CODIMENSION 2 EMBEDDINGS, ALGEBRAIC SURGERY AND SEIFERT FORMS

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Abstract. We study the cobordism of manifolds with boundary, and its applications to codimension 2 embeddings $M^m \subset N^{m+2}$, using the method of the algebraic theory of surgery. The first main result is a splitting theorem for cobordisms of algebraic Poincaré pairs, which is then applied to describe the behaviour on the chain level of Seifert surfaces of embeddings $M^{2n-1} \subset S^{2n+1}$ under isotopy and cobordism. The second main result is that the S-equivalence class of a Seifert form is an isotopy invariant of the embedding, generalizing the Murasugi–Levine result for knots and links. The third main result is a generalized Murasugi–Kawauchi inequality giving an upper bound on the difference of the Levine–Tristram signatures of cobordant embeddings.

Introduction

This is one of a trilogy of papers by the authors concerned with Morse theory for manifolds with boundary [BNR1], Seifert forms and signature invariants of codimension 2 embeddings $M^m \subset N^{m+2}$ (in the current paper), and the applications in the case $M^{2n-1} \subset S^{2n+1}$ to the mod 2 spectrum of isolated hypersurface singularities [BNR3].

The common feature of these papers is the use of the relative cobordism theory of manifolds with boundary, and in this paper also the relative cobordism theory of symmetric Poincaré pairs in the algebraic theory of surgery [Ra1, Ra2]. A symmetric Poincaré complex is a chain complex with Poincaré duality; a symmetric Poincaré pair is a chain complex pair with Poincaré-Lefschetz duality. A closed manifold determines a complex; a manifold with boundary determines a pair.

In the first instance, the relative cobordism theories appear to be trivial, since every manifold with boundary is null-cobordant, and similarly for a symmetric Poincaré pair. Nevertheless, it is possible to extract nontrivial applications!

In our applications of relative cobordism, the manifolds with boundary are the Seifert surfaces Σ^{m+1} of codimension 2 embeddings $M^m \subset S^{m+2}$, with $\partial\Sigma = M$. We shall be particularly concerned with the case $m = 2n-1$. In order to understand the behaviour of the Seifert form on $H_n(\Sigma)$ under relative cobordism it is necessary to extend the homology methods pioneered by Kervaire [Ke] and Levine [Le1, Le2] for $M = S^{2n-1} \subset S^{2n+1}$ to chain complexes.

0.1. **Background.** Seifert [Sei] proved that every knot $M = S^1 \subset S^3$ is the boundary $M = \partial\Sigma$ of a Seifert surface $\Sigma^2 \subset S^3$, using linking numbers of disjoint cycles in S^3 to define the Seifert form $(H_1(\Sigma), A(\Sigma))$, and to express the Alexander polynomial as

$$\Delta(t) \;=\; \det(tA(\Sigma) - A(\Sigma)^* : H_1(\Sigma)[t,t^{-1}] \to H_1(\Sigma)^*[t,t^{-1}]) \in \mathbb{Z}[t,t^{-1}] \;.$$

Date: November 27, 2012.

2010 *Mathematics Subject Classification.* primary: 57Q45 secondary: 57Q60, 19J25, 57R65.

Key words and phrases. Codimension 2 embedding, cobordism, algebraic surgery, symmetric Poincaré complex, Seifert surface, Seifert form, S-equivalence, Levine–Tristram signature, variation structures, high-dimensional knots and links, Murasugi-Kawauchi inequality.

The second author is partially supported by OTKA Grant K67928.

Murasugi [Mu] introduced the S-equivalence relation on Seifert forms, proving that the Seifert forms of a knot are S-equivalent, and that the S-equivalence class is an isotopy invariant. Kervaire [Ke] initiated the classification theory of m-dimensional knots $S^m \subset S^{m+2}$ for all $m \geqslant 1$, extending the construction of Seifert surfaces and forms to high-dimensional knots. For $n > 1$ Levine [Le1] identified the cobordism group C_{2n-1} of $(2n-1)$-dimensional knots $S^{2n-1} \subset S^{2n+1}$ with the cobordism group of Seifert forms, and showed that C_{2n-1} is determined modulo 2-primary torsion by the signatures

$$\sigma_{S^{2n-1}\subset S^{2n+1}}(\xi) \;=\; \text{signature}((1-\xi)A(\Sigma) + (-1)^{n+1}(1-\xi^{-1})A(\Sigma)^*) \in \mathbb{Z}$$

for $\xi \in S^1$ with $\Delta(\xi) \neq 0 \in \mathbb{C}$. Furthermore, Levine [Le2] proved that for $n > 1$ the isotopy classes of simple (= $(n-1)$-connected) knots $S^{2n-1} \subset S^{2n+1}$ are in one-one correspondence with the S-equivalence classes of Seifert forms. Tristram [Tr] extended the constructions of [Le1] to Seifert surfaces and matrices of spherical links

$$M \;=\; S^{2n-1} \sqcup S^{2n-1} \sqcup \cdots \sqcup S^{2n-1} \subset S^{2n+1}$$

with signatures $\sigma_M(\xi)$ for $\xi \in S^1$, which are cobordism invariant for $\Delta(\xi) \neq 0 \in \mathbb{C}$. The papers [Le1, Le2, Tr] made essential use of the behaviour of Seifert surfaces and matrices under cobordisms and isotopies of knots and links.

The algebraic theory of surgery of [Ra1, Ra2, Ra3] deals with the cobordism of symmetric Poincaré complexes (= chain complexes with abstract Poincaré duality), which has many features in common with the cobordism theory of manifolds. In this paper we shall consider isotopy and cobordism invariants of codimension 2 embeddings $M^{2n-1} \subset S^{2n+1}$ via their Seifert surfaces. Our results depend on the descriptions of the isotopies and cobordisms given by the surgery theory of manifolds with boundary, and on the algebraic analogues for symmetric Poincaré pairs. In fact, much of the methodology of the paper applies to codimension 2 embeddings $M^m \subset N^{m+2}$ for arbitrary m, N.

0.2. **Outline of the paper.** In §1 *The cobordism of manifolds* we extend the surgery and handlebody theory for cobordisms of closed manifolds to the half-surgery and half-handlebody theory of relative cobordisms, i.e. the cobordisms of manifolds with boundary. (Half-surgeries and half-handlebodies have already appeared in [BNR1], in the context of Morse theory for manifolds with boundary). Also, we introduce the notion of split relative cobordisms, which are the unions of left and right product cobordisms: the homological properties of split cobordisms are the key to our algebraic descriptions of isotopies and cobordisms of codimension 2 embeddings. The main result of §1 is Theorem 1.3.4, which gives a complete description of the relationships between the homology groups of a relative cobordism with a half-handle decomposition.

In §2 *Forms and their enlargements* we study the algebraic properties of $(-1)^n$-symmetric forms (F, B), with F a f.g. free abelian group and B a $(-1)^n$-symmetric bilinear pairing

$$B \;:\; F \times F \to \mathbb{Z} \;;\; (x, y) \mapsto B(x, y) = (-1)^n B(y, x) \;,$$

which can also be a viewed as a self-$(-1)^n$-dual morphism of abelian groups

$$B \;=\; (-1)^n B^* \;:\; F \to F^* \;=\; \text{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) \;;\; x \mapsto (y \mapsto B(x, y)) \;.$$

We introduce the notion of a *rank ℓ enlargement* of a $(-1)^n$-symmetric form (F, B) over $\mathbb{Z}$ as a form of the type

$$(F', B') \;=\; (F \oplus L, \begin{pmatrix} B & C \\ (-1)^n C^* & D \end{pmatrix}) \;,$$

with (L, D) a $(-1)^n$-symmetric form, $\dim_{\mathbb{Z}} L = \ell$ and $C \in \text{Hom}_{\mathbb{Z}}(L, F^*)$. Call (F, B) a *rank ℓ reduction* of (F', B'). Every form (F, B) is a rank $\dim_{\mathbb{Z}} F$ enlargement of $(0, 0)$, but the notion is useful all the same.

A 1-symmetric form (F,B) has a signature $\sigma(F,B)\in\mathbb{Z}$ and a nullity $n(F,B)\geqslant 0$.
The main result of §2 is :

Theorem 2.2.7. *If n is even and (F',B') is a rank ℓ enlargement of (F,B) the signatures and nullities are related by the inequality*

$$|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\leqslant \ell\ .$$

For a rank 1 enlargement there is an equality

$$|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\ =\ 1\ .$$

In §3 *The intersection form of a manifold with boundary* we consider the $(-1)^n$-symmetric intersection form $(F_n(\Sigma),B(\Sigma))$ of a $2n$-dimensional manifold with boundary (Σ,M), with $F_n(\Sigma)=H_n(\Sigma)/\text{torsion}$ a f.g. free abelian group. The effect on the intersection form of a surgery on the interior $\Sigma\backslash M$ is a rank ℓ enlargement or reduction with $\ell\in\{0,2\}$, so $\dim_{\mathbb{Z}}F_n(\Sigma)$ changes by one of $\{-2,0,2\}$. The effect on the intersection form of a half-surgery on (Σ,M) is a rank ℓ enlargement or reduction with $\ell\in\{0,1\}$, so $\dim_{\mathbb{Z}}F_n(\Sigma)$ changes by one of $\{-1,0,1\}$.
The main result of §3 is :

Theorem 3.4.2. *A half-handle decomposition for a $(2n+1)$-dimensional relative cobordism $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ determines a sequence of enlargements and reductions taking $(F_n(\Sigma_0),B(\Sigma_0))$ to $(F_n(\Sigma_1),B(\Sigma_1))$.*

In §4 *The cobordism of symmetric Poincaré complexes* we develop the algebraic theory of surgery of [Ra1, Ra2, Ra3] further, to include algebraic versions of splitting and half handle decompositions for relative cobordisms of symmetric Poincaré pairs. Again, a relative cobordism is split if it is a union of a left and a right product cobordism. A splitting (resp. half-handle decomposition) of a relative cobordism of manifolds with boundary determines on the chain level an algebraic splitting (resp. half-handle decomposition) of a relative symmetric Poincaré cobordism.

The main result of §4, and indeed the first main result of the paper, is the Algebraic Poincaré Splitting Theorem :

Main Theorem 1. (4.5.6) *Every relative symmetric Poincaré cobordism is algebraically split.*

This theorem is an algebraic converse to a standard construction in cobordism theory: the union of three manifolds at a common boundary component is a stratified set, with a thickening which is a manifold with boundary.

As a consequence of Main Theorem 1, every relative cobordism of manifolds with boundary has at least an algebraic half-handle decomposition. In [BNR1] such an algebraic half-handle decomposition is realized geometrically as a half-handle decomposition of a relative cobordism, under the hypothesis that all the manifolds involved are non-empty and connected, but we only need an algebraic half-handle decomposition here.

In §5 *Codimension q embeddings, especially for $q=2$* we consider the general properties of codimension q embeddings $M^m\subset N^{m+q}$ and the particular properties in the case $q=2$. A Seifert surface for $M^m\subset N^{m+2}$ is a codimension 1 embedding $\Sigma^{m+1}\subset N$ such that $\partial\Sigma=M$.
The main result of §5 is :

Theorem 5.4.8. *A codimension 2 embedding $M^m\subset N^{m+2}$ admits a Seifert surface if and only if $[M]=0\in H_m(N)$, if and only if it is framed (i.e. the normal 2-plane bundle $\nu_{M\subset N}$ is trivial).*

§6 *Codimension 2 embeddings* $M^m \subset S^{m+2}$ is the core of the paper. We extend to such embeddings the standard notions associated to high-dimensional knots and links, concentrating on the odd dimensions $m = 2n-1$. In many cases the extensions are already in the literature. The notions include *cobordism*, *H-cobordism*, *Seifert surface* Σ, *Seifert form* $(F_n(\Sigma), A(\Sigma))$ with $B(\Sigma) = A(\Sigma) + (-1)^n A(\Sigma)^*$, *variation map*, *Blanchfield pairing* etc. In particular, we introduce the notion of *enlargement* and *reduction* of a Seifert form, by analogy with the enlargement and reductions of forms, as well as the *S-equivalence* and *H-equivalence* of Seifert forms (Definition 6.2.1).

Main Theorem 2. (6.2.6) *Let $M^{2n-1} \subset S^{2n+1}$ be a codimension 2 embedding with a Seifert surface Σ, and Seifert form $(F_n(\Sigma), A(\Sigma))$. Then the S-equivalence class of $(F_n(\Sigma), A(\Sigma))$ depends only on the isotopy class of the embedding. Furthermore, if $M_0, M_1 \subset S^{2n+1}$ are H-cobordant codimension 2 embeddings, then the corresponding Seifert forms are H-equivalent.*

The Main Theorem 2 has a deep consequence: all the classical knot and link invariants that can be derived from the Seifert forms $(F_n(\Sigma), A(\Sigma))$ also give rise to invariants of arbitrary framed codimension 2 embeddings $M^{2n-1} \subset S^{2n+1}$.

The *Alexander polynomial* of $M^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface Σ is defined by

$$\Delta_{M,\Sigma}(t) \;=\; \det(tA(\Sigma) + (-1)^n A(\Sigma)^* : F_n(\Sigma)[t,t^{-1}] \to F_n(\Sigma)^*[t,t^{-1}]) \in \mathbb{Z}[t,t^{-1}]\ .$$

The Main Theorem 2 implies that this definition does not depend on the choice of a Seifert surface Σ, so $\Delta_{M,\Sigma}(t)$ can be denoted $\Delta_M(t)$.

For $\xi \in S^1$ the *nullity* of $M^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface Σ is defined by

$$n_{M,\Sigma}(\xi) \;=\; \text{nullity}(\xi A(\Sigma) + (-1)^n A(\Sigma)^* : H_n(\Sigma;\mathbb{C}) \to H_n(\Sigma;\mathbb{C})^*) \geqslant 0\ ,$$

and the *Levine–Tristram signature* is defined by

$$\sigma_{M,\Sigma}(\xi) \;=\; \text{signature}(H_n(\Sigma;\mathbb{C}), (1-\xi)A(\Sigma) + (-1)^{n+1}(1-\overline{\xi})A(\Sigma)^*) \in \mathbb{Z}$$

with $H_n(\Sigma;\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} F_n(\Sigma)$. Corollary 6.4.10 (i) to the Main Theorem 2 states that $n_{M,\Sigma}(\xi)$ and $\sigma_{M,\Sigma}(\xi)$ are invariants of the isotopy class of $M \subset S^{2n+1}$, and so may be denoted $n_M(\xi)$, $\sigma_M(\xi)$. Corollary 6.4.10 (ii) shows that $n_\Sigma(\xi)$, $\sigma_M(\xi)$ are invariants of the H-equivalence class of $(F_n(\Sigma), A(\Sigma))$ for $\xi \in S^1$ such that $\Delta_M(\xi) \neq 0$, and hence invariants of the H-cobordism class (in the case $n = 1$ this is the concordance class) of $M \subset S^{2n+1}$.

We obtain in §6 a generalization of the inequality of Murasugi [Mu] and Kawauchi [Ka] for the signatures of cobordant links $\bigcup S^1 \subset S^3$ to the Levine–Tristram signatures $\sigma_M(\xi), \sigma_{M'}(\xi)$ of cobordant codimension 2 embeddings $M, M' \subset S^{2n+1}$ in all dimensions :

Main Theorem 3 (6.5.1) *Let $(W^{2n}; M_0, M_1) \subset S^{2n+1} \times (I; \{0\}, \{1\})$ be a cobordism of codimension 2 embeddings $M_0, M_1 \subset S^{2n+1}$. Given Seifert surfaces Σ_0, Σ_1 for M_0, M_1 define the closed $2n$-dimensional manifold*

$$\Sigma^{2n} \;=\; \Sigma_0 \cup_{M_0} W \cup_{M_1} -\Sigma_1 \subset S^{2n+1} \times I\ .$$

For any $\xi \neq 1 \in S^1$

$$|\sigma_{M_0}(\xi) - \sigma_{M_1}(\xi)| \leqslant b_n(\Sigma) - b_n(\Sigma_0) - b_n(\Sigma_1) + n_{M_0}(\xi) + n_{M_1}(\xi)\ ,$$

where $b_n(X)$ denotes the n-th Betti number of a topological space X.

In fact, a proof of the Main Theorem 3 is also available in [BNR3], using a slightly different method. It should be noted that it is neither assumed that $(W; M_0, M_1)$ is an H-cobordism, nor that $\Delta_{M_0}(\xi), \Delta_{M_1}(\xi) \neq 0$. The inequality plays a key role in [BNR3], where it is used to obtain a topological proof of the semicontinuity of the mod 2 spectrum of isolated hypersurface singularities.

The literature devoted to codimension 2 embedding theory is vast, both in the classical case $m = 1$ and in the higher dimensions $m \geqslant 2$. The survey of Kervaire and Weber [KW] and the book of Ranicki [Ra3] have many references on high-dimensional knots. The work of Cappell and Shaneson [CS], Blanloeil and Michel [BM], Blanloeil and Saeki [BS] on high-dimensional links is only a sample. But to date ours is the most general in dealing with the homological properties of cobordisms of manifolds with boundary, both intrinsically and in codimension 2.

1. The cobordism of manifolds

We shall be working with oriented smooth manifolds, denoting by $-M$ the manifold M with the opposite orientation.

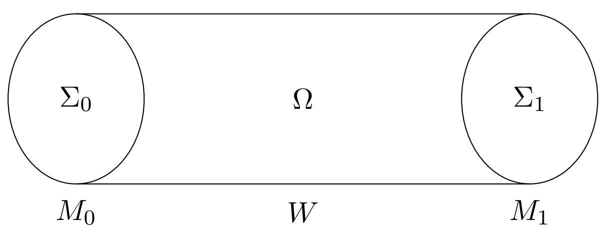
1.1. **Absolute and relative cobordisms.**

Definition 1.1.1. (i) An $(m+1)$-*dimensional (absolute) cobordism* $(W; M_0, M_1)$ consists of closed m-dimensional manifolds M_0, M_1 and an $(m+1)$-dimensional manifold W with boundary

$$\partial W \;=\; M_0 \sqcup -M_1 \;.$$

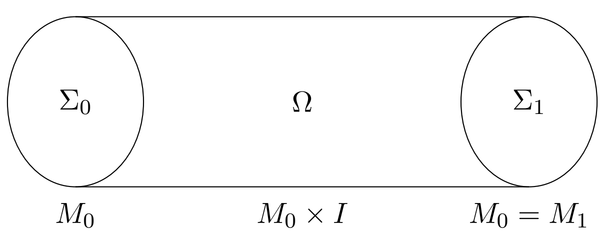
(ii) An $(m+2)$-*dimensional (relative) cobordism* $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ consists of $(m+1)$-dimensional manifolds with boundary (Σ_0, M_0), (Σ_1, M_1), an absolute cobordism $(W; M_0, M_1)$, and an $(m+2)$-dimensional manifold Ω with boundary

$$\partial\Omega \;=\; \Sigma_0 \cup_{M_0} W \cup_{M_1} -\Sigma_1 \;.$$



(iii) A relative cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a *boundary product* if

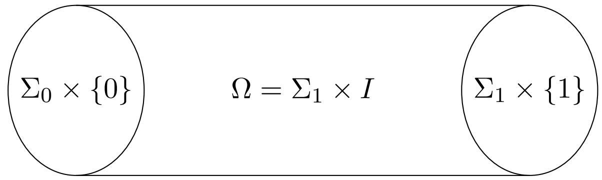
$$(W; M_0, M_1) \;=\; M_0 \times (I; \{0\}, \{1\}) \;.$$



(iv) A relative cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a *right product* if

$$\begin{aligned}&(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)\\ &= \;(\Sigma_1 \times I; \Sigma_0 \times \{0\}, (\Sigma_0 \cup_{M_0} W) \times \{1\}, W \times \{0\} \cup M_1 \times I; M_0 \times \{0\}, M_1 \times \{1\})\end{aligned}$$

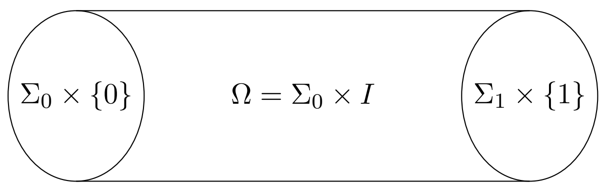
with $\Sigma_1 = \Sigma_0 \cup_{M_0} W$.



(v) A relative cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a *left product* if it is the reverse of a right product

$$\begin{aligned}&(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)\\ &= (\Sigma_0 \times I; (W \cup_{M_1} \Sigma_1) \times \{0\}, \Sigma_1 \times \{1\}, M_0 \times I \cup W \times \{1\}; M_0 \times \{0\}, M_1 \times \{1\})\end{aligned}$$

with $\Sigma_0 = W \cup_{M_1} \Sigma_1$.



The trace of an interior surgery on a manifold with boundary is a boundary is a boundary product relative cobordism. We now develop the notions of half-surgeries on manifolds with boundary, and their trace relative products. We refer to [BNR1] for the Morse theory approach to half-surgeries and half-handles.

Definition 1.1.2. (i) Given an m-dimensional manifold with boundary $(M_0, \partial M_0)$ and an embedding

$$\bigcup_{\ell} S^r \times D^{m-r} \subset M_0 \backslash \partial M_0$$

for some $\ell \geqslant 1$ define the m-dimensional manifold with boundary obtained by ℓ *index* $r+1$ *surgeries*

$$(M_1, \partial M_1) \ = \ (\mathrm{cl.}(M_0 \backslash \bigcup_{\ell} S^r \times D^{m-r}) \cup \bigcup_{\ell} D^{r+1} \times S^{m-r-1}, \partial M_0)\ .$$

Call M_1 the *effect* of the surgeries on M_0. Note that M_0 is the effect of the ℓ index $m-r$ surgeries on M_1 by $\bigcup_{\ell} D^{r+1} \times S^{m-r-1} \subset M_1 \backslash \partial M_1$.

(ii) Given an $(m+1)$-dimensional manifold with boundary (Σ_0, M_0) and an embedding

$$\bigcup_{\ell} S^r \times D^{m-r} \subset M_0$$

define the $(m+1)$-dimensional manifold with boundary obtained by ℓ *index* $r+1$ *right half-surgeries*

$$(\Sigma_1, M_1) \ = \ (\Sigma_0 \cup_{\bigcup_{\ell} S^r \times D^{m-r}} \bigcup_{\ell} D^{r+1} \times D^{m-r}, \mathrm{cl.}(M_0 \backslash \bigcup_{\ell} S^r \times D^{m-r}) \cup \bigcup_{\ell} D^{r+1} \times S^{m-r-1})\ .$$

Note that M_1 is the effect of the ℓ index $r+1$ surgeries on $\bigcup_{\ell} S^r \times D^{m-r} \subset M_0$.

(iii) The *trace* of the ℓ surgeries in (i) is the boundary product cobordism

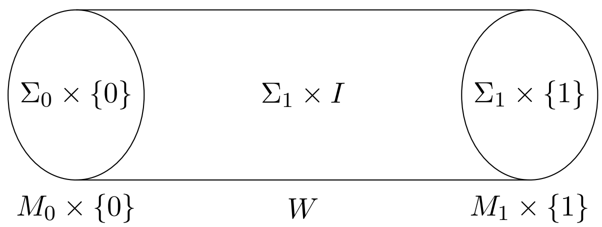
$$(W; M_0, M_1, \partial M_0 \times I; \partial M_0 \times \{0\}, \partial M_0 \times \{1\})$$

obtained by attaching ℓ *index* $(r+1)$ *handles* to $M_0 \times I$ at $\bigcup_{\ell} S^r \times D^{m-r} \subset M_0 \times \{1\}$ with $W = M_0 \times I \cup \bigcup_{\ell} D^{r+1} \times D^{m-r}$.

(iv) The *trace* of the right half-surgeries in (ii) is the right product cobordism of *index* $r+1$, with

$$(\Sigma_1 \times I; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$$

with $W = M_0 \times I \cup \bigcup_{\ell} D^{r+1} \times D^{m-r}$ the trace (in the sense of (iii)) of the ℓ index $r+1$ surgeries on $\bigcup_{\ell} S^r \times D^{m-r} \subset M_0$.



(v) Given an $(m+1)$-dimensional manifold with boundary (Σ_0, M_0) and an embedding

$$(\bigcup_{\ell} D^{r+1} \times D^{m-r}, \bigcup_{\ell} S^r \times D^{m-r}) \subset (\Sigma_0, M_0)$$

define the $(m+1)$-dimensional manifold with boundary obtained by ℓ *index* $r+1$ *left half-surgeries*

$$(\Sigma_1, M_1) \;=\; (\mathrm{cl.}(\Sigma_0 \backslash \bigcup_{\ell} D^{r+1} \times D^{m-r}), \mathrm{cl.}(M_0 \backslash \bigcup_{\ell} S^r \times D^{m-r}) \cup \bigcup_{\ell} D^{r+1} \times S^{m-r-1}) \;.$$

In particular, M_1 is the effect of the ℓ index $r+1$ surgeries on M_0 by $\bigcup_{\ell} S^r \times D^{m-r} \subset M_0$. The *trace* of the left half-surgeries is the left product cobordism of

$$(\Sigma_0 \times I; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$$

with $W = M_0 \times I \cup \bigcup_{\ell} D^{r+1} \times D^{m-r}$ the trace (in the sense of (iii)) of the ℓ index $r+1$ surgeries on $\bigcup_{\ell} S^r \times D^{m-r} \subset M_0$. Note that (Σ_0, M_0) is obtained from (Σ_1, M_1) by ℓ index $m-r$ right half-surgeries.

Remark 1.1.3. In terms of homotopy theory, if (Σ_1, M_1) is the $(m+1)$-dimensional manifold with boundary obtained by an index $r+1$ left half-surgery then Σ_1 is obtained from Σ_0 by detaching an $(m-r)$-cell, and M_1 is obtained from M_0 by attaching an $(r+1)$-cell and detaching an $(m-r)$-cell, so that the Euler characteristics are related by

$$\chi(\Sigma_1) \;=\; \chi(\Sigma_0) - (-1)^{m-r} \;,\; \chi(M_1) \;=\; \chi(M_0) + (-1)^{r+1} - (-1)^{m-r} \;.$$

Example 1.1.4. Here are the two key examples of the effects of surgeries and half-surgeries on the intersection form, which both start with a closed $2n$-dimensional manifold Σ. Define the $2n$-dimensional manifold with boundary

$$(\Sigma_0, M_0) \;=\; (\mathrm{cl.}(\Sigma \backslash D^{2n}), S^{2n-1})$$

with intersection form

$$(F_n(\Sigma_0), B(\Sigma_0)) \;=\; (F_n(\Sigma), B(\Sigma)) \;.$$

(i) Surgery on a trivial embedding $S^{n-1} \times D^{n+1} \subset D^{2n} \subset \Sigma$ results in the connected sum of Σ and $S^n \times S^n$

$$\Sigma' \;=\; \mathrm{cl.}(\Sigma \backslash S^{n-1} \times D^{n+1}) \cup D^n \times S^n \;=\; \Sigma_0 \cup_{S^{2n-1}} (S^n \times S^n)_0 \;=\; \Sigma \# S^n \times S^n \;.$$

The intersection form of Σ' is the rank 2 enlargement of the intersection form of Σ

$$\begin{aligned}(F_n(\Sigma'),B(\Sigma')) &= (F_n(\Sigma)\oplus\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}B(\Sigma) & 0 & 0\\ 0 & 0 & 1\\ 0 & (-1)^n & 0\end{pmatrix})\\ &= (F_n(\Sigma),B(\Sigma))\oplus(\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}0 & 1\\ (-1)^n & 0\end{pmatrix})\end{aligned}$$

given by adding the hyperbolic form

$$(F_n(S^n\times S^n),B(S^n\times S^n)) = (\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}0 & 1\\ (-1)^n & 0\end{pmatrix})\ .$$

(ii) The effect of an index n left half-surgery on

$$(D^n\times D^n,S^{n-1}\times D^n)\subset(D^{2n},S^{2n-1})\subset(\Sigma_0,M_0)$$

is the $2n$-dimensional manifold with boundary

$$(\Sigma_{1/2},M_{1/2}) \ = \ (\Sigma\# S^n\times D^n,M\# S^n\times S^{n-1})$$

with intersection form a rank 1 enlargement of the intersection form of (Σ,M)

$$(F_n(\Sigma_{1/2}),B(\Sigma_{1/2})) \ = \ (F_n(\Sigma)\oplus\mathbb{Z},\begin{pmatrix}B(\Sigma) & 0\\ 0 & 0\end{pmatrix})\ .$$

The effect of an index $n+1$ right half-surgery on $(\Sigma_{1/2},M_{1/2})$ by $S^n\times D^{n-1}\subset M_{1/2}$ is the $2n$-dimensional manifold with boundary

$$(\Sigma_1,M_1) \ = \ (\mathrm{cl.}(\Sigma'\backslash D^{2n}),S^{2n-1})$$

(Σ' as in (i)) with intersection form a rank 1 enlargement of the intersection form of $(\Sigma_{1/2},M_{1/2})$ and a rank 2 enlargement of the intersection form of (Σ_0,M_0)

$$\begin{aligned}(F_n(\Sigma_1),B(\Sigma_1)) &= (F_n(\Sigma_{1/2})\oplus\mathbb{Z},\begin{pmatrix}B(\Sigma_{1/2}) & \begin{pmatrix}0\\1\end{pmatrix}\\ \begin{pmatrix}0 & (-1)^n\end{pmatrix} & 0\end{pmatrix})\\ &= (F_n(\Sigma)\oplus\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}B(\Sigma) & 0 & 0\\ 0 & 0 & 1\\ 0 & (-1)^n & 0\end{pmatrix}) \ = \ (F_n(\Sigma'),B(\Sigma'))\ .\end{aligned}$$

The traces $(W^-;M_0,M_{1/2})$, $(W^+;M_{1/2},M_1)$ are attachments of a cancelling pair of handles, with

$$\begin{aligned}&\mathrm{intersection}(\{0\}\times S^{n-1},S^n\times\{0\}\subset M_{1/2}) \ = \ 1\ ,\\ &(W^-;M_0,M_{1/2})\cup(W^+;M_{1/2},M_1) \ = \ M_0\times(I;\{0\},\{1\})\ .\end{aligned}$$

The Witt group stabilization in (i) is matched up to the cancellation of handles in (ii).

Proposition 1.1.5. *Let (Σ_0,M_0) be an $(m+1)$-dimensional manifold with boundary.*
(i) *If $(\Omega;\Sigma_0,\Sigma_1,M_0\times I;M_0\times\{0\},M_0\times\{1\})$ is the trace boundary product cobordism of ℓ index $r+1$ surgeries on $\bigcup_{\ell}S^r\times D^{m-r+1}\subset\Sigma_0\backslash M_0$ there are homotopy equivalences*

$$\Omega \ = \ \Sigma_0\times I\cup\bigcup_{\ell}D^{r+1}\times D^{m-r+1} \ \simeq \ \Sigma_0\cup\bigcup_{\ell}D^{r+1} \ \simeq \ \Sigma_1\cup\bigcup_{\ell}D^{m-r+1}\ ,$$

so that

$$H_q(\Omega,\Sigma_0) \ = \ \begin{cases}\mathbb{Z}^\ell & \textit{if } q=r+1\\ 0 & \textit{otherwise}\end{cases}\ ,\ H_q(\Omega,\Sigma_1) \ = \ \begin{cases}\mathbb{Z}^\ell & \textit{if } q=m-r+1\\ 0 & \textit{otherwise}\end{cases}\ .$$

(ii) *If*

$$\begin{aligned}&(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1) \ = \\ &(\Sigma_1\times I;\Sigma_0\times\{0\},\Sigma_1\times\{1\},W\times\{0\}\cup_{M_1\times\{0\}}M_1\times I;M_0\times\{0\},M_1\times\{1\})\end{aligned}$$

is the trace right product relative cobordism of ℓ *index* $r+1$ *right half-surgeries on* $\bigcup\limits_{\ell}S^r\times D^{m-r}\subset M_0$ *with*

$$\begin{aligned}(\Sigma_1,M_1) &= (\Sigma_0\cup_{M_0}W,M_1)\\ &= (\Sigma_0\cup_{\bigcup\limits_{\ell}S^r\times D^{m-r}}\bigcup_{\ell}D^{r+1}\times D^{m-r},\mathrm{cl.}(M_0\backslash\bigcup_{\ell}S^r\times D^{m-r})\bigcup_{\ell}D^{r+1}\times S^{m-r-1})\end{aligned}$$

there are defined homotopy equivalences

$$\begin{aligned}&\Omega \ = \ \Sigma_1\times I \ \simeq \ \Sigma_1 \ = \ \Sigma_0\cup_{\bigcup\limits_{\ell}S^r\times D^{m-r}}\bigcup_{\ell}D^{r+1}\times D^{m-r} \ \simeq \ \Sigma_0\cup\bigcup_{\ell}D^{r+1}\ ,\\ &W \ \simeq \ M_0\cup\bigcup_{\ell}D^{r+1} \ \simeq \ M_1\cup\bigcup_{\ell}D^{m-r}\end{aligned}$$

so that

$$\begin{aligned}&H_q(\Sigma_1\times I,\Sigma_0\times\{0\}) \ = \ H_q(W,M_0) \ = \ \begin{cases}\mathbb{Z}^{\ell} & \text{if } q=r+1\\ 0 & \text{otherwise}\end{cases}\ ,\\ &H_q(W,M_1) \ = \ \begin{cases}\mathbb{Z}^{\ell} & \text{if } q=m-r\\ 0 & \text{otherwise}\end{cases}\ .\end{aligned}$$

(iii) *If*

$$\begin{aligned}&(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1) \ = \\ &(\Sigma_0\times I;\Sigma_0\times\{0\},\Sigma_1\times\{1\},W\times\{0\}\cup_{M_1\times\{0\}}M_1\times I;M_0\times\{0\},M_1\times\{1\})\end{aligned}$$

is the trace left product relative cobordism of ℓ *index* $r+1$ *left half-surgeries on*

$$(\bigcup_{\ell}D^{r+1}\times D^{m-r},\bigcup_{\ell}S^r\times D^{m-r})\subset(\Sigma_0,M_0)$$

with

$$(\Sigma_1,M_1) \ = \ (\mathrm{cl.}(\Sigma_0\backslash\bigcup_{\ell}D^{r+1}\times D^{m-r}),\mathrm{cl.}(M_0\backslash\bigcup_{\ell}S^r\times D^{m-r})\cup\bigcup_{\ell}D^{r+1}\times S^{m-r-1})$$

there are defined homotopy equivalences

$$\begin{aligned}&\Omega \ = \ \Sigma_0\times I \ \simeq \ \Sigma_0 \ = \ \Sigma_1\cup_{\bigcup\limits_{\ell}D^{r+1}\times S^{m-r-1}}\bigcup_{\ell}D^{r+1}\times D^{m-r} \ \simeq \ \Sigma_1\cup\bigcup_{\ell}D^{m-r}\ ,\\ &W \ \simeq \ M_0\cup\bigcup_{\ell}D^{r+1} \ \simeq \ M_1\cup\bigcup_{\ell}D^{m-r}\end{aligned}$$

so that

$$\begin{aligned}&H_q(\Sigma_0\times I,\Sigma_1\times\{0\}) \ = \ H_q(W,M_1) \ = \ \begin{cases}\mathbb{Z}^{\ell} & \text{if } q=m-r\\ 0 & \text{otherwise}\end{cases}\ ,\\ &H_q(W,M_0) \ = \ \begin{cases}\mathbb{Z}^{\ell} & \text{if } q=r+1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

1.2. **Handle and half-handle decompositions.** Let us recall the standard theory of handle decompositions of absolute cobordisms:

Proposition 1.2.1. (Thom, Milnor) *Every absolute cobordism* $(\Omega; \Sigma, \Sigma')$ *has a handlebody decomposition, i.e. is the union of traces of surgeries.*

Proof. By the standard translation between Morse functions and handlebody decompositions. □

Similarly for boundary and right product relative cobordisms :

Proposition 1.2.2. (i) *Every boundary product relative cobordism has a handlebody decomposition, i.e. is the union of traces of interior surgeries.*
(ii) *Every right product relative cobordism has a right half-handlebody decomposition, i.e. is the union of traces of right half-surgeries. Likewise, a left product relative cobordism has a left half-handlebody decomposition.*

Proof. (i) Work as in the proof of Proposition 1.2.1. An $(m+2)$-dimensional boundary product relative cobordism $(\Omega; \Sigma, \Sigma', M \times I; M \times \{0\}, M \times \{1\})$ admits a Morse function $U : \Omega \to I$ such that

$$U^{-1}(0) \;=\; \Sigma \;,\; U^{-1}(1) \;=\; \Sigma' \;,\; U| \;=\; \text{projection} \;:\; M_0 \times I \to I \;.$$

Let $0 = c_{-1} < c_0 < \cdots < c_{m+1} < c_{m+2} = 1$ be defined by

$$c_r \;=\; \frac{r+1}{m+3} \;(-1 \leqslant r \leqslant m+2) \;.$$

The Morse function can be arranged for there to be ℓ_{r+1} critical values of index $r+1$ in $(c_r, c_{r+1}) \subset (0,1)$ for $r = -1, 0, \ldots, m+2$, so that the boundary product relative cobordism

$$(\Omega_r; \Sigma_r, \Sigma_{r+1}) \;=\; U^{-1}([r, r+1]; \{c_r\}, \{c_{r+1}\})$$

is the trace of ℓ_{r+1} index $r+1$ surgeries on

$$\bigcup_{\ell_{r+1}} S^r \times D^{m-r+1} \subset \Sigma_r \backslash \partial \Sigma_r \;,$$

with

$$(\Omega; \Sigma, \Sigma') \;=\; \bigcup_{r=-1}^{m+2} (\Omega_r; \Sigma_r, \Sigma_{r+1}) \;.$$

(ii) By definition, the right product relative cobordism $(\Omega; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$ has

$$\Omega \;=\; \Sigma_1 \times I \;,\; \Sigma_1 \;=\; \Sigma_0 \cup_{M_0} W \;.$$

A decomposition of the absolute cobordism $(W; M_0, M_1)$ as a union of adjoining absolute cobordisms

$$(W; M_0, M_1) \;=\; (W_0; M_0, M_{1/2}) \cup (W_1; M_{1/2}, M_1)$$

extends to a decomposition of $(\Omega; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$ as a union of right product cobordisms

$$\begin{aligned}&(\Omega; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)\\ &= (\Omega_0; \Sigma_0 \times \{0\}, \Sigma_{1/2} \times \{1\}, W_0; M_0, M_{1/2}) \cup (\Omega_1; \Sigma_{1/2} \times \{0\}, \Sigma_1 \times \{1\}, W_1; M_{1/2}, M_1)\end{aligned}$$

with

$$\Sigma_{1/2} \;=\; \Sigma_0 \cup_{M_0} W_0 \;,\; \Omega_0 \;=\; \Sigma_{1/2} \times I \;,\; \Omega_1 \;=\; \Sigma_1 \times I \;.$$

Now apply Proposition 1.2.1 to $(W; M_0, M_1)$: each index $r+1$ handle in $(W; M_0, M_1)$ determines an index $r+1$ right half-handle in of $(\Omega; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$.

Similarly for a left product relative cobordism. □

The situation is more complicated for relative cobordisms $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ which are not boundary products. Of course it is possible to treat the interior and the boundary separately: it is not hard to show that every relative cobordism is a union of a right product and a boundary product, i.e. the union of traces of surgeries on the boundary and interior. But for the applications to the cobordism of Seifert surfaces of codimension 2 embeddings we need a closer connection between the interior and boundary surgeries.

1.3. **Split relative cobordisms.**

Definition 1.3.1. An $(m+2)$-dimensional relative cobordism $\Gamma=(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ is *split* if it is expressed as a union $\Gamma=\Gamma^-\cup\Gamma^+$ of a left and a right product

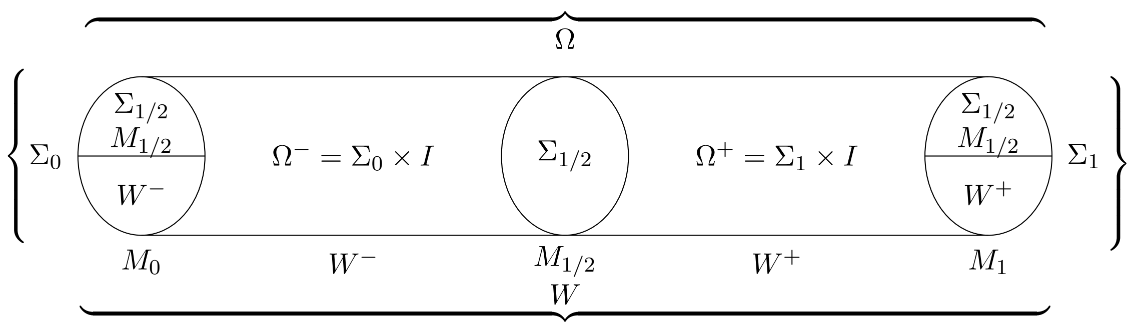
$$\Gamma^- \ = \ (\Omega^-;\Sigma_0,\Sigma_{1/2},W^-;M_0,M_{1/2})\ ,\ \Gamma^+ \ = \ (\Omega^+;\Sigma_{1/2},\Sigma_1,W^+;M_{1/2},M_1)\ ,$$

with embeddings

$$(W^-,M_0)\subset(\Sigma_0,M_0)\ ,\ (W^+,M_1)\subset(\Sigma_1,M_1)$$

and

$$\Omega^- \ = \ \Sigma_0\times I\ ,\ \Omega^+ \ = \ \Sigma_1\times I\ ,$$
$$\Sigma_0 \ = \ \Sigma_{1/2}\cup_{M_{1/2}}W^-\ ,\ \Sigma_1 \ = \ \Sigma_{1/2}\cup_{M_{1/2}}W^+\ ,\ \Sigma_{1/2} \ = \ \text{cl.}(\Sigma_0\backslash W^-) \ = \ \text{cl.}(\Sigma_1\backslash W^+)\ .$$



Remark 1.3.2. A split $(m+2)$-dimensional relative cobordism $\Gamma=\Gamma^-\cup\Gamma^+$ has three ingredients:

(i) An $(m+1)$-dimensional manifold with boundary $(\Sigma_{1/2},M_{1/2})$,
(ii) An $(m+1)$-dimensional cobordism $(W^-;M_0,M_{1/2})$,
(iii) An $(m+1)$-dimensional cobordism $(W^+;M_{1/2},M_1)$.

The inclusion

$$W\cup_{M_{1/2}}\Sigma_{1/2} \ = \ W^-\cup\Sigma_{1/2}\cup W^+\subset\Omega$$

is a homotopy equivalence, with isomorphisms

$$H_*(\Sigma_{1/2}) \ = \ H_{*+1}(\Omega,\Sigma_0\sqcup\Sigma_1)\ ,\ H_*(\Sigma_{1/2},M_{1/2}) \ \cong \ H_*(W\cup_{M_{1/2}}\Sigma_{1/2},W) \ \cong \ H_*(\Omega,W)\ .$$

Definition 1.3.3. (i) A split $(m+2)$-dimensional relative cobordism

$$(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1) = \ (\Omega^-;\Sigma_0,\Sigma_{1/2},W^-;M_0,M_{1/2})\cup(\Omega^+;\Sigma_{1/2},\Sigma_1,W^+;M_{1/2},M_1)$$

is *elementary of index* $r+1$ (with $-1\leqslant r\leqslant m+1$) and of *rank* (ℓ^-_r,ℓ^+_{r+1}) if $(\Omega^-;\Sigma_0,\Sigma_{1/2},W^-;M_0,M_{1/2})$ is the left product obtained as in Definition 1.1.2 (v) from (Σ_0,M_0) by ℓ^-_r left half-surgeries

$$(\bigcup_{\ell^-_r}D^r\times D^{m-r+1},\bigcup_{\ell^-_r}S^{r-1}\times D^{m-r+1})\subset(\Sigma_0,M_0)$$

and $(\Omega^+;\Sigma_{1/2},\Sigma_1,W^+;M_{1/2},M_1)$ is the trace right product obtained from $(\Sigma_{1/2},M_{1/2})$ by ℓ^+_{r+1} right half-surgeries $\bigcup\limits_{\ell^+_{r+1}} S^r\times D^{m-r}\subset M_{1/2}$ (1.1.2 (iii))

$$\begin{aligned} W^- &= M_0\times I\cup_{\bigcup\limits_{\ell^-_r} S^{r-1}\times D^{m-r+1}} \bigcup_{\ell^-_r} D^r\times D^{m-r+1}\ ,\\ W^+ &= M_{1/2}\times I\cup_{\bigcup\limits_{\ell^+_{r+1}} S^r\times D^{m-r}} \bigcup_{\ell^+_{r+1}} D^{r+1}\times D^{m-r}\ . \end{aligned}$$

the traces of the surgeries on $\bigcup\limits_{\ell^-_r} S^{r-1}\times D^{m-r+1}\subset M_0$ and $\bigcup\limits_{\ell^+_{r+1}} S^r\times D^{m-r}\subset M_{1/2}$, and

$$\begin{aligned} \Sigma_0 &= W^-\cup_{M_{1/2}}\Sigma_{1/2}\ ,\ \Sigma_1\ =\ \Sigma_{1/2}\cup_{M_{1/2}}W^+\ ,\\ \Omega^- &= \Sigma_0\times I\ =\ (\Sigma_0\cup_{M_0}W^-)\times I\cup_{\bigcup\limits_{\ell^-_r} S^r\times D^{m-r+1}}\bigcup_{\ell^-_r} D^{r+1}\times D^{m-r+1}\ ,\\ \Omega^+ &= \Sigma_1\times I\ =\ \Sigma_{1/2}\times I\cup_{\bigcup\limits_{\ell^+_{r+1}} S^{r+1}\times D^{m-r}}\bigcup_{\ell^+_{r+1}} D^{r+2}\times D^{m-r}\ . \end{aligned}$$

In the case $r=-1$ it is understood that $\ell^-_{-1}=0$, and in the case $r=m+1$ it is understood that $\ell^+_{m+2}=0$. Note that

$$(\Omega;\Sigma_1,\Sigma_0,W;M_1,M_0)\ =\ (\Omega^+;\Sigma_1,\Sigma_{1/2},W^-;M_1,M_{1/2})\cup(\Omega^-;\Sigma_{1/2},\Sigma_0,W^+;M_{1/2},M_0)$$

is an elementary splitting of index $m-r$, with $(\Sigma_{1/2},M_{1/2})$ is obtained from (Σ_1,M_1) by ℓ^+_{r+1} left half-surgeries

$$(\bigcup_{\ell^+_{r+1}} D^{r+1}\times D^{m-r},\bigcup_{\ell^+_{r+1}} D^{r+1}\times S^{m-r-1})\subset(\Sigma_1,M_1)\ ,$$

and (Σ_0,M_0) is obtained from $(\Sigma_{1/2},M_{1/2})$ by ℓ^-_r right half-surgeries $\bigcup\limits_{\ell^-_r} D^r\times S^{m-r}\subset M_{1/2}$.

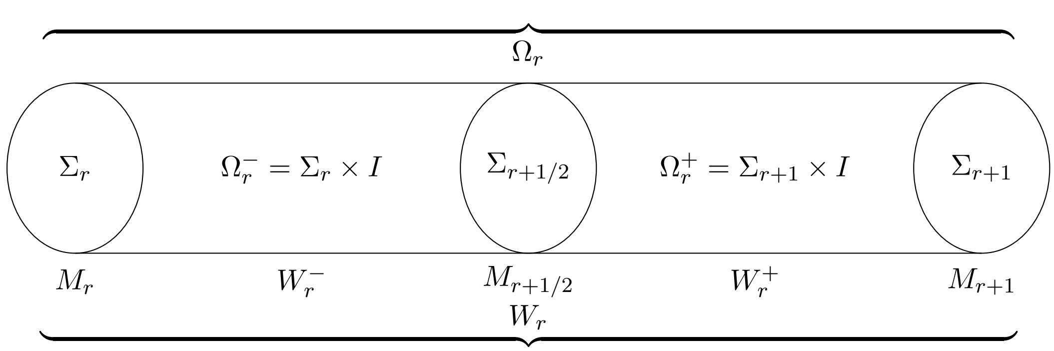
(ii) A *half-handle decomposition* of an $(m+2)$-dimensional relative cobordism $(\Omega;\Sigma,\Sigma',W;M,M')$ is an expression as a union of relative cobordisms

$$(\Omega;\Sigma,\Sigma',W;M,M')\ =\ \bigcup_{r=-1}^{m+1}(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1})$$

with each

$$\begin{aligned} &(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1})\\ &=\ (\Omega^-_r;\Sigma_r,\Sigma_{r+1/2},W^-_r;M_r,M_{r+1/2})\cup(\Omega^+_r;\Sigma_{r+1/2},\Sigma_{r+1},W^+_r;M_{r+1/2},M_{r+1}) \end{aligned}$$

an elementary splitting of index $r+1$.



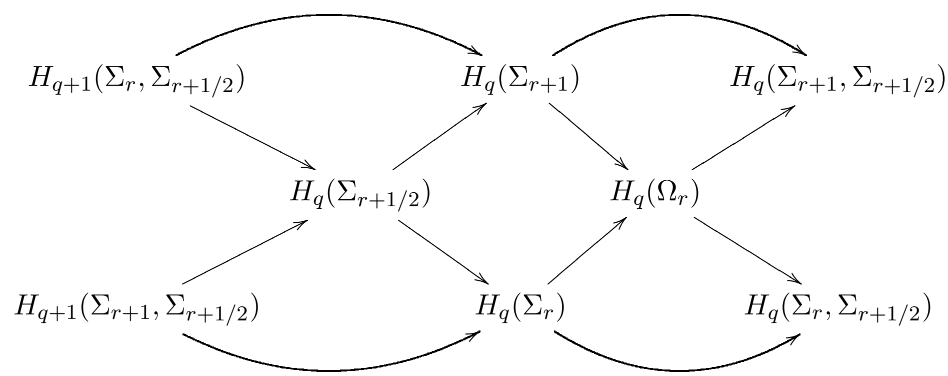
Theorem 1.3.4. (i) *Let*

$$\begin{aligned}&(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1})\\ &=\ (\Omega_r^-;\Sigma_r,\Sigma_{r+1/2},W_r^-;M_r,M_{r+1/2})\cup(\Omega_r^+;\Sigma_{r+1/2},\Sigma_{r+1},W_r^+;M_{r+1/2},M_{r+1})\end{aligned}$$

be an index $r+1$ rank (ℓ_r^-,ℓ_{r+1}^+) elementary splitting of an $(m+2)$-dimensional relative cobordism. The homology and cohomology groups are such that

$$\begin{aligned}&H_q(W_r^-,M_r)\ =\ H^{m+1-q}(W_r^-,M_{r+1/2})\ =\ H_{q+1}(\Omega_r^-,\Sigma_r\cup_{M_r}W_r^-)\\ &=\ H^{m+1-q}(\Omega_r^-,\Sigma_{r+1/2})\ =\ H^{m+1-q}(\Omega_r,\Sigma_{r+1})\ =\ \begin{cases}\mathbb{Z}^{\ell_r^-} & \text{if } q=r\\ 0 & \text{if } q\neq r\ ,\end{cases}\\ &H_q(W_r^+,M_{r+1/2})\ =\ H^{m+1-q}(W_r^+,M_{r+1})\ =\ H^{m+2-q}(\Omega_r^+,W_r^+\cup_{M_{r+1}}\Sigma_{r+1})\\ &=\ H_q(\Omega_r^+,\Sigma_{r+1/2})\ =\ H_q(\Omega_r,\Sigma_r)\ =\ \begin{cases}\mathbb{Z}^{\ell_{r+1}^+} & \text{if } q=r+1\\ 0 & \text{if } q\neq r+1\ .\end{cases}\end{aligned}$$

The homology groups of $\Sigma_r,\Sigma_{r+1/2},\Sigma_{r+1},\Omega_r$ fit into a commutative braid of exact sequences



with

$$H_q(\Sigma_r,\Sigma_{r+1/2})\ =\ H_q(\Omega_r^-,\Sigma_{r+1/2})\ =\ \begin{cases}\mathbb{Z}^{\ell_r^-} & \text{if } q=m-r+1\\ 0 & \text{if } q\neq m-r+1\ ,\end{cases}$$

$$H_q(\Sigma_{r+1},\Sigma_{r+1/2})\ =\ H_q(\Omega_r^+,\Sigma_{r+1/2})\ =\ \begin{cases}\mathbb{Z}^{\ell_{r+1}^+} & \text{if } q=r+1\\ 0 & \text{if } q\neq r+1\ .\end{cases}$$

The $\ell_r^-\times\ell_{r+1}^+$ matrix of the boundary map

$$d\ :\ H_{r+1}(W_r,W_r^-)\ =\ H_{r+1}(W_r^+,M_{r+1/2})\ =\ \mathbb{Z}^{\ell_{r+1}^+}\to H_r(W_r^-,M_r)\ =\ \mathbb{Z}^{\ell_r^-}$$

has entries the homological intersection numbers of the cores

$$(\bigcup_{\ell_r^-}\{0\}\times S^{m-r})\cap(\bigcup_{\ell_{r+1}^+}S^r\times\{0\})\subset M_{r+1/2}\ .$$

It follows from the exact sequence

$$\dots\to H_{q+1}(W_r,M_r)\to H_{q+1}(W_r,W_r^-)\xrightarrow{\ d\ }H_q(W_r^-,M_r)\to H_q(W_r,M_r)\to\dots$$

that

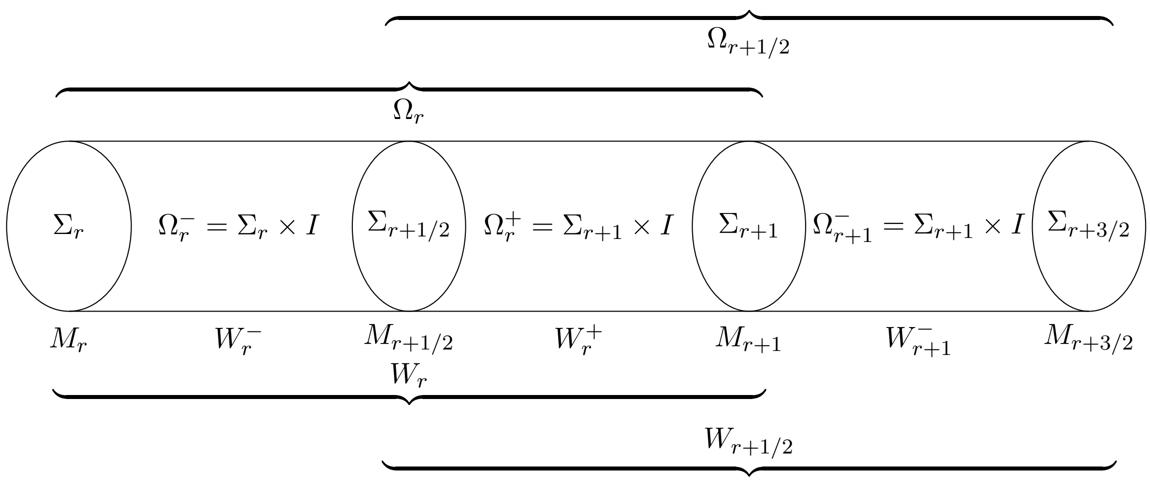
$$H_q(W_r,M_r)\ =\ \begin{cases}\ker(d:\mathbb{Z}^{\ell_{r+1}^+}\to\mathbb{Z}^{\ell_r^-}) & \text{if } q=r+1\\ \operatorname{coker}(d:\mathbb{Z}^{\ell_{r+1}^+}\to\mathbb{Z}^{\ell_r^-}) & \text{if } q=r\\ 0 & \text{otherwise}\ .\end{cases}$$

In particular, $(W_r; M_r, M_{r+1})$ *is an* H*-cobordism if and only if* d *is an isomorphism.*
(ii) *Let*

$$\begin{aligned}(\Omega;\Sigma,\Sigma',W;M,M') \ &= \ \bigcup_{r=0}^{m+1}(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1}) \\ &= \ \bigcup_{r=0}^{m+1}(\Omega_r^-;\Sigma_r,\Sigma_{r+1/2},W_r^-;M_r,M_{r+1/2})\cup(\Omega_r^+;\Sigma_{r+1/2},\Sigma_{r+1},W_r^+;M_{r+1/2},M_{r+1})\end{aligned}$$

be an $(m+2)$*-dimensional relative cobordism with a half-handle decomposition. Define the* $(m+2)$*-dimensional relative cobordisms*

$$\begin{aligned}&(\Omega_{r+1/2};\Sigma_{r+1/2},\Sigma_{r+3/2},W_{r+1/2};M_{r+1/2},M_{r+3/2}) \\ &= \ (\Omega_r^+;\Sigma_{r+1/2},\Sigma_{r+1},W_r^+;M_{r+1/2},M_{r+1})\cup(\Omega_{r+1}^-;\Sigma_{r+1},\Sigma_{r+3/2},W_{r+1}^-;M_{r+1},M_{r+3/2}) \ .\end{aligned}$$



We use the handle decompositions to define three chain complexes C, C^+, C^- *such that*

$$H_*(C) \ = \ H_*(W,M) \ , \ H_*(C^+) \ = \ H_*(\Omega,\Sigma) \ , \ H_*(C^-) \ = \ H_{*+1}(\Omega,\Sigma\cup_M W) \ .$$

The connecting maps of the triples

$$(\,W_{r-3/2}\cup W_{r-1/2}\cup W_{r+1/2}\,,\,W_{r-3/2}\cup W_{r-1/2}\,,\,W_{r-3/2})$$

are the differentials

$$\begin{aligned}\partial \ : \ C_r \ &= \ H_r(W_{r+1/2},M_{r+1/2}) \\ &= \ H_r(W_{r-3/2}\cup W_{r-1/2}\cup W_{r+1/2},W_{r-3/2}\cup W_{r-1/2}) \ = \ \mathbb{Z}^{\ell_r^+ + \ell_{r+1}^-} \\ \to C_{r-1} \ &= \ H_{r-1}(W_{r-1/2},M_{r-1/2}) \ = \ H_{r-1}(W_{r-3/2}\cup W_{r-1/2},W_{r-3/2}) \ = \ \mathbb{Z}^{\ell_{r-1}^+ + \ell_r^-}\end{aligned}$$

of the cellular chain complex $(C,\partial)=C(W,M)$ *of the absolute handlebody decomposition*

$$(W;M,M') \ = \ \bigcup_{r=-1}^{m+1}(W_{r+1/2};M_{r+1/2},M_{r+3/2})$$

as in [Ra4, 2.22]*. The connecting maps of the triples*

$$(\,\Omega_{r-3/2}\cup\Omega_{r-1/2}\cup\Omega_{r+1/2}\,,\,\Omega_{r-3/2}\cup\Omega_{r-1/2}\,,\,\Omega_{r-3/2})$$

are the differentials

$$\begin{aligned} d^+ \ : \ C_r^+ \ &= \ H_r(W_r^+, M_{r+1/2}) \\ &= \ H_r(\Omega_{r-3/2} \cup \Omega_{r-1/2} \cup \Omega_{r+1/2}, \Omega_{r-3/2} \cup \Omega_{r-1/2}) \ = \ \mathbb{Z}^{\ell_r^+} \\ \to C_{r-1}^+ \ &= \ H_{r-1}(W_{r-1}^+, M_{r-1/2}) \ = \ H_{r-1}(\Omega_{r-3/2} \cup \Omega_{r-1/2}, \Omega_{r-3/2}) \ = \ \mathbb{Z}^{\ell_{r-1}^+} \end{aligned}$$

of the cellular chain complex $(C^+, d^+) = C(\Omega, \Sigma)$. The connecting maps of the triples

$$(\, \Omega_{r-3/2} \cup \Omega_{r-1/2} \cup \Omega_{r+1/2} \, , \, \Omega_{r-3/2} \cup \Omega_{r-1/2} \cup W_{r+1/2} \, , \, \Omega_{r-3/2} \cup W_{r-1/2} \cup W_{r+1/2}) \ .$$

are the differentials

$$\begin{aligned} d^- \ : \ C_r^- \ &= \ H_r(W_{r+1}^-, M_{r+1}) \\ &= \ H_{r+1}(\Omega_{r-3/2} \cup \Omega_{r-1/2} \cup \Omega_{r+1/2}, \Omega_{r-3/2} \cup \Omega_{r-1/2} \cup W_{r+1/2}) \ = \ \mathbb{Z}^{\ell_{r+1}^-} \\ \to C_{r-1}^- \ &= \ H_{r-1}(W_r^-, M_r) \\ &= \ H_r(\Omega_{r-3/2} \cup \Omega_{r-1/2} \cup W_{r+1/2} \, , \, \Omega_{r-3/2} \cup W_{r-1/2} \cup W_{r+1/2}) \ = \ \mathbb{Z}^{\ell_r^-} \end{aligned}$$

of the cellular chain complex $(C^-, d^-) = C(\Omega, \Sigma \cup_M W)_{*+1}$. The chain complex (C, ∂) is the algebraic mapping cone of a chain map $d : (C^+, d^+) \to (C^-, d^-)_{*-1}$

$$\begin{aligned} C_r \ &= \ C(W, M)_r \ = \ H_r(W_{r+1/2}, M_{r+1/2}) \ = \ C_r^+ \oplus C_r^- \ = \ \mathbb{Z}^{\ell_r^+ + \ell_{r+1}^-} \ , \\ \partial \ &= \ \begin{pmatrix} d^+ & 0 \\ (-1)^r d & d^- \end{pmatrix} \ : \ C_r \ = \ C_r^+ \oplus C_r^- \to C_{r-1} \ = \ C_{r-1}^+ \oplus C_{r-1}^- \end{aligned}$$

with an exact sequence

$$\dots \longrightarrow H_{r+1}(C^+, d^+) \xrightarrow{\ d\ } H_r(C^-, d^-) \longrightarrow H_r(C, \partial) \longrightarrow H_r(C^+, d^+) \longrightarrow \dots \ .$$

It follows from $H_*(W, M) = H_*(C, \partial) = H_*(d)$ that $(W; M, M')$ is an H-cobordism if and only if d is a chain equivalence.

Remark 1.3.5. See [Ra4, Proposition 8.17] for a proof of the identification of the algebraic and geometric intersection numbers in Theorem 1.3.4.

2. Forms and their enlargements

We describe some algebraic properties of $(-1)^n$-symmetric forms over a ring with involution R, which will be applied to topology in subsequent sections. In the case $R \subseteq \mathbb{C}$ we obtain various estimates for the signature, in terms of ranks and nullities. In particular, we obtain an estimate for the difference of signatures of a form and its enlargement. Later on, in §3 we shall deal with the intersection $(-1)^n$-symmetric form of a $2n$-dimensional manifold with boundary (Σ, M), showing that a $(2n+1)$-dimensional relative cobordism $(\Omega, \Sigma, \Sigma'; W, M, M')$ determines a common enlargement of the intersection forms of (Σ, M) and (Σ', M'). In §6 we use the algebra to obtain a generalized Murasugi–Kawauchi inequality estimating the difference in the Levine–Tristram signatures $|\sigma_M(\xi) - \sigma_{M'}(\xi)|$ for a codimension 1 relative cobordism

$$(\Omega, \Sigma, \Sigma'; W, M, M') \subset S^{2n+1} \times (I; \{0\}, \{1\})$$

of Seifert surfaces $\Sigma, \Sigma' \subset S^{2n+1}$ of codimension 2 embeddings $M, M' \subset S^{2n+1}$.

2.1. **Forms.** We fix a ring R with an involution $R \to R; a \mapsto \bar{a}$.

Notation 2.1.1. The transpose of a $k \times \ell$ matrix

$$A \ = \ (a_{ij}) \ (a_{ij} \in R,\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant \ell)$$

is the $\ell \times k$ matrix $A^* = (\bar{a}_{ij})$.

Definition 2.1.2. Fix a sign $\epsilon = 1$ or -1.
(i) An *ϵ-symmetric form over R (F, B)* is a f.g. free R-module F together with an ϵ-symmetric pairing

$$B \ : \ F \times F \to R \ ; \ (x, y) \mapsto B(x, y) \ = \ \epsilon\overline{B(y, x)} \ .$$

(ii) The *adjoint* of (F, B) is the R-module morphism

$$B \ = \ \epsilon B^* \ : \ F \to F^* \ = \ \mathrm{Hom}_R(F, R) \ ; \ x \mapsto (y \mapsto B(x, y)) \ .$$

(iii) A form (F, B) is *nonsingular* if $B : F \to F^*$ is an isomorphism.

A form (F, B) with $\dim_R F = k$ is essentially the same as a $k \times k$ matrix B over R such that $B = \epsilon B^*$.

Next, let us define the morphisms of forms.

Definition 2.1.3. (i) A *morphism* of ϵ-symmetric forms $j : (F', B') \to (F, B)$ over R is an R-module morphism $j : F' \to F$ such that $j^* B j = B'$, or equivalently if the following diagram commutes:

$$\begin{array}{ccc} F' & \xrightarrow{\ j\ } & F \\ {\scriptstyle B'}\big\downarrow & & \big\downarrow{\scriptstyle B} \\ F'^* & \xleftarrow{\ j^*\ } & F^* \end{array}$$

(ii) A *subform* $(F', B') \subseteq (F, B)$ of an ϵ-symmetric form (F, B) is the ϵ-symmetric form defined on a f.g. free R-submodule $F' \subseteq F$ by the restriction $B' = B|_{F'}$, so that the inclusion $j : F' \to F$ defines a morphism $j : (F', B') \to (F, B)$.
(iii) The *annihilator* of a subform $(F', B') \subseteq (F, B)$ is the subform $(F'^{\perp}, B'^{\perp}) \subseteq (F, B)$ defined by

$$F'^{\perp} \ = \ \{x \in F \,|\, B(x, y) = 0 \in R \text{ for all } y \in F'\} \ = \ \ker(j^* B : F \to F'^*) \subseteq F \ .$$

(iv) The *radical* of an ϵ-symmetric form (F, B) is the annihilator of (F, B) itself

$$(F^{rad}, B^{rad}) \ = \ (\ker(B : F \to F^*), 0) \subseteq (F, B) \ .$$

Proposition 2.1.4. (i) *If $j : (F', B') \to (F, B)$ is a morphism of ϵ-symmetric forms over R then* $\ker j \subseteq F'^{rad}$. *In particular, if (F', B') is nonsingular then* $\ker j = \{0\}$.
(ii) *If (F, B) is an ϵ-symmetric form over R such that $F^{rad} \subseteq F$ is a direct summand (automatically the case if R is a principal ideal domain) then the radical quotient $(F/F^{rad}, [B])$ is a well-defined nonsingular ϵ-symmetric form over R, such that up to isomorphism*

$$(F, B) \ = \ (F^{rad} \oplus (F/F^{rad}), \begin{pmatrix} 0 & 0 \\ 0 & [B] \end{pmatrix}) \ .$$

As usual, isotropic subforms are important for the applications of forms to topology.

Definition 2.1.5. (i) A *sublagrangian* of an ϵ-symmetric form (F, B) is a subform $(L, 0) \subset (F, B)$ such that L is a direct summand of F and $j^* B : F \to L^*$ is onto, with $j : L \to F$ the inclusion. In particular, $L \subseteq L^{\perp}$.
(ii) A *lagrangian* L of (F, B) is a sublagrangian L such that $L^{\perp} = L$.
(iii) A form (F, B) which admits a lagrangian is called *metabolic*.

Example 2.1.6. Every nonsingular (-1)-symmetric form over $\mathbb{Z}$ is metabolic.

Proposition 2.1.7. (i) *The inclusion of a sublagrangian* $j : (L,0) \to (F,B)$ *in an* ϵ*-symmetric form* (F,B) *over* R *extends to an isomorphism of forms*

$$(L^{\perp}/L,[B]) \oplus \mathcal{H} \;\cong\; (F,B)$$

with $\mathcal{H}$ *a metabolic form with lagrangian* L.
(ii) *An* ϵ*-symmetric form* (F,B) *is metabolic with lagrangian* L *if and only if it is isomorphic to* $(L\oplus L^*,\begin{pmatrix} 0 & I \\ \epsilon I & C\end{pmatrix})$ *for some* ϵ*-symmetric form* (L^*,C).
(iii) *Let* $j : (F',B') \to (F,B)$ *be the inclusion of a subform. If* $j^*B : F \to F'^*$ *is onto (e.g. if* (F,B) *is nonsingular and* j *is a split injection) there is defined an isomorphism of forms*

$$(F'^{\perp},B'^{\perp}) \oplus \mathcal{H} \;\cong\; (F',-B') \oplus (F,B)$$

with $\mathcal{H}$ *a metabolic form with lagrangian* F'.

Proof. (i)+(ii) Standard.
(iii) Apply (i) to the inclusion of a sublagrangian

$$\begin{pmatrix} 1 \\ j \end{pmatrix} \;:\; (L,0) \;=\; (F',0) \to (F',-B') \oplus (F,B)\ ,$$

noting that $(L^{\perp}/L,[B]) = (F'^{\perp},B'^{\perp})$. □

Definition 2.1.8. Let (F,B) be an ϵ-symmetric form over R.
(i) A *rank* ℓ *enlargement* of (F,B) is an ϵ-symmetric form over R of the type

$$(F',B') \;=\; (F\oplus L,\begin{pmatrix} B & C \\ \epsilon C^* & D\end{pmatrix})$$

with (L,D) an ϵ-symmetric form over R with $\dim_R L = \ell$ and $C : L \to F^*$ an R-module morphism. Note that (F,B) is a subform of (F',B').
(ii) A *rank* (ℓ^-,ℓ^+) *enlargement* of (F,B) is a rank $\ell^+ + \ell^-$ enlargement of the type

$$(F',B') \;=\; (F\oplus L^-\oplus L^+,\begin{pmatrix} B & 0 & C \\ 0 & 0 & D \\ \epsilon C^* & \epsilon D^* & E\end{pmatrix})$$

with L^+,L^- f.g. free R-modules with $\dim_R L^{\pm} = \ell^{\pm}$ and C,D,E R-module morphisms

$$C \;:\; L^+ \to F^*\ ,\ D \;:\; L^+ \to (L^-)^*\ ,\ E \;=\; \epsilon E^* \;:\; L^+ \to (L^+)^*\ .$$

(iii) An *H-enlargement* is a rank (ℓ,ℓ) enlargement as in (ii) in which D is an R-module isomorphism.

Proposition 2.1.9. (i) *If* L^- *is a sublagrangian of an* ϵ*-symmetric form* (F',B') *then (up to isomorphism)* (F',B') *is an* H*-enlargement of* $(F,B) = ((L^-)^{\perp}/L^-,[B'])$*, with*

$$L^+ \;=\; (L^-)^*\ ,\ (L^-)^{\perp} \;=\; F\oplus L^-\ ,\ (F',B') \;=\; (F\oplus L^-\oplus L^+,\begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & E\end{pmatrix})$$

for some ϵ*-symmetric form* (L^+,E).
(ii) *If* (F',B') *is an* H*-enlargement of* (F,B) *then* L_- *is a sublagrangian of* (F',B') *such that*

$$(L^-)^{\perp} \;=\; F'\oplus L^-\ ,\ ((L^-)^{\perp}/L^-,[B_1]) \;=\; (F,B)\ ,$$

and there is defined an isomorphism of forms $f : (F', B') \cong (F, B) \oplus \mathcal{H}$ *with*

$$\mathcal{H} \;=\; (L^- \oplus L^+, \begin{pmatrix} 0 & D \\ \epsilon D^* & E \end{pmatrix})$$

metabolic with lagrangian L^- *and*

$$f \;=\; \begin{pmatrix} 1 & 0 & 0 \\ (D^*)^{-1}C^* & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;:\; F' \;=\; F \oplus L^- \oplus L^+ \to F \oplus L^- \oplus L^+ \;.$$

Proof. (i) Immediate from Proposition 2.1.7.
(ii) By construction. □

2.2. **Nullity and signature.** In this subsection we work over a subring $R \subseteq \mathbb{C}$. We shall make use of the fact that a submodule $F' \subseteq F$ of a f.g. free R-module F is a f.g. free R-module, with $\mathbb{C} \otimes_R F' \subseteq \mathbb{C} \otimes_R F$ a direct summand.

Definition 2.2.1. The *nullity* of a morphism $B : F \to G$ of f.g. free R-modules is

$$n(B) \;=\; \dim_R \ker B \;.$$

We shall make use of the following standard properties of nullity:

$$n(B : F \to G) \;=\; \dim_R F - \dim_R \operatorname{im} B \;,$$
$$n(B^* : G^* \to F^*) \;=\; \dim_R G - \dim_R F + n(B).$$

The induced morphism of f.g. free $\mathbb{C}$-modules $1 \otimes j : \mathbb{C} \otimes_R F \to \mathbb{C} \otimes_R G$ is such that

$$\ker(1 \otimes B) \;=\; \mathbb{C} \otimes_R \ker(B) \;,\; \operatorname{im}(1 \otimes B) \;=\; \mathbb{C} \otimes_R \operatorname{im}(B) \;,\; n(1 \otimes B) \;=\; n(B) \;.$$

Notation 2.2.2. A 1-symmetric form over $\mathbb{C}$ is called *hermitian*.

Definition 2.2.3. Given an ϵ-symmetric form (F, B) over R define the hermitian form over $\mathbb{C}$

$$(F, B)_{\mathbb{C}} \;=\; \begin{cases} \mathbb{C} \otimes_R (F, B) & \text{if } \epsilon = 1 \\ \mathbb{C} \otimes_R (F, iB) & \text{if } \epsilon = -1 \;. \end{cases}$$

The hermitian form is diagonalizable, and has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R} \subset \mathbb{C}$ with $k = \dim_R F = \dim_{\mathbb{C}}(\mathbb{C} \otimes_R F)$.
(i) The *nullity* of (F, B) is

$$n(F, B) \;=\; n(B : F \to F^*) \;=\; \dim_{\mathbb{C}} F^{rad} \;=\; |\{i \,|\, \lambda_i = 0\}| \geqslant 0$$

with $F^{rad} = \ker(B)$ the radical of (F, B) (2.1.3).
(ii) The *positive* and *negative rank* of (F, B) are

$$r_+(F, B) \;=\; \dim_{\mathbb{C}} F_+ \;=\; |\{i \,|\, \lambda_i > 0\}| \;,$$
$$r_-(F, B) \;=\; \dim_{\mathbb{C}} F_- \;=\; |\{i \,|\, \lambda_i < 0\}| \geqslant 0 \;,$$

with $F_+, F_- \subseteq F$ maximal subspaces such that $(F_+, B|)$ is positive definite and $(F_-, B|)$ is negative definite.
(iii) The *signature* of (F, B) is

$$\sigma(F, B) \;=\; r_+(F, B) - r_-(F, B) \in \mathbb{Z} \;.$$

Note that

$$n(F, B) \;=\; n(\mathbb{C} \otimes_R (F, B)) \;,\; r_\pm(F, B) \;=\; r_\pm(\mathbb{C} \otimes_R (F, B)) \;,\; \sigma(F, B) \;=\; \sigma(\mathbb{C} \otimes_R (F, B)) \;.$$

Example 2.2.4. (i) An ϵ-symmetric form (F,B) over R is nonsingular if and only if $n(F,B)=0$.
(ii) A hermitian form (F,B) over $\mathbb{C}$ is metabolic if and only if it is nonsingular and $\sigma(F,B)=0$.

Remark 2.2.5. The signature is 0 for $(R,\epsilon)=(\mathbb{Z},-1)$ or $(\mathbb{R},-1)$, so is not very helpful in these cases!

Proposition 2.2.6. *Let (F,B) be an ϵ-symmetric form over R.*
(i) *The nullity, ranks and signature of (F,B) are related by the equality*

$$n(F,B)+r_+(F,B)+r_-(F,B) \;=\; \dim_R F$$

and the inequality

$$|\sigma(F,B)|+n(F,B)\leqslant \dim_R F\ .$$

(ii) *The radical quotient nonsingular ϵ-symmetric form $(F/F^{rad},[B])$ has the same ranks and signature as (F,B)*

$$r_+(F/F^{rad}) \;=\; r_+(F,B)\ ,\ r_-(F/F^{rad}) \;=\; r_-(F,B)\ ,\ \sigma(F/F^{rad},[B]) \;=\; \sigma(F,B)\ .$$

(iii) *If L is a sublagrangian of (F,B) then*

$$n(F,B) \;=\; n(L^\perp/L,[B])\ ,\ \sigma(F,B) \;=\; \sigma(L^\perp/L,[B])$$

and

$$|\sigma(F,B)|+n(F,B)\leqslant \dim_R(L^\perp/L) \;=\; \dim_R F-2\dim_R L\ .$$

In particular, if L is a lagrangian then $\sigma(F,B)=0$.
(iv) *For any morphism $j:(F',B')\to(F,B)$*

$$\sigma(F',B') \;=\; \sigma(\mathrm{im}(j),B|) \;=\; \sigma(F,B)-\sigma(F'^\perp,B'^\perp)\in\mathbb{Z}$$

with $(F'^\perp,B'^\perp)=(\ker(j^*B:F\to F'^*),B|)\subseteq(F,B)$.
(v) *For any morphism $j:(F',B')\to(F,B)$ with (F,B) metabolic*

$$|\sigma(F',B')|\leqslant \dim_R F-\dim_R F'+n(F',B')\ .$$

Proof. There is no loss of generality in taking $(R,\epsilon)=(\mathbb{C},1)$.
(i) Immediate from a decomposition

$$(F,B) \;=\; (F_+,B_+)\oplus(F_-,B_-)\oplus(F^{rad},0)$$

with (F_+,B_+) positive definite and (F_-,B_-) negative definite.
(ii)+(iii) Immediate from Proposition 2.1.7.
(iv) Choose a direct complement to $\ker(j)\subseteq F'$, so that up to isomorphism

$$j \;=\; 0\oplus i\ :\ (F',B') \;=\; (\ker(j),0)\oplus(\mathrm{im}(j),B|)\to(F,B)\ .$$

It follows that

$$\sigma(F',B') \;=\; \sigma(\ker(j),0)+\sigma(\mathrm{im}(j),B|) \;=\; \sigma(\mathrm{im}(j),B|)\in\mathbb{Z}\ .$$

The identity $\sigma(F,B)=\sigma(F',B')+\sigma(F'^\perp,B'^\perp)$ holds in the special case when (F,B) is nonsingular: writing $L=F'$, note that

$$\begin{pmatrix}1\\ j\end{pmatrix}:(L,0)\to(F',-B')\oplus(F,B)$$

is the inclusion of a sublagrangian with annihilator

$$\begin{aligned} L^\perp \;&=\; \ker((B'\ \ -j^*B):F'\oplus F\to F'^*)\\ &=\; L\oplus\ker(j^*B:F\to F'^*) \;=\; L\oplus F'^\perp\ , \end{aligned}$$

so that by Proposition 2.1.7 (i)

$$\sigma(F,B)-\sigma(F',B') \;=\; \sigma(L^{\perp}/L,[-B'\oplus B]) \;=\; \sigma(F'^{\perp},B'^{\perp})\in\mathbb{Z}\ .$$

In the general case apply the special case to the morphism of forms

$$J \;=\; \begin{pmatrix} j & 1 \\ 0 & -B\end{pmatrix} \;:\; (F'\oplus F,\begin{pmatrix} B' & 0 \\ 0 & -B\end{pmatrix})\to(F'',B'') \;=\; (F\oplus F^*,\begin{pmatrix} B & 1 \\ 1 & 0\end{pmatrix})$$

with (F'',B'') metabolic, using

$$\begin{array}{l} J^*B'' \;=\; \begin{pmatrix} j^*B & j^* \\ 0 & 1\end{pmatrix} \;:\; F'' \;=\; F\oplus F^*\to F'^*\oplus F^*\ , \\ ((F'\oplus F)^{\perp},(B'\oplus -B)^{\perp}) \;=\; (\ker(J^*B''),B''|) \;=\; (F'^{\perp},B'^{\perp}) \end{array}$$

to obtain

$$\begin{array}{rl} \sigma(F'',B'') & = \; \sigma(F'\oplus F,B'\oplus -B)+\sigma((F'\oplus F)^{\perp},(B'\oplus -B)^{\perp}) \\ & = \; \sigma(F,B)-\sigma(F',B')+\sigma(F^{\perp},B^{\perp}) \;=\; 0\ \in\mathbb{Z}\ . \end{array}$$

(v) Let $i:(L,0)\to(F,B)$ be the inclusion of a lagrangian, so that $L=\ker(i^*B:F\to L^*)$, and define

$$i' \;=\; \text{inclusion} \;:\; L' \;=\; j^{-1}(L)\to F'\ ,\ j' \;=\; j| \;:\; L'\to L$$

such that

$$ji' \;=\; ij' \;:\; L' \;=\; \ker(i^*B'j:F'\to L^*)\to F\ .$$

Then

(a) $F'^{rad} = \ker j' \subseteq L'$, since (F,B) is nonsingular, $j(F'^{rad})\subseteq F^{rad}=\{0\}$ and $L=\ker i^*B$,
(b) $[i]:(L'/F'^{rad},0)\to(F'/F'^{rad},[B])$ is the inclusion of a sublagrangian in the nonsingular ϵ-symmetric form $(F'/F'^{rad},[B'])$,
(c) $i^*Bji'=i^*Bij'=0:L'\to L^*$, so that

$$\begin{array}{rl} \dim_R L' & = \; \dim_R F'-\dim_R \mathrm{im}(i^*Bj:F'\to L^*) \\ & \geqslant \dim_R F'-\dim_R L^* \;=\; \dim_R F'-(\dim_R F)/2\ . \end{array}$$

It now follows from (ii) and (iii) that

$$\begin{array}{rl} \sigma(F',B') \;=\; \sigma(F'/F'^{rad},[B']) & \leqslant \dim_R(F'/F'^{rad})-2\dim_R(L'/F'^{rad}) \\ & = \; \dim_R F'-2\dim_R L'+\dim_R F'^{rad} \\ & \leqslant \dim_R F-\dim_R F'+n(F',B')\ . \end{array}$$

□

Theorem 2.2.7. *Let $(R,\epsilon)=(\mathbb{Z},1)$ or $(\mathbb{R},1)$ or $(\mathbb{C},\pm1)$, and let (F,B) be an ϵ-symmetric form over R with a rank ℓ enlargement*

$$(F',B') \;=\; (F\oplus L,\begin{pmatrix} B & C \\ \epsilon C^* & D\end{pmatrix})\ .$$

(i) *The signatures of (F,B), (F',B') differ by*

$$\sigma(F',B')-\sigma(F,B) \;=\; \sigma(F^{\perp},B^{\perp})$$

with $(F^{\perp},B^{\perp})\subseteq(F',B')$ the annihilator of $(F,B)\subseteq(F',B')$, given by

$$\begin{array}{l} F^{\perp} \;=\; \ker((B\ C):F\oplus L\to F^*)\ , \\ B^{\perp}((x_1,y_1),(x_2,y_2)) \;=\; B(x_1,x_2)+D(y_1,y_2)\ . \end{array}$$

(ii) *The nullities and signatures are related by the inequality*

$$|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\leqslant \ell\ .$$

(iii) *If* $\ell=1$ *then*

$$|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\ =\ 1\ .$$

(iv) *If* (F',B') *is nonsingular then*

$$n(F,B)\ =\ n(F^{\perp},B^{\perp})\ .$$

(v) *If* (F',B') *is metabolic then*

$$\sigma(F',B')\ =\ \sigma(F,B)+\sigma(F^{\perp},B^{\perp})\ =\ 0\ ,\ |\sigma(F,B)|\leqslant \ell-n(F,B)\ .$$

Proof. (i) A special case of Proposition 2.2.6 (iii).
(ii) Without loss of generality may assume $(R,\epsilon)=(\mathbb{C},1)$, so that

$$\begin{aligned}(F,B)\ &=\ (F_+,B_+)\oplus(F_-,B_-)\oplus(F^{rad},0)\\ (F',B')\ &=\ (F'_+,B'_+)\oplus(F'_-,B'_-)\oplus(F'^{rad},0)\end{aligned}$$

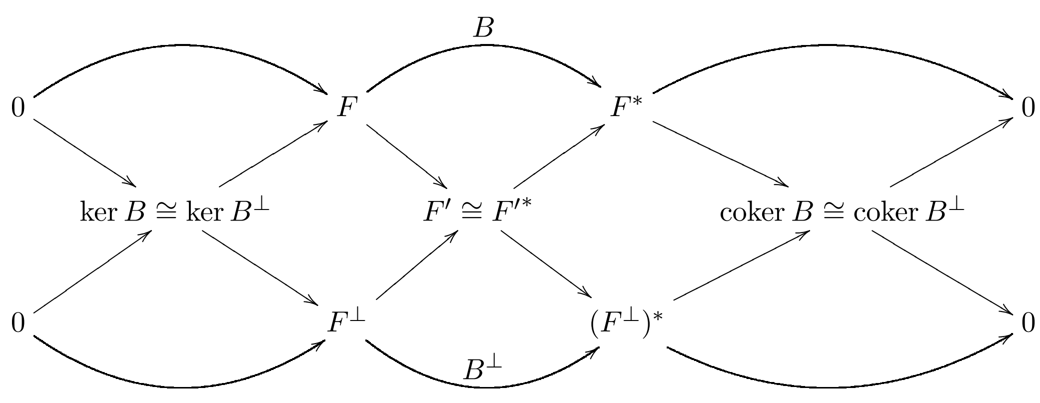
with $(F_+,B_+)\subseteq(F'_+,B'_+)$ positive definite, $(F_-,B_-)\subseteq(F'_-,B'_-)$ negative definite, and $F^{rad}\subseteq F'^{rad}$. It follows from $\dim_{\mathbb{C}}F_\pm=r_\pm(F,B)$, $\dim_{\mathbb{C}}F'_\pm=r_\pm(F',B')$ that

$$\begin{aligned}&|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\\ &=\ \dim_{\mathbb{C}}(F'_+/F_+)-\dim_{\mathbb{C}}(F'_-/F_-)+\dim_{\mathbb{C}}(F'^{rad}/F^{rad})\leqslant\dim_{\mathbb{C}}(F'/F)\ =\ \dim_{\mathbb{C}}(L)\ =\ \ell\ .\end{aligned}$$

(iii) If $L=R$ then

$$n(F',B')\ =\ \begin{cases}1 & \text{if } B^{\perp}=0\in\mathbb{R}\\ 0 & \text{if } B^{\perp}\neq 0\in\mathbb{R}\end{cases}\ ,\ \sigma(F^{\perp},B^{\perp})\ =\ \begin{cases}0 & \text{if } B^{\perp}=0\in\mathbb{R}\\ \mathrm{sign}(B^{\perp}) & \text{if } B^{\perp}\neq 0\in\mathbb{R}\end{cases}$$

(iv) Consider the commutative braid of exact sequences



(v) follows from (i) and (ii), noting that if (F,B) is metabolic then $\sigma(F,B)=0$, $n(F,B)=0$. □

Corollary 2.2.8. *Let* $(R,\epsilon)=(\mathbb{Z},1)$ *or* $(\mathbb{C},\pm1)$, *let* (F,B) *be an* ϵ*-symmetric form over* R, *and let*

$$(F',B')\ =\ (F\oplus L_-\oplus L_+,\begin{pmatrix}B&0&C\\0&0&D\\ \epsilon C^*&\epsilon D^*&E\end{pmatrix})$$

be a rank (ℓ^-,ℓ^+) *enlargement.*
(i) *The signatures of* (F,B), (F',B') *differ by*

$$\sigma(F',B')-\sigma(F,B) ~=~ \sigma(F'',B'')\in \mathbb{Z}$$

with

$$\begin{aligned}(F'',B'') ~&=~ (\ker\begin{pmatrix} B & C \\ 0 & D\end{pmatrix}/\ker(B),[B'])\\ &=~ (\{x\in L_+\,|\,D(x)=0\in L_-^*~,~C(x)\in \operatorname{im}(B)\subseteq F^*\},E|)~.\end{aligned}$$

(ii) *The nullities and signatures are related by the inequalities*

$$\begin{aligned}&|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\leqslant \dim_R L_- + \dim_R L_+~,\\ &|\sigma(F',B')-\sigma(F,B)|\leqslant \min(n(D),n(F,B)+n(F',B'))~.\end{aligned}$$

If (F',B') *is an* H*-enlargement (i.e. if* D *is an isomorphism) then* $F''=0$ *and*

$$n(F,B) ~=~ n(F',B')\geqslant 0~,~\sigma(F',B') ~=~ \sigma(F,B)\in\mathbb{Z}~.$$

(iii) *The determinants of* B *and* B' *with respect to bases for* F,L_-,L_+ *and the dual bases for* F^*,L_-^*,L_+^* *are related by*

$$\det(B') ~=~ -\epsilon\det(B)\det(D)\det(D^*)\in R~.$$

(F',B') *is nonsingular if and only if* (F,B) *is nonsingular and* (F',B') *is an* H*-enlargement of* (F,B) *(i.e.* B' *is an isomorphism if and only if both* B *and* D *are isomorphisms), in which case*

$$\sigma(F,B) ~=~ \sigma(F',B')\in\mathbb{Z}~.$$

Thus any H*-enlargement of a nonsingular form is a nonsingular form with the same signature.*

Proof. (i) By Proposition 2.2.6 with $j=$ inclusion : $(F,B)\to(F',B')$ we have

$$\sigma(F',B')-\sigma(F,B) ~=~ \sigma(F^\perp,B^\perp)\in\mathbb{Z}$$

with

$$\begin{aligned}(F^\perp,B^\perp) ~&=~ (\ker((B~0~C):F\oplus L_-\oplus L_+\to F^*),B'|) ~=~ (L_-\oplus L'_+,\begin{pmatrix} 0 & C' \\ \epsilon C'^* & E'\end{pmatrix})~,\\ L'_+ ~&=~ \ker((B~C):F\oplus L_+\to F^*)~,\\ C' ~&:~ L'_+\to L_-^*~;~(x,y)\mapsto (z\mapsto D(y,z))~,\\ E' ~&:~ L'_+\to L'^*_+~;~(x',y')\mapsto ((x_2,y_2)\mapsto B(x',x_2)+E(y',y_2))~.\end{aligned}$$

The annihilator of $(\ker(B)\oplus L_-,0)\subseteq(F^\perp,B^\perp)$ is

$$((\ker(B)\oplus L_-)^\perp,0^\perp) ~=~ (\ker(B)\oplus L_-,0)\oplus(F'',B'')$$

so that applying Lemma 2.1.7 again

$$\begin{aligned}\sigma(F',B')-\sigma(F,B) ~&=~ \sigma(F^\perp,B^\perp)\\ &=~ \sigma(\ker(B)\oplus L_-,0)+\sigma(F'',B'')\\ &=~ \sigma(F'',B'')\in\mathbb{Z}~.\end{aligned}$$

(ii) The inequality

$$|\sigma(F',B')-\sigma(F,B)|+|n(F',B')-n(F,B)|\leqslant \dim_R L_- + \dim_R L_+$$

is a special case of Theorem 2.2.7 (ii). It is immediate from (i) and $F''\subseteq\ker(D)$ that

$$|\sigma(F'',B'')|\leqslant \dim_R F''\leqslant \dim_R\ker(D) ~=~ n(D)~.$$

The ϵ-symmetric form

$$(H,\theta) \;=\; (F\oplus F^*\oplus L_+\oplus L_+^*, \begin{pmatrix} -B & 1 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & E & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix})$$

is metabolic. Let $(P,\phi)\subseteq(H,\theta)$ be the image of the morphism of forms

$$f \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \epsilon C^* & \epsilon D^* & 0 \end{pmatrix} \;:\; (\mathcal{F},\mathcal{B}) \;=\; (F\oplus F', -B\oplus B') \to (H,\theta)$$

and let

$$\begin{aligned} J \;&=\; \ker(f) \;\cong\; \ker(\begin{pmatrix} B & 0 \\ \epsilon C^* & \epsilon D^* \end{pmatrix} : F\oplus L_- \to F^*\oplus L_+^*)\ , \\ K \;&=\; \ker(\mathcal{B}) \;=\; \ker(B)\oplus\ker(B')\ . \end{aligned}$$

By Proposition 2.2.6

$$\sigma(F',B') - \sigma(F,B) \;=\; \sigma(\mathcal{F},\mathcal{B}) \;=\; \sigma(P,\phi)\in\mathbb{Z}\ .$$

The image of the injection

$$\begin{aligned} g \;&=\; \begin{pmatrix} 1 & 0 \\ B & 0 \\ 0 & 1 \\ 0 & -E \end{pmatrix}| \\ &:\; (Q,\psi) \;=\; (\ker(\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} : F\oplus L_+ \to F^*\oplus L_-^*), \begin{pmatrix} B & 0 \\ 0 & -E \end{pmatrix}|) \to (H,\theta) \end{aligned}$$

is such that

$$(Q,\psi) \;=\; (P,\phi)^{\perp}\subseteq(H,\theta)\ ,$$

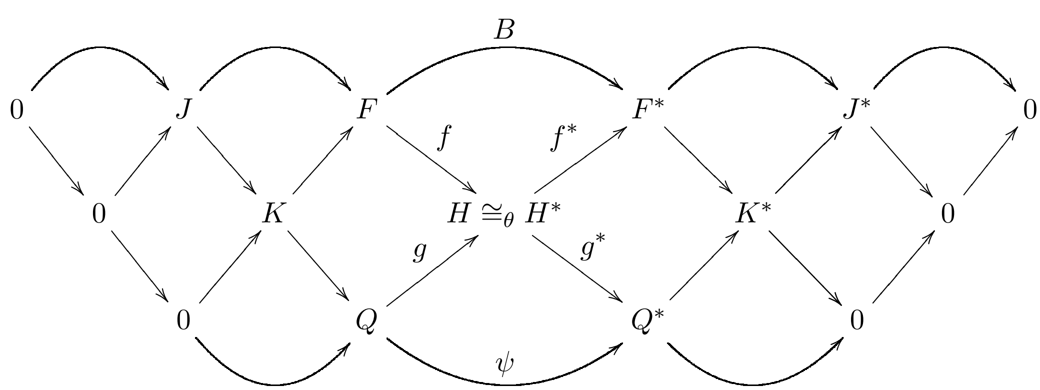
so that

$$\dim_R P + \dim_R Q \;=\; \dim_R H$$

and by Proposition 2.2.6

$$\sigma(P,\phi)+\sigma(Q,\psi) \;=\; \sigma(H,\theta) \;=\; 0\in\mathbb{Z}\ .$$

Consider the commutative braid of exact sequences



It follows from the short exact sequences

$$0\to J\to F\to P\to 0\ ,\ 0\to Q\to H\to P^*\to 0\ ,\ 0\to J\to K\to \ker(\psi)\to 0\ ,$$

and $\dim_R F = \dim_R H$ that

$$\dim_R J \;=\; \dim_R F - \dim_R P \;=\; \dim_R H - \dim_R P \;=\; \dim_R Q \leqslant \dim_R K$$

so that by Proposition 2.2.6

$$|\sigma(F',B') - \sigma(F,B)| \;=\; |\sigma(Q,\psi)| \leqslant \dim_R Q \leqslant \dim_R K \;=\; n(F,B) + n(F',B') \ .$$

(iii) Immediate from (ii). □

3. The intersection form of a manifold with boundary

3.1. **Torsion-free homology.** The intersection $(-1)^n$-symmetric form $(F_n(\Sigma), B(\Sigma))$ of a $2n$-dimensional manifold with boundary $(\Sigma, \partial\Sigma)$ is defined on the torsion-free quotient $F_n(\Sigma) = H_n(\Sigma)/\text{torsion}$. In order to study the behaviour of the intersection form under cobordism we bring together some of the properties of the torsion-free quotients of homology groups.

Definition 3.1.1. The *torsion subgroup* and the *torsion-free quotient* of an abelian group A are the abelian groups

$$\begin{aligned} T(A) \;&=\; \{a \in A \,|\, ka = 0 \in A \text{ for some } k \neq 0 \in \mathbb{Z}\} \subseteq A \ , \\ F(A) \;&=\; A/T(A) \ . \end{aligned}$$

If A is f.g. then $T(A)$ is finite, and $F(A)$ is f.g. free.

Proposition 3.1.2. (i) *The short exact sequence*

$$0 \to T(A) \to A \to F(A) \to 0$$

is natural. A $\mathbb{Z}$-module morphism $f : A \to B$ induces a natural transformation of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(A) & \longrightarrow & A & \longrightarrow & F(A) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle T(f)} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle F(f)} & & \\ 0 & \longrightarrow & T(B) & \longrightarrow & B & \longrightarrow & F(B) & \longrightarrow & 0 \end{array}$$

with a snake lemma exact sequence

$$0 \to \ker(T(f)) \to \ker(f) \to \ker(F(f)) \to \operatorname{coker}(T(f)) \to \operatorname{coker}(f) \to \operatorname{coker}(F(f)) \to 0 \ .$$

In particular, if $F(f) : F(A) \to F(B)$ is an isomorphism then $f : A \to B$ is an isomorphism modulo torsion, i.e. $\ker(f)$ *and* $\operatorname{coker}(f)$ *are both torsion modules, with*

$$\ker(f) \;=\; \ker(T(f)) \ , \ \operatorname{coker}(f) \;=\; \operatorname{coker}(T(f)) \ .$$

(ii) *If*

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

is an exact sequence of $\mathbb{Z}$-modules then

$$\ker(F(g) : F(B) \to F(C)) \;=\; \{x \in F(B) \,|\, kx \in \operatorname{im}(F(f)) \textit{ for some } k \neq 0 \in \mathbb{Z}\}$$

and $\ker(F(g))/\operatorname{im}(F(f))$ *is a torsion $\mathbb{Z}$-module.*
(iii) *If B is a f.g. free $\mathbb{Z}$-module and $A \subseteq B$ is a submodule (not necessarily a direct summand), then A is a f.g. free $\mathbb{Z}$-module, and*

$$A_0 \;=\; \{x \in B \,|\, kx \in A \textit{ for some } k \neq 0 \in \mathbb{Z}\} \subseteq B$$

is a direct summand such that $A_0 \subseteq A$, A/A_0 is finite, $\dim_{\mathbb{Z}} A_0 = \dim_{\mathbb{Z}} A$, *and*

$$\mathbb{Q} \otimes_{\mathbb{Z}} A_0 \;=\; \mathbb{Q} \otimes_{\mathbb{Z}} A \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} B \ .$$

(iv) *If $f : A \to B$ is a morphism of f.g. free $\mathbb{Z}$-modules then* $\mathrm{im}(f) \subseteq B$ *is a f.g. free submodule, and* $\ker(f) \subseteq A$ *is a f.g. free submodule which is a direct summand. Furthermore*

$$\begin{aligned}\mathrm{im}(1\otimes f : \mathbb{Q}\otimes_{\mathbb{Z}} A \to \mathbb{Q}\otimes_{\mathbb{Z}} B) &\cong \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{im}(f) ~,\\ \ker(1\otimes f : \mathbb{Q}\otimes_{\mathbb{Z}} A \to \mathbb{Q}\otimes_{\mathbb{Z}} B) &\cong \mathbb{Q}\otimes_{\mathbb{Z}} \ker(f) ~.\end{aligned}$$

Proof. Standard homological algebra. □

Definition 3.1.3. Given a $\mathbb{Z}$-module chain complex C let

$$F_r(C) ~=~ F(H_r(C)) ~,~ F^r(C) ~=~ F(H^r(C))$$

with $H^r(C) = H_{-r}(\mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z}))$.

Proposition 3.1.4. (i) *For a finite f.g. free $\mathbb{Z}$-module chain complex C the groups $H_r(C)$, $H^r(C)$ are f.g. $\mathbb{Z}$-modules, and*

$$\begin{aligned}F(H^r(C)) &\cong H_r(C)^* \cong F(H_r(C))^*\\ H_r(\mathbb{Q}\otimes_{\mathbb{Z}} C) &\cong \mathbb{Q}\otimes_{\mathbb{Z}} H_r(C) \cong \mathbb{Q}\otimes_{\mathbb{Z}} F_r(C) ~.\end{aligned}$$

(ii) *For a chain map $f : C \to D$ of finite f.g. free $\mathbb{Z}$-module chain complexes the following conditions are equivalent:*

(a) *the induced $\mathbb{Z}$-module morphisms $f_* : H_*(C) \to H_*(D)$ are isomorphisms modulo torsion*

(b) *the induced $\mathbb{Z}$-module morphisms $F(f_*) : F(H_*(C)) \to F(H_*(D))$ are isomorphisms,*

(c) *the induced $\mathbb{Q}$-module morphisms $(1\otimes f)_* : H_*(\mathbb{Q}\otimes_{\mathbb{Z}} C) \to H_*(\mathbb{Q}\otimes_{\mathbb{Z}} D)$ are isomorphisms.*

Proof. Standard. □

The homology groups of a CW complex K

$$H_r(K) ~=~ H_r(C(K))$$

are the homology $\mathbb{Z}$-modules of the cellular chain complex $C(K)$, the $\mathbb{Z}$-module chain complex with

$$C_r(K) ~=~ H_r(K^{(r)}, K^{(r-1)})$$

the free $\mathbb{Z}$-module generated by the r-cells of K. The cohomology groups

$$H^r(K) ~=~ H_{-r}(\mathrm{Hom}_{\mathbb{Z}}(C(K),\mathbb{Z}))$$

are such that there are defined $\mathbb{Z}$-module morphisms

$$\begin{aligned}&H^r(K) \to H_r(K)^* ~=~ \mathrm{Hom}_{\mathbb{Z}}(H_r(K),\mathbb{Z}) ~;~ f \mapsto (x \mapsto f(x)) ~,\\ &H_n(K)\otimes_{\mathbb{Z}} H^r(K) \to H_{n-r}(K) ~;~ x\otimes y \mapsto x\cap y\end{aligned}$$

and a bilinear pairing

$$H_r(K)\times H^r(K) \to \mathbb{Z} ~;~ (x,f) \mapsto f(x) ~.$$

Write the torsion-free quotients as

$$F_r(K) ~=~ F(H_r(K)) ~,~ F^r(K) ~=~ F(H^r(K)) ~.$$

Proposition 3.1.5. (i) *For a map $f : K \to L$ of CW complexes*

$$\begin{aligned}&\ker(f_* : F_n(K) \to F_n(L))\\ &\quad\cong \{x \in H_n(K) \,|\, kx \in \ker(f_* : H_n(K) \to H_n(L)) \text{ for some } k \neq 0 \in \mathbb{Z}\}/T_n(K) ~,\\ &\mathrm{im}(f_* : F_n(K) \to F_n(L))\\ &\quad\cong \{y \in H_n(L) \,|\, \ell y \in \mathrm{im}(f_* : H_n(K) \to H_n(L)) \text{ for some } \ell \neq 0 \in \mathbb{Z}\}/T_n(L) ~.\end{aligned}$$

The following conditions are equivalent:

(a) *the induced* $\mathbb{Z}$*-module morphisms* $f_* : H_*(K) \to H_*(L)$ *are isomorphisms modulo torsion,*
(b) *the induced* $\mathbb{Z}$*-module morphisms* $f_* : F_*(K) \to F_*(L)$ *are isomorphisms,*
(c) *the induced* $\mathbb{Q}$*-module morphisms* $f_* : H_*(K;\mathbb{Q}) \to H_*(L;\mathbb{Q})$ *are isomorphisms.*

(ii) *For a finite CW complex K the groups* $H_r(K)$, $H^r(K)$ *are f.g.* $\mathbb{Z}$*-modules, and*

$$F^r(K) \cong F_r(K)^* \ , \ H_r(K;\mathbb{Q}) \cong \mathbb{Q}\otimes_{\mathbb{Z}} F_r(K) \ , \ H^r(K;\mathbb{Q}) \cong \mathbb{Q}\otimes_{\mathbb{Z}} F^r(K) \ .$$

Proof. Immediate from Propositions 3.1.2, 3.1.4. □

3.2. **The intersection form of a manifold with boundary.** Let (Σ, M) be an oriented $(m+1)$-dimensional manifold with boundary. Cap product with the fundamental class $[\Sigma] \in H_{m+1}(\Sigma, M)$ defines the Poincaré-Lefschetz duality isomorphisms

$$[\Sigma]\cap - \ : \ H^r(\Sigma, M) \cong H_{m+1-r}(\Sigma) \ , \ [\Sigma]\cap - \ : \ H^r(\Sigma) \cong H_{m+1-r}(\Sigma, M)$$

in the usual manner.

Definition 3.2.1. The *intersection pairing* of an $(m+1)$-dimensional manifold with boundary (Σ, M) is the pairing

$$B(\Sigma) \ : \ H_r(\Sigma) \times H_{m+1-r}(\Sigma) \to \mathbb{Z}$$

with adjoint the composite

$$H_r(\Sigma) \longrightarrow H_r(\Sigma, M) \cong H^{m+1-r}(\Sigma) \longrightarrow H_{m+1-r}(\Sigma)^* \ ,$$

corresponding to the evaluation of the cup product pairing on the fundamental class $[\Sigma] \in H_{m+1}(\Sigma, M)$

$$H^{m+1-r}(\Sigma, M) \times H^r(\Sigma, M) \xrightarrow{\ \cup\ } H^{m+1}(\Sigma, M) \xrightarrow{\langle[\Sigma], -\rangle} \mathbb{Z} \ .$$

The intersection pairing is such that

$$B(\Sigma)(x, y) = (-1)^{r(m+1-r)} B(\Sigma)(y, x) \in \mathbb{Z} \ (x \in H_r(\Sigma), y \in H_{m+1-r}(\Sigma)) \ .$$

The intersection pairing takes 0 values on torsion homology classes, so there is induced an intersection pairing on the torsion-free quotients

$$B(\Sigma) \ : \ F_r(\Sigma) \times F_{m+1-r}(\Sigma) \to \mathbb{Z} \ .$$

It follows from the exact sequence of $\mathbb{Z}$-modules

$$\dots \to H_r(M) \to H_r(\Sigma) \to H_r(\Sigma, M) \to H_{r-1}(M) \to \dots$$

that there is defined an exact sequence of $\mathbb{Q}$-modules

$$\dots \to \mathbb{Q}\otimes_{\mathbb{Z}} F_r(M) = H_r(M;\mathbb{Q}) \to \mathbb{Q}\otimes_{\mathbb{Z}} F_r(\Sigma) = H_r(\Sigma;\mathbb{Q})$$

$$\xrightarrow{1\otimes B(\Sigma)} \mathbb{Q}\otimes_{\mathbb{Z}} F_r(\Sigma)^* = H_r(\Sigma, M;\mathbb{Q}) \to \mathbb{Q}\otimes_{\mathbb{Z}} F_{r-1}(M) = H_{r-1}(M;\mathbb{Q}) \to \dots \ .$$

If $M = \emptyset$ or S^m the adjoint $\mathbb{Z}$-module morphisms

$$B(\Sigma) \ : \ F_r(\Sigma) \to F_{m+1-r}(\Sigma)^* \ ; \ x \mapsto (y \mapsto B(x, y))$$

are isomorphisms.

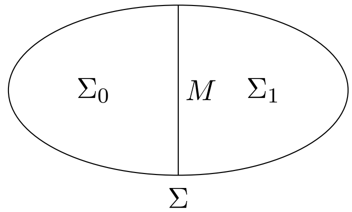
Definition 3.2.2. The *intersection form* of a $2n$-dimensional manifold with boundary (Σ, M) is the $(-1)^n$-symmetric form $(F_n(\Sigma), B(\Sigma))$ over $\mathbb{Z}$.

Example 3.2.3. The intersection $(-1)^n$-symmetric form $(F_n(\Sigma), B(\Sigma))$ is nonsingular in either of the following cases

(i) Σ is closed, i.e. $M = \emptyset$,

(ii) $M = \bigsqcup_{\ell} S^{2n-1}$ for any $n, \ell \geqslant 1$, with $\ell = 1$ if $n = 1$.

Proposition 3.2.4. *Let Σ be a closed $2n$-dimensional manifold such that $\Sigma = \Sigma_0 \cup_M \Sigma_1$ for codimension 0 submanifolds with boundary (Σ_0, M), $(\Sigma_1, M) \subset \Sigma$.*



(i) *The restrictions of the nonsingular $(-1)^n$-symmetric intersection form $(F, B) = (F_n(\Sigma), B(\Sigma))$ over $\mathbb{Z}$ to the direct summands of F*

$$\begin{aligned} F_0 &= \ker(F \to F_n(\Sigma, \Sigma_0)) = \{x \in F \,|\, kx \in \mathrm{im}(F_n(\Sigma_0)) \textit{ for some } k \neq 0 \in \mathbb{Z}\} , \\ F_1 &= \ker(F \to F_n(\Sigma, \Sigma_1)) = \{x \in F \,|\, kx \in \mathrm{im}(F_n(\Sigma_1)) \textit{ for some } k \neq 0 \in \mathbb{Z}\} \end{aligned}$$

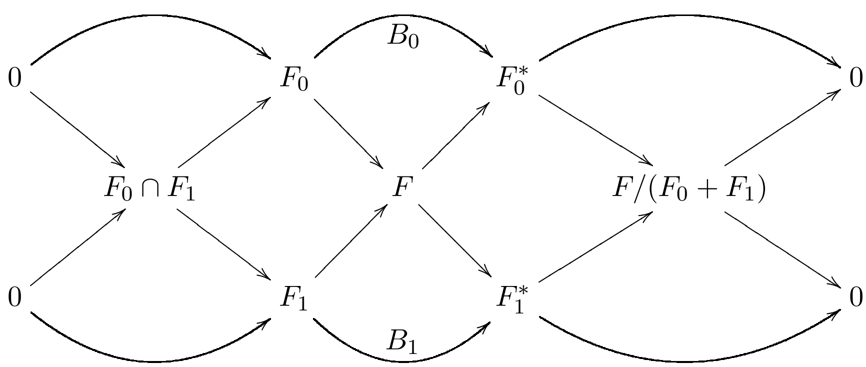
define $(-1)^n$-symmetric forms (F_0, B_0), (F_1, B_1) over $\mathbb{Z}$ such that the morphisms of forms over $\mathbb{Z}$

$$(F_n(\Sigma_0), B(\Sigma_0)) \to (F_0, B_0) \ , \ (F_n(\Sigma_1), B(\Sigma_1)) \to (F_1, B_1)$$

induce surjections of forms over $\mathbb{Q}$

$$\begin{aligned} \mathbb{Q} \otimes_{\mathbb{Z}} (F_n(\Sigma_0), B(\Sigma_0)) &\twoheadrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (F_0, B_0) \ , \\ \mathbb{Q} \otimes_{\mathbb{Z}} (F_n(\Sigma_1), B(\Sigma_1)) &\twoheadrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (F_1, B_1) \ . \end{aligned}$$

(ii) *There is defined a commutative braid of exact sequences*



so that

$$(F_0, B_0)^{\perp} = (F_1, B_1) \ , \ (F_1, B_1)^{\perp} = (F_0, B_0) \subseteq (F, B)$$

and by Proposition 2.1.7 (iii) there are defined isomorphism of forms

$$\begin{aligned} (F_1, B_1) \oplus \mathcal{H}_0 &\cong (F_0, -B_0) \oplus (F, B) \ , \\ (F_0, B_0) \oplus \mathcal{H}_1 &\cong (F_1, -B_1) \oplus (F, B) \end{aligned}$$

with $\mathcal{H}_i$ metabolic with lagrangian F_i.

3.3. **The signature of a manifold with boundary.**

Definition 3.3.1. The *signature* of a $4k$-dimensional manifold with boundary (Σ, M) is the signature of the intersection symmetric form over $\mathbb{Z}$

$$\sigma(\Sigma) \;=\; \sigma(F_{2k}(\Sigma), B(\Sigma)) \in \mathbb{Z}\ .$$

Example 3.3.2. If $(\Sigma, M) = (K \times I, \partial(K \times I))$ for a $(4k-1)$-dimensional manifold with boundary $(K, \partial K)$ then the intersection form is

$$B(\Sigma) \;=\; 0 \;:\; F_{2k}(\Sigma) \;=\; F_{2k}(K) \to F_{2k}(\Sigma, M) \;=\; F_{2k-1}(K) \;=\; F_{2k}(K)^*$$

and the signature is

$$\sigma(\Sigma) \;=\; \sigma(F_{2k}(\Sigma), 0) \;=\; 0 \in \mathbb{Z}\ .$$

Proposition 3.3.3. (i) (Thom [Th]) *For a $(2n+1)$-dimensional manifold with boundary (Ω, Σ) the $(-1)^n$-symmetric intersection form $(F_n(\Sigma), B(\Sigma))$ has a lagrangian* $\ker(F_n(\Sigma) \to F_n(\Omega))$. *If $n = 2k$ then Σ^{4k} has signature*

$$\sigma(\Sigma) \;=\; 0 \in \mathbb{Z}\ .$$

(ii) (Novikov additivity) *If Σ is a closed $4k$-dimensional manifold such that $\Sigma = \Sigma_0 \cup_\partial -\Sigma_1$ for codimension 0 submanifolds with boundary (Σ_0, M), $(\Sigma_1, M) \subset \Sigma$ with $\Sigma_0 \cap \Sigma_1 = M$ then*

$$\sigma(\Sigma) \;=\; \sigma(\Sigma_0) - \sigma(\Sigma_1) \in \mathbb{Z}\ .$$

(iii) *If $(\Omega^{4k+1}; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a $(4k+1)$-dimensional relative cobordism then*

$$\sigma(\Sigma_0) - \sigma(\Sigma_1) \;=\; \sigma(W) \in \mathbb{Z}\ .$$

If $(W; M_0, M_1)$ is an H-cobordism then $\sigma(W) = 0$ and

$$\sigma(\Sigma_0) \;=\; \sigma(\Sigma_1) \in \mathbb{Z}\ .$$

Proof. (i) $\ker(F_{2k}(\Sigma) \to F_{2k}(\Omega))$ is a lagrangian of $(F_{2k}(\Sigma), B(\Sigma))$ (Proposition 2.2.6 (iii)).
(ii) The subforms

$$(F_0, B_0) \;=\; \mathrm{im}(F_{2k}(\Sigma_0), B(\Sigma_0))\ ,\ (F_1, B_1) \;=\; \mathrm{im}(F_{2k}(\Sigma_1), B(\Sigma_1)) \subseteq (F_{2k}(\Sigma), B(\Sigma))$$

are such that $(F_0^\perp, B_0^\perp) = (F_1, B_1)$, $(F_1^\perp, B_1^\perp) = (F_0, B_0)$.
(iii) By (i) and (ii) the signature of $\partial\Omega = \Sigma_0 \cup_{M_0} W \cup_{M_1} -\Sigma_1$ is

$$\sigma(\partial\Omega) \;=\; \sigma(\Sigma_0) + \sigma(W) - \sigma(\Sigma_1) \;=\; 0 \in \mathbb{Z}\ .$$

If $(W; M_0, M_1)$ is an H-cobordism then $B(W) = 0$ (as in 3.3.2). □

Example 3.3.4. (i) Let $\Gamma = (\Omega; \Sigma, \Sigma', W; M, M')$ be a split $(2n+2)$-dimensional relative cobordism. with $\Gamma = \Gamma^- \cup \Gamma^+$ the union of a left and a right product cobordism

$$\Gamma^- \;=\; (\Omega^-; \Sigma, \Sigma'', W^-; M, M'')\ ,\ \Gamma^+ \;=\; (\Omega^+; \Sigma', \Sigma'', W^+; M'', M')\ .$$

The $(-)^n$-symmetric intersection forms $(F_n(M), B(M))$, $(F_n(M'), B(M'))$, $(F_n(M''), B(M''))$ have lagrangians

$$\begin{aligned} L \;&=\; \ker(F_n(M) \to F_n(\Sigma))\ ,\\ L' \;&=\; \ker(F_n(M') \to F_n(\Sigma'))\ ,\\ L'' \;&=\; \ker(F_n(M'') \to F_n(\Sigma'')) \end{aligned}$$

and $(F_n(M''), B(M''))$ also has lagrangians

$$\begin{aligned} L^- \;&=\; \ker(F_n(M'') \to F_n(\Sigma \cup_M W^-))\ ,\\ L^+ \;&=\; \ker(F_n(M'') \to F_n(W^+ \cup_{M'} \Sigma'))\ . \end{aligned}$$

The homology group $H_{n+1}(\Omega)$ fits into an exact sequence

$$0 \to \mathrm{coker}(H_{n+2}(W^-, M'') \oplus H_{n+2}(W^+, M'') \to H_{n+1}(\Sigma'')) \to H_{n+1}(\Omega)$$
$$\to \ker(H_{n+1}(W^-, M'') \oplus H_{n+1}(W^+, M'') \to H_n(\Sigma'')) \to 0 \ .$$

If the condition

$$\mathrm{coker}(H_{n+2}(W^-, M'') \oplus H_{n+2}(W^+, M'') \to H_{n+1}(\Sigma'')) \text{ is torsion} \qquad (*)$$

is satisfied then the $(-1)^{n+1}$-symmetric intersection form $(F_{n+1}(\Omega), B(\Omega))$ is the form $(F'', B''; L'', L^-, L^+)$ determined (by the even-dimensional algebraic analogue of Remark 1.3.2) by the $(-1)^n$-symmetric intersection form $(F'', B'') = (F_n(M''), B(M''))$ and the inclusions of the three lagrangians

$$j^- : (L^-, 0) \to (F'', B'') \ , \ j'' : (L'', 0) \to (F'', B'') \ , \ j^+ : (L^+, 0) \to (F'', B'') \ .$$

The $(-1)^{n+1}$-symmetric form defined by

$$(F'', B''; L'', L^-, L^+) \ =$$
$$(\ker((j^- \ j'' \ j^+) : L^- \oplus L'' \oplus L^+ \to F''), \begin{pmatrix} 0 & j^{-*}B''j'' & j^{-*}B''j^+ \\ -j''^*B''j^- & 0 & j''^*B''j^+ \\ -j^{+*}B''j^- & -j^{+*}B''j'' & 0 \end{pmatrix})$$

is such that

$$(F_{n+1}(\Omega), B(\Omega)) \ = \ (F'', B''; L^-, L'', L^+) \ .$$

In the case $n = 2k-1$ the signature of the $4k$-dimensional manifold with boundary $(\Omega, \partial\Omega)$ is

$$\begin{aligned} \sigma(\Omega) \ &= \ \sigma(\Omega^+) + \sigma(\Omega^-) + \sigma(F'', B''; L^-, L'', L^+) \\ &= \ \sigma(F'', B''; L^-, L'', L^+) \in \mathbb{Z} \end{aligned}$$

since $\sigma(\Omega^+) = \sigma(\Omega^-) = 0$ by Example 3.3.2. The signature $\sigma(F'', B''; L^-, L'', L^+)$ is the invariant of Wall [Wa] for the non-additivity of the signature.

(ii) Let $(\Omega, \partial\Omega)$ be a $(2n+2)$-dimensional manifold with non-empty boundary, choose an embedding $D^{2n+1} \times \{0,1\} \subset \partial\Omega$ and let $\Gamma = (\Omega; \Sigma, \Sigma', W; M, M')$ be the $(2n+2)$-dimensional relative cobordism defined by

$$\begin{aligned} (\Sigma, M) \ &= \ (D^{2n+1}, S^{2n}) \times \{0\} \ , \\ (\Sigma', M') \ &= \ (D^{2n+1}, S^{2n}) \times \{1\} \ , \\ W \ &= \ \mathrm{cl.}(\partial\Omega \backslash (D^{2n+1} \times \{0,1\})) \end{aligned}$$

Assume that Ω is n-connected, so that Σ'', M'', $\partial\Omega$ and W are $(n-1)$-connected (meaning that each component is $(n-1)$-connected) and the condition $(*)$ in (i) is satisfied, since $H_{n+1}(\Sigma'') = 0$. Write the $(-1)^{n+1}$-symmetric intersection form as

$$(F_{n+1}(\Omega), B(\Omega)) \ = \ (F, B)$$

with $F = F_{n+1}(\Omega) = H_{n+1}(\Omega)$ a f.g. free $\mathbb{Z}$-module of rank $b_{n+1}(\Omega)$, with an exact sequence

$$0 \to H_{n+1}(\partial\Omega) = H_{n+1}(W) \to F \xrightarrow{\ B\ } F^* \to H_n(\partial\Omega) = H_n(W) \to 0 \ .$$

The $(n-1)$-connected $2n$-dimensional 'manifold' M'' has the homological properties of $\#_{b_{n+1}(\Omega)} S^n \times S^n$ with $(-1)^n$-symmetric intersection form

$$(F'', B'') \ = \ (H_n(M''), B(M'')) \ = \ (F \oplus F^*, \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix})$$

and the inclusions of the three lagrangians in (i) given by

$$\begin{aligned} j^- &= \begin{pmatrix} 1 \\ B \end{pmatrix} : L^- = H_{n+1}(W^-, M'') = F \to F'' = F \oplus F^* , \\ j^+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : L^+ = H_{n+1}(W^+, M'') = F \to F'' = F \oplus F^* , \\ j'' &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} : L'' = H_{n+1}(\Sigma'', M'') = F^* \to F'' = F \oplus F^* . \end{aligned}$$

In fact, the construction only makes use of the $(-1)^{n+1}$-symmetric form (F, B).
(iii) Let $n = 2$ with

$$(\Omega, \partial\Omega) = (\mathrm{cl.}(\mathbb{CP}^2 \backslash D^4), S^3)$$

in (ii), so that $(F, B) = (\mathbb{Z}, 1)$. Now $(\Omega, \partial\Omega)$ is the total pair of the Hopf bundle

$$(D^2, S^1) \to (\Omega, \partial\Omega) \to S^2 .$$

The decomposition in (ii) is realized geometrically by the decomposition of Ω as a union of two contractible spaces given by the inverse images of the upper and lower hemispheres of $S^2 = D^2 \cup D^2$, as in Wall [Wa], with signature

$$\begin{aligned} \sigma(\mathbb{CP}^2) = \sigma(\Omega) &= \sigma(F'', B''; L^-, L'', L^+) \\ &= \sigma\left(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \{(x, x) \,|\, x \in \mathbb{Z}\}, 0 \oplus \mathbb{Z}, \mathbb{Z} \oplus 0\right) = 1 \in \mathbb{Z} . \end{aligned}$$

3.4. **The intersection forms of cobordant manifolds.** We investigate the intersection forms in a $(2n + 1)$-dimensional relative cobordism:

Proposition 3.4.1. *Let* $(\Omega; \Sigma, \Sigma', W; M, M')$ *be a* $(2n + 1)$*-dimensional boundary product relative cobordism, with*

$$(W; M, M') = M \times (I; \{0\}, \{1\}) , \ \partial\Omega = \Sigma \cup_M W \cup_{M'} -\Sigma' .$$

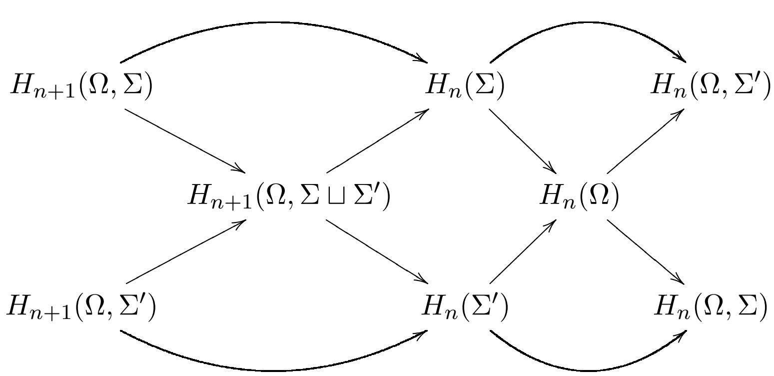
Let (F, B), (F', B') *be the restrictions of the* $(-1)^n$*-symmetric intersection form* $(F_n(\partial\Omega), B(\partial\Omega))$ *over* $\mathbb{Z}$ *to the direct summands*

$$\begin{aligned} F &= \{x \in F_n(\partial\Omega) \,|\, kx \in \mathrm{im}(F_n(\Sigma)) \text{ for some } k \neq 0 \in \mathbb{Z}\} , \\ F' &= \{x \in F_n(\partial\Omega) \,|\, kx \in \mathrm{im}(F_n(\Sigma')) \text{ for some } k \neq 0 \in \mathbb{Z}\} \subseteq F \ (i = 0, 1) , \end{aligned}$$

with surjections

$$\begin{aligned} \mathbb{Q} \otimes_{\mathbb{Z}} (F_n(\Sigma), B(\Sigma)) &\twoheadrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (F, B) , \\ \mathbb{Q} \otimes_{\mathbb{Z}} (F_n(\Sigma'), B(\Sigma')) &\twoheadrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (F', B') . \end{aligned}$$

(i) *The homology groups of* Σ, Σ', Ω *fit into the commutative braid of exact sequences*



The $(-1)^n$-symmetric intersection forms (F,B), (F',B') have sublagrangians

$$\begin{aligned}L &= \ker(F\to F_n(\Omega)) = \{x\in F\,|\,kx\in \mathrm{im}(H_{n+1}(\Omega,\Sigma)) \text{ for some } k\neq 0\in\mathbb{Z}\}\subseteq F\ ,\\ L' &= \ker(F'\to F_n(\Omega)) = \{x\in F'\,|\,kx\in \mathrm{im}(H_{n+1}(\Omega,\Sigma')) \text{ for some } k\neq 0\in\mathbb{Z}\}\subseteq F'\end{aligned}$$

such that there are defined isomorphisms

$$\begin{aligned}&(L^\perp/L,[B]) \cong (L'^\perp/L',[B'])\ ,\\ &(F,B)\cong (L^\perp/L,[B])\oplus\mathcal{H}\ ,\ (F',B')\cong (L'^\perp/L',[B'])\oplus\mathcal{H}'\ ,\\ &(F,B)\oplus\mathcal{H}'\cong (F',B')\oplus\mathcal{H}\end{aligned}$$

with $\mathcal{H},\mathcal{H}'$ metabolic forms with lagrangians L,L'.
(ii) *Suppose that there is given a handle decomposition*

$$(\Omega;\Sigma,\Sigma') = \bigcup_{r=-1}^{2n}(\Omega_r;\Sigma_r,\Sigma_{r+1})$$

with Ω_r the trace of ℓ_{r+1} surgeries of index $r+1$ on $\bigcup_{\ell_{r+1}} S^r\times D^{2n-r}\subset \Sigma_r\backslash M_0$, so that

$$\Omega_r = \Sigma_r\times I\cup\bigcup_{\ell_{r+1}} D^{r+1}\times D^{2n-r}\ .$$

The boundary product relative cobordisms

$$(\Omega^-;\Sigma,\Sigma_n) = \bigcup_{r=-1}^{n-1}(\Omega_r;\Sigma_r,\Sigma_{r+1})\ ,\ (\Omega^+;\Sigma_n,\Sigma') = \bigcup_{r=n}^{2n}(\Omega_r;\Sigma_r,\Sigma_{r+1})$$

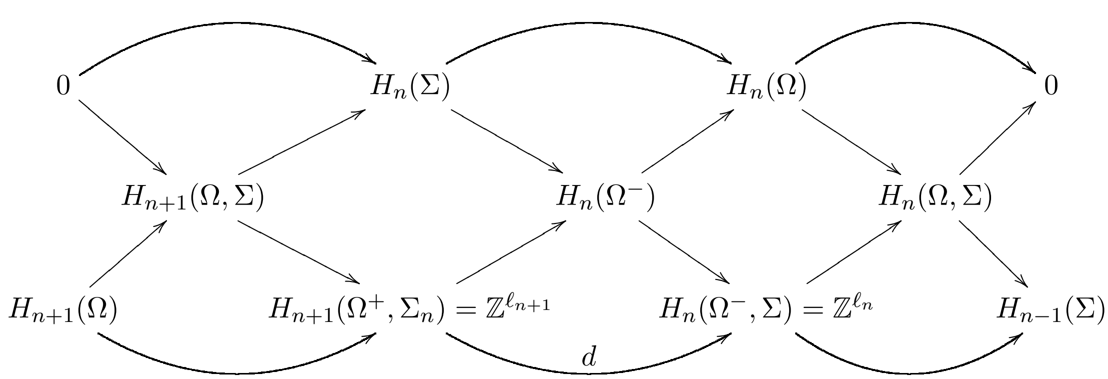
are such that

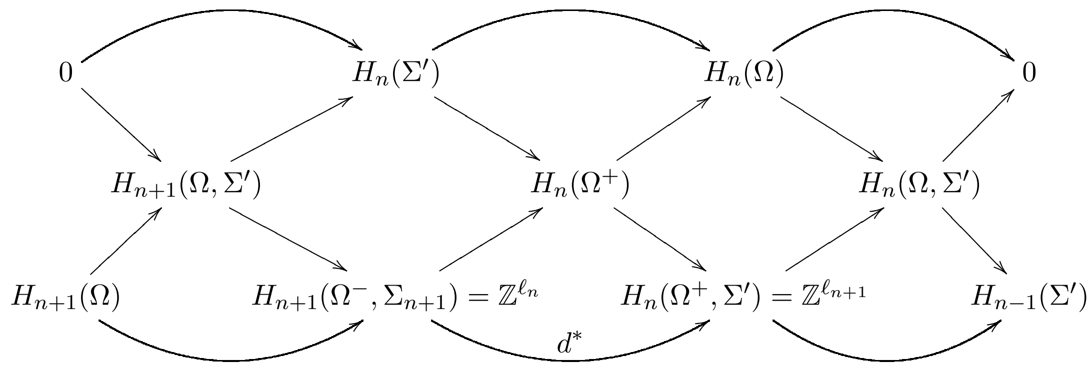
$$(\Omega;\Sigma,\Sigma') = (\Omega^-;\Sigma,\Sigma_n)\cup(\Omega^+;\Sigma_n,\Sigma')$$

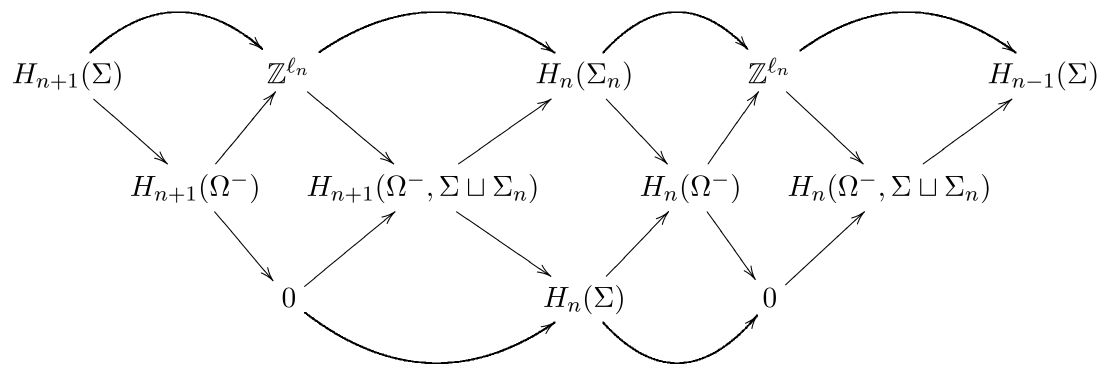
with intersection $(-1)^n$-symmetric forms

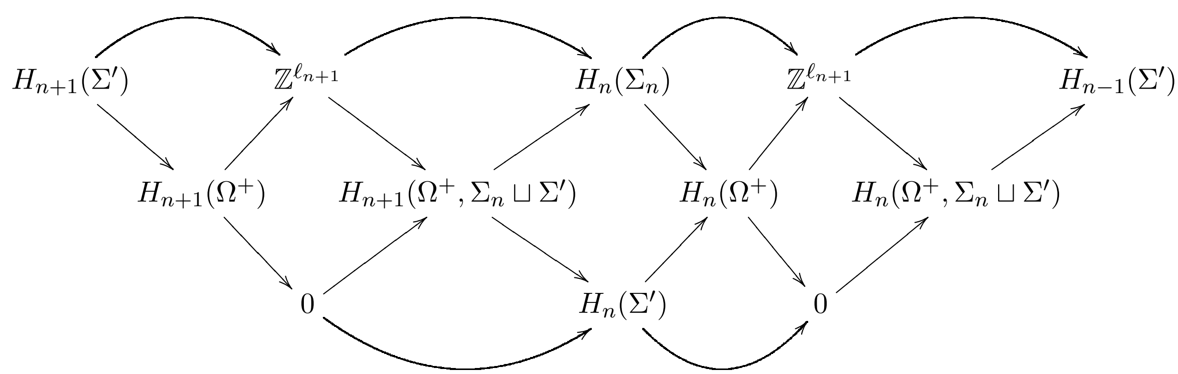
$$\begin{aligned}&(F_n(\Sigma),B(\Sigma)) = (F_n(\Sigma_{-1}),B(\Sigma_{-1})) = \dots = (F_n(\Sigma_{n-1}),B(\Sigma_{n-1}))\ ,\\ &(F_n(\Sigma_{n+1}),B(\Sigma_{n+1})) = \dots = (F_n(\Sigma_{2n+1}),B(\Sigma_{2n+1})) = (F_n(\Sigma'),B(\Sigma'))\ .\end{aligned}$$

The homology groups of $\Sigma,\Sigma_n,\Sigma',\Omega^-,\Omega^+,\Omega$ fit into commutative braids of exact sequences









The $(-1)^n$-symmetric intersection form

$$(F_{1/2},B_{1/2}) \;=\; (F_n(\Sigma_n),B(\Sigma_n))$$

has sublagrangians

$$\begin{aligned} L_{1/2} \;&=\; \{x\in F_{1/2}\,|\,kx\in \mathrm{im}(H_{n+1}(\Omega^-,\Sigma_n))=\mathrm{im}(\mathbb{Z}^{\ell_n}) \textit{ for some } k\neq 0\in\mathbb{Z}\}\ ,\\ L'_{1/2} \;&=\; \{x\in F_{1/2}\,|\,kx\in \mathrm{im}(H_{n+1}(\Omega^+,\Sigma_n))=\mathrm{im}(\mathbb{Z}^{\ell_{n+1}}) \textit{ for some } k\neq 0\in\mathbb{Z}\}\subseteq F_{1/2}\end{aligned}$$

with isomorphisms

$$(L_{1/2}^\perp/L_{1/2},[B_{1/2}]) \;\cong\; (F_0,B_0)\ ,\ ((L'_{1/2})^\perp/L'_{1/2},[B_{1/2}]) \;\cong\; (F_1,B_1)\ ,$$

so that

$$(F_{1/2},B_{1/2}) \;\cong\; (F,B)\oplus\mathcal{H}_{1/2} \;\cong\; (F',B')\oplus\mathcal{H}'_{1/2}$$

with $\mathcal{H}_{1/2}, \mathcal{H}'_{1/2}$ *metabolic with lagrangians* $L_{1/2}, L'_{1/2}$. *The sublagrangians in* (i) *are given by*

$$\begin{aligned} L &= \ker(F \to F_n(\Omega)) \\ &= \{x \in F \,|\, kx \in \mathrm{im}(H_{n+1}(\Omega,\Sigma)) = \mathrm{im}(\ker(d : \mathbb{Z}^{\ell_{n+1}} \to \mathbb{Z}^{\ell_n})) \text{ for some } k \neq 0 \in \mathbb{Z}\} \subseteq F \ , \\ L' &= \ker(F' \to F_n(\Omega)) \\ &= \{x \in F' \,|\, kx \in \mathrm{im}(H_{n+1}(\Omega,\Sigma')) = \mathrm{im}(\ker(d^* : \mathbb{Z}^{\ell_n} \to \mathbb{Z}^{\ell_{n+1}})) \text{ for some } k \neq 0 \in \mathbb{Z}\} \subseteq F' \ . \end{aligned}$$

Finally, we have the following compendium of results on the intersection forms in an odd-dimensional relative cobordism:

Theorem 3.4.2. *Let* $(\Omega,\Sigma,\Sigma';W,M,M')$ *be a* $(2n+1)$*-dimensional relative cobordism with a half-handle decomposition*

$$(\Omega;\Sigma,\Sigma',W;M,M') \ = \ \bigcup_{r=-1}^{2n}(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1})$$

with each

$$\begin{aligned} &(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1}) \\ &= \ (\Omega_r^-;\Sigma_r,\Sigma_{r+1/2},W_r^-;M_r,M_{r+1/2}) \cup_{\Sigma_{r+1/2}} (\Omega_r^+;\Sigma_{r+1/2},\Sigma_{r+1},W_r^+;M_{r+1/2},M_{r+1}) \end{aligned}$$

an index $r+1$ *splitting* (1.3.3 I(i)), *so that*

$$\begin{aligned} W_r^- &= M_r \times I \cup_{\bigcup_{\ell_r^-} S^{r-1}\times D^{2n-r}} \bigcup_{\ell_r^-} D^r \times D^{2n-r} \\ &= M_{r+1/2} \times I \cup_{\bigcup_{\ell_r^-} D^r\times S^{2n-r-1}} \bigcup_{\ell_r^-} D^r \times D^{2n-r} \ , \\ W_r^+ &= M_{r+1/2} \times I \cup_{\bigcup_{\ell_{r+1}^+} S^r\times D^{2n-r-1}} \bigcup_{\ell_{r+1}^+} D^{r+1} \times D^{2n-r-1} \\ &= M_{r+1} \times I \cup_{\bigcup_{\ell_{r+1}^+} D^{r+1}\times S^{2n-r-2}} \bigcup_{\ell_{r+1}^+} D^{r+1} \times D^{2n-r-1} \ , \\ \Sigma_r &= \Sigma_{r+1/2} \cup_{M_{r+1/2}} W_r^- \ = \ \Sigma_{r+1/2} \cup_{\bigcup_{\ell_r^-} D^r\times S^{2n-r-1}} \bigcup_{\ell_r^-} D^r \times D^{2n-r} \ , \\ \Sigma_{r+1} &= \Sigma_{r+1/2} \cup_{M_{r+1/2}} W_r^+ \ = \ \Sigma_{r+1/2} \cup_{\bigcup_{\ell_{r+1}^+} S^r\times D^{2n-r-1}} \bigcup_{\ell_{r+1}^+} D^{r+1} \times D^{2n-r-1} \ , \\ \Omega_r^- &= \Sigma_r \times I \ = \ (\Sigma_r \cup_{M_r} W_r^-) \times I \cup_{\bigcup_{\ell_r^-} S^r\times D^{2n-r}} \bigcup_{\ell_r^-} D^{r+1} \times D^{2n-r} \ , \\ \Omega_r^+ &= \Sigma_{r+1} \times I \ = \ (W_r^+ \cup_{M_{r+1}} \Sigma_{r+1}) \times I \cup_{\bigcup_{\ell_{r+1}^+} S^r\times D^{2n-r}} \bigcup_{\ell_{r+1}^+} D^{r+1} \times D^{2n-r} \ . \end{aligned}$$

(i) *The homology groups are such that*

$$\begin{aligned} H_n(\Sigma) &= H_n(\Sigma_{-1}) \ = \ H_n(\Sigma_{-1/2}) \ = \ \dots \ = \ H_n(\Sigma_{n-1}) \ , \\ H_n(\Sigma_{n+1}) &= H_n(\Sigma_{n+3/2}) \ = \ \dots \ = \ H_n(\Sigma_{2n}) \ = \ H_n(\Sigma') \ , \end{aligned}$$

so that the $(-1)^n$*-symmetric intersection forms are given by*

$$\begin{aligned} (F_n(\Sigma),B(\Sigma)) &= (F_n(\Sigma_{-1}),B(\Sigma_{-1})) \ = \ (F_n(\Sigma_{-1/2}),B(\Sigma_{-1/2})) \\ &= \ \dots \ = \ F_n(\Sigma_{n-1}),B(\Sigma_{n-1})) \ , \\ (F_n(\Sigma_{n+1}),B(\Sigma_{n+1})) &= (F_n(\Sigma_{n+3/2}),B(\Sigma_{n+3/2})) \\ &= \ \dots \ = \ (F_n(\Sigma_{2n},B(\Sigma_{2n})) \ = \ (F_n(\Sigma'),B(\Sigma')) \ . \end{aligned}$$

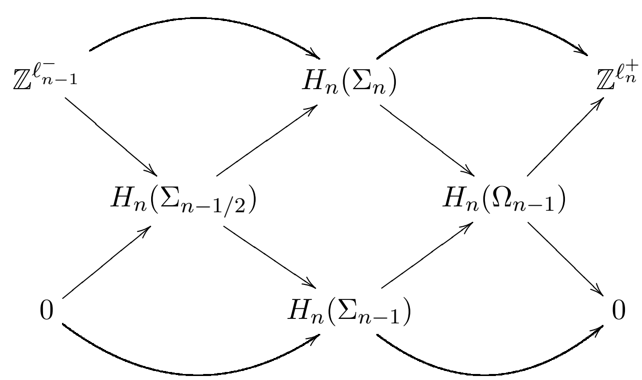
(ii) *For a relative cobordism with an index n splitting*

$$\begin{aligned}&(\Omega_{n-1};\Sigma_{n-1},\Sigma_n,W_{n-1};M_{n-1},M_n)\\ &=\ (\Omega^-_{n-1};\Sigma_{n-1},\Sigma_{n-1/2},W^-_{n-1};M_{n-1},M_{n-1/2})\cup_{\Sigma_{n-1/2}}(\Omega^+_{n-1};\Sigma_{n-1/2},\Sigma_n,W^+_{n-1};M_{n-1/2},M_n)\end{aligned}$$

we have

$$\begin{aligned}W^-_{n-1}\ &=\ M_{n-1}\times I\cup_{\bigcup\limits_{\ell^-_{n-1}}S^{n-2}\times D^{n+1}}\bigcup_{\ell^-_{n-1}}D^{n-1}\times D^{n+1}\\ &=\ M_{n-1/2}\times I\cup_{\bigcup\limits_{\ell^-_{n-1}}D^{n-1}\times S^{n}}\bigcup_{\ell^-_{n-1}}D^{n-1}\times D^{n+1}\ ,\\ W^+_{n-1}\ &=\ M_{n-1/2}\times I\cup_{\bigcup\limits_{\ell^+_{n}}S^{n-1}\times D^{n}}\bigcup_{\ell^+_{n}}D^{n}\times D^{n}\\ &=\ M_{n}\times I\cup_{\bigcup\limits_{\ell^+_{n}}D^{n}\times S^{n-1}}\bigcup_{\ell^+_{n}}D^{n}\times D^{n}\ ,\\ \Sigma_{n-1}\ &=\ \Sigma_{n-1/2}\cup_{M_{n-1/2}}W^-_{n-1}\ =\ \Sigma_{n-1/2}\cup_{\bigcup\limits_{\ell^-_{n-1}}D^{n-1}\times S^{n}}\bigcup_{\ell^-_{n-1}}D^{n-1}\times D^{n+1}\ ,\\ \Sigma_{n}\ &=\ \Sigma_{n-1/2}\cup_{M_{n-1/2}}W^+_{n-1}\ =\ \Sigma_{n-1/2}\cup_{\bigcup\limits_{\ell^+_{n}}S^{n-1}\times D^{n}}\bigcup_{\ell^+_{n}}D^{n}\times D^{n}\ ,\\ \Omega^-_{n-1}\ &=\ \Sigma_{n-1}\times I\ =\ (\Sigma_{n-1}\cup_{M_{n-1}}W^-_{n-1})\times I\cup_{\bigcup\limits_{\ell^-_{n-1}}S^{n-1}\times D^{n+1}}\bigcup_{\ell^-_{n-1}}D^{n}\times D^{n+1}\ ,\\ \Omega^+_{n-1}\ &=\ \Sigma_{n}\times I\ =\ (W^+_{n-1}\cup_{M_n}\Sigma_n)\times I\cup_{\bigcup\limits_{\ell^+_{n}}S^{n-1}\times D^{n+1}}\bigcup_{\ell^+_{n}}D^{n}\times D^{n+1}\ .\end{aligned}$$

There is defined a commutative braid of exact sequences



The intersection form of $\Sigma_{n-1/2}$ is a rank $(\ell^-,0)$ enlargement of the intersection form of Σ_{n-1}

$$(F_n(\Sigma_{n-1/2}),B(\Sigma_{n-1/2}))\ =\ (F_n(\Sigma_{n-1})\oplus L^-_{n-1},\begin{pmatrix}B(\Sigma_{n-1})&0\\0&0\end{pmatrix})$$

and the intersection form of Σ_n is a rank $(0,\ell^+)$ enlargement of the intersection form of $\Sigma_{n-1/2}$

$$(F_n(\Sigma_n),B(\Sigma_n))\ =\ (F_n(\Sigma_{n-1/2})\oplus L^+_n,\begin{pmatrix}B(\Sigma_{n-1/2})&A_n\\(-1)^nA^*_n&E_n\end{pmatrix})$$

with

$$\begin{aligned}L^-_{n-1}\ &=\ \mathrm{im}(\mathbb{Z}^{\ell^-_{n-1}})\ =\ \ker(F_n(\Sigma_{n-1/2})\to F_n(\Sigma_{n-1}))\subseteq F_n(\Sigma_{n-1/2})\subseteq F_n(\Sigma_n)\ ,\\ (L^+_n)^*\ &=\ \mathrm{im}((\mathbb{Z}^{\ell^+_n})^*)\ =\ \ker(F_n(\Sigma_n)^*\to F_n(\Sigma_{n-1/2})^*)\subseteq F_n(\Sigma_n)^*\ =\ F^n(\Sigma_n)\end{aligned}$$

f.g. free $\mathbb{Z}$*-modules such that* $\dim L^-_{n-1} \leqslant \ell^-_{n-1}$, $\dim L^+_n \leqslant \ell^+_n$, *for some* $\mathbb{Z}$*-module morphism*

$$\begin{aligned} A_n &= \begin{pmatrix} C_n \\ D_n \end{pmatrix} : L^+_n \to F_n(\Sigma_{n-1/2})^* = F_n(\Sigma_{n-1})^* \oplus (L^-_{n-1})^* , \\ E_n &= (-1)^n E^*_n : L^+_n \to (L^+_n)^* . \end{aligned}$$

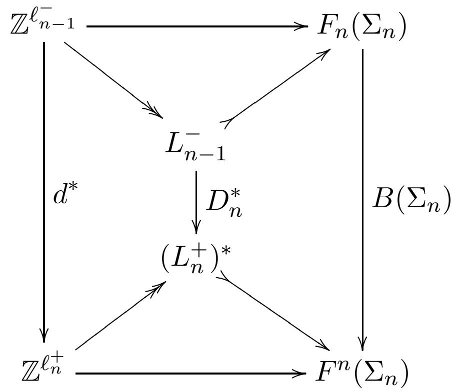
The intersection form of Σ_n *is thus a rank* (ℓ^-_{n-1}, ℓ^+_n) *enlargement of the intersection form of* Σ_{n-1}

$$(F_n(\Sigma_n), B(\Sigma_n)) = (F_n(\Sigma_{n-1}) \oplus L^-_{n-1} \oplus L^+_n, \begin{pmatrix} B(\Sigma_{n-1}) & 0 & C_n \\ 0 & 0 & D_n \\ (-1)^n C^*_n & (-1)^n D^*_n & E_n \end{pmatrix}) .$$

The $\mathbb{Z}$*-module morphism* $D_n : L^+_n \to (L^-_{n-1})^*$ *is related to the boundary map*

$$d : H_n(W^+_{n-1}, M_{n-1/2}) = \mathbb{Z}^{\ell^+_n} \to H_{n-1}(W^-_{n-1}, M_{n-1}) = \mathbb{Z}^{\ell^-_{n-1}}$$

by a commutative diagram



Consider the following conditions

1. $(W_{n-1}; M_{n-1}, M_n)$ *is an* H*-cobordism,*
2. d *is an isomorphism,*
3. D_n *is an isomorphism,*
4. *the intersection form* $(F_n(\Sigma_n), B(\Sigma_n))$ *is an* H*-enlargement of the intersection form* $(F_n(\Sigma_{n-1}), B(\Sigma_{n-1}))$.

(1) *is equivalent to* (2). (3) *is equivalent to* (4). *If* (1) *holds and*

$$\dim_{\mathbb{Z}} L^-_{n-1} = \dim_{\mathbb{Z}} L^+_n$$

then (3) *holds. If the* $\mathbb{Z}$*-module morphisms* $\mathbb{Z}^{\ell^-_{n-1}} \to L^-_{n-1}$, $\mathbb{Z}^{\ell^+_n} \to (L^+_n)^*$ *are isomorphisms then* (1) *is equivalent to* (3). *If* (1) *holds and* $(F_n(\Sigma_n), B(\Sigma_n))$ *is nonsingular then* (3) *holds.*

(iii) *The reverse of the* $(2n+1)$*-dimensional relative cobordism with an index* $n+1$ *splitting*

$$\begin{aligned} &(\Omega_n; \Sigma_n, \Sigma_{n+1}, W_n; M_n, M_{n+1}) \\ &= (\Omega^-_n; \Sigma_n, \Sigma_{n+1/2}, W^-_n; M_n, M_{n+1/2}) \cup (\Omega^+_n; \Sigma_{n+1/2}, \Sigma_{n+1}, W^+_n; M_{n+1/2}, M_{n+1}) \end{aligned}$$

is a $(2n+1)$*-dimensional relative cobordism* $(\Omega_n; \Sigma_{n+1}, \Sigma_n, W_n; M_{n+1}, M_n)$ *with an index* n *splitting, so* (ii) *applies, showing that* $(F_n(\Sigma_n), B(\Sigma_n))$ *is also a rank* (ℓ^+_{n+1}, ℓ^-_n) *enlargement of* $(F_n(\Sigma_{n+1}), B(\Sigma_{n+1}))$

$$(F_n(\Sigma_n), B(\Sigma_n)) = (F_n(\Sigma_{n+1}) \oplus L^+_{n+1} \oplus L^-_n, \begin{pmatrix} B(\Sigma_{n+1}) & 0 & C_{n+1} \\ 0 & 0 & D_{n+1} \\ (-1)^n C^*_{n+1} & (-1)^n D^*_{n+1} & E_{n+1} \end{pmatrix}) .$$

Example 3.4.3. Suppose given a $(2n+1)$-dimensional relative cobordism with an index n splitting

$$\begin{aligned}&(\Omega_{n-1};\Sigma_{n-1},\Sigma_n,W_{n-1};M_{n-1},M_n)\\ &=\ (\Omega^-_{n-1};\Sigma_{n-1},\Sigma_{n-1/2},W^-_{n-1};M_{n-1},M_{n-1/2})\cup(\Omega^+_{n-1};\Sigma_{n-1/2},\Sigma_n,W^+_{n-1};M_{n-1/2},M_n)\end{aligned}$$

as in Theorem 3.4.2. Suppose that

$$(W_{n-1};M_{n-1},M_n)\ \cong\ M_{n-1}\times(I;\{0\},\{1\})$$

with

$$d\ =\ 1\ :\ \mathbb{Z}^{\ell^+_n}\ =\ \mathbb{Z}\to\mathbb{Z}^{\ell^-_{n-1}}\ =\ \mathbb{Z}\ ,$$

so that Σ_n is obtained from Σ_{n-1} by a surgery on $S^{n-1}\times D^{n+1}\subset\Sigma_{n-1}\backslash M_{n-1}$. Suppose also that $M_{n-1}=S^{2n-1}$, so that the $(-1)^n$-symmetric intersection forms $(F_n(\Sigma_{n-1}),B(\Sigma_{n-1}))$, $(F_n(\Sigma_n),B(\Sigma_n))$ are nonsingular. Let $\alpha\in H_n(\Sigma_n)$ be the homology class of $D^n\times S^n\subset\Sigma_n$, the image of the generator $1\in H_{n+1}(\Omega_{n-1},\Sigma_n)=\mathbb{Z}$. If α is of finite order then

$$(F_n(\Sigma_{n-1}),B(\Sigma_{n-1}))\ =\ (F_n(\Sigma_n),B(\Sigma_n))$$

by Theorem 3.4.2 (ii), exactly as in Levine [Le2, §6], with

$$\dim_{\mathbb{Z}}L^-_{n-1}\ =\ \dim_{\mathbb{Z}}L^+_n\ =\ 0\ .$$

If α is of infinite order then

$$(F_n(\Sigma_n),B(\Sigma_n))\ =\ (F_n(\Sigma_{n-1})\oplus\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}B(\Sigma_{n-1}) & 0 & C_n\\ 0 & 0 & 1\\ (-1)^nC_n^* & (-1)^n & E_n\end{pmatrix})$$

is an H-enlargement of $(F_n(\Sigma_{n-1}),B(\Sigma_{n-1}))$ by Theorem 3.4.2 (ii), exactly as in Levine [Le2, §7], with

$$\dim_{\mathbb{Z}}L^-_{n-1}\ =\ \dim_{\mathbb{Z}}L^+_n\ =\ 1\ .$$

Example 3.4.4. Let $\Gamma=(\Omega;\Sigma,\Sigma',W;M,M')$ be a split $(2n+1)$-dimensional relative cobordism, with $\Gamma=\Gamma^-\cup\Gamma^+$ the union of a left and a right product cobordism

$$\Gamma^-\ =\ (\Omega^-;\Sigma,\Sigma'',W^-;M,M'')\ ,\ \Gamma^+\ =\ (\Omega^+;\Sigma',\Sigma'',W^+;M'',M')\ .$$

For simplicity assume that Ω,Σ,Σ',W are all $(n-1)$-connected, so that M,M' are also $(n-1)$-connected. We refer to Ranicki [Ra2, §§1.6,1.7] for the glueing of forms using boundary formations. Write the $(-1)^n$-symmetric intersection forms as

$$\begin{aligned}&(F,B)\ =\ (F_n(\Sigma),B(\Sigma))\ ,\ (F',B')\ =\ (F_n(\Sigma'),B(\Sigma'))\ ,\ (F'',B'')\ =\ (F_n(\Sigma''),B(\Sigma'))\ ,\\ &(G^-,C^-)\ =\ (F_n(W^-),B(W^-))\ ,\ (G^+,C^+)\ =\ (F_n(W^+),B(W^+))\ ,\\ &(G,C)\ =\ (F_n(W),B(W))\ .\end{aligned}$$

The corresponding boundary $(-1)^{n-1}$-symmetric formations are such that

$$\begin{aligned}&\partial(F,B)\oplus\partial(F''-,B'')\ \simeq\ \partial(G^-,C^-)\ ,\ \partial(F'',B'')\oplus\partial(F',-B')\ \simeq\ \partial(G^+,C^+)\ ,\\ &\partial(F,B)\oplus\partial(F',-B')\ \simeq\ \partial(G,C)\ ,\end{aligned}$$

and there are defined isomorphisms of $(-1)^n$-symmetric forms

$$\begin{aligned}&(F,B)\ \cong\ (F'',B'')\cup(G^-,C^-)\ ,\ (F',B')\ \cong\ (F'',B'')\cup(G^+,C^+)\ ,\\ &(G,C)\ \cong\ (G^-,C^-)\cup(G^+,C^+)\ .\end{aligned}$$

The nonsingular $(-1)^n$-symmetric intersection form of the geometric union

$$\partial\Omega\ =\ (\Sigma\sqcup-\Sigma')\cup_\partial W$$

is the algebraic union

$$(F_n(\partial\Omega), B(\partial\Omega)) \;=\; ((F,B)\oplus(F',B'))\cup_\partial (G,C)$$

with lagrangian $L = \ker(F_n(\partial\Omega)\to F_n(\Omega))$, such that

$$(F'',B'') \;=\; (\ker(F\oplus F'\to L^*), (B\oplus -B')|)\ .$$

The $(n-1)$-connected $(2n-1)$-dimensional manifolds M, M', M'' correspond to the boundary $(-1)^n$-symmetric formations $\partial(F,B)$, $\partial(F',B')$, $\partial(F'',B'')$ respectively, by the odd-dimensional algebraic analogue of Remark 1.3.2 .

4. The cobordism of algebraic Poincaré complexes

We now apply the algebraic theory of surgery on symmetric Poincaré complexes of [Ra1] to obtain algebraic half-handle decompositions. The main result of this section is the Algebraic Poincaré Splitting Theorem 4.5.6 : up to homotopy equivalence every relative cobordism of symmetric Poincaré pairs is split, i.e. the union of a left product and a right product. This will be used in Theorem 4.7.1 below to prove that a relative cobordism $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ has a half-handle decomposition on the chain level, and in Theorem 6.2.6 that a relative cobordism of codimension 2 embeddings has an embedded half-handle decomposition on the chain level.

4.1. **The Q-groups.** Let R be a ring with an involution $R\to R; a\mapsto \bar{a}$. In our applications either $\mathbb{Z}\subseteq R\subseteq\mathbb{R}$ with the identity involution, or $R=\mathbb{C}$ with the complex conjugation involution.

Regard a left R-module as a right R-module by

$$F\times R\to F\ ;\ (x,a)\mapsto \bar{a}x\ .$$

The tensor product over R of (left) R-modules F,G is the $\mathbb{Z}$-module

$$F\otimes_R G \;=\; F\otimes_{\mathbb{Z}} G/\{ax\otimes y - x\otimes \bar{a}y \,|\, x\in F, y\in G, a\in R\}\ .$$

For any R-modules F,G there are defined a *transposition* isomorphism

$$T_{F,G}\ :\ F\otimes_R G\to G\otimes_R F\ ;\ x\otimes y\mapsto y\otimes x\ .$$

The *dual* of an R-module F is the R-module $F^* = \mathrm{Hom}_R(F,R)$ with

$$R\times F^*\to F^*\ ;\ (a,f)\mapsto (x\mapsto f(x)\bar{a})\ .$$

If F is f.g. free then so is F^*, and the natural R-module morphism

$$F\to F^{**}\ ;\ x\mapsto (f\mapsto \overline{f(x)})$$

is an isomorphism, which will be used to identify $F^{**}=F$.

For any R-modules F,G there is defined a $\mathbb{Z}$-module morphism

$$F\otimes_R G\to \mathrm{Hom}_R(F^*,G)\ ;\ x\otimes y\mapsto (f\mapsto \overline{f(x)(y)})\ .$$

This morphism is an isomorphism if F is f.g. projective, in which case it will be used as an identification.

The *duality* morphism defined for any R-modules F,G by

$$D_{F,G}\ :\ \mathrm{Hom}_R(F,G)\to \mathrm{Hom}_R(G^*,F^*)\ ;\ f\mapsto (f^*: g\mapsto (x\mapsto g(f(x))))$$

is an isomorphism for f.g. projective F,G, with

$$D_{F,G} \;=\; T_{F^*,G}\ :\ \mathrm{Hom}_R(F,G) \;=\; F^*\otimes_R G\to \mathrm{Hom}_R(G^*,F^*) \;=\; G\otimes_R F^*\ .$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$\dots \longrightarrow W_3=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T} W_2=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1+T} W_1=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T} W_0=\mathbb{Z}[\mathbb{Z}_2]\longrightarrow \mathbb{Z}\ .$$

The *symmetric Q-groups* of a f.g. free R-module chain complex C are defined by

$$Q^m(C) \ = \ H_m(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes_R C)) \ (m \geqslant 0) \ .$$

An element $\phi \in Q^m(C)$ is represented by a sequence of higher chain homotopies $\phi_{s+1} : \phi_s \simeq T\phi_s$ $(s \geqslant 0)$ with

$$T \ : \ \mathrm{Hom}_R(C^p, C_q) \to \mathrm{Hom}_R(C^q, C_p) \ ; \ \theta \mapsto (-1)^{pq}\theta^* \ .$$

Thus

$$\phi_s \ : \ C^r \ = \ C_r^* \ = \ \mathrm{Hom}_R(C_r, R) \to C_{m-r+s}$$

with

$$d\phi_s + (-1)^r\phi_s d^* + (-1)^{m-r+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) \ = \ 0 \ : \\ C^{m-r+s-1} \to C_r \ (s \geqslant 0, \phi_{-1} = 0) \ .$$

In particular, there is defined an R-module chain map

$$\phi_0 \ : \ C^{m-*} \ = \ \mathrm{Hom}_R(C, R)_{m-*} \to C$$

with C^{m-*} the dual f.g. free R-module chain complex defined by

$$d_{C^{m-*}} \ = \ (-1)^r d^* \ : \ (C^{m-*})_r \ = \ C^{m-r} \to (C^{m-*})_{r-1} \ = \ C^{m-r+1} \ .$$

Proposition 4.1.1. ([Ra1, Prop.I.1.4]) *The Q-groups are not additive with respect to the direct sum. The Q-groups of a direct sum given by*

$$Q^m(C \oplus C') \ = \ Q^m(C) \oplus Q^m(C') \oplus H_m(C \otimes_R C') \ .$$

The *algebraic mapping cone* of an R-module chain map $f : C \to D$ is the R-module chain complex $\mathcal{C}(f)$ with

$$d_{\mathcal{C}(f)} \ = \ \begin{pmatrix} d_D & (-1)^{r-1}f \\ 0 & d_C \end{pmatrix} \ : \ \mathcal{C}(f)_r \ = \ D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} \ = \ D_{r-1} \oplus C_{r-2} \ .$$

As usual, the relative homology R-modules are defined by

$$H_*(f) \ = \ H_*(\mathcal{C}(f))$$

with an exact sequence

$$\dots \longrightarrow H_{m+1}(C) \xrightarrow{f_*} H_{m+1}(D) \longrightarrow H_{m+1}(f) \longrightarrow H_m(C) \longrightarrow \dots \ .$$

An R-module chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f \otimes f : C \otimes_R C \to D \otimes_R D$ and hence a $\mathbb{Z}$-module chain map

$$f^{\%} \ : \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, D \otimes_R D)$$

inducing a long exact sequence of Q-groups

$$\dots \longrightarrow Q^{m+1}(C) \xrightarrow{f^{\%}} Q^{m+1}(D) \longrightarrow Q^{m+1}(f) \longrightarrow Q^m(C) \longrightarrow \dots$$

with $Q^{m+1}(f) = H_{m+1}(f^{\%})$ the *relative symmetric Q-groups.* An element $(\delta\phi, \phi) \in Q^{m+1}(f)$ is represented by an element $\phi \in Q^m(C)$, and higher chain homotopies $\delta\phi_s : D^r \to D_{m-r+s+1}$ $(s \geqslant 1)$ such that

$$f\phi_s f^* \ = \ d_D\delta\phi_s + (-1)^r\delta\phi_s d_D^* + (-1)^{m+s}(\delta\phi_{s-1} + (-1)^s T\delta\phi_{s-1}) \ : \\ D^{m-r+s} \to D_r \ (s \geqslant 0, \delta\phi_{-1} = 0) \ .$$

In particular, there is defined a chain homotopy

$$\delta\phi_0 \ : \ f\phi_0 f^* \ \simeq \ 0 \ : \ D^{m-*} \to D \ .$$

Proposition 4.1.2. ([Ra1, Prop I.1.1], [BR, Prop. 18])
(i) *The Q-groups are chain homotopy invariant: if $f : C \to D$ is an R-module chain equivalence then the induced morphisms $f^{\%} : Q^*(C) \to Q^*(D)$ are isomorphisms, and $Q^*(f) = 0$.*
(ii) *The Q-groups of $\mathcal{C}(f)$ and the relative Q-groups of f are related by an exact sequence*

$$\cdots \to H_m(C \otimes_R \mathcal{C}(f)) \to Q^m(f) \to Q^m(\mathcal{C}(f)) \to H_{m-1}(C \otimes_R \mathcal{C}(f)) \to \dots$$

with

$$Q^m(f) \to Q^m(\mathcal{C}(f)) \; ; \; (\delta\phi, \phi) \mapsto \delta\phi/\phi \; , \; (\delta\phi/\phi)_s \; = \; \begin{pmatrix} \delta\phi_s & 0 \\ (-1)^{n-r-s}\phi_s f^* & (-1)^{n-r+s} T\phi_{s-1} \end{pmatrix}$$

the algebraic Thom construction.

4.2. **The disjoint union.** The disjoint union of chain complexes is a construction akin to (but not the same as) the direct sum, with respect to which the Q-groups are additive. In working with the algebraic Poincaré version of a manifold cobordism $(W; M, M')$ it is essential to deal with the Q-groups of the disjoint union $C(M) \sqcup C(M')$ rather than the Q-groups of the direct sum $C(M \sqcup M') = C(M) \oplus C(M')$, to avoid having any terms in

$$H_m(M \times M') \subseteq Q^m(C(M) \oplus C(M')) \; = \; Q^m(C(M)) \oplus Q^m(C(M')) \oplus H_m(M \times M') \; .$$

The product of rings with involution R, S is the ring with involution $R \times S$.

Definition 4.2.1. The *disjoint union* of an R-module F and an S-module G is

$$F \sqcup G \; = \; F \oplus G \text{ regarded as an } R \times S \text{ module} \; .$$

If F is a f.g. projective R-module and G is a f.g. projective S-module then $F \sqcup G$ is a f.g. projective $R \times S$-module, with a natural $R \times S$-module isomorphism

$$F^* \sqcup G^* \to (F \sqcup G)^* \; ; \; (f, g) \mapsto ((x, y) \mapsto (f(x), g(y))) \; .$$

(However, if F, G are f.g. free of different rank then $F \sqcup G$ is not a f.g. free $R \times S$-module.) The functor

$$\{\text{f.g. projective } R\text{-modules}\} \times \{\text{f.g. projective } S\text{-module}\}$$
$$\to \{\text{f.g. projective } R \times S\text{-modules}\} \; ; \; (F, G) \mapsto F \sqcup G$$

is an equivalence of categories with involution.

We have the obvious but useful property of the disjoint union:

Proposition 4.2.2. *For any R-modules F, F' and any S-modules G, G' there is an identity of $\mathbb{Z}$-modules*

$$(F \sqcup G) \otimes_{R\times S} (F' \sqcup G') \; = \; (F \otimes_R F') \oplus (G \otimes_S G') \; .$$

We shall only be concerned with the disjoint union in the case $R = S$. By contrast with the result of Proposition 4.2.2, for any R-modules F, F', G, G'

$$(F \oplus G) \otimes_R (F' \oplus G') \; = \; (F \otimes_R F') \oplus (G \otimes_R G') \oplus (F \otimes_R G') \oplus (G \otimes_R F') \; .$$

The symmetric Q-groups of an $R \times R$-module chain complex C are defined by

$$Q^m(C) \; = \; H_m(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{R\times R} C)) \; .$$

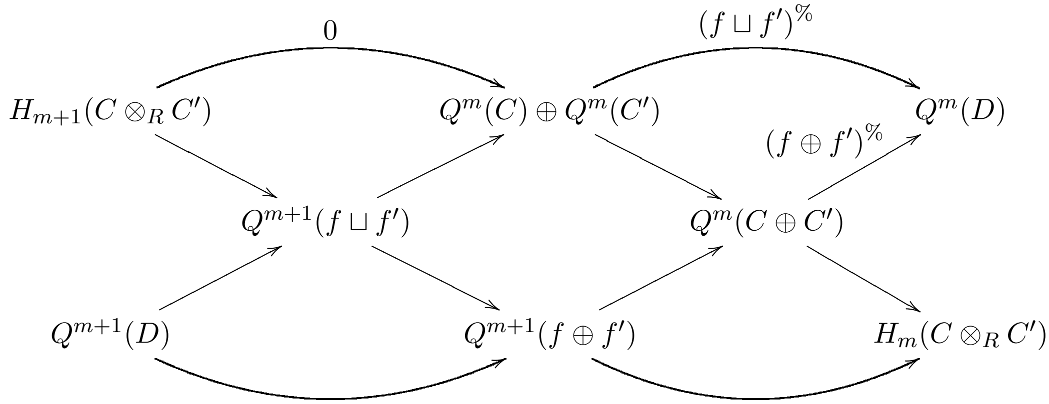
Proposition 4.2.3. (i) *The symmetric Q-groups of $R \times R$-module chain complexes are additive with respect to the disjoint union. For any R-module chain complexes C, C'*

$$Q^m(C \sqcup C') \; = \; Q^m(C) \oplus Q^m(C') \; .$$

(ii) *Given R-module chain maps $f : C \to D$, $f' : C' \to D$ there are defined $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps*

$$\begin{aligned}(f \oplus f')^{\%} &: \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, (C \oplus C') \otimes_R (C \oplus C')) \to \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, D \otimes_R D) \ , \\ (f \sqcup f')^{\%} = f^{\%} \oplus f'^{\%} &: \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, (C \otimes_R C) \oplus (C' \otimes_R C')) \to \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, D \otimes_R D)\end{aligned}$$

with a commutative braid of exact sequences



Proof. Immediate from the identity

$$(C \sqcup C') \otimes_{R \times R} (C \sqcup C') \ = \ (C \otimes_R C) \oplus (C' \otimes_R C') \ .$$

□

It is easy to extend the disjoint union construction to k-fold disjoint unions, for all $k \geqslant 2$. Define the k-fold product ring

$$\prod_k R \ = \ R \times R \times \cdots \times R$$

and given R-modules $F_1, F_2, \ldots, F_k$ define the k-fold disjoint union $\prod_k R$-module

$$F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k \ = \ F_1 \oplus F_2 \oplus \cdots \oplus F_k \ .$$

For R-module chain complexes $C_1, C_2, \ldots, C_k$ there is then defined a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$(C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k) \otimes_{\prod_k R} (C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k) \ = \ (C_1 \otimes_R C_1) \oplus (C_2 \otimes_R C_2) \oplus \cdots \oplus (C_k \otimes_R C_k)$$

such that

$$\begin{aligned}Q^*(C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k) &= H_*(\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, (C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k) \otimes_{\prod_k R} (C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k))) \\ &= Q^*(C_1) \oplus Q^*(C_2) \oplus \cdots \oplus Q^*(C_k) \ .\end{aligned}$$

4.3. **Symmetric Poincaré complexes, pairs and triads.** An *m-dimensional symmetric complex* (C, ϕ) over R is a bounded f.g. projective R-module chain complex C together with an element $\phi \in Q^m(C)$. The symmetric complex (C, ϕ) is *Poincaré* if the R-module chain map $\phi_0 : C^{m-*} \to C$ is a chain equivalence. In the next subsection we recall from [Ra1] how an m-dimensional manifold M determines an m-dimensional symmetric Poincaré complex $(C(M), \phi)$ over $\mathbb{Z}$.

A *homotopy equivalence* of symmetric complexes $f : (C, \phi) \to (C', \phi')$ is a chain equivalence $f : C \to C'$ such that $f^{\%}(\phi) = \phi' \in Q^m(C')$.

Example 4.3.1. Given a f.g. free R-module F and $n \geqslant 0$ let C be the $2n$-dimensional f.g. free R-module chain complex defined by

$$C_r \;=\; \begin{cases} F^* & \text{if } r = n \\ 0 & \text{otherwise} . \end{cases}$$

Then

$$Q^{2n}(C) \;=\; \ker(1 - (-1)^n T : \mathrm{Hom}_R(F, F^*) \to \mathrm{Hom}_R(F, F^*)) ,$$

so that a $2n$-dimensional symmetric (Poincaré) complex (C, ϕ) is the same as a (nonsingular) $(-1)^n$-symmetric form (F, B), with $B = \phi_0$. A homotopy equivalence of such complexes is the same as an isomorphism of forms.

An $(m+1)$-*dimensional symmetric pair* $(f : C \to D, (\delta\phi, \phi))$ over R is an R-module chain map $f : C \to D$ together with an element $(\delta\phi, \phi) \in Q^{m+1}(f)$. The symmetric pair is *Poincaré* if the chain map $D^{m+1-*} \to \mathcal{C}(f)$ given by

$$\begin{pmatrix} \delta\phi_0 \\ \phi_0 f^* \end{pmatrix} \;:\; D^{m-r+1} \to \mathcal{C}(f)_r \;=\; D_r \oplus C_{r-1}$$

is a chain equivalence. We refer to [Ra2, p.45] for the *homotopy equivalence* of symmetric pairs. We refer to [Ra1, Prop.I.3.4] for the proof that the algebraic Thom complex construction

$$(f : C \to D, (\delta\phi, \phi)) \mapsto (\mathcal{C}(f), \delta\phi/\phi)$$

defines a one-one correspondence between the homotopy equivalence classes of $(m+1)$-dimensional symmetric Poincaré pairs and $(m+1)$-dimensional symmetric complexes.

An $(m+1)$-*dimensional symmetric (Poincaré) cobordism* is an $(m+1)$-dimensional symmetric (Poincaré) pair of the type

$$(f \sqcup f' : C \sqcup C' \to D, (\delta\phi, \phi \sqcup -\phi') \in Q^{m+1}(f \sqcup f')) .$$

In [Ra1, §I.3] the relative Q-groups $Q^{m+1}(f \oplus f')$ were used instead of $Q^{m+1}(f \sqcup f')$; although the difference is slight, it is significant here. In the following subsection we recall how a cobordism of manifolds determines a cobordism of symmetric Poincaré complexes.

Definition 4.3.2. The *union* of chain complexes D, D' along chain maps $f : C \to D$, $f' : C \to D'$ is the chain complex

$$D \cup_C D' \;=\; \mathcal{C}(\begin{pmatrix} f \\ f' \end{pmatrix} : C \to D \oplus D') .$$

We refer to [Ra2, §1.7] for the construction of the *union* of adjoining $(m+1)$-dimensional symmetric Poincaré cobordisms, the $(m+1)$-dimensional symmetric Poincaré cobordism

$$\begin{aligned} &(f_C \sqcup f_{C'} : C \sqcup C' \to D, (\delta\phi, \phi \sqcup -\phi')) \cup (f'_{C'} \sqcup f'_{C''} : C' \sqcup C'' \to D', (\delta\phi', \phi' \sqcup -\phi'')) \\ &\qquad = \; (f''_C \sqcup f''_{C''} : C \sqcup C'' \to D'', (\delta\phi'', \phi \sqcup -\phi'')) \end{aligned}$$

with $D'' = D \cup_{C'} D'$.

The theory of chain complex triads is developed in [Ra2, §§1.3,2.1].

Definition 4.3.3. A *chain complex triad* Γ is a commutative square of R-module chain complexes and chain maps

$$\begin{array}{ccc} B & \longrightarrow & E \\ \big\downarrow & \Gamma & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

In the general theory Γ is a chain homotopy commutative diagram, with an explicit chain homotopy, but we only consider the commutative case here.

The homology R-modules of Γ are defined by

$$H_*(\Gamma) \;=\; H_*(C(\Gamma))$$

with $C(\Gamma) = \mathcal{C}(E \cup_B C \to D)$, to fit into the exact sequence

$$\dots \to H_m(E \cup_B C) \to H_m(D) \to H_m(\Gamma) \to H_{m-1}(E \cup_B C) \to \dots\ .$$

We shall also use $C(\Gamma)$ to denote the $R \times R$-module chain complex

$$C(\Gamma) \;=\; \mathcal{C}(\mathcal{C}(B \to E \sqcup C) \to D)\ .$$

The symmetric Q-groups of Γ

$$Q^*(\Gamma) \;=\; H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\Gamma) \otimes_{R \times R} C(\Gamma)))$$

fit into the exact sequence

$$\dots \to Q^m(B \to E \sqcup C) \to Q^m(D) \to Q^m(\Gamma) \to Q^{m-1}(B \to E \sqcup C) \to \dots\ .$$

An $(m+2)$-*dimensional symmetric triad* (Γ, Φ) over R is a triad Γ of bounded f.g. projective R-module chain complexes, together with an element $\Phi \in Q^{m+2}(\Gamma)$. The symmetric triad is *Poincaré* if Φ determines abstract Poincaré-Lefschetz duality isomorphisms

$$\begin{aligned} &H^{m-*}(B) \;\cong\; H_*(B)\ ,\ H^{m+1-*}(B \to C) \;\cong\; H_*(C)\ ,\\ &H^{m+1-*}(B \to E) \;\cong\; H_*(E)\ ,\ H^{m+2-*}(C \to D) \;\cong\; H_*(E \to D)\ . \end{aligned}$$

An $(m+2)$-*dimensional relative symmetric Poincaré cobordism* (Γ, Φ) is an $(m+2)$-dimensional symmetric Poincaré triad with Γ of the form

$$\begin{array}{ccc} B \sqcup B' & \longrightarrow & E \\ \big\downarrow & \Gamma & \big\downarrow \\ C \sqcup C' & \longrightarrow & D \end{array}$$

with

$$\Phi \in Q^{m+2}(\Gamma) \;=\; H_{m+2}(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\Gamma) \otimes_{R \times R} C(\Gamma)))\ .$$

There are defined abstract Poincaré-Lefschetz duality isomorphisms

$$\begin{aligned} &H^{m-*}(B) \;\cong\; H_*(B)\ ,\ H^{m-*}(B') \;\cong\; H_*(B')\ ,\\ &H^{m+1-*}(B \to C) \;\cong\; H_*(B' \to C)\ ,\ H^{m+1-*}(B \to E) \;\cong\; H_*(B' \to E)\ ,\\ &H^{m+2-*}(C \oplus C' \to D) \;\cong\; H_*(E \to D)\ . \end{aligned}$$

This is just a cobordism of symmetric Poincaré pairs in the sense of [Ra2, §2.1], with a corresponding union construction. In the following subsection we recall how an $(m+2)$-dimensional relative cobordism of manifolds determines an $(m+2)$-dimensional relative symmetric Poincaré cobordism over $\mathbb{Z}$.

4.4. **The symmetric construction.** We refer to [Ra1, §II.1] for the *symmetric construction* on a space X, the natural transformation induced by the Alexander-Whitney-Steenrod diagonal chain approximation

$$\phi_X \ : \ H_*(X) \to Q^*(C(X))$$

with $R = \mathbb{Z}$, such that for any homology class $[X] \in H_m(X)$

$$\phi_X[X]_0 \ = \ [X] \cap - \ : \ C(X)^{m-*} \to C(X) \ .$$

In general, X is an arbitrary space and $C(X)$ is the singular $\mathbb{Z}$-module chain complex. We shall only be concerned with spaces X which are finite CW complexes, with $C(X)$ the cellular chain complex.

Example 4.4.1. An m-dimensional manifold M determines an m-dimensional symmetric Poincaré complex $(C(M), \phi_M[M] \in Q^m(C(M)))$, with $[M] \in H_m(M)$ the fundamental class of M and $\phi_M[M]_0 = [M] \cap - : C(M)^{m-*} \to C(M)$ the Poincaré duality chain equivalence.

We shall abbreviate $\phi_M[M]$ to ϕ_M.

There is also a relative symmetric construction for a map of spaces $f : X \to Y$

$$\phi_f \ : \ H_*(f) \to Q^*(f : C(X) \to C(Y))$$

such that the symmetric construction on the geometric mapping cone $\mathcal{C}(f) = X \times I \cup_f Y$ is given by the composite with the algebraic Thom construction

$$\begin{aligned}\phi_{\mathcal{C}(f)} \ : \ H_*(\mathcal{C}(f)) \ = \ H_*(f) \ &\xrightarrow{\phi_f} \ Q^*(f) \\ &\longrightarrow \ Q^*(C(\mathcal{C}(f))) \ = \ Q^*(\mathcal{C}(f : C(X) \to C(Y))) \ .\end{aligned}$$

For the inclusion of a subspace $f : X \subseteq Y$ and a homology class $[Y] \in H_{m+1}(f) = H_{m+1}(Y, X)$ with image $[X] = \partial[Y] \in H_m(X)$, we have a relative Q-group class

$$\phi_f[Y] \ = \ (\phi_Y[Y], \phi_X[X]) \in Q^{m+1}(f)$$

with image

$$\phi_{Y/X}[Y] \ = \ \phi_Y[Y]/\phi_X[X] \in Q^{m+1}(\mathcal{C}(f)) \ = \ Q^{m+1}(C(Y, X)) \ .$$

Example 4.4.2. An $(m+1)$-dimensional manifold with boundary (Σ, M) determines an $(m+1)$-dimensional symmetric Poincaré pair

$$(i : C(M) \to C(\Sigma), (\phi_\Sigma, \phi_M) \in Q^{m+1}(i))$$

with $i : M \to \Sigma$ the inclusion, and with a Poincaré-Lefschetz chain equivalence $\mathcal{C}(i)^{m-*} \simeq C(\Sigma)$.

In order to deal with cobordisms we need to know the symmetric construction on a disjoint union:

Proposition 4.4.3. *The symmetric construction on a disjoint union of spaces $X \sqcup Y$ is the disjoint union of the symmetric constructions on X and Y*

$$\begin{aligned}&\phi_{X \sqcup Y} \ = \ \phi_X \sqcup \phi_Y \ : \ H_*(X \sqcup Y) \ = \ H_*(X) \oplus H_*(Y) \\ &\xrightarrow{\phi_X \oplus \phi_Y} \ Q^*(C(X) \sqcup C(Y)) \ = \ Q^*(C(X)) \oplus Q^*(C(Y)) \\ &\subseteq Q^*(C(X \sqcup Y)) \ = \ Q^*(C(X) \oplus C(Y)) \ = \ Q^*(C(X)) \oplus Q^*(C(Y)) \oplus H_*(X \times Y) \ .\end{aligned}$$

Example 4.4.4. An $(m+1)$-dimensional absolute cobordism $(W; M_0, M_1)$ determines an $(m+1)$-dimensional symmetric Poincaré pair

$$(i : C(M_0) \sqcup C(M_1) \to C(W), (\phi_W, \phi_{M_0} \sqcup -\phi_{M_1}) \in Q^{m+1}(i))$$

with $i : M_0 \sqcup M_1 \to W$ the inclusion.

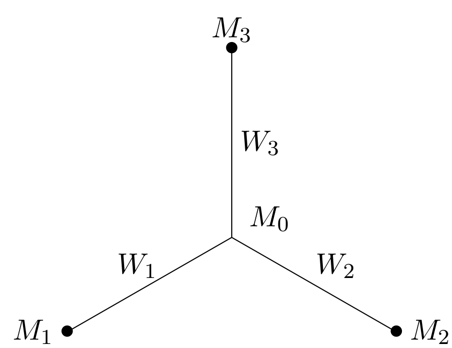
There is also a triad symmetric construction $\phi_\Gamma : H_*(\Gamma) \to Q^*(C(\Gamma))$ for a commutative square of spaces and maps Γ.

Example 4.4.5. An $(m+2)$-dimensional relative manifold cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ determines an $(m+2)$-dimensional symmetric Poincaré triad

$$(C(\Gamma), \phi_\Gamma) \;=\; \left(\begin{array}{ccc} C(M_0) \sqcup C(M_1) & \longrightarrow & C(W) \\ \downarrow & & \downarrow \\ C(\Sigma_0) \sqcup C(\Sigma_1) & \longrightarrow & C(\Omega) \end{array} \;,\; \begin{array}{ccc} \phi_{M_0} \sqcup -\phi_{M_1} & \longrightarrow & \phi_W \\ \downarrow & & \downarrow \\ \phi_{\Sigma_0} \sqcup -\phi_{\Sigma_1} & \longrightarrow & \phi_\Omega \end{array} \right) .$$

4.5. **The Algebraic Poincaré Splitting Theorem.** We shall now prove a splitting theorem for relative cobordisms of symmetric Poincaré pairs which is an algebraic converse to the following construction of split relative cobordisms of manifolds with boundary. We start by recalling the standard thickening construction of a manifold with boundary.

Definition 4.5.1. (i) An $(m+1)$-dimensional *trinity* $(W_1, W_2, W_3; M_0, M_1, M_2, M_3)$ is a stratified set $W_1 \cup W_2 \cup W_3$ which is the union at M_0 of $(m+1)$-dimensional cobordisms $(W_k; M_k, M_0)$ $(k = 1, 2, 3)$.



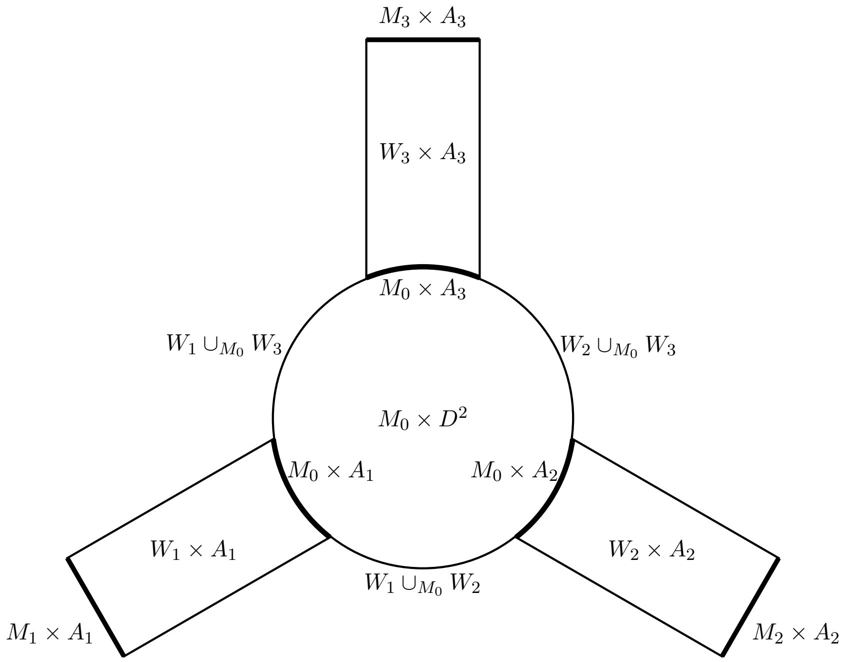
(ii) The trinity has strata $M_0 \sqcup M_1 \sqcup M_2 \sqcup M_3$ and $\mathrm{int}(W_1) \sqcup \mathrm{int}(W_2) \sqcup \mathrm{int}(W_3)$. The *thickening* of the trinity is the $(m+2)$-dimensional manifold with boundary $(\Omega, \partial\Omega)$ defined by choosing disjoint closed arcs $A_1, A_2, A_3 \subset S^1$ and setting

$$\Omega \;=\; (M_0 \times D^2) \cup_{M_0 \times (A_1 \sqcup A_2 \sqcup A_3)} ((W_1 \times A_1) \sqcup (W_2 \times A_2) \sqcup (W_3 \times A_3)) \;,$$

so that

$$\partial\Omega \;=\; (W_1 \cup_{M_0} W_2) \cup (W_2 \cup_{M_0} W_3) \cup (W_1 \cup_{M_0} W_3) \;.$$

The inclusion $W \to \Omega$ is a homotopy equivalence.



Remark 4.5.2. (i) Trinities first appeared in the work of Wall [Wa] on the nonadditivity of the signature. If V is an $(m+2)$-dimensional manifold with boundary expressed as a union $V = V_1 \cup V_2 \cup V_3$ of transverse codimension 0 submanifolds $V_1, V_2, V_3 \subset V$ then there is defined an $(m+1)$-dimensional trinity

$$(W_1, W_2, W_3; M_0, M_1, M_2, M_3) \;=\; (V_2 \cap V_3, V_3 \cap V_1, V_1 \cap V_2; V_1 \cap V_2 \cap V_3, \emptyset, \emptyset, \emptyset)$$

with thickening $\Omega \subset V$ such that $\mathrm{cl.}(V \backslash \Omega) \cong V_1 \sqcup V_2 \sqcup V_3$. If $m = 4k-2$ the signature of V is given by Novikov additivity (3.3.3 (ii)) to be

$$\sigma(V) \;=\; \sigma(V_1) + \sigma(V_2) + \sigma(V_3) + \sigma(\Omega) \in \mathbb{Z}$$

with $\sigma(\Omega)$ the nonadditivity invariant (= Maslov index) determined by the three lagrangians

$$L_j \;=\; \ker(F_{2k-1}(M_0) \to F_{2k-1}(W_j)) \subset F_{2k-1}(M_0) \;(j = 1, 2, 3)$$

of the (-1)-symmetric intersection form $(F_{2k-1}(M_0), B(M_0))$ (cf. Example 3.3.4).
(ii) The thickening Ω of the 1-dimensional trinity

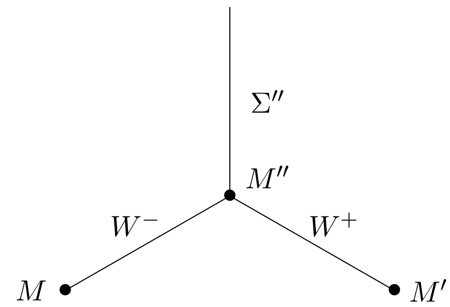
$$(W_1, W_2, W_3; M_0, M_1, M_2, M_3) \;=\; (I, I, I; \{0, 1\}, \emptyset, \emptyset, \emptyset)$$

is the 2-dimensional 'pair of pants' cobordism, with boundary $S^1 \sqcup S^1 \sqcup S^1$.

We now formalize Remark 1.3.2, that a trinity with $M_3 = \emptyset$ determines a split relative cobordism.

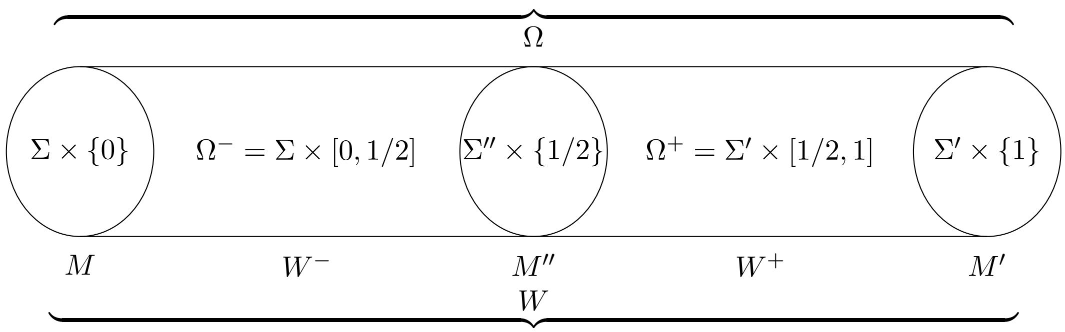
Proposition 4.5.3. *Let $(W_0, W_1, W_2; M_0, M_1, M_2, M_3)$ be an $(m+1)$-dimensional trinity such that $M_3 = \emptyset$, writing*

$$\begin{aligned}(W_0; M_1, M_0) \;&=\; (W^-; M, M'')\ ,\ (W_1; M_2, M_0) \;=\; (W^+; M', M'')\ ,\\ (W_2; M_3, M_0) \;&=\; (\Sigma''; \emptyset, M'')\ .\end{aligned}$$



Then the thickening of the trinity $W^- \cup \Sigma'' \cup W^+$ *is a split* $(m+2)$*-dimensional relative cobordism* $(\Omega; \Sigma, \Sigma', W; M, M')$ *given by*

$$\begin{aligned} \Sigma &= W^- \cup_{M''} \Sigma'' \ , \ \Sigma' \ = \ \Sigma'' \cup_{M''} W^+ \ , \\ \Omega &= \Sigma \times [0,1/2] \cup_{\Sigma'' \times \{1/2\}} \Sigma' \times [1/2,1] \ , \ W \ = \ W^- \cup_{M'' \times \{1/2\}} W^+ \ . \end{aligned}$$



Proof. By inspection. □

From now on, we shall only consider trinities with $M_3 = \emptyset$.

We shall now prove that up to homotopy equivalence every relative symmetric Poincaré cobordism is the thickening of an algebraic trinity.

Definition 4.5.4. An $(m+1)$*-dimensional symmetric Poincaré trinity* is defined by three $(m+1)$-dimensional symmetric Poincaré cobordisms of the type

$$(C_0 \oplus C_k \to D_k, (\delta\phi_k, \phi_0 \oplus -\phi_k)) \ (k = 1, 2, 3) \ .$$

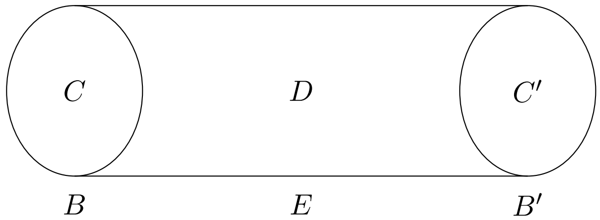
The *thickening* of the trinity is the $(m+2)$-dimensional symmetric Poincaré pair $(\partial D \to D, (\phi, \partial\phi))$ defined by algebraic mimicry of the thickening of a geometric trinity in Definition 4.5.1, using the glueing construction of [Ra1, p.135], with $D = \mathcal{C}(C_0 \to D_1 \oplus D_2 \oplus D_3)$.

Again, we shall only be concerned with algebraic trinities with $D_3 = 0$, in which case the thickening is a split relative symmetric Poincaré cobordism, exactly as in Proposition 4.5.3.

Let (Γ, Φ) be an $(m+2)$-dimensional relative symmetric Poincaré cobordism, with

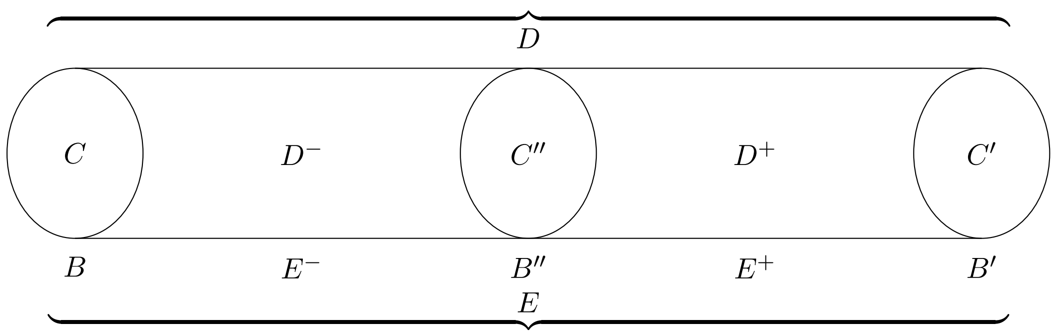
$$\Gamma \ = \ \begin{array}{ccc} B \sqcup B' & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ C \sqcup C' & \longrightarrow & D \end{array}$$

We can draw this as if it were a relative cobordism of manifolds:



Definition 4.5.5. (i) (Γ,Φ) is a *left product* if the chain map $C\to D$ is a chain equivalence, in which case $E\cup_{B'}C'\to D$ is also a chain equivalence.
(ii) (Γ,Φ) is a *right product* if the chain map $C'\to D$ is a chain equivalence, in which case $C\cup_B E\to D$ is also a chain equivalence.
(iii) (Γ,Φ) is *split* if it is homotopy equivalent to the union $(\Gamma^-,\Phi^-)\cup(\Gamma^+,\Phi^+)$ of a left product and a right product, with

$$\Gamma^- \ = \ \begin{array}{ccc} B\sqcup B'' & \longrightarrow & E^- \\ \big\downarrow & & \big\downarrow \\ C\sqcup C'' & \longrightarrow & D^- \end{array} \ , \ \Gamma^+ \ = \ \begin{array}{ccc} B''\sqcup B' & \longrightarrow & E^+ \\ \big\downarrow & & \big\downarrow \\ C''\sqcup C' & \longrightarrow & D^+ \end{array} \ .$$



Theorem 4.5.6. (Algebraic Poincaré Splitting)
Every relative symmetric Poincaré cobordism (Γ,Φ) *is homotopy equivalent to a split relative cobordism*

$$(\Gamma,\Phi) \ \simeq \ (\Gamma^-,\Phi^-)\cup(\Gamma^+,\Phi^+) \ ,$$

the thickening of an algebraic trinity.

Proof. The chain complex triads Γ^-,Γ^+ are defined by

$$C'' \ = \ \mathcal{C}(C\oplus C'\to D)_{*+1} \ , \ B'' \ = \ \mathcal{C}(C\oplus E\oplus C'\to D\oplus D)_{*+1} \ ,$$

$$D^- \ = \ C \ , \ E^- \ = \ \mathcal{C}(C\oplus E\to D)_{*+1} \ , \ D^+ \ = \ C' \ , \ E^+ \ = \ \mathcal{C}(E\oplus C'\to D)_{*+1}$$

with

$$D\simeq C\cup_{C''}C' \ , \ E\simeq E^-\cup_{B''}E^+ \ , \ C\simeq E^-\cup_{B''}C'' \ , \ C'\simeq C''\cup_{B''}E^+ \ .$$

It follows from the identity

$$\begin{array}{l} C(\Gamma)\otimes_{\mathbb{Z}\times\mathbb{Z}}C(\Gamma) \\ = \ \mathcal{C}((B\sqcup B''\sqcup B')\otimes_{\mathbb{Z}\times\mathbb{Z}}(B\sqcup B''\sqcup B')\to(E^-\sqcup C''\sqcup E^+)\otimes_{\mathbb{Z}\times\mathbb{Z}}(E^-\sqcup C''\sqcup E^+))_{*-1} \end{array}$$

that

$$Q^{m+2}(\Gamma) \ = \ Q^{m+1}(B\sqcup B''\sqcup B'\to E^-\sqcup C''\sqcup E^+) \ ,$$

and so

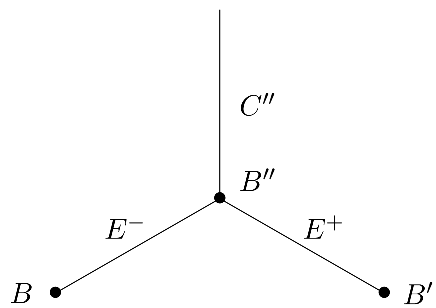
$$\Phi \ = \ \Phi^- \cup \Phi^+ \in Q^{m+2}(\Gamma)$$

for $\Phi^- \in Q^{m+2}(\Gamma^-)$, $\Phi^+ \in Q^{m+2}(\Gamma^+)$ determined uniquely by Φ. The $(m+2)$-dimensional relative symmetric Poincaré cobordism (Γ, Φ) thus determines an algebraic trinity, consisting of

(i) an $(m+1)$-dimensional symmetric Poincaré pair $(B'' \to C'', (\phi_{C''}, \phi_{B''}))$
(ii) $(m+1)$-dimensional symmetric Poincaré cobordisms

$$(B \oplus B'' \to E^-, (\phi_{E^-}, \phi_B \oplus -\phi_{B''}))\ ,\ (B'' \oplus B' \to E^+, (\phi_{E^+}, \phi_{B''} \oplus -\phi_{B'}))\ .$$

with union (Γ, Φ).



□

4.6. **Algebraic surgery.** We recall from [Ra1, §I.4] the essentials of the algebraic theory of surgery. An algebraic surgery on an m-dimensional symmetric Poincaré complex (C, ϕ) has input an $(m+1)$-dimensional symmetric pair $(e : C \to \delta C, (\delta\phi, \phi))$. The trace is the $(m+1)$-dimensional symmetric Poincaré cobordism $(C \sqcup C' \to \delta C', (0, \phi \sqcup -\phi'))$ with

$$d_{C'} \ = \ \begin{pmatrix} d_C & 0 & (-1)^{m+1}\phi_0 e^* \\ (-1)^r e & d_{\delta C} & (-)^r \delta\phi_0 \\ 0 & 0 & (-1)^r d^*_{\delta C} \end{pmatrix} :$$
$$C'_r \ = \ C_r \oplus \delta C_{r+1} \oplus \delta C^{m-r+1} \to C'_{r-1} \ = \ C_{r-1} \oplus \delta C_r \oplus \delta C^{m-r+2}\ ,$$

$$d_{\delta C'} \ = \ \begin{pmatrix} d_C & (-1)^{m+1}\phi_0 e^* \\ 0 & (-1)^r d^*_{\delta C} \end{pmatrix} \ : \ \delta C'_r \ = \ C_r \oplus \delta C^{m-r+1} \to \delta C'_{r-1} \ = \ C_{r-1} \oplus \delta C^{m-r+2}\ .$$

The effect of the algebraic surgery is the m-dimensional symmetric Poincaré complex (C', ϕ') cobordant to (C, ϕ). By definition, the surgery is of rank ℓ and index $r+1$ if

$$\delta C_s \ = \ \begin{cases} \mathbb{Z}^\ell & \text{for } s = m - r \\ 0 & \text{for } s \neq m - r\ . \end{cases}$$

An $(m+1)$-dimensional manifold cobordism $(W; M_0, M_1)$ determines an $(m+1)$-dimensional symmetric Poincaré cobordism

$$\Gamma \ = \ (C(M_0) \sqcup C(M_1) \to C(W), (\phi_W, \phi_{M_0} \sqcup -\phi_{M_1}))$$

such that $(C(M_1), \phi_{M_1})$ is homotopy equivalent to the effect of algebraic surgery on $(C(M_0), \phi_{M_0})$ by the symmetric pair $(C(M_0) \to C(W, M_1), (\phi_W/\phi_{M_1}, \phi_{M_0}))$. If $(W; M_0, M_1)$ is the trace of ℓ index $r+1$ surgeries on $\bigcup_\ell D^{r+1} \times D^{m-r} \subset M_0$ then Γ is the trace of an algebraic surgery of rank ℓ and index $r+1$.

By [Ra1, Prop.I.4.3] every algebraic surgery is composed of a sequence of surgeries of varying ranks, and increasing index, by analogy with the Thom-Milnor geometric handlebody theorem (1.2.1).

There is a corresponding algebraic theory of relative surgery on algebraic Poincaré pairs. Given an $(m+1)$-dimensional symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ and an input $(m+2)$-dimensional symmetric triad

$$\left(\begin{array}{ccc} C & \xrightarrow{e} & \delta C \\ \Big\downarrow f & & \Big\downarrow \delta f \\ D & \xrightarrow{g} & \delta D \end{array} \ , \ \begin{array}{ccc} \phi & \longrightarrow & \delta\phi \\ \Big\downarrow & & \Big\downarrow \\ \phi\delta & \longrightarrow & \delta\phi\delta \end{array} \right)$$

the effect of the algebraic relative surgery is an $(m+1)$-dimensional symmetric Poincaré pair $(f' : C' \to D', (\delta\phi', \phi'))$ with

$$\begin{aligned} f' &= \begin{pmatrix} f & 0 & 0 \\ 0 & \delta f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \\ C'_r &= C_r \oplus \delta C_{r+1} \oplus \delta C^{m-r+1} \to D'_r = D_r \oplus \delta D_{r+1} \oplus \delta D^{m-r+2} \oplus \delta C^{m-r+1} \ , \\ d_{D'} &= \begin{pmatrix} d_D & 0 & (-1)^{m+2}\phi\delta_0 g^* & 0 \\ (-1)^r e & d_{\delta D} & (-)^r \delta\phi\delta_0 & 0 \\ 0 & 0 & (-1)^r d^*_{\delta D} & 0 \\ 0 & 0 & \delta f^* & (-1)^r d^*_{\delta C} \end{pmatrix} : \\ D'_r &= D_r \oplus \delta D_{r+1} \oplus \delta D^{m-r+2} \oplus \delta C^{m-r+1} \to D'_{r-1} = D_{r-1} \oplus \delta D_r \oplus \delta D^{m-r+3} \oplus \delta C^{m-r+2} \end{aligned}$$

The algebraic effect of a half-surgery on a manifold with boundary is an algebraic surgery on a symmetric Poincaré pair, as follows. (See also [BNR1, Section2]).

As in Example 4.4.5 an $(m+2)$-dimensional relative manifold cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ determines an $(m+2)$-dimensional relative symmetric Poincaré cobordism

$$\left(\begin{array}{ccc} C(M_0) \sqcup C(M_1) & \longrightarrow & C(W) \\ \Big\downarrow & & \Big\downarrow \\ C(\Sigma_0) \sqcup C(\Sigma_1) & \longrightarrow & C(\Omega) \end{array} \ , \ \begin{array}{ccc} \phi_{M_0} \sqcup -\phi_{M_1} & \longrightarrow & \phi_W \\ \Big\downarrow & & \Big\downarrow \\ \phi_{\Sigma_0} \sqcup -\phi_{\Sigma_1} & \longrightarrow & \phi_\Omega \end{array} \right) .$$

The $(m+1)$-dimensional symmetric Poincaré pair $(C(M_1) \to C(\Sigma_1), (\phi_{\Sigma_1}, \phi_{M_1}))$ is obtained from $(C(M_0) \to C(\Sigma_0), (\phi_{\Sigma_0}, \phi_{M_0}))$ by a sequence of algebraic half-surgeries, corresponding to a relative algebraic surgery on the $(m+2)$-dimensional symmetric triad

$$\left(\begin{array}{ccc} C(M_0) & \longrightarrow & C(W, M_1) \\ \Big\downarrow & & \Big\downarrow \\ C(\Sigma_0) & \longrightarrow & C(\Omega, \Sigma_1) \end{array} \ , \ \begin{array}{ccc} \phi_{M_0} & \longrightarrow & \phi_W/\phi_{M_1} \\ \Big\downarrow & & \Big\downarrow \\ \phi_{\Sigma_0} & \longrightarrow & \phi_\Omega/\phi_{\Sigma_1} \end{array} \right)$$

(non-Poincaré in general) with

$$\begin{aligned} C(M_1)_r &= C(M_0)_r \oplus C(W, M_1)_{r+1} \oplus C(W, M_1)^{m+1-r} \ , \\ C(\Sigma_1)_r &= C(\Sigma_0)_r \oplus C(\Omega, \Sigma_1)_{r+1} \oplus C(\Omega, \Sigma_1)^{m+2-r} \oplus C(W, M_1)^{m+1-r} \ . \end{aligned}$$

If $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a boundary product then $C(W, M_1)$ is chain contractible. If $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a left product then $C(W, M_1) \to C(\Omega, \Sigma_1)$ is a chain equivalence. If $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a right product then $C(\Omega, \Sigma_1)$ is chain contractible. If $C(W, M_1)_{m-r} = \mathbb{Z}^{\ell_r}$ then $(C(M_1), \phi_{M_1})$ is obtained from $(C(M_0), \phi_{M_0})$ by a sequence of algebraic surgeries, with ℓ_r index $r+1$ surgeries on

$$\bigcup_{\ell_r} S^r \times D^{m-r} \subset \text{result of surgeries of index} \leqslant r \text{ on } M_0 \ (-1 \leqslant r \leqslant m)\ .$$

Furthermore, if $C(\Omega,\Sigma_1)_{m+1-r} = \mathbb{Z}^{k_r}$ then $(C(\Sigma_1),\phi_{\Sigma_1})$ is obtained from $(C(\Sigma_0 \cup_{M_0} W),\phi_{\Sigma_0\cup_{M_0}W})$ by a sequence of algebraic surgeries, with k_r index $r+1$ surgeries on

$$\bigcup_{k_r} S^r \times D^{m+1-r} \subset \text{result of surgeries of index} \leqslant r \text{ on } \Sigma_0 \cup_{M_0} W\backslash M_1 \ (-1 \leqslant r \leqslant m+1)\ .$$

Let (Σ_0, M_0) be an $(m+1)$-dimensional manifold with boundary. Given an embedding

$$(\bigcup_{\ell_r^-} D^r \times D^{m-r+1}, \bigcup_{\ell_r^-} S^{r-1} \times D^{m-r+1}) \subset (\Sigma_0, M_0)$$

there is defined a left product cobordism $(\Omega_0;\Sigma_0,\Sigma_{1/2},W^-;M_0,M_{1/2})$ as in Proposition 1.1.5 (ii), with

$$\begin{aligned}
\Omega_0 \ &= \ \Sigma_0 \times I\ ,\\
(W_0;M_0,M_{1/2}) \ &= \ (M_0 \times I \cup \bigcup_{\ell_r^-} D^r \times D^{m-r+1};\\
&\qquad M_0 \times \{0\}, \mathrm{cl.}(M_0 \backslash \bigcup_{\ell_r^-} S^{r-1} \times D^{m-r+1}) \cup \bigcup_{\ell_r^-} D^r \times S^{m-r})\ ,\\
\Sigma_{1/2} \ &= \ \mathrm{cl.}(\Sigma_0 \backslash \bigcup_{\ell_r^-} D^r \times D^{m-r+1})\ .
\end{aligned}$$

Given also an embedding $\bigcup_{\ell_{r+1}^+} S^r \times D^{m-r} \subset M_{1/2}$ there is defined a right product cobordism $(\Omega_1;\Sigma_{1/2},\Sigma_1,W_1;M_{1/2},M_1)$ as in Proposition 1.1.5 (iii), with

$$\begin{aligned}
\Omega_1 \ &= \ \Sigma_1 \times I\ ,\\
(W_1;M_{1/2},M_1) \ &= \ (M_{1/2} \times I \cup \bigcup_{\ell_{r+1}^+} D^{r+1} \times D^{m-r};\\
&\qquad M_{1/2} \times \{0\}, \mathrm{cl.}(M_{1/2} \backslash \bigcup_{\ell_{r+1}^+} S^r \times D^{m-r}) \cup \bigcup_{\ell_{r+1}^+} D^{r+1} \times S^{m-r-1})\ .
\end{aligned}$$

The $(m+2)$-dimensional relative cobordism

$$(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1) \ = \ (\Omega_0;\Sigma_0,\Sigma_{1/2},W_0;M_0,M_{1/2}) \cup (\Omega_1;\Sigma_{1/2},\Sigma_1,W_1;M_{1/2},M_1)$$

is an index $r+1$ elementary splitting (Definition 1.3.3 (i)) realizing geometrically the algebraic splitting of Theorem 4.5.6. The corresponding $(m+2)$-dimensional symmetric triad

$$\left(\begin{array}{ccc} C(M_0) & \longrightarrow & C(W,M_1) \\ \big\downarrow & & \big\downarrow \\ C(\Sigma_0) & \longrightarrow & C(\Omega,\Sigma_1) \end{array}\ ,\ \begin{array}{ccc} \phi_{M_0} & \longrightarrow & \phi_W/\phi_{M_1} \\ \big\downarrow & & \big\downarrow \\ \phi_{\Sigma_0} & \longrightarrow & \phi_\Omega/\phi_{\Sigma_1} \end{array}\right)$$

has

$$\begin{array}{ccccccccccccc}
C(W,M_1) & : & \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^{\ell_r^-} & \xrightarrow{d^*} & \mathbb{Z}^{\ell_{r+1}^+} & \longrightarrow & 0 & \longrightarrow & \dots \\
\big\downarrow & & & & & & \big\downarrow 1 & & \big\downarrow & & & & \\
C(\Omega,\Sigma_1) & : & \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^{\ell_r^-} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

with

$$C(W,M_1)_{m-r+1} \ = \ C(\Omega,\Sigma_1)_{m-r+1} \ = \ \mathbb{Z}^{\ell_r^-}\ ,\ C(W,M_1)_{m-r} \ = \ \mathbb{Z}^{\ell_{r+1}^+}\ .$$

For an $(m+2)$-dimensional relative cobordism with a half-handle decomposition as in 1.3.3 (ii)

$$(\Omega;\Sigma,\Sigma',W;M,M') \;=\; \bigcup_{r=0}^{m+1}(\Omega_r;\Sigma_r,\Sigma_{r+1},W_r;M_r,M_{r+1})$$

the $(m+2)$-dimensional symmetric triad

$$\left(\begin{array}{ccc} C(M) & \longrightarrow & C(W,M') \\ \big\downarrow & & \big\downarrow \\ C(\Sigma) & \longrightarrow & C(\Omega,\Sigma') \end{array}\;,\;\begin{array}{ccc} \phi_M & \longrightarrow & \phi_W/\phi_{M'} \\ \big\downarrow & & \big\downarrow \\ \phi_\Sigma & \longrightarrow & \phi_\Omega/\phi_{\Sigma'} \end{array}\right)$$

has

$$\begin{array}{ccccccc} C(W,M') \;:\; \dots & \longrightarrow & \mathbb{Z}^{\ell_r^+}\oplus\mathbb{Z}^{\ell_r^-} & \xrightarrow{\begin{pmatrix}(d^+)^* & \pm d^* \\ 0 & (d^-)^*\end{pmatrix}} & \mathbb{Z}^{\ell_{r+1}^+}\oplus\mathbb{Z}^{\ell_{r+1}^-} & \longrightarrow & \dots \\ \big\downarrow & & \big\downarrow{\scriptstyle (0\ 1)} & & \big\downarrow{\scriptstyle (0\ 1)} & & \\ C(\Omega,\Sigma') \;:\; \dots & \longrightarrow & \mathbb{Z}^{\ell_r^-} & \xrightarrow{(d^-)^*} & \mathbb{Z}^{\ell_{r+1}^-} & \longrightarrow & \dots \end{array}$$

with

$$\begin{aligned} C(W,M')_{m-r+1} \;&=\; C(W,M)^r \;=\; (C^+)^r\oplus(C^-)^r \;=\; \mathbb{Z}^{\ell_r^+}\oplus\mathbb{Z}^{\ell_r^-}\ , \\ C(\Omega,\Sigma')_{m-r+1} \;&=\; C(\Omega,\Sigma\cup_M W)^{r+1} \;=\; (C^-)^r \;=\; \mathbb{Z}^{\ell_r^-}\ . \end{aligned}$$

4.7. **Homological half-handle decompositions.** We now study the half-handle decompositions of relative cobordisms. The basic idea is that a manifold determines a symmetric Poincaré complex, a cobordism of manifolds determines a cobordism of symmetric Poincaré complexes, and similarly for relative cobordism. In each case, the handle (or half-handle) decompositions in topology determine symmetric Poincaré analogues in algebra. In fact, algebraic handle and half-handle decompositions can be constructed purely algebraically, as in :

Theorem 4.7.1. *Let $\Gamma=(\Omega;\Sigma,\Sigma',W;M,M')$ be an $(m+2)$-dimensional relative cobordism.*
(i) *The relative cobordism of $(m+1)$-dimensional symmetric Poincaré pairs*

$$C(\Gamma) \;=\; \begin{array}{ccccc} C(M) & \longrightarrow & C(W) & \longleftarrow & C(M') \\ \big\downarrow & & \big\downarrow{\scriptstyle k} & & \big\downarrow \\ C(\Sigma) & \xrightarrow{j} & C(\Omega) & \xleftarrow{j'} & C(\Sigma') \end{array}$$

is split in an essentially canonical manner, in the sense that there is a chain equivalence

$$C(\Gamma) \;\simeq\; C(\Gamma^-)\cup C(\Gamma^+)$$

of left and right product relative symmetric Poincaré cobordisms $C(\Gamma^-)$, $C(\Gamma^+)$ *with the algebraic properties of*

$$C(\Gamma^-) \;=\; \begin{array}{ccccc} C(M) & \longrightarrow & C(W^-) & \longleftarrow & C(M'') \\ \downarrow & & \downarrow & & \downarrow \\ C(\Sigma) & = & C(\Sigma) & \longleftarrow & C(\Sigma'') \end{array}$$

$$C(\Gamma^+) \;=\; \begin{array}{ccccc} C(M'') & \longrightarrow & C(W^+) & \longleftarrow & C(M') \\ \downarrow & & \downarrow & & \downarrow \\ C(\Sigma'') & \longrightarrow & C(\Sigma') & = & C(\Sigma') \end{array}$$

(*even though there may not be actual left and right product cobordisms* $\Gamma^- = (\Omega^-;\Sigma,\Sigma'',W^-;M,M'')$, $\Gamma^+ = (\Omega^+;\Sigma'',\Sigma',W^+;M'',M')$). *The chain complexes in* $C(\Gamma^-), C(\Gamma^+)$ *are defined by*

$$\begin{aligned} C(\Sigma'') &= \mathcal{C}((j\ j') : C(\Sigma)\oplus C(\Sigma') \to C(\Omega))_{*+1}\ , \\ C(M'') &= \mathcal{C}(\begin{pmatrix} j & k & 0 \\ 0 & k & j' \end{pmatrix} : C(\Sigma)\oplus C(W)\oplus C(\Sigma') \to C(\Omega)\oplus C(\Omega))_{*+1}\ , \\ C(W^-) &= \mathcal{C}((j\ k) : C(\Sigma)\oplus C(W) \to C(\Omega))_{*+1}\ , \\ C(W^+) &= \mathcal{C}((k\ j') : C(W)\oplus C(\Sigma') \to C(\Omega))_{*+1} \end{aligned}$$

with

$$\begin{aligned} &C(\Omega) \simeq \mathcal{C}(C(\Sigma'') \to C(\Sigma)\oplus C(\Sigma'))\ ,\ C(W) \simeq \mathcal{C}(C(M'') \to C(W^-)\oplus C(W^+))\ , \\ &C(\Sigma) \simeq \mathcal{C}(C(M'') \to C(W^-)\oplus C(\Sigma''))\ ,\ C(\Sigma') \simeq \mathcal{C}(C(M'') \to C(\Sigma'')\oplus C(W^+))\ , \\ &C(\Omega,\Sigma) \simeq C(W^+,M'')\ ,\ C(\Omega,\Sigma') \simeq C(W^-,M'')\ ,\ C(\Omega,\Sigma\cup_M W) \simeq C(W^-,M)_{*-1}\ . \end{aligned}$$

(ii) *Let* $d : (C^+,d^+) \to (C^-,d^-)_{*-1}$ *be a chain map of finite f.g. free* $\mathbb{Z}$*-module chain complexes such that*

$$d\ :\ (C^+,d^+) \simeq C(\Omega,\Sigma) \to (C^-,d^-)_{*-1} \simeq C(\Omega,\Sigma\cup_M W)$$

is in the chain homotopy class of the chain map induced by the inclusion $(\Omega,\Sigma)\subset(\Omega,\Sigma\cup_M W)$. *Then* $C(\Gamma)$ *has an algebraic half-handle decomposition*

$$C(\Gamma) \;=\; \bigcup_{r=-1}^{m+1} (C(\Omega_r);C(\Sigma_r),C(\Sigma_{r+1}),C(W_r);C(M_r),C(M_{r+1}))$$

with

$$\begin{aligned} &(C(\Omega_r);C(\Sigma_r),C(\Sigma_{r+1}),C(W_r);C(M_r),C(M_{r+1})) \\ &=\ (C(\Omega_r^+);C(\Sigma_r),C(\Sigma_{r+1/2}),C(W_r^-);C(M_r),C(M_{r+1/2})) \\ &\qquad \cup(C(\Omega_r^-);C(\Sigma_{r+1/2}),C(\Sigma_{r+1}),C(W_r^+);C(M_{r+1/2}),C(M_{r+1})) \end{aligned}$$

the canonical splitting given by (i), *elementary of index* $r+1$ *and rank*

$$(\ell_r^-,\ell_{r+1}^+) \;=\; (\dim_{\mathbb{Z}} C_r^-,\dim_{\mathbb{Z}} C_{r+1}^+)\ ,$$

with

$$\begin{aligned} H_q(W_r^-,M_r) &= H_{q+1}(\Omega_r,\Sigma_r\cup_{M_r} W_r) = H^{m-q+1}(\Omega_r,\Sigma_{r+1}) = \begin{cases} \mathbb{Z}^{\ell_r^-} & \text{if } q=r \\ 0 & \text{if } q\neq r\ , \end{cases} \\ H_q(W_r^+,M_{r+1/2}) &= H_q(\Omega_r,\Sigma_r) = \begin{cases} \mathbb{Z}^{\ell_{r+1}^+} & \text{if } q=r+1 \\ 0 & \text{if } q\neq r+1\ . \end{cases} \end{aligned}$$

(iii) *The relative homology groups of* (W, M) *are given by*

$$H_*(W, M) \;=\; H_{*+1}(d : (C^+, d^+) \to (C^-, d^-)_{*-1})$$

so that $(W; M, M')$ *is an* H*-cobordism if and only if* d *is a chain equivalence.*
(iv) *The relative homology groups of* (W_r, M_r) *are given by*

$$H_q(W_r, M_r) \;=\; \begin{cases} \ker(d : \mathbb{Z}^{\ell^+_{r+1}} \to \mathbb{Z}^{\ell^-_r}) & \textit{if } q = r+1 \\ \operatorname{coker}(d : \mathbb{Z}^{\ell^+_{r+1}} \to \mathbb{Z}^{\ell^-_r}) & \textit{if } q = r \\ 0 & \textit{otherwise} \end{cases}$$

so that $(C(W_r); C(M_r), C(M_{r+1}))$ *is an* H*-cobordism if and only if the* $\mathbb{Z}$*-module morphism* $d : C^+_{r+1} = \mathbb{Z}^{\ell^+_{r+1}} \to C^-_r = \mathbb{Z}^{\ell^-_r}$ *is an isomorphism. In particular, if* $(W; M, M') \cong M \times (I; \{0\}, \{1\})$ *it is possible to realize the chain map* $d : (C^+, d^+) \to (C^-, d^-)_{*-1}$ *by an isomorphism, and each* $(C(W_r); C(M_r), C(M_{r+1}))$ *is an* H*-cobordism.*

Proof. This is a direct application of the Algebraic Poincaré Splitting Theorem 4.5.6. Every cobordism of symmetric Poincaré pairs is the union of traces of algebraic half-surgeries. □

Remark 4.7.2. (i) In general, the algebraic splitting of Theorem 4.7.1 (i) is not realized geometrically. For example, if $\Gamma = (\Omega; \emptyset, \emptyset, \emptyset; \emptyset, \emptyset)$ is the relative cobordism determined by a non-empty closed $(m+2)$-dimensional manifold Ω there do not exist left and right product relative cobordisms

$$\Gamma^- \;=\; (\Omega^-; \Sigma, \Sigma'', W^-; M, M'') \;,\; \Gamma^+ \;=\; (\Omega^+; \Sigma'', \Sigma', W^+; M'', M')$$

such that $\Gamma = \Gamma^- \cup \Gamma^+$, with homology groups

$$H_*(\Sigma'') \;=\; H_*(W^-) \;=\; H_*(W^+) \;=\; H_{*+1}(\Omega) \;,\; H_*(M'') \;=\; H_{*+1}(\Omega) \oplus H_{*+1}(\Omega) \;.$$

(ii) See Example 3.3.4 below for even $m = 2n$ (resp. 3.4.4 for odd $m = 2n+1$) for the algebraic splitting of Theorem 4.7.1 (i) for the relative $(m+2)$-dimensional symmetric Poincaré cobordism $C(\Gamma) = (D; 0, 0, C; 0, 0)$ determined by an n-connected m-dimensional symmetric Poincaré pair (D, C), corresponding to a $(-1)^{n+1}$-symmetric form (resp. formation).
(iii) By [BNR1, Theorem 4.18] every relative cobordism $(\Omega; \Sigma, \Sigma', W; M, M')$ consisting of non-empty connected manifolds admits a half-handle decomposition, as a union of right and left product cobordisms. We shall not actually need geometric half-handle decompositions in this paper, only the algebraic half-handle decompositions of Theorem 4.7.1 (ii).

5. Codimension q embeddings, especially for $q = 2$.

5.1. **Codimension q embeddings.**

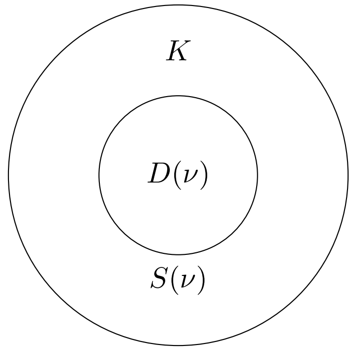
Definition 5.1.1. Let $q \geqslant 0$.
(i) A *codimension* q *embedding* $M^m \subset N^n$ is a proper embedding of an m-dimensional manifold M in an n-dimensional manifold N, such that $n - m = q$.
(ii) The *normal bundle* of $M \subset N$ is the normal q-plane bundle $\nu = \nu_{M \subset N} : M \to BSO(q)$. By the tubular neighbourhood theorem $M \subset N$ extends to a codimension 0 embedding $D(\nu) \subset N$, with

$$(D^q, S^{q-1}) \longrightarrow (D(\nu), S(\nu)) \xrightarrow{(p, \partial p)} M$$

the total (D^q, S^{q-1})-pair of ν, and $M \subset D(\nu)$ the zero section.
(iii) The *complement* of $M \subset N$ is the n-dimensional manifold with boundary

$$(K, \partial K) \;=\; (\mathrm{cl.}(N \backslash D(\nu)), S(\nu)) \;.$$

The inclusion $K \subset N\backslash M$ is a homotopy equivalence.



$$N = D(\nu) \cup_{S(\nu)} K$$

(iv) The *Thom class* $U \in H^q(D(\nu), S(\nu))$ of ν is characterized by the property that for every $x \in M$ the restriction of U along the inclusion of the fibre

$$(i_x, \partial i_x) \ : \ (p, \partial p)^{-1}(x) \ = \ (D^q, S^{q-1}) \subset (D(\nu), S(\nu))$$

is the generator

$$(i_x, \partial i_x)^*(U) \ = \ 1 \in H^q(D^q, S^{q-1}) \ = \ \mathbb{Z} \ .$$

The Thom isomorphism

$$U \cup - \ : \ H^r(M) \ \cong \ H^{r+q}(D(\nu), S(\nu))$$

sends the canonical element $(1, 1, \ldots, 1) \in H^0(M) = \mathbb{Z}[\pi_0(M)]$ to $U \in H^q(D(\nu), S(\nu))$.
(v) The *Euler class* of ν is the image

$$e \ = \ [U] \in H^q(D(\nu)) \ = \ H^q(M)$$

of the Thom class $U \in H^q(D(\nu), S(\nu))$. This is also the image of the fundamental class $[M] \in H_m(M)$ under the composite

$$H_m(M) \xrightarrow{\text{inclusion}_*} H_m(N) \ \cong \ H^q(N) \xrightarrow{\text{inclusion}^*} H^q(M) \ .$$

(vi) The embedding $M \subset N$ is *framed* if there is given a trivialization $\delta\nu : \nu \cong \epsilon^q$, in which case

$$(D(\nu), S(\nu)) \ \cong \ (M \times D^q, M \times S^{q-1}) \ , \ e \ = \ 0 \in H^q(M) \ .$$

Note that if $M^m \subset N^n$ is a codimension q embedding such that $[M] = 0 \in H_m(N)$ then $e = 0 \in H^q(M)$.

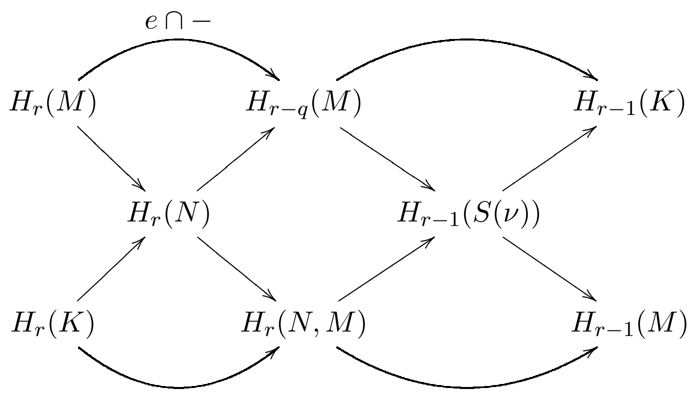
Proposition 5.1.2. *For a codimension q embedding $M^m \subset N^n$ with complement K the homology groups are such that*

$$H_r(N, K) \ \cong \ H_r(D(\nu), S(\nu)) \ \cong \ H_{r-q}(M) \ , \ H_r(N, M) \ \cong \ H_r(K, S(\nu)) \ .$$

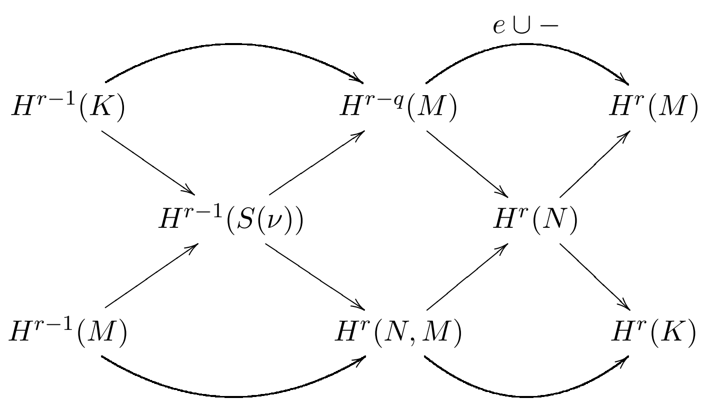
The Umkehr morphisms

$$H_r(N) \ \cong \ H^{n-r}(N) \xrightarrow{\text{inclusion}^*} H^{n-r}(M) \ \cong \ H_{r-q}(M) \ (n - q = m)$$

fit into a commutative braid of exact sequences



Similarly for cohomology, with a commutative braid of exact sequences



5.2. **Cobordism and surgery for codimension q embeddings.**

Definition 5.2.1. Let $M_0, M_1 \subset N$ be framed codimension q embeddings.
(i) A *cobordism* of $M_0, M_1 \subset N$ is a framed codimension q embedding

$$(W; M_0, M_1) \subset N \times (I; \{0\}, \{1\})$$

such that $W \cap (N \times \{j\}) = M_j$ $(j = 0, 1)$. The complements

$$(J; K_0, K_1) \;=\; (\mathrm{cl.}(N \times I \backslash W \times D^q), \mathrm{cl.}(N \times \{0\} \backslash M_0 \times D^q), \mathrm{cl.}(N \times \{1\} \backslash M_1 \times D^q))$$

are such that

$$\partial J \;=\; K_0 \cup_{M_0 \times S^{q-1}} W \times S^{q-1} \cup_{M_1 \times S^{q-1}} K_1 \ ,$$

and

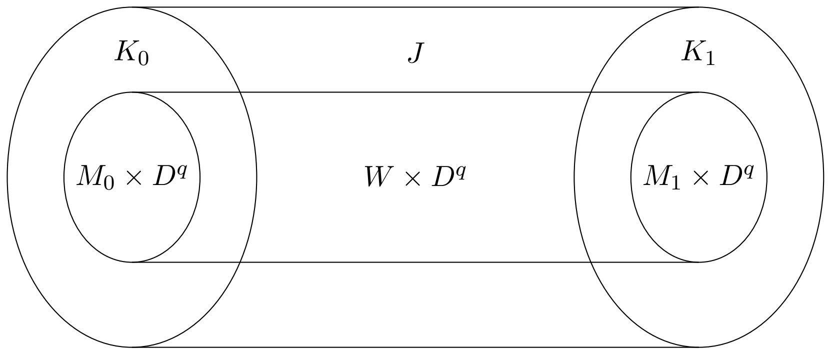
$$N \times (I; \{0\}, \{1\}) \;=\; (W; M_0, M_1) \times D^q \cup_{(W; M_0, M_1) \times S^{q-1}} (J; K_0, K_1) \ .$$

(ii) An *h-cobordism* of $M_0, M_1 \subset N$ is a cobordism $(W; M_0, M_1) \subset N \times (I; \{0\}, \{1\})$ such that $(W; M_0, M_1)$ is an h-cobordism.
(iii) An *isotopy* of $M_0, M_1 \subset N$ is a level-preserving h-cobordism of the type

$$(W; M_0, M_1) \;=\; M \times (I; \{0\}, \{1\}) \subset N \times (I; \{0\}, \{1\}) \ ,$$

so that for each $s \in I$ there is defined a framed codimension q embedding

$$M_s \;=\; M \times \{s\} \subset N \times \{s\} \ .$$



Proposition 5.2.2. *Let N be a closed n-dimensional manifold, and let $1 \leqslant q \leqslant n$.*
(i) *The Pontrjagin-Thom map of a framed codimension q submanifold $M^m \subset N^n$ ($m = n - q$)*

$$U_M \ : \ N \to N/\text{cl.}(N\backslash M \times D^q) \ = \ M \times D^q/M \times S^{q-1} \to D^q/S^{q-1} \ = \ S^q$$

is transverse regular at $0 \in S^q$, with $U_M^{-1}(0) = M$.
(ii) *A cobordism of framed codimension q submanifolds $(W; M, M') \subset N \times (I; \{0\}, \{1\})$ determines a homotopy $U_W : U_M \simeq U_{M'} : N \to S^q$.*
(iii) *Every map $U : N \to S^q$ is homotopic to a map which is transverse regular at $0 \in S^q$, with $M^m = U^{-1}(0) \subset N^n$ a framed codimension q submanifold. Homotopic maps determine cobordant submanifolds.*
(iv) *The Pontrjagin-Thom construction defines a bijection between the set of cobordism classes of framed codimension q submanifolds $M \subset N$ and the qth cohomotopy group $[N, S^q]$.*

Proof. Standard (e.g. [Ra4, 6.10]), noting that $S^q = T(\epsilon^q)$ is the Thom space of the unique q-plane bundle $\epsilon^q = \mathbb{R}^q$ over a point. □

Remark 5.2.3. There is also a rel ∂ version of Pontrjagin-Thom theory, with $(K, \partial K)$ an $(m+q)$-dimensional manifold with boundary $\partial K = M^m \times S^{q-1}$, such as the complement of a framed codimension q embedding $M^m \subset N^{m+q}$.
(i) There exists a framed codimension $(q-1)$ embedding $\Sigma^{m+1} \subset K$ with

$$M \ = \ \Sigma \cap \partial K \ = \ M \times \{*\}$$

for some $* \in S^{q-1}$ if and only if the projection $\partial K \to S^{q-1}$ extends to a map $U : K \to S^{q-1}$, in which case U may be taken to be regular at $* \in S^{q-1}$ and $\Sigma = U^{-1}(*) \subset K$ will do.
(ii) For any framed codimension $(q-1)$ embedding $\Sigma^{m+1} \subset K$ with $M = M \times \{*\}$ there is a rel ∂ Pontrjagin-Thom map

$$U \ : \ K \to K/\text{cl.}(K\backslash\Sigma \times D^{q-1}) \ = \ (\Sigma \times D^{q-1})/(\Sigma \times S^{q-1}) \to D^{q-1}/S^{q-1} \ = \ S^{q-1} \ .$$

For two such embeddings $\Sigma_0, \Sigma_1 \subset K$ there exists a framed codimension $(q-1)$ embedding $(\Omega^{m+2}; \Sigma_0, \Sigma_1) \subset K \times (I; \{0\}, \{1\})$ with $\partial\Omega = \Sigma_0 \cup (M \times I) \cup \Sigma_1$ if and only if the rel ∂ Pontrjagin-Thom maps are homotopic, $U_0 = U_1 \in [K, S^{q-1}]$.

5.3. **Embedded split relative cobordisms.** We shall now extend the splitting theory of relative cobordisms in §1.3 to relative cobordisms which are embedded in $N \times (I; \{0\}, \{1\})$, for use in §6.

Definition 5.3.1. An *embedded splitting* of a framed codimension $q - 1$ embedding of an $(m+2)$-dimensional relative cobordism

$$\Gamma \ = \ (\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1) \subset N \times (I; \{0\}, \{1\})$$

is a splitting (in the sense of Definition 1.3.1) as a union $\Gamma = \Gamma^- \cup \Gamma^+$ of left and right relative cobordisms

$$\Gamma^- = (\Omega^-; \Sigma_0, \Sigma_{1/2}, W^-; M_0, M_{1/2}) \ , \ \Gamma^+ = (\Omega^+; \Sigma_{1/2}, \Sigma_1, W^+; M_{1/2}, M_1)$$

together with framed codimension $q-1$ embeddings

$$\Gamma^- \subset N \times ([0,1/2]; \{0\}, \{1/2\}) \ , \ \Gamma^+ \subset N \times ([1/2,1]; \{1/2\}, \{1\})$$

involving the same embedding $(\Sigma_{1/2}, M_{1/2}) \subset N \times \{1/2\}$.

We shall construct embedded split relative cobordisms using the following general method:

Lemma 5.3.2. *Let N be a closed n-dimensional manifold which is a union*

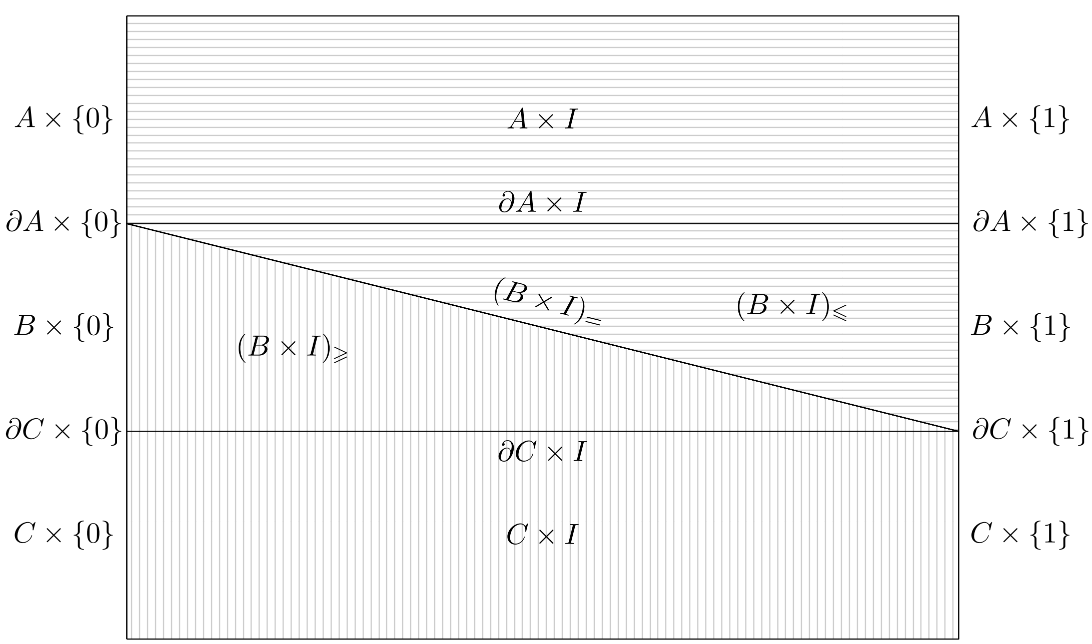
$$N = A \cup B \cup C$$

with $(A, \partial A)$, $(C, \partial C)$ n-dimensional manifolds with boundary, and $(B; \partial A, \partial C)$ an n-dimensional cobordism. Given a Morse function $f : (B; \partial A, \partial C) \to (I; \{0\}, \{1\})$ define submanifolds

$$\begin{aligned} (B \times I)_{\leqslant} &= \{(x,y) \in B \times I \,|\, f(x) \leqslant y\} \ , \\ (B \times I)_{\geqslant} &= \{(x,y) \in B \times I \,|\, f(x) \geqslant y\} \ , \\ (B \times I)_{=} &= \{(x,y) \in B \times I \,|\, f(x) = y\} \end{aligned}$$

such that

$$\begin{aligned} B \times I &= (B \times I)_{\leqslant} \cup_{(B\times I)_=} (B \times I)_{\geqslant} \ , \\ N \times I &= (A \times I \cup_{\partial A \times I} (B \times I)_{\leqslant}) \cup_{(B \times I)_=} ((B \times I)_{\geqslant} \cup_{\partial C \times I} C \times I) \end{aligned}$$



The right and left product relative cobordisms

$$\begin{aligned} \Gamma_{\leqslant} &= (A \times I \cup (B \times I)_{\leqslant}; A \times \{0\}, (A \cup B) \times \{1\}, (B \times I)_=; \partial A \times \{0\}, \partial C \times \{1\}) \ , \\ \Gamma_{\geqslant} &= ((B \times I)_{\geqslant} \cup C \times I; (B \cup C) \times \{0\}, C \times \{1\}, (B \times I)_=; \partial A \times \{0\}, \partial C \times \{1\}) \end{aligned}$$

are such that

$$N \times (I; \{0\}, \{1\}) = \Gamma_{\leqslant} \cup_{\Gamma_=} \Gamma_{\geqslant}$$

with $\Gamma_= = ((B \times I)_=; \partial A \times \{0\}, \partial C \times \{1\})$.

Proposition 5.3.3. (i) *Suppose given an $(m+1)$-dimensional manifold with boundary (Σ_0, M_0) with a decomposition*

$$\Sigma_0 \;=\; W^- \cup_{M^-_{1/2}} \Sigma_{1/2}$$

for an $(m+1)$-dimensional manifold with boundary $(\Sigma^-_{1/2}, M^-_{1/2})$ and an $(m+1)$-dimensional cobordism $(W^-; M_0, M^-_{1/2})$. Let $\Gamma^- = (\Omega^-; \Sigma_0, \Sigma^-_{1/2}, W^-; M_0, M^-_{1/2})$ be the corresponding $(m+2)$-dimensional left product relative cobordism defined by $\Omega^- = \Sigma_0 \times [0,1/2]$, and define the $(m+q+1)$-dimensional left product relative cobordism

$$\begin{aligned}\Gamma^- \times D^{q-1} \;=\; (&\Omega^- \times D^{q-1}; \Sigma_0 \times D^{q-1}, \Sigma^-_{1/2} \times D^{q-1}, \Omega^- \times S^{q-2} \cup W^- \times D^{q-1};\\ &(\Sigma_0 \times S^{q-2} \cup M_0 \times D^{q-1}), (\Sigma^-_{1/2} \times S^{q-2} \cup M^-_{1/2} \times D^{q-1}))\ .\end{aligned}$$

A framed codimension $q-1$ embedding $\Sigma_0 \subset N^{m+q}$ extends to a framed codimension $q-1$ embedding $\Gamma^- \subset N \times ([0,1/2]; \{0\}, \{1/2\})$ with complements

$$\begin{aligned}J_{\Omega^-} \;&=\; \mathrm{cl.}(N \times [0,1/2] \backslash \Omega^- \times D^{q-1})\ ,\ J_{W^-} \;=\; \mathrm{cl.}(N \times [0,1/2] \backslash W^- \times D^q)\ ,\\ K_{\Sigma_0} \;&=\; \mathrm{cl.}(N \times \{0\} \backslash \Sigma_0 \times D^{q-1})\ ,\ K_{\Sigma^-_{1/2}} \;=\; \mathrm{cl.}(N \times \{1/2\} \backslash \Sigma^-_{1/2} \times D^{q-1})\end{aligned}$$

such that

$$\begin{aligned}\Delta^- \;=\; (&J_{\Omega^-}; K_{\Sigma_0}, K_{\Sigma^-_{1/2}}, \Omega^- \times S^{q-2} \cup W^- \times D^{q-1};\\ &(\Sigma_0 \times S^{q-2} \cup M_0 \times D^{q-1}), (\Sigma^-_{1/2} \times S^{q-2} \cup M^-_{1/2} \times D^{q-1}))\end{aligned}$$

is an $(m+q+1)$-dimensional right product cobordism with

$$(\Gamma^- \times D^{q-1}) \cup \Delta^- \;=\; N \times ([0,1/2]; \{0\}, \{1/2\})\ .$$

(ii) *Suppose given an $(m+1)$-dimensional manifold with boundary (Σ_1, M_1) with a decomposition*

$$\Sigma_1 \;=\; \Sigma^+_{1/2} \cup W^+$$

for an $(m+1)$-dimensional manifold with boundary $(\Sigma^+_{1/2}, M^+_{1/2})$ and an $(m+1)$-dimensional cobordism $(W^+; M^+_{1/2}, M_1)$. Let $\Gamma^+ = (\Omega^+; \Sigma^+_{1/2}, \Sigma_1, W^+; M^+_{1/2}, M_1)$ be the corresponding $(m+2)$-dimensional right product relative cobordism, with $\Omega^+ = \Sigma_1 \times [1/2, 1]$, and define the $(m+q+1)$-dimensional left product relative cobordism

$$\begin{aligned}\Gamma^+ \times D^{q-1} \;=\; (&\Omega^+ \times D^{q-1}; \Sigma^+_{1/2} \times D^{q-1}, \Sigma_1 \times D^{q-1}, \Omega^+ \times S^{q-2} \cup W^+ \times D^{q-1};\\ &(\Sigma^+_{1/2} \times S^{q-2} \cup M^+_{1/2} \times D^{q-1}), (\Sigma_1 \times S^{q-2} \cup M_1 \times D^{q-1}))\ .\end{aligned}$$

Given a framed codimension $q-1$ embedding $\Sigma_1 \subset N$ there is defined an extension to a framed codimension $q-1$ embedding $\Gamma^- \subset N \times ([1/2,1]; \{1/2\}, \{1\})$ with complements

$$\begin{aligned}J_{\Omega^+} \;&=\; \mathrm{cl.}(N \times [1/2,1] \backslash \Omega^+ \times D^{q-1})\ ,\ J_{W^+} \;=\; \mathrm{cl.}(N \times [1/2,1] \backslash W^+ \times D^q)\ ,\\ K_{\Sigma_0} \;&=\; \mathrm{cl.}(N \times \{1/2\} \backslash \Sigma^+_{1/2} \times D^{q-1})\ ,\ K_{\Sigma_1} \;=\; \mathrm{cl.}(N \times \{1\} \backslash \Sigma_1 \times D^{q-1})\end{aligned}$$

such that

$$\begin{aligned}\Delta^+ \;=\; (&J_{\Omega^+}; K_{\Sigma^+_{1/2}}, K_{\Sigma_1}, \Omega^+ \times S^{q-1} \cup W^+ \times D^q;\\ &(\Sigma_1 \times S^{q-2} \cup M_1 \times D^{q-1})m, (\Sigma^+_{1/2} \times S^{q-2} \cup M^+_{1/2} \times D^{q-1}))\end{aligned}$$

is an $(m+q+1)$-dimensional left product cobordism with

$$(\Gamma^+ \times D^{q-1}) \cup \Delta^+ \;=\; N \times ([1/2,1]; \{1/2\}, \{1\})\ .$$

(iii) *If* $\Gamma = (\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ *is an* $(m+2)$*-dimensional relative cobordism which is split, i.e. expressed as a union of a left and a right product cobordism*

$$\Gamma \;=\; \Gamma^- \cup \Gamma^+ \;=\; (\Omega^-;\Sigma_0,\Sigma_{1/2},W^-;M_0,M_{1/2}) \cup (\Omega^+;\Sigma_{1/2},\Sigma_1,W^+;M_{1/2},M_1)$$

restricting to the same embedding $\Sigma_{1/2} \subset N$, *and* Δ^-,Δ^+ *are obtained from* Γ^-,Γ^+ *as in* (i) *and* (ii) *then* $\Delta = \Delta^- \cup \Delta^+$ *is a union of a right and a left product cobordism such that*

$$(\Gamma \times D^{q-1}) \cup \Delta \;=\; N \times (I;\{0\},\{1\})\ ,$$

so that $\Gamma \subset N \times (I;\{0\},\{1\}$ *is embedded split.*

Proof. (i) Apply Lemma 5.3.2 with $N = A^- \cup B^- \cup C^-$ for

$$A^- \;=\; K_{\Sigma_0}\ ,\ B^- \;=\; W^- \times D^{q-1}\ ,\ C^- \;=\; \Sigma^-_{1/2} \times D^{q-1}\ .$$

(ii) Apply Lemma 5.3.2 with $N = A^+ \cup B^+ \cup C^+$ for

$$A^+ \;=\; K_{\Sigma_1}\ ,\ B^+ \;=\; W^+ \times D^{q-1}\ ,\ C^+ \;=\; \Sigma^+_{1/2} \times D^{q-1}\ .$$

(iii) Combine (i) and (ii). □

Proposition 5.3.4. *Let* (Σ,M) *be an* $(m+1)$*-dimensional manifold with boundary. The complement of a framed codimension* $q-1$ *embedding* $(\Sigma,M) \subset N^{m+q}$ *is the* $(m+q)$*-dimensional manifold with boundary*

$$(K_\Sigma,\partial K_\Sigma) \;=\; (\mathrm{cl.}(N\backslash\Sigma \times D^{q-1}),\partial(\Sigma \times D^{q-1}))$$

such that there are defined chain equivalences

$$\begin{aligned}&\mathcal{C}(C(\partial(\Sigma \times D^{q-1})) \to C(\Sigma \times D^{q-1}) \oplus C(K_\Sigma)) \;\simeq\; C(N)\ ,\\ &C(K_\Sigma) \;\simeq\; \mathcal{C}(C(N) \to C(\Sigma,M)_{*-q+1})_{*+1}\ .\end{aligned}$$

Proof. Immediate from the excision chain equivalence

$$C(N,K_\Sigma) \;\simeq\; C(\Sigma \times D^{q-1},\partial(\Sigma \times D^{q-1})) \;=\; C(\Sigma,M)_{*-q+1}\ .$$

□

We do not actually need topological embedded half-handle decompositions in this paper, relying instead on algebraic half-handle decompositions.

Using the language of algebraic surgery recalled in §4 we formulate:

Definition 5.3.5. An $(m+1)$-dimensional symmetric Poincaré pair (A,B) has an *algebraic codimension* $q-1$ *embedding* $(A,B) \subset C$ in an $(m+q)$-dimensional symmetric Poincaré complex C if there are given a *complement* $(m+q)$-dimensional symmetric Poincaré pair $(D,\partial D)$ with boundary $(m+q-1)$-dimensional symmetric Poincaré complex

$$\begin{aligned}\partial D \;&=\; (A \otimes C(S^{q-2})) \cup (B \otimes C(D^{q-1}))\\ &=\; \mathcal{C}(B \otimes C(S^{q-2}) \to (A \otimes C(S^{q-2})) \oplus (B \otimes C(D^{q-1})))\end{aligned}$$

and a chain equivalence

$$(A \otimes C(D^{q-1})) \cup D \;=\; \mathcal{C}(B \otimes C(S^{q-2}) \to (A \otimes C(D^{q-1})) \oplus D) \xrightarrow{\;\simeq\;} C\ .$$

Proposition 5.3.4 gives the prime example of an algebraic codimension $q-1$ embedding

$$(A,B) \;=\; (C(\Sigma),C(M)) \subset C \;=\; C(N)$$

with complement $D = C(K_\Sigma)$.

The algebraic theory of surgery extends to symmetric Poincaré complexes with a subcomplex modelled on a framed codimension q embedding $M^m \subset N^{m+q}$ with complement $K = \text{cl.}(N \backslash M \times D^q)$, so that

$$N \;=\; M \times D^q \cup_{M \times S^{q-1}} K$$

(cf. [Ra3, §7]). The $(m+q)$-dimensional symmetric Poincaré complex of N has an algebraic decomposition as a union

$$(C(N), \phi_N) \;=\; (C(M \times D^q) \cup_{C(M \times S^{q-1})} C(K), \phi_{M \times D^q} \cup_{\phi_{M \times S^{q-1}}} \phi_K) \qquad (*)$$

of the $(m+q)$-dimensional symmetric Poincaré pairs of $(M \times D^q, M \times S^{q-1})$ and $(K, M \times S^{q-1})$, with

$$\begin{aligned}&(C(M \times S^{q-1}) \to C(M \times D^q), (\phi_{M \times D^q}, \phi_{M \times S^{q-1}}))\\ &\quad = \; (C(M), \phi_M) \otimes (C(S^{q-1}) \to C(D^q), (\phi_{D^q}, \phi_{S^{q-1}}))\end{aligned}$$

For an $(m+q)$-dimensional symmetric pair $(f : C(N) \to D, (\phi_D, \phi_N))$ there is defined an algebraic surgery with trace an $(m+q)$-dimensional symmetric Poincaré cobordism $(C(N) \oplus C' \to D', (\phi_{D'}, \phi_N \oplus -\phi_C)$. If f, D and ϕ_D have codimension q decompositions as (algebraic) unions compatible with $(*)$ then so does $(C(N) \oplus C' \to D', (\phi_{D'}, \phi_N \oplus -\phi_C)$.

Theorem 5.3.6. *For any framed codimension $q-1$ embedding*

$$\Gamma \;=\; (\Omega; \Sigma, \Sigma', W; M, M') \subset N \times (I; \{0\}, \{1\})$$

the algebraic half-handle decomposition of the relative symmetric Poincaré cobordism $C(\Gamma)$ given by Theorem 4.7.1 can be realized by a union of algebraic codimension $q-1$ embeddings

$$C(\Gamma) \;=\; \bigcup_{r=-1}^{m+1} (C(\Omega_r); C(\Sigma_r), C(\Sigma_{r+1}), C(W_r); C(M_r), C(M_{r+1})) \subset C(N) \otimes C(I; \{0\}, \{1\}) \;.$$

Proof. Given a splitting of Γ as a union of left and right product cobordisms

$$(\Omega; \Sigma, \Sigma', W; M, M') \;=\; (\Omega^-; \Sigma, \Sigma_{1/2}, W^-; M, M_{1/2}) \cup (\Omega^+; \Sigma_{1/2}, \Sigma', W^+; M_{1/2}, M')$$

Proposition 5.3.3 (i) and (ii) gives extensions to framed codimension $q-1$ embeddings

$$\begin{aligned}\Gamma^- \;&=\; (\Omega^-; \Sigma, \Sigma_{1/2}, W^-; M, M_{1/2}) \subset N \times ([0,1/2]; \{0\}, \{1/2\}) \;,\\ \Gamma^+ \;&=\; (\Omega^+; \Sigma_{1/2}, \Sigma', W^+; M_{1/2}, M') \subset N \times ([1/2,1]; \{1/2\}, \{1\})\end{aligned}$$

with right and left product complements

$$\begin{aligned}&\Delta^- \;=\; (J_{\Omega^-}; K_\Sigma, K_{\Sigma^-_{1/2}}, \Omega^- \times S^{q-2} \cup W^- \times D^{q-1};\\ &(\Sigma \times S^{q-2} \cup M \times D^{q-1}), (\Sigma_{1/2} \times S^{q-2} \cup M_{1/2} \times D^{q-1})) \subset N \times ([0,1/2]; \{0\}, \{1/2\}) \;,\\ &\Delta^+ \;=\; (J_{\Omega^+}; K_{\Sigma_{1/2}}, K_{\Sigma'}, \Omega^+ \times S^{q-2} \cup W^+ \times D^{q-1};\\ &(\Sigma_{1/2} \times S^{q-2} \cup M_{1/2} \times D^{q-1}), (\Sigma' \times S^{q-2} \cup M' \times D^{q-1})) \subset N \times ([1/2,1]; \{1/2\}, \{1\})\end{aligned}$$

such that

$$\begin{aligned}(\Gamma^- \times D^{q-1}) \cup \Delta^- \;&=\; N \times ([0,1/2]; \{0\}, \{1/2\}) \;,\\ (\Gamma^+ \times D^{q-1}) \cup \Delta^+ \;&=\; N \times ([1/2,1]; \{1/2\}, \{1\}) \;.\end{aligned}$$

Next, recall the terminology of 4.7.1, and the relative cobordism of $(m+1)$-dimensional symmetric Poincaré pairs

$$C(\Gamma) \;=\; \begin{array}{ccccc} C(M) & \longrightarrow & C(W) & \longleftarrow & C(M') \\ \downarrow & & \downarrow{\scriptstyle k} & & \downarrow \\ C(\Sigma) & \xrightarrow{\;j\;} & C(\Omega) & \xleftarrow{\;j'\;} & C(\Sigma') \end{array}$$

involving chain complexes with Poincaré duality. There is defined a chain equivalence

$$C(\Gamma) \simeq C(\Gamma^-) \cup C(\Gamma^+)$$

with $C(\Gamma^-)$, $C(\Gamma^+)$ the left and right product symmetric Poincaré cobordisms defined by

$$C(\Gamma^-) = \begin{array}{ccccc} C(M) & \longrightarrow & C(W^-) & \longleftarrow & C(M_{1/2}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ C(\Sigma) & = & C(\Sigma) & \longleftarrow & C(\Sigma_{1/2}) \end{array}$$

$$C(\Gamma^+) = \begin{array}{ccccc} C(M_{1/2}) & \longrightarrow & C(W^+) & \longleftarrow & C(M') \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ C(\Sigma_{1/2}) & \longrightarrow & C(\Sigma') & = & C(\Sigma') \end{array}$$

with

$$\begin{aligned} C(\Sigma_{1/2}) &= \mathcal{C}((j\ j') : C(\Sigma) \oplus C(\Sigma') \to C(\Omega))_{*+1} , \\ C(M_{1/2}) &= \mathcal{C}(\begin{pmatrix} j & k & 0 \\ 0 & k & j' \end{pmatrix} : C(\Sigma) \oplus C(W) \oplus C(\Sigma') \to C(\Omega) \oplus C(\Omega))_{*+1} , \\ C(W^-) &= \mathcal{C}((j\ k) : C(\Sigma) \oplus C(W) \to C(\Omega))_{*+1} , \\ C(W^+) &= \mathcal{C}((k\ j') : C(W) \oplus C(\Sigma') \to C(\Omega))_{*+1} \end{aligned}$$

such that

$$\begin{array}{l} C(\Omega) \simeq \mathcal{C}(C(\Sigma_{1/2}) \to C(\Sigma) \oplus C(\Sigma')) ,\ C(W) \simeq \mathcal{C}(C(M_{1/2}) \to C(W^-) \oplus C(W^+)) , \\ C(\Sigma) \simeq \mathcal{C}(C(M_{1/2}) \to C(W^-) \oplus C(\Sigma_{1/2})) ,\ C(\Sigma') \simeq \mathcal{C}(C(M_{1/2}) \to C(\Sigma_{1/2}) \oplus C(W^+)) , \\ C(\Omega, \Sigma) \simeq C(W^+, M_{1/2}) ,\ C(\Omega, \Sigma') \simeq C(W^-, M_{1/2}) ,\ C(\Omega, \Sigma \cup_M W) \simeq C(W^-, M)_{*-1} . \end{array}$$

(Again, there may not be actual left and right product cobordisms $(\Omega^-; \Sigma, \Sigma_{1/2}, W^-; M, M_{1/2})$, $(\Omega^+; \Sigma_{1/2}, \Sigma', W^+; M_{1/2}, M'))$. Define right and left $(m+q+1)$-dimensional symmetric Poincaré relative cobordisms

$$\Delta^- = \begin{array}{ccccc} C(K_\Sigma) & \longrightarrow & C(J_{\Omega^-}) & \longleftarrow & C(K_{\Sigma_{1/2}}) \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ D & \longrightarrow & E^- & \longleftarrow & D_{1/2} \end{array} ,\ \Delta^+ = \begin{array}{ccccc} C(K_{\Sigma_{1/2}}) & \longrightarrow & C(J_{\Omega^+}) & \longleftarrow & C(K_{\Sigma'}) \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ D_{1/2} & \longrightarrow & E^+ & \longleftarrow & D' \end{array}$$

by

$$\begin{aligned} C(J_{\Omega^-}) = C(J_{\Omega^+}) = C(K_{\Sigma_{1/2}}) &= \mathcal{C}(C(N) \to C(\Sigma_{1/2}, M_{1/2})_{*-q+1})_{*+1} \\ &\simeq \mathcal{C}(C(N) \to C(\Omega, W)_{*-q})_{*+1} , \\ D_i &= C(\partial(\Sigma_i \times D^{q-1})) = C(\Sigma_i \times S^{q-2} \cup M_i \times D^{q-1})\ (i = 0, 1/2, 1) , \\ E^\pm &= C(\Omega^\pm \times S^{q-2} \cup W^\pm \times D^{q-1}) . \end{aligned}$$

By Proposition 5.3.3 (iii) the unions

$$C(\Gamma) = C(\Gamma^-) \cup C(\Gamma^+) ,\ \Delta = \Delta^- \cup \Delta^+$$

are such that there is defined a chain equivalence

$$\Delta \otimes C(D^{q-1}) \cup C(\Gamma) \simeq C(N) \otimes C(I; \{0\}, \{1\})$$

giving an embedded splitting of $C(\Gamma)$. This gives the embedded analogue of Theorem 4.7.1 (i). Apply the analogue successively to the relative cobordisms in the algebraic half-handle decomposition of 4.7.1 (ii), to obtain an algebraic codimension $q-1$ embedded half-handle decomposition. □

5.4. **Codimension 2 embeddings.**

We shall be mainly concerned with codimension 2 embeddings $M^m \subset N^{m+2}$, particularly for $N = S^{m+2}$. We shall assume that N is connected, but not that M is connected, so the theory will apply to links as well as knots.

Example 5.4.1. (i) An *m-dimensional knot* is a codimension 2 embedding $S^m \subset S^{m+2}$.
(ii) An *m-dimensional link* is a codimension 2 embedding $S^m \cup S^m \cup \cdots \cup S^m \subset S^{m+2}$.

We shall make much use of Seifert surfaces for a codimension 2 embedding $M^m \subset N^{m+2}$, which are the codimension 1 framed embeddings $\Sigma^{m+1} \subset N^{m+2}$ with $\partial\Sigma = M$. In Theorem 5.4.8 below we shall prove that a codimension 2 embedding $M \subset N$ admits a Seifert surface if and only if $[M] = 0 \in H_m(N)$, in which case the embedding can be framed.

Proposition 5.4.2. *Let $M^m \subset N^{m+2}$ be a codimension 2 embedding, with normal 2-plane bundle $\nu = \nu_{M\subset N} : M \to BSO(2) = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.*
(i) *The Thom class of ν is*

$$(1, 1, \ldots, 1) \in H^2(D(\nu), S(\nu)) \;=\; H^0(M) \;=\; \mathbb{Z}[\pi_0(M)] \;,$$

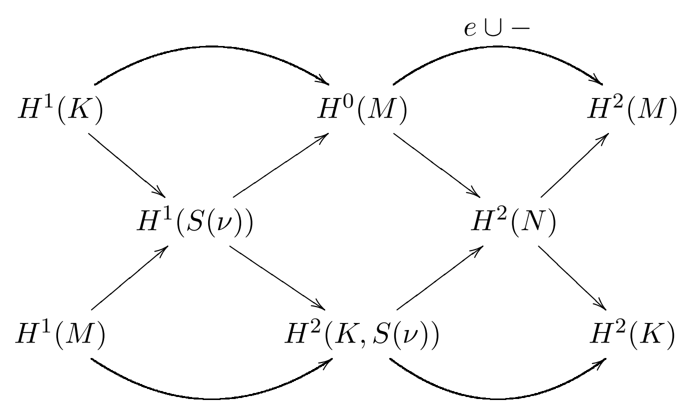
and ν is classified by the Euler class

$$e \;=\; [1, 1, \ldots, 1] \in H^2(D(\nu)) \;=\; H^2(M) \;=\; [M, BSO(2)] \;.$$

The Euler class e is the image of the fundamental class $[M] \in H_m(M)$ under the composite

$$H_m(M) \xrightarrow{\text{inclusion}_*} H_m(N) \;\cong\; H^2(N) \xrightarrow{\text{inclusion}^*} H^2(M) \;.$$

In particular, $M \subset N$ can be framed (i.e. ν is trivial) if and only if $e = 0 \in H^2(M)$.
(ii) *There is defined a commutative braid of exact sequences*



noting that $H^*(N, M) = H^*(N, D(\nu)) = H^*(K, S(\nu))$. In particular

$$\ker(H^0(M) \to H^2(N)) \;=\; \operatorname{im}(H^1(K) \to H^0(M)) \;,$$
$$\ker(e \cup - : H^0(M) \to H^2(M)) \;=\; \operatorname{im}(H^1(S(\nu)) \to H^0(M)) \;.$$

(iii) *The Euler class $e \in H^2(D(\nu)) = H^2(M)$ is the image of the Thom class*

$$(1, 1, \ldots, 1) \in H^2(N, K) \;=\; H^2(D(\nu), S(\nu)) \;=\; H^0(M) \;,$$

so that a framing $\delta\nu : \nu \cong \epsilon^2$ corresponds to a lift of $(1, 1, \ldots, 1) \in H^0(M)$ to an element $U_{\delta\nu} \in H^1(S(\nu)) = [S(\nu), S^1]$. The map

$$U_{\delta\nu} \;=\; \text{projection} \;:\; \partial K \;=\; S(\nu) \;\cong\; M \times S^1 \to S^1$$

induces the surjection

$$\begin{aligned}(U_{\delta\nu})_* \ :\ H_1(S(\nu)) \ =\ H_1(M)\oplus H_0(M) &\xrightarrow{\text{projection}} H_0(M) \ =\ \mathbb{Z}[\pi_0(M)] \\ &\xrightarrow{\text{augmentation}} H_1(S^1)=\mathbb{Z}\ .\end{aligned}$$

The section $s : M \to S(\nu)$ corresponding to $\epsilon \oplus 0 \subset \nu$ is such that

$$\begin{aligned} s^*(U_{\delta\nu}) \ &=\ U_{\delta\nu}\circ s \in H^1(M) \ =\ [M,S^1]\ , \\ U_{\delta\nu} \ &=\ s_*[M] \ =\ ([M], s^*(U_{\delta\nu})\cap [M]) \\ &\in H^1(S(\nu)) \ =\ H_m(S(\nu)) \ =\ H_m(M\times S^1) \ =\ H_m(M)\oplus H_{m-1}(M)\ .\end{aligned}$$

Proof. (i) This follows from the commutative square

$$\begin{array}{ccc} H_m(M) & \xrightarrow{\text{inclusion}_*} & H_m(N) \\ \big\downarrow \cong & & \big\downarrow \cong \\ H^0(M) & \xrightarrow{\text{Umkehr}} & H^2(N) \\ \big\downarrow & & \big\downarrow \text{inclusion}^* \\ \cong & & H^2(M) \\ \big\downarrow & & \big\downarrow \cong \\ H^2(D(\nu),S(\nu)) & \xrightarrow{\text{inclusion}^*} & H^2(D(\nu)) \end{array}$$

(ii)+(iii) By construction. □

Definition 5.4.3. The *canonical framing* of a codimension 2 embedding $M^m \subset N^{m+2}$ with $e = 0 \in H^2(M)$ is the unique framing $\delta\nu : \nu \cong \epsilon^2$ with $s^*(U_{\delta\nu}) = 0 \in H^1(M)$ and

$$s_* \ =\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ :\ H_m(M) \to H_m(S(\nu)) \ =\ H_m(M)\oplus H_{m-1}(M)\ .$$

The canonical framing is obtained from an arbitrary framing $\delta\nu$ by subtracting

$$s^*(U_{\delta\nu}) \in H^1(M) = [M, SO(2)]\ .$$

Remark 5.4.4. For a knot $M = S^1 \subset N = S^3$ the canonical framing of Proposition 5.4.3 corresponds to choosing a preferred longitude with linking number 0 with the knot itself. This is the preferred framing of Rolfsen [Ro, p.31].

Definition 5.4.5. A *Seifert surface* for a codimension 2 embedding $M^m \subset N^{m+2}$ is a framed codimension 1 embedding $\Sigma^{m+1} \subset N$ such that $\partial\Sigma = M$.

Proposition 5.4.6. *Let $M^m \subset N^{m+2}$ be a codimension 2 embedding with a Seifert surface $\Sigma^{m+1} \subset N^{m+2}$.*
(i) *The fundamental class $[M] \in H_m(M)$ has image $[M] = 0 \in H_m(N)$, so that $e = 0 \in H^2(M)$ (by 5.4.2 (i)) and $M \subset N$ is framed.*
(ii) *The composite $M \subset \Sigma \subset N$ expresses the normal 2-plane bundle $\nu = \nu_{M\subset N}$ as a sum of two line bundles*

$$\nu \ =\ \nu_{M\subset\Sigma} \oplus \nu_{\Sigma\subset N}|_M \ :\ M \to BSO(2)\ .$$

The framings $\nu_{M\subset\Sigma} \cong \epsilon$, $\nu_{\Sigma\subset N} \cong \epsilon$ given by the orientations add up to the canonical framing $\delta\nu : \nu \cong \epsilon^2$, with the section

$$s \ :\ M \to S(\nu_{M\subset\Sigma}) \ \cong\ M\times S^0 \to S(\nu) \ \cong\ M \times S^1$$

such that $s_*[M] = ([M], 0) \in H_m(M\times S^1) = H_m(M)\oplus H_{m-1}(M)$.

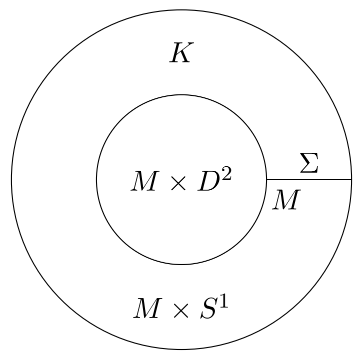
There is no loss of generality in assuming that

$$(\Sigma \cap D(\nu); \partial\Sigma, s(M)) \;=\; M \times (I; \{0\}, \{1\})$$

is a collar of $\partial\Sigma = M \subset \Sigma$, so that $(\mathrm{cl.}(\Sigma \backslash \Sigma \cap D(\nu)), s(M))$ is a copy of (Σ, M). From now on we identify $D(\nu) = M \times D^2$ using the canonical framing, and denote the copy by (Σ, M), so that $\Sigma \subset K$ with

$$\partial\Sigma \;=\; s(M) \;=\; M \times \{*\} \subset \partial K \;=\; S(\nu) \;=\; M \times S^1 \;.$$

We shall denote the 'preferred longitude' $s(M)$ by M also, as in the figure:



$$N = M \times D^2 \cup_{M \times S^1} K$$

Definition 5.4.7. (i) A *Seifert class* for a canonically framed codimension 2 embedding $M^m \subset N^{m+2}$ is a lift of $U_{\delta\nu} \in H^1(S(\nu))$ to a cohomology class $U \in H^1(K) = [K, S^1]$.
(ii) A *Seifert map* $U : K \to S^1$ is a map representing a Seifert class $U \in H^1(K)$.

There is no loss of generality in assuming that the Seifert map $U : K \to S^1$ restricts to the projection

$$U_{\delta\nu} \;:\; \partial K \;=\; S(\nu) \;=\; M \times S^1 \to S^1 \;.$$

Theorem 5.4.8. *The following conditions on a codimension 2 embedding $M^m \subset N^{m+2}$ are equivalent:*

1. $[M] = 0 \in H_m(N)$.
2. *$M \subset N$ admits a Seifert surface $\Sigma^{m+1} \subset N^{m+2}$.*
3. *$M \subset N$ is framed and admits a Seifert class $U \in H^1(K)$.*
4. *$M \subset N$ is framed and*

$$U_{\delta\nu} \in \ker(H^1(M \times S^1) \to H^2(K, M \times S^1)) \;=\; \mathrm{im}(H^1(K) \to H^1(M \times S^1)) \;.$$

Proof. 1. $\Longrightarrow$ 3. Immediate form the exact sequence

$$H^1(K) \to H^2(N, K) \;=\; H^0(M) \to H^2(N) \;=\; H_m(N) \;,$$

noting that $U_{\delta\nu} = (1, 1, \ldots, 1) \in H^0(M) = \mathbb{Z}[\pi_0(M)]$ has image $[M] \in H_m(N)$.
2. $\Longrightarrow$ 1. Obvious.
2. $\Longrightarrow$ 3. The Euler class $e \in H^2(M)$ is the image of $[M] \in H_m(M)$ under the composite $H_m(M) \to H_m(N) \cong H^2(N) \to H^2(M)$. If there exists a Seifert surface $\Sigma^{m+1} \subset N$ then $[M] = 0 \in H^2(N)$, so that $e = 0$ and ν can be framed. The Pontrjagin-Thom construction gives a Seifert map

$$U \;:\; K \to K/\mathrm{cl.}(K \backslash \Sigma \times D^1) \;=\; \Sigma \times D^1/\Sigma \times S^0 \to D^1/S^0 \;=\; S^1 \;.$$

3. $\Longrightarrow$ 2. It may be assumed that the Seifert map $U : K \to S^1$ is smooth and transverse regular at $* \in S^1$, in which case $\Sigma^{m+1} = U^{-1}(*)$ is a Seifert surface for $M \subset N$.
3. $\Longleftrightarrow$ 4. Immediate from the exact sequence $H^1(K) \to H^1(M \times S^1) \to H^2(K, M \times S^1)$. $\square$

Corollary 5.4.9. *Let $M^m \subset N^{m+2}$ be a canonically framed codimension 2 embedding, with complement K.*
(i) *The Pontrjagin-Thom map (5.2.2 (i))*

$$U_M \ : \ N \to N/\mathrm{cl.}(N\backslash M \times D^2) \ = \ M \times D^2/M \times S^1 \to D^2/S^1 \ = \ S^2$$

has inverse image $(U_M)^{-1}(*) = M \subset N$. The composite

$$H^1(M \times S^1) \to H^2(K, M \times S^1) \ = \ H^2(N, M) \to H^2(N)$$

sends the canonical class $U_{\delta\nu} = (0,1) \in H^1(M \times S^1) = H^1(M) \oplus H^0(M)$ to the Hurewicz image in $H^2(N)$ of the cohomotopy class $U_M \in [N, S^2]$. There exists a framed codimension 2 embedding $(\Sigma^{m+1}; M, \emptyset) \subset N \times (I; \{0\}, \{1\})$ if and only if $U_M = 0 \in [N, S^2]$.
(ii) *The rel ∂ homotopy classes of Seifert maps are in one-one correspondence with the cobordism classes of Seifert surfaces, corresponding to the cosets of*

$$\mathrm{im}(H^1(N) \to H^1(K)) \ = \ \ker(H^1(K) \to H^2(N, K)) \subseteq H^1(K)$$

defined by the inverse image of $(1, 1, \ldots, 1) \in H^2(N, K)$, noting the exact sequence

$$\begin{aligned} H^1(N, K) \ = \ H^{-1}(M) \ = \ 0 \to H^1(N) \to H^1(K) \\ \to H^2(N, K) \ = \ H^0(M) \to H^2(N) \ = \ H_m(N) \end{aligned}$$

and that $(1, 1, \ldots, 1) \in H^0(M)$ has image $[M] \in H_m(N)$.
(iii) *A Seifert surface $\Sigma = U^{-1}(*) \subset K$ can be pushed into $N \times I$ rel $M \times \{0\}$ to obtain a framed codimension 2 embedding $(\Sigma; M, \emptyset) \subset N \times (I; \{0\}, \{1\})$ as in* (i)*, so that $U_M = 0 \in [N, S^2]$.*
(iv) *A Seifert map $U \in H^1(K) = [K, S^1]$ determines an infinite cyclic cover $\overline{K} = U^*\mathbb{R}$ of K, and a cobordism class of Seifert surfaces (as in 5.2.2 (iv)), namely all the inverse images $\Sigma^{m+1} = U^{-1}(*) \subset K$ of regular values $* \in S^1$ of all representative smooth maps $U : K \to S^1$.*
(v) *If $H^1(N) = 0$ there is a unique homotopy class of Seifert maps $U \in H^1(K)$, and all the Seifert surfaces $\Sigma \subset K$ for $M \subset N$ are cobordant (in the sense of Definition 5.2.1 (i)).*

Proof. Immediate from Theorem 5.4.8. □

Remark 5.4.10. For $m \geqslant 1$ every codimension 2 embedding $M^m \subset N^{m+2} = S^{m+2}$ is framed, since the Euler number $e \in H^2(M)$ is the image of $[M] \in H_m(M)$ under the composite

$$H_m(M) \to H_m(S^{m+2}) \cong H^2(S^{m+2}) = 0 \to H^2(M) \ .$$

In this case, the existence of Seifert surfaces obtained in Theorem 5.4.8 goes back to Erle [Er].

Remark 5.4.11. (Continuation of Remark 4.7.2). Let Σ^{m+1} be a Seifert surface for $M^m \subset N^{m+2}$, corresponding to a Seifert map $U : K \to S^1$. Suppose that Σ is disconnected, and that Σ_1 is a closed connected component of Σ. Then $\Sigma = \Sigma_0 \sqcup \Sigma_1$ with Σ_0 a Seifert surface with the same Seifert map $U_0 = U \in [K, S^1] = H^1(K)$. The Pontrjagin-Thom map of $\Sigma_1 \subset K\backslash\partial K$ is

$$U_1 \ = \ [U - U_0] \ = \ 0 \in \mathrm{im}(H^1(K) \to H^1(K\backslash\partial K)) \ ,$$

so that $\Sigma_1 = \partial\Omega_1$ is the boundary of a codimension 1 submanifold $\Omega_1 \subset K\backslash\partial K$. In dealing with Seifert surfaces, there is thus no loss of generality in assuming that Σ has no closed components.

5.5. **Infinite cyclic covers.**

Definition 5.5.1. (i) Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial extension ring of $\mathbb{Z}$, with elements the finite polynomials $\sum_j a_j t^j$ $(a_j \in \mathbb{Z})$. Use $\overline{t} = t^{-1}$ to define the *canonical involution*

$$^- : \Lambda \to \Lambda ; \ p(t) = \sum_j a_j t^j \mapsto \overline{p(t)} = \sum_j a_j t^{-j} .$$

Use the involution to regard a right Λ-module K as a left Λ-module by

$$K \times \Lambda \to K : (x, p(t)) \mapsto \overline{p(t)} x .$$

(ii) The Λ-*dual* of a (left) Λ-module L is the Λ-module

$$L^* = \mathrm{Hom}_\Lambda(L, \Lambda) , \ \Lambda \times L^* \to L^* ; \ (p(t), f) \mapsto (x \mapsto f(x)\overline{p(t)})$$

with a pairing

$$\langle \ , \ \rangle : L \otimes_\Lambda L^* \to \Lambda ; \ x \otimes f \mapsto f(x) .$$

such that

$$\langle p(t)x, q(t)y \rangle = \overline{p(t)} q(t) \langle x, y \rangle \in \Lambda .$$

Proposition 5.5.2. (Reidemeister [Re], Blanchfield [Bl], Levine [Le3])
Let K be a CW complex with an infinite cyclic cover $\overline{K}$, and let $t : \overline{K} \to \overline{K}$ be a generating covering translation.
(i) *$C(\overline{K})$ is a free Λ-module chain complex, and the homology groups $H_*(\overline{K}) = H_*(C(\overline{K}))$ are Λ-modules.*
(ii) *The Λ-cohomology modules*

$$H^r_\Lambda(\overline{K}) = H_{-r}(\mathrm{Hom}_\Lambda(C(\overline{K}), \Lambda))$$

are such that there are defined Λ-module morphisms

$$H^r_\Lambda(\overline{K}) \to H_r(\overline{K})^* = \mathrm{Hom}_\Lambda(H_r(\overline{K}), \Lambda) ; \ f \mapsto (x \mapsto f(x)) ,$$
$$H_n(K) \otimes_\mathbb{Z} H^r_\Lambda(\overline{K}) \to H_{n-r}(\overline{K}) ; \ x \otimes y \mapsto x \cap y$$

and a Λ-hermitian pairing

$$H_r(\overline{K}) \otimes_\Lambda H^r_\Lambda(\overline{K}) \to \Lambda ; \ x \otimes f \mapsto f(x) .$$

(iii) *The augmentation $\Lambda \to \mathbb{Z}; t \mapsto 1$ is such that*

$$\mathbb{Z} \otimes_\Lambda C(\overline{K}) = \mathrm{coker}(t - 1 : C(\overline{K}) \to C(\overline{K})) = C(K) ,$$

with an exact sequence

$$\dots \longrightarrow H_r(\overline{K}) \xrightarrow{t-1} H_r(\overline{K}) \longrightarrow H_r(K) \longrightarrow H_{r-1}(\overline{K}) \longrightarrow \dots .$$

(iv) *If K is finite, the homology $H_*(\overline{K})$ and the cohomology $H^*_\Lambda(\overline{K})$ are f.g. Λ-modules.*
(v) *If $(K, \partial K)$ is an (oriented) n-dimensional manifold with boundary and $(\overline{K}, \overline{\partial K})$ is an infinite cyclic cover, cap product with the fundamental class $[K] \in H_n(K, \partial K)$ defines the Poincaré-Lefschetz Λ-module duality isomorphisms*

$$[K] \cap - : H^r_\Lambda(\overline{K}, \overline{\partial K}) \cong H_{n-r}(\overline{K}) , \ [K] \cap - : H^r_\Lambda(\overline{K}) \cong H_{n-r}(\overline{K}, \overline{\partial K})$$

in the usual manner. The natural Λ-module morphism $\overline{B} : H_r(\overline{K}) \to H_r(\overline{K}, \overline{\partial K})$ is such that the composite

$$H_r(\overline{K}) \xrightarrow{\overline{B}} H_r(\overline{K}, \overline{\partial K}) \cong H^{n-r}_\Lambda(\overline{K}) \longrightarrow H_{n-r}(\overline{K})^*$$

is the adjoint of the hermitian (up to sign) intersection pairing

$$\overline{B} \ : \ H_r(\overline{K}) \times H_{n-r}(\overline{K}) \to \Lambda \ ; \ (x,y) \mapsto \overline{B}(x,y)$$

with

$$\begin{aligned}\overline{B}(x,y) &= (-1)^{r(n-r)}\overline{\overline{B}(y,x)} \in \Lambda \ (x \in H_r(\overline{K}), y \in H_{n-r}(\overline{K})) \ , \\ \overline{B}(\lambda x, \mu y) &= \overline{\lambda}\mu \overline{B}(x,y) \in \Lambda \ (\lambda, \mu \in \Lambda) \ .\end{aligned}$$

Remark 5.5.3. We shall call the Λ-module intersection pairing of 5.5.2

$$\overline{B} \ : \ H_r(\overline{K}) \times H_{n-r}(\overline{K}) \to \Lambda$$

the *Blanchfield pairing*, even though it was first introduced by Reidemeister [Re]. The closely related Λ-module linking pairing of Blanchfield [Bl] is defined for the canonical infinite cyclic cover $\overline{K}$ of the complement K of a knot $S^{n-2} \subset S^n$

$$\overline{Bl} \ : \ H_r(\overline{K}) \times H_{n-r-1}(\overline{K}) \to S^{-1}\Lambda/\Lambda \ (1 \leqslant r \leqslant n-2)$$

with $S^{-1}\Lambda$ the localization of Λ inverting the multiplicative subset

$$S \ = \ \{p(t) \,|\, p(1) = \pm 1\} \subset \Lambda \ .$$

The natural Λ-module morphisms

$$H^{r+1}_{\Lambda}(\overline{K}) \to \mathrm{Hom}_{\Lambda}(H_r(\overline{K}), S^{-1}\Lambda/\Lambda)$$

are isomorphisms, by [Le3, Corollary 4.4].

The pullback along any map $U : K \to S^1$ of the universal cover $\mathbb{R} \to S^1; y \mapsto e^{2\pi i y}$ is an infinite cyclic cover of K

$$\overline{K} \ = \ U^*\mathbb{R} \ = \ \{(x,y) \in K \times \mathbb{R} \,|\, U(x) = e^{2\pi i y} \in S^1\} \to K \ ; \ (x,y) \mapsto x \ ,$$

with generating covering translation

$$t \ : \ \overline{K} \to \overline{K} \ ; \ (x,y) \mapsto (x, y+1)$$

and a $\mathbb{Z}$-equivariant lift of U

$$\overline{U} \ : \ \overline{K} \to \mathbb{R} \ ; \ (x,y) \mapsto y \ .$$

Proposition 5.5.4. *Let $M^m \subset N^{m+2}$ be a framed codimension 2 embedding with a Seifert map $U : K = \mathrm{cl.}(N\backslash M \times D^2) \to S^1$ which is transverse regular at $* \in S^1$ with*

$$U| \ = \ \text{projection} \ : \ \partial K \ = \ M \times S^1 \to S^1 \ ,$$

so that

$$(\Sigma, \partial\Sigma) \ = \ (U^{-1}(*), (U|)^{-1}(*)) \subset N^{m+2}$$

is a Seifert surface for $\partial\Sigma = M \subset N$ with trivial normal bundle $\Sigma \times I \subset N$. It is convenient to remove a collar neighbourhood from M, so that $\Sigma \times I \subset K$ with

$$(\Sigma \times I) \cap (M \times D^2) \ = \ M \times \{(x,y) \in S^1 \,|\, y \geqslant 0\} \ .$$

The complement of $\Sigma \subset K$

$$K_\Sigma \ = \ \mathrm{cl.}(K \backslash (\Sigma \times I))$$

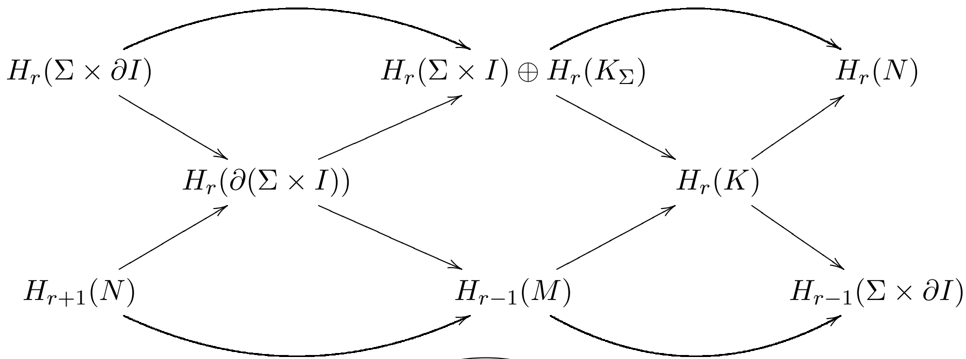
has boundary

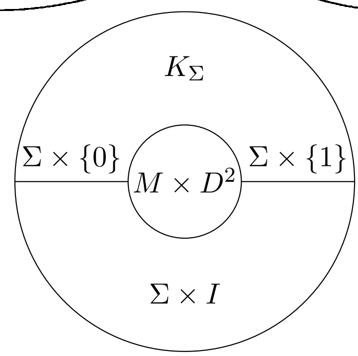
$$\partial K_\Sigma \ = \ \partial(\Sigma \times I) \ = \ \Sigma \times \{0\} \cup M \times I \cup \Sigma \times \{1\} \ ,$$

and

$$\begin{aligned}N &= (M \times D^2) \cup_{M \times S^1} K \\ &= (M \times D^2) \cup (\Sigma \times I) \cup K_\Sigma \ \cong \ (\Sigma \times I) \cup_{\partial(\Sigma \times I)} K_\Sigma\end{aligned}$$

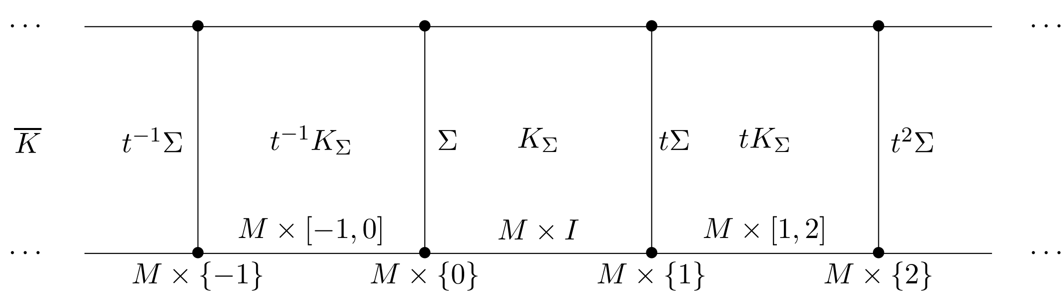
with a commutative braid of exact sequences





Cutting K along Σ there is obtained a fundamental domain $(K_\Sigma;\Sigma,t\Sigma)$ for the infinite cyclic cover $\overline{K}$ of K, with

$$\overline{K} \;=\; \bigcup_{r=-\infty}^{\infty} t^r(K_\Sigma;\Sigma,t\Sigma)\ ,\ \overline{\partial K} \;=\; M\times\mathbb{R}\ .$$



5.6. **Alexander modules and variation maps.** The Alexander modules and the variation maps are defined for framed codimension 2 embeddings $M^m \subset N^{m+2}$, with a view to applying them in §6 in the special case $N = S^{m+2}$.

Definition 5.6.1. The *Alexander modules* of a framed codimension 2 embedding $M^m \subset N^{m+2}$ with a Seifert class $U \in H^1(K)$ are the homology Λ-modules $H_*(\overline{K})$ of the corresponding infinite cyclic cover $\overline{K} = U^*\mathbb{R}$ of the complement K.

Proposition 5.6.2. *For a Seifert surface $\Sigma^{m+1} \subset N$ the two inclusions*

$$i_j\ :\ \Sigma \to K_\Sigma\ ;\ x \mapsto (x,j)\ (j=0,1)$$

are such that there are defined Mayer-Vietoris exact sequences in homology

$$\dots \longrightarrow H_r(\Sigma)[t,t^{-1}] \xrightarrow{ti_1-i_0} H_r(K_\Sigma)[t,t^{-1}] \longrightarrow H_r(\overline{K}) \longrightarrow H_{r-1}(\Sigma)[t,t^{-1}] \longrightarrow \dots\ ,$$

$$\dots \longrightarrow H_r(\Sigma) \xrightarrow{i_1-i_0} H_r(K_\Sigma) \longrightarrow H_r(K) \longrightarrow H_{r-1}(\Sigma) \longrightarrow \dots$$

and also in cohomology

$$\dots \longrightarrow H^r(K_\Sigma)[t,t^{-1}] \xrightarrow{ti_0^* - i_1^*} H^r(\Sigma)[t,t^{-1}] \longrightarrow H^{r+1}_\Lambda(\overline{K}) \longrightarrow H^{r+1}(K_\Sigma)[t,t^{-1}] \longrightarrow \dots ,$$

$$\dots \longrightarrow H^r(K_\Sigma) \xrightarrow{i_0^* - i_1^*} H^r(\Sigma) \longrightarrow H^{r+1}(K) \longrightarrow H^{r+1}(K_\Sigma) \longrightarrow \dots .$$

The inclusion

$$f ~:~ M \times S^1 ~=~ \partial K \to K$$

lifts to a $\mathbb{Z}$-equivariant inclusion

$$\overline{f} ~:~ M \times \mathbb{R} ~=~ \overline{\partial K} \to \overline{K} .$$

The homotopy $V : f \simeq f : M \times S^1 \to K$ defined by

$$V ~:~ M \times S^1 \times I \to K ~;~ (x, e^{2\pi i y}, z) \mapsto f(x, e^{2\pi i(y+z)}).$$

lifts to the $\mathbb{Z}$-equivariant homotopy $\overline{V} : \overline{f} \simeq t\overline{f}$ defined by

$$\overline{V} ~:~ M \times \mathbb{R} \times I \to \overline{K} ~;~ (x,y,z) \mapsto \overline{f}(x, y+z) .$$

The restriction of $\overline{f}$

$$\begin{aligned} V_\Sigma ~=~ \overline{f}| ~=~ \overline{V}| ~:~ M \times I ~=~ M \times \{0\} \times I \to K_\Sigma ~; \\ (x,z) ~=~ (x,0,z) \mapsto \overline{f}(x,z) ~=~ \overline{V}(x,0,z) \end{aligned}$$

defines a homotopy $V_\Sigma : i_0|_M \simeq i_1|_M$ between the restrictions

$$i_j|_M ~:~ M \longrightarrow \Sigma \xrightarrow{i_j} K_\Sigma ~(j ~=~ 0,1)$$

such that there is defined a commutative diagram

$$\begin{array}{ccc} M \times I & \xrightarrow{V_\Sigma} & K_\Sigma \\ \big\downarrow & & \big\downarrow \\ M \times \mathbb{R} \times I & \xrightarrow{\overline{V}} & \overline{K} \end{array}$$

The following variation chain maps are motivated by the variation maps in homology constructed by Lamotke [La].

Definition 5.6.3. (i) The *Blanchfield variation* chain map of $M^m \subset N^{m+2}$ is the Λ-module chain map

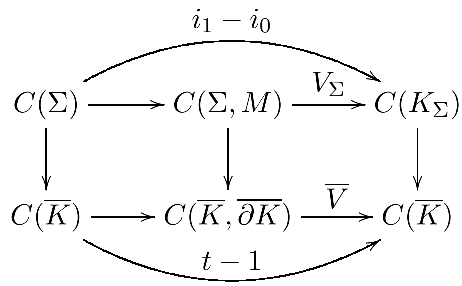
$$\overline{V} ~:~ C(\overline{K}, \overline{\partial K}) \to C(\overline{K})$$

induced by $\overline{V} : \overline{f} \simeq t\overline{f}$.
(ii) The *Seifert variation* chain map of $M^m \subset N^{m+2}$ with respect to a Seifert surface Σ is the $\mathbb{Z}$-module chain map

$$V_\Sigma ~:~ C(\Sigma, M) \to C(K_\Sigma)$$

induced by $V_\Sigma : i_0|_M \simeq i_1|_M$.

Proposition 5.6.4. (i) *The Blanchfield and Seifert variation chain maps are such that there are defined chain homotopy commutative diagrams of $\mathbb{Z}$-module chain maps*

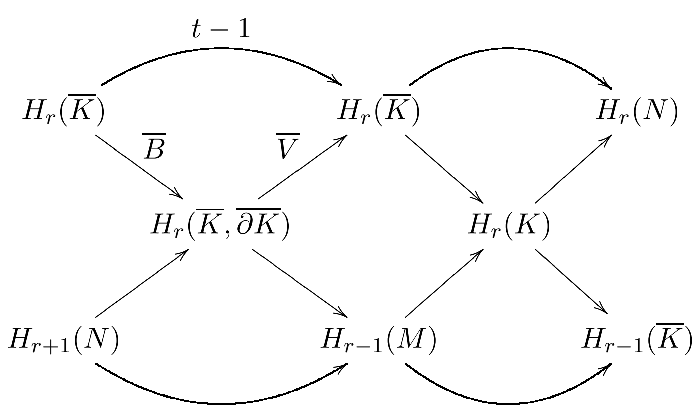


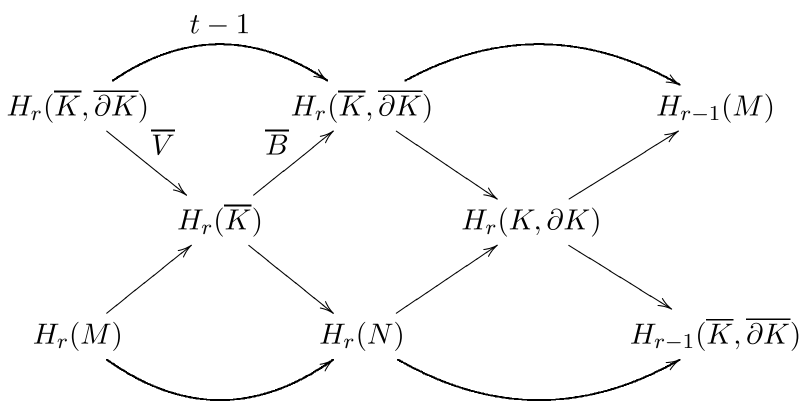
$$\begin{array}{ccc}
C(K_\Sigma)^{m+1-*} & \xrightarrow{\ i_j^*\ } & C(\Sigma)^{m+1-*} \simeq C(\Sigma, M) \\
{\scriptstyle V_\Sigma^*}\big\downarrow & & \big\downarrow{\scriptstyle V_\Sigma} \\
C(\Sigma, M)^{m+1-*} \simeq C(\Sigma) & \xrightarrow{\ -i_{1-j}\ } & C(K_\Sigma)
\end{array}$$

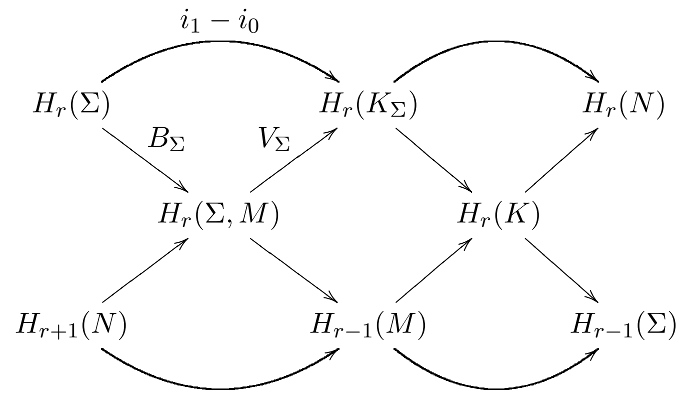
and also a chain homotopy commutative diagram of Λ-module chain maps

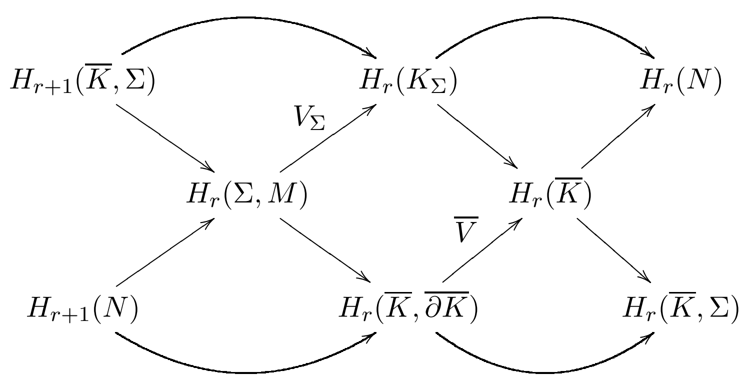
$$\begin{array}{ccc}
C(\overline{K})^{m+2-*} & \underset{\simeq}{\xrightarrow{\ [K]\cap -\ }} & C(\overline{K}, \overline{\partial K}) \\
{\scriptstyle \overline{V}^*}\big\downarrow & & \big\downarrow{\scriptstyle \overline{V}} \\
C(\overline{K}, \overline{\partial K})^{m+2-*} & \underset{\simeq}{\xrightarrow{\ -t([K]\cap -)\ }} & C(\overline{K})
\end{array}$$

(ii) *The Blanchfield and Seifert variation morphisms fit into commutative braids of exact sequences*



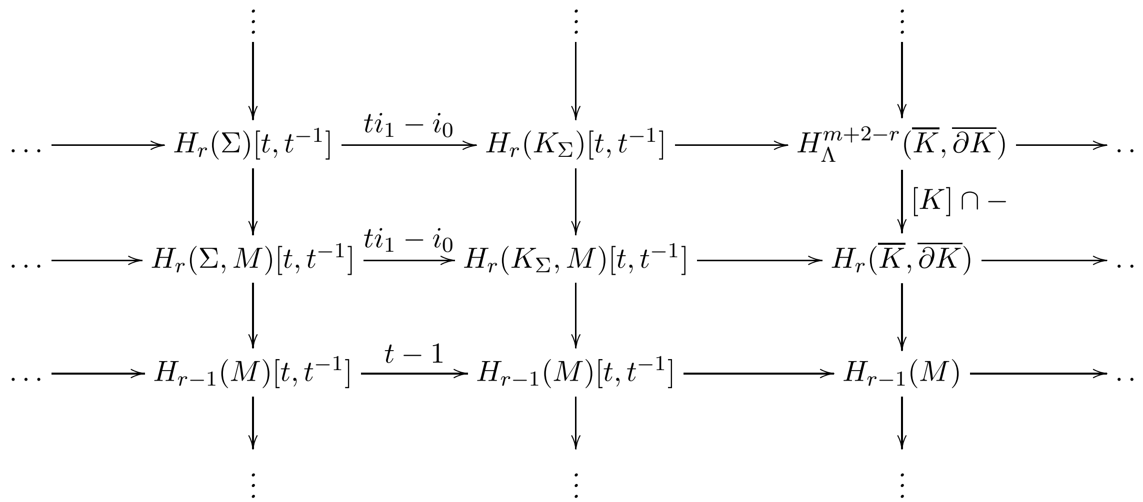


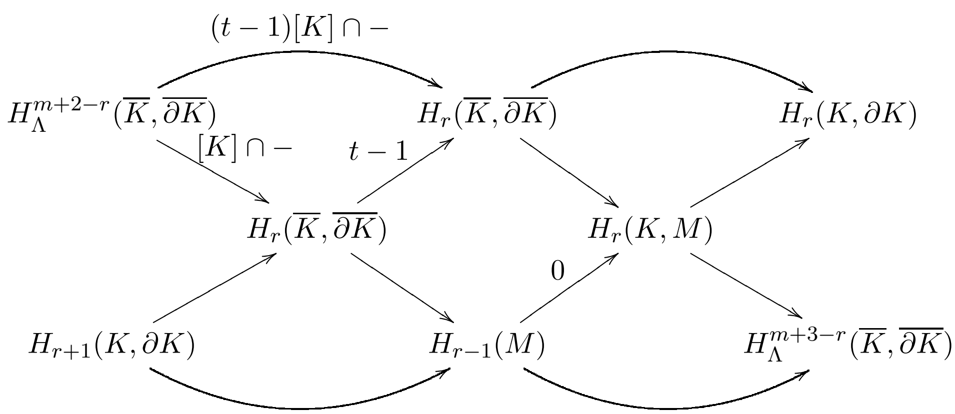


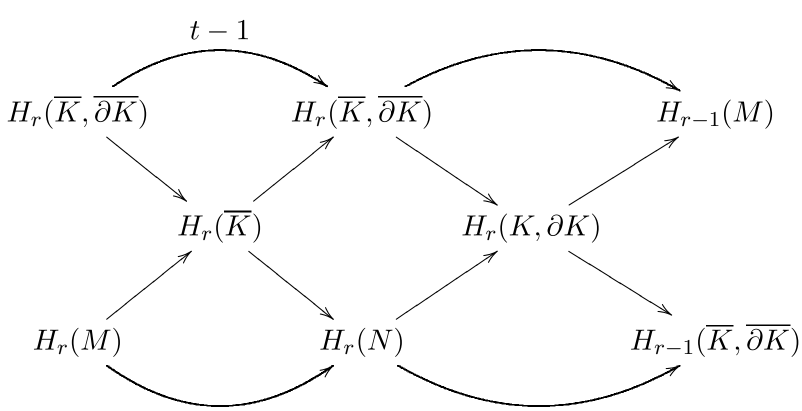


identifying $H_*(\overline{K}, \overline{\partial K}) = H_*(\overline{K}, M \times \mathbb{R}) = H_*(\overline{K}, M)$.
(iii) *The Mayer-Vietoris exact sequences of* (ii) *and the Blanchfield and Seifert variation morphisms fit into the following commutative diagrams of exact sequences of* Λ*-modules*

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H^{m+1-r}(K_\Sigma)[t,t^{-1}] & \xrightarrow{-ti_0^* + i_1^*} & H^{m+1-r}(\Sigma)[t,t^{-1}] & \longrightarrow & H_\Lambda^{m+2-r}(\overline{K}) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle V_\Sigma^*} & & \downarrow{\scriptstyle V_\Sigma} & & \downarrow{\scriptstyle \overline{V}} & & \\
\dots & \longrightarrow & H_r(\Sigma)[t,t^{-1}] & \xrightarrow{ti_1 - i_0} & H_r(K_\Sigma)[t,t^{-1}] & \longrightarrow & H_r(\overline{K}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_r(N)[t,t^{-1}] & \xrightarrow{t-1} & H_r(N)[t,t^{-1}] & \longrightarrow & H_r(N) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$







(iv) *The following conditions are equivalent:*

(a) $H_*(N) = H_*(S^{m+2})$,
(b) $\overline{V} : H_r(\overline{K}, M) \to H_r(\overline{K})$ *is an isomorphism for* $0 < r < m+1$,
(c) $V_\Sigma : H_r(\Sigma, M) \to H_r(K_\Sigma)$ *is an isomorphism for* $0 < r < m+1$.

□

Definition 5.6.5. (i) The *mapping torus* of a map $h : \Sigma \to \Sigma$ is the identification space

$$T(h) \ = \ (\Sigma \times I)/\{(x,0) \sim (h(x),1)\,|\, x \in \Sigma\}\ .$$

If h is an automorphism there is defined a fibre bundle $\Sigma \to T(h) \to S^1$ with projection

$$U \ : \ T(h) \to S^1 \ ; \ (x,y) \mapsto e^{2\pi i y}\ .$$

If $M \subset \Sigma$ is a subspace such that $h| = 1 : M \to M$ write

$$t(h) \ = \ M \times D^2 \cup_{M \times S^1} T(h)\ .$$

(ii) For an $(m+1)$-dimensional manifold with boundary (Σ, M) and an automorphism $(h,1) : (\Sigma, M) \to (\Sigma, M)$ there is a defined a framed codimension 2 embedding

$$M^m \ = \ M \times \{0\} \subset N^{m+2} \ = \ t(h)$$

with complement $K = T(h)$, Seifert surface Σ, and Seifert map $U : K \to S^1$ as in (i). The $(m+2)$-dimensional manifold N has an *open book decomposition* with *page* Σ, *monodromy* h and *binding* M.

Remark 5.6.6. (i) See Winkelnkemper's Appendix to [Ra3] for a historical account of open book decompositions.
(ii) For the codimension 2 embedding of the binding of an open book

$$M^m \subset N^{m+2} \ = \ t(h : \Sigma \to \Sigma)$$

the complement $(K, \partial K) = (T(h), M \times S^1)$ is the mapping torus of the monodromy, and the corresponding infinite cyclic cover is given by

$$t \ : \ \overline{K} \ = \ \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R} \ ; \ (x,y) \mapsto (h(x), y+1)\ ,$$

with

$$t| \ : \ \overline{\partial K} \ = \ M \times \mathbb{R} \to M \times \mathbb{R} \ ; \ (x,y) \mapsto (x, y+1)$$

and

$$\begin{aligned} i_0 \ &: \ \Sigma \to K_\Sigma \ = \ \Sigma \times I \ ; \ x \mapsto (x,0)\ , \\ i_1 \ &: \ \Sigma \to K_\Sigma \ = \ \Sigma \times I \ ; \ x \mapsto (h(x),1)\ . \end{aligned}$$

The Blanchfield and Seifert variation maps coincide

$$\overline{V} \ = \ V_\Sigma \ : \ H_*(\overline{K}, \overline{\partial K}) \ = \ H_*(\Sigma, M) \to H_*(\overline{K}) \ = \ H_*(\Sigma)$$

See Lamotke [La] (and also [BNR3]) for the application of this variation map to the classical Picard-Lefschetz theory.

5.7. **Cobordism of framed codimension 2 embeddings.** Recall from Theorem 5.4.8 that to every framed codimension 2 embedding $M^m \subset N^{m+2}$ with $[M] = 0 \in H_m(M)$ there are associated an infinite cyclic cover $\overline{K}$ of the complement K, and an equivalence class of Seifert surfaces $\Sigma^{m+1} \subset K$. This also applies to a cobordism of framed codimension 2 embeddings $(W; M_0, M_1) \subset N \times (I; \{0\}, \{1\})$, as follows.

Proposition 5.7.1. *Let $(W^{m+1}; M, M') \subset N^{m+2} \times (I; \{0\}, \{1\})$ be a cobordism of canonically framed codimension 2 embeddings.*
(i) *The complement of the cobordism is the cobordism of the complements $K, K' \subset N$*

$$\begin{aligned} (J; K, K') \ = \ &(\mathrm{cl.}(N \times I \backslash W \times D^2); \mathrm{cl.}(N \times \{0\} \backslash M \times D^2), \mathrm{cl.}(N \times \{1\} \backslash M' \times D^2)) \\ &\subset N \times (I; \{0\}, \{1\}) \end{aligned}$$

such that

(a) $\partial J = K \cup_{M \times S^1} W \times S^1 \cup_{M' \times S^1} K'$.
(b) $N \times (I; \{0\}, \{1\}) = (W; M, M') \times D^2 \cup_{(W;M,M') \times S^1} (J; K, K')$.

(c) *The fundamental classes have the same images*

$$[W] \;=\; [M] \;=\; [M'] \in H_{m+1}(N\times I, N\times\{0,1\}) \;=\; H_m(N)\ .$$

If these images are all 0 *there exists a Seifert map* $U \in H^1(J) = [J, S^1]$ *of* $J \subset N\times I$, *a lift of the canonical class*

$$U_{\delta\nu} \;=\; (0,1) \in H^1(S(\nu)) \;=\; H^1(J\times S^1) \;=\; H^1(J)\oplus H^0(J)$$

of $\nu = \nu_{J\subset N\times I}$, *which restricts to Seifert maps* $U \in H^1(K)$, $U' \in H^1(K')$.

(d) *If* $(W; M, M')$ *is an isotopy then the inclusions* $K \to J$, $K' \to J$ *are homotopy equivalences, inducing isomorphisms in homotopy groups*

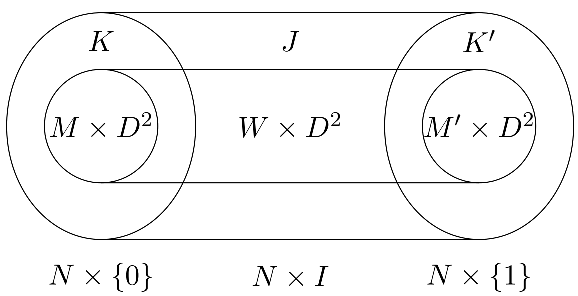
$$\pi_*(K) \;\cong\; \pi_*(J)\ ,\ \pi_*(K') \;\cong\; \pi_*(J)$$

and also in the homology Λ*-modules*

$$H_*(\overline{K}) \;\cong\; H_*(\overline{J})\ ,\ H_*(\overline{K}') \;\cong\; H_*(\overline{J})\ .$$

(e) *If* $(W; M, M')$ *is an h-cobordism then the inclusions* $K \to J, K' \to J$ *are homology equivalences, inducing isomorphisms in the homology* $\mathbb{Z}$*-modules*

$$H_*(K) \;\cong\; H_*(J)\ ,\ H_*(K') \;\cong\; H_*(J)\ .$$



(ii) *A Seifert map* $V \in H^1(J) = [J, S^1]$ *is represented by a smooth map* $(V; U, U') : (J; K, K') \to S^1$ *classifying an infinite cyclic cover* $(\overline{J}; \overline{K}, \overline{K}')$ *of* $(J; K, K')$, *with*

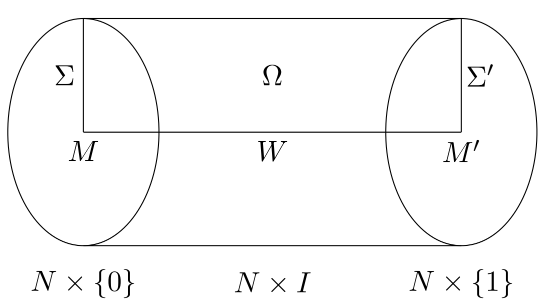
$$(V; U, U')| \;=\; \text{projection} \;:\; (W; M, M')\times S^1 \to S^1\ .$$

The inverse image of a regular value $* \in S^1$

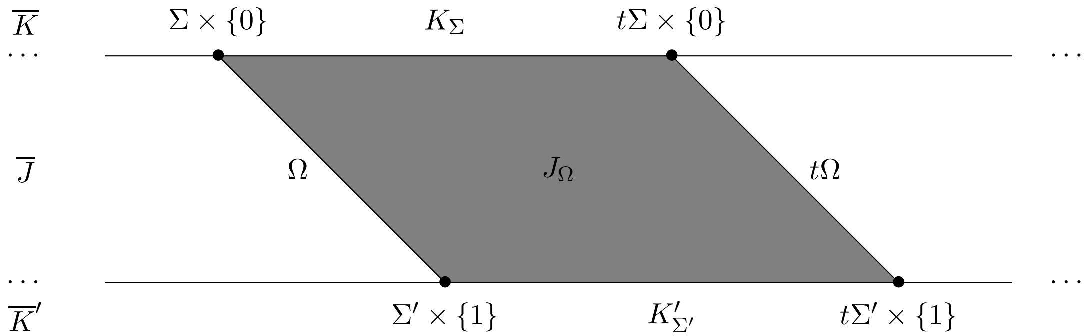
$$(V; U, U')^{-1}(*) \;=\; (\Omega; \Sigma, \Sigma') \subset (J; K, K')$$

is a Seifert surface for $(W; M, M')$, *such that*

$$\partial\Omega \;=\; \Sigma \cup_M W \cup_{M'} -\Sigma' \subset N\times I\ ,\ \Omega\cap(N\times\{0\}) \;=\; \Sigma\ ,\ \Omega\cap(N\times\{1\}) \;=\; \Sigma'\ .$$



Cutting the complement $(J;K,K')$ *along* $(\Omega;\Sigma,\Sigma')$ *there is obtained a fundamental domain* $(J_\Omega;K_\Sigma,K'_{\Sigma'})$ *for the infinite cyclic cover* $(\overline{J};\overline{K},\overline{K}')$ *of* $(J;K,K')$.



The various homology Λ*-modules fit into a commutative diagram of exact rows and columns*

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_r(\Sigma)[t,t^{-1}] & \xrightarrow{ti_1-i_0} & H_r(K_\Sigma)[t,t^{-1}] & \longrightarrow & H_r(\overline{K}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_r(\Omega)[t,t^{-1}] & \xrightarrow{tj_1-j_0} & H_r(J_\Omega)[t,t^{-1}] & \longrightarrow & H_r(\overline{J}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_r(\Omega,\Sigma)[t,t^{-1}] & \longrightarrow & H_r(J_\Omega,K_\Sigma)[t,t^{-1}] & \longrightarrow & H_r(\overline{J},\overline{K}) & \longrightarrow & \dots
\end{array}$$

Define the $(m+3)$*-dimensional cobordism of manifolds with boundary*

$$(P;Q,Q') \ = \ \big(\ J_\Omega\ ;\ \Omega\cup_{\Sigma\times\{0\}}K_\Sigma\times\{0\}\ ,\ K'_{\Sigma'}\times\{1\}\cup_{t\Sigma'\times\{1\}}t\Omega\ \big)\ .$$

If $(W;M,M')$ *is an isotopy then* $(P;Q,Q')$ *is an* h*-cobordism (i.e.* $\pi_*(Q)=\pi_*(P)=\pi_*(Q')$*), and the* Λ*-module morphisms*

$$H_*(\overline{K})\to H_*(\overline{J})\ ,\ t(j_1,i_1)-(j_0,i_0)\ :\ H_*(\Omega,\Sigma)[t,t^{-1}]\to H_*(J_\Omega,K_\Sigma)[t,t^{-1}]$$

are isomorphisms.
If $(W;M,M')$ *is an* h*-cobordism then* $(P;Q,Q')$ *is an* H*-cobordism (i.e.* $H_*(Q)=H_*(P)=H_*(Q')$*), and the* $\mathbb{Z}$*-module morphisms*

$$H_*(K)\to H_*(J)\ ,\ (j_1,i_1)-(j_0,i_0)\ :\ H_*(\Omega,\Sigma)\to H_*(J_\Omega,K_\Sigma)$$

are isomorphisms.
(iii) *Any two Seifert surfaces* $\Sigma,\Sigma'\subset N$ *for* $M\subset N$ *are cobordant: there exists a Seifert surface* $(\Omega;\Sigma,\Sigma')$ *for* $(W;M_0,M')=M\times(I;\{0\},\{1\})\subset N\times I$.

6. Codimension 2 embeddings $M^m\subset S^{m+2}$

We generalize the Seifert forms, Blanchfield forms and Alexander polynomials familiar in the spherical case $S^m\subset S^{m+2}$ to arbitrary framed codimension 2 embeddings $M^m\subset S^{m+2}$.

6.1. **The Seifert and Blanchfield pairings.**

Proposition 6.1.1. *Let* $M^m\subset S^{m+2}$ *be a codimension 2 embedding with complement* K *and normal bundle* ν. *Suppose that either* $m\geqslant 1$ *or* $m=0$ *with* $[M]=0\in H_0(S^2)=\mathbb{Z}$.
(i) *The Thom class of* ν *determines a unique Seifert map for the embedding*

$$U\ =\ (1,1,...,1)\in H^2(D(\nu),S(\nu))\ =\ H^0(M)\ =\ H^1(K)\ =\ \mathbb{Z}[\pi_0(M)]\ ,$$

so that $\nu \cong \epsilon^2$ *and*

$$S^{m+2} \;=\; M \times D^2 \cup_{M \times S^1} K \ .$$

The Pontrjagin-Thom construction for the Seifert map determines a cobordism class of Seifert surfaces $\Sigma \subset K$ *for* $M \subset S^{m+2}$.
(ii) *The homology groups of* K *are given by*

$$H_r(K) \;=\; \begin{cases} H_{r+1}(D(\nu), S(\nu)) = H_{r-1}(M) & \text{if } r \neq 0, m+1 \\ \mathbb{Z} & \text{if } r = 0 \\ H_m(M)/\mathbb{Z} & \text{if } r = m+1 \ , \end{cases}$$

with $\mathbb{Z} \subseteq H_m(M) = \mathbb{Z}[\pi_0(M)]$ *the infinite cyclic subgroup generated by* $(1, 1, \dots, 1) \in \mathbb{Z}[\pi_0(M)]$. *In particular,* K *is connected, and the composite*

$$U_* \ : \ \pi_1(K) \xrightarrow{\text{Hurewicz}} H_1(K) = H_0(M) = \mathbb{Z}[\pi_0(M)] \xrightarrow{\text{augmentation}} \pi_1(S^1) \;=\; \mathbb{Z}$$

is surjective, so that the infinite cyclic cover $\overline{K} = U^*\mathbb{R}$ *of* K *is also connected. For any Seifert surface* $\Sigma \subset K$ *there is an isomorphism* $H_0(\Sigma, M) \cong H_1(S^{m+2}, K_\Sigma)$, *so that* K_Σ *is connected if and only if* Σ *has one component for each component of* M *(i.e.* $M \subset S^{m+2}$ *is a boundary link).*
(iii) *The Blanchfield variation morphisms* $\overline{V}$ *are such that*

$$\overline{V} : H_r(\overline{K}, M) \to H_r(\overline{K}) \text{ is an isomorphism for } 0 < r < m+1,$$

$$0 \to H_0(\overline{K}, M) = 0 \xrightarrow{\overline{V}} H_0(\overline{K}) \to H_0(S^{m+2}) = \mathbb{Z} \to 0 \text{ is exact},$$

$$0 \to H_{m+2}(S^{m+2}) = \mathbb{Z} \to H_{m+1}(\overline{K}, M) \xrightarrow{\overline{V}} H_{m+1}(\overline{K}) \to 0 \text{ is exact}$$

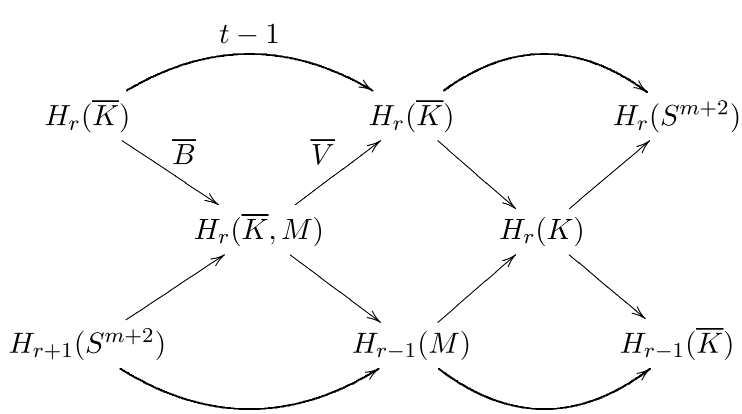
and the Seifert variation morphisms V_Σ *are such that*

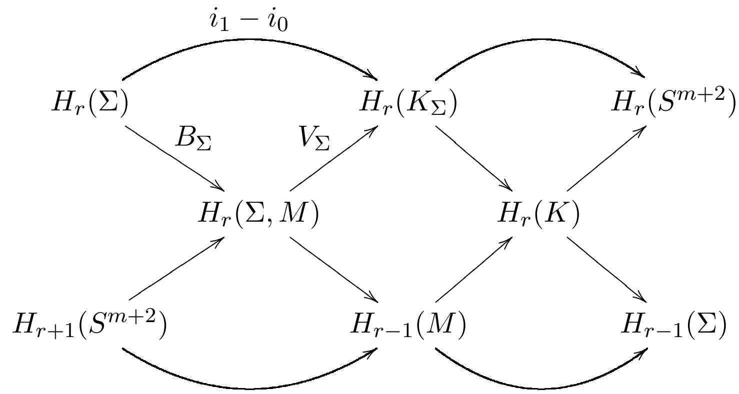
$$V_\Sigma : H_r(\Sigma, M) \to H_r(K_\Sigma) \text{ is an isomorphism for } 0 < r < m+1,$$

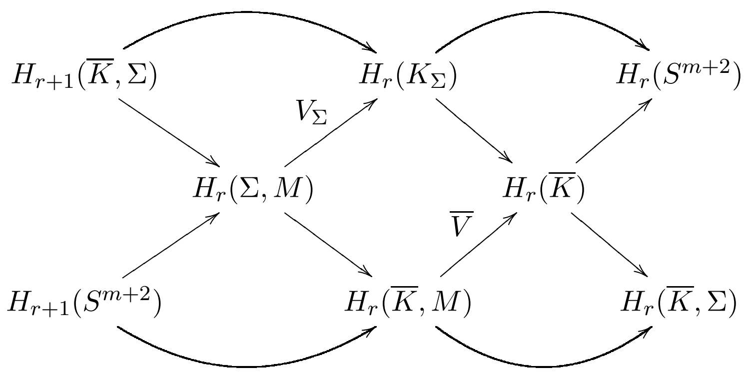
$$0 \to H_0(\Sigma, M) \xrightarrow{V_\Sigma} H_0(K_\Sigma) \to H_0(S^{m+2}) = \mathbb{Z} \to 0 \text{ is exact},$$

$$0 \to H_{m+2}(S^{m+2}) = \mathbb{Z} \to H_{m+1}(\Sigma, M) \xrightarrow{V_\Sigma} H_{m+1}(K_\Sigma) \to 0 \text{ is exact.}$$

There are defined commutative braids of exact sequences







Proof. Apply Propositions 5.4.2 and 5.6.4, noting that

$$[M] \;=\; 0 \in H_m(S^{m+2}) \;(=\; 0 \text{ for } m \geqslant 1)$$

and that there is a Mayer-Vietoris exact sequence

$$\cdots \to H_{r+1}(S^{m+2}) \to H_r(S(\nu)) \to H_r(K) \oplus H_r(D(\nu)) \to H_r(S^{m+2}) \to \dots \;.$$

□

Example 6.1.2. The complement K of a spherical knot $M = S^m \subset S^{m+2}$ is a homology circle, with $U : K \to S^1$ inducing isomorphisms $U_* : H_*(K) \cong H_*(S^1)$.

Remark 6.1.3. Proposition 6.1.1 works also for codimension 2 embeddings $M^m \subset S^{m+1} \times I$ with essentially the same proof.

Definition 6.1.4. The *Seifert morphisms* $A(\Sigma)_0, A(\Sigma)_1$ of a framed codimension 2 embedding $M^m \subset S^{m+2}$ with respect to a Seifert surface $\Sigma^{m+1} \subset S^{m+2}$ are the composites

$$A(\Sigma)_j \;:\; H_r(\Sigma) \xrightarrow{\;i_j\;} H_r(K_\Sigma) \xrightarrow[\cong]{(V_\Sigma)^{-1}} H_r(\Sigma, M) \;\cong\; H^{m+1-r}(\Sigma) \;(j = 0, 1)$$

for $0 < r < m+1$. The *Seifert pairings* are the adjoint bilinear pairings on the torsion-free quotients

$$A(\Sigma)_j \;:\; F_r(\Sigma) \times F_{m+1-r}(\Sigma) \to \mathbb{Z} \;(j = 0, 1) \;.$$

Proposition 6.1.5. (i) *The Seifert morphisms $A(\Sigma)_0, A(\Sigma)_1$ are such that*

$$A(\Sigma)_1 - A(\Sigma)_0 \;=\; B(\Sigma) \;:\; F_r(\Sigma) \to F_r(\Sigma, M) \;\cong\; F^{m+1-r}(\Sigma)$$

with $B(\Sigma)$ the natural map induced by the inclusion $(\Sigma,\emptyset)\subset(\Sigma,M)$ with adjoint the intersection pairing. There is an exact sequence of $\mathbb{Z}$-modules

$$\dots\to H_r(M)\to H_r(\Sigma)\xrightarrow{A(\Sigma)_1-A(\Sigma)_0} H^{m+1-r}(\Sigma)\to H_{r-1}(M)\to\dots$$

and also an exact sequence of Λ-modules

$$\dots\to H_{r+1}(\overline{K})\to H_r(\Sigma)[t,t^{-1}]\xrightarrow{tA(\Sigma)_1-A(\Sigma)_0} H^{m+1-r}(\Sigma)[t,t^{-1}]\to H_r(\overline{K})\to\dots$$

(ii) *The Seifert pairings are such that $A(\Sigma)_0$ is the transpose of $A(\Sigma)_1$ up to sign*

$$A(\Sigma)_0 \;=\; -(-1)^{r(m+1-r)}A(\Sigma)_1^* \;:\; F_r(\Sigma)\times F_{m+1-r}(\Sigma)\to\mathbb{Z}\ ,$$

so that

$$\begin{aligned} B(\Sigma)(x,y) &= (A(\Sigma)_1-A(\Sigma)_0)(x,y)\\ &= A(\Sigma)_1(x,y)+(-1)^{r(m+1-r)}A(\Sigma)_1(y,x)\in\mathbb{Z}. \end{aligned}$$

for $x\in F_r(\Sigma)$, $y\in F_{m+1-r}(\Sigma)$.

From now on, we shall write $A(\Sigma)_1=A(\Sigma)$.

6.2. **The Seifert form and Alexander polynomial of a codimension 2 embedding $M^{2n-1}\subset S^{2n+1}$.** For us, a Seifert form (F,A) is a f.g. free $\mathbb{Z}$-module together with a $\mathbb{Z}$-module morphism $A:F\to F^*$ (or equivalently a bilinear pairing $A:F\times F\to A$) together with a choice of sign $\epsilon=1$ or -1. This is more general than the usual notion of a Seifert form in which it is also required that $A+\epsilon A^*:F\to F^*$ be an isomorphism, such as arises for a Seifert surface Σ of a spherical knot $S^{2n-1}\subset S^{2n+1}$ with $F=F_n(\Sigma)$, $\epsilon=(-1)^n$.

We consider equivalence relations on Seifert forms called S*-equivalence* and H*-equivalence*. We shall also define such equivalence relations on the Laurent polynomial extension ring $\Lambda=\mathbb{Z}[t,t^{-1}]$. For a framed codimension 2 embedding $M^{2n-1}\subset S^{2n+1}$ a choice of Seifert surface $\Sigma^{2n}\subset S^{2n+1}$ determines a Seifert form $(F_n(\Sigma),A(\Sigma))$ with sign $\epsilon=(-1)^n$, and hence an Alexander polynomial

$$\Delta_{M,\Sigma}(t) \;=\; \Delta_{A(\Sigma)}(t) \;=\; \det(tA(\Sigma)+(-1)^nA(\Sigma)^*)\in\Lambda\ .$$

We shall prove that

(i) the S-equivalence classes of $(F_n(\Sigma),A(\Sigma))$ and $\Delta_{M,\Sigma}(t)$ are isotopy invariants of $M\subset S^{2n+1}$,
(ii) the H-equivalence classes of $(F_n(\Sigma),A(\Sigma))$ and $\Delta_{M,\Sigma}(t)$ are h-cobordism invariants of $M\subset S^{2n+1}$

generalizing the results of Levine [Le1, Le2] for spherical knots $M=S^{2n-1}\subset S^{2n+1}$. In the next section we shall consider the Levine–Tristram signatures for an arbitrary framed codimension 2 embedding $M\subset S^{2n+1}$, and investigate their cobordism properties.

We now proceed to define Seifert forms, and the S- and H-equivalence relations.

Definition 6.2.1. Fix a sign $\epsilon=1$ or -1.
(i) A *Seifert form* (F,A) is a f.g. free $\mathbb{Z}$-module F with a bilinear pairing

$$A\;:\;F\times F\to\mathbb{Z}\;;\;(x,y)\mapsto A(x,y)\ .$$

An *isomorphism* $h:(F,A)\to(F',A')$ of Seifert forms $(F,A),(F',A')$ is a $\mathbb{Z}$-module isomorphism $h:F\to F'$ such that

$$A \;=\; h^*A'h\;:\;F\to F^*\ .$$

(ii) A Seifert form (F,A) is ϵ*-nonsingular* if the $\mathbb{Z}$-module morphism

$$B \;=\; A+\epsilon A^*\;:\;F \;=\; F^{**}\to F^*$$

is an isomorphism.
(iii) A *sublagrangian* L of a Seifert form (F, A) is a sublagrangian of the ϵ-symmetric form $(F, A + \epsilon A^*)$ such that

$$A(x, y) \ = \ 0 \text{ for all } x, y \in L \ .$$

The *sublagrangian quotient* is the Seifert form $(L^\perp/L, [A])$. A *lagrangian* L is a sublagrangian such that $L^\perp = L$.
(iv) A *rank* (ℓ^-, ℓ^+) *enlargement* of a Seifert form (F, A) is a Seifert form (F', A') of the type

$$A' \ = \ \begin{pmatrix} A & 0 & \alpha \\ 0 & 0 & x \\ \beta & y & z \end{pmatrix} \ : \ F' \ = \ F \oplus \mathbb{Z}^{\ell^-} \oplus \mathbb{Z}^{\ell^+} \to F'^* \ = \ F^* \oplus \mathbb{Z}^{\ell^-} \oplus \mathbb{Z}^{\ell^+} \ .$$

The submodule

$$L' \ = \ 0 \oplus \mathbb{Z}^{\ell^-} \oplus 0 \subset F' \ = \ F \oplus \mathbb{Z}^{\ell^-} \oplus \mathbb{Z}^{\ell^+} \text{ (where } F = \mathbb{Z}^k)$$

is such that

$$L' \subseteq L'^\perp \ = \ F \oplus \mathbb{Z}^{\ell^-} \oplus \ker(x + \epsilon y^* : \mathbb{Z}^{\ell^+} \to \mathbb{Z}^{\ell^-}) \subseteq F' \ ,$$

$$\text{inclusion}^*(A' + \epsilon A'^*) \ = \ (0\ 0\ x + \epsilon y^*) \ : \ F' \ = \ F \oplus \mathbb{Z}^{\ell^-} \oplus \mathbb{Z}^{\ell^+} \to L'^* \ = \ \mathbb{Z}^{\ell^-} \ .$$

The ϵ-symmetric form $(F', A' + \epsilon A'^*)$ with

$$A' + \epsilon A'^* \ = \ \begin{pmatrix} A + \epsilon A^* & 0 & \alpha + \epsilon\beta^* \\ 0 & 0 & x + \epsilon y^* \\ \beta + \epsilon\alpha^* & y + \epsilon x^* & z + \epsilon z^* \end{pmatrix}$$

is then a rank (ℓ^-, ℓ^+) enlargement of $(F, A + \epsilon A^*)$ in the sense of 2.1.8 (ii).
(v) An H*-enlargement* of (F, A) is a rank (ℓ^-, ℓ^+) enlargement (F', A') such that $x + \epsilon y^*$ is an invertible $\ell^+ \times \ell^+$ matrix, or equivalently if L' is a sublagrangian of (F', A'), in which case

$$(L'^\perp/L', [A']) \ = \ (F, A)$$

and the ϵ-symmetric form $(F', A' + \epsilon A'^*)$ is an H-enlargement of $(F, A + \epsilon A^*)$ in the sense of 2.1.8. Two Seifert forms (F_0, A_0), (F_1, A_1) are H*-equivalent* if they have isomorphic H-enlargements (F'_0, A'_0), (F'_1, A'_1).
(vi) An S*-enlargement* of (F, A) is a rank $(1, 1)$ H-enlargement (F', A') with $(x, y) = (\pm 1, 0)$ or $(0, \pm 1)$. (F, A) is an S*-reduction* of any of its S-enlargements. Two Seifert forms are S*-equivalent* if they can be connected by a chain of S-enlargements, reductions and congruences.

Remark 6.2.2. For a fixed sign $\epsilon = \pm 1$ Seifert forms (F_0, A_0), (F_1, A_1) are S-equivalent if and only if they have isomorphic H-enlargements (F'_0, A'_0), (F'_1, A'_1) such that

$$A'_j \ = \ \begin{pmatrix} A_j & 0 & \alpha_j \\ 0 & 0 & x_j \\ \beta_j & y_j & z_j \end{pmatrix} \ (j = 0, 1)$$

with $tx_j + \epsilon y_j$ invertible over Λ. We shall not actually need this result, which makes use of the Higman [Hi] computation $Wh(\mathbb{Z}) = 0$ of the Whitehead group of an infinite cyclic group $\mathbb{Z}$; it follows from $\tau(tx_j + \epsilon y_j) = 0 \in Wh(\mathbb{Z}) = 0$ that $tx_j + \epsilon y_j$ is stably a product of elementary matrices over Λ. This is the algebraic analogue of the following special case of the s-cobordism theorem: if $(W; M_0, M_1) \subset N^{m+2} \times (I; \{0\}, \{1\})$ is an h-cobordism of framed codimension 2 embeddings with complement an h-cobordism $(J; K_0, K_1)$, and $m > 4$, $\pi_1(W) = \{1\}$, $\pi_1(J) = \mathbb{Z}$ then $(W; M_0, M_1) \subset N \times (I; \{0\}, \{1\})$ can be deformed rel ∂ to an isotopy.

Definition 6.2.3. The *Seifert form* $(F_n(\Sigma), A(\Sigma))$ of a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface Σ is the Seifert pairing

$$A(\Sigma) \ : \ F_n(\Sigma) \times F_n(\Sigma) \to \mathbb{Z}$$

of Definition 6.1.4.

By Proposition 6.1.5 the Seifert from $A(\Sigma)$ determines the intersection pairing by

$$B(\Sigma) \ = \ A(\Sigma) + (-1)^n A(\Sigma)^* \ : \ F_n(\Sigma) \to F_n(\Sigma)^* \ .$$

The linking number interpretation of the Seifert form of a spherical knot $S^{2n-1} \subset S^{2n+1}$ ([Le2]) extends to the Seifert form of an arbitrary framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$:

Proposition 6.2.4. (i) *Let $M^{2n-1} \subset S^{2n+1}$ be a framed codimension 2 embedding with Seifert surface $\Sigma^{2n} \subset S^{2n+1}$. If $x, y \in C_n(\Sigma)$ are cycles then*

$$A(\Sigma)(x,y) \ = \ \mathrm{lk}(x, y^+) \in \mathbb{Z}$$

is the linking number in S^{2n+1} of x and the disjoint cycle $y^+ \in C_n(S^{2n+1} \backslash \Sigma)$ defined by y pushed slightly off Σ in the positive normal direction.
(ii) *For a framed codimension 1 embedding*

$$(\Omega^{2n+1}, \Sigma, \Sigma'; W^{2n}, M, M') \subset S^{2n+1} \times (I; \{0\}, \{1\})$$

the subgroup $L = \ker(F_n(\Sigma) \oplus F_n(\Sigma') \to F_n(\Omega)) \subseteq F_n(\Sigma) \oplus F_n(\Sigma')$ is such that

$$(A(\Sigma) \oplus -A(\Sigma'))(x,y) \ = \ 0 \in \mathbb{Z} \textit{ for all } x, y \in L \ .$$

Proof. (i) Exactly as for the spherical case in [Le2].
(ii) For any $x, y \in L$ there exist chains Γ_x, $\Gamma_y \in C_{n+1}(\Omega)$ such that

$$\partial \Gamma_x \ = \ x \ , \ \partial \Gamma_y \ = \ y \in C_n(\Sigma) \oplus C_n(\Sigma') \ .$$

By pushing Γ_y along the normal vector to Ω in a positive direction, we get a chain $\Gamma_y^+ \in C_{n+1}(S^{2n+1} \times I \backslash \Omega)$ such that

$$\partial \Gamma_y^+ \ = \ y^+ \ , \ \Gamma_x \cap \Gamma_y^+ \ = \ \emptyset \ .$$

It follows that

$$(A(\Sigma) \oplus -A(\Sigma'))(x,y) \ = \ \mathrm{lk}(x, y^+) \ = \ \Gamma_x \cdot \Gamma_y^+ \ = \ 0 \in \mathbb{Z} \ .$$

□

Remark 6.2.5. The Seifert form $(F_n(\Sigma), A(\Sigma))$ for a spherical knot $\mathcal{K} : S^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface $\Sigma^{2n} \subset S^{2n+1}$ is $(-1)^n$-nonsingular, with

$$B(\Sigma) \ = \ A(\Sigma) + (-1)^n A(\Sigma)^* \ : \ F_n(\Sigma) \to F_n(\Sigma)^*$$

an isomorphism. We have the following classic results of Levine [Le1, Le2] for spherical knots $\mathcal{K}$.

(i) The S-equivalence class of $(F_n(\Sigma), A(\Sigma))$ is an isotopy invariant of $\mathcal{K}$, and in particular independent of the choice of Seifert surface Σ. The function

$$\{\text{isotopy classes of simple knots } \mathcal{K} : S^{2n-1} \subset S^{2n+1}\} \to$$
$$\{S\text{-equivalence classes of } (-1)^n\text{-nonsingular Seifert forms}\} \ ; \ \mathcal{K} \mapsto A(\Sigma)$$

is a bijection for $n > 1$. By definition, a knot $\mathcal{K}$ is *simple* if the knot complement K is such that $\pi_r(K) = 0$ for $1 \leqslant r \leqslant n-1$.
(ii) The S-equivalence relation of 6.2.1 (vi) is generated by the S-enlargements of the type

$$A' = \begin{pmatrix} A & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ \beta & 0 & 0 \end{pmatrix}$$

proved exactly as in [Le1]. This is immediate from the congruences

$$\begin{pmatrix} 1 & \mp\beta & 0 \\ 0 & \pm 1 & 0 \\ 0 & \mp z & 1 \end{pmatrix} \begin{pmatrix} A & 0 & \alpha \\ 0 & 0 & 0 \\ \beta & \pm 1 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mp\beta & \pm 1 & \mp z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \mp\alpha & 0 \\ 0 & \pm 1 & 0 \\ 0 & \mp z & 1 \end{pmatrix} \begin{pmatrix} A & 0 & \alpha \\ 0 & 0 & \pm 1 \\ \beta & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mp\alpha & \pm 1 & \mp z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ \beta & 0 & 0 \end{pmatrix} .$$

(iii) The H-equivalence class of $A(\Sigma)$ is an h-cobordism invariant of $\mathcal{K} : S^{2n-1} \subset S^{2n+1}$. If $\mathcal{K}$ is h-cobordant to the trivial knot $\mathcal{K}_0 : S^{2n-1} \subset S^{2n+1}$, via an h-cobordism

$$(W^{2n}; \mathcal{K}(S^{2n-1}), \mathcal{K}_0(S^{2n-1})) \subset S^{2n+1} \times (I; \{0\}, \{1\}) ,$$

then for any Seifert surface $\Sigma^{2n} \subset S^{2n+1}$ for $\mathcal{K}$ there exists a cobordism of framed codimension 1 embeddings

$$(\Omega^{2n+1}; \Sigma^{2n}, D^{2n}) \subset S^{2n+1} \times (I; \{0\}, \{1\}) .$$

The closed $2n$-dimensional submanifold

$$\partial\Omega = \Sigma^{2n} \cup_{\mathcal{K}(S^{2n-1})} W \cup_{\mathcal{K}_0(S^{2n-1})} D^{2n} \subset S^{2n+1} \times I$$

is such that $F_n(\partial\Omega) = F_n(\Sigma)$ and

$$L = \ker(F_n(\Sigma) \to F_n(\Omega)) \subset F_n(\Sigma)$$

is a lagrangian of the Seifert form $(F_n(\Sigma), A(\Sigma))$. Thus

$$A(\Sigma) = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} : F_n(\Sigma) = L \oplus L^* \to F_n(\Sigma)^* = L^* \oplus L$$

with $x + (-1)^n y^* : L^* \to L^*$ an isomorphism, so that $(F_n(\Sigma), A(\Sigma))$ is H-equivalent to $(0, 0)$. The function

$$\begin{aligned} C_{2n-1} = \ & \{h\text{-cobordism classes of knots } \mathcal{K} : S^{2n-1} \subset S^{2n+1}\} \to \\ & \{H\text{-equivalence classes of } (-1)^n\text{-nonsingular Seifert forms}\} \ ; \ \mathcal{K} \mapsto A \end{aligned}$$

is a bijection for $n > 1$.

We now generalize (ii) and (iii) in Remark 6.2.5 to arbitrary framed codimension 2 embeddings $M^{2n-1} \subset S^{2n+1}$.

Theorem 6.2.6. *Let $(W; M, M') \subset S^{2n+1} \times (I; \{0\}, \{1\})$ be a cobordism of framed codimension 2 embeddings, with a relative cobordism of Seifert surfaces*

$$\Gamma = (\Omega; \Sigma, \Sigma') \subset S^{2n+1} \times (I; \{0\}, \{1\}) .$$

(i) *A decomposition of $C(\Gamma)$ as union of algebraic codimension 1 embeddings*

$$C(\Gamma) = \bigcup_{r=-1}^{m+1} (C(\Omega_r); C(\Sigma_r), C(\Sigma_{r+1}), C(W_r); C(M_r), C(M_{r+1})) \subset C(N) \otimes C(I; \{0\}, \{1\})$$

given by Theorem 5.3.6 determines a sequence of enlargements and reductions taking the Seifert form $(F_n(\Sigma), A(\Sigma))$ *to the Seifert form* $(F_n(\Sigma'), A(\Sigma'))$.
(ii) *If* $(W; M, M')$ *is an* H*-cobordism the Seifert forms* $(F_n(\Sigma), A(\Sigma))$, $(F_n(\Sigma'), A(\Sigma'))$ *are* H*-equivalent.*
(iii) *If* $(W; M, M')$ *is an isotopy the Seifert forms* $(F_n(\Sigma), A(\Sigma))$, $(F_n(\Sigma'), A(\Sigma'))$ *are* S*-equivalent.*

Proof. (i) We extend to the Seifert forms $(F_n(\Sigma_q), A(\Sigma_q))$ the computation in Theorem 3.4.2 of the intersection forms $(F_n(\Sigma_q), B(\Sigma_q))$ of the Seifert surfaces Σ_q of the framed codimension 2 embeddings $M_q \subset S^{2n+1}$ which arise in a half-handle decomposition

$$(\Omega; \Sigma, \Sigma', W; M, M') \;=\; \bigcup_{r=-1}^{m+1} (\Omega_r; \Sigma_r, \Sigma_{r+1}, W_r; M_r, M_{r+1})$$

(noting that we only need an embedded algebraic decomposition), with each

$$\begin{aligned}&(\Omega_r; \Sigma_r, \Sigma_{r+1}, W_r; M_r, M_{r+1})\\ &=\; (\Omega_r^-; \Sigma_r, \Sigma_{r+1/2}, W_r^-; M_r, M_{r+1/2}) \cup_{\Sigma_{r+1/2}} (\Omega_r^+; \Sigma_{r+1/2}, \Sigma_{r+1}, W_r^+; M_{r+1/2}, M_{r+1})\end{aligned}$$

an elementary embedded splitting of index $r+1$ and index (ℓ_r^-, ℓ_{r+1}^+). As in 3.4.2 only the case $r = n-1$ need be considered, so we identify

$$(\Omega; \Sigma, \Sigma', W; M, M') \;=\; (\Omega_{n-1}; \Sigma_{n-1}, \Sigma_n, W_{n-1}; M_{n-1}, M_n)$$

and write

$$(\Sigma'', M'') \;=\; (\Sigma_{n-1/2}, M_{n-1/2}) \;.$$

By 3.4.2 the intersection forms

$$\begin{aligned}(F_n(\Sigma''), B(\Sigma'')) \;&=\; (F_n(\Sigma) \oplus L^-, \begin{pmatrix} B(\Sigma) & 0 \\ 0 & 0 \end{pmatrix}) \;,\\ (F_n(\Sigma'), B(\Sigma')) \;&=\; (F_n(\Sigma'') \oplus L^+, \begin{pmatrix} B(\Sigma') & A \\ (-1)^n A^* & E \end{pmatrix})\end{aligned}$$

are enlargements of rank $(\ell^-, 0)$, $(0, \ell^+)$, with

$$\begin{aligned}&L^- \;=\; \mathrm{im}(\mathbb{Z}^{\ell^-}) \;=\; \ker(F_n(\Sigma'') \to F_n(\Sigma)) \subseteq F_n(\Sigma'') \subseteq F_n(\Sigma) \;,\\ &(L^+)^* \;=\; \mathrm{im}((\mathbb{Z}^{\ell^+})^*) \;=\; \ker(F_n(\Sigma')^* \to F_n(\Sigma'')^*) \subseteq F_n(\Sigma')^* \;=\; F^n(\Sigma') \;,\\ &\ell^- \;=\; \dim_{\mathbb{Z}} L^- \;,\;\; \ell^+ \;=\; \dim_{\mathbb{Z}} L^+ \;.\end{aligned}$$

Let $(J; K, K')$ be the complement of $(\Omega; \Sigma, \Sigma') \subset S^{n+2} \times (I; \{0\}, \{1\})$, and as in Proposition 5.7.1 let $(J_\Omega; K_\Sigma, K'_{\Sigma'})$ be the fundamental domain for the canonical infinite cyclic cover $(\overline{J}; \overline{K}, \overline{K}')$. The inclusions induce a commutative diagram of homology groups with exact rows and columns

$$\begin{array}{ccccc}
F_n(\Sigma) & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & F_n(\Omega) = F_n(\Sigma) \oplus L^- & \xleftarrow{\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}} & F_n(\Sigma') = F_n(\Sigma) \oplus L^- \oplus L^+ \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle i'_0} \\
F_n(K_\Sigma) = F^n(\Sigma) & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & F_n(J_\Omega) = F^n(\Sigma) \oplus L^- & \xleftarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} & F_n(K'_{\Sigma'}) = F^n(\Sigma) \oplus L^- \oplus L^+ \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle i'_1} \\
F_n(t\Sigma) = tF_n(\Sigma) & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & F_n(t\Omega) = t(F_n(\Sigma) \oplus L^-) & \xleftarrow{\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}} & F_n(t\Sigma') = t(F_n(\Sigma) \oplus L^- \oplus L^+)
\end{array}$$

with

$$i_0 \;=\; A(\Sigma)\ ,\ i_1 \;=\; (-1)^{n+1}A(\Sigma)^*\ ,$$

$$j_0 \;=\; \begin{pmatrix} A(\Sigma) & \alpha \\ 0 & x \end{pmatrix}\ ,\ j_1 \;=\; (-1)^{n+1}\begin{pmatrix} A(\Sigma)^* & \beta^* \\ 0 & y^* \end{pmatrix}\ ,$$

$$i'_0 \;=\; A(\Sigma') \;=\; \begin{pmatrix} A(\Sigma) & 0 & \alpha \\ 0 & 0 & x \\ \beta & y & z \end{pmatrix}\ ,\ i'_1 \;=\; (-1)^{n+1}A(\Sigma')^* \;=\; (-1)^{n+1}\begin{pmatrix} A(\Sigma)^* & 0 & \beta^* \\ 0 & 0 & y^* \\ \alpha^* & x^* & z^* \end{pmatrix}\ .$$

The Seifert form $(F_n(\Sigma'),A(\Sigma'))$ is a rank (ℓ^-,ℓ^+) enlargement of the Seifert form $(F_n(\Sigma),A(\Sigma))$.
(ii) It follows from the exact sequence

$$\begin{array}{l} 0\to H_n(W,M)\to H_n(\Omega,\Sigma) \;=\; \mathbb{Z}^{\ell^-} \xrightarrow{\ x+(-1)^n y^*\ } \\ \qquad\qquad H_n(\Omega,\Sigma\cup_M W) \;=\; H^{n+1}(\Omega,\Sigma') \;=\; \mathbb{Z}^{\ell^+}\to H_{n-1}(W,M)\to 0 \end{array}$$

that $(W;M,M')$ is an H-cobordism if and only if $x+(-1)^n y^*$ is an invertible $\ell\times\ell$ matrix (with $\ell=\ell^+=\ell^-$) if and only if $(F_n(\Sigma'),A(\Sigma'))$ is an H-enlargement of $(F_n(\Sigma),A(\Sigma))$.
(iii) If $(W;M,M')$ is an isotopy then the trace of each of the ℓ individual surgeries is an isotopy, and we can assume $\ell=1$. Consider the commutative diagram of Λ-modules with exact rows and columns

$$\begin{array}{ccccccccc}
\dots\longrightarrow & H_r(\Sigma)[t,t^{-1}] & \xrightarrow{ti_0-i_1} & H_r(K_\Sigma)[t,t^{-1}] & \longrightarrow & H_r(\overline{K}) & \longrightarrow\dots \\
& \downarrow & & \downarrow & & \downarrow & \\
\dots\longrightarrow & H_r(\Omega)[t,t^{-1}] & \xrightarrow{tj_0-j_1} & H_r(J_\Omega)[t,t^{-1}] & \longrightarrow & H_r(\overline{J}) & \longrightarrow\dots \\
& \downarrow & & \downarrow & & \downarrow & \\
\dots\longrightarrow & H_r(\Omega,\Sigma)[t,t^{-1}] & \longrightarrow & H_r(J_\Omega,K_\Sigma)[t,t^{-1}] & \longrightarrow & H_r(\overline{J},\overline{K})=0 & \longrightarrow\dots
\end{array}$$

The Λ-module morphism

$$\begin{array}{l} t(j_0,i_0)-(j_1,i_1) \;=\; tx+(-1)^n y^*\ : \\ \qquad\qquad H_n(\Omega,\Sigma)[t,t^{-1}] \;=\; \Lambda\to H_n((K\times I)_\Omega,K_\Sigma)[t,t^{-1}] \;=\; \Lambda \end{array}$$

is an isomorphism, with $(x,y)=$ either $(0,\pm1)$ or $(\pm1,0)$, so that $(F_n(\Sigma'),A(\Sigma'))$ is an S-enlargement of $(F_n(\Sigma),A(\Sigma))$. □

6.3. **The Alexander polynomial.** We now define the Alexander polynomial of a Seifert form, and the S- and H-equivalence relations for polynomials.

Definition 6.3.1. The *Alexander polynomial* of a Seifert form (F,A) is

$$\Delta_A(t) \;=\; \det(tA+\epsilon A^* : F[t,t^{-1}]\to F^*[t,t^{-1}])\in\Lambda$$

using any choice of basis for F.

Remark 6.3.2. By convention, the Alexander polynomial of the zero matrix $A=0$ is $\Delta_0(t)=1$.

Definition 6.3.3. (i) Two polynomials $p_0(t),p_1(t)\in\Lambda$ are *H-equivalent* if

$$p_0(t)q_0(t)q_0(t^{-1}) \;\sim_S\; p_1(t)q_1(t)q_1(t^{-1})$$

for some $q_0(t), q_1(t) \in \Lambda$ with $q_0(1), q_1(1) \in \{-1, 1\}$. Written as $p_0(t) \sim_H p_1(t)$.
(ii) Two polynomials $p_0(t), p_1(t) \in \Lambda$ are *S-equivalent* if

$$p_1(t) \;=\; \pm t^k p_0(t) \in \Lambda$$

for some $k \in \mathbb{Z}$. Written as $p_0(t) \sim_S p_1(t)$.

Example 6.3.4. A Seifert form $(\mathbb{Z}^\ell \oplus \mathbb{Z}^\ell, A)$ with

$$A \;=\; \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}$$

and $x + (-1)^n y^*$ an invertible $\ell \times \ell$ matrix is H-equivalent to 0, with Alexander polynomial

$$\Delta_A(t) \;=\; -\det(tx + (-1)^n y^*)\det(ty + (-1)^n x^*) \sim_H \Delta_0(t) \;=\; 1 \;.$$

Proposition 6.3.5. (i) *The Alexander polynomial of the transpose A^* of a Seifert $k \times k$ matrix A is such that*

$$\Delta_{A^*}(t) \;=\; (\epsilon t)^k \Delta_A(t^{-1}) \;\sim_S\; \Delta_A(t^{-1}) \;.$$

For any $\ell \times \ell$ matrices x, y

$$\det(ty + \epsilon x^*) \;=\; \det(ty^* + \epsilon x) \;=\; (\epsilon t)^\ell \det(t^{-1}x + \epsilon y^*) \;\sim_S\; \det(t^{-1}x + \epsilon y^*) \;.$$

(ii) *The Alexander polynomial of an enlargement (F', A') of a Seifert form (F, A) is*

$$\Delta_{A'}(t) \;=\; -\det(tx + \epsilon y^*)\det(ty + \epsilon x^*)\Delta_A(t) \;.$$

(iii) *If A_0, A_1 are H-equivalent then $\Delta_{A_0}(t) \sim_H \Delta_{A_1}(t)$.*
(iv) *If A_0, A_1 are S-equivalent then $\Delta_{A_0}(t) \sim_S \Delta_{A_1}(t)$.*

Proof. (i) The Alexander polynomial of a $k \times k$ matrix A is of the form

$$\Delta_A(t) \;=\; \det(tA^* + \epsilon A) \;=\; \sum_{j=0}^{k} a_j t^j \in \Lambda$$

and

$$\begin{aligned} \Delta_{A^*}(t) \;&=\; \det(tA^* + \epsilon A) \;=\; (\epsilon t)^k \det(t^{-1}A + \epsilon A^*) \\ &=\; (\epsilon t)^k \sum_{j=0}^{k} a_j t^{-j} \;=\; (\epsilon t)^k \Delta_A(t^{-1}) \in \Lambda \;. \end{aligned}$$

(ii)+(iii)+(iv) By construction. □

Definition 6.3.6. The *Alexander polynomial* of a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface Σ is

$$\Delta_{M,\Sigma}(t) \;=\; \Delta_{A(\Sigma)}(t) \;=\; \det(tA(\Sigma) + (-1)^n A(\Sigma)^*) \in \Lambda$$

with $(F_n(\Sigma), A(\Sigma))$ the Seifert form.

Proposition 6.3.7. (i) *The Alexander polynomials $\Delta_{M,\Sigma}(t), \Delta_{M',\Sigma'}(t)$ of h-cobordant framed codimension 2 embeddings $M^{2n-1}, M'^{2n-1} \subset S^{2n+1}$ with respect to Seifert surfaces $\Sigma, \Sigma' \subset S^{2n+1}$ are H-equivalent (6.3.3). The H-equivalence class $\Delta_{M,\Sigma}(t)$ is thus an isotopy invariant of $M^{2n-1} \subset S^{2n+1}$.*
(ii) *The Alexander polynomials $\Delta_{M,\Sigma}(t), \Delta_{M',\Sigma'}(t)$ of isotopic framed codimension 2 embeddings $M^{2n-1}, M'^{2n-1} \subset S^{2n+1}$ with respect to Seifert surfaces $\Sigma, \Sigma' \subset S^{2n+1}$ are S-equivalent (6.3.3). The S-equivalence class of $\Delta_{M,\Sigma}(t)$ is thus an isotopy invariant of $M^{2n-1} \subset S^{2n+1}$.*

Proof. Immediate from Theorem 6.2.6 (ii)+(iii) and Proposition 6.3.5 (iii)+(iv). □

In view of Proposition 6.3.7 (ii) we set:

Definition 6.3.8. The *Alexander polynomial* of a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$ is

$$\Delta_M(t) \ = \ \Delta_{M,\Sigma}(t) \in \Lambda/S\text{-equivalence}$$

for any Seifert surface Σ.

Remark 6.3.9. Let M be a closed m-dimensional manifold, and let $(K, \partial K)$ be an $m+2$-dimensional manifold with boundary $\partial K = M \times S^1$, such that the projection $\partial K \to S^1$ extends to a map $U : K \to S^1$ transverse regular at $* \in S^1$. Then $\Sigma^{m+1} = U^{-1}(*) \subset N$ is a framed codimension 1 submanifold, and cutting K along Σ there is obtained a relative cobordism

$$(K_\Sigma; \Sigma, t\Sigma, M \times I; M \times \{0\}, M \times \{1\})$$

which is a fundamental domain for the infinite cyclic cover $\overline{K} = U^*\mathbb{R}$ of K. The Mayer-Vietoris exact sequence of

$$\overline{K} \ = \ \bigcup_{j=-\infty}^{\infty} t^j K_\Sigma$$

is an exact sequence of Λ-modules

$$\dots \to H_n(\Sigma)[t,t^{-1}] \xrightarrow{\ ti_0 - i_1\ } H_n(K_\Sigma)[t,t^{-1}] \to H_n(\overline{K}) \xrightarrow{\ \partial\ } H_{n-1}(\Sigma)[t,t^{-1}] \to \dots \ ,$$

with $i_0, i_1 : \Sigma \to K_\Sigma$ the two inclusions. For the complement K of a spherical knot $S^{2n-1} \subset S^{2n+1}$ with Seifert surface Σ and Seifert form $A(\Sigma) = A$ we can identify

$$i_0 \ = \ A \ , \ i_1 \ = \ (-1)^{n+1}A^* \ : \ H_n(\Sigma) \to H^n(\Sigma) \ = \ H_n(K_\Sigma) \ ,$$

so the exact sequence can be written as

$$\dots \to H_n(\Sigma)[t,t^{-1}] \xrightarrow{\ tA + (-1)^n A^*\ } H^n(\Sigma)[t,t^{-1}] \to H_n(\overline{K}) \xrightarrow{\ \partial\ } H_{n-1}(\Sigma)[t,t^{-1}] \to \dots \ ,$$

The $\mathbb{Z}$-module morphism $A + (-1)^n A^* : H_n(\Sigma) \to H^n(\Sigma)$ is an isomorphism. The Alexander polynomial

$$\Delta_A(t) \ = \ \det(tA + (-1)^n A^*) \in \Lambda/S\text{-equivalence}$$

is an isotopy invariant of the knot such that $\Delta_A(1) = \pm 1$ and $\Delta_A(t) H_n(\overline{K}) = 0$. In the non–spherical case $M^{2n-1} \subset S^{2n+1}$ the exact sequence $(*)$ and $\Delta_A(t) H_n(\overline{K}) = 0$ are known only under additional assumptions which ensure that $\partial = 0 : H_r(\overline{K}) \to H_{r-1}(\Sigma)[t,t^{-1}]$ for $r = n, n+1$ (see [Er]).

6.4. **The generalized Levine–Tristram signatures.** Having defined the Seifert form we pass to the definition of the generalized Levine–Tristram signatures $\sigma_\xi(M) \in \mathbb{Z}$ $(\xi \in S^1)$ of a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$, using the ξ-twisted $(-1)^{n+1}$-hermitian intersection form $(H_{n+1}(X;\xi), B(X;\xi))$ on the ξ-twisted homology of the complement $X = \mathrm{cl.}(D^{2n+2} \backslash \Sigma \times D^2)$ of a Seifert surface $\Sigma^{2n} \subset S^{2n+1}$ pushed into D^{2n+2}.

Definition 6.4.1. For any space X with infinite cyclic cover $\overline{X}$ and $\xi \in S^1$ use the morphism of rings with involution

$$\mathbb{C}[t,t^{-1}] \to \mathbb{C} \ ; \ t \mapsto \xi \ (\overline{t} = t^{-1})$$

to define the ξ*-twisted* $\mathbb{C}$*-coefficient homology*

$$H_*(X;\xi) \ = \ H_*(\mathbb{C} \otimes_{\mathbb{C}[t,t^{-1}]} C(\overline{X};\mathbb{C})) \ .$$

Example 6.4.2. For $\xi = 1 \in S^1$

$$H_*(X;1) \ = \ H_*(X;\mathbb{C}) \ .$$

Proposition 6.4.3. *For any $\xi \in S^1$ the short exact sequence of $\mathbb{C}$-module chain complexes*

$$0 \longrightarrow C(\overline{X};\mathbb{C}) \xrightarrow{t-\xi} C(\overline{X};\mathbb{C}) \longrightarrow C(X;\xi) \longrightarrow 0$$

induces a long exact sequence of homology $\mathbb{C}$-modules

$$\dots \longrightarrow H_r(\overline{X};\mathbb{C}) \xrightarrow{t-\xi} H_r(\overline{X};\mathbb{C}) \longrightarrow H_r(X;\xi) \longrightarrow H_{r-1}(\overline{X};\mathbb{C}) \longrightarrow \dots\ .$$

Example 6.4.4. For $\xi \neq 1 \in S^1$ and $X = Y \times S^1$, $\overline{X} = Y \times \mathbb{R}$

$$H_*(X;\xi) \ = \ H_*(1-\xi : C(Y;\mathbb{C}) \to C(Y;\mathbb{C})) \ = \ 0\ .$$

Definition 6.4.5. Let $(X, \partial X)$ be a m-dimensional manifold with boundary and an infinite cyclic cover $(\overline{X}, \overline{\partial X})$.
(i) The $\mathbb{C}[t,t^{-1}]$-module chain map

$$C(\overline{X};\mathbb{C}) \to C(\overline{X},\overline{\partial X};\mathbb{C}) \underset{\simeq}{\xrightarrow{([X]\cap -)^{-1}}} C(\overline{X};\mathbb{C})^{m-*} \ = \ \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C(\overline{X}),\mathbb{C}[t,t^{-1}])_{m-*}$$

induces the Blanchfield intersection pairings (Remark 5.5.3)

$$B(\overline{X}) \ : \ H_r(\overline{X};\mathbb{C}) \times H_{m-r}(\overline{X};\mathbb{C}) \to \mathbb{C}[t,t^{-1}]$$

such that

$$B(\overline{X})(x,y) \ = \ (-1)^{r(m-r)}\overline{B(\overline{X})(y,x)} \in \mathbb{C}[t,t^{-1}]\ ,$$

with an exact sequence

$$\dots \longrightarrow H_r(\partial\overline{X};\mathbb{C}) \longrightarrow H_r(\overline{X};\mathbb{C}) \xrightarrow{B(\overline{X})} H_{m-r}(\overline{X};\mathbb{C})^* \longrightarrow H_{r-1}(\partial\overline{X};\mathbb{C}) \longrightarrow \dots\ .$$

(ii) For $\xi \in S^1$ there is also induced the ξ-*twisted intersection pairings*

$$B(X;\xi) \ : \ H_r(X;\xi) \times H_{m-r}(X;\xi) \to \mathbb{C}$$

such that

$$B(X;\xi)(x,y) \ = \ (-1)^{r(m-r)}\overline{B(X;\xi)(y,x)} \in \mathbb{C}\ ,$$

with an exact sequence

$$\dots \longrightarrow H_r(\partial X;\xi) \longrightarrow H_r(X;\xi) \xrightarrow{B(X;\xi)} H_{m-r}(X;\xi)^* \longrightarrow H_{r-1}(\partial X;\xi) \longrightarrow \dots\ .$$

Example 6.4.6. For $m = 2n$ there is defined a $(-1)^n$-hermitian intersection form $(H_n(X;\xi), B(X;\xi))$ over $\mathbb{C}$, which is nonsingular if $\partial X = \emptyset$. In particular, if $\partial X = \emptyset$, $X = \partial Y$ for a $(2n+1)$-dimensional manifold Y with infinite cyclic cover $\overline{Y} \supset \overline{X}$, then $\mathrm{im}(H_{n+1}(Y,X;\xi) \to H_n(X;\xi))$ is a lagrangian of $(H_n(X;\xi), B(X;\xi))$.

Given a framed codimension 2 embedding $M^m \subset S^{m+2}$ with complement K and a Seifert surface $\Sigma^{m+1} \subset S^{m+2}$ push Σ rel ∂ into a framed codimension 2 embedding $(\Sigma, M) \subset (D^{m+3}, S^{m+2})$ (cf. [Ra3, Proposition 27.8]). The complement

$$(L; K, \Sigma \times S^1) \ = \ (\mathrm{cl.}(D^{m+3}\backslash \Sigma \times D^2); \mathrm{cl.}(S^{m+2}\backslash M \times D^2), \Sigma \times S^1)$$

has boundary the closed $(m+2)$-dimensional manifold

$$\partial L \ = \ K \cup_{M \times S^1} -\Sigma \times S^1\ .$$

The canonical infinite cyclic cover $(\overline{L}; \overline{K}, \Sigma \times \mathbb{R})$ of $(L; K, \Sigma \times S^1)$ is classified by the Seifert map $U : L \to S^1$, and

$$H_r(\overline{L};\mathbb{C}) \ = \ H_{r-1}(\Sigma;\mathbb{C})[t,t^{-1}]\ ,\ H_r(\partial L;\xi) \ = \ H_r(K;\xi)\ (1 \leqslant r \leqslant m)\ .$$

The map U has regular inverse image $U^{-1}(*) = \Sigma \times I$, and cutting along the codimension 1 framed embedding

$$\Sigma \times (I; \{0\}, \{1\}) \subset (L; K, \Sigma \times S^1)$$

gives a fundamental domain

$$(L_{\Sigma \times I}; K_\Sigma, (\Sigma \times S^1)_\Sigma) \cong (D^{m+3}; K_\Sigma, \Sigma \times I)$$

for the canonical infinite cyclic cover $(\overline{L}; \overline{K}, \Sigma \times \mathbb{R})$ of $(L; K, \Sigma \times S^1)$, with an exact sequence of $\mathbb{C}[t, t^{-1}]$-modules

$$\cdots \to H_{r+1}(\overline{L}; \mathbb{C}) = H_r(\Sigma; \mathbb{C})[t, t^{-1}] \xrightarrow{(1-t)A + (-1)^{(r+1)(m+2-r)}(1-t^{-1})A^*}$$
$$H_{r+1}(\overline{L}, \overline{\partial L}; \mathbb{C}) = H_{m+1-r}(\Sigma; \mathbb{C})^*[t, t^{-1}]) \to H_r(\overline{\partial L}; \mathbb{C}) \to H_r(\overline{L}; \mathbb{C}) = H_{r-1}(\Sigma; \mathbb{C})[t, t^{-1}] \to \dots$$

with $A : H_r(\Sigma; \mathbb{C}) \to H_{m+1-r}(\Sigma; \mathbb{C})^*$ the Seifert form.

Lemma 6.4.7. *For $m = 2n - 1$ the $(-1)^{n+1}$-hermitian intersection form over $\mathbb{C}[t, t^{-1}]$ of $\overline{L}$ is determined by the Seifert form $(F_n(\Sigma), A(\Sigma))$*

$$(H_{n+1}(\overline{L}; \mathbb{C}), B(\overline{L})) = (H_n(\Sigma; \mathbb{C})[t, t^{-1}], B_{A(\Sigma)}(t))$$

with

$$B_{A(\Sigma)}(t) = (1-t)A(\Sigma) + (-1)^{n+1}(1-t^{-1})A(\Sigma)^* ,$$

and an exact sequence of $\mathbb{C}[t, t^{-1}]$-modules

$$\cdots \to H_{n+1}(\overline{L}; \mathbb{C}) = H_n(\Sigma; \mathbb{C})[t, t^{-1}] \xrightarrow{B_{A(\Sigma)}(t)} H_{n+1}(\overline{L}, \overline{\partial L}; \mathbb{C}) = H_n(\Sigma; \mathbb{C})^*[t, t^{-1}])$$
$$\to H_n(\overline{\partial L}; \mathbb{C}) \to H_n(\overline{L}; \mathbb{C}) = H_{n-1}(\Sigma; \mathbb{C})[t, t^{-1}] \to \dots .$$

The ξ-twisted $(-1)^{n+1}$-hermitian intersection form over $\mathbb{C}$ is

$$(H_{n+1}(L; \xi), B(L; \xi)) = (H_n(\Sigma; \mathbb{C}), B_{A(\Sigma)}(\xi)) .$$

For $\xi \neq 1$ there is an exact sequence

$$\cdots \to H_{n+1}(K; \xi) = H_{n+1}(\partial L; \xi) \to H_{n+1}(L; \xi) = H_n(\Sigma; \mathbb{C})$$
$$\xrightarrow{B_{A(\Sigma)}(\xi)} H_{n+1}(L, K; \xi) = H_n(\Sigma; \mathbb{C})^* \to \dots$$

Definition 6.4.8. Let $\epsilon = 1$ or -1, let (F, A) be a Seifert form, and let $\xi \neq 1 \in S^1 \subset \mathbb{C}$.
(i) The $-\epsilon$-hermitian form $(\mathbb{C} \otimes_{\mathbb{Z}} F, B_A(\xi))$ over $\mathbb{C}$ is defined by

$$B_A(\xi) = (1-\xi)A - \epsilon(1-\overline{\xi})A^* .$$

(ii) The *nullity* of (F, A) at ξ is

$$n_A(\xi) = \text{nullity}(B_A(\xi)) = \dim_{\mathbb{C}} \left(\ker(\xi A + \epsilon A^*)\right) \geqslant 0 .$$

(iii) The *signature* of (F, A) at ξ is

$$\sigma_A(\xi) = \sigma(\mathbb{C} \otimes_{\mathbb{Z}} F, B_A(\xi)) \in \mathbb{Z} .$$

(iv) Given a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$ with a choice of Seifert surface Σ and Seifert form $(F_n(\Sigma), A(\Sigma))$, set $\epsilon = (-1)^n$ and define the *nullity* and *Levine–Tristram signature* to be

$$n_{M,\Sigma}(\xi) = n_{A(\Sigma)}(\xi) \geqslant 0 , \ \sigma_{M,\Sigma}(\xi) = \sigma_{A(\Sigma)}(\xi) \in \mathbb{Z} .$$

We shall now investigate the behaviour of the nullities and signatures under the h-cobordism and isotopy of framed codimension 2 embeddings, using the following algebraic lemma.

Proposition 6.4.9. *Let (F_0, A_0) be a Seifert form, and let*

$$(F_1, A_1) \;=\; (F_0 \oplus \mathbb{Z}^\ell \oplus \mathbb{Z}^\ell, \begin{pmatrix} A_0 & 0 & \alpha \\ 0 & 0 & x \\ \beta & y & z \end{pmatrix})$$

be a rank (ℓ, ℓ) enlargement of (F_0, A_0). Set $\epsilon \;=\; (-1)^n$.
(i) *The evaluations of the Alexander polynomials of (F_0, A_0), (F_1, A_1) at $\xi \in \mathbb{C}$ are related by*

$$\begin{aligned}\Delta_{A_1}(\xi) \;&=\; \det(\xi A_1 + (-1)^n A_1^*) \\ &=\; -\det(\xi x + (-1)^n y^*)\det(\xi y + (-1)^n x^*)\det(\xi A_0 + (-1)^n A_0^*) \in \mathbb{C}\ .\end{aligned}$$

(ii) *If $\xi \neq 1 \in S^1$ is such that $\det(\xi x + (-1)^n y^*) \neq 0 \in \mathbb{C}$ (e.g. if $\Delta_{A_1}(\xi) \neq 0 \in \mathbb{C}$), the nullities and signatures of the $(-1)^{n+1}$-hermitian forms $(\mathbb{C}\otimes_{\mathbb{Z}} F_0, B_{A_0}(\xi))$, $(\mathbb{C}\otimes_{\mathbb{Z}} F_1, B_{A_1}(\xi))$ are the same*

$$n_{A_0}(\xi) \;=\; n_{A_1}(\xi) \geqslant 0\ ,\ \sigma_{A_0}(\xi) \;=\; \sigma_{A_1}(\xi) \in \mathbb{Z}\ .$$

In particular, this is the case if $\xi - 1 \in \mathbb{C}$ is not an algebraic integer.
(iii) *For any $\xi \neq 1 \in S^1$*

$$\sigma_{A_1}(\xi) - \sigma_{A_0}(\xi) \;=\; \sigma(F, B) \in \mathbb{Z}$$

with

$$\begin{aligned}F \;&=\; \{w \in \mathbb{C}^\ell \,|\, (\xi x + (-1)^n y^*)(w) \;=\; 0 \in \mathbb{C}^\ell\,,\, (\xi\alpha + (-1)^n\beta^*)(w) \in \operatorname{im}(B_{A_0}(\xi)) \subseteq \mathbb{C}^k\}\ , \\ B \;&=\; (1-\xi)z + (-1)^{n+1}(1-\overline{\xi})z^*|\ .\end{aligned}$$

Proof. Apply Theorem 2.2.7 to

$$\begin{aligned}B_{A_1}(\xi) \;&=\; \begin{pmatrix} B_{A_0}(\xi) & 0 & C \\ 0 & 0 & D \\ (-1)^{n+1}C^* & (-1)^{n+1}D^* & E \end{pmatrix}\ , \\ C \;&=\; (1-\xi)\alpha + (-1)^{n+1}(1-\overline{\xi})\beta^* \;=\; (\overline{\xi}-1)(\xi\alpha + (-1)^n\beta^*)\ , \\ D \;&=\; (1-\xi)x + (-1)^{n+1}(1-\overline{\xi})y^* \;=\; (\overline{\xi}-1)(\xi x + (-1)^n y^*)\ , \\ E \;&=\; (1-\xi)z + (-1)^{n+1}(1-\overline{\xi})z^*\ .\end{aligned}$$

□

Corollary 6.4.10. *Let $M^{2n-1} \subset S^{2n+1}$ be a framed codimension 2 embedding, with a Seifert surface $\Sigma^{2n} \subset S^{2n+1}$ and Seifert form $(F_n(\Sigma), A(\Sigma))$.*
(i) *For any $\xi \neq 1 \in S^1$ the nullities $n_{M,\Sigma}(\xi)$ and the Levine–Tristram signatures $\sigma_{M,\Sigma}(\xi)$ are isotopy invariants, meaning that if there exists an isotopy $(W; M, M') \subset S^{2n+1} \times (I; \{0\}, \{1\})$ then for any Seifert surfaces Σ, Σ' for $M, M' \subset S^{2n+1}$*

$$n_{M,\Sigma}(\xi) \;=\; n_{M',\Sigma'}(\xi)\ ,\ \sigma_{M,\Sigma}(\xi) \;=\; \sigma_{M',\Sigma'}(\xi)\ .$$

(ii) *For any $\xi \neq 1 \in S^1$ such that $\det(\xi A(\Sigma) + (-1)^n A(\Sigma)^*) \neq 0$ the nullities $n_{M,\Sigma}(\xi)$ and the Levine–Tristram signatures $\sigma_{M,\Sigma}(\xi)$ are h-cobordism invariants, meaning that if there exists an h-cobordism $(W; M, M') \subset S^{2n+1} \times (I; \{0\}, \{1\})$ then for any Seifert surfaces Σ, Σ' for $M, M' \subset S^{2n+1}$*

$$n_{M,\Sigma}(\xi) \;=\; n_{M',\Sigma'}(\xi)\ ,\ \sigma_{M,\Sigma}(\xi) \;=\; \sigma_{M',\Sigma'}(\xi)\ .$$

Proof. Immediate from Theorem 6.2.6 and Proposition 6.4.9. □

Definition 6.4.11. Let $M^{2n-1} \subset S^{2n+1}$ be a framed codimension 2 embedding.
(i) The *nullity* of M for $\xi \neq 1 \in S^1$ is

$$n_M(\xi) \ = \ n_{M,\Sigma}(\xi) \geqslant 0$$

defined using any Seifert surface $\Sigma^{2n} \subset S^{2n+1}$. This is an isotopy invariant of $M \subset S^{2n+1}$, independent of the choice of Σ, by 6.4.10 (i).
(ii) The *Levine–Tristram signature* of M for $\xi \neq 1 \in S^1$ is

$$\sigma_M(\xi) \ = \ \sigma_{M,\Sigma}(\xi) \in \mathbb{Z}$$

defined using any Seifert surface $\Sigma^{2n} \subset S^{2n+1}$. This is an isotopy invariant of $M \subset S^{2n+1}$, independent of the choice of Σ, by 6.4.10 (i). For $\Delta_{M,\Sigma}(\xi) \neq 0$ it is an h-cobordism invariant, by 6.4.10 (ii).

The Levine–Tristram signatures of a framed codimension 2 embedding satisfy the same algebraic properties as the signatures of knots.

Definition 6.4.12. (i) Two complex numbers $z_0, z_1 \in \mathbb{C}$ are *S-equivalent* if $|z_0| = |z_1|$. The function

$$\mathbb{C}/\{S\text{-equivalence}\} \to [0,\infty) \ ; \ z \mapsto |z|$$

is a bijection.
(ii) Given a framed codimension 2 embedding $M^{2n-1} \subset S^{2n+1}$ and $\xi \neq 1 \in S^1$ let

$$\Delta_M(\xi) \ = \ \Delta_A(\xi) \ = \ \det(\xi A + (-1)^n A^*) \in \mathbb{C}/\{S\text{-equivalence}\} \ = \ [0,\infty)$$

be the evaluation of the Alexander polynomial $\Delta_M(t) \in \Lambda/\{S\text{-equivalence}\}$ at $t = \xi$. This is an isotopy invariant of $M \subset S^{2n+1}$, by Proposition 6.3.6.

Lemma 6.4.13. *$n_M(\xi) > 0$ if and only if $\Delta_M(\xi) = 0$. In particular, if $\dim(\ker A \cap \ker A^*) = 0$, the nullity is equal to zero for all but finitely many values of $\xi \in S^1$.*

Proof. As $\xi\overline{\xi} = 1$, we can rewrite $B_A(\xi) = (\overline{\xi} - 1)(\xi A + (-1)^n A^*)$, hence $\det B_A(\xi) = 0$ if and only if $\Delta_M(\xi) = 0$. If $\dim(\ker A \cap \ker A^*) > 0$, then $\Delta_M(\xi) = 0$ and $n_M(\xi) > 0$. So assume that $\dim(\ker A \cap \ker A^*) = 0$. By the result of Keef [Keef] (see also [Ka, Theorem 12.2.9] and [BN, Section 3.1]), A is S-equivalent over $\mathbb{Q}$ to a matrix with $\det(A) \neq 0 \in \mathbb{Q}$, and we can write

$$B_A(\xi) \ = \ (\overline{\xi} - 1)\left(\xi I + (-1)^n A^* A^{-1}\right) A \ ,$$

and so $\det B_A(\xi) = 0 \in \mathbb{C}$ if and only if $\det(\xi I + (-1)^n A^* A^{-1}) = 0 \in \mathbb{C}$. □

6.5. **The Murasugi–Kawauchi inequality in higher dimensions.** We obtain a Murasugi–Kawauchi like theorem giving an upper bound for the difference between the Levine–Tristram signatures $\sigma_{M_0}(\xi), \sigma_{M_1}(\xi)$ of cobordant framed codimension 2 embeddings M_0^{2n-1}, $M_1^{2n-1} \subset S^{2n+1}$. See Kawauchi [Ka, Theorem 12.3.1] for the classical case of signatures for links $M_k = \bigcup_{\mu_k} S^1 \subset S^3$ $(k = 0, 1)$, the special case $\xi = -1$, $n = 1$.

Theorem 6.5.1. *Let $(W^{2n}; M_0, M_1) \subset S^{2n+1} \times (I; \{0\}, \{1\})$ be a cobordism of framed codimension 2 embeddings M_0, $M_1 \subset S^{2n+1}$, let Σ_0, Σ_1 be Seifert surfaces for M_0, M_1, so that*

$$\Sigma^{2n} \ = \ \Sigma_0 \cup_{M_0} W \cup_{M_1} -\Sigma_1 \subset S^{2n+1} \times I$$

is a closed $2n$-dimensional manifold. For any $\xi \neq 1 \in S^1$

$$|\sigma_{M_0}(\xi) - \sigma_{M_1}(\xi)| \leqslant b_n(\Sigma) - b_n(\Sigma_0) - b_n(\Sigma_1) + n_{M_0}(\xi) + n_{M_1}(\xi) \ ,$$

where $b_n(X)$ denotes the n-th Betti number of a topological space X.

Proof. The complement

$$(J;K_0,K_1) \;=\; (\text{cl.}(S^{2n+1}\times I\backslash W\times D^2);\text{cl.}(S^{2n+1}\times\{0\}\backslash M_0\times D^2),\text{cl.}(S^{2n+1}\times\{1\}\backslash M_1\times D^2))$$

has boundary the closed $(2n+1)$-dimensional manifold

$$\partial J \;=\; K_0 \cup_{M_0\times S^1} W\times S^1 \cup_{M_1\times S^1} -K_1 \;.$$

The Seifert map $V : J \to S^1$ (Proposition 5.7.1) has regular inverse image $V^{-1}(*) = \Omega^{2n+1} \subset J$ for a cobordism of framed codimension 1 embeddings

$$(\Omega^{2n+1};\Sigma_0,\Sigma_1) \subset S^{2n+1}\times(I;\{0\},\{1\})$$

with boundary $\partial\Omega = \Sigma$. It is possible to push Ω rel Σ into a cobordism of framed codimension 2 embeddings

$$(\Omega^{2n+1};\Sigma_0,\Sigma_1) \subset D^{2n+2}\times(I;\{0\},\{1\}) \;,$$

which restricts to a framed codimension 2 embedding

$$\Sigma^{2n} \subset D^{2n+2}\times\{0\}\cup S^{2n+1}\times I\cup -D^{2n+2}\times\{1\} \;=\; S^{2n+2} \;.$$

The complement

$$(Y;L_0,L_1) \;=\; (\text{cl.}(D^{2n+2}\times I\backslash\Omega\times D^2);\text{cl.}(D^{2n+2}\times\{0\}\backslash\Sigma_0\times D^2);\text{cl.}(D^{2n+2}\times\{1\}\backslash\Sigma_1\times D^2))$$

has boundary the closed $(2n+2)$-dimensional manifold

$$X \;=\; \partial Y \;=\; \text{cl.}(S^{2n+2}\backslash\Sigma\times D^2)\cup -\Omega\times S^1 \;=\; (L_0\cup_{K_0} J\cup_{K_1} -L_1)\cup_{\Sigma\times S^1} -\Omega\times S^1$$

with canonical infinite cyclic cover $\overline{X}$ such that

$$H_{n+1}(\overline{X};\mathbb{C}) \;=\; H_n(\Sigma;\mathbb{C})[t,t^{-1}] \;,\; H_{n+1}(\overline{Y};\mathbb{C}) \;=\; H_n(\Omega;\mathbb{C})[t,t^{-1}] \;.$$

Let $\Sigma_{01} = \Sigma_0 \sqcup -\Sigma_1$, and use the inclusion $j : \Sigma_{01} \to \Sigma$ to induce a morphism of $(-1)^{n+1}$-hermitian forms over $\mathbb{C}$

$$\begin{aligned} j \;:\; (F',B') \;=\; (H_n(\Sigma_{01};\mathbb{C}),B(\Sigma_{01};\xi)) \;&=\; (H_n(\Sigma_0;\mathbb{C}),B_{A_0}(\xi))\oplus(H_n(\Sigma_1;\mathbb{C}),-B_{A_1}(\xi)) \\ &\to (F,B) \;=\; (H_n(\Sigma;\mathbb{C}),B(\Sigma;\xi)) \;. \end{aligned}$$

The $(-1)^{n+1}$-hermitian intersection form over $\mathbb{C}[t,t^{-1}]$ $(H_{n+1}(\overline{X}),B(\overline{X}))$ is metabolic, with lagrangian $\ker(H_{n+1}(\overline{X};\mathbb{C}) \to H_{n+1}(\overline{Y};\mathbb{C}))$. Use the morphism $\mathbb{C}[t,t^{-1}] \to \mathbb{C}; t \mapsto \xi$ to induce a $(-1)^{n+1}$-hermitian form $\mathbb{C}\otimes_{\mathbb{C}[t,t^{-1}]}(H_{n+1}(\overline{X};\mathbb{C}),B(\overline{X})) = (F,B)$ over $\mathbb{C}$ which is also metabolic. By Proposition 2.2.6 (v)

$$|\sigma(F',B')| \leqslant \dim_{\mathbb{C}} F - \dim_{\mathbb{C}} F' + n(F',B')$$

which is precisely

$$|\sigma_{M_0}(\xi) - \sigma_{M_1}(\xi)| \leqslant b_n(\Sigma) - b_n(\Sigma_0) - b_n(\Sigma_1) + n_{M_0}(\xi) + n_{M_1}(\xi) \;.$$

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