ON THE SEMICONTINUITY OF THE MOD 2 SPECTRUM OF HYPERSURFACE SINGULARITIES

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Abstract. We use purely topological methods to prove the semicontinuity of the mod 2 spectrum of local isolated hypersurface singularities in $\mathbb{C}^{n+1}$, using Seifert matrices of high-dimensional non-spherical links, the Levine–Tristram signatures and the generalized Kawauchi–Murasugi inequality obtained in earlier work for cobordisms of links.

1. Introduction

The present article extends the results of [BoNe12], valid for plane curves, to arbitrary dimensions. The main message is that the semicontinuity of the mod 2 Hodge spectrum (associated with local isolated singularities, or with affine polynomials with some 'tameness' condition) is topological in nature, although its very definition and all known 'traditional' proofs sit deeply in the analytic/algebraic theory. Usually, the spectrum cannot be deduced from the topological data. In order to make clear these differences, let us review in short the involved invariants of a local isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$.

The 'homological package of the embedded topological type' of an isolated hypersurface singularity contains the information about the vanishing cohomology (cohomology of the Milnor fiber), the algebraic monodromy acting on it, and different polarizations of it including the intersection form and the (linking) Seifert form. In fact, the Seifert form determines all this homological data. See e.g. [AGV84].

On the other hand, the vanishing cohomology carries a mixed Hodge structure polarized by the intersection form and the Seifert form, and it is compatible with the monodromy action too. It has several definitions, but all of them are analytic [Arn81, Stee76, Stee85, Var83a, Var83b]. The equivariant Hodge numbers were codified by Steenbrink in the spectral pairs; if one deletes the information about the weight filtration one gets the spectrum/spectral numbers $Sp(f)$. They are (in some normalization) rational numbers in the interval $(0, n+1)$. Arnold conjectured [Arn81], and Varchenko [Var83a, Var83b] and Steenbrink [Stee85] proved that the spectrum behaves semicontinuously under deformations. In this way it becomes a very strong tool e.g. in the treatment of the adjacency problem of singularities.

More precisely, in the presence of a deformation f_t, where t is the deformation parameter $t \in (\mathbb{C}, 0)$, the semicontinuity guarantees that $|Sp(f_0) \cap I| \geqslant |Sp(f_{t\neq 0}) \cap I|$ for certain semicontinuity domains I. Arnold conjectured that $I = (-\infty, \alpha]$ is a semicontinuity domain for any $\alpha \in \mathbb{R}$, Steenbrink and Varchenko proved the statement for $I = (\alpha, \alpha + 1]$, which implies Arnold's conjecture. Additionally, for some cases, Varchenko verified the stronger version, namely semicontinuity for $I = (\alpha, \alpha + 1)$ [Var83a].

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The relation between the Hodge invariants and the Seifert form was established in [Nem95], proving that the collection of mod 2 spectral pairs are equivalent with the real Seifert form. Therefore, the real Seifert form determines the mod 2 spectrum, that is, the collection of numbers α mod 2 in $(0,2]$, where α run over $Sp(f)$. Clearly, for plane curve singularities, i.e. when $n=1$, by taking mod 2 reduction we loose no information.

Surprisingly, this correspondence can be continued: in [BoNe12] is proved that if $n=1$ then the *semicontinuity property of the mod 2 spectrum too is topological*: it can be proved independently of Hodge theoretical tools, it follows from classical link theory. Its precise statement is the following: length one 'intervals' intersected by the mod 2 spectrum, namely sets of type $Sp \cap (\alpha, \alpha+1)$ and $(Sp \cap (0,\alpha)) \cup (Sp \cap (\alpha+1, 2])$, for $\alpha \in [0,1]$, satisfy semicontinuity properties, whenever this question is well–posed (and under certain mild extra assumptions regarding the roots of the monodromy operator). The tools needed in this topological proof were the following: properties of the *Tristram–Levine signature*, its connection with the spectrum, and the *Kawauchi–Murasugi inequality* valid for it.

It was very natural to ask for a possible generalization of this fact for arbitrary dimensions. This was seriously obstructed by two facts: the definition of the *existence of the Seifert form up to an S–equivalence* and of the analogue of the *Kawauchi–Murasugi inequality* valid for any 2–codimension embedded manifold of S^{2n+1}.

The proof of these two facts involving the study of higher dimensional links is rather long and the needed tools are different from those needed to verify the semicontinuity. Therefore, we decided two separate them in another note [BNR12]. In the present manuscript we collect and prove all the other needed steps.

The article is organized as follows. In section 2 we review the theory of higher dimensional links, Seifert forms and the generalization of the Kawauchi–Murasugi inequality from [BNR12]. In its last subsection we show how they can be applied for complex affine hypersurfaces. Sections 3 reviews facts about hermitian variation structures, the tool which connects Seifert forms with the spectral numbers via Tristram–Levine signature. In fact, all the arguments can be applied to the global affine hypersurfaces as well, provided that the corresponding polynomial map satisfies some regularity conditions which guarantees similar properties at infinity which are valid for local isolated singularities (the 'Milnor package'). In this section we review some needed facts about these 'tameness' conditions as well. Section 4 contains the three semicontinuity results: (a) the local case of deformation of isolated singularities, (b) semicontinuity of the mod 2 spectrum of the mixed Hodge structure at infinity of 'nice' polynomials, and (c) an inequality which compares the mod 2 spectrum at infinity of an affine fiber with the local spectrum of its singular points. Certain proofs and parts of the note show similarities with [BoNe12], therefore some arguments are shortened, although we tried to provide a presentation emphasizing all the basic steps of the proof.

Notation. For a finite set A, we denote by $|A|$ the cardinality of A, for a subset X of $\mathbb{C}^{n+1}$, $\operatorname{int} X$ denotes its interior and $\overline{X}$ its closure.

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2. High dimensional links and their signatures

2.1. **A quick trip trough the theory of high dimensional links.** Here we present, in a condensed form, results about *high dimensional links*, that is, framed codimension 2 embeddings $M \subset S^{2n+1}$, where M is a closed oriented $(2n-1)$–dimensional manifold, for any $n \geqslant 1$. The theory resembles the 'classical' theory of embeddings of copies of S^1's into S^3, the special case $n=1$, although the proofs of the corresponding statements are more involved. Unless specified otherwise, the results are from [BNR12]. The first one, however, is older. It dates back to Erle [Erle69], see also [BNR12, Theorem 1.2.4].[1]

Proposition 2.1.1. *Let $M^{2n-1} \subset S^{2n+1}$ be a link. Then there exists a compact, oriented, connected $2n$–dimensional manifold $\Sigma \subset S^{2n+1}$, such that $\partial\Sigma = M$.*

We shall call such Σ a *Seifert surface* for M : in general, there is not a canonical choice of Σ, although there is a canonical choice for the link of a singularity. Given a Seifert surface Σ, let FH_n be the torsion free part of $H_n(\Sigma, \mathbb{Z})$. Let $\alpha_1, \dots, \alpha_m$ be the basis of FH_n and let us represent it by cycles denoted also by $\alpha_1, \dots, \alpha_m$. We can define a matrix $S = \{s_{ij}\}_{i,j=1}^m$ with $s_{ij} = \mathrm{lk}(\alpha_i, \alpha_j^+)$, where α_j^+ is a cycle α_j pushed slightly off Σ in the positive direction. S is called the Seifert matrix of M relative to Σ. We refer to [BNR12] for the S-equivalence of Seifert matrices. This is one of the main results of [BNR12].

Proposition 2.1.2 (See [BNR12, Theorem 1.2.6])**.** *Any two Seifert matrices of a given link M, corresponding to possibly different Seifert surfaces, are S-equivalent.*

This is a standard result for knots $S^{2n-1} \subset S^{2n+1}$ (when $b = S + (-1)^n S^T$ is invertible), but is less familiar for links $M^{2n-1} \subset S^{2n+1}$.

Let us fix a link M and let Σ be one of its Seifert surfaces, with Seifert matrix S. Following an algebraic result of Keef [Keef83] (see also [BoNe11, Section 3]), over the field of rational numbers changing S by a matrix S-equivalent to it we can write

$$S = S_{ndeg} \oplus S_0, \tag{2.1.3}$$

where $\det S_{ndeg} \neq 0$ and all the entries of S_0 are zero. Let us define

$$n_0 := \text{size of } S_0 = \dim(\ker S \cap \ker S^T). \tag{2.1.4}$$

Moreover, the right hand side of (2.1.3) does not depend on a particular choice of the Seifert matrix in its S-equivalence class, more precisely, the integer n_0 and the rational congruence class of the matrix S_{ndeg} are well defined.

Definition 2.1.5. Let $M^{2n-1} \subset S^{2n+1}$ be a link with a Seifert surface $\Sigma^{2n} \subset S^{2n+1}$, S the Seifert matrix and S^T the transposed matrix. The *Alexander polynomial* of M is defined by

$$\Delta_M(t) := \det(S_{ndeg} \cdot t + (-1)^n S_{ndeg}^T).$$

For any $\xi \in S^1 \setminus \{1\}$ define the *Levine–Tristram signature* and *nullity* of M at ξ

$$\sigma_M(\xi) := \text{signature}\,[\,(1-\xi)S + (-1)^{n+1}(1-\overline{\xi})S^T\,]$$
$$n_M(\xi) := \text{nullity}\,[\,(1-\xi)S + (-1)^{n+1}(1-\overline{\xi})S^T\,]\;.$$

Remark 2.1.6. Strictly speaking, $\Delta_M(t)$ is the classical Alexander polynomial only if $n_0 = 0$, i.e. if $S = S_{ndeg}$. Otherwise, if $n_0 > 0$, then $\Delta_M(t)$ is the (n_0+1)–st classical Alexander polynomial. In this note we shall not insist on this distinction.

The following result is completely analogous to its one-dimensional version.

Lemma 2.1.7. *If $\xi \in S^1 \setminus \{1\}$ is not a root of the Alexander polynomial, then $n_M(\xi) = n_0$. Otherwise $n_M(\xi) > n_0$. For $\xi, \eta \in S^1 \setminus \{1\}$ if there exists an arc in $S^1 \setminus \{1\}$ connecting them and not containing any root of the Alexander polynomial, then $\sigma_M(\xi) = \sigma_M(\eta)$.*

[1][MB] This reference might be changed, we have to be careful. This applies to all further references to [BNR12].

For the completeness of the argument we present the straightforward proof.

Proof. Let $S = S_{ndeg} \oplus S_0$. Then $\sigma_M(\xi) = \text{signature}[(1-\xi)S_{ndeg} + (-1)^{n+1}(1-\overline{\xi})S^T_{ndeg}]$ and $n_M(\xi) = \text{null}[(1-\xi)S_{ndeg} + (-1)^{n+1}(1-\overline{\xi})S^T_{ndeg}] + n_0$. Hence we can assume $S = S_{ndeg}$. But then,

$$(1-\xi)S + (-1)^{n+1}(1-\overline{\xi})S^T = (\overline{\xi}-1)(\xi S + (-1)^n S^T).$$

Hence $\Delta_M(\xi) \neq 0$ if and only if $(1-\xi)S+(-1)^{n+1}(1-\overline{\xi})S^T$ is non-degenerate. Furthermore, if the arc γ joins ξ and η, then the matrix $[(1-\alpha)S + (-1)^{n+1}(1-\overline{\alpha})S^T]_{\alpha\in\gamma}$ is a path in the space of non-degenerate sesquilinear forms along which the signature is constant. □

Our main tool is the following generalization of the classical *Kawauchi–Murasugi inequality.*

Theorem 2.1.8 (see [BNR12, Theorem 1.2.8])**.** *Let $(Y; M_0, M_1) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\})$ be a cobordism of links $M_0, M_1 \subset S^{2n+1}$. For any Seifert surfaces $\Sigma_0, \Sigma_1 \subset S^{2n+1}$ for M_0, M_1 and $\xi \in S^1 \setminus \{1\}$ we have*

$$|\sigma_{M_0}(\xi) - \sigma_{M_1}(\xi)| \leqslant b_n(\Sigma_0 \cup_{M_0} Y \cup_{M_1} \Sigma_1) - b_n(\Sigma_0) - b_n(\Sigma_1) + n_{M_0}(\xi) + n_{M_1}(\xi), \tag{2.1.9}$$

where $b_n(\cdot)$ denotes the $n-$th Betti number.

Remark 2.1.10. In the classical case, the inequality looks slightly different (compare [Kaw96, Theorem 12.3.1]), namely one has $|n_{M_0}(\xi) - n_{M_1}(\xi)|$ on the left hand side, with a plus sign; hence the classical inequality is stronger. The reason for it is that in the $n = 1$ case one has a better interplay between the topology and nullities (see for example the quantities w_L and u_L in [Boro11, Section 5]). We do not know, whether the stronger inequality holds in higher dimensions, the approach of [BNR12] seems to be insufficient to prove that.

For a convenience of a reader we present a sketch of proof of Theorem 2.1.8 in a special case. In the proof we shall use specific choice of a Seifert surface, so we do not use Proposition 2.1.2. In fact, the following proof is completely independent from [BNR12].

Proof. Let us assume that the spheres $C_0 := S^{2n+1} \times \{0\}$ and $C_1 := S^{2n+1} \times \{1\}$ from assumptions of Theorem 2.1.8 are boundaries of balls $B_0, B_1 \subset \mathbb{C}^{n+1}$ and $S^{2n+1} \times [0,1]$ from the assumptions is embedded in $\mathbb{C}^{n+1}$ as $U := \overline{B_1 \setminus B_0}$.

Suppose furthermore that $X = f^{-1}(0)$ is a complex hypersurface with $M_i = \Sigma_i \cap X$ for $i = 0, 1$ and $\arg f$ induces a fibration of $C_0 \setminus M_0$ and $C_1 \setminus M_1$ with the base S^1 and the fiber Σ_0 (respectively Σ_1). We put $Y = U \cap X$. In this case the proof of Theorem 2.1.8 is the following.

The map $\arg f \colon U \setminus Y \to S^1$ is a well-defined surjection. Let $\delta \in S^1$ be a non-critical value. The inverse image $\Omega = (\arg f)^{-1}(\delta) \cap U$ is a compact manifold. Let $\Sigma_0 = (\arg f)^{-1}(\delta) \cap C_0$ and $\Sigma_1 = (\arg f)^{-1}(\delta) \cap C_1$. Then we have

$$\partial\Omega = \Sigma_0 \cup_{M_0} Y \cup_{M_1} \Sigma_1.$$

Let V_0 and V_1 be the Seifert matrices for M_0 and M_1 related to the Seifert surfaces Σ_0 and Σ_1. On $(-C_0) \cup C_1$ (the minus sign denotes that we reverse the orientation) we can consider the linking form: if α, β are two n-dimensional cycles on $C_0 \cup C_1$ we define

$$\text{lk}(\alpha, \beta) = -\text{lk}(\alpha_0, \beta_0) + \text{lk}(\alpha_1, \beta_1),$$

where $\alpha_i = \alpha \cap C_i$, $\beta_i = \beta \cap C_i$, $i = 0, 1$. This definition allows us to define the Seifert pairing for $M_0 \cup M_1$ with respect to the Seifert surface $-\Sigma_0 \cup \Sigma_1$ by $S(\alpha, \beta) = \text{lk}(\alpha, \beta^+)$, where β^+ is the cycle β pushed off slightly from $\Sigma_0 \cup \Sigma_1$ in a positive normal direction. If S_0 and S_1 denote the Seifert pairings for M_0 and M_1, then clearly $S = (-S_0) \oplus S_1$.

Let $i : C_0 \cup C_1 \to \Omega$ be the inclusion map. Note that $i = j \circ k$, where $k\colon C_0 \cup C_1 \to \partial\Omega$ and $j\colon \partial\Omega \to \Omega$ are the inclusions.

Claim. If $\alpha, \beta \in \ker i_*\colon H_n(C_0 \cup C_1; \mathbb{Q}) \to H_n(\Omega; \mathbb{Q})$, then $S(\alpha, \beta) = 0$.

To prove the claim, we assume that $\alpha, \beta \in \ker i_*$. Then, there exist $(n+1)$-dimensional cycles $A, B \subset \Omega$, such that $\partial A = \alpha$, $\partial B = \beta$. Let B^+ be the cycle in $\overline{B_1 \setminus B_0}$ obtained by pushing B off Ω in a positive normal direction. Clearly $\partial B^+ = \beta^+$. But then

$$\operatorname{lk}(\alpha, \beta^+) = A \cdot B^+ = 0,$$

because A and B^+ are disjoint. This proves the claim.

Let now k_* and j_* denote the induced maps on n-th homology with rational coefficients. By a standard Poincaré duality argument $\dim\ker j_* = \frac{1}{2} b_n(\partial\Omega)$. Therefore

$$\dim\ker i_* \geqslant \dim\ker k_* + (\dim\ker j_* - \dim\operatorname{coker} k_*) = \frac{1}{2} b_n(\partial\Omega) - (b_n(\partial\Omega) - b_n(C_0 \cup C_1)).$$

Consider now $\xi \in S^1 \setminus \{1\}$ and assume for simplicity that $n_{M_0}(\xi) = n_{M_1}(\xi) = 0$ (the proof in general case is only slightly more complicated). This means that the form $(1-\xi)S + (1-\overline{\xi})S^T$ is non-degenerate. By the claim, it vanishes on a space of dimension $\dim\ker i_*$, therefore the absolute value of its signature is bounded by

$$b_n(C_0 \cup C_1) - 2\dim\ker i_* = b_n(\partial\Omega) - b_n(C_0 \cup C_1).$$

□

The argument used in the proof provides also a proof of Proposition 2.2.1 below. That enables us to prove all the results from Section 4 without referring to surgery theory from [BNR12]. However, in this approach, the signatures and their properties depend *a priori* on the function f and on the specific choice of the Seifert surface. It is not clear whether these properties are of topological nature, in particular, whether the results from Section 4 are purely topological, or not.

The theorems from [BNR12] clarify this. The Levine–Tristram signatures are defined even if the link in question is not fibered and depend only on the isotopy of the link. Moreover, the behaviour of Levine–Tristram signatures under cobordism depends only on homological properties of the manifold which realizes the cobordism between links.

2.2. **Application to affine or local complex hypersurfaces.** Let $X \subset \mathbb{C}^{n+1}$ be a complex hypersurface with at most isolated singularities. If $x \in X$ is a singular point of X, consider a sufficiently small sphere $S_x \simeq S^{2n+1}$ centered at x. The intersection $X \cap S_x$ embedded in S_x is called the *link* of the hypersurface singularity $(X, x) \subset (\mathbb{C}^{n+1}, x)$. We shall denote it by $M_x \subset S_x$.

Let now B be any ball in $\mathbb{C}^{n+1}$ such that $S = \partial B$ is transverse to X. Let $M := S \cap X \subset S$ be the corresponding link. Let $x_1, \dots, x_k$ be those singular points of X, that lie inside B. We wish to relate the signatures of M and $M_{x_1}, \dots, M_{x_k}$. We have the following result.

Proposition 2.2.1. *Let X^s be the smoothing of X inside B (whose boundary will be identified via an isotopy with M too) and Σ a Seifert surface of $M \subset S$. Moreover, for $j = 1, \dots, k$, let Σ_j be the Milnor fiber at x_j. Then for all $\xi \in S^1 \setminus \{1\}$*

$$\Big|\sigma_M(\xi) - \sum_j \sigma_{M_{x_j}}(\xi)\Big| \leqslant b_n(X^s \cup_M \Sigma) - b_n(\Sigma) - \sum_j b_n(\Sigma_j) + n_M(\xi) + \sum_j n_{M_{x_j}}(\xi).$$

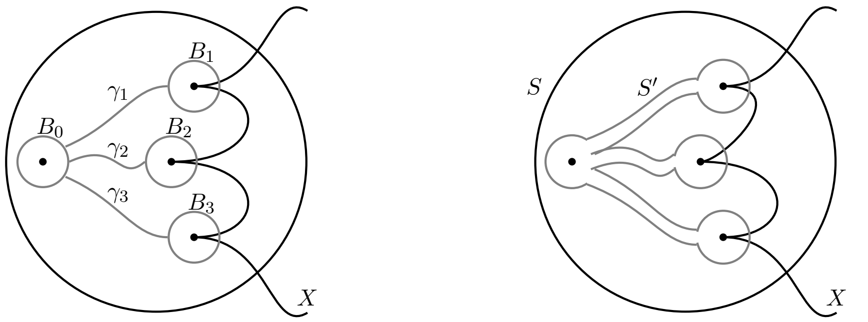

FIGURE 1. The cobordism between local and global links

Proof. For any $j = 1, \dots, k$ let us pick small Milnor balls B_j around x_j such that they are pairwise disjoint; hence $\partial B_j \pitchfork X$, $\partial B_j \cap X = M_{x_j}$ and $\Sigma_j \subset \partial B_j$. Let $B_0 \subset B$ be a ball disjoint from $X \cup B_1 \cup \dots \cup B_k$. For each $j = 1, \dots, k$, let γ_j be a smooth, closed curve joining ∂B_0 with ∂B_k, such that γ_j is disjoint from Σ_j and from all other balls B_l and other curves γ_l for $l \neq j$. We also assume that the relative interior of γ_j is disjoint from B_0 and B_j (see Figure 1). Let U_j be a small tubular neighbourhood of γ_j and consider

$$U = B_0 \cup \bigcup_j (B_j \cup U_j), \quad \text{and} \quad Y = X \cap B \setminus \bigcup_j \operatorname{int} B_j.$$

The assumptions on γ_j's guarantee that

- U is diffeomorphic to a standard ball;
- $S' := \partial U$ (after possibly smoothing corners) is a sphere transverse to X;
- $M' := S' \cap X$ is a disjoint union $M_{x_1} \sqcup \dots \sqcup M_{x_k}$ of the local links;
- $\Sigma' := \Sigma_1 \sqcup \dots \sqcup \Sigma_k$ is a Seifert surface for M'.

In particular

$$\sigma_{M'}(\xi) = \sum_j \sigma_{M_{x_j}}(\xi) \ \text{ and } \ n_{M'}(\xi) = \sum_j n_{M_{x_j}}(\xi).$$

We say that the cobordism of links

$$(Y; M, M') \subset (B \setminus \operatorname{int} U; S, S') \ \approx \ S^{2n+1} \times ([0,1]; \{0\}, \{1\})$$

is constructed by the 'boleadoras' trick. The generalized Kawauchi–Murasugi inequality of Theorem 2.1.8 gives

$$\left|\sigma_M(\xi) - \sum_j \sigma_{M_{x_j}}(\xi)\right| \leqslant b_n(Y \cup \Sigma' \cup \Sigma) - b_n(\Sigma') - b_n(\Sigma) + n_M(\xi) + \sum_j n_{M_{x_j}}(\xi).$$

Now $\Sigma_j \approx X^s \cap B_j$ by [Miln68], hence $Y \cup \Sigma' \approx X^s$, and the statement follows. □

Remark 2.2.2. The space U, as drawn in the picture above, resembles the South American throwing weapon *boleadoras* [Bol], hence the name of the cobordism construction in 2.2.1.

Remark 2.2.3. Proposition 2.2.1 is the main tool in the proof of the semicontinuity of the mod 2 spectrum, cf. Section 4. We remark that if $n = 1$, this proposition allows to prove semicontinuity of the spectrum without referring to [Boro11] and [BNR12]. (The article [BoNe12] uses [Boro11].)

3. Hermitian Variation Structures and Mixed Hodge Structures

3.1. **Generalities about hermitian variation structures.** Variation structures were introduced in [Nem95]. As it was shown in [BoNe11] and [BoNe12] they form a bridge between knot theory and Hodge theory. Let us recall shortly the definition, referring to [Nem95] or [BoNe11, Section 2] for all details and further references.

Definition 3.1.1. Fix a sign $\varepsilon = \pm 1$. An *ε–hermitian variation structure* (in short: HVS) consist of the quadruple $(U; b, h, V)$, where U is a complex linear space, $b\colon U \to U^*$ is an ε–hermitian endomorphism (it can be regarded as a ε–symmetric pairing on $U \times U$), $h\colon U \to U$ is an automorphism preserving b, and $V\colon U^* \to U$ is an endomorphism such that

$$V \circ b = h - I \text{ and } \overline{V}^* = -\varepsilon V \circ \overline{h}^*.$$

Here $\bar{\cdot}$ denotes the complex conjugate and $*$ the duality.

We shall call a HVS *simple* if V is an isomorphism. In this case V determines b and h completely by the formulae $h = -\varepsilon V(\overline{V}^*)^{-1}$ and $b = -V^{-1} - \varepsilon \overline{V}^{*-1}$. It was proved in [Nem95] that each simple variation structure is a direct sum of indecomposable ones, moreover the decomposable ones can be completely classified. More precisely, for any $k \geqslant 1$ and any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, there are two structures $\mathcal{W}_\lambda^k(\pm 1)$ (up to an isomorphism). In their case h is a single Jordan block of size k with eigenvalue λ. Furthermore, for any $k \geqslant 1$ and any $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$ there exists a single structure $\mathcal{V}_\lambda^k$. In this case h is a direct sum of two Jordan blocks of size k: one block has eigenvalue λ, the other $1/\bar{\lambda}$. In particular for each HVS $\mathcal{V}$ there exists a unique decomposition

$$\mathcal{V} = \bigoplus_{\substack{0<|\lambda|<1 \\ k\geqslant 1}} q_\lambda^k \cdot \mathcal{V}_\lambda^{2k} \oplus \bigoplus_{\substack{|\lambda|=1 \\ k\geqslant 1,\ u=\pm 1}} p_\lambda^k(u) \cdot \mathcal{W}_\lambda^k(u), \tag{3.1.2}$$

where q_λ^k and $p_\lambda^k(\pm 1)$ are certain non–negative integers. Here we write $m \cdot \mathcal{V}$ for a direct sum of m copies of $\mathcal{V}$. Next we recall the definition of the spectrum associated with a HVS.

Definition 3.1.3. Let $\mathcal{V}$ be a HVS. Let $p_\lambda^k(\pm 1)$ and q_λ^k be the integers defined by (3.1.2). The *extended spectrum* ESp is the union $ESp = Sp \cup ISp$, where

(a) Sp, the *spectrum*, is a finite set of real numbers from the interval $(0, 2]$ with integral multiplicities such that any real number α occurs in Sp precisely $s(\alpha)$ times, where

$$s(\alpha) = \sum_{n=1}^{\infty} \sum_{u=\pm 1} \left(\frac{2n - 1 - u(-1)^{\lfloor \alpha \rfloor}}{2} p_\lambda^{2n-1}(u) + n p_\lambda^{2n}(u) \right), \quad (e^{2\pi i \alpha} = \lambda).$$

(b) ISp is the set of complex numbers from $(0, 2] \times i\mathbb{R}$, $ISp \cap \mathbb{R} = \emptyset$, where $z = \alpha + i\beta$ occurs in ISp precisely $s(z)$ times, where

$$s(z) = \begin{cases} \sum k \cdot q_\lambda^k & \text{if } \alpha \leqslant 1,\ \beta > 0 \text{ and } e^{2\pi i z} = \lambda \\ \sum k \cdot q_\lambda^k & \text{if } \alpha > 1,\ \beta < 0 \text{ and } e^{2\pi i z} = 1/\bar{\lambda} \\ 0 & \text{if } \alpha \leqslant 1 \text{ and } \beta < 0, \text{ or } \alpha > 1 \text{ and } \beta > 0. \end{cases}$$

We have the following relation

$$|ESp| = \dim U = \deg \det(h - t\,\mathrm{Id}).$$

Main motivation for introducing HVS comes from singularity theory, cf. [AGV84, Nem95].

Lemma 3.1.4. *Let $X \subset \mathbb{C}^{n+1}$ be a complex hypersurface with an isolated singularity $x \in X$. Let S_x be a small sphere centered at x and let $\pi\colon S_x \setminus X \to S^1$ be the Milnor fibration with fiber Σ. Let $U = H_n(\Sigma; \mathbb{C})$, b be the intersection form on U and $h\colon U \to U$ be the homological* *monodromy. Finally, let* $V\colon H_n(\Sigma,\partial\Sigma;\mathbb{C})\to H_n(\Sigma;\mathbb{C})$ *be the Picard–Lefschetz variation operator. Then the quadruple* $(U;b,h,V)$ *constitutes a simple HVS with* $\varepsilon=(-1)^n$*.*

Definition 3.1.5. The HVS defined in Lemma 3.1.4 is called the HVS of $(X,x)\subset(\mathbb{C}^{n+1},x)$.

Remark 3.1.6. The Seifert matrix (associated with Σ) is the inverse transpose of V.

Conversely, any non-degenerate Seifert matrix S associated with any (topologically defined) link $M\subset S^{2n+1}$ determines a simple HVS via $V=(S^{-1})^T$.

In the above algebraic/analytic case, by the Monodromy Theorem, h has all eigenvalues on the unit circle. In particular $ESp=Sp$. The spectrum associated with the *hermitian variation structure* of an isolated singularity will be denoted by Sp^{HVS}.

On the other hand, for an isolated singular point x one can define a *mixed Hodge structure* on the n-th cohomology of its Milnor fiber, which determines a spectrum denoted by Sp^{MHS}, cf. work of Steenbrink and Varchenko [Stee76, Stee85, Var83a]. (We will use the same normalization as in [BoNe12, 2.3]. In particular, Sp^{HVS} is a subset of $(0,n+1)$.)

The relation between the two sets of spectral numbers is given in Proposition 3.1.8. Before we state it, we need to fix some terminology.[2]

Definition 3.1.7. We denote the mod 2 reduction of the spectrum Sp^{MHS} by $Sp^{\text{MHS}/2}$. This means that $Sp^{\text{MHS}/2}$ is a finite set of rational numbers from the interval $(0,2]$ with integral multiplicities, and for any $\alpha\in(0,2]$ the multiplicity of the spectral number in $Sp^{\text{MHS}/2}$ is the sum of the multiplicities of the spectral numbers $\{\alpha+2j\}_{j\in\mathbb{Z}}$ in Sp^{MHS}.

Proposition 3.1.8 ([Nem95, Theorem 6.5])**.** Sp^{HVS} *is the mod 2 reduction of* Sp^{MHS} *for any isolated hypersurface singularity.*

The point is that the spectrum Sp^{HVS}, by its very definition, can be fully recovered from Seifert form of the link. Indeed, one has the following result.

Proposition 3.1.9 ([BoNe12, Corollary 2.4.6])**.** *Let x be an isolated singular point of the hypersurface $X\subset\mathbb{C}^{n+1}$, and $\mathcal{V}=(U;b,h,V)$ the corresponding HVS. Let $\alpha\in[0,1)$ be such that $\xi=e^{2\pi i\alpha}$ is not an eigenvalue of the monodromy operator h. Then we have*

$$|Sp^{\text{HVS}}\cap(\alpha,\alpha+1)|=\frac{1}{2}(\dim U-\sigma_{M_x}(\xi))$$
$$|Sp^{\text{HVS}}\setminus[\alpha,\alpha+1]|=\frac{1}{2}(\dim U+\sigma_{M_x}(\xi)).$$

Remark 3.1.10. The dimension $\dim U=b_n(\Sigma)$ is the Milnor number of the singularity. The condition that ξ is not eigenvalue of the monodromy implies, by Lemma 2.1.7, that $n_{M_x}(\xi)=0$, since the Seifert matrix of the link corresponding to the Milnor fiber is non-degenerate.

In Proposition 3.1.9 we assume that $\alpha\in[0,1)$. If $\alpha=1$, the statement of proposition still holds. In fact $Sp^{\text{HVS}}\cap(1,2)=Sp^{\text{HVS}}\setminus[0,1]$ and $Sp^{\text{HVS}}\setminus[1,2]=Sp^{\text{HVS}}\cap(0,1)$. Hence the case $\alpha=1$ is equivalent to the case $\alpha=0$.[3]

3.2. **Spectrally tame polynomials.** We introduce now a new class of tame polynomials, namely *spectrally tame*. We add this new terminology (to the rather big variety of different versions of 'tame' polynomials) in order to make precise, what assumptions are needed to obtain a topological proof for the semicontinuity of the mod 2 reduction of the MHS-spectrum at infinity associated with polynomial maps. Below, we shall give some examples of spectrally tame polynomials as well.

[2][M] Corrected, check if ok.

[3][M] I added this little remark, we may delete it as well.

Let $P\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map. Let $\mathcal{B}$ be the bifurcation set of P, a finite subset $\mathcal{B} \subset \mathbb{C}$ such that the restriction of P is a C^∞ locally trivial fibration over $\mathbb{C} \setminus \mathcal{B}$. Let $D \subset \mathbb{C}$ be a sufficiently large closed disc centered at the origin so that $\mathcal{B} \subset D$. Finally, take R sufficiently large, such that for any $R' \geqslant R$ the boundary of any closed ball $B_{R'} \subset \mathbb{C}^{n+1}$ centered at the origin intersects any $P^{-1}(c)$, $c \in \partial D$, transversally.

For a fixed value $c \in \partial D$ set $X_c := P^{-1}(c)$, the generic fiber of P, and let M_c be the link at infinity $X_c \cap \partial B_R$. Notice that by the above choices, the restriction of P on $P^{-1}(\partial D)$, or on $P^{-1}(\partial D) \cap \partial B_R$ is the fibration of P at infinity.

Since X_c is Stein, $H_i(X_c, \mathbb{Z}) = 0$ for $i > n$ and $H_n(X_c, \mathbb{Z})$ is free. Moreover, $H^n(X_c, \mathbb{Q})$ carries a MHS, the *'mixed Hodge structure of P at infinity'*, see e.g. [BoNe12, Dim00, NeSa99, Sab99]. We add to these facts the following.

Proposition 3.2.1. *Consider* $\phi(z) := P(z)/|P(z)| : \partial B_R \setminus P^{-1}(\operatorname{int} D) \to S^1$, *the Milnor map restricted on the complement of* $P^{-1}(\operatorname{int} D)$. *Then*

$$\phi : (\partial B_R \setminus P^{-1}(\operatorname{int} D), \partial B_R \cap P^{-1}(\partial D)) \to S^1$$

is a C^∞ *locally trivial fibration of pairs of spaces over* S^1. *This fibration is* C^∞ *equivalent with the fibration at infinity associated with* P

$$P : (P^{-1}(\partial D) \cap B_R, P^{-1}(\partial D) \cap \partial B_R) \to \partial D.$$

In particular, $M_c \subset \partial B_R$ *admits a Seifert surface, namely* $\Sigma := \phi^{-1}(c/|c|)$, *which is diffeomorphic to* $X_c \cap B_R$.

Proof. The proof is similar to the proofs of Theorems 10 and 11 from [NeZa92], with the only modification that the arbitrary disc D used in [loc.cit.] for semitame polynomials should be replaced by a sufficiently large disc D containing all the bifurcation values (as above). $\square$

Next, we analyze the Seifert form S associated with the Seifert surface Σ defined above.

Proposition 3.2.2. (a) *Set* $\mathcal{D} := P^{-1}(D) \cap \partial B_R$ *and* $\Phi_I := \bigcup_{t\in I} \phi^{-1}(e^{2\pi i t})$ *for any subset* $I \subset [0,1]$. *We write* Φ_1 *for* $\Phi_{\{1\}}$ *(in particular* Φ_1 *is a Seifert surface). Then the groups* $H_n(\Phi_1 \cup \mathcal{D}, \mathbb{Z})$ *and* $H_n(\Phi_1, \mathbb{Z})^*$ *are isomorphic. In fact one has the following sequence of isomorphisms, denoted by s:*

$$H_n(\Phi_1 \cup \mathcal{D}) \xrightarrow{\partial^{-1}} H_{n+1}(S^{2n+1}, \Phi_1 \cup \mathcal{D}) \xrightarrow{(1)} H_{n+1}(S^{2n+1}, \Phi_{[0,\frac{1}{2}]} \cup \mathcal{D}) \xrightarrow{(2)}$$

$$H_{n+1}(\Phi_{[\frac{1}{2},1]},\ \partial\Phi_{[\frac{1}{2},1]}) \xrightarrow{(3)} H_n(\operatorname{int} \Phi_{[\frac{1}{2},1]})^* \xrightarrow{(4)} H_n(\Phi_1)^*,$$

where ∂^{-1} *comes from the exact sequence of the pair,* (1) *and* (4) *are induced by deformation retracts,* (2) *is an excision, while* (3) *is provided by Lefschetz duality.*
(b) *Let* $j : H_n(\Phi_1, \mathbb{Z}) \to H_n(\Phi_1 \cup \mathcal{D}, \mathbb{Z})$ *be induced by the inclusion. Then the composition*

$$H_n(\Phi_1, \mathbb{Z}) \xrightarrow{j} H_n(\Phi_1 \cup \mathcal{D}, \mathbb{Z}) \xrightarrow{s} H_n(\Phi_1, \mathbb{Z})^*$$

can be identified with the Seifert form S *associated with* $\Phi_1 = \Sigma \subset \partial B_R$.
(c) *Let* b_∞ *and* h_∞ *be the intersection form and monodromy of* $H_n(\Phi_1, \mathbb{Z}) = H_n(X_c, \mathbb{Z})$, *cf. 3.2.1. Then, in matrix notation,*

$$b_\infty = -\varepsilon S - S^T \quad \text{and} \quad -\varepsilon S^T h_\infty = S.$$

In particular, h_∞ *is an automorphism of* S, *that is* $h_\infty^T S h_\infty = S$.

Proof. Part (a) is clear, while part (b) and (c) follow by similar argument as in the classical case, see e.g. the survey [Nem96, (3.15)]. $\square$

Definition 3.2.3. P is called *spectrally tame* if the following conditions are satisfied:

(S1) $\widetilde{H}_{n-1}(X_c,\mathbb{Q})=0$;
(S2) the natural inclusion induces an isomorphism

$$H_n(X_c\cap B_R)\to H_n((X_c\cap B_R)\cup(P^{-1}(D)\cap\partial B_R));$$

(S3) the spectrum of the HVS associated with the link M_c (and the Seifert form S) is the mod 2 reduction of the MHS spectrum of P at infinity.

Note that the conditions are 'only' (co)homological, a fact which allows more possibilities for their verifications and for applications. By excision and the long homological exact sequence of the pair, $(S2)$ is the consequence of the vanishings (for $c\in\partial D$)

(S2′) $$H_q(P^{-1}(D)\cap\partial B_R,P^{-1}(c)\cap\partial B_R)=0 \quad \text{for } q=n,n+1.$$

By Proposition 3.2.2(b), (S2) is equivalent with the non–degeneracy of the Seifert form S.

Remark 3.2.4. If all the fibers of P have only isolated singularities, and P is regular at infinity then conditions (S1) and (S2′) are satisfied, see e.g. [Nem99]. For (S3) one needs additionally the fact that the MHS at infinity is polarized by the intersection form (for monodromy eigenvalues $\lambda\neq 1$) and by the Seifert form (for eigenvalue $\lambda=1$). This for $\lambda\neq 1$ usually follows from the standard properties of a projectivization/compactification of the fibers of P and the Hodge–Riemann polarization properties, while for $\lambda=1$ one needs an additional Thom–Sebastiani type argument (that is adding e.g. z^N, where z is a new variable) in order to reduce the situation to the $\lambda\neq 1$ case. This means that $P+z^N$ also should satisfy certain/similar regularity conditions at infinity.

Proposition 3.2.5. *The conditions (S1)–(S3) are guaranteed for several 'tame' polynomials present in the literature:*
(a) $*$–polynomials [GaNe96]*;*
(b) M–tame polynomials [NeZa90]*;*
(c) cohomologically tame polynomials [Sab99]*.*

Proof. (a) The topological part follows from [GaNe96], (S3) from [GaNe99, section 5]. (b) The topological part follows from [NeZa90, NeZa92], the Hodge theoretical part from [NeSa99]. (c) follows from [Sab99, NeSa99]. □

An immediate application of (S1)–(S2) is the following

Lemma 3.2.6. *With the notations of 3.2.3, let $\Sigma\cup X_c$ be the smooth closed manifold obtained from Σ and $X_c\cap B_R$ by gluing them together along their boundary M_c. Then (S1) and Proposition 3.2.1 imply $b_n(\Sigma\cup X_c)=b_n(\Sigma)+b_n(X_c)$.*

Proof. In the long homological exact sequence of the pair $(\Sigma\cup X_c,\Sigma)$ one has $H_{n+1}(\Sigma\cup X_c,\Sigma)=H_{n+1}(X_c,M_c)=H^{n-1}(X_c)=0$ and $H_{n-1}(\Sigma)=H_{n-1}(X_c)=0$. □

4. Semicontinuity results

4.1. **Local case.** Let $P_t\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a family of germs of analytic maps depending smoothly on a parameter $t\in(\mathbb{C},0)$. Assume that $P_0^{-1}(0)$ has an isolated singularity at $0\in\mathbb{C}^{n+1}$. Let us fix a small Milnor ball $B\subset\mathbb{C}^{n+1}$ centered at 0, and let h_0 be the homological monodromy operator of the Milnor fibration of P_0.

Theorem 4.1.1. *For any t with $0 < |t| \ll 1$, let $x_1, \dots, x_k$ be all the singular points of $P_t^{-1}(0) \cap B$, and $\alpha \in [0,1]$ is chosen so that $\xi = e^{2\pi i \alpha}$ is not an eigenvalue of h_0. Then*

$$
\begin{aligned}
|Sp_0^{MHS/2} \cap (\alpha, \alpha+1)| &\geqslant \sum_j |Sp_j^{MHS/2} \cap (\alpha, \alpha+1)|, \\
|Sp_0^{MHS/2} \setminus [\alpha, \alpha+1]| &\geqslant \sum_j |Sp_j^{MHS/2} \setminus [\alpha, \alpha+1]|,
\end{aligned} \tag{4.1.2}
$$

where $Sp_0^{MHS/2}$ (respectively $Sp_j^{MHS/2}$) is the mod 2 reduction of the MHS–spectrum associated with the singularity at 0 of P_0 (respectively with the singularity x_j of P_t).

Proof. Let M_{x_j} be the link of $(P_t^{-1}(0), x_j)$ with Seifert surface (i.e., Milnor fiber) Σ_j, and let M be the link of $(P_0^{-1}(0), 0)$ with Seifert surface Σ in ∂B.

First assume that α is chosen so that $\xi = e^{2\pi i \alpha}$ is not an eigenvalue of the homological monodromy operators h_j of $(P_t^{-1}(0), x_j)$ for neither $j = 1, \dots, k$. Then, the nullities $n_{M_{x_j}}(\xi)$ and $n_M(\xi)$ are all zero (see Remark 3.1.10). Then we apply Proposition 2.2.1 to obtain

$$
\Big|\sigma_M(\xi) - \sum_j \sigma_{M_{x_j}}(\xi)\Big| \leqslant b_n(X^s \cup \Sigma) - b_n(\Sigma) - \sum_j b_n(\Sigma_j),
$$

where X^s is a smoothing (nearby smooth fiber) of $f_0^{-1}(0) \cap B$. By removing the absolute value sign, and using $b_n(X^s) = b_n(\Sigma)$ (by [Miln68]) and $b_n(X^s) + b_n(\Sigma) = b_n(X^s \cup \Sigma)$ (by the local analogue of Lemma 3.2.6 proved in the same way) we obtain

$$
-\sigma_M(\xi) + b_n(\Sigma) \geqslant \sum_j (-\sigma_{M_{x_j}}(\xi) + b_n(\Sigma_j)). \tag{4.1.3}
$$

In this way, via Proposition 3.1.9, we prove the first inequality of (4.1.2). The second one is proved if we remove the absolute value sign in the other way.

If $\xi = e^{2\pi i \alpha}$ happens to be an eigenvalue of h_j for some j, but it is not an eigenvalue of h_0, then for all α' sufficiently close to α, $e^{2\pi i \alpha'}$ is not an eigenvalue neither of h_0, nor of any h_j. We can deduce the inequality (4.1.2) for α from the fact that it holds for α' and that the function $\alpha \to |Sp_j^{\mathrm{MHS}/2} \cap (\alpha, \alpha+1)|$ is lower semicontinuous. □

4.2. **Semicontinuity at infinity.** We prove topologically that the mod 2 reduction of the result of [NeSa99] is valid for polynomial maps which are 'tame' at infinity.

Theorem 4.2.1. *Let P_t be a smooth family of spectrally tame polynomial maps, where $t \in (\mathbb{C}, 0)$. Then, for any t with $0 < |t| \ll 1$, and for any $\alpha \in [0,1]$ such that $\xi = e^{2\pi i \alpha}$ is not a root of the homological monodromy operator at infinity of P_t, we have*

$$
\begin{aligned}
|Sp_t^{MHS/2} \cap (\alpha, \alpha+1)| &\geqslant |Sp_0^{MHS/2} \cap (\alpha, \alpha+1)| \\
|Sp_t^{MHS/2} \setminus [\alpha, \alpha+1]| &\geqslant |Sp_0^{MHS/2} \setminus [\alpha, \alpha+1]|,
\end{aligned}
$$

where $Sp_t^{MHS/2}$ denotes the mod 2 spectrum of the MHS at infinity associated with P_t.

Proof. We fix a regular fiber $X_0 = P_0^{-1}(c)$ of P_0. Let B be a large ball so that $X_0 \cap \partial B = M_{\Sigma_0}$ is the link at infinity of P_0. For any t with $0 < |t| \ll 1$, the intersection of $X_t = P_t^{-1}(c)$ with ∂B is isotopic to M_{Σ_0}. After perturbing c, if necessary, we can assume that X_t is smooth. Let $B_t \supset B$ be a ball such that $X_t \cap \partial B_t$ is the link at infinity of X_t. Then

$$
Y = X_t \cap (B_t \setminus \operatorname{int} B)
$$

realizes a cobordism between M_{Σ_0} and M_{Σ_t}. The proof is similar to the proof of Theorem 4.1.1 based on Assumptions (S1)–(S4) and Proposition 2.2.1. □

4.3. **Local to global case.** We can also compare the mod 2 spectrum of all the local singularities of a fixed fiber with the spectrum at infinity of a spectrally tame polynomial.

Theorem 4.3.1. *Let $P\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a spectrally tame polynomial. Let X be one of its fiber with (isolated) singular points $x_1, \dots, x_k$. We denote by $M_{x_1}, \dots, M_{x_k}$ the corresponding links of local singularities and $Sp_1^{MHS/2}, \dots, Sp_k^{MHS/2}$ the mod 2 reduction of their MHS–spectrum. Let $Sp_\infty^{MHS/2}$ be the mod 2 reduction of the MHS–spectrum at infinity of P. Then, if $\alpha \in [0,1]$ is chosen so that $\xi = e^{2\pi i\alpha}$ is not an eigenvalue of the monodromy of P at infinity, then*

$$|Sp_\infty^{MHS/2} \cap (\alpha, \alpha+1)| \geqslant \sum_j |Sp_j^{MHS/2} \cap (\alpha, \alpha+1)|$$
$$|Sp_\infty^{MHS/2} \setminus [\alpha, \alpha+1]| \geqslant \sum_j |Sp_j^{MHS/2} \setminus [\alpha, \alpha+1]|.$$

Proof. We follow closely the proof of Theorem 4.1.1 with the modification that now B is a large ball such that $X \cap \partial B$ is the link at infinity of P. □

References

[Arn81] V.I. Arnold, *On some problems in singularity theory*, In: Geometry and Analysis, Papers dedicated to the memory of V.K.Patodi, Bombay, 1981, 1-10.

[AGV84] V.I. Arnold, S.M. Gussein–Zade, A.N. Varchenko, *Singularities of differentiable maps II.*, "Nauka", Moscow, 1984. English translation, Monographs in Mathematics 83, Birkhäuser, 1988.

[Bol] http://en.wikipedia.org/wiki/Boleadoras

[Boro11] M. Borodzik, *Morse theory for plane algebraic curves*, arxiv:1101.1870, to appear in J. of Topology.

[BoNe11] M. Borodzik, A. Némethi, *Hodge-type structures as link invariants*, arxiv: 1005.2084, to appear in Ann. Inst. Fourier.

[BoNe12] M. Borodzik, A. Némethi, *Spectrum of plane curves via knot theory*, J. London Math. Soc. 2012, doi: 10.1112/jlms/jdr078

[BoNe13] M. Borodzik, A. Némethi, *Spectra of singularities of curves in normal surfaces*, in preparation.

[BNR12] M. Borodzik, A. Némethi, A. Ranicki, *Codimension 2 embeddings, Seifert matrices and signatures*, in preparation.

[Dim00] A. Dimca, *Monodromy and Hodge theory of regular functions*, New Developments in Singularity Theory, Cambridge 2000, 257–278.

[DiNe02] A. Dimca, A, Némethi, *Thom-Sebastiani construction and monodromy of polynomials*, Proceedings of the Steklov Institute, **238** (2002), 1-18; Monodromy in problems of algebraic geometry and differential equations, Editors A.A. Bolibrukh and C. Sabbah.

[GaNe96] R. García López, A. Némethi, *On the monodromy at infinity of a polynomial map*, Compos. Math. **100** (1996), no. 2, 205–231

[GaNe99] R. García López, A. Némethi, *Hodge numbers attached to a polynomial map*, Ann. Inst. Fourier **49** (1999), no. 5, 1547–1579.

[Erle69] D. Erle, *Quadratische Formen als Invarianten von Einbettungen der Kodimension 2*, Topology **8**(1969) 99–114.

[Kaw96] A. Kawauchi, *A survey on knot theory*, Birkhäuser—Verlag, Basel, Boston, Berlin 1996.

[Keef83] P.W. Keef, *On the S–equivalence of some general sets of matrices*, Rocky Mountain J. of Math. **13**(3) (1983), 541–551.

[Miln68] J. Milnor, *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. **61** Princeton University Press, Princeton, N.J, 1968.

[Mur65] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.

[Nem95] A. Némethi, *The real Seifert form and the spectral pairs of isolated hypersurface singularities*, Compositio Mathematica, **98** (1995), 23–41.

[Nem96] A. Némethi, *Some topological invariants of isolated hypersurface singularities*, Five lectures of the EMS-Summer School, Eger (Hungary) 1996, Proceedings of the Summer school, Bolyai Society Mathematical Studies 8, Low Dimensional Topology, (1999), 353-413.

[Nem99] A. Némethi, *On the Seifert form at infinity associated with polynomial maps*, Journal of the Math. Soc. of Japan, **51**, no. 1, 63-70, 1999.

[NeSa99] A. Némethi, C. Sabbah, *Semicontinuity of the spectrum at infinity*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 25–35.

[NeZa90] A. Némethi, A. Zaharia, *On the bifurcation set of a polynomial function and Newton boundary*, Publ. RIMS. Kyoto Univ., **26**, 681-689, 1990.

[NeZa92] A. Némethi, A. Zaharia, *Milnor fibration at infinity*, Indag. Mathem., N.S. **3** (3), 323-335, 1992.

[Neu89] W. D. Neumann, *Complex algebraic plane curves via their links at infinity*, Invent. Math. **98** (1989), no. 3, 445–489.

[Neu99] W. D. Neumann, *Irregular links at infinity of complex affine plane curves*, Quart. J. Math. Oxford Ser. (2) **50**(1999), no. 199, 301–320.

[Ra] A. Ranicki, High-dimensional knot theory. Algebraic surgery in codimension 2. Springer-Verlag, New York, 1998.

[Sab99] C. Sabbah, *Hypergeometric periods for a tame polynomial*, C. R. Acad. Sci. Paris Ser. I Math. **328** (1999), no. 7, 603–608.

[Stee76] J.H.M. Steenbrink, *Mixed Hodge Structures on the Vanishing Cohomology*, Nordic Summer School/NAVF, Symposium in Math., Oslo 1976.

[Stee85] J.H.M. Steenbrink, *Semicontinuity of the singularity spectrum*, Invent. Math **79** (1985), 557–565.

[Var83a] A.N. Varchenko, *On the semicontinuity of the spectrum and an upper bound for the number of singular points of projective hypersurfaces*, Doklady Akad. Nauk. **270**(6) (1983), 1294–1297.

[Var83b] A.N. Varchenko, *On change of discrete invariants of critical points of functions under deformation*, Uspehi Mat. Nauk. **5** (1983), 126-127.

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