Introduction to **Ends of Complexes**
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We take 'complex' to mean both a CW (or simplicial) complex in topology and a chain complex in algebra. An 'end' of a complex is a subcomplex with a particular type of infinite behaviour, involving non-compactness in topology and infinite generation in algebra. The ends of manifolds are of greatest interest; we regard the ends of CW and chain complexes as tools in the investigation of manifolds and related spaces, such as stratified sets. The interplay of the topological properties of the ends of manifolds, the homotopy theoretic properties of the ends of CW complexes and the algebraic properties of the ends of chain complexes has been an important theme in the classification theory of high dimensional manifolds for over 35 years. However, the gaps in the literature mean that there are still some loose ends to wrap up! Our aim in this book is to present a systematic exposition of the various types of ends relevant to manifold classification, closing the gaps as well as obtaining new results. The book is intended to serve both as an account of the existing applications of ends to the topology of high dimensional manifolds and as a foundation for future developments.

We assume familiarity with the basic language of high dimensional manifold theory, and the standard applications of algebraic K- and L-theory to manifolds, but otherwise we have tried to be as self contained as possible.

The algebraic topology of finite CW complexes suffices for the combinatorial topology of compact manifolds. However, in order to understand the difference between the topological and combinatorial properties it is necessary to deal with infinite CW complexes and non-compact manifolds. The classic cases include the Hauptvermutung counterexamples of Milnor [96], the topological invariance of the rational Pontrjagin classes proved by Novikov [103], the topological manifold structure theory of Kirby and Siebenmann [84], and the topological invariance of Whitehead torsion proved by Chapman [22]. The algebraic and geometric topology of non-compact manifolds has been a prominent feature in much of the recent work on the Novikov conjectures – see Ferry, Ranicki and Rosenberg [59] for a survey. (In these applications the non-compact manifolds arise as the universal covers of aspherical compact manifolds, e.g. the Euclidean space $\mathbb{R}^i$ covering the torus $T^i = S^1 \times S^1 \times \ldots \times S^1 = B\mathbb{Z}^i$.) In fact, many current developments in topology, operator theory, differential geometry, hyperbolic geometry, and group theory are concerned with the asymptotic properties of non-compact manifolds and infinite groups – see Gromov [65], Connes [33] and Roe [135] for example.

What is an end of a topological space? Roughly speaking, an end of a non-compact space W is a component of $W\backslash K$ for arbitrarily large compact subspaces $K \subseteq W$. More precisely:

Definition 1. (i) A *neighbourhood of an end* in a non-compact space W is a subspace $U \subset W$ which contains a component of $W\backslash K$ for a non-empty compact subspace $K \subset W$.

(ii) An *end* ϵ of W is an equivalence class of sequences of connected open neighbourhoods $W \supset U_1 \supset U_2 \supset \ldots$ such that

$$\bigcap_{i=1}^{\infty} \mathrm{cl}\,(U_i) \;=\; \emptyset$$

subject to the equivalence relation

$$(W \supset U_1 \supset U_2 \supset \ldots) \sim (W \supset V_1 \supset V_2 \supset \ldots)$$

if for each U_i there exists j with $U_i \subseteq V_j$, and for each V_j there exists i with $V_j \subseteq U_i$.

(iii) The *fundamental group* of an end ϵ is the inverse limit

$$\pi_1(\epsilon) \;=\; \varprojlim_i \pi_1(U_i)\ . \qquad \square$$

The theory of ends was initiated by Freudenthal [61] in connection with topological groups. The early applications of the theory concerned the ends of open 3-dimensional manifolds, and the ends of discrete groups (which are the ends of the universal covers of their classifying spaces).

We are especially interested in the ends of manifolds which are ‘tame’, and in extending the notion of tameness to other types of ends. An end of a manifold is tame if it has a system of neighbourhoods satisfying certain strong restrictions on the fundamental group and chain homotopy type. Any non-compact space W can be compactified by adding a point at infinity, $W^\infty = W \cup \{\infty\}$. A manifold end is ‘collared’ if it can be compactified by a manifold, i.e. if the point at infinity can be replaced by a closed manifold boundary, allowing the end to be identified with the interior of a compact manifold with boundary. A high dimensional tame manifold end can be collared if and only if an algebraic K-theory obstruction vanishes. The theory of tame ends has found wide application in the surgery classification theory of high dimensional compact manifolds and stratified spaces, and in the related controlled topology and algebraic K- and L-theory.

Example 2. Let K be a connected compact space.

(i) $K \times [0, \infty)$ has one end ϵ, with connected open neighbourhoods

$$U_i \;=\; K \times (i, \infty) \subset K \times [0, \infty) \;,$$

such that $\pi_1(\epsilon) = \pi_1(K)$.

(ii) $K \times \mathbb{R}$ has two ends ϵ^+, ϵ^-, with connected open neighbourhoods

$$U_i^+ \;=\; K \times (i, \infty) \;\;,\;\; U_i^- \;=\; K \times (-\infty, -i) \subset K \times \mathbb{R} \;,$$

such that $\pi_1(\epsilon^\pm) = \pi_1(K)$.

(iii) $K \times \mathbb{R}^2$ has one end ϵ, with connected open neighbourhoods

$$U_i \;=\; K \times \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 > i^2\} \;,$$

such that $\pi_1(\epsilon) = \pi_1(K) \times \mathbb{Z}$. □

Example 3. (i) Let W be a space with a proper map $d : W \longrightarrow [0, \infty)$ which is onto, and such that the inverse images $U_t = d^{-1}(t, \infty) \subseteq W$ $(t \geq 1)$ are connected. Then W has one end ϵ with connected open neighbourhoods $W \supset U_1 \supset U_2 \supset \dots$ such that $\mathrm{cl}(U_t) = d^{-1}[t, \infty)$, $\bigcap\limits_{i=0}^{\infty} \mathrm{cl}(U_i) = \emptyset$.

(ii) Let $(W, \partial W)$ be a connected open n-dimensional manifold with connected compact boundary. Then W has one end ϵ if and only if there exists a proper map $d : (W, \partial W) \longrightarrow ([0, \infty), \{0\})$ which is transverse regular at $\mathbb{N} = \{0, 1, 2, \dots\} \subset [0, \infty)$, with the inverse images

$$(W_i; M_i, M_{i+1}) \;=\; d^{-1}([i, i+1]; \{i\}, \{i+1\}) \;\; (i \in \mathbb{N})$$

connected compact n-dimensional cobordisms such that

$$(W, \partial W) \;=\; (\bigcup_{i=0}^{\infty} W_i, M_0) \;.$$

(iii) Given connected compact n-dimensional cobordisms $(W_i; M_i, M_{i+1})$ $(i \in \mathbb{N})$ there is defined a connected open n-dimensional manifold with compact boundary $(W, \partial W) = (\bigcup\limits_{i=0}^{\infty} W_i, M_0)$. The union of Morse functions $d_i : (W_i; M_i, M_{i+1}) \longrightarrow ([i, i+1]; \{i\}, \{i+1\})$ defines a proper map $d : (W, \partial W) \longrightarrow ([0, \infty), \{0\})$, and as in (ii) W has one end ϵ. If the inclusions $M_i \longrightarrow W_i$, $M_{i+1} \longrightarrow W_i$ induce isomorphisms in π_1 then

$$\pi_1(M_0) \;=\; \pi_1(W_0) \;=\; \pi_1(M_1) \;=\; \dots \;=\; \pi_1(W) \;=\; \pi_1(\epsilon) \;.$$ □

Definition 4. An end ϵ of an open n-dimensional manifold W can be *collared* if it has a neighbourhood of the type $M\times[0,\infty)\subset W$ for a connected closed $(n-1)$-dimensional manifold M. □

Example 5. (i) An open n-dimensional manifold with one end ϵ is (homeomorphic to) the interior of a closed n-dimensional manifold if and only if ϵ can be collared. More generally, if W is an open n-dimensional manifold with compact boundary ∂W and one end ϵ, then there exists a compact n-dimensional cobordism $(L;\partial W,M)$ with $L\backslash M$ homeomorphic to W rel ∂W if and only if ϵ can be collared.

(ii) If $(V,\partial V)$ is a compact n-dimensional manifold with boundary then for any $x\in V\backslash\partial V$ the complement $W=V\backslash\{x\}$ is an open n-dimensional manifold with a collared end ϵ and $\partial W=\partial V$, with a neighbourhood $M\times[0,\infty)\subset W$ for $M=S^{n-1}$. The one-point compactification of W is $W^\infty=V$. The compactification of W provided by (i) is $L=\mathrm{cl}(V\backslash D^n)$, for any neighbourhood $D^n\subset V\backslash\partial V$ of x, with $(L;\partial W,M)=(W\cup S^{n-1};\partial V,S^{n-1})$. □

Stallings [154] used engulfing to prove that if W is a contractible open n-dimensional PL manifold with one end ϵ such that $\pi_1(\epsilon)=\{1\}$ and $n\geq 5$ then W is PL homeomorphic to $\mathbb{R}^n$ – in particular, the end ϵ can be collared.

Let $(W,\partial W)$ be an open n-dimensional manifold with compact boundary and one end ϵ. Making a proper map $d:(W,\partial W)\longrightarrow([0,\infty),\{0\})$ transverse regular at some $t\in(0,\infty)$ gives a decomposition of $(W,\partial W)$ as

$$(W,\partial W) \;=\; (L;\partial W,M)\cup_M (N,M)$$

with $(L;\partial W,M)=d^{-1}([0,t];\{0\},\{t\})$ a compact n-dimensional cobordism and $N=d^{-1}[t,\infty)$ non-compact. The end ϵ can be collared if and only if N can be chosen such that there exists a homeomorphism $N\cong M\times[0,\infty)$ rel $M=M\times\{0\}$, in which case $L\backslash M\cong L\cup_{M\times\{0\}}M\times[0,\infty)\cong W$ rel ∂W. In terms of Morse theory: it is possible to collar ϵ if and only if $(W,\partial W)$ admits a proper Morse function d with only a finite number of critical points. Browder, Levine and Livesay [14] used codimension 1 surgery on $M\subset W$ to show that if $\pi_1(W)=\pi_1(\epsilon)=\{1\}$ and $n\geq 6$ then ϵ can be collared if and only if the homology groups $H_*(W)$ are finitely generated (with $H_r(W)=0$ for all but finitely many values of r). Siebenmann [140] combined codimension 1 surgery with the finiteness obstruction theory of Wall [163] for finitely dominated spaces, proving that in dimensions ≥ 6 a tame manifold end can be collared if and only if an algebraic K-theory obstruction vanishes.

Definition 6. A space X is *finitely dominated* if there exist a finite CW complex K and maps $f:X\longrightarrow K$, $g:K\longrightarrow X$ with $gf\simeq 1:X\longrightarrow X$. □

Example 7. Any space homotopy equivalent to a finite CW complex is finitely dominated. □

Example 8. A connected CW complex X with $\pi_1(X) = \{1\}$ is finitely dominated if and only if $H_*(X)$ is finitely generated, if and only if X is homotopy equivalent to a finite CW complex. □

For non-simply-connected X the situation is more complicated:

Theorem 9. (Wall [163, 164]) *A connected CW complex X is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex P. The reduced projective class of a finitely dominated X*

$$[X] \;=\; [P] \;=\; \sum_{r=0}^{\infty}(-)^r[P_r] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

is the **finiteness obstruction** *of X, such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.* □

Definition 10. An end ϵ of an open manifold W is *tame* if it admits a sequence $W \supset U_1 \supset U_2 \supset \dots$ of finitely dominated neighbourhoods with

$$\bigcap_{i=1}^{\infty} \mathrm{cl}(U_i) \;=\; \emptyset \;\;,\;\; \pi_1(U_1) \;=\; \pi_1(U_2) \;=\; \dots \;=\; \pi_1(\epsilon)\;. \qquad \square$$

Example 11. If an end ϵ of an open manifold W can be collared then it is tame: if $M \times [0, \infty) \subset W$ is a neighbourhood of ϵ then the open neighbourhoods $W \supset U_1 = M \times (1, \infty) \supset U_2 = M \times (2, \infty) \supset \dots$ satisfy the conditions of Definition 10, with $\mathrm{cl}(U_i) = M \times [i, \infty)$, $\pi_1(\epsilon) = \pi_1(M)$. □

Tameness is a geometric condition which ensures stable (as opposed to wild) behaviour in the topology at infinity of a non-compact space W. The fundamental example is $W = K \times [0, \infty)$ for a compact space K, in which the topology at infinity is that of K.

Theorem 12. (Siebenmann [140]) *A tame end ϵ of an open n-dimensional manifold W has a reduced projective class invariant, the* **end obstruction**

$$[\epsilon] \;=\; \varprojlim_i\, [U_i] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)]) \;=\; \varprojlim_i\, \widetilde{K}_0(\mathbb{Z}[\pi_1(U_i)])$$

such that $[\epsilon] = 0$ if (and for $n \geq 6$ only if) ϵ can be collared. □

Even if a tame manifold end ϵ can be collared, the collarings need not be unique. The various collarings of a tame end ϵ in an open manifold W of dimension ≥ 6 with $[\epsilon] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$ are classified by the Whitehead group $Wh(\pi_1(\epsilon))$: if $M \times [0,\infty)$, $M' \times [0,\infty) \subset W$ are two collar neighbourhoods of ϵ then for sufficiently large $t \geq 0$ there exists an h-cobordism $(N; M, M')$ between $M \times \{0\}$ and $M' \times \{t\} \subset W$, with

$$M \times [0,\infty) \ = \ N \cup_{M' \times \{t\}} M' \times [t,\infty) \subset W \ .$$

By the s-cobordism theorem $(N; M, M')$ is homeomorphic to the product $M \times (I; \{0\}, \{1\})$ if and only if $\tau(M \simeq N) = 0 \in Wh(\pi_1(\epsilon))$. The non-uniqueness of collarings of PL manifold ends was used by Milnor [96] in the construction of homeomorphisms of compact polyhedra which are not homotopic to a PL homeomorphism, disproving the Hauptvermutung for compact polyhedra. The end obstruction theory played an important role in the disproof of the manifold Hauptvermutung by Casson and Sullivan (Ranicki [131]) – the manifold case also requires surgery and L-theory.

Quinn [114, 115, 116] developed a controlled version of the Siebenmann end obstruction theory, and applied it to stratified spaces. (See Ranicki and Yamasaki [132] for a treatment of the controlled finiteness obstruction, and Connolly and Vajiac [34] for an end theorem for stratified spaces.) The tameness condition of Definition 10 for manifold ends was extended by Quinn to stratified spaces, distinguishing *two* tameness conditions for ends of non-compact spaces, involving maps pushing *forward* along the end and in the *reverse* direction. We shall only consider the two-stratum case of a one-point compactification, with the lower stratum the point at infinity. In Chapters 7, 8 we state the definitions of forward and reverse tameness. The original tameness condition of Siebenmann [140] appears in Chapter 8 as *reverse* π_1*-tameness*, so called since it is a combination of reverse tameness and π_1-stability. In general, forward and reverse tameness are independent of each other, but for π_1-stable manifold ends ϵ with finitely presented $\pi_1(\epsilon)$ the two kinds of tameness are equivalent by a kind of Poincaré duality.

Definition 13. (Quinn [116]) The *end space* $e(W)$ of a space W is the space of proper paths $\omega : [0,\infty) \longrightarrow W$. □

We refer to Appendix B for a brief history of end spaces.

The end space $e(W)$ is a homotopy model for the 'space at infinity' of W, playing a role similar to the ideal boundary in hyperbolic geometry. The topology at infinity of a space W is the inverse system of complements of compact subspaces (i.e. cocompact subspaces or neighbourhoods of infinity) of W, which are the open neighbourhoods of the point ∞ in the one-point compactification $W^\infty = W \cup \{\infty\}$. The homology at infinity $H^\infty_*(W)$ is defined to fit into an exact sequence

$$\ldots \longrightarrow H^{\infty}_r(W) \longrightarrow H_r(W) \longrightarrow H^{lf}_r(W) \longrightarrow H^{\infty}_{r-1}(W) \longrightarrow \ldots ,$$

and $H^{lf}_*(W) = H_*(W^\infty, \{\infty\})$ for reasonable W. The end space $e(W)$ is the 'link of infinity in W^∞'. There is a natural passage from the algebraic topology at infinity of W to the algebraic topology of $e(W)$, which is a one-to-one correspondence for forward tame W, with $H_*(e(W)) = H^{\infty}_*(W)$.

If $(W, \partial W)$ is an open n-dimensional manifold with compact boundary and one tame end ϵ the end space $e(W)$ is a finitely dominated $(n-1)$-dimensional Poincaré space with $\pi_1(e(W)) = \pi_1(\epsilon)$, and $(W; \partial W, e(W))$ is a finitely dominated n-dimensional Poincaré cobordism, regarding $e(W)$ as a subspace of W via the evaluation map

$$e(W) \longrightarrow W \; ; \; (\omega : [0,\infty) \longrightarrow W) \longrightarrow \omega(0) .$$

The non-compact spaces of greatest interest to us are the infinite cyclic covers of 'bands':

Definition 14. A *band* (M, c) is a compact space M with a map $c : M \longrightarrow S^1$ such that the infinite cyclic cover $\overline{M} = c^*\mathbb{R}$ of M is finitely dominated, and such that the projection $\overline{M} \longrightarrow M$ induces a bijection of path components $\pi_0(\overline{M}) \cong \pi_0(M)$. □

Example 15. A connected finite CW complex M with a map $c : M \longrightarrow S^1$ inducing an isomorphism $c_* : \pi_1(M) \cong \mathbb{Z}$ defines a band (M, c) (i.e. the infinite cyclic cover $\overline{M} = c^*\mathbb{R}$ is finitely dominated) if and only if the homotopy groups $\pi_*(M) = H_*(\overline{M})$ $(* \geq 2)$ are finitely generated. □

The infinite cyclic cover $\overline{M}$ of a connected manifold band (M, c) has two ends. The projection $c : M \longrightarrow S^1$ lifts to a proper map $\overline{c} : \overline{M} \longrightarrow \mathbb{R}$, such that the inverse images

$$\overline{M}^+ \; = \; \overline{c}^{-1}[0,\infty) \;\; , \;\; \overline{M}^- \; = \; \overline{c}^{-1}(-\infty, 0] \subset \overline{M}$$

are closed neighbourhoods of the two ends. In Chapter 15 we shall prove that the two ends of $\overline{M}$ are tame, with homotopy equivalences

$$e(\overline{M}^+) \; \simeq \; e(\overline{M}^-) \; \simeq \; \overline{M} .$$

The problem of deciding if an open manifold is the interior of a compact manifold with boundary is closely related to the problem of deciding if a compact manifold M fibres over S^1, i.e. if a map $c : M \longrightarrow S^1$ is homotopic to the projection of a fibre bundle. In the first instance, it is necessary for (M, c) to be a band:

Example 16. Suppose given a fibre bundle $F \longrightarrow M \xrightarrow{c} S^1$ with F a closed $(n-1)$-dimensional manifold and $M = T(h)$ the mapping torus of a monodromy self homeomorphism $h : F \longrightarrow F$. If h preserves the path components then (M, c) is an n-dimensional manifold band, with the infinite cyclic cover $\overline{M} = F \times \mathbb{R}$ homotopy equivalent to a finite CW complex. □

Stallings [153] used codimension 1 surgery on a surface $c^{-1}(*) \subset M$ to prove that a map $c : M \longrightarrow S^1$ from a compact irreducible 3-dimensional manifold M with $\ker(c_*: \pi_1(M) \longrightarrow \mathbb{Z}) \ncong \mathbb{Z}_2$ is homotopic to the projection of a fibre bundle if and only if $\ker(c_*)$ is finitely generated, in which case $\ker(c_*) = \pi_1(F)$ is the fundamental group of the fibre F. In particular, the complement of a knot $k : S^1 \subset S^3$

$$(M, \partial M) \;=\; (\mathrm{cl}(S^3 \backslash (k(S^1) \times D^2)), S^1 \times S^1)$$

fibres over S^1 if and only if the commutator subgroup $[\pi, \pi]$ of the fundamental group $\pi = \pi_1(M)$ is finitely generated. Browder and Levine [13] used codimension 1 surgery in higher dimensions to prove that for $n \geq 6$ a compact n-dimensional manifold band (M, c) with $c_* : \pi_1(M) \cong \mathbb{Z}$ fibres. Thus a high-dimensional knot $k : S^{n-2} \subset S^n$ $(n \geq 6)$ with $\pi_1(S^n \backslash k(S^{n-2})) = \mathbb{Z}$ fibres (i.e. the knot complement fibres over S^1) if and only if the higher homotopy groups $\pi_*(S^n \backslash k(S^{n-2}))$ $(* \geq 2)$ are finitely generated. More generally:

Theorem 17. (Farrell [46], Siebenmann [145]) *An n-dimensional manifold band (M, c) has a Whitehead torsion invariant, the* **fibring obstruction**

$$\Phi(M, c) \in Wh(\pi_1(M)) \;,$$

such that $\Phi(M, c) = 0$ if (and for $n \geq 6$ only if) M fibres over S^1, with $c : M \longrightarrow S^1$ homotopic to a fibre bundle projection. □

In the main text we shall actually be dealing with the two fibring obstructions $\Phi^+(M, c)$, $\Phi^-(M, c) \in Wh(\pi_1(M))$ defined for a CW band (M, c). For an n-dimensional manifold band (M, c) the two obstructions determine each other by Poincaré duality

$$\Phi^+(M, c) \;=\; (-)^{n-1} \Phi^-(M, c)^* \in Wh(\pi_1(M)) \;,$$

and in the Introduction we write $\Phi^+(M, c)$ as $\Phi(M, c)$.

Example 18. For any n-dimensional manifold band (M, c) the $(n+1)$-dimensional manifold band $(M \times S^1, d)$ with $d(x, t) = c(x)$ has fibring obstruction

$$\Phi(M \times S^1, d) \;=\; 0 \in Wh(\pi_1(M) \times \mathbb{Z}) \;.$$

For $n \geq 5$ the geometric construction of Theorem 19 below actually gives a canonical fibre bundle

$$F \longrightarrow M \times S^1 \xrightarrow{p} S^1$$

with p homotopic to d. The fibre F is the 'wrapping up' of the tame end $\overline{M}^+$ of $\overline{M}$, a closed n-dimensional manifold such that there are defined homeomorphisms

$$F \times \mathbb{R} \cong \overline{M} \times S^1 \quad , \quad M \times S^1 \cong T(h)$$

for a monodromy self homeomorphism $h : F \longrightarrow F$. The fibring obstruction $\Phi(M,c) \in Wh(\pi_1(M))$ is the obstruction to splitting off an S^1-factor from $h : F \longrightarrow F$, so that for $n \geq 6$ $\Phi(M,c) = 0$ if and only if up to isotopy

$$h = h_1 \times 1 : F = F_1 \times S^1 \longrightarrow F = F_1 \times S^1$$

with $h_1 : F_1 \longrightarrow F_1$ a self homeomorphism such that $M \cong T(h_1)$. □

Bands are of interest in their own right. For example, the fibring obstruction theory for bands gives a geometric interpretation of the 'fundamental theorem' of algebraic K-theory of Bass [4]

$$Wh(\pi \times \mathbb{Z}) = Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])$$

– see Ranicki [124] for a recent account. The following uniformization theorem shows that every tame manifold end of dimension ≥ 6 has an open neighbourhood which is the infinite cyclic cover of a manifold band. It was announced by Siebenmann [141], and is proved here in Chapter 17.

Theorem 19. *Let $(W, \partial W)$ be a connected open n-dimensional manifold with compact boundary and one end ϵ, with $n \geq 6$.*

(i) *The end ϵ is tame if and only if it has a neighbourhood $X = \overline{M} \subset W$ which is the finitely dominated infinite cyclic cover of a compact n-dimensional manifold band $\widehat{X} = (M, c)$, the* **wrapping up** *of ϵ, such that*

$$\pi_1(\overline{M}) = \pi_1(\epsilon) \quad , \quad \pi_1(M) = \pi_1(\epsilon) \times \mathbb{Z} \quad , \quad e(W) \simeq \overline{M} \ ,$$

$$\Phi(M,c) = [\epsilon] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)]) \subseteq Wh(\pi_1(\epsilon) \times \mathbb{Z}) \ ,$$

and such that the covering translation $\zeta : \overline{M} \longrightarrow \overline{M}$ is isotopic to the identity. The $(n+1)$-dimensional manifold band $(M \times S^1, d)$ with $d(x,t) = c(x)$ fibres over S^1 : the map $d : M \times S^1 \longrightarrow S^1$ is homotopic to the projection of a fibre bundle with fibre M, with a homeomorphism

$$\overline{M} \times S^1 \cong M \times \mathbb{R} \ .$$

Thus $\epsilon \times S^1$ can be collared with boundary M : there exists a compact $(n+1)$-dimensional cobordism $(N; \partial W \times S^1, M)$ with a rel ∂ *homeomorphism*

$$(N \backslash M, \partial W \times S^1) \cong (W, \partial W) \times S^1 \ .$$

(ii) *For tame ϵ the Siebenmann end obstruction of ϵ is the Wall finiteness obstruction of* $\overline{M}^+$

$$[\epsilon] \;=\; [\overline{M}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)]) \;,$$

with $[\epsilon] = 0$ *if and only if* ϵ *can be collared, in which case there exists a compact* n*-dimensional cobordism* $(K;\partial W, L)$ *with a* rel ∂ *homeomorphism*

$$(K\backslash L, \partial W) \;\cong\; (W, \partial W)$$

and a homeomorphism

$$(K;\partial W, L)\times S^1 \;\cong\; (N;\partial W\times S^1, M)$$

(N as in (i)), and (M,c) *fibres over* S^1 *with* $M\cong L\times S^1$ *and* $\overline{M}\cong L\times\mathbb{R}$.

□

A CW complex X is finitely dominated if and only if $X\times S^1$ is homotopy equivalent to a finite CW complex, by a result of M. Mather [91]. A manifold end ϵ of dimension ≥ 6 is tame if and only if $\epsilon\times S^1$ can be collared – this was already proved by Siebenmann [140], but the wrapping up procedure of Theorem 19 actually gives a canonical collaring of $\epsilon\times S^1$.

In principle, Theorem 19 could be proved using the canonical regular neighbourhood theory of Siebenmann [148] and Siebenmann, Guillou and Hähl [149]. We prefer to give a more elementary approach, using a combination of the geometric, homotopy theoretic and algebraic methods which have been developed in the last 25 years to deal with non-compact spaces. While the wrapping up construction has been a part of the folklore, the new aspect of our approach is that we rely on the end space and the extensively developed theory of *manifold approximate fibrations* rather than ad hoc engulfing methods. An approximate fibration is a map with an approximate lifting property. (Of course, manifold approximate fibration theory relies on engulfing, but we prefer to subsume the details of the engulfing in the theory.) We do not assume previous acquaintance with approximate fibrations and engulfing.

The proof of Theorem 19 occupies most of Parts One and Two (Chapters 1–20). There are three main steps in passing from a tame end ϵ of W to the wrapping up band (M,c) such that the infinite cyclic cover $\overline{M}\subseteq W$ is a neighbourhood of ϵ:

(i) in Chapter 9 we show that tameness conditions on a space W imply that the end space $e(W)$ is finitely dominated and that, near infinity, W looks like the product $e(W)\times[0,\infty)$;

(ii) in Chapter 16 we use (i) to prove that every tame manifold end ϵ of dimension ≥ 5 has a neighbourhood X which is the total space of a manifold approximate fibration $d : X \longrightarrow \mathbb{R}$;

(iii) in Chapter 17 we show that for every manifold approximate fibration $d : X \longrightarrow \mathbb{R}$ of dimension ≥ 5 there exists a manifold band (M, c) such that $X = \overline{M}$, with a proper homotopy $d \simeq \overline{c} : X \longrightarrow \mathbb{R}$.

The construction in (iii) of the wrapping up (M, c) of (X, d) is by the manifold 'twist glueing' due to Siebenmann [145]. The twist glueing construction of manifold bands is extended to the CW category in Chapters 19 and 20.

In Part Three (Chapters 21–27) we study the algebraic properties of tame ends in the context of chain complexes over a polynomial extension ring and also in bounded algebra. We obtain an abstract version of Theorem 19, giving a chain complex account of wrapping up : manifold wrapping up induces a CW complex wrapping up, which in turn induces a chain complex wrapping up, and similarly for the various types of twist glueing.

In Chapter 15 we introduce the notion of a *ribbon* (X, d), which is a non-compact space X with a proper map $d : X \longrightarrow \mathbb{R}$ with the homotopy theoretic and homological end properties of the infinite cyclic cover $(\overline{W}, \overline{c})$ of a band (W, c). Ribbons are the homotopy analogues of manifold approximate fibration over $\mathbb{R}$. In Chapter 25 we develop the chain complex versions of CW ribbons as well as algebraic versions of tameness.

The study of ends of complexes is particularly relevant to stratified spaces. A *topologically stratified space* is a space X together with a filtration

$$\emptyset \ = \ X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X^{n-1} \subseteq X^n \ = \ X$$

by closed subspaces such that the *strata* $X^j \backslash X^{j-1}$ are open topological manifolds which satisfy certain tameness conditions and a homotopy link condition. These spaces were first defined by Quinn [116] in order to study purely topological stratified phenomena as opposed to the smoothly stratified spaces of Whitney [170], Thom [161] and J. Mather [90], and the piecewise linear stratified spaces of Akin [1] and Stone [159]. Quinn's paper should be consulted for more precise definitions. Our results only apply directly to the very special case obtained from the one-point compactification $W^\infty = X$ of an open manifold W, regarded as a filtered space by $X^0 = \{\infty\} \subseteq W^\infty = X$. Then X is a topologically stratified space with two strata if and only if W is tame. (The general case requires controlled versions of our results.) Earlier, Siebenmann [147] had studied a class of topologically stratified spaces called *locally conelike stratified spaces*. The one-point compactification of an open manifold W with one end is locally conelike stratified if and only if the end of W can be collared. Hence, Quinn's stratified spaces are much more general than Siebenmann's. The conditions required of topologically stratified spaces by Quinn are designed to imply that strata have neighbourhoods which are homotopy equivalent to mapping cylinders of fibrations, whereas in the classical cases the strata have neighbourhoods which are homeomorphic to mapping cylinders of bundle projections in the appropriate category: fibre bundle projections in the smooth case, block bundle projections in the piecewise linear case. Strata in Siebenmann's locally conelike stratified spaces have neighbourhoods which are locally homeomorphic to mapping cylinders of fibre bundle projections, but not necessarily globally.

A *stratified homotopy equivalence* is a homotopy equivalence in the stratified category (maps must preserve strata, not just the filtration). In the special case of one-point compactifications, stratified homotopy equivalences $(W^\infty, \{\infty\}) \longrightarrow (V^\infty, \{\infty\})$ are exactly the proper homotopy equivalences $W \longrightarrow V$. Weinberger [166] has developed a stratified surgery theory which classifies topologically stratified spaces up to stratified homotopy equivalence in the same sense that classical surgery theory classifies manifolds up to homotopy equivalence. Weinberger outlines two separate proofs of his theory. The first proof [166, pp. 182–188] involves stabilizing a stratified space by crossing with high dimensional tori in order to get a nicer stratified space which is amenable to the older stratified surgery theory of Browder and Quinn [15]. The obstruction to codimension i destabilization involves the codimension i lower K-group $K_{1-i}(\mathbb{Z}[\pi]) \subseteq Wh(\pi \times \mathbb{Z}^i)$. (Example 18 and Theorem 19 treat the special case $i = 1$.) The second proof outlined in [166, Remarks p. 189] uses more directly the existence of appropriate tubular neighbourhoods of strata called *teardrop neighbourhoods*. These neighbourhoods were shown to exist in the case of two strata by Hughes, Taylor, Weinberger and Williams [76] and in general by Hughes [74]. In 16.13 we give a complete proof of the existence of teardrop neighbourhoods in the special case of the topologically stratified space $(W^\infty, \{\infty\})$ determined by an open manifold W with a tame end. The result asserts that W contains an open cocompact subspace $X \subseteq W$ which admits a manifold approximate fibration $X \longrightarrow \mathbb{R}$. In the more rigid smoothly stratified spaces, the tubular neighbourhoods would be given by a genuine fibre bundle projection. The point is that Quinn's definition gives information on the neighbourhoods of strata only up to homotopy. The existence of teardrop neighbourhoods means there is a much stronger geometric structure given in terms of manifold approximate fibrations.

We use the theory of manifold approximate fibrations to perform geometric wrapping up constructions. This is analogous to Weinberger's second approach to stratified surgery, in which teardrop neighbourhoods of strata are used in order to be able to draw on manifold approximate fibration theory rather than stabilization and destabilization. We expect that the general theory of teardrop neighbourhoods will likewise allow generalizations of the wrapping up construction to arbitrary topologically stratified spaces, using the homotopy theoretic and algebraic properties of the ribbons introduced in this book. Such a combination of geometry, homotopy theory and algebra will be necessary to fully understand the algebraic K- and L-theory of stratified spaces.

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Chapter summaries

Part One, *Topology at infinity*, is devoted to the basic theory of the general, geometric and algebraic topology at infinity of non-compact spaces. Various models for the topology at infinity are introduced and compared.

Chapter 1, *End spaces*, begins with the definition of the end space $e(W)$ of a non-compact space W. The set of path components $\pi_0(e(W))$ is shown to be in one-to-one correspondence with the set of ends of W (in the sense of Definition 1 above) for a wide class of spaces.

Chapter 2, *Limits*, reviews the basic constructions of homotopy limits and colimits of spaces, and the related inverse, direct and derived limits of groups and chain complexes. The end space $e(W)$ is shown to be weak homotopy equivalent to the homotopy inverse limit of cocompact subspaces of W and the homotopy inverse limit is compared to the ordinary inverse limit. The 'fundamental group at infinity' $\pi_1^\infty(W)$ of W is defined and compared to $\pi_1(e(W))$.

Chapter 3, *Homology at infinity*, contains an account of locally finite singular homology, which is the homology based on infinite chains. The homology at infinity $H_*^\infty(W)$ of a space W is the difference between ordinary singular homology $H_*(W)$ and locally finite singular homology $H_*^{lf}(W)$.

Chapter 4, *Cellular homology*, reviews locally finite cellular homology, although the technical proof of the equivalence with locally finite singular homology is left to Appendix A.

Chapter 5, *Homology of covers*, concerns ordinary and locally finite singular and cellular homology of the universal cover (and other covers) $\widetilde{W}$ of W. The version of the Whitehead theorem for detecting proper homotopy equivalences of CW complexes is stated.

Chapter 6, *Projective class and torsion*, recalls the Wall finiteness obstruction and Whitehead torsion. A locally finite finiteness obstruction is introduced, which is related to locally finite homology in the same way that the Wall finiteness obstruction is related to ordinary homology, and the difference between the two obstructions is related to homology at infinity.

Chapter 7, *Forward tameness*, concerns a tameness property of ends, which is stated in terms of the ability to push neighbourhoods towards infinity. It is proved that for forward tame W the singular chain complex of the end space $e(W)$ is chain equivalent to the singular chain complex at infinity of W, and that the homotopy groups of $e(W)$ are isomorphic to the inverse limit of the homotopy groups of cocompact subspaces of W. There is a related concept of forward collaring.

Chapter 8, *Reverse tameness*, deals with the other tameness property of ends, which is stated in terms of the ability to pull neighbourhoods in from infinity. It is closely related to finite domination properties of cocompact subspaces of W. There is a related concept of reverse collaring.

Chapter 9, *Homotopy at infinity*, gives an account of proper homotopy theory at infinity. It is shown that the homotopy type of the end space, the two types of tameness, and other end phenomena are invariant under proper homotopy equivalences at infinity. It is also established that in most cases of interest a space W is forward and reverse tame if and only if W is bounded homotopy equivalent at ∞ to $e(W) \times [0, \infty)$, in which case $e(W)$ is finitely dominated.

Chapter 10, *Projective class at infinity*, introduces two finiteness obstructions which the two types of tameness allow to be defined. The finiteness obstruction at infinity of a reverse tame space is an obstruction to reverse collaring. Likewise, the locally finite finiteness obstruction at infinity of a forward tame space is an obstruction to forward collaring. For a space W which is both forward and reverse tame, the end space $e(W)$ is finitely dominated and its Wall finiteness obstruction is the difference of the two finiteness obstructions at infinity. It is also proved that for a manifold end forward and reverse tameness are equivalent under certain fundamental group conditions.

Chapter 11, *Infinite torsion*, contains an account of the infinite simple homotopy theory of Siebenmann for locally finite CW complexes. The infinite Whitehead group of a forward tame CW complex is described algebraically as a relative Whitehead group. The infinite torsion of a proper homotopy equivalence is related to the locally finite finiteness obstruction at infinity. A CW complex W is forward (resp. reverse) tame if and only if $W \times S^1$ is infinite simple homotopy equivalent to a forward (resp. reverse) collared CW complex.

Chapter 12, *Forward tameness is a homotopy pushout*, deals with Quinn's characterization of forward tameness for a σ-compact metric space W in terms of a homotopy property, namely that the one-point compactification W^∞ is the homotopy pushout of the projection $e(W) \longrightarrow W$ and $e(W) \longrightarrow \{\infty\}$, or equivalently that W^∞ is the homotopy cofibre of $e(W) \longrightarrow W$.

Part Two, *Topology over the real line*, concerns spaces W with a proper map $d : W \longrightarrow \mathbb{R}$.

Chapter 13, *Infinite cyclic covers*, proves that a connected infinite cyclic cover $\overline{W}$ of a connected compact ANR W has two ends $\overline{W}^+$, $\overline{W}^-$, and establishes a duality between the two types of tameness: $\overline{W}^+$ is forward tame if and only if $\overline{W}^-$ is reverse tame. A similar duality holds for forward and reverse collared ends.

Chapter 14, *The mapping torus*, works out the end theory of infinite cyclic covers of mapping tori.

Chapter 15, *Geometric ribbons and bands*, presents bands and ribbons. It is proved that $(M, c : M \longrightarrow S^1)$ with M a finite CW complex defines a band (i.e. the infinite cyclic cover $\overline{M} = c^*\mathbb{R}$ of M is finitely dominated) if and only if the ends $\overline{M}^+$, $\overline{M}^-$ are both forward tame, or both reverse tame. The Siebenmann twist glueing construction of a band is formulated for a ribbon $(X, d : X \longrightarrow \mathbb{R})$ and an end-preserving homeomorphism $h : X \longrightarrow X$.

Chapter 16, *Approximate fibrations*, presents the main geometric tool used in the proof of the uniformization Theorem 19 (every tame manifold end of dimension ≥ 5 has a neighbourhood which is the infinite cyclic cover of a manifold band). It is proved that an open manifold W of dimension ≥ 5 is forward and reverse tame if and only if there exists an open cocompact subspace $X \subseteq W$ which admits a manifold approximate fibration $X \longrightarrow \mathbb{R}$.

Chapter 17, *Geometric wrapping up*, uses the twist glueing construction with $h = 1 : X \longrightarrow X$ to prove that the total space X of a manifold approximate fibration $d : X \longrightarrow \mathbb{R}$ is the infinite cyclic cover $X = \overline{M}$ of a manifold band (M, c).

Chapter 18, *Geometric relaxation*, uses the twist glueing construction with $h =$ covering translation $: \overline{M} \longrightarrow \overline{M}$ to pass from a manifold band (M, c) to an h-cobordant manifold band (M', c') such that $c' : M' \longrightarrow S^1$ is a manifold approximate fibration.

Chapter 19, *Homotopy theoretic twist glueing*, and Chapter 20, *Homotopy theoretic wrapping up and relaxation*, extend the geometric constructions for manifolds in Chapters 17 and 18 to CW complex bands and ribbons. Constructions in this generality serve as a bridge to the algebraic theory of Part Three. Moreover, it is shown that any CW ribbon is infinite simple homotopy equivalent to the infinite cyclic cover of a CW band, thereby justifying the concept.

Part Three, *The algebraic theory*, translates most of the geometric, homotopy theoretic and homological constructions of Parts One and Two into an appropriate algebraic context, thereby obtaining several useful algebraic characterizations.

Chapter 21, *Polynomial extensions*, gives background information on chain complexes over polynomial extension rings, motivated by the fact that the cellular chain complex of an infinite cyclic cover of a CW complex is defined over a Laurent polynomial extension.

Chapter 22, *Algebraic bands*, discusses chain complexes over Laurent polynomial extensions which have the algebraic properties of cellular chain complexes of CW complex bands.

Chapter 23, *Algebraic tameness*, develops the algebraic analogues of forward and reverse tameness for chain complexes over polynomial extensions. This yields an algebraic characterization of forward (and reverse) tameness for an end of an infinite cyclic cover of a finite CW complex. End complexes are also defined in this algebraic setting.

Chapter 24, *Relaxation techniques*, contains the algebraic analogues of the constructions of Chapters 18 and 20. When combined with the geometry of Chapter 18 this gives an algebraic characterization of manifold bands which admit approximate fibrations to S^1.

Chapter 25, *Algebraic ribbons*, explores the algebraic analogue of CW ribbons in the context of bounded algebra. The algebra is used to prove that CW ribbons are infinite simple homotopy equivalent to infinite cyclic covers of CW bands.

Chapter 26, *Algebraic twist glueing*, proves that algebraic ribbons are simple chain equivalent to algebraic bands.

Chapter 27, *Wrapping up in algebraic K- and L-theory*, describes the effects of the geometric constructions of Part Two on the level of the algebraic K- and L-groups.

Part Four consists of the three appendices:

Appendix A, *Locally finite homology with local coefficients*, contains a technical treatment of ordinary and locally finite singular and cellular homology theories with local coefficients. This establishes the equivalence of locally finite singular and cellular homology for regular covers of CW complexes.

Appendix B, *A brief history of end spaces*, traces the development of end spaces as homotopy theoretic models for the topology at infinity.

Appendix C, *A brief history of wrapping up*, outlines the history of the wrapping up compactification procedure.