^{(n+1+k)} \widetilde{m}$, and $\widetilde{L}/(\widetilde{L} \cap L) = \bigoplus_{k=1}^{n} \mathbb{A} f^{(k)} \overline{m}$ where $\overline{m} = \widetilde{m} + q(\widetilde{L} \cap L)$. Thus the conditions (i)–(iii) of Definition 3.4 are satisfied. It is now easy to check condition (2) of Definition 3.5.

Uniqueness follows from parts (i) and (ii) of Definition 3.4. □

Remark.

(i) Considering the lattice $\widetilde{L}$ from the proof of previous proposition, notice that $\widetilde{L} \cap M = \bigoplus_{k \geqslant 0} \mathbb{A} f^{(n+1+k)} \widetilde{m}$ which is a proper $\mathbb{A}$-submodule of the crystal lattice $L = \bigoplus_{k \geqslant 0} \mathbb{A} q^{-k(n+1)} f^{(n+1+k)} \widetilde{m}$ of M, i.e., in order to complete L to a crystal lattice of $C(M)$ that is invariant under Kashiwara operators $\widetilde{e}_{C(M)}$ and $\widetilde{f}_{C(M)}$, L has to be made "thinner" as an $\mathbb{A}$-lattice in a particular way.

(ii) Using the proof of the previous proposition, we can also see the following. Let $n \in \mathbb{Z}^+$ and let $M = M(n)$ be the Verma module with highest weight n and M' its unique Verma submodule. Let L be any crystal lattice of M and L' any crystal lattice of M'. Then, there exists $r \in \mathbb{Z}^+$ such that $L/(L \cap q^{-r}L')$ is isomorphic to a crystal lattice of the irreducible U_q-module M/M'.

3.7.

Proposition. *Direct sum of completions of crystal bases is a completion of direct sum of crystal bases.*

Proof. For $i = 1, 2$, let (L_i, B_i) be a crystal basis of M_i and $(\widetilde{L_i}, \widetilde{B_i})$ its completion. We claim that $(\widetilde{L_1}, \widetilde{B_1}) \oplus (\widetilde{L_2}, \widetilde{B_2})$ is a completion of $(L_1, B_1) \oplus (L_2, B_2)$. Since both crystal bases and completions of modules in $\widetilde{\mathcal{I}}$ respect direct sums, $(\widetilde{L_1}, \widetilde{B_1}) \oplus (\widetilde{L_2}, \widetilde{B_2}) = (\widetilde{L_1} \oplus \widetilde{L_2}, \widetilde{B_1} \sqcup \widetilde{B_2})$ is a crystal basis of $C(M_1) \oplus C(M_2) = C(M_1 \oplus M_2)$. Condition (i) of Definition 3.4 is evident since $(L_1 \oplus L_2)^e = L_1^e \oplus L_2^e \subseteq \widetilde{L_1} \oplus \widetilde{L_2}$. Furthermore, $\widetilde{L_1} \oplus \widetilde{L_2} \not\subset L_1 \oplus L_2$. By definition, $\widetilde{L_i} \cap M_i = \widetilde{L_i} \cap L_i$. Clearly, for $i \neq j$, $M_i \cap \widetilde{L_j} = 0$. Therefore, $(\widetilde{L_1} \oplus \widetilde{L_2}) \cap (M_1 \oplus M_2) = (\widetilde{L_1} \oplus \widetilde{L_2}) \cap (L_1 \oplus L_2)$, so condition (ii) is satisfied, as well. Next,

$$\begin{aligned}(\widetilde{L_1} \oplus \widetilde{L_2})/\big((\widetilde{L_1} \oplus \widetilde{L_2}) \cap (L_1 \oplus L_2)\big) &= (\widetilde{L_1} \oplus \widetilde{L_2})/\big((\widetilde{L_1} \cap L_1) \oplus (\widetilde{L_2} \cap L_2)\big) \\ &\cong \widetilde{L_1}/(\widetilde{L_1} \cap L_1) \oplus \widetilde{L_2} \setminus (\widetilde{L_2} \cap L_2).\end{aligned}$$

On the other hand,

$$C(M_1 \oplus M_2)/(M_1 \oplus M_2) = \big(C(M_1) \oplus C(M_2)\big)/(M_1 \oplus M_2) \cong C(M_1)/M_1 \oplus C(M_2)/M_2.$$

Condition (iii) now follows from crystal lattices respecting direct sums. Furthermore, it is easy to see that $\overline{\varphi}(\widetilde{B_1} \sqcup \widetilde{B_2}) = \overline{\varphi}(\widetilde{B_1}) \sqcup \overline{\varphi}(\widetilde{B_2})$ proving condition (2) of Definition 3.5. □

Lemma. *Let M be a module in $\widetilde{\mathcal{I}}$ and suppose $L_i, L_i', i = 1, \dots, k$, are crystal lattices of indecomposable submodules of M such that $L_1 \oplus L_2 \oplus \dots \oplus L_k = L_1' \oplus L_2' \oplus \dots \oplus L_k'$. If $\widetilde{L_i}, \widetilde{L_i'}$ are the completions of L_i, L_i', respectively, then $\widetilde{L_1} \oplus \widetilde{L_2} \oplus \dots \oplus \widetilde{L_k} = \widetilde{L_1'} \oplus \widetilde{L_2'} \oplus \dots \oplus \widetilde{L_k'}$.*

Proof. It suffices to consider the case when M is a generalized eigenspace of the Casimir element with eigenvalue c_n. If all of the L_i are complete, then so is $L_1 \oplus L_2 \oplus \dots \oplus L_k$ and we are done using Proposition 3.7 and Corollary 3.5. To simplify the notation we consider only the case when $k = 2$, L_1 is a crystal lattice of a Verma submodule of M with highest weight n, and L_2 is a crystal lattice of a Verma submodule of M of highest weight $-n-2$. The proof of the general case is similar, but with more involved notation. It is clear that either L_1' or L_2' is a crystal lattice of a Verma submodule of M with highest weight n, so we assume it is L_1'. Furthermore, it is then obvious from the condition $L_1 \oplus L_2 = L_1' \oplus L_2'$ that we must have $L_1 = L_1' = \widetilde{L_1} = \widetilde{L_1'}$. Let $v \in \operatorname{Ker} e' \cap M_n$ and $u, u' \in \operatorname{Ker} e' \cap M_{-n-2}$ be such that L_1, L_2, L_2' are the $\mathbb{A}$-spans of $f^{(i)}v, f^{(i)}u, f^{(i)}u'$ $(i \geqslant 0)$, respectively. Clearly $u' = af^{(n+1)}v + bu$ where $a, b \in \mathbb{A}$. Since $u \in L_1' \oplus L_2'$, then $b \in \mathbb{A}^\times$. Let m, m' be the unique elements in $C(M)$ such that $f^{(n+1)}m = u$ and $f^{(n+1)}m' = u'$. Then $\widetilde{L_2} = \sum_{i \geqslant 0} \mathbb{A} f^{(i)}m$ and $\widetilde{L_2'} = \sum_{i \geqslant 0} \mathbb{A} f^{(i)}m'$ by the proof of Proposition 3.6. Also, $f^{(n+1)}m' = f^{(n+1)}(av + bm)$. Since M is U_q^--torsion free, then $m' = av + bm \in \widetilde{L_1} \oplus \widetilde{L_2}$ and $m = b^{-1}(m' - av) \in \widetilde{L_2'} \oplus \widetilde{L_1'}$ proving the lemma in this case. □

3.8. It is convenient to have the following definition.

Definition. Let M_1, M_2 be modules in $\widetilde{\mathcal{I}}$ and let (L_1, B_1), (L_2, B_2) be crystal bases of M_1, M_2, respectively. We say that (L_1, B_1) is strongly isomorphic to (L_2, B_2) if there exists a B_q-isomorphism $\varphi : M_1 \to M_2$ such that

(i) φ induces an isomorphism of the crystal bases (L_1, B_1) and (L_2, B_2);
(ii) φ is weight-preserving;
(iii) $\varphi(\operatorname{Ker} e|_{M_1}) = \operatorname{Ker} e|_{M_2}$.

For example, any two standard crystal basis of a module in $\widetilde{\mathcal{I}}$ are strongly isomorphic.

Proof of Theorem 3.6. By Definition 3.3, a standard crystal basis is a direct sum of crystal bases of the form $(L^{(r_i)}, B^{(r_i)})$, for $i = 1, \ldots, m$, and crystal bases of the form $(L^{\{n_j\}}, B^{\{n_j\}})$, for $j = 1, \ldots, t$, where the integers r_i and n_j depend on M, i.e., these are the highest weights of Verma and T modules present in a decomposition of M. The first claim of the theorem now follows from Propositions 3.6 and 3.7.

We now prove the uniqueness of completion of L. Since L is a standard crystal lattice of M, then $L = L_1 \oplus \cdots \oplus L_k$ for some crystal lattices $L_i, i = 1, \ldots, k$, of indecomposable U_q-submodules M_i of M such that $M = M_1 \oplus \cdots \oplus M_k$. If $\widetilde{L_i}$ denotes the completion of L_i, then $\widetilde{L_1} \oplus \cdots \oplus \widetilde{L_k}$ is a completion of L. Let $\widetilde{L}'$ denote another standard completion of L. Then $\widetilde{L}'$ is a crystal lattice of $C(M) = C(M_1) \oplus \cdots \oplus C(M_k)$, thus it is a direct sum of standard crystal lattices $L_i'', i = 1, \ldots, k$, of indecomposable submodules M_i'' of $C(M)$ such that $C(M) = M_1'' \oplus \cdots \oplus M_k''$ and such that L_i'' is strongly isomorphic to $\widetilde{L}_i, i = 1, \ldots, k$ (up to reordering). Since $L, L_1'', \ldots, L_k''$ are standard crystal lattices and $L_1'' \oplus \cdots \oplus L_k''$ is a completion of L, it is easily seen that there exist crystal lattices $L_i', i = 1, \ldots, k$, strongly isomorphic to $L_i, i = 1, \ldots, k$, respectively, such that L_i'' is a completion of L_i' and $L = L_1' \oplus \cdots \oplus L_k'$. The conclusion now follows from the previous lemma. □

4. Constructions of completions of crystal lattices

We now construct a completion of a crystal lattice modifying Deodhar's construction of completion for modules in $\widetilde{\mathcal{I}}$. By Theorems 2.4 and 3.5 and Proposition 3.7, it suffices to consider crystal bases of Verma modules $M(-n-2)$ for $n \in \mathbb{Z}^+$.

4.1. We start by recalling Deodhar's construction for $U_q(\mathfrak{sl}_2)$ (cf. [2,15]). Denote by $\mathcal{A}(U_q)$ the category of U_q-weight modules that are U_q^--torsion free. For M in $\mathcal{A}(U_q)$, let S_M be the set of formal symbols $S_M = \{f^{-r}m \mid r \in \mathbb{Z}^+,\ m \in M\}$. Define an equivalence relation $\sim$ on S_M by $f^{-r}m \sim f^{-k}m'$ iff $f^k m = f^r m'$. Set $D(M) = S_M/\sim$. Then $D(M)$ has a vector space structure. For $z \in U_q$ and $r \in \mathbb{Z}^+$, there exist $u \in U_q$ and $s \in \mathbb{Z}^+$ such that $f^s z = uf^r$. Now, for $f^{-r}m \in D(M)$ and $z \in U_q$, define $z \cdot f^{-r}m = f^{-s}um$. This action makes $D(M)$ a U_q-module.

Any nonzero element $v \in D(M)$ can be uniquely expressed as $f^{-n}m$ ($n \in \mathbb{Z}^+, m \in M$) where n is minimal with respect to this property. This expression is called the minimal expression for v.

Clearly, $f^{-n}m$ is a minimal expression iff $m \notin fM$. Also, $v \in M$ iff $n = 0$ in the minimal expression for v.

Set $C(M) = \{v \in D(M) \mid e^k v = 0 \text{ for some } k > 0\}$.

Theorem. *(Cf. [2,15].) Let M be a module in the category $\mathcal{I}$.*

(i) *$C(M)$ is a completion of M.*
(ii) *Let $v \in D(M) \setminus M$ be a weight vector such that $tv = q^a v$ for some $a \in \mathbb{Z}$ and let $v = f^{-n}m$ be the minimal expression for v. Then $v \in C(M)$ iff the following two conditions are satisfied*:
  (a) *$a = n - j$ for some $j > 0$, and*
  (b) *$e^k f^{k-j} m = 0$ for some $k \geqslant j$.*

The next corollary gives basis vectors for the completion $C(M)$ of an irreducible Verma module M.

Corollary. *Let M be the Verma module with highest weight $-n-2$ for $n \in \mathbb{Z}^+$ and let m_0 be a highest weight vector of M. Then the completion of M is $C(M) = \bigoplus_{k \geqslant -n-1} \mathbb{Q}(q) f^k m_0$.*

Proof. Applying Deodhar's construction on $M = \bigoplus_{k \geqslant 0} \mathbb{Q}(q) f^k m_0$, we obtain $D(M) = \bigoplus_{k \in \mathbb{Z}} \mathbb{Q}(q) f^k m_0$ because f is injective and $f^{-r} f^k m_0 = f^{k-r} m_0$ for $r, k \in \mathbb{Z}^+$. Since $M \subset C(M)$, we need to consider only $f^{-k} m_0$ for $k > 0$. By Theorem 4.1(ii), $f^{-k} m_0 \in C(M)$ iff (a) $-n-2+2k = k-j$ for some $j > 0$, and (b) $e^p f^{p-j} m_0 = 0$ for some $p \geqslant j$. This is equivalent to $k < n+2$ ($p = j = n+2-k$). □

4.2. It is convenient to introduce the following formal notation. Set $[-k]! = [-1] \ldots [-k]$ and $f^{(-k)} m = \dfrac{f^{-k} m}{[-k]!}$ for $k > 0$ and $m \in M$. Then $[-k]! = (-1)^k [k]!$ and $f^{(k)} m = \dfrac{f^k m}{[k]!}$ for $k \in \mathbb{Z}$ and $m \in M$.

For $i, k \in \mathbb{Z}^+$ and $m \in M$,

$$f^{(i)} f^{(-k)} m = \frac{f^i}{[i]!} \frac{f^{-k} m}{[-k]!} = \frac{f^{i-k} m}{[i]! [-k]!} = \frac{[i-k]!}{[i]! [-k]!} f^{(i-k)} m. \tag{4.1}$$

Thus,

$$C(M) = \bigoplus_{k \geqslant -n-1} \mathbb{Q}(q) f^{(k)} m_0 = \bigoplus_{i \geqslant 0} \mathbb{Q}(q) f^{(i)} f^{(-n-1)} m_0,$$

where we follow the notation of Corollary 4.1.

4.3. We now construct directly from the crystal lattice of an irreducible Verma module a crystal lattice of its completion.

Lemma.

(a) *If $i \leqslant n$, then $1 = \dfrac{1}{1 + q^{-(i-n-1)}} a^*$ for some $a^* \in \mathbb{A}^\times$.*
(b) *If $i > n$, then $q^{i-n-1} = \dfrac{1}{1 + q^{-(i-n-1)}} a^*$ for some $a^* \in \mathbb{A}^\times$.*

Proof. (a) For $i \leqslant n$, $1+q^{-i+n+1} \in \mathbb{A}^{\times}$.

(b) For $i > n$, $\dfrac{1}{1+q^{-(i-n-1)}} = \dfrac{q^{i-n-1}}{q^{i-n-1}+1}$, so let $a^* = 1+q^{i-n-1} \in \mathbb{A}^{\times}$. $\square$

Proposition. *Let M be the Verma module with highest weight $-n-2$ for some $n \in \mathbb{Z}^+$ and let m_0 be a highest weight vector of M. Then*

(i) *the $\mathbb{A}$-module $\widetilde{L}$ generated by $\{\dfrac{2q^{n(i-n-1)}}{1+q^{-(i-n-1)}} f^{(i-n-1)} m_0 \mid i \geqslant 0\}$ is a crystal lattice of $C(M)$;*

(ii) *$\widetilde{L}$ is a completion of the crystal lattice $L = \bigoplus_{k\geqslant 0} \mathbb{A} f^{(k)} m_0$ of M.*

Proof. (1) By Corollary 4.1, $C(M)$ is a Verma module with a highest weight vector $f^{(-n-1)} m_0$. Thus, a crystal lattice of $C(M)$ is of the form $\widetilde{L} = \bigoplus_{i\geqslant 0} \mathbb{A} r(q) f^{(i)} f^{(-n-1)} m_0$ for $r(q) \in \mathbb{Q}(q)$. Utilizing (3.3), we have the following:

$$\begin{aligned} f^{(i)} f^{(-n-1)} m_0 &= (-1)^{n+1} \frac{[i-n-1]!}{[i]![n+1]!} f^{(i-n-1)} m_0 \\ &= (-1)^{n+1} \frac{[i-n-1]!}{q^{-\frac{1}{2}[(i-1)i+n(n+1)]}(1+aq)} f^{(i-n-1)} m_0 \quad \text{for some } a \in \mathbb{A}. \end{aligned}$$

Case 1. $i \geqslant n+1$.

$$\begin{aligned} f^{(i)} f^{(-n-1)} m_0 &= (-1)^{n+1} \frac{q^{-\frac{1}{2}(i-n-2)(i-n-1)}(1+a'q)}{q^{-\frac{1}{2}[(i-1)i+n(n+1)]}(1+aq)} f^{(i-n-1)} m_0 \quad \text{for some } a' \in \mathbb{A} \\ &= a^* q^{(i-1)(n+1)} f^{(i-n-1)} m_0 \quad \text{for some } a^* \in \mathbb{A}^{\times}. \end{aligned}$$

Case 2. $0 \leqslant i \leqslant n$.

$$\begin{aligned} f^{(i)} f^{(-n-1)} m_0 &= (-1)^{n+1}(-1)^{n+1-i} \frac{q^{-\frac{1}{2}(n-i)(n+1-i)}(1+a'q)}{q^{-\frac{1}{2}[(i-1)i+n(n+1)]}(1+aq)}, f^{(i-n-1)} m_0 \quad \text{for some } a' \in \mathbb{A} \\ &= a^* q^{ni} f^{(i-n-1)} m_0 \quad \text{for some } a^* \in \mathbb{A}^{\times}. \end{aligned} \tag{4.2}$$

Hence, for $r(q) = q^{-n(n+1)}$, we have by Lemmas 3.1 and 4.3

$$\begin{aligned} \widetilde{L} &= \bigoplus_{i\geqslant 0} \mathbb{A} q^{-n(n+1)} f^{(i)} f^{(-n-1)} m_0 \\ &= \left(\bigoplus_{i=0}^{n} \mathbb{A} q^{n(i-n-1)} f^{(i-n-1)} m_0 \right) \oplus \left(\bigoplus_{i\geqslant n+1} \mathbb{A} q^{(n+1)(i-n-1)} f^{(i-n-1)} m_0 \right) \\ &= \bigoplus_{i\geqslant 0} \mathbb{A} \frac{q^{n(i-n-1)}}{1+q^{-(i-n-1)}} f^{(i-n-1)} m_0 = \bigoplus_{i\geqslant 0} \mathbb{A} \frac{2q^{n(i-n-1)}}{1+q^{-(i-n-1)}} f^{(i-n-1)} m_0. \end{aligned}$$

(ii) Evidently, $L^e = \mathbb{A}m_0 \subseteq \widetilde{L}$, $\widetilde{L} \not\subset L$, and $\widetilde{L} \cap M = \bigoplus_{j\geqslant 0} \mathbb{A}q^{j(n+1)} f^{(j)} m_0 = \widetilde{L} \cap L$. Next,

$$\widetilde{L}/(\widetilde{L} \cap L) = \bigoplus_{i=0}^{n} \mathbb{A}q^{-n(n+1-i)} f^{(-n-1+i)} \overline{m_0} \quad \text{where } \overline{m_0} = m_0 + \widetilde{L} \cap L$$
$$= \bigoplus_{i=0}^{n} \mathbb{A}q^{-n(n+1)} f^{(i)} f^{(-n-1)} \overline{m_0} \quad \text{by (4.2).}$$

Since $q^{-n(n+1)} f^{(-n-1)} \overline{m_0}$ (in fact, its image under $\widetilde{L}/(\widetilde{L} \cap L) \hookrightarrow C(M)/M$) is a highest weight vector of $C(M)/M$, then $\widetilde{L}/(\widetilde{L} \cap L)$ is a crystal lattice of $C(M)/M$. Thus, $\widetilde{L}$ is a completion of L. □

Remark.

(i) The above proposition gives the basis vectors of a crystal lattice of $C(M)$ in a closed form. However, if we remove the $\mathbb{A}^\times$-multiples from the given basis vectors of $\widetilde{L}$, they come down to

$$\begin{cases} q^{n(i-n-1)} f^{(i-n-1)} m_0 & \text{for } 0 \leqslant i \leqslant n, \\ q^{(n+1)(i-n-1)} f^{(i-n-1)} m_0 & \text{for } i > n, \end{cases} \tag{4.3}$$

and they generate the same crystal lattice $\widetilde{L}$.

(ii) If we start with the lattice from Proposition 4.3 and apply $f^{(n+1)}$ to its highest weight vector, we get:

$$f^{(n+1)} \left(\frac{2q^{-n(n+1)}}{1+q^{n+1}} f^{(-n-1)} m_0 \right) = \frac{2q^{-n(n+1)}}{1+q^{n+1}} \frac{[0]!}{[n+1]![-n-1]!} m_0$$
$$= a^* m_0 \quad \text{for some } a^* \in \mathbb{A}^\times.$$

In other words, $m_0 \in \widetilde{L}$.

4.4. In the last subsection, we show how an operator defined by Kashiwara in [7] can be applied to a certain $\mathbb{A}$-lattice in order to deform it to a crystal lattice of the completion.

We consider the $\mathbb{A}$-lattices L in $M = M(-n-2)$ and $L^\sharp$ in $C(M)$ where $L = \bigoplus_{k\geqslant 0} \mathbb{A} f^{(k)} m_0$ and $L^\sharp = \bigoplus_{k\geqslant -n-1} \mathbb{A} f^{(k)} m_0$. Note that L is a crystal lattice of M, but $L^\sharp$ is *not* its completion. Actually, $L^\sharp$ is not even a crystal lattice since $\widetilde{e}(L^\sharp) \nsubseteq L^\sharp$ where $\widetilde{e}$ denotes the Kashiwara operator on $C(M)$. However, we will directly transform $L^\sharp$ to a completion of L.

Let $\Delta = qt + q^{-1}t^{-1} + (q - q^{-1})^2 fe - 2$ be a central element of U_q and consider the action of $qt\Delta$ on $C(M)$. For $k > 0$, $f^k \Delta = \Delta f^k$, thus $\Delta f^{-k} m_0 = f^{-k} \Delta m_0$. Hence $\Delta f^{(k)} m_0 = f^{(k)} \Delta m_0 = (q^{-n-1} + q^{n+1} - 2) f^{(k)} m_0$ for $k \geqslant -n-1$.

Now, $qt\Delta f^{(k)} m_0 = (q^{-n-1} + q^{n+1} - 2) q^{-n-1-2k} f^{(k)} m_0 = [(q^{-n-1} - 1) q^{-k}]^2 f^{(k)} m_0$. We define $(qt\Delta)^{\frac{1}{2}} \in \mathrm{End}(C(M))$ as follows: $(qt\Delta)^{\frac{1}{2}} f^{(k)} m_0 = q^{-k} (q^{-n-1} - 1) f^{(k)} m_0$ for $k \geqslant -n-1$ (cf. [7]). Thus, $(qt\Delta)^{-\frac{1}{2}} \in \mathrm{End}(C(M))$ is defined by

$$(qt\Delta)^{-\frac{1}{2}} f^{(k)} m_0 = q^k \left(q^{-n-1} - 1\right)^{-1} f^{(k)} m_0 \tag{4.4}$$

for $k \geqslant -n-1$. Let $S_n \in \mathrm{End}(C(M))$ be given by

$$S_n = \begin{cases} q^{-n-1}(qt\Delta)^{-\frac{1}{2}}, & \text{on } C(M)_\ell \text{ for } \ell \leqslant -n-2, \\ \mathrm{id}, & \text{on } C(M)_\ell \text{ for } \ell > -n-2. \end{cases} \tag{4.5}$$

Proposition. *Let $L = \bigoplus_{k\geqslant 0} \mathbb{A} f^{(k)} m_0$ be a crystal lattice of the Verma module M with highest weight $-n-2$, and let $L^\sharp = \bigoplus_{k\geqslant -n-1} \mathbb{A} f^{(k)} m_0$. Then $q^{-n(n+1)} S_n (qt\Delta)^{-\frac{n}{2}} L^\sharp$ is a completion of L.*

Proof. We note that $q^{-n-1} - 1 = aq^{-n-1}$ where $a = 1 - q^{n+1} \in \mathbb{A}^\times$. It follows from (4.4) that $(qt\Delta)^{-\frac{n}{2}} L^\sharp = \bigoplus_{k\geqslant -n-1} \mathbb{A} q^{nk}(q^{-n-1})^{-n} f^{(k)} m_0 = \bigoplus_{k\geqslant -n-1} \mathbb{A} q^{n(k+n+1)} f^{(k)} m_0$. Also, for $k \geqslant 0$,

$$S_n f^{(k)} m_0 = q^{-n-1} q^k \big(q^{-n-1} - 1\big)^{-1} f^{(k)} m_0 = q^k a^{-1} f^{(k)} m_0. \tag{4.6}$$

Therefore,

$$\begin{aligned} q^{-n(n+1)} S_n (qt\Delta)^{-\frac{n}{2}} L^\sharp &= S_n \Bigg(\bigoplus_{k\geqslant -n-1} \mathbb{A} q^{nk} f^{(k)} m_0 \Bigg) \\ &= \Bigg(\bigoplus_{k\geqslant 0} \mathbb{A} q^{(n+1)k} f^{(k)} m_0 \Bigg) \oplus \Bigg(\bigoplus_{-n-1\leqslant k<0} \mathbb{A} q^{nk} f^{(k)} m_0 \Bigg). \end{aligned}$$

Hence, $q^{-n(n+1)} S_n (qt\Delta)^{-\frac{n}{2}} L^\sharp$ is the $\mathbb{A}$-lattice generated by the same vectors as in (4.3). Therefore, it is equal to the crystal lattice $\widetilde{L}$ from the Proposition 4.3 which is a completion of L. $\square$

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