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Generalized Arf invariants in algebraic L-theory

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Abstract

The difference between the quadratic L-groups $L_*(A)$ and the symmetric L-groups $L^*(A)$ of a ring with involution A is detected by generalized Arf invariants. The special case $A = \mathbb{Z}[x]$ gives a complete set of invariants for the Cappell UNil-groups $\mathrm{UNil}_*(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ for the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, extending the results of Connolly and Ranicki [Adv. Math. 195 (2005) 205–258], Connolly and Davis [Geom. Topol. 8 (2004) 1043–1078, e-print http://arXiv.org/abs/math/0306054].

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0. Introduction

The invariant of Arf [1] is a basic ingredient in the isomorphism classification of quadratic forms over a field of characteristic 2. The algebraic L-groups of a ring with involution A are Witt groups of quadratic structures on A-modules and A-module chain complexes, or equivalently the cobordism groups of algebraic Poincaré complexes over A.

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The cobordism formulation of algebraic L-theory is used here to obtain generalized Arf invariants detecting the difference between the quadratic and symmetric L-groups of an arbitrary ring with involution A, with applications to the computation of the Cappell UNil-groups.

The (projective) *quadratic L-groups* of Wall [20] are 4-periodic groups

$$L_n(A) = L_{n+4}(A).$$

The $2k$-dimensional L-group $L_{2k}(A)$ is the Witt group of nonsingular $(-1)^k$-quadratic forms (K, ψ) over A, where K is a f.g. projective A-module and ψ is an equivalence class of A-module morphisms

$$\psi : K \to K^* = \mathrm{Hom}_A(K, A)$$

such that $\psi + (-1)^k\psi^* : K \to K^*$ is an isomorphism, with

$$\psi \sim \psi + \chi + (-1)^{k+1}\chi^* \quad \text{for } \chi \in \mathrm{Hom}_A(K, K^*).$$

A lagrangian L for (K, ψ) is a direct summand $L \subset K$ such that $L^\perp = L$, where

$$\begin{aligned} &L^\perp = \{x \in K \mid (\psi + (-1)^k\psi^*)(x)(y) = 0 \text{ for all } y \in L\},\\ &\psi(x)(x) \in \{a + (-1)^{k+1}\overline{a} \mid a \in A\} \quad \text{for all } x \in L. \end{aligned}$$

A form (K, ψ) admits a lagrangian L if and only if it is isomorphic to the hyperbolic form $H_{(-1)^k}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, in which case

$$(K, \psi) = H_{(-1)^k}(L) = 0 \in L_{2k}(A).$$

The $(2k + 1)$-dimensional L-group $L_{2k+1}(A)$ is the Witt group of $(-1)^k$-quadratic formations $(K, \psi; L, L')$ over A, with $L, L' \subset K$ lagrangians for (K, ψ).

The *symmetric L-groups* $L^n(A)$ of Mishchenko [13] are the cobordism groups of *n-dimensional symmetric Poincaré complexes* (C, ϕ) over A, with C an n-dimensional f.g. projective A-module chain complex

$$C : \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0 \to \cdots$$

and $\phi \in Q^n(C)$ an element of the n-dimensional symmetric Q-group of C (about which more in §1 below) such that $\phi_0 : C^{n-*} \to C$ is a chain equivalence. In particular, $L^0(A)$ is the Witt group of nonsingular symmetric forms (K, ϕ) over A, with

$$\phi = \phi^* : K \to K^*$$

an isomorphism, and $L^1(A)$ is the Witt group of symmetric formations $(K,\phi;L,L')$ over A. An analogous cobordism formulation of the quadratic L-groups was obtained in [15], expressing $L_n(A)$ as the cobordism group of *n-dimensional quadratic Poincaré complexes* (C,ψ), with $\psi\in Q_n(C)$ an element of the n-dimensional quadratic Q-group of C such that $(1+T)\psi_0 : C^{n-*}\to C$ is a chain equivalence. The *hyperquadratic L-groups* $\widehat{L}^n(A)$ of [15] are the cobordism groups of n-dimensional (symmetric, quadratic) Poincaré pairs $(f : C\to D,(\delta\phi,\psi))$ over A such that

$$(\delta\phi_0,(1+T)\psi_0) : D^{n-*}\to \mathcal{C}(f)$$

is a chain equivalence, with $\mathcal{C}(f)$ the algebraic mapping cone of *f*. The various L-groups are related by an exact sequence

$$\cdots \longrightarrow L_n(A) \xrightarrow{1+T} L^n(A) \longrightarrow \widehat{L}^n(A) \xrightarrow{\partial} L_{n-1}(A) \longrightarrow \cdots .$$

The symmetrization maps $1+T : L_*(A)\to L^*(A)$ are isomorphisms modulo 8-torsion, so that the hyperquadratic L-groups $\widehat{L}^*(A)$ are of exponent 8. The symmetric and hyperquadratic L-groups are not 4-periodic in general. However, there are defined natural maps

$$L^n(A)\to L^{n+4}(A),\ \ \widehat{L}^n(A)\to \widehat{L}^{n+4}(A)$$

(which are isomorphisms modulo 8-torsion), and there are 4-periodic versions of the L-groups

$$L^{n+4*}(A)=\lim_{k\to\infty} L^{n+4k}(A),\ \ \widehat{L}^{n+4*}(A)=\lim_{k\to\infty}\widehat{L}^{n+4k}(A).$$

The 4-periodic symmetric L-group $L^{n+4*}(A)$ is the cobordism group of n-dimensional symmetric Poincaré complexes (C,ϕ) over A with C a finite (but not necessarily n-dimensional) f.g. projective A-module chain complex, and similarly for $\widehat{L}^{n+4*}(A)$.

The Tate $\mathbb{Z}_2$-cohomology groups of a ring with involution A,

$$\widehat{H}^n(\mathbb{Z}_2;A)=\frac{\{x\in A\,|\,\overline{x}=(-1)^n x\}}{\{y+(-1)^n\overline{y}\,|\,y\in A\}}\quad (n(\mathrm{mod}\,2))$$

are A-modules via

$$A\times \widehat{H}^n(\mathbb{Z}_2;A)\to \widehat{H}^n(\mathbb{Z}_2;A);\ \ (a,x)\mapsto ax\overline{a}.$$

The Tate $\mathbb{Z}_2$-cohomology A-modules give an indication of the difference between the quadratic and symmetric L-groups of A. If $\widehat{H}^*(\mathbb{Z}_2; A) = 0$ (e.g. if $\frac{1}{2} \in A$) then the symmetrization maps $1 + T : L_*(A) \to L^*(A)$ are isomorphisms and $\widehat{L}^*(A) = 0$. If A is such that $\widehat{H}^0(\mathbb{Z}_2; A)$ and $\widehat{H}^1(\mathbb{Z}_2; A)$ have one-dimensional f.g. projective A-module resolutions then the symmetric and hyperquadratic L-groups of A are 4-periodic (Proposition 30).

For any ring A define

$$A_2 = A/2A,$$

an additive group of exponent 2.

We shall say that a ring with the involution A is *r-even* for some $r \geqslant 1$ if

(i) A is commutative with the identity involution, so that $\widehat{H}^0(\mathbb{Z}_2; A) = A_2$ as an additive group with

$$A \times \widehat{H}^0(\mathbb{Z}_2; A) \to \widehat{H}^0(\mathbb{Z}_2; A); \ \ (a, x) \mapsto a^2 x$$

and

$$\widehat{H}^1(\mathbb{Z}_2; A) = \{a \in A \mid 2a = 0\},$$

(ii) $2 \in A$ is a nonzero divisor, so that $\widehat{H}^1(\mathbb{Z}_2; A) = 0$,

(iii) $\widehat{H}^0(\mathbb{Z}_2; A)$ is a f.g. free A_2-module of rank r with a basis $\{x_1 = 1, x_2, \ldots, x_r\}$.

If A is r-even then $\widehat{H}^0(\mathbb{Z}_2; A)$ has a one-dimensional f.g. free A-module resolution

$$0 \to A^r \xrightarrow{\ 2\ } A^r \xrightarrow{\ x\ } \widehat{H}^0(\mathbb{Z}_2; A) \to 0,$$

so that the symmetric and hyperquadratic L-groups of A are 4-periodic (30).

Theorem A. *The hyperquadratic L-groups of a* 1*-even ring with involution A are given by*

$$\widehat{L}^n(A) = \begin{cases} \dfrac{\{a \in A \mid a - a^2 \in 2A\}}{\{8b + 4(c - c^2) \mid b, c \in A\}} & \text{if } n \equiv 0 \pmod 4, \\ \dfrac{\{a \in A \mid a - a^2 \in 2A\}}{2A} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ \dfrac{A}{\{2a + b - b^2 \mid a, b \in A\}} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

The boundary maps $\partial : \widehat{L}^n(A) \to L_{n-1}(A)$ *are given by*

$$\partial : \widehat{L}^0(A) \to L_{-1}(A);\ a \mapsto \left(A \oplus A, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ A, \operatorname{im}\left(\begin{pmatrix} 1-a \\ a \end{pmatrix} : A \to A \oplus A\right)\right),$$

$$\partial : \widehat{L}^1(A) \to L_0(A);\ a \mapsto \left(A \oplus A, \begin{pmatrix} (a-a^2)/2 & 1-2a \\ 0 & -2 \end{pmatrix}\right),$$

$$\partial : \widehat{L}^3(A) \to L_2(A);\ a \mapsto \left(A \oplus A, \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}\right).$$

The map

$$L^0(A) \to \widehat{L}^0(A);\ (K, \phi) \mapsto \phi(v, v)$$

is defined using any element $v \in K$ *such that*

$$\phi(u, u) = \phi(u, v) \in A_2 \quad (u \in K).$$

For any commutative ring A the squaring function on A_2:

$$\psi^2 : A_2 \to A_2;\ a \mapsto a^2$$

is a morphism of additive groups. If $2 \in A$ is a nonzero divisor than A is 1-even if and only if ψ^2 is an isomorphism, with

$$\begin{aligned} \widehat{L}^1(A) &= \ker(\psi^2 - 1 : A_2 \to A_2), \\ \widehat{L}^3(A) &= \operatorname{coker}(\psi^2 - 1 : A_2 \to A_2). \end{aligned}$$

In particular, if $2 \in A$ is a nonzero divisor and $\psi^2 = 1 : A_2 \to A_2$ (or equivalently $a - a^2 \in 2A$ for all $a \in A$) then A is 1-even. In this case Theorem A gives

$$\widehat{L}^n(A) = \begin{cases} A_8 & \text{if } n \equiv 0 (\operatorname{mod} 4), \\ A_2 & \text{if } n \equiv 1, 3 (\operatorname{mod} 4), \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4). \end{cases}$$

Thanks to Liam O'Carroll and Frans Clauwens for examples of 1-even rings A such that $\psi^2 \neq 1$, e.g. $A = \mathbb{Z}[x]/(x^3 - 1)$ with

$$\psi^2 : A_2 = \mathbb{Z}_2[x]/(x^3 - 1) \to A_2;\ a + bx + cx^2 \mapsto (a + bx + cx^2)^2 = a + cx + bx^2.$$

Theorem A is proved in §2 (Corollary 61). In particular, $A = \mathbb{Z}$ is 1-even with $\psi^2 = 1$, and in this case Theorem A recovers the computation of $\widehat{L}^*(\mathbb{Z})$ obtained in [15]—the algebraic L-theory of $\mathbb{Z}$ is recalled further below in the Introduction.

Theorem B. *If A is 1-even with $\psi^2 = 1$ then the polynomial ring $A[x]$ is 2-even, with $A[x]_2$-module basis $\{1, x\}$ for $\widehat{H}^0(\mathbb{Z}_2; A[x])$. The hyperquadratic L-groups of $A[x]$ are given by*

$$\widehat{L}^n(A[x]) = \begin{cases} A_8 \oplus A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 0 (\operatorname{mod} 4), \\ A_2 & \text{if } n \equiv 1 (\operatorname{mod} 4), \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4), \\ A_2[x] & \text{if } n \equiv 3 (\operatorname{mod} 4). \end{cases}$$

Theorems A and B are special cases of the following computation:

Theorem C. *The hyperquadratic L-groups of an r-even ring with involution A are given by*

$$\widehat{L}^n(A) = \begin{cases} \dfrac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N + N^t) - 4N^tXN \mid N \in M_r(A)\}} & \text{if } n = 0, \\ \dfrac{\{N \in M_r(A) \mid N + N^t - 2N^tXN \in 2\mathrm{Quad}_r(A)\}}{2M_r(A)} & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{L - LXL \mid L \in \mathrm{Sym}_r(A)\}} & \text{if } n = 3, \end{cases}$$

with $\mathrm{Sym}_r(A)$ *the additive group of symmetric $r \times r$ matrices* $(a_{ij}) = (a_{ji})$ in A, $\mathrm{Quad}_r(A) \subset \mathrm{Sym}_r(A)$ *the subgroup of the matrices such that $a_{ii} \in 2A$, and*

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)$$

for an A_2-module basis $\{x_1 = 1, x_2, \dots, x_r\}$ of $\widehat{H}^0(\mathbb{Z}_2; A)$. The boundary maps $\partial : \widehat{L}^n(A) \to L_{n-1}(A)$ are given by

$$\partial : \widehat{L}^0(A) \to L_{-1}(A);\ \ M \mapsto \left(H_-(A^r); A^r, \operatorname{im}\left(\begin{pmatrix} 1 - XM \\ M \end{pmatrix} : A^r \to A^r \oplus (A^r)^*\right)\right),$$

$$\partial : \widehat{L}^1(A) \to L_0(A);\ \ N \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^tXN) & 1 - 2NX \\ 0 & -2X \end{pmatrix}\right),$$

$$\partial : \widehat{L}^3(A) \to L_2(A);\ \ M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix}\right).$$

In §1,2 we recall and extend the Q-groups and algebraic chain bundles of Ranicki [15,18] and Weiss [21]. Theorem C is proved in Theorem 60.

We shall be dealing with two types of generalized Arf invariant: for forms on f.g. projective modules, and for linking forms on homological dimension 1 torsion modules, which we shall be considering separately.

In §3 we define the *generalized Arf invariant* of a nonsingular $(-1)^k$-quadratic form (K,ψ) over an arbitrary ring with involution A with a lagrangian $L \subset K$ for $(K, \psi + (-)^k\psi^*)$ to be an element

$$(K,\psi;L) \in \widehat{L}^{4*+2k+1}(A),$$

with image

$$\begin{aligned}(K,\psi) \in \; & \mathrm{im}(\partial : \widehat{L}^{4*+2k+1}(A) \to L_{2k}(A)) \\ & = \ker(1+T : L_{2k}(A) \to L^{4*+2k}(A)).\end{aligned}$$

Theorem 70 gives an explicit formula for the generalized Arf invariant $(K,\psi;L) \in \widehat{L}^3(A)$ for an r-even A. Generalizations of the Arf invariants in L-theory have been previously studied by Clauwens [7], Bak [2] and Wolters [22].

In §4 we consider a ring with involution A with a localization $S^{-1}A$ inverting a multiplicative subset $S \subset A$ of central nonzero divisors such that $\widehat{H}^*(\mathbb{Z}_2; S^{-1}A) = 0$ (e.g. if $2 \in S$). The relative L-group $L_{2k}(A,S)$ in the localization exact sequence

$$\cdots \to L_{2k}(A) \to L_{2k}(S^{-1}A) \to L_{2k}(A,S) \to L_{2k-1}(A) \to L_{2k-1}(S^{-1}A) \to \cdots$$

is the Witt group of nonsingular $(-1)^k$-quadratic linking forms (T,λ,μ) over (A,S), with T a homological dimension 1 S-torsion A-module, λ an A-module isomorphism

$$\lambda = (-1)^k\lambda\widehat{\ } : T \to T\widehat{\ } = \mathrm{Ext}^1_A(T,A) = \mathrm{Hom}_A(T, S^{-1}A/A)$$

and

$$\mu : T \to Q_{(-1)^k}(A,S) = \frac{\{b \in S^{-1}A \,|\, \overline{b} = (-1)^k b\}}{\{a + (-1)^k\overline{a} \,|\, a \in A\}}$$

a $(-1)^k$-quadratic function for λ. The *linking Arf invariant* of a nonsingular $(-1)^k$-quadratic linking form (T,λ,μ) over (A,S) with a lagrangian $U \subset T$ for (T,λ) is defined to be an element

$$(T,\lambda,\mu;U) \in \widehat{L}^{4*+2k}(A),$$

with properties analogous to the generalized Arf invariant defined for forms in §3. Theorem 80 gives an explicit formula for the linking Arf invariant $(T,\lambda,\mu;U) \in \widehat{L}^{2k}(A)$

for an r-even A, using

$$S = (2)^{\infty} = \{2^i \mid i \geqslant 0\} \subset A, \quad S^{-1}A = A[1/2].$$

In §5 we apply the generalized and linking Arf invariants to the algebraic L-groups of a polynomial extension $A[x]$ ($\overline{x} = x$) of a ring with involution A, using the exact sequence

$$\cdots \longrightarrow L_n(A[x]) \xrightarrow{1+T} L^n(A[x]) \longrightarrow \widehat{L}^n(A[x]) \longrightarrow L_{n-1}(A[x]) \longrightarrow \cdots .$$

For a Dedekind ring A the quadratic L-groups of $A[x]$ are related to the UNil-groups $\mathrm{UNil}_*(A)$ of Cappell [4] by the splitting formula of Connolly and Ranicki [10]

$$L_n(A[x]) = L_n(A) \oplus \mathrm{UNil}_n(A)$$

and the symmetric and hyperquadratic L-groups of $A[x]$ are 4-periodic, and such that

$$L^n(A[x]) = L^n(A), \;\; \widehat{L}^{n+1}(A[x]) = \widehat{L}^{n+1}(A) \oplus \mathrm{UNil}_n(A).$$

Any computation of $\widehat{L}^*(A)$ and $\widehat{L}^*(A[x])$ thus gives a computation of $\mathrm{UNil}_*(A)$. Combining the splitting formula with Theorems A, B gives:

Theorem D. *If A is a* 1-*even Dedekind ring with $\psi^2 = 1$ then*

$$\begin{aligned}\mathrm{UNil}_n(A) &= \widehat{L}^{n+1}(A[x])/\widehat{L}^{n+1}(A)\\ &= \begin{cases} 0 & \text{if } n \equiv 0, 1 (\mathrm{mod}\, 4),\\ xA_2[x] & \text{if } n \equiv 2 (\mathrm{mod}\, 4),\\ A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 3 (\mathrm{mod}\, 4).\end{cases}\end{aligned}$$

In particular, Theorem D applies to $A = \mathbb{Z}$. The twisted quadratic Q-groups were first used in the partial computation of

$$\mathrm{UNil}_n(\mathbb{Z}) = \widehat{L}^{n+1}(\mathbb{Z}[x])/\widehat{L}^{n+1}(\mathbb{Z})$$

by Connolly and Ranicki [10]. The calculation in [10] was almost complete, except that $\mathrm{UNil}_3(\mathbb{Z})$ was only obtained up to extensions. The calculation was first completed by Connolly and Davis [8], using linking forms. We are grateful to them for sending us a preliminary version of their paper. The calculation of $\mathrm{UNil}_3(\mathbb{Z})$ in [8] uses the results of [10] and the classifications of quadratic and symmetric linking forms over $(\mathbb{Z}[x], (2)^{\infty})$. The calculation of $\mathrm{UNil}_3(\mathbb{Z})$ here uses the linking Arf invariant measuring the difference between the Witt groups of quadratic and symmetric linking forms over $(\mathbb{Z}[x], (2)^\infty)$, developing further the Q-group strategy of [10].

The algebraic L-groups of $A = \mathbb{Z}_2$ are given by

$$L^n(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \text{ (rank } (\text{mod } 2)) & \text{if } n \equiv 0 (\text{mod } 2), \\ 0 & \text{if } n \equiv 1 (\text{mod } 2), \end{cases}$$

$$L_n(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 0 (\text{mod } 2), \\ 0 & \text{if } n \equiv 1 (\text{mod } 2), \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $1 + T = 0 : L_n(\mathbb{Z}_2) \to L^n(\mathbb{Z}_2)$. The classical Arf invariant is defined for a nonsingular quadratic form (K, ψ) over $\mathbb{Z}_2$ and a lagrangian $L \subset K$ for the symmetric form $(K, \psi + \psi^*)$ to be

$$(K, \psi; L) = \sum_{i=1}^{\ell} \psi(e_i, e_i).\psi(e_i^*, e_i^*) \in \widehat{L}^1(\mathbb{Z}_2) = L_0(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $e_1, e_2, \ldots, e_\ell$ any basis for $L \subset K$, and $e_1^*, e_2^*, \ldots, e_\ell^*$ a basis for a direct summand $L^* \subset K$ such that

$$(\psi + \psi^*)(e_i^*, e_j^*) = 0, \;\; (\psi + \psi^*)(e_i^*, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The Arf invariant is independent of the choices of L and L^*.

The algebraic L-groups of $A = \mathbb{Z}$ are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0 (\text{mod } 4), \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1 (\text{mod } 4), \\ 0 & \text{otherwise}, \end{cases}$$

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } n \equiv 0 (\text{mod } 4), \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 2 (\text{mod } 4), \\ 0 & \text{otherwise}, \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}) = \begin{cases} \mathbb{Z}_8 \text{ (signature } (\text{mod } 8)) & \text{if } n \equiv 0 (\text{mod } 4), \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1 (\text{mod } 4), \\ 0 & \text{if } n \equiv 2 (\text{mod } 4), \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

Given a nonsingular symmetric form (K, ϕ) over $\mathbb{Z}$ there is a congruence [19,12, Theorem 3.10]

$$\text{signature}(K, \phi) \equiv \phi(v, v) \ (\text{mod}\, 8),$$

with $v \in K$ any element such that

$$\phi(u, v) \equiv \phi(u, u) \ (\text{mod}\, 2) \quad (u \in K),$$

so that

$$\begin{aligned}(K, \phi) &= \text{signature}(K, \phi) = \phi(v, v) \\ &\in \text{coker}(1 + T : L_0(\mathbb{Z}) \to L^0(\mathbb{Z})) = \widehat{L}^0(\mathbb{Z}) = \text{coker}(8 : \mathbb{Z} \to \mathbb{Z}) \\ &= \mathbb{Z}_8.\end{aligned}$$

The projection $\mathbb{Z} \to \mathbb{Z}_2$ induces an isomorphism $L_2(\mathbb{Z}) \cong L_2(\mathbb{Z}_2)$, so that the Witt class of a nonsingular (-1)-quadratic form (K, ψ) over $\mathbb{Z}$ is given by the Arf invariant of the mod 2 reduction

$$(K, \psi; L) = \mathbb{Z}_2 \otimes_{\mathbb{Z}} (K, \psi; L) \in L_2(\mathbb{Z}) = L_2(\mathbb{Z}_2) = \mathbb{Z}_2,$$

with $L \subset K$ a lagrangian for the (-1)-symmetric form $(K, \psi - \psi^*)$. Again, this is independent of the choice of L.

The Q-groups are defined for an A-module chain complex C to be $\mathbb{Z}_2$-hyperhomology invariants of the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $C \otimes_A C$. The involution on A is used to define the tensor product over A of left A-module chain complexes C, D, the abelian group chain complex

$$C \otimes_A D = \frac{C \otimes_{\mathbb{Z}} D}{\{ax \otimes y - x \otimes \overline{a} y \,|\, a \in A, x \in C, y \in D\}}.$$

Let $C \otimes_A C$ denote the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex defined by $C \otimes_A C$ via the transposition involution

$$T : C_p \otimes_A C_q \to C_q \otimes_A C_p; \ \ x \otimes y \mapsto (-1)^{pq} y \otimes x.$$

The $\left\{\begin{array}{l}\textit{symmetric}\\ \textit{quadratic}\\ \textit{hyperquadratic}\end{array}\right.$ *Q-groups* of C are defined by

$$\left\{\begin{array}{l} Q^n(C) = H^n(\mathbb{Z}_2; C \otimes_A C), \\ Q_n(C) = H_n(\mathbb{Z}_2; C \otimes_A C), \\ \widehat{Q}^n(C) = \widehat{H}^n(\mathbb{Z}_2; C \otimes_A C). \end{array}\right.$$

The Q-groups are covariant in C, and are chain homotopy invariant. The Q-groups are related by an exact sequence

$$\cdots \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \longrightarrow \cdots .$$

A *chain bundle* (C, γ) over A is an A-module chain complex C together with an element $\gamma \in \widehat{Q}^0(C^{-*})$. The *twisted quadratic Q-groups* $Q_*(C, \gamma)$ were defined in [21] using simplicial abelian groups, to fit into an exact sequence

$$\cdots \longrightarrow Q_n(C, \gamma) \xrightarrow{N_\gamma} Q^n(C) \xrightarrow{J_\gamma} \widehat{Q}^n(C) \xrightarrow{H_\gamma} Q_{n-1}(C, \gamma) \longrightarrow \cdots ,$$

with

$$J_\gamma : Q^n(C) \to \widehat{Q}^n(C); \;\; \phi \mapsto J(\phi) - (\widehat{\phi_0})^{\%}(\gamma).$$

An *n-dimensional algebraic normal complex* $(C, \phi, \gamma, \theta)$ over A is an n-dimensional symmetric complex (C, ϕ) together with a chain bundle $\gamma \in \widehat{Q}^0(C^{-*})$ and an element $(\phi, \theta) \in Q_n(C, \gamma)$ with image $\phi \in Q^n(C)$. Every n-dimensional symmetric Poincaré complex (C, ϕ) has the structure of an algebraic normal complex $(C, \phi, \gamma, \theta)$: the *Spivak normal chain bundle* (C, γ) is characterized by

$$(\widehat{\phi_0})^{\%}(\gamma) = J(\phi) \in Q^n(C),$$

with

$$(\widehat{\phi_0})^{\%} : \widehat{Q}^0(C^{-*}) = \widehat{Q}^n(C^{n-*}) \to \widehat{Q}^n(C),$$

the isomorphism induced by the Poincaré duality chain equivalence $\phi_0 : C^{n-*} \to C$, and the *algebraic normal invariant* $(\phi, \theta) \in Q_n(C, \gamma)$ is such that

$$N_\gamma(\phi, \theta) = \phi \in Q^n(C).$$

See [18, §7] for the one–one correspondence between the homotopy equivalence classes of n-dimensional (symmetric, quadratic) Poincaré pairs and n-dimensional algebraic normal complexes. Specifically, an n-dimensional algebraic normal complex $(C, \phi, \gamma, \theta)$ determines an n-dimensional (symmetric, quadratic) Poincaré pair $(\partial C \to C^{n-*}, (\delta\phi, \psi))$ with

$$\partial C = \mathcal{C}(\phi_0 : C^{n-*} \to C)_{*+1}.$$

Conversely, an n-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ determines an n-dimensional algebraic normal complex $(\mathcal{C}(f), \phi, \gamma, \theta)$, with $\gamma \in \widehat{Q}^0\ (\mathcal{C}(f)^{-*})$ the Spivak normal chain bundle and $\phi = \delta\phi/(1+T)\psi$; the class $(\phi, \theta) \in Q_n(\mathcal{C}(f), \gamma)$ is the algebraic normal invariant of $(f : C \to D, (\delta\phi, \psi))$. Thus $\widehat{L}^n(A)$ is the cobordism group of n-dimensional normal complexes over A.

Weiss [21] established that for any ring with involution A there exists a universal chain bundle (B^A, β^A) over A, such that every chain bundle (C, γ) is classified by a chain bundle map

$$(g, \chi) : (C, \gamma) \to (B^A, \beta^A),$$

with

$$H_*(B^A) = \widehat{H}^*(\mathbb{Z}_2; A).$$

The function

$$\widehat{L}^{n+4*}(A) \to Q_n(B^A, \beta^A);\ \ (C, \phi, \gamma, \theta) \mapsto (g, \chi)_\%(\phi, \theta)$$

was shown in [21] to be an isomorphism. Since the Q-groups are homological in nature (rather than of the Witt type) they are in principle effectively computable. The algebraic normal invariant defines the isomorphism

$$\begin{array}{c} \ker(1+T : L_n(A) \to L^{n+4*}(A)) \xrightarrow{\ \cong\ } \operatorname{coker}(L^{n+4*+1}(A) \to Q_{n+1}(B^A, \beta^A)), \\ (C, \psi) \mapsto (g, \chi)_\%(\phi, \theta), \end{array}$$

with $(\phi, \theta) \in Q_{n+1}(\mathcal{C}(f), \gamma)$ the algebraic normal invariant of any $(n+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$, with classifying chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \to (B^A, \beta^A)$. For $n = 2k$ such a pair with $H_i(C) = H_i(D) = 0$ for $i \neq k$ is just a nonsingular $(-1)^k$-quadratic form $(K = H^k(C), \psi)$ with a lagrangian

$$L = \operatorname{im}(f^* : H^k(D) \to H^k(C)) \subset K = H^k(C)$$

for $(K, \psi + (-1)^k\psi^*)$, such that the generalized Arf invariant is the image of the algebraic normal invariant

$$(K, \psi; L) = (g, \chi)_\%(\phi, \theta) \in \widehat{L}^{4*+2k+1}(A) = Q_{2k+1}(B^A, \beta^A).$$

For $A=\mathbb{Z}_2$ and $n=0$ this is just the classical Arf invariant isomorphism

$$\begin{array}{rcl} L_0(\mathbb{Z}_2) & = & \ker(1+T=0 : L_0(\mathbb{Z}_2)\to L^0(\mathbb{Z}_2)) \\ & \xrightarrow{\cong} & \operatorname{coker}(L^1(\mathbb{Z}_2)=0\to Q_1(B^{\mathbb{Z}_2},\beta^{\mathbb{Z}_2}))=\mathbb{Z}_2, \\ & & (K,\psi)\mapsto (K,\psi;L), \end{array}$$

with $L\subset K$ an arbitrary lagrangian of $(K,\psi+\psi^*)$. The isomorphism

$$\operatorname{coker}(1+T : L_n(A)\to L^{n+4*}(A)) \xrightarrow{\cong} \ker(\partial : Q_n(B^A,\beta^A)\to L_{n-1}(A))$$

is a generalization from $A=\mathbb{Z}$, $n=0$ to arbitrary A, n of the identity signature$(K,\phi)\equiv \phi(v,v) \pmod 8$ described above.

(Here is some of the geometric background. Chain bundles are algebraic analogues of vector bundles and spherical fibrations, and the twisted Q-groups are the analogues of the homotopy groups of the Thom spaces. A $(k-1)$-spherical fibration $v : X\to BG(k)$ over a connected CW complex X determines a chain bundle $(C(\widetilde{X}),\gamma)$ over $\mathbb{Z}[\pi_1(X)]$, with $C(\widetilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex of the universal cover $\widetilde{X}$, and there are defined Hurewicz-type morphisms

$$\pi_{n+k}(T(v))\to Q_n(C(\widetilde{X}),\gamma),$$

with $T(v)$ the Thom space. An n-dimensional normal space $(X, v : X\to BG(k), \rho : S^{n+k}\to T(v))$ [14] determines an n-dimensional algebraic normal complex $(C(\widetilde{X}),\phi,\gamma,\theta)$ over $\mathbb{Z}[\pi_1(X)]$. An n-dimensional geometric Poincaré complex X has a Spivak normal structure (v,ρ) such that the composite of the Hurewicz map and the Thom isomorphism

$$\pi_{n+k}(T(v))\to \widetilde{H}_{n+k}(T(v))\cong H_n(X)$$

sends ρ to the fundamental class $[X]\in H_n(X)$, and there is defined an n-dimensional symmetric Poincaré complex $(C(\widetilde{X}),\phi)$ over $\mathbb{Z}[\pi_1(X)]$, with

$$\phi_0=[X]\cap - : C(\widetilde{X})^{n-*}\to C(\widetilde{X}).$$

The symmetric signature of X is the symmetric Poincaré cobordism class

$$\sigma^*(X)=(C(\widetilde{X}),\phi)\in L^n(\mathbb{Z}[\pi_1(X)]),$$

which is both a homotopy and a $K(\pi_1(X),1)$-bordism invariant. The algebraic normal invariant of a normal space (X,ν,ρ),

$$[\rho] = (\phi,\theta) \in Q_n(C(\widetilde{X}),\gamma)$$

is a homotopy invariant. The classifying chain bundle map

$$(g,\chi) : (C(\widetilde{X}),\gamma) \to (B^{\mathbb{Z}[\pi_1(X)]},\beta^{\mathbb{Z}[\pi_1(X)]})$$

sends $[\rho]$ to the hyperquadratic signature of X:

$$\widehat{\sigma}^*(X) = [\phi,\theta] \in Q_n(B^{\mathbb{Z}[\pi_1(X)]},\beta^{\mathbb{Z}[\pi_1(X)]}) = \widehat{L}^{n+4*}(\mathbb{Z}[\pi_1(X)]),$$

which is both a homotopy and a $K(\pi_1(X),1)$-bordism invariant. The (simply-connected) symmetric signature of a $4k$-dimensional geometric Poincaré complex X is just the signature

$$\sigma^*(X) = \text{signature}(X) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$$

and the hyperquadratic signature is the mod 8 reduction of the signature

$$\widehat{\sigma}^*(X) = \text{signature}(X) \in \widehat{L}^{4k}(\mathbb{Z}) = \mathbb{Z}_8.$$

See [18] for a more extended discussion of the connections between chain bundles and their geometric models.)

1. The Q- and L-groups

1.1. Duality

Let $T \in \mathbb{Z}_2$ be the generator. The *Tate $\mathbb{Z}_2$-cohomology* groups of a $\mathbb{Z}[\mathbb{Z}_2]$-module M are given by

$$\widehat{H}^n(\mathbb{Z}_2;M) = \frac{\{x \in M \mid T(x) = (-1)^n x\}}{\{y + (-1)^n T(y) \mid y \in M\}}$$

and the $\begin{cases} \mathbb{Z}_2\text{-}cohomology \\ \mathbb{Z}_2\text{-}homology \end{cases}$ groups are given by

$$H^n(\mathbb{Z}_2; M) = \begin{cases} \{x \in M \mid T(x) = x\} & \text{if } n = 0, \\ \widehat{H}^n(\mathbb{Z}_2; M) & \text{if } n > 0, \\ 0 & \text{if } n < 0, \end{cases}$$

$$H_n(\mathbb{Z}_2; M) = \begin{cases} M/\{y - T(y) \mid y \in M\} & \text{if } n = 0, \\ \widehat{H}^{n+1}(\mathbb{Z}_2; M) & \text{if } n > 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We recall some standard properties of $\mathbb{Z}_2$-(co)homology:

Proposition 1. *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module.*

(i) *There is defined an exact sequence*

$$\cdots \to H_n(\mathbb{Z}_2; M) \xrightarrow{N} H^{-n}(\mathbb{Z}_2; M) \to \widehat{H}^n(\mathbb{Z}_2; M) \to H_{n-1}(\mathbb{Z}_2; M) \to \cdots ,$$

with

$$N = 1 + T : H_0(\mathbb{Z}_2; M) \to H^0(\mathbb{Z}_2; M); \ \ x \mapsto x + T(x).$$

(ii) *The Tate $\mathbb{Z}_2$-cohomology groups are 2-periodic and of exponent 2,*

$$\widehat{H}^*(\mathbb{Z}_2; M) = \widehat{H}^{*+2}(\mathbb{Z}_2; M), \ \ 2\widehat{H}^*(\mathbb{Z}_2; M) = 0.$$

(iii) $\widehat{H}^*(\mathbb{Z}_2; M) = 0$ if *M is a free $\mathbb{Z}[\mathbb{Z}_2]$-module.*

Let A be an associative ring with 1, and with an involution

$$\bar{\ } : A \to A; \ \ a \mapsto \bar{a},$$

such that

$$\overline{a+b} = \bar{a} + \bar{b}, \ \ \overline{ab} = \bar{b}.\bar{a}, \ \ \bar{1} = 1, \quad \bar{\bar{a}} = a.$$

When a ring A is declared to be commutative it is given the identity involution.

Definition 2. For a ring with involution A and $\varepsilon = \pm 1$ let (A, ε) denote the $\mathbb{Z}[\mathbb{Z}_2]$-module given by A with $T \in \mathbb{Z}_2$ acting by

$$T_\varepsilon : A \to A;\ \ a \mapsto \varepsilon \overline{a}.$$

For $\varepsilon = 1$ we shall write

$$\begin{aligned}&\widehat{H}^*(\mathbb{Z}_2; A, 1) = \widehat{H}^*(\mathbb{Z}_2; A),\\ &H^*(\mathbb{Z}_2; A, 1) = H^*(\mathbb{Z}_2; A),\ \ H_*(\mathbb{Z}_2; A, 1) = H_*(\mathbb{Z}_2; A).\end{aligned}$$

The *dual* of a f.g. projective (left) A-module P is the f.g. projective A-module

$$P^* = \mathrm{Hom}_A(P, A),\ \ A \times P^* \to P^*;\ \ (a, f) \mapsto (x \mapsto f(x)\overline{a}).$$

The natural A-module isomorphism

$$P \to P^{**};\ \ x \mapsto (f \mapsto \overline{f(x)})$$

is used to identify

$$P^{**} = P.$$

For any f.g. projective A-modules P, Q there is defined an isomorphism

$$P \otimes_A Q \to \mathrm{Hom}_A(P^*, Q);\ \ x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

regarding Q as a right A-module by

$$Q \times A \to Q;\ \ (y, a) \mapsto \overline{a}y.$$

There is also a duality isomorphism

$$T : \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(Q^*, P^*);\ \ f \mapsto f^*,$$

with

$$f^* : Q^* \to P^*;\ \ g \mapsto (x \mapsto g(f(x))).$$

Definition 3. For any f.g. projective A-module P and $\varepsilon = \pm 1$ let $(S(P), T_\varepsilon)$ denote the $\mathbb{Z}[\mathbb{Z}_2]$-module given by the abelian group

$$S(P) = \mathrm{Hom}_A(P, P^*),$$

with $\mathbb{Z}_2$-action by the ε-duality involution

$$T_\varepsilon : S(P) \to S(P);\ \ \phi \mapsto \varepsilon\phi^*.$$

Furthermore, let

$$\begin{aligned}\mathrm{Sym}(P,\varepsilon) &= H^0(\mathbb{Z}_2; S(P), T_\varepsilon) = \{\phi \in S(P) \,|\, T_\varepsilon(\phi) = \phi\},\\ \mathrm{Quad}(P,\varepsilon) &= H_0(\mathbb{Z}_2; S(P), T_\varepsilon) = \frac{S(P)}{\{\theta \in S(P) \,|\, \theta - T_\varepsilon(\theta)\}}.\end{aligned}$$

An element $\phi \in S(P)$ can be regarded as a sesquilinear form

$$\phi : P \times P \to A;\ \ (x, y) \mapsto \langle x, y\rangle_\phi = \phi(x)(y)$$

such that

$$\langle ax, by\rangle_\phi = b\langle x, y\rangle_\phi \overline{a} \in A \quad (x, y \in P, a, b \in A),$$

with

$$\langle x, y\rangle_{T_\varepsilon(\phi)} = \varepsilon\overline{\langle y, x\rangle}_\phi \in A.$$

An A-module morphism $f : P \to Q$ induces contravariantly a $\mathbb{Z}[\mathbb{Z}_2]$-module morphism

$$S(f) : (S(Q), T_\varepsilon) \to (S(P), T_\varepsilon);\ \ \theta \mapsto f^*\theta f.$$

For a f.g. free A-module $P = A^r$ we shall use the A-module isomorphism

$$A^r \to (A^r)^*;\ \ (a_1, a_2, \ldots, a_r) \mapsto \left((b_1, b_2, \ldots, b_r) \mapsto \sum_{i=1}^{r} b_i\overline{a}_i\right)$$

to identify

$$(A^r)^* = A^r,\ \ \mathrm{Hom}_A(A^r, (A^r)^*) = M_r(A),$$

noting that the duality involution T corresponds to the conjugate transposition of a matrix. We can thus identify

$$\begin{aligned}
&M_r(A) = S(A^r) = \text{additive group of } r \times r \text{ matrices } (a_{ij}) \text{ with } a_{ij} \in A,\\
&T : M_r(A) \to M_r(A);\;\; M = (a_{ij}) \mapsto M^t = (\overline{a}_{ji}),\\
&\mathrm{Sym}_r(A, \varepsilon) = \mathrm{Sym}(A^r, \varepsilon) = \{(a_{ij}) \in M_r(A) \,|\, a_{ij} = \varepsilon\overline{a}_{ji}\},\\
&\mathrm{Quad}_r(A, \varepsilon) = \mathrm{Quad}(A^r, \varepsilon) = \frac{M_r(A)}{\{(a_{ij} - \varepsilon\overline{a}_{ji}) \,|\, (a_{ij}) \in M_r(A)\}},\\
&1 + T_\varepsilon : \mathrm{Quad}_r(A, \varepsilon) \to \mathrm{Sym}_r(A, \varepsilon);\;\; M \mapsto M + \varepsilon M^t.
\end{aligned}$$

The homology of the chain complex

$$\cdots \longrightarrow M_r(A) \xrightarrow{1-T} M_r(A) \xrightarrow{1+T} M_r(A) \xrightarrow{1-T} M_r(A) \longrightarrow \cdots$$

is given by

$$\frac{\ker(1 - (-1)^n T : M_r(A) \to M_r(A))}{\mathrm{im}(1 + (-1)^n T : M_r(A) \to M_r(A))} = \widehat{H}^n(\mathbb{Z}_2; M_r(A)) = \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A).$$

The $(-1)^n$-symmetrization map $1 + (-1)^n T : \mathrm{Sym}_r(A) \to \mathrm{Quad}_r(A)$ fits into an exact sequence

$$\begin{aligned}
0 \to \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2; A) \quad &\to \quad \mathrm{Quad}_r(A, (-1)^n)\\
&\xrightarrow{1+(-1)^n T} \quad \mathrm{Sym}_r(A, (-1)^n) \to \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A) \to 0.
\end{aligned}$$

For $\varepsilon = 1$ we abbreviate

$$\begin{aligned}
&\mathrm{Sym}(P, 1) = \mathrm{Sym}(P), \quad \mathrm{Quad}(P, 1) = \mathrm{Quad}(P),\\
&\mathrm{Sym}_r(A, 1) = \mathrm{Sym}_r(A), \quad \mathrm{Quad}_r(A, 1) = \mathrm{Quad}_r(A).
\end{aligned}$$

Definition 4. An involution on a ring A is *even* if

$$\widehat{H}^1(\mathbb{Z}_2; A) = 0,$$

that is if

$$\{a \in A \,|\, a + \overline{a} = 0\} = \{b - \overline{b} \,|\, b \in A\}.$$

Proposition 5. (i) *For any f.g. projective A-module P there is defined an exact sequence*

$$0 \to \widehat{H}^1(\mathbb{Z}_2; S(P), T) \to \mathrm{Quad}(P) \xrightarrow{1+T} \mathrm{Sym}(P),$$

with

$$1 + T : \mathrm{Quad}(P) \to \mathrm{Sym}(P);\ \ \psi \mapsto \psi + \psi^*.$$

(ii) *If the involution on A is even the symmetrization $1 + T : \mathrm{Quad}(P) \to \mathrm{Sym}(P)$ is injective, and*

$$\widehat{H}^n(\mathbb{Z}_2; S(P), T) = \begin{cases} \dfrac{\mathrm{Sym}(P)}{\mathrm{Quad}(P)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

identifying $\mathrm{Quad}(P)$ *with* $\mathrm{im}(1 + T) \subseteq \mathrm{Sym}(P)$.

Proof. (i) This is a special case of 1(i).

(ii) If Q is a f.g. projective A-module such that $P \oplus Q = A^r$ is f.g. free then

$$\begin{aligned} \widehat{H}^1(\mathbb{Z}_2; S(P), T) \oplus \widehat{H}^1(\mathbb{Z}_2; S(Q), T) &= \widehat{H}^1(\mathbb{Z}_2; S(P \oplus Q), T) \\ &= \bigoplus_r \widehat{H}^1(\mathbb{Z}_2; A, -T) = 0 \end{aligned}$$

and so $\widehat{H}^1(\mathbb{Z}_2; S(P), T) = 0$. $\square$

In particular, if the involution on A is even there is defined an exact sequence

$$0 \to \mathrm{Quad}_r(A) \xrightarrow{1+T} \mathrm{Sym}_r(A) \to \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \to 0$$

with

$$\mathrm{Sym}_r(A) \to \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A);\ \ (a_{ij}) \mapsto (a_{ii}).$$

For any involution on A, $\mathrm{Sym}_r(A)$ is the additive group of symmetric $r \times r$ matrices $(a_{ij}) = (\overline{a}_{ji})$ with $a_{ij} \in A$. For an even involution $\mathrm{Quad}_r(A) \subseteq \mathrm{Sym}_r(A)$ is the subgroup of the matrices such that the diagonal terms are of the form $a_{ii} = b_i + \overline{b}_i$

for some $b_i \in A$, with

$$\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A).$$

Definition 6. A ring A is *even* if $2 \in A$ is a nonzero divisor, i.e. $2 : A \to A$ is injective.

Example 7. (i) An integral domain A is even if and only if it has characteristic $\neq 2$.

(ii) The identity involution on a commutative ring A is even (4) if and only if the ring A is even (6), in which case

$$\widehat{H}^n(\mathbb{Z}_2; A) = \begin{cases} A_2 & \text{if } n \equiv 0 (\mathrm{mod}\, 2), \\ 0 & \text{if } n \equiv 1 (\mathrm{mod}\, 2) \end{cases}$$

and

$$\mathrm{Quad}_r(A) = \{(a_{ij}) \in \mathrm{Sym}_r(A) \mid a_{ii} \in 2A\}.$$

Example 8. For any group π there is defined an involution on the group ring $\mathbb{Z}[\pi]$:

$$- : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]; \ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1}.$$

If π has no 2-torsion this involution is even.

1.2. The hyperquadratic Q-groups

Let C be a finite (left) f.g. projective A-module chain complex. The dual of the f.g. projective A-module C_p is written

$$C^p = (C_p)^* = \mathrm{Hom}_A(C_p, A).$$

The dual A-module chain complex C^{-*} is defined by

$$d_{C^{-*}} = (d_C)^* : (C^{-*})_r = C^{-r} \to (C^{-*})_{r-1} = C^{-r+1}.$$

The n-dual A-module chain complex C^{n-*} is defined by

$$d_{C^{n-*}} = (-1)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1}.$$

Identify

$$C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C),$$

noting that a cycle $\phi \in (C \otimes_A C)_n$ is a chain map $\phi : C^{n-*} \to C$. For $\varepsilon = \pm 1$ the ε-transposition involution T_ε on $C \otimes_A C$ corresponds to the ε-duality involution on $\mathrm{Hom}_A(C^{-*}, C)$,

$$T_\varepsilon : \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p);\ \ \phi \mapsto (-1)^{pq}\varepsilon\phi^*.$$

Let $\widehat{W}$ be the complete resolution of the $\mathbb{Z}[\mathbb{Z}_2]$-module $\mathbb{Z}$:

$$\widehat{W} : \cdots \longrightarrow \widehat{W}_1 = \mathbb{Z}[\mathbb{Z}_2] \stackrel{1-T}{\longrightarrow} \widehat{W}_0 = \mathbb{Z}[\mathbb{Z}_2] \stackrel{1+T}{\longrightarrow} \widehat{W}_{-1} = \mathbb{Z}[\mathbb{Z}_2] \stackrel{1-T}{\longrightarrow} \widehat{W}_{-2} = \mathbb{Z}[\mathbb{Z}_2] \longrightarrow \cdots.$$

If we set

$$\widehat{W}^{\%}C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, \mathrm{Hom}_A(C^{-*}, C)),$$

then an *n-dimensional ε-hyperquadratic structure on C* is a cycle $\theta \in (\widehat{W}^{\%}C)_n$, which is just a collection $\{\theta_s \in \mathrm{Hom}_A(C^{n-r+s}, C_r)\,|\, r, s \in \mathbb{Z}\}$ such that

$$d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s-1}(\theta_{s-1} + (-1)^s T_\varepsilon \theta_{s-1}) = 0 : C^{n-r+s-1} \to C_r.$$

Definition 9. The *n-dimensional ε-hyperquadratic Q-group* $\widehat{Q}^n(C, \varepsilon)$ is the abelian group of equivalence classes of n-dimensional ε-hyperquadratic structures on C, that is,

$$\widehat{Q}^n(C, \varepsilon) = H_n(\widehat{W}^{\%}C).$$

The ε-hyperquadratic Q-groups are 2-periodic and of exponent 2

$$\widehat{Q}^*(C, \varepsilon) \cong \widehat{Q}^{*+2}(C, \varepsilon),\ \ 2\widehat{Q}^*(C, \varepsilon) = 0.$$

More precisely, there are defined isomorphisms

$$\widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C, \varepsilon);\ \ \{\theta_s\} \mapsto \{\theta_{s+2}\}$$

and for any n-dimensional ε-hyperquadratic structure $\{\theta_s\}$,

$$2\theta_s = d\chi_s + (-1)^r \chi_s d^* + (-1)^{n+s}(\chi_{s-1} + (-1)^s T_\varepsilon \chi_{s-1}) : C^{n-r+s} \to C_r,$$

with $\chi_s = (-1)^{n+s-1}\theta_{s+1}$. There are also defined suspension isomorphisms

$$S : \widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+1}(C_{*-1}, \varepsilon);\ \{\theta_s\} \mapsto \{\theta_{s-1}\}$$

and skew-suspension isomorphisms

$$\overline{S} : \widehat{Q}^n(C, \varepsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C_{*-1}, -\varepsilon);\ \{\theta_s\} \mapsto \{\theta_s\}.$$

Proposition 10. *Let C be a f.g. projective A-module chain complex which is concentrated in degree k*

$$C : \cdots \to 0 \to C_k \to 0 \to \cdots\ .$$

The ε-hyperquadratic Q-groups of C are given by

$$\widehat{Q}^n(C, \varepsilon) = \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon)$$

(*with* $S(C^k) = \mathrm{Hom}_A(C^k, C_k)$).

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \mathrm{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \to V_{2k+1} = 0 \to V_{2k} = S(C^k) \to V_{2k-1} = 0 \to \cdots$$

and

$$(\widehat{W}^{\%}C)_j = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, V_{2k}) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, S(C^k)).$$

Thus the chain complex $\widehat{W}^{\%}C$ is of the form

$$\begin{array}{ll}
(\widehat{W}^{\%}C)_{2k+1} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{-1}, V_{2k}) = S(C^k), \\
\quad\Big\downarrow{\scriptstyle d_{2k+1}=1+(-1)^k T_\varepsilon} & \\
(\widehat{W}^{\%}C)_{2k} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_0, V_{2k}) = S(C^k), \\
\quad\Big\downarrow{\scriptstyle d_{2k}=1+(-1)^{k+1} T_\varepsilon} & \\
(\widehat{W}^{\%}C)_{2k-1} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_1, V_{2k}) = S(C^k), \\
\quad\Big\downarrow{\scriptstyle d_{2k-1}=1+(-1)^k T_\varepsilon} & \\
(\widehat{W}^{\%}C)_{2k-2} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_2, V_{2k}) = S(C^k) \\
\quad\Big\downarrow &
\end{array}$$

and

$$\widehat{Q}^n(C, \varepsilon) = H_n(\widehat{W}^{\%}C) = \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon). \qquad \square$$

Example 11. The ε-hyperquadratic Q-groups of a zero-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_0 = A^r \to 0 \to \cdots$$

are given by

$$\widehat{Q}^n(C, \varepsilon) = \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon).$$

The *algebraic mapping cone* $\mathcal{C}(f)$ of a chain map $f : C \to D$ is the chain complex defined as usual by

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-1)^{r-1} f \\ 0 & d_C \end{pmatrix} : \ \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

The relative homology groups

$$H_n(f) = H_n(\mathcal{C}(f))$$

fit into an exact sequence

$$\cdots \to H_n(C) \xrightarrow{f_*} H_n(D) \to H_n(f) \to H_{n-1}(C) \to \cdots .$$

An A-module chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$\begin{aligned} f \otimes f &= \mathrm{Hom}_A(f^*, f) : C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C) \to D \otimes_A D \\ &= \mathrm{Hom}_A(D^{-*}, D) \end{aligned}$$

and hence a $\mathbb{Z}$-module chain map

$$\widehat{f}^{\%} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_{\widehat{W}}, f \otimes_A f) : \widehat{W}^{\%}C \longrightarrow \widehat{W}^{\%}D,$$

which induces

$$\widehat{f}^{\%} : \widehat{Q}^n(C, \varepsilon) \longrightarrow \widehat{Q}^n(D, \varepsilon)$$

on homology. The *relative ε-hyperquadratic Q-group*

$$\widehat{Q}^n(f, \varepsilon) = H_n(\widehat{f}^{\%} : \widehat{W}^{\%}C \to \widehat{W}^{\%}D)$$

fits into a long exact sequence

$$\cdots \longrightarrow \widehat{Q}^n(C, \varepsilon) \xrightarrow{\widehat{f}^{\%}} \widehat{Q}^n(D, \varepsilon) \longrightarrow \widehat{Q}^n(f, \varepsilon) \longrightarrow \widehat{Q}^{n-1}(C, \varepsilon) \longrightarrow \cdots .$$

As in [15, §1] define a $\mathbb{Z}_2$*-isovariant chain map* $f : C \to D$ of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes C, D to be a collection

$$\{f_s \in \mathrm{Hom}_{\mathbb{Z}}(C_r, D_{r+s}) | r \in \mathbb{Z}, s \geqslant 0\}$$

such that

$$\begin{aligned} & d_D f_s + (-1)^{s-1} f_s d_C + (-1)^{s-1}(f_{s-1} + (-1)^s T_D f_{s-1} T_C) \\ & \quad = 0 : C_r \to D_{r+s-1} \quad (f_{-1} = 0), \end{aligned}$$

so that $f_0 : C \to D$ is a $\mathbb{Z}$-module chain map, $f_1 : f_0 \simeq T_D f_0 : C \to D$ is a $\mathbb{Z}$-module chain map, etc. There is a corresponding notion of $\mathbb{Z}_2$-isovariant chain homotopy.

For any A-module chain complexes C, D a $\mathbb{Z}_2$-isovariant chain map $F : C \otimes_A C \to D \otimes_A D$ induces morphisms of the ε-hyperquadratic Q-groups

$$\widehat{F}^{\%} : \widehat{Q}^n(C, \varepsilon) \to \widehat{Q}^n(D, \varepsilon);\ \ \theta \mapsto \widehat{F}^{\%}(\theta),\ \ \widehat{F}^{\%}(\theta)_s = \sum_{r=0}^{\infty} \pm F_r(T^r \theta_{s-r}).$$

If F_0 is a chain equivalence the morphisms $\widehat{F}^{\%}$ are isomorphisms. An A-module chain map $f : C \to D$ determines a $\mathbb{Z}_2$-isovariant chain map

$$f \otimes_A f : C \otimes_A C \to D \otimes_A D,$$

with $(f \otimes_A f)_s = 0$ for $s \geqslant 1$.

Proposition 12 (*Ranicki [15, Propositions 1.1,1.4] Weiss [21, Theorem 1.1]*). (i) *The relative ε-hyperquadratic Q-groups of an A-module chain map $f : C \to D$ are isomorphic to the absolute ε-hyperquadratic Q-groups of the algebraic mapping cone $\mathcal{C}(f)$,*

$$\widehat{Q}^*(f, \varepsilon) \cong \widehat{Q}^*(\mathcal{C}(f), \varepsilon).$$

(ii) *The ε-hyperquadratic Q-groups are additive*: *for any collection $\{C(i) \,|\, i \in \mathbb{Z}\}$ of f.g. projective A-module chain complexes $C(i)$,*

$$\widehat{Q}^n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i \widehat{Q}^n(C(i), \varepsilon).$$

(iii) *If $f : C \to D$ is a chain equivalence the morphisms $\widehat{f}^{\%} : \widehat{Q}^*(C, \varepsilon) \to \widehat{Q}^*(D, \varepsilon)$ are isomorphisms*, *and*

$$\widehat{Q}^*(f, \varepsilon) = 0.$$

Proof. (i) The $\mathbb{Z}_2$-isovariant chain map $t : \mathcal{C}(f \otimes_A f) \to \mathcal{C}(f) \otimes_A \mathcal{C}(f)$ defined by

$$t_0(\theta, \partial\theta) = \theta + (f \otimes 1)\partial\theta,\ \ t_1(\theta, \partial\theta) = \partial\theta,\ \ t_s = 0\ \ (s \geqslant 2)$$

induces the algebraic Thom construction maps

$$\widehat{t}^{\%} : \widehat{Q}^n(f, \varepsilon) \to \widehat{Q}^n(\mathcal{C}(f), \varepsilon);\ \ (\theta, \partial\theta) \mapsto \theta/\partial\theta,$$

with

$$\begin{aligned}(\theta/\partial\theta)_s &= \begin{pmatrix} \theta_s & 0 \\ \pm\,\partial\theta_s f^* & \pm T_\varepsilon \partial\theta_{s-1} \end{pmatrix} : \\ \mathcal{C}(f)^{n-r+s} &= D^{n-r+s} \oplus C^{n-r+s-1} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1} \quad (r, s \in \mathbb{Z}).\end{aligned}$$

Define a free $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$E = (C_{*-1} \otimes_A \mathcal{C}(f)) \oplus (\mathcal{C}(f) \otimes_A C_{*-1}),$$

with

$$T : E \to E;\ \ (a \otimes b, x \otimes y) \mapsto (y \otimes x, b \otimes a),$$

such that

$$H_*(\widehat{W} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} E) = H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, E)) = 0.$$

Let $p : \mathcal{C}(f) \to C_{*-1}$ be the projection. The chain map

$$\begin{pmatrix} p \otimes 1 \\ 1 \otimes p \end{pmatrix} : \mathcal{C}(f) \otimes_A \mathcal{C}(f) \to E$$

induces a chain equivalence

$$\mathcal{C}(t_0 : \mathcal{C}(f \otimes f) \to \mathcal{C}(f) \otimes_A \mathcal{C}(f)) \ \simeq \ E$$

so that the morphisms $\widehat{t}^{\%} : \widehat{Q}^*(f, \varepsilon) \cong \widehat{Q}^*(\mathcal{C}(f), \varepsilon)$ are isomorphisms.

(ii) $\widehat{Q}^*(C(1) \oplus C(2)) = \widehat{Q}^*(C(1)) \oplus \widehat{Q}^*(C(2))$ is the special case of (i) with $f = 0 : C(1)_{*+1} \to C(2)$.

(iii) An A-module chain homotopy $g : f \simeq f' : C \to D$ determines a $\mathbb{Z}_2$-isovariant chain homotopy

$$h : f \otimes_A f \simeq f' \otimes_A f' : C \otimes_A C \to D \otimes_A D,$$

with

$$h_0 = f \otimes_A g \pm g \otimes_A f,\ \ h_1 = \pm g \otimes_A g,\ \ h_s = 0\ \ (s \geqslant 2),$$

so that

$$\widehat{f}^{\%} = \widehat{f'}^{\%} : \widehat{Q}^n(C, \varepsilon) \to \widehat{Q}^n(D, \varepsilon).$$

(See the proof of [15, Proposition 1.1(ii)] for the signs.) In particular, if f is a chain equivalence the morphisms $\widehat{f}^{\%}$ are isomorphisms. $\square$

Proposition 13. *Let C be a f.g. projective A-module chain complex which is concentrated in degrees $k, k+1$,*

$$C : \cdots \to 0 \to C_{k+1} \xrightarrow{\;d\;} C_k \to 0 \to \cdots .$$

(i) *The ε-hyperquadratic Q-groups of C are the relative Tate $\mathbb{Z}_2$-cohomology groups in the exact sequence*

$$\begin{aligned}\cdots \to \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) &\xrightarrow{\widehat{d}^{\%}} \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon)\\ &\to \widehat{Q}^n(C,\varepsilon) \to \widehat{H}^{n-2k-1}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \to \cdots\end{aligned}$$

that is

$$\widehat{Q}^n(C,\varepsilon) = \frac{\{(\phi,\theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi^* = (-1)^{n+k-1}\varepsilon\phi,\ d\phi d^* = \theta + (-1)^{n+k-1}\varepsilon\theta^*\}}{\{(\sigma + (-1)^{n+k-1}\varepsilon\sigma^*, d\sigma d^* + \tau + (-1)^{n+k}\varepsilon\tau^*) \mid (\sigma,\tau) \in S(C^{k+1}) \oplus S(C^k)\}},$$

with (ϕ,θ) corresponding to the cycle $\beta \in (\widehat{W}^{\%}C)_n$ given by

$$\begin{aligned}&\beta_{2k-n+2} = \theta : C^{k+1} \to C_{k+1},\quad \beta_{2k-n} = \phi : C^k \to C_k,\\ &\beta_{2k-n+1} = \begin{cases} d\phi : C^{k+1} \to C_k,\\ 0 : C^k \to C_{k+1}.\end{cases}\end{aligned}$$

(ii) *If the involution on A is even then*

$$\widehat{Q}^n(C) = \begin{cases} \operatorname{coker}\left(\widehat{d}^{\%} : \dfrac{\operatorname{Sym}(C^{k+1})}{\operatorname{Quad}(C^{k+1})} \to \dfrac{\operatorname{Sym}(C^k)}{\operatorname{Quad}(C^k)}\right) & \text{if } n-k \text{ is even},\\[2ex] \ker\left(\widehat{d}^{\%} : \dfrac{\operatorname{Sym}(C^{k+1})}{\operatorname{Quad}(C^{k+1})} \to \dfrac{\operatorname{Sym}(C^k)}{\operatorname{Quad}(C^k)}\right) & \text{if } n-k \text{ is odd}.\end{cases}$$

Proof. (i) Immediate from Proposition 12.
(ii) Combine (i) and the vanishing $\widehat{H}^1(\mathbb{Z}_2; S(P), T) = 0$ given by Proposition 5(ii). $\square$

For $\varepsilon = 1$ we write

$$T_\varepsilon = T,\quad \widehat{Q}^n(C,\varepsilon) = \widehat{Q}^n(C),\quad \varepsilon\text{-hyperquadratic} = \text{hyperquadratic}.$$

Example 14. Let A be a ring with an involution which is even (6), i.e. such that $2 \in A$ is a nonzero divisor.

(i) The hyperquadratic Q-groups of a one-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_1 = A^q \xrightarrow{d} C_0 = A^r \to 0 \to \cdots$$

are given by

$$\widehat{Q}^n(C) = \frac{\{(\phi, \theta) \in M_q(A) \oplus M_r(A) \mid \phi^* = (-1)^{n-1}\phi, d\phi d^* = \theta + (-1)^{n-1}\theta^*\}}{\{(\sigma + (-1)^{n-1}\sigma^*, d\sigma d^* + \tau + (-1)^n\tau^* \mid (\sigma, \tau) \in M_q(A) \oplus M_r(A)\}}.$$

Example 11 and Proposition 13 give an exact sequence

$$\begin{aligned} \widehat{H}^1(\mathbb{Z}_2; S(C^1), T) \quad &= \quad 0 \to \widehat{Q}^1(C) \\ &\longrightarrow \quad \widehat{H}^0(\mathbb{Z}_2; S(C^1), T) = \bigoplus_q \widehat{H}^0(\mathbb{Z}_2; A) \\ &\xrightarrow{\widehat{d}^{\%}} \quad \widehat{H}^0(\mathbb{Z}_2; S(C^0), T) = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \\ &\longrightarrow \quad \widehat{Q}^0(C) \to \widehat{H}^{-1}(\mathbb{Z}_2; S(C^1), T) = 0. \end{aligned}$$

(ii) If A is an even commutative ring and

$$d = 2 : C_1 = A^r \to C_0 = A^r,$$

then $\widehat{d}^{\%} = 0$ and there are defined isomorphisms

$$\begin{aligned} \widehat{Q}^0(C) \quad &\xrightarrow{\cong} \quad \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} = \bigoplus_r A_2; \quad (\phi, \theta) \mapsto \theta = (\theta_{ii})_{1 \leqslant i \leqslant r}, \\ \widehat{Q}^1(C) \quad &\xrightarrow{\cong} \quad \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} = \bigoplus_r A_2; \quad (\phi, \theta) \mapsto \phi = (\phi_{ii})_{1 \leqslant i \leqslant r}. \end{aligned}$$

1.3. The symmetric Q-groups

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$:

$$W : \cdots \longrightarrow W_3 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_2 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} W_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} W_0 = \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0.$$

Given a f.g. projective A-module chain complex C we set

$$W^{\%}C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^{-*}, C)),$$

with $T \in \mathbb{Z}_2$ acting on $C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C)$ by the ε-duality involution T_ε. An *n-dimensional ε-symmetric structure on C* is a cycle $\phi \in (W^{\%}C)_n$, which is just a collection $\{\phi_s \in \mathrm{Hom}_A(C^r, C_{n-r+s}) \,|\, r \in \mathbb{Z}, s \geqslant 0\}$ such that

$$d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T_\varepsilon \phi_{s-1}) = 0 : C^r \to C_{n-r+s-1}$$
$$(r \in \mathbb{Z}, s \geqslant 0, \phi_{-1} = 0).$$

Definition 15. The *n-dimensional ε-symmetric Q-group* $Q^n(C, \varepsilon)$ is the abelian group of equivalence classes of n-dimensional ε-symmetric structures on C, that is,

$$Q^n(C, \varepsilon) = H_n(W^{\%}C).$$

Note that there are defined skew-suspension isomorphisms

$$\overline{S} : Q^n(C, \varepsilon) \xrightarrow{\cong} Q^{n+2}(C_{*-1}, -\varepsilon); \;\; \{\phi_s\} \mapsto \{\phi_s\}.$$

Proposition 16. *The ε-symmetric Q-groups of a f.g. projective A-module chain complex concentrated in degree k,*

$$C : \cdots \to 0 \to C_k \to 0 \to \cdots$$

are given by

$$\begin{aligned} Q^n(C, \varepsilon) &= H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\ &= \begin{cases} \widehat{H}^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n \leqslant 2k-1, \\ H^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n = 2k, \\ 0 & \text{if } n \geqslant 2k+1. \end{cases} \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \mathrm{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \to V_{2k+1} = 0 \to V_{2k} = S(C^k) \to V_{2k-1} = 0 \to \cdots$$

and

$$(W^{\%}C)_j = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, V_{2k}) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, S(C^k)),$$

which vanishes for $j > 2k$. Thus the chain complex $W^{\%}C$ is of the form

$$\begin{array}{ll}
(W^{\%}C)_{2k+1} & = 0, \\
\Big\downarrow {\scriptstyle d_{2k+1}} & \\
(W^{\%}C)_{2k} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_0, V_{2k}) = S(C^k), \\
\Big\downarrow {\scriptstyle d_{2k}=1+(-1)^{k+1}T_\varepsilon} & \\
(W^{\%}C)_{2k-1} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_1, V_{2k}) = S(C^k), \\
\Big\downarrow {\scriptstyle d_{2k-1}=1+(-1)^{k}T_\varepsilon} & \\
(W^{\%}C)_{2k-2} & = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_2, V_{2k}) = S(C^k) \\
\Big\downarrow &
\end{array}$$

and

$$Q^n(C, \varepsilon) = H_n(W^{\%}C) = H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon). \qquad \square$$

For $\varepsilon = 1$ we write

$$T_\varepsilon = T, \ \ Q^n(C, \varepsilon) = Q^n(C), \ \ \varepsilon\text{-symmetric} = \text{symmetric}.$$

Example 17. The symmetric Q-groups of a zero-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_0 = A^r \to 0 \to \cdots$$

are given by

$$Q^n(C) = \begin{cases} \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A) & \text{if } n < 0, \\ \mathrm{Sym}_r(A) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An A-module chain map $f : C \to D$ induces a chain map

$$\mathrm{Hom}_A(f^*, f) : \mathrm{Hom}_A(C^{-*}, C) \to \mathrm{Hom}_A(D^{-*}, D);\ \ \phi \mapsto f\phi f^*$$

and thus a chain map

$$f^{\%} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_W, \mathrm{Hom}_A(f^*, f)) : W^{\%}C \longrightarrow W^{\%}D,$$

which induces

$$f^{\%} : Q^n(C, \varepsilon) \longrightarrow Q^n(D, \varepsilon)$$

on homology. The *relative ε-symmetric Q-group*

$$Q^n(f, \varepsilon) = H_n(f^{\%} : W^{\%}C \to W^{\%}D)$$

fits into a long exact sequence

$$\cdots \longrightarrow Q^n(C, \varepsilon) \xrightarrow{f^{\%}} Q^n(D, \varepsilon) \longrightarrow Q^n(f, \varepsilon) \longrightarrow Q^{n-1}(C, \varepsilon) \longrightarrow \cdots .$$

Proposition 18. (i) *The relative ε-symmetric Q-groups of an A-module chain map $f : C \to D$ are related to the absolute ε-symmetric Q-groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \to H_n(\mathcal{C}(f) \otimes_A C) \xrightarrow{F} Q^n(f, \varepsilon) \xrightarrow{t} Q^n(\mathcal{C}(f), \varepsilon) \to H_{n-1}(\mathcal{C}(f) \otimes_A C) \to \cdots ,$$

with

$$t : Q^n(f, \varepsilon) \to Q^n(\mathcal{C}(f), \varepsilon);\ \ (\phi, \partial\phi) \mapsto \phi/\partial\phi$$

the algebraic Thom construction

$$(\phi/\partial\phi)_s = \begin{pmatrix} \phi_s & 0 \\ \pm\,\partial\phi_s f^* & \pm T_\varepsilon \partial\phi_{s-1} \end{pmatrix} :$$
$$\mathcal{C}(f)^{n-r+s} = D^{n-r+s} \oplus C^{n-r+s-1} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1} \quad (r \in \mathbb{Z}, s \geqslant 0, \phi_{-1} = 0).$$

An element $(g, h) \in H_n(\mathcal{C}(f) \otimes_A C)$ is represented by a chain map $g : C^{n-1-*} \to C$ together with a chain homotopy $h : fg \simeq 0 : C^{n-1-*} \to D$, and

$$F : H_n(\mathcal{C}(f) \otimes_A C) \to Q^n(f, \varepsilon);\ \ (g, h) \mapsto (\phi, \partial\phi),$$

with

$$\partial\phi_s = \begin{cases} (1+T_\varepsilon)g & \text{if } s=0, \\ 0 & \text{if } s \geqslant 1, \end{cases} \qquad \phi_s = \begin{cases} (1+T_\varepsilon)hf^* & \text{if } s=0, \\ 0 & \text{if } s \geqslant 1. \end{cases}$$

The map

$$Q^n(\mathcal{C}(f), \varepsilon) \to H_{n-1}(\mathcal{C}(f) \otimes_A C); \;\; \phi \mapsto p\phi_0$$

is defined using $p =$ projection : $\mathcal{C}(f) \to C_{*-1}$.

(ii) *If* $f : C \to D$ *is a chain equivalence the morphisms* $f^{\%} : Q^*(C, \varepsilon) \to Q^*(D, \varepsilon)$ *are isomorphisms*, *and*

$$Q^*(\mathcal{C}(f), \varepsilon) = Q^*(f, \varepsilon) = 0.$$

(iii) *For any collection* $\{C(i) \mid i \in \mathbb{Z}\}$ *of f.g. projective A-module chain complexes* $C(i)$

$$Q^n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i Q^n(C(i), \varepsilon) \oplus \bigoplus_{i<j} H_n(C(i) \otimes_A C(j)).$$

Proof. (i) As in Proposition 12 there is defined a chain equivalence

$$\mathcal{C}(t_0 : \mathcal{C}(f \otimes f) \to \mathcal{C}(f) \otimes_A \mathcal{C}(f)) \simeq E,$$

with

$$\begin{aligned} &E = (C_{*-1} \otimes_A \mathcal{C}(f)) \oplus (\mathcal{C}(f) \otimes_A C_{*-1}), \\ &H_*(W^{\%}E) = H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, E)) = H_{*-1}(C \otimes_A \mathcal{C}(f)). \end{aligned}$$

(ii)+(iii) See [15, Propositions 1.1,1.4]. □

Proposition 19. *Let C be a f.g. projective A-module chain complex which is concentrated in degrees* $k, k+1$:

$$C : \cdots \to 0 \to C_{k+1} \xrightarrow{\;d\;} C_k \to 0 \to \cdots .$$

The absolute ε-symmetric Q-groups $Q^*(C, \varepsilon)$ *and the relative ε-symmetric Q-groups* $Q^*(d, \varepsilon)$ *of* $d : C_{k+1} \to C_k$ *regarded as a morphism of chain complexes concentrated* *in degree k are given as follows*:

(i) *For* $n \neq 2k, 2k+1, 2k+2$:

$$Q^n(C,\varepsilon) = Q^n(d,\varepsilon) = \begin{cases} \widehat{Q}^n(d,\varepsilon) = \widehat{Q}^n(C,\varepsilon) & \textit{if } n \leqslant 2k-1, \\ 0 & \textit{if } n \geqslant 2k+3, \end{cases}$$

with $\widehat{Q}^n(C,\varepsilon)$ *as given by Proposition* 13.

(ii) *For* $n = 2k, 2k+1, 2k+2$ *there are exact sequences*

$$\begin{aligned} 0 \quad & \to Q^{2k+1}(d,\varepsilon) \longrightarrow Q^{2k}(C_{k+1},\varepsilon) = H^0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\ & \xrightarrow{d^{\%}} Q^{2k}(C_k,\varepsilon) = H^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \longrightarrow Q^{2k}(d,\varepsilon) \\ & \longrightarrow Q^{2k-1}(C_{k+1},\varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\ & \xrightarrow{d^{\%}} Q^{2k-1}(C_k,\varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon), \end{aligned}$$

$$\begin{aligned} Q^{2k+2}(d,\varepsilon) \quad & = 0 \to Q^{2k+2}(C,\varepsilon) \to C_{k+1} \otimes_A H_{k+1}(C) \xrightarrow{F} Q^{2k+1}(d,\varepsilon) \\ & \xrightarrow{t} Q^{2k+1}(C,\varepsilon) \to C_{k+1} \otimes_A H_k(C) \xrightarrow{F} Q^{2k}(d,\varepsilon) \\ & \xrightarrow{t} Q^{2k}(C,\varepsilon) \to 0. \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \operatorname{Hom}_A(C^{-*}, C)$ is such that

$$V_n = \begin{cases} S(C^k) & \text{if } n = 2k, \\ \operatorname{Hom}_A(C^k, C_{k+1}) \oplus \operatorname{Hom}_A(C^{k+1}, C_k) & \text{if } n = 2k+1, \\ S(C^{k+1}) & \text{if } n = 2k+2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(W^{\%}C)_n = \sum_{s=0}^{\infty} \operatorname{Hom}_A(W_s, V_{n+s}) = 0 \quad \text{for } n \geqslant 2k+3. \qquad \square$$

Example 20. Let C be a one-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_1 = A^q \xrightarrow{d} C_0 = A^r \to 0 \to \cdots ,$$

so that $C = \mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d : C_1 \to C_0$ of zero-dimensional complexes, with

$$d^{\%} : \mathrm{Hom}_A(C^1, C_1) = M_q(A) \to \mathrm{Hom}_A(C^0, C_0) = M_r(A);\ \ \phi \mapsto d\phi d^*.$$

Example 17 and Proposition 19 give exact sequences

$$\begin{aligned}
&Q^1(C_0) = 0 \to Q^1(d) \longrightarrow Q^0(C_1) = \mathrm{Sym}_q(A) \xrightarrow{d^{\%}} Q^0(C_0) = \mathrm{Sym}_r(A)\\
&\longrightarrow Q^0(d) \to Q^{-1}(C_1) = \bigoplus_q \widehat{H}^1(\mathbb{Z}_2; A) \xrightarrow{d^{\%}} Q^{-1}(C_0) = \bigoplus_r \widehat{H}^1(\mathbb{Z}_2; A)\\
&H_1(C) \otimes_A C_1 \xrightarrow{F} Q^1(d) \xrightarrow{t} Q^1(C) \to H_0(C) \otimes_A C_1 \xrightarrow{F} Q^0(d) \xrightarrow{t}\\
&\quad Q^0(C) \to 0.
\end{aligned}$$

In particular, if A is an even commutative ring and

$$d = 2 : C_1 = A^r \to C_0 = A^r,$$

then $d^{\%} = 4$ and

$$\begin{aligned}
Q^0(d) &= \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)},\ \ Q^1(d) = 0,\\
Q^0(C) &= \mathrm{coker}\left(2(1+T) : M_r(A) \to \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)}\right) = \frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)},\\
Q^1(C) &= \ker\left(2(1+T) : \frac{M_r(A)}{2M_r(A)} \to \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)}\right)\\
&= \frac{\{(a_{ij}) \in M_r(A) \mid a_{ij} + a_{ji} \in 2A\}}{2M_r(A)} = \frac{\mathrm{Sym}_r(A)}{2\mathrm{Sym}_r(A)}.
\end{aligned}$$

We refer to [15] for the one–one correspondence between highly-connected algebraic Poincaré complexes/pairs and forms, lagrangians and formations.

1.4. The quadratic Q-groups

Given a f.g. projective A-module chain complex C we set

$$W_{\%}C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^{-*}, C),$$

with $T \in \mathbb{Z}_2$ acting on $C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C)$ by the ε-duality involution T_ε. An *n-dimensional ε-quadratic structure on C* is a cycle $\psi \in (W_{\%}C)_n$, a collection

$\{\psi_s \in \mathrm{Hom}_A(C^r, C_{n-r-s}) \mid r \in \mathbb{Z}, s \geqslant 0\}$ such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1} T_\varepsilon \psi_{s+1}) = 0 : C^r \to C_{n-r-s-1}.$$

Definition 21. The *n-dimensional ε-quadratic Q-group* $Q_n(C, \varepsilon)$ is the abelian group of equivalence classes of n-dimensional ε-quadratic structures on C, that is,

$$Q_n(C, \varepsilon) = H_n(W_\% C).$$

Note that there are defined skew-suspension isomorphisms

$$\overline{S} : Q_n(C, \varepsilon) \xrightarrow{\cong} Q_{n+2}(C_{*-1}, -\varepsilon);\ \{\psi_s\} \mapsto \{\psi_s\}.$$

Proposition 22. *The ε-quadratic Q-groups of a f.g. projective A-module chain complex concentrated in degree k,*

$$C : \cdots \to 0 \to C_k \to 0 \to \cdots$$

are given by

$$\begin{aligned} Q_n(C, \varepsilon) &= H_{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\ &= \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \textit{if } n \geqslant 2k+1, \\ H_0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \textit{if } n = 2k, \\ 0 & \textit{if } n \leqslant 2k-1. \end{cases} \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \mathrm{Hom}_A(C^{-*}, C)$ is given by

$$V : \cdots \to V_{2k+1} = 0 \to V_{2k} = \mathrm{Hom}_A(C^k, C_k) \to V_{2k-1} = 0 \to \cdots$$

and

$$(W_\% C)_j = W_{j-2k} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, S(C^k)),$$

which vanishes for $j < 2k$. Thus the chain complex $W_{\%}C$ is of the form

$$\begin{array}{lcl}
(W_{\%}C)_{2k+2} & = & W_2 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
\Big\downarrow {\scriptstyle d_{2k+2}=1+(-1)^k T_\varepsilon} & & \\
(W_{\%}C)_{2k+1} & = & W_1 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
\Big\downarrow {\scriptstyle d_{2k+1}=1+(-1)^{k+1} T_\varepsilon} & & \\
(W_{\%}C)_{2k} & = & W_0 \otimes_{\mathbb{Z}[\mathbb{Z}_2]} V_{2k} = S(C^k), \\
\Big\downarrow & & \\
(W_{\%}C)_{2k-1} & = & 0
\end{array}$$

and

$$Q_n(C, \varepsilon) = H_n(W_{\%}C) = H_{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon). \qquad \square$$

Example 23. The ε-quadratic Q-groups of the zero-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_0 = A^r \to 0 \to \cdots$$

are given by

$$Q_n(C) = \begin{cases} \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2; A) & \text{if } n > 0, \\ \mathrm{Quad}_r(A) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An A-module chain map $f : C \to D$ induces a chain map

$$f_{\%} = 1_W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(f^*, f) : W_{\%}C \longrightarrow W_{\%}D,$$

which induces

$$f_{\%} : Q_n(C, \varepsilon) \longrightarrow Q_n(D, \varepsilon)$$

on homology. The *relative ε-quadratic Q-group* $Q_n(f, \varepsilon)$ is designed to fit into a long exact sequence

$$\cdots \longrightarrow Q_n(C, \varepsilon) \xrightarrow{f_\%} Q_n(D, \varepsilon) \longrightarrow Q_n(f, \varepsilon) \longrightarrow Q_{n-1}(C, \varepsilon) \longrightarrow \cdots ,$$

that is, $Q_n(f, \varepsilon)$ is defined as the nth homology group of the mapping cone of $f_\%$,

$$Q_n(f, \varepsilon) = H_n(f_\% : W_\% C \longrightarrow W_\% D).$$

Proposition 24. (i) *The relative ε-quadratic Q-groups of $f : C \to D$ are related to the absolute ε-quadratic Q-groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \to H_n(\mathcal{C}(f) \otimes_A C) \xrightarrow{F} Q_n(f, \varepsilon) \xrightarrow{t} Q_n(\mathcal{C}(f), \varepsilon) \to H_{n-1}(\mathcal{C}(f) \otimes_A C) \to \cdots .$$

(ii) *If $f : C \to D$ is a chain equivalence the morphisms $f_\% : Q_*(C) \to Q_*(D)$ are isomorphisms, and*

$$Q_*(\mathcal{C}(f), \varepsilon) = Q_*(f, \varepsilon) = 0.$$

(iii) *For any collection $\{C(i) \mid i \in \mathbb{Z}\}$ of f.g. projective A-module chain complexes $C(i)$*

$$Q_n\left(\sum_i C(i), \varepsilon\right) = \bigoplus_i Q_n(C(i), \varepsilon) \oplus \bigoplus_{i<j} H_n(C(i) \otimes_A C(j)).$$

Proposition 25. *Let C be a f.g. projective A-module chain complex which is concentrated in degrees $k, k+1$:*

$$C : \cdots \to 0 \to C_{k+1} \xrightarrow{d} C_k \to 0 \to \cdots .$$

The absolute ε-quadratic Q-groups $Q_*(C, \varepsilon)$ and the relative ε-quadratic Q-groups $Q_*(d, \varepsilon)$ of $d : C_{k+1} \to C_k$ regarded as a morphism of chain complexes concentrated in degree k are given as follows:

(i) *For $n \neq 2k, 2k+1, 2k+2$*

$$Q_n(C, \varepsilon) = Q_n(d, \varepsilon) = \begin{cases} \widehat{Q}^{n+1}(d, \varepsilon) = \widehat{Q}^{n+1}(C, \varepsilon) & \text{if } n \geqslant 2k+3, \\ 0 & \text{if } n \leqslant 2k-1, \end{cases}$$

with $\widehat{Q}^n(C, \varepsilon)$ as given by Proposition 13.

(ii) *For $n = 2k, 2k+1, 2k+2$ there are exact sequences*

$$
\begin{aligned}
&Q_{2k+2}(C_{k+1}, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \xrightarrow{d_\%} Q_{2k+2}(C_k, \varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \\
&\longrightarrow Q_{2k+2}(d, \varepsilon) = \widehat{Q}^{2k+3}(C, \varepsilon) \xrightarrow{d_\%} Q_{2k+1}(C_{k+1}, \varepsilon) \\
&= \widehat{H}^0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\
&\xrightarrow{d_\%} Q_{2k+1}(C_k, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \longrightarrow Q_{2k+1}(d, \varepsilon) \\
&\longrightarrow Q_{2k}(C_{k+1}, \varepsilon) = H_0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) \\
&\xrightarrow{d_\%} Q_{2k}(C_k, \varepsilon) = H_0(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) \longrightarrow Q_{2k}(d, \varepsilon) \\
&\longrightarrow Q_{2k-1}(C_{k+1}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&0 \to Q_{2k+2}(d, \varepsilon) \xrightarrow{t} Q_{2k+2}(C, \varepsilon) \longrightarrow H_{k+1}(C) \otimes_A C_{k+1} \\
&\xrightarrow{F} Q_{2k+1}(d, \varepsilon) \\
&\xrightarrow{t} Q_{2k+1}(C, \varepsilon) \longrightarrow C_{k+1} \otimes_A H_k(C) \xrightarrow{F} Q_{2k}(d, \varepsilon) \\
&\xrightarrow{t} Q_{2k}(C, \varepsilon) \to 0.
\end{aligned}
$$

For $\varepsilon = 1$ we write

$$T_\varepsilon = T, \quad Q_n(C, \varepsilon) = Q_n(C), \quad \varepsilon\text{-quadratic} = \text{quadratic}.$$

Example 26. Let C be a one-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_1 = A^q \xrightarrow{d} C_0 = A^r \to 0 \to \cdots ,$$

so that $C = \mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d : C_1 \to C_0$ of zero-dimensional complexes, with

$$d^\% : \mathrm{Hom}_A(C^1, C_1) = M_q(A) \to \mathrm{Hom}_A(C^0, C_0) = M_r(A); \quad \phi \mapsto d\phi d^*.$$

Example 23 and Proposition 25 give exact sequences

$$\begin{array}{llllll} Q_1(C_1) & = & \bigoplus_q \widehat{H}^0(\mathbb{Z}_2;A) & \xrightarrow{\widehat{d}^{\%}} & Q_1(C_0)=\bigoplus_r \widehat{H}^0(\mathbb{Z}_2;A)\to Q_1(d) & \\ & \longrightarrow & Q_0(C_1)=\mathrm{Quad}_q(A) & \xrightarrow{d_{\%}} & Q_0(C_0)=\mathrm{Quad}_r(A) \quad \longrightarrow & Q_0(d) \\ & \to & \multicolumn{4}{l}{Q_{-1}(C_1)=0,\ H_1(C)\otimes_A C_1\to Q_1(d)} \\ & \to & \multicolumn{4}{l}{Q_1(C)\to H_0(C)\otimes_A C_1\to Q_0(d)\to Q_0(C)\to 0.} \end{array}$$

In particular, if A is an even commutative ring and

$$d=2:C_1=A^r\to C_0=A^r,$$

then $d_{\%}=4$ and

$$\begin{aligned} Q_0(d)&=\frac{\mathrm{Quad}_r(A)}{4\mathrm{Quad}_r(A)},\\ Q_1(d)&=\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)+4\mathrm{Sym}_r(A)},\\ Q_0(C)&=\mathrm{coker}\left(2(1+T):\frac{M_r(A)}{2M_r(A)}\to\frac{\mathrm{Quad}_r(A)}{4\mathrm{Quad}_r(A)}\right)=\frac{\mathrm{Quad}_r(A)}{2\mathrm{Quad}_r(A)},\\ Q_1(C)&=\frac{\{(\psi_0,\psi_1)\in M_r(A)\oplus M_r(A)\,|\,2\psi_0=\psi_1-\psi_1^*\}}{\{(2(\chi_0-\chi_0^*),4\chi_0+\chi_2+\chi_2^*)\,|\,(\chi_0,\chi_2)\in M_r(A)\oplus M_r(A)\}}=\bigoplus_{\frac{r(r+1)}{2}}A_2. \end{aligned}$$

1.5. L-groups

An *n-dimensional* $\begin{cases}\varepsilon\text{-}symmetric\\ \varepsilon\text{-}quadratic\end{cases}$ *Poincaré complex* $\begin{cases}(C,\phi)\\ (C,\psi)\end{cases}$ over A is an n-dimensional f.g. projective A-module chain complex

$$C:\cdots\to 0\to C_n\to C_{n-1}\to\cdots\to C_1\to C_0\to 0\to\cdots$$

together with an element $\begin{cases}\phi\in Q^n(C,\varepsilon)\\ \psi\in Q_n(C,\varepsilon)\end{cases}$ such that the A-module chain map

$$\begin{cases}\phi_0:C^{n-*}\to C\\ (1+T_\varepsilon)\psi_0:C^{n-*}\to C\end{cases}$$

is a chain equivalence. We refer to [18] for the detailed definition of the *n-dimensional* $\begin{cases}\varepsilon\text{-}symmetric\\ \varepsilon\text{-}quadratic\end{cases}$ *L-group* $\begin{cases}L^n(A,\varepsilon)\\ L_n(A,\varepsilon)\end{cases}$ as the cobordism group of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes over A.

Definition 27. (i) The *relative* (ε*-symmetric*, ε*-quadratic*) *Q-group* $Q_n^n(f, \varepsilon)$ of a chain map $f : C \to D$ of f.g. projective A-module chain complexes is the relative group in the exact sequence

$$\cdots \to Q_n(C, \varepsilon) \xrightarrow{(1+T_\varepsilon) f_\%} Q^n(D, \varepsilon) \to Q_n^n(f, \varepsilon) \to Q_{n-1}(C, \varepsilon) \to \cdots .$$

An element $(\delta\phi, \psi) \in Q_n^n(f, \varepsilon)$ is an equivalence class of pairs

$$(\delta\phi, \psi) \in (W^\% D)_n \oplus (W_\% C)_{n-1},$$

such that

$$d(\psi) = 0 \in (W_\% C)_{n-2}, \ \ (1 + T_\varepsilon) f_\% \psi = d(\delta\phi) \in (W^\% D)_{n-1}.$$

(ii) An *n-dimensional* (ε*-symmetric*, ε*-quadratic*) *pair over* A $(f : C \to D, (\delta\phi, \psi))$ is a chain map f together with a class $(\delta\phi, \psi) \in Q_n^n(f, \varepsilon)$ such that the chain map

$$(\delta\phi, (1 + T_\varepsilon)\psi)_0 : D^{n-*} \to \mathcal{C}(f)$$

defined by

$$(\delta\phi, (1 + T_\varepsilon)\psi)_0 = \begin{pmatrix} \delta\phi_0 \\ (1 + T_\varepsilon)\psi_0 f^* \end{pmatrix} : D^{n-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is a chain equivalence.

Proposition 28. *The relative* (ε*-symmetric*, ε*-quadratic*) *Q-groups* $Q_n^n(f, \varepsilon)$ *of a chain map* $f : C \to D$ *fit into a commutative braid of exact sequences*

$$\begin{tikzcd}
Q_n(C,\epsilon) \arrow[dr, "f_\%"] \arrow[rr, bend left, "(1+T_\epsilon)f_\%"] & & Q^n(D,\epsilon) \arrow[dr] \arrow[rr, bend left, "J"] & & \widehat{Q}^n(D,\epsilon) \\
& Q_n(D,\epsilon) \arrow[ur, "1+T_\epsilon"] \arrow[dr] & & Q_n^n(f,\epsilon) \arrow[ur, "J_f"] \arrow[dr] & \\
\widehat{Q}^{n+1}(D,\epsilon) \arrow[ur, "H"] \arrow[rr, bend right] & & Q_n(f,\epsilon) \arrow[ur] \arrow[rr, bend right] & & Q_{n-1}(C,\epsilon)
\end{tikzcd}$$

with

$$
\begin{aligned}
&J_f : Q_n^n(f,\varepsilon) \to \widehat{Q}^n(D,\varepsilon);\ \ (\delta\phi,\psi) \mapsto \alpha,\\
&\alpha_s = \begin{cases} \delta\phi_s & \textit{if } s \geqslant 0 \\ f\psi_{-s-1}f^* & \textit{if } s \leqslant -1 \end{cases} : D^r \to D_{n-r+s}.
\end{aligned}
$$

The *n-dimensional ε-hyperquadratic L-group* $\widehat{L}^n(A,\varepsilon)$ is the cobordism group of n-dimensional (ε-symmetric, ε-quadratic) Poincaré pairs $(f : C \to D, (\phi,\psi))$ over A. As in [15], there is defined an exact sequence

$$\cdots \longrightarrow L_n(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L^n(A,\varepsilon) \longrightarrow \widehat{L}^n(A,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \cdots .$$

The skew-suspension maps in the $\pm\varepsilon$-quadratic L-groups are isomorphisms

$$\overline{S} : L_n(A,\varepsilon) \xrightarrow{\cong} L_{n+2}(A,-\varepsilon);\ \ (C,\{\psi_s\}) \mapsto (C_{*-1},\{\psi_s\}),$$

so the ε-quadratic L-groups are 4-periodic

$$L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon) = L_{n+4}(A,\varepsilon).$$

The skew-suspension maps in ε-symmetric and ε-hyperquadratic L-groups and $\pm\varepsilon$-hyperquadratic L-groups

$$
\begin{aligned}
&\overline{S} : L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon);\ \ (C,\{\phi_s\}) \mapsto (C_{*-1},\{\phi_s\}),\\
&\overline{S} : \widehat{L}^n(A,\varepsilon) \to \widehat{L}^{n+2}(A,-\varepsilon);\ \ (f : C \to D,\{\psi_s,\phi_s\})\\
&\quad \mapsto (f : C_{*-1} \to D_{*-1},\{(\psi_s,\phi_s)\})
\end{aligned}
$$

are not isomorphisms in general, so the ε-symmetric and ε-hyperquadratic L-groups need not be 4-periodic. We shall write the 4-periodic versions of the ε-symmetric and ε-hyperquadratic L-groups of A as

$$L^{n+4*}(A,\varepsilon) = \lim_{k\to\infty} L^{n+4k}(A,\varepsilon),\ \ \widehat{L}^{n+4*}(A,\varepsilon) = \lim_{k\to\infty} \widehat{L}^{n+4k}(A,\varepsilon),$$

noting that there is defined an exact sequence

$$\cdots \to L_n(A,\varepsilon) \to L^{n+4*}(A,\varepsilon) \to \widehat{L}^{n+4*}(A,\varepsilon) \to L_{n-1}(A,\varepsilon) \to \cdots .$$

Definition 29. The *Wu classes* of an n-dimensional ε-symmetric complex (C, ϕ) over A are the A-module morphisms

$$\widehat{v}_k(\phi) : H^{n-k}(C) \to \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon);\ \ x \mapsto \phi_{n-2k}(x)(x) \quad (k \in \mathbb{Z}).$$

For an n-dimensional ε-symmetric Poincaré complex (C, ϕ) over A the evaluation of the Wu class $\widehat{v}_k(\phi)(x) \in \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)$ is the obstruction to killing $x \in H^{n-k}(C) \cong H_k(C)$ by algebraic surgery [15, §4].

Proposition 30. (i) *If $\widehat{H}^0(\mathbb{Z}_2; A, \varepsilon)$ has a one-dimensional f.g. projective A-module resolution then the skew-suspension maps*

$$\overline{S} : L^{n-2}(A, -\varepsilon) \to L^n(A, \varepsilon),\ \ \overline{S} : \widehat{L}^{n-2}(A, -\varepsilon) \to \widehat{L}^n(A, \varepsilon) \quad (n \geqslant 2)$$

are isomorphisms. Thus if $\widehat{H}^1(\mathbb{Z}_2; A, \varepsilon)$ also has a one-dimensional f.g. projective A-module resolution the ε-symmetric and ε-hyperquadratic L-groups of A are 4*-periodic*

$$\begin{aligned} L^n(A, \varepsilon) &= L^{n+2}(A, -\varepsilon) = L^{n+4}(A, \varepsilon), \\ \widehat{L}^n(A, \varepsilon) &= \widehat{L}^{n+2}(A, -\varepsilon) = \widehat{L}^{n+4}(A, \varepsilon). \end{aligned}$$

(ii) *If A is a Dedekind ring then the ε-symmetric L-groups are 'homotopy invariant'*

$$L^n(A[x], \varepsilon) = L^n(A, \varepsilon)$$

and the ε-symmetric and ε-hyperquadratic L-groups of A and $A[x]$ are 4*-periodic.*

Proof. (i) Let D be a one-dimensional f.g. projective A-module resolution of $\widehat{H}^0(\mathbb{Z}_2; A, \varepsilon)$:

$$0 \to D_1 \to D_0 \to \widehat{H}^0(\mathbb{Z}_2; A) \to 0.$$

Given an n-dimensional ε-symmetric Poincaré complex (C, ϕ) over A resolve the A-module morphism

$$\widehat{v}_n(\phi)(\phi_0)^{-1} : H_0(C) \cong H^n(C) \to H_0(D) = \widehat{H}^0(\mathbb{Z}_2; A, \varepsilon);\ \ u \mapsto (\phi_0)^{-1}(u)(u)$$

by an A-module chain map $f : C \to D$, defining an $(n+1)$-dimensional ε-symmetric pair $(f : C \to D, (\delta\phi, \phi))$. The effect of algebraic surgery on (C, ϕ) using $(f : C \to D, (\delta\phi, \phi))$ is a cobordant n-dimensional ε-symmetric Poincaré complex (C', ϕ') such that there are defined an exact sequence:

$$0 \to H^n(C') \to H^n(C) \xrightarrow{\widehat{v}_n(\phi)} \widehat{H}^0(\mathbb{Z}_2; A, \varepsilon) \to H^{n+1}(C') \to 0$$

and an $(n+1)$-dimensional ε-symmetric pair $(f' : C' \to D', (\delta\phi', \phi'))$ with f' the projection onto the quotient complex of C' defined by

$$D' : \cdots \to 0 \to D'_{n+1} = C'_{n+1} \to D'_n = C'_n \to 0 \to \cdots \ .$$

The effect of algebraic surgery on (C', ϕ') using $(f' : C' \to D', (\delta\phi', \phi'))$ is a cobordant n-dimensional ε-symmetric Poincaré complex (C'', ϕ'') with $H_n(C'') = 0$, so that it is (homotopy equivalent to) the skew-suspension of an $(n-2)$-dimensional $(-\varepsilon)$-symmetric Poincaré complex.

(ii) The 4-periodicity $L^*(A, \varepsilon) = L^{*+4}(A, \varepsilon)$ was proved in [15, §7]. The 'homotopy invariance' $L^*(A[x], \varepsilon) = L^*(A, \varepsilon)$ was proved in [17, 41.3]; [10, 2.1]. The 4-periodicity of the ε-symmetric and ε-hyperquadratic L-groups for A and $A[x]$ now follows from the 4-periodicity of the ε-quadratic L-groups $L_*(A, \varepsilon) = L_{*+4}(A, \varepsilon)$. $\square$

2. Chain bundle theory

2.1. Chain bundles

Definition 31. (i) An *ε-bundle* over an A-module chain complex C is a zero-dimensional ε-hyperquadratic structure γ on C^{0-*}, that is, a cycle

$$\gamma \in (\widehat{W}^{\%} C^{0-*})_0$$

as given by a collection of A-module morphisms

$$\{\gamma_s \in \mathrm{Hom}_A(C_{r-s}, C^{-r}) \,|\, r, s \in \mathbb{Z}\},$$

such that

$$(-1)^{r+1} d^* \gamma_s + (-1)^s \gamma_s d + (-1)^{s-1}(\gamma_{s-1} + (-1)^s T_\varepsilon \gamma_{s-1}) = 0 : C_{r-s+1} \to C^{-r}.$$

(ii) An *equivalence of ε-bundles* over C,

$$\chi : \gamma \longrightarrow \gamma'$$

is an equivalence of ε-hyperquadratic structures.

(iii) A *chain ε-bundle* (C, γ) over A is an A-module chain complex C together with an ε-bundle $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$.

Let (D, δ) be a chain ε-bundle and $f : C \to D$ a chain map. The dual of f

$$f^* : D^{0-*} \longrightarrow C^{0-*}$$

induces a map

$$\widehat{(f^*)}^{\%}_0 : (\widehat{W}^{\%} D^{0-*})_0 \longrightarrow (\widehat{W}^{\%} C^{0-*})_0.$$

Definition 32. (i) The *pullback chain ε-bundle* $(C, f^*\delta)$ is defined to be

$$f^*\delta = \widehat{(f^*)}^{\%}_0(\delta) \in (\widehat{W}^{\%} C^{0-*})_0.$$

(ii) A *map of chain ε-bundles*

$$(f, \chi) : (C, \gamma) \longrightarrow (D, \delta)$$

is a chain map $f : C \to D$ together with an equivalence of ε-bundles over C:

$$\chi : \gamma \longrightarrow f^*\delta.$$

The ε-hyperquadratic Q-group $\widehat{Q}^0(C^{0-*}, \varepsilon)$ is thus the group of equivalence classes of chain ε-bundles on the chain complex C, the algebraic analogue of the topological K-group of a space. The Tate $\mathbb{Z}_2$-cohomology groups

$$\widehat{H}^n(\mathbb{Z}_2; A, \varepsilon) = \frac{\{a \in A \,|\, \overline{a} = (-1)^n \varepsilon a\}}{\{b + (-1)^n \varepsilon \overline{b} \,|\, b \in A\}}$$

are A-modules via

$$A \times \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon) \to \widehat{H}^n(\mathbb{Z}_2; A, \varepsilon);\ \ (a, x) \mapsto ax\overline{a}.$$

Definition 33. The *Wu classes* of a chain ε-bundle (C, γ) are the A-module morphisms

$$\widehat{v}_k(\gamma) : H_k(C) \to \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon);\ \ x \mapsto \gamma_{-2k}(x)(x) \quad (k \in \mathbb{Z}).$$

An n-dimensional ε-symmetric Poincaré complex (C, ϕ) with Wu classes (29)

$$\widehat{v}_k(\phi) : H^{n-k}(C) \to \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon);\ \ y \mapsto \phi_{n-2k}(y)(y) \quad (k \in \mathbb{Z})$$

has a Spivak normal ε-bundle [15]

$$\gamma = S^{-n}(\phi_0^{\%})^{-1}(J(\phi)) \in \widehat{Q}^0(C^{0-*}, \varepsilon),$$

such that

$$\widehat{v}_k(\phi) = \widehat{v}_k(\gamma)\phi_0 : H^{n-k}(C) \cong H_k(C) \to \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon) \quad (k \in \mathbb{Z}),$$

the abstract analogue of the formulae of Wu and Thom.

For any A-module chain map $f : C \to D$ Proposition 12(i) gives an exact sequence

$$\cdots \to \widehat{Q}^1(C^{0-*}, \varepsilon) \to \widehat{Q}^0(\mathcal{C}(f)^{0-*}, \varepsilon) \to \widehat{Q}^0(D^{0-*}, \varepsilon) \xrightarrow{(\widehat{f}^*)^{\%}} \widehat{Q}^0(C^{0-*}, \varepsilon) \to \cdots ,$$

motivating the following construction of chain ε-bundles:

Definition 34. The *cone* of a chain ε-bundle map $(f, \chi) : (C, 0) \to (D, \delta)$ is the chain ε-bundle

$$(B, \beta) = \mathcal{C}(f, \chi),$$

with $B = \mathcal{C}(f)$ the algebraic mapping cone of $f : C \to D$ and

$$\beta_s = \begin{pmatrix} \delta_s & 0 \\ f^*\delta_{s+1} & \chi_{s+1} \end{pmatrix} : B_{r-s} = D_{r-s} \oplus C_{r-s-1} \to B^{-r} = D^{-r} \oplus C^{-r-1}.$$

Note that $(D, \delta) = g^*(B, \beta)$ is the pullback of (B, β) along the inclusion $g : D \to B$.

Proposition 35. *For a f.g. projective A-module chain complex concentrated in degree k*:

$$C : \cdots \to 0 \to C_k \to 0 \to \cdots ,$$

the kth Wu class defines an isomorphism

$$\widehat{v}_k : \widehat{Q}^0(C^{0-*}, \varepsilon) \xrightarrow{\cong} \mathrm{Hom}_A(C_k, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)); \;\; \gamma \mapsto \widehat{v}_k(\gamma).$$

Proof. By construction. $\square$

Proposition 36. *For a f.g. projective A-module chain complex concentrated in degrees* $k, k+1$,

$$C : \cdots \to 0 \to C_{k+1} \xrightarrow{d} C_k \to 0 \to \cdots$$

there is defined an exact sequence

$$\begin{array}{llll} & \mathrm{Hom}_A(C_k, \widehat{H}^{k+1}(\mathbb{Z}_2; A, \varepsilon)) & \xrightarrow{d^*} & \mathrm{Hom}_A(C_{k+1}, \widehat{H}^{k+1}(\mathbb{Z}_2; A, \varepsilon)) \\ \longrightarrow & \widehat{Q}^0(C^{0-*}, \varepsilon) & \xrightarrow{p^*\widehat{v}_k} & \mathrm{Hom}_A(C_k, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)) \\ \xrightarrow{d^*} & \mathrm{Hom}_A(C_{k+1}, \widehat{H}^k(\mathbb{Z}_2; A, \varepsilon)), & & \end{array}$$

with $p : C_k \to H_k(C)$ the projection. Thus every chain ε-bundle (C, γ) is equivalent to the cone $\mathcal{C}(d, \chi)$ (34) of a chain ε-bundle map $(d, \chi) : (C_{k+1}, 0) \to (C_k, \delta)$, regarding $d : C_{k+1} \to C_k$ as a map of chain complexes concentrated in degree k, with

$$\begin{aligned} \delta^* &= (-1)^k \delta : C_k \to C^k, \ \ d^* \delta d = \chi + (-1)^k \chi^* : C_{k+1} \to C^{k+1}, \\ \gamma_{-2k} &= \delta : C_k \to C^k, \ \ \gamma_{-2k-1} = \begin{cases} d^*\delta : C_k \to C^{k+1} \\ 0 : C_{k+1} \to C^k \end{cases}, \\ \gamma_{-2k-2} &= \chi : C_{k+1} \to C^{k+1}. \end{aligned}$$

Proof. This follows from Proposition 35 and the algebraic Thom isomorphisms

$$\widehat{t} : \widehat{Q}^*(d, \varepsilon) \cong \widehat{Q}^*(C, \varepsilon)$$

of Proposition 12. □

2.2. The twisted quadratic Q-groups

For any f.g. projective A-module chain complex C there is defined a $\mathbb{Z}$-module chain map

$$\begin{aligned} & 1 + T_\varepsilon : W^\% C; \ \ \psi \mapsto (1 + T_\varepsilon)\psi, \\ & ((1 + T_\varepsilon)\psi)_s = \begin{cases} (1 + T_\varepsilon)(\psi_0) & \text{if } s = 0, \\ 0 & \text{if } s \geqslant 1, \end{cases} \end{aligned}$$

with algebraic mapping cone

$$\mathcal{C}(1 + T_\varepsilon) = \widehat{W}^\% C.$$

Write the inclusion as

$$J : W^\% C \to \widehat{W}^\% C; \ \ \phi \mapsto J\phi, \ \ (J\phi)_s = \begin{cases} \phi_s & \text{if } s \geqslant 0, \\ 0 & \text{if } s \leqslant -1. \end{cases}$$

The sequence of $\mathbb{Z}$-module chain complexes

$$0 \to W_{\%}C \xrightarrow{1+T_\varepsilon} W^{\%}C \xrightarrow{J} \widehat{W}^{\%}C \to 0$$

induces the long exact sequence of Ranicki [15] relating the ε-symmetric, ε-quadratic and ε-hyperquadratic Q-groups of C,

$$\cdots \to \widehat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H} Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{J} \widehat{Q}^n(C,\varepsilon) \to \cdots ,$$

with

$$H : \widehat{W}^{\%}C \to (W_{\%}C)_{*-1};\ \theta \mapsto H\theta,\ (H\theta)_s = \theta_{-s-1} \quad (s \geqslant 0).$$

Weiss [21] used simplicial abelian groups to defined the twisted quadratic Q-groups $Q_*(C,\gamma,\varepsilon)$ of a chain ε-bundle (C,γ), to fit into the exact sequence

$$\cdots \to \widehat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H_\gamma} Q_n(C,\gamma,\varepsilon) \xrightarrow{N_\gamma} Q^n(C,\varepsilon) \xrightarrow{J_\gamma} \widehat{Q}^n(C,\varepsilon) \to \cdots .$$

The morphisms

$$J_\gamma : Q^n(C,\varepsilon) \longrightarrow \widehat{Q}^n(C,\varepsilon);\ \phi \mapsto J_\gamma\phi,\ (J_\gamma\phi)_s = J(\phi) - (\phi_0)^{\%}(S^n\gamma)$$

are induced by a morphism of simplicial abelian groups, where

$$S^n : \widehat{Q}^0(C^{0-*},\varepsilon) \xrightarrow{\cong} \widehat{Q}^n(C^{n-*},\varepsilon);\ \{\theta_s\} \mapsto \{(S^n\theta)_s = \theta_{s-n}\}$$

are the n-fold suspension isomorphisms.

The Kan-Dold theory associates to a chain complex C a simplicial abelian group $K(C)$ such that

$$\pi_*(K(C)) = H_*(C).$$

For any chain complexes C, D a simplicial map $f : K(C) \to K(D)$ has a mapping fibre $K(f)$. The relative homology groups of f are defined by

$$H_*(f) = \pi_{*-1}(K(f))$$

and the fibration sequence of simplicial abelian groups

$$K(f) \longrightarrow K(C) \xrightarrow{f} K(D)$$

induces a long exact sequence in homology

$$\cdots \to H_n(C) \to H_n(D) \to H_n(f) \to H_{n-1}(C) \to \cdots \ .$$

For a chain map $f : C \to D$,

$$K(f) = K(\mathcal{C}(f)).$$

The applications involve simplicial maps which are not chain maps, and the *triad homology groups*: given a homotopy-commutative square of simplicial abelian groups

$$\Phi \ : \quad \begin{tikzcd} K(C) \arrow[r] \arrow[d] \arrow[dr, squiggly] & K(D) \arrow[d] \\ K(E) \arrow[r] & K(F) \end{tikzcd}$$

(with $\rightsquigarrow$ denoting an explicit homotopy) the triad homology groups of Φ are the homotopy groups of the mapping fibre of the map of mapping fibres

$$H_*(\Phi) = \pi_{*-1}(K(C \to D) \to K(E \to F)),$$

which fit into a commutative diagram of exact sequences

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & H_{n+1}(D) \arrow[r] \arrow[d] & H_{n+1}(C \to D) \arrow[r] \arrow[d] & H_n(C) \arrow[r] \arrow[d] & H_n(D) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1}(F) \arrow[r] \arrow[d] & H_{n+1}(E \to F) \arrow[r] \arrow[d] & H_n(E) \arrow[r] \arrow[d] & H_n(F) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1}(D \to F) \arrow[r] \arrow[d] & H_{n+1}(\Phi) \arrow[r] \arrow[d] & H_n(C \to E) \arrow[r] \arrow[d] & H_n(D \to F) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(D) \arrow[r] \arrow[d] & H_n(C \to D) \arrow[r] \arrow[d] & H_{n-1}(C) \arrow[r] \arrow[d] & H_{n-1}(D) \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{tikzcd}$$

If $H_*(\Phi) = 0$ there is a commutative braid of exact sequences

$$\begin{tikzcd}
H_{n+1}(C \to D) \arrow[dr] \arrow[rr, bend left] & & H_n(E) \arrow[dr] \arrow[rr, bend left] & & H_n(C \to E) \\
& H_n(C) \arrow[ur] \arrow[dr] & & H_n(F) \arrow[ur] \arrow[dr] & \\
H_{n+1}(C \to E) \arrow[ur] \arrow[rr, bend right] & & H_n(D) \arrow[ur] \arrow[rr, bend right] & & H_n(C \to D)
\end{tikzcd}$$

The twisted ε-quadratic Q-groups were defined in [21] to be the relative homology groups of a simplicial map

$$J_\gamma : K(W^{\%}C) \to K(\widehat{W}^{\%}C),$$

with

$$Q_n(C, \gamma, \varepsilon) = \pi_{n+1}(J_\gamma).$$

A more explicit description of the twisted quadratic Q-groups was then obtained in [18], as equivalence classes of ε-symmetric structures on the chain ε-bundle.

Definition 37. (i) An *ε-symmetric structure on a chain ε-bundle* (C, γ) is a pair (ϕ, θ) with $\phi \in (W^{\%}C)_n$ a cycle and $\theta \in (\widehat{W}^{\%}C)_{n+1}$ such that

$$d\theta = J_\gamma(\phi),$$

or equivalently

$$\begin{aligned}
& d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T_\varepsilon \phi_{s-1}) = 0 : C^r \to C_{n-r+s-1}, \\
& \phi_s - \phi_0^* \gamma_{s-n} \phi_0 = d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s}(\theta_{s-1} + (-1)^s T_\varepsilon \theta_{s-1}) : C^r \to C_{n-r+s} \\
& \qquad (r, s \in \mathbb{Z}, \phi_s = 0 \text{ for } s \leqslant -1).
\end{aligned}$$

(ii) Two structures (ϕ, θ) and (ϕ', θ') are *equivalent* if there exist $\xi \in (W^{\%}C)_{n+1}$, $\eta \in (\widehat{W}^{\%}C)_{n+2}$ such that

$$d\xi = \phi' - \phi, \;\; d\eta = \theta' - \theta + J(\xi) + (\xi_0, \phi_0, \phi'_0)^{\%}(S^n\gamma),$$

where $(\xi_0, \phi_0, \phi'_0)^{\%} : (\widehat{W}^{\%}C^{-*})_n \to (\widehat{W}^{\%}C)_{n+1}$ is the chain homotopy from $(\phi_0)^{\%}$ to $(\phi'_0)^{\%}$ induced by ξ_0. (See [15, 1.1] for the precise formula.)

(iii) The *n-dimensional twisted ε-quadratic Q-group* $Q_n(C,\gamma,\varepsilon)$ is the abelian group of equivalence classes of n-dimensional ε-symmetric structures on (C,γ) with addition by

$$(\phi,\theta)+(\phi',\theta')=(\phi+\phi',\theta+\theta'+\zeta),\ \text{ where } \zeta_s=\phi_0\gamma_{s-n+1}\phi'_0.$$

As for the ±ε-symmetric and ±ε-quadratic Q-groups, there are defined skew-suspension isomorphisms of twisted ±ε-quadratic Q-groups

$$\overline{S}:Q_n(C,\gamma,\varepsilon)\xrightarrow{\cong}Q_{n+2}(C_{*-1},\gamma,-\varepsilon);\ \ (\{\phi_s\},\{\theta_s\})\mapsto(\{\phi_s\},\{\theta_s\}).$$

Proposition 38. (i) *The twisted ε-quadratic Q-groups* $Q_*(C,\gamma,\varepsilon)$ *are related to the ε-symmetric Q-groups* $Q^*(C,\varepsilon)$ *and the ε-hyperquadratic Q-groups* $\widehat{Q}^*(C,\varepsilon)$ *by the exact sequence*

$$\cdots\to\widehat{Q}^{n+1}(C,\varepsilon)\xrightarrow{H_\gamma}Q_n(C,\gamma,\varepsilon)\xrightarrow{N_\gamma}Q^n(C,\varepsilon)\xrightarrow{J_\gamma}\widehat{Q}^n(C,\varepsilon)\to\cdots\ ,$$

with

$$H_\gamma:\widehat{Q}^{n+1}(C,\varepsilon)\to Q_n(C,\gamma,\varepsilon);\ \ \theta\mapsto(0,\theta),$$
$$N_\gamma:Q_n(C,\gamma,\varepsilon)\to Q^n(C,\varepsilon);\ \ (\phi,\theta)\mapsto\phi.$$

(ii) *For a chain ε-bundle* (C,γ) *such that* C *splits as*

$$C=\sum_{i=-\infty}^{\infty}C(i),$$

the ε-hyperquadratic Q-groups split as

$$\widehat{Q}^n(C,\varepsilon)=\sum_{i=-\infty}^{\infty}\widehat{Q}^n(C(i),\varepsilon)$$

and

$$\gamma=\sum_{i=-\infty}^{\infty}\gamma(i)\in\widehat{Q}^0(C^{-*},\varepsilon)=\sum_{i=-\infty}^{\infty}\widehat{Q}^0(C(i)^{-*},\varepsilon).$$

The twisted ε-quadratic Q-groups of (C,γ) fit into the exact sequence

$$\cdots \to \sum_i Q_n(C(i),\gamma(i),\varepsilon) \xrightarrow{\quad q\quad} Q_n(C,\gamma,\varepsilon) \xrightarrow{\quad p\quad} \sum_{i<j} H_n(C(i)\otimes_A C(j))$$
$$\xrightarrow{\quad \partial\quad} \sum_i Q_{n-1}(C(i),\gamma(i),\varepsilon) \to \cdots\ ,$$

with

$$\begin{aligned}
p &: Q_n(C,\gamma,\varepsilon) \to \sum_{i<j} H_n(C(i)\otimes_A C(j));\ \ (\phi,\theta)\mapsto \sum_{i<j}(p(i)\otimes p(j))(\phi_0)\\
&\qquad\qquad (p(i) = \text{projection} : C \to C(i)),\\
q &= \sum_i q(i)_\% : \sum_i Q_n(C(i),\gamma(i),\varepsilon) \to Q_n(C,\gamma,\varepsilon)\\
&\qquad\qquad (q(i) = \text{inclusion} : C(i) \to C)),\\
\partial &: \sum_{i<j} H_n(C(i)\otimes_A C(j)) \to \sum_i Q_{n-1}(C(i),\gamma(i),\varepsilon);\\
&\qquad\qquad \sum_{i<j} h(i,j) \mapsto \left(0, \sum_{i\neq j} \widehat{h(i,j)}^{\%}(S^n\gamma(j))\right)\quad (h(i,j) : C(j)^{n-*} \to C(i)),
\end{aligned}$$

with $h(j,i) = h(i,j)^*$ for $i<j$.

Proof. (i) See [21].
(ii) See [18, p. 26]. $\square$

Example 39. The twisted ε-quadratic Q-groups of the zero chain ε-bundle $(C,0)$ are just the ε-quadratic Q-groups of C, with isomorphisms

$$Q_n(C,\varepsilon) \to Q_n(C,0,\varepsilon);\ \ \psi \mapsto ((1+T)\psi,\theta)$$

defined by

$$\theta_s = \begin{cases} \psi_{-s-1} : C^{n-r+s+1} \to C_r & \text{if } s \leqslant -1,\\ 0 & \text{if } s \geqslant 0\end{cases}$$

and with an exact sequence

$$\cdots \to \widehat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H} Q_n(C,\varepsilon) \xrightarrow{N} Q^n(C,\varepsilon) \xrightarrow{J} \widehat{Q}^n(C,\varepsilon) \to \cdots\ .$$

For $\varepsilon = 1$ we write

$$\text{chain 1-bundle} = \text{chain bundle},\ \ Q_n(C,\gamma,1) = Q_n(C,\gamma).$$

2.3. The algebraic normal invariant

Fix a chain ε-bundle (B, β) over A.

Definition 40. (i) An *algebraic normal structure* (γ, ϕ, θ) on an n-dimensional (ε-symmetric, ε-quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ is a chain ε-bundle $(\mathcal{C}(f), \gamma)$ together with an ε-symmetric structure (ϕ, θ), where $\phi = \delta\phi/(1 + T_\varepsilon)\psi \in (W^{\%}\mathcal{C}(f))_n$ is the ε-symmetric structure on $\mathcal{C}(f)$ given by the algebraic Thom construction on $(\delta\phi, (1 + T_\varepsilon)\psi)$ (18).

(ii) A (B, β)-*structure* $(\gamma, \phi, \theta, g, \chi)$ on an n-dimensional (ε-symmetric, ε-quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ is an algebraic normal structure (γ, ϕ, θ) with $\phi = \delta\phi/(1 + T_\varepsilon)\psi$, together with a chain ε-bundle map

$$(g, \chi) : (\mathcal{C}(f), \gamma) \to (B, \beta).$$

(iii) The *n-dimensional (B, β)-structure ε-symmetric L-group* $L\langle B, \beta\rangle^n(A, \varepsilon)$ is the cobordism group of n-dimensional ε-symmetric Poincaré complexes $(D, \delta\phi)$ over A together with a (B, β)-structure $(\gamma, \delta\phi, \theta, g, \chi)$ (so $(C, \psi) = (0, 0)$).

(iv) The *n-dimensional (B, β)-structure ε-hyperquadratic L-group* $\widehat{L}\langle B, \beta\rangle^n(A, \varepsilon)$ is the cobordism group of n-dimensional (ε-symmetric, ε-quadratic) Poincaré pairs $(f : C \to D, (\delta\phi, \psi))$ over A together with a (B, β)-structure $(\gamma, \delta\phi/(1 + T_\varepsilon)\psi, \theta, g, \chi)$.

There are defined skew-suspension maps in the (B, β)-structure ε-symmetric and ε-hyperquadratic L-groups

$$\overline{S} : L\langle B, \beta\rangle^n(A, \varepsilon) \to L\langle B_{*-1}, \beta_{*-1}\rangle^{n+2}(A, -\varepsilon),$$
$$\overline{S} : \widehat{L}\langle B, \beta\rangle^n(A, \varepsilon) \to \widehat{L}\langle B_{*-1}, \beta_{*-1}\rangle^{n+2}(A, -\varepsilon)$$

given by $C \mapsto C_{*-1}$ on the chain complexes, with (B_{*-1}, β_{*-1}) a chain $(-\varepsilon)$-bundle. We shall write the 4-periodic versions of the (B, β)-structure L-groups as

$$L\langle B, \beta\rangle^{n+4*}(A, \varepsilon) = \lim_{k\to\infty} L\langle B, \beta\rangle^{n+4k}(A, \varepsilon),$$
$$\widehat{L}\langle B, \beta\rangle^{n+4*}(A, \varepsilon) = \lim_{k\to\infty} \widehat{L}\langle B, \beta\rangle^{n+4k}(A, \varepsilon).$$

Example 41. An (ε-symmetric, ε-quadratic) Poincaré pair with a $(0, 0)$-structure is essentially the same as an ε-quadratic Poincaré pair. In particular, an ε-symmetric Poincaré complex with a $(0, 0)$-structure is essentially the same as an ε-quadratic Poincaré complex. The (0,0)-structure L-groups are given by

$$L\langle 0, 0\rangle^n(A, \varepsilon) = L_n(A, \varepsilon), \quad \widehat{L}\langle 0, 0\rangle^n(A, \varepsilon) = 0.$$

Proposition 42 (*Ranicki [18, §7]*). (i) *An n-dimensional ε-symmetric structure* $(\phi, \theta) \in Q_n(B, \beta, \varepsilon)$ *on a chain ε-bundle* (B, β) *determines an n-dimensional* (*ε-symmetric*, *ε-quadratic*) *Poincaré pair* $(f : C \to D, (\delta\phi, \psi))$ *with*

$$f = \text{proj.} : C = \mathcal{C}(\phi_0 : B^{n-*} \to B)_{*+1} \to D = B^{n-*},$$

$$\psi_0 = \begin{pmatrix} \theta_0 & 0 \\ 1 + \beta_{-n}\phi_0^* & \beta_{-n-1}^* \end{pmatrix} : C^r = B^{r+1} \oplus B_{n-r} \to C_{n-r-1} = B_{n-r} \oplus B^{r+1},$$

$$\psi_s = \begin{pmatrix} \theta_{-s} & 0 \\ \beta_{-n-s}\phi_0^* & \beta_{-n-s-1}^* \end{pmatrix} : C^r = B^{r+1} \oplus B_{n-r} \to C_{n-r-s-1} = B_{n-r-s} \oplus B^{r+s+1} \quad (s \geqslant 1),$$

$$\delta\phi_s = \beta_{s-n} : D^r = B_{n-r} \to D_{n-r+s} = B^{r-s} \quad (s \geqslant 0)$$

(*up to signs*) *such that* $(\mathcal{C}(f), \gamma) \simeq (B, \beta)$.

(ii) *An n-dimensional* (*ε-symmetric*, *ε-quadratic*) *Poincaré pair* $(f : C \to D, (\delta\phi, \psi) \in Q_n^n(f, \varepsilon))$ *has a canonical equivalence class of* '*algebraic Spivak normal structures*' (γ, ϕ, θ) *with* γ *a chain ε-bundle over* $\mathcal{C}(f)$ *and* (ϕ, θ) *an n-dimensional ε-symmetric structure on* γ *representing an element*

$$(\phi, \theta) \in Q_n(\mathcal{C}(f), \gamma, \varepsilon),$$

with $\phi = \delta\phi/(1+T_\varepsilon)\psi$. *The construction of* (i) *applied to* (ϕ, θ) *gives an n-dimensional* (*ε-symmetric*, *ε-quadratic*) *Poincaré pair homotopy equivalent to* $(f : C \to D, (\delta\phi, \psi) \in Q_n^n(f, \varepsilon))$.

Proof. (i) By construction.

(ii) The equivalence class $\phi = \delta\phi/(1 + T_\varepsilon)\psi \in Q^n(\mathcal{C}(f))$ is given by the algebraic Thom construction

$$\phi_s = \begin{cases} \begin{pmatrix} \delta\phi_0 & 0 \\ (1 + T_\varepsilon)\psi_0 f^* & 0 \end{pmatrix} & \text{if } s = 0, \\ \begin{pmatrix} \delta\phi_1 & 0 \\ 0 & (1 + T_\varepsilon)\psi_0 \end{pmatrix} & \text{if } s = 1, \\ \begin{pmatrix} \delta\phi_s & 0 \\ 0 & 0 \end{pmatrix} & \text{if } s \geqslant 2, \end{cases}$$

$$: \ \mathcal{C}(f)^r = D^r \oplus C^{r-1} \to \mathcal{C}(f)_{n-r+s} = D_{n-r+s} \oplus C_{n-r+s-1},$$

such that

$$\phi_0 : \mathcal{C}(f)^{n-*} \to D^{n-*} \xrightarrow[\simeq]{(\delta\phi,(1+T_\varepsilon)\psi)_0} \mathcal{C}(f).$$

The equivalence class $\gamma \in \widehat{Q}^0(\mathcal{C}(f)^{0-*}, \varepsilon)$ of the Spivak normal chain bundle is the image of $(\delta\phi, \psi) \in Q_n^n(f, \varepsilon)$ under the composite

$$Q_n^n(f,\varepsilon) \xrightarrow{J_f} \widehat{Q}^n(D,\varepsilon) \xrightarrow[\cong]{((\delta\phi,(1+T_\varepsilon)\psi)_0^{\%})^{-1}} \widehat{Q}^n(\mathcal{C}(f)^{n-*},\varepsilon) \xrightarrow[\cong]{S^{-n}} \widehat{Q}^0(\mathcal{C}(f)^{0-*},\varepsilon).$$

$\square$

Definition 43. (i) The *boundary* of an n-dimensional ε-symmetric structure $(\phi, \theta) \in Q_n(B, \beta, \varepsilon)$ on a chain ε-bundle (B, β) over A is the ε-symmetric null-cobordant $(n-1)$-dimensional ε-quadratic Poincaré complex over A:

$$\partial(\phi, \theta) = (C, \psi)$$

defined in Proposition 42(i) above, with $C = \mathcal{C}(\phi_0 : B^{n-*} \to B)_{*+1}$.

(ii) The *algebraic normal invariant* of an n-dimensional (ε-symmetric, ε-quadratic) Poincaré pair over A $(f : C \to D, (\delta\phi, \psi) \in Q_n^n(f, \varepsilon))$ is the class

$$(\phi, \theta) \in Q_n(\mathcal{C}(f), \gamma, \varepsilon)$$

defined in Proposition 42(ii) above.

Proposition 44. *Let (B, β) be a chain ε-bundle over A such that B is concentrated in degree k,*

$$B : \cdots \to 0 \to B_k \to 0 \to \cdots \ .$$

The boundary map $\partial : Q_{2k}(B, \beta, \varepsilon) \to L_{2k-1}(A, \varepsilon)$ sends an ε-symmetric structure $(\phi, \theta) \in Q_{2k}(B, \beta, \varepsilon)$ to the Witt class of the $(-1)^{k-1}\varepsilon$-quadratic formation

$$\partial(\phi, \theta) = \left(H_{(-1)^{k-1}\varepsilon}(B^k); B^k, \operatorname{im}\left(\begin{pmatrix} 1 - \beta\phi \\ \phi \end{pmatrix} : B^k \to B^k \oplus B_k \right) \right),$$

with

$$H_{(-1)^{k-1}\varepsilon}(B^k) = \left(B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right),$$

the hyperbolic $(-1)^{k-1}\varepsilon$-quadratic form.

Proof. The chain ε-bundle (equivalence class)

$$\beta \in \widehat{Q}^0(B^{0-*}, \varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B_k), (-1)^k\varepsilon)$$

is represented by a $(-1)^k\varepsilon$-symmetric form (B_k, β). An ε-symmetric structure $(\phi, \theta) \in Q_{2k}(B, \beta, \varepsilon)$ is represented by an $(-1)^k\varepsilon$-symmetric form (B^k, ϕ) together with $\theta \in S(B_k)$ such that

$$\phi - \phi\beta\phi = \theta + (-1)^k\varepsilon\theta^* \in H^0(\mathbb{Z}_2; S(B^k), (-1)^k\varepsilon).$$

The boundary of (ϕ, θ) is the ε-symmetric null-cobordant $(2k-1)$-dimensional ε-quadratic Poincaré complex $\partial(\phi, \theta) = (C, \psi)$ concentrated in degrees $k-1, k$ corresponding to the formation in the statement. $\square$

Proposition 45. *Let (B, β) be a chain ε-bundle over A such that B is concentrated in degrees $k, k+1$,*

$$B : \cdots \to 0 \to B_{k+1} \xrightarrow{d} B_k \to 0 \to \cdots .$$

The boundary map $\partial : Q_{2k+1}(B, \beta, \varepsilon) \to L_{2k}(A, \varepsilon)$ sends an ε-symmetric structure $(\phi, \theta) \in Q_{2k+1}(B, \beta, \varepsilon)$ to the Witt class of the nonsingular $(-1)^k\varepsilon$-quadratic form over A

$$\left(\operatorname{coker}\left(\begin{pmatrix} -d^* \\ \phi_0^* \\ 1 - \beta_{-2k}d\phi_0^* \end{pmatrix} : B^k \to B^{k+1} \oplus B_{k+1} \oplus B^k\right), \begin{pmatrix} \theta_0 & 0 & \phi_0 \\ 1 & \beta^*_{-2k-2} & d^* \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Proof. This is an application of the instant surgery obstruction of [15, 4.3], which identifies the cobordism class $(C, \psi) \in L_{2k}(A, \varepsilon)$ of a $2k$-dimensional ε-quadratic Poincaré complex (C, ψ) with the Witt class of the nonsingular ε-quadratic form

$$I(C, \psi) = \left(\operatorname{coker}\left(\begin{pmatrix} d^* \\ (-1)^{k+1}(1 + T_\varepsilon)\psi_0 \end{pmatrix} : C^{k-1} \to C^k \oplus C_{k+1}\right), \begin{pmatrix} \psi_0 & d \\ 0 & 0 \end{pmatrix}\right).$$

By Proposition 36 the chain ε-bundle β can be taken to be the cone of a chain ε-bundle map

$$(d, \beta_{-2k-2}) : (B_{k+1}, 0) \to (B_k, \beta_{-2k}),$$

with

$$\begin{aligned}
&\beta^*_{-2k} = (-1)^k \varepsilon \beta_{-2k} : B_k \to B^k, \\
&d^* \beta_{-2k} d = \beta_{-2k-2} + (-1)^k \varepsilon \beta^*_{-2k-2} : B_{k+1} \to B^{k+1}, \\
&\beta_{-2k-1} = \begin{cases} \beta_{-2k} d : B_{k+1} \to B^k, \\ 0 : B_k \to B^{k+1}. \end{cases}
\end{aligned}$$

An ε-symmetric structure $(\phi, \theta) \in Q_{2k+1}(B, \beta, \varepsilon)$ is represented by A-module morphisms

$$\begin{aligned}
&\phi_0 : B^k \to B_{k+1},\ \ \widetilde{\phi}_0 : B^{k+1} \to B_k,\ \ \phi_1 : B^{k+1} \to B_{k+1}, \\
&\theta_0 : B^{k+1} \to B_{k+1},\ \ \theta_{-1} : B^k \to B_{k+1},\ \ \widetilde{\theta}_{-1} : B^{k+1} \to B_k,\ \ \theta_{-2} : B^k \to B_k
\end{aligned}$$

such that

$$\begin{aligned}
&d\phi_0 + (-1)^k \widetilde{\phi}_0 d^* = 0 : B^k \to B_k, \\
&\phi_0 - \varepsilon \widetilde{\phi}^*_0 + (-1)^{k+1} \phi_1 d^* = 0 : B^k \to B_{k+1}, \\
&\phi_1 + (-1)^{k+1} \varepsilon \phi^*_1 = 0 : B^{k+1} \to B_{k+1}, \\
&\phi_0 - \phi_0 \beta_{-2k} d \widetilde{\phi}^*_0 = (-1)^k \theta_0 d^* - \theta_{-1} - \varepsilon \widetilde{\theta}^*_{-1} : B^k \to B_{k+1}, \\
&\widetilde{\phi}_0 = d\theta_0 - \widetilde{\theta}_{-1} - \varepsilon \theta^*_{-1} : B^{k+1} \to B_k, \\
&-\widetilde{\phi}_0 \beta_{-2k-2} \widetilde{\phi}^*_0 = \theta_{-2} + (-1)^{k+1} \varepsilon \theta^*_{-2} : B^k \to B_k, \\
&\phi_1 - \phi_0 \beta_{-2k} \phi^*_0 = \theta_0 + (-1)^k \varepsilon \theta^*_0 : B^{k+1} \to B_{k+1}.
\end{aligned}$$

The boundary of (ϕ, θ) given by 43(i) is an ε-symmetric null-cobordant $2k$-dimensional ε-quadratic Poincaré complex $\partial(\phi, \theta) = (C, \psi)$ concentrated in degrees $k-1, k, k+1$, with $I(C, \psi)$ the instant surgery obstruction form (45) in the statement. □

The ε-quadratic L-groups and the (B, β)-structure L-groups fit into an evident exact sequence

$$\cdots \to L_n(A, \varepsilon) \to L\langle B, \beta\rangle^n(A, \varepsilon) \to \widehat{L}\langle B, \beta\rangle^n(A, \varepsilon) \xrightarrow{\partial} L_{n-1}(A, \varepsilon) \to \cdots$$

and similarly for the 4-periodic versions

$$\cdots \to L_n(A,\varepsilon) \to L\langle B,\beta\rangle^{n+4*}(A,\varepsilon) \to \widehat{L}\langle B,\beta\rangle^{n+4*}(A,\varepsilon) \xrightarrow{\partial} L_{n-1}(A,\varepsilon) \to \cdots .$$

Proposition 46 (*Weiss [21]*). (i) *The function*

$$Q_n(B,\beta,\varepsilon) \to \widehat{L}\langle B,\beta\rangle^{n+4*}(A,\varepsilon);\ (\phi,\theta) \mapsto (f: C \to D, (\delta\phi,\psi)) \quad (42(\text{ii}))$$

is an isomorphism, with inverse given by the algebraic normal invariant. The ε*-quadratic* L*-groups of* A*, the* 4*-periodic* (B,β)*-structure* ε*-symmetric* L*-groups of* A *and the twisted* ε*-quadratic* Q*-groups of* (B,β) *are thus related by an exact sequence*

$$\cdots \to L_n(A,\varepsilon) \xrightarrow{1+T} L\langle B,\beta\rangle^{n+4*}(A,\varepsilon) \to Q_n(B,\beta,\varepsilon) \xrightarrow{\partial} L_{n-1}(A,\varepsilon) \to \cdots .$$

(ii) *The cobordism class of an* n*-dimensional* (ε*-symmetric,* ε*-quadratic*) *Poincaré pair* $(f: C \to D, (\delta\phi,\psi))$ *over* A *with a* (B,β)*-structure* $(\gamma,\phi,\theta,g,\chi)$ *is the image of the algebraic normal invariant* $(\phi,\theta) \in Q_n(\mathcal{C}(f),\gamma,\varepsilon)$

$$(f: C \to D, (\delta\phi,\psi)) = (g,\chi)_\%(\phi,\theta) \in Q_n(B,\beta,\varepsilon).$$

Proof. The ε-symmetrization of an n-dimensional ε-quadratic Poincaré complex (C,ψ) is an n-dimensional ε-symmetric Poincaré complex $(C,(1+T_\varepsilon)\psi)$ with (B,β)-structure $(0,(1+T)\psi,\theta,0,0)$ given by

$$\theta_s = \begin{cases} \psi_{-s-1} \in \mathrm{Hom}_A(C^{-*},C)_{n+s+1} & \text{if } s \leqslant -1, \\ 0 & \text{if } s \geqslant 0. \end{cases}$$

The relative groups of the symmetrization map

$$1+T_\varepsilon : L_n(A,\varepsilon) \to L\langle B,\beta\rangle^n(A,\varepsilon);\ (C,\psi) \mapsto (C,(1+T_\varepsilon)\psi)$$

are the cobordism groups of n-dimensional (ε-symmetric, ε-quadratic) Poincaré pairs $(f: C \to D, (\delta\phi,\psi))$ together with a (B,β)-structure $(\gamma,\phi,\theta,g,\chi)$. □

Proposition 47. *Let* (B,β) *be a chain* ε*-bundle over* A *with* B *concentrated in degree* k

$$B : \cdots \to 0 \to B_k \to 0 \to \cdots$$

so that $\beta \in \widehat{Q}^0(B^{0-*},\varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon)$ *is represented by an element*

$$\beta_{-2k} = (-1)^k \varepsilon \beta^*_{-2k} \in S(B^k).$$

The twisted ε-quadratic Q-groups $Q_n(B,\beta,\varepsilon)$ *are given as follows*:

(i) *For* $n \neq 2k-1, 2k$:

$$\begin{aligned} Q_n(B,\beta,\varepsilon) &= Q_n(B,\varepsilon) \\ &= \begin{cases} \widehat{Q}^{n+1}(B,\varepsilon) = \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) & \text{if } n \geqslant 2k+1, \\ 0 & \text{if } n \leqslant 2k-2. \end{cases} \end{aligned}$$

(ii) For $n = 2k$:

$$Q_{2k}(B,\beta,\varepsilon) = \frac{\{(\phi,\theta) \in S(B^k) \oplus S(B^k) \mid \phi = (-1)^k \varepsilon \phi^*, \phi - \phi\beta_{-2k}\phi^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(0, \eta + (-1)^{k+1}\varepsilon\eta^*) \mid \eta \in S(B^k)\}},$$

with addition by

$$(\phi,\theta) + (\phi',\theta') = (\phi+\phi', \theta+\theta'+\phi'\beta_{-2k}\phi^*).$$

The boundary of $(\phi,\theta) \in Q_{2k}(B,\beta,\varepsilon)$ *is the* $(2k-1)$*-dimensional ε-quadratic Poincaré complex over A concentrated in degrees* $k-1, k$ *corresponding to the* $(-1)^{k+1}\varepsilon$*-quadratic formation over* A,

$$\partial(\phi,\theta) = \left(H_{(-1)^{k+1}\varepsilon}(B^k); B^k, \operatorname{im}\left(\begin{pmatrix} 1-\beta_{-2k}\phi \\ \phi \end{pmatrix} : B^k \to B^k \oplus B_k \right) \right).$$

(iii) *For* $n = 2k-1$:

$$\begin{aligned} Q_{2k-1}(B,\beta,\varepsilon) &= \operatorname{coker}(J_\beta : Q^{2k}(B,\varepsilon) \to \widehat{Q}^{2k}(B,\varepsilon)) \\ &= \frac{\{\sigma \in S(B^k) \mid \sigma = (-1)^k \varepsilon \sigma^*\}}{\{\phi - \phi\beta_{-2k}\phi^* - (\theta + (-1)^k\varepsilon\theta^*) \mid \phi = (-1)^k\varepsilon\phi^*, \theta \in S(B^k)\}}. \end{aligned}$$

The boundary of $\sigma \in Q_{2k-1}(B,\beta,\varepsilon)$ *is the* $(2k-2)$*-dimensional ε-quadratic Poincaré complex over A concentrated in degree* $k-1$ *corresponding to the* $(-1)^{k+1}\varepsilon$*-quadratic form over* A,

$$\partial(\sigma) = \left(B^k \oplus B_k, \begin{pmatrix} \sigma & 1 \\ 0 & \beta_{-2k} \end{pmatrix} \right),$$

with

$$(1 + T_{(-1)^{k+1}\varepsilon})\partial(\sigma) = \left(B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\varepsilon & 0 \end{pmatrix} \right).$$

(iv) *The maps in the exact sequence*

$$\begin{array}{lllll} 0 \to \widehat{Q}^{2k+1}(B,\varepsilon) & \xrightarrow{H_\beta} & Q_{2k}(B,\beta,\varepsilon) & \xrightarrow{N_\beta} & Q^{2k}(B,\varepsilon) \\ & & & \xrightarrow{J_\beta} & \widehat{Q}^{2k}(B,\varepsilon) \xrightarrow{H_\beta} Q_{2k-1}(B,\beta,\varepsilon) \to 0 \end{array}$$

are given by

$$\begin{aligned} &H_\beta : \widehat{Q}^{2k+1}(B,\varepsilon) = \widehat{H}^1(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \to Q_{2k}(B,\beta,\varepsilon);\ \ \theta \mapsto (0,\theta), \\ &N_\beta : Q_{2k}(B,\beta,\varepsilon) \to Q^{2k}(B,\varepsilon) = H^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon);\ \ (\phi,\theta) \mapsto \phi, \\ &J_\beta : Q^{2k}(B,\varepsilon) = H^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \to \widehat{Q}^{2k}(B,\varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon); \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \phi \mapsto \phi - \phi\beta_{-2k}\phi^*, \\ &H_\beta : \widehat{Q}^{2k}(B,\varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(B^k), (-1)^k T_\varepsilon) \to Q_{2k-1}(B,\beta,\varepsilon);\ \ \sigma \mapsto \sigma. \end{aligned}$$

Example 48. Let (K,λ) be a nonsingular ε-symmetric form over A, which may be regarded as a zero-dimensional ε-symmetric Poincaré complex (D,ϕ) over A with

$$\phi_0 = \lambda : D^0 = K \to D_0 = K^*.$$

The composite

$$Q^0(D,\varepsilon) = H^0(\mathbb{Z}_2; S(K),\varepsilon) \xrightarrow{J} \widehat{Q}^0(D,\varepsilon) \xrightarrow{(\phi_0)^{-1}} \widehat{Q}^0(D^{0-*},\varepsilon)$$

sends $\phi \in Q^0(D,\varepsilon)$ to the algebraic Spivak normal chain bundle

$$\gamma \in \widehat{Q}^0(D^{0-*},\varepsilon) = \widehat{H}^0(\mathbb{Z}_2; S(K^*),\varepsilon),$$

with

$$\gamma_0 = \varepsilon\lambda^{-1} : D_0 = K^* \to D^0 = K.$$

By Proposition 47

$$Q_0(D,\gamma,\varepsilon) = \frac{\{(\kappa,\theta) \in S(K) \oplus S(K) \mid \kappa = \varepsilon\kappa^*, \kappa - \kappa\gamma_0\kappa^* = \theta + \varepsilon\theta^*\}}{\{(0,\eta - \varepsilon\eta^*) \mid \eta \in S(K)\}},$$

with addition by

$$(\kappa, \theta) + (\kappa', \theta') = (\kappa + \kappa', \theta + \theta' + \kappa' \gamma_0 \kappa^*).$$

The algebraic normal invariant of (D, ϕ) is given by

$$(\phi, 0) \in Q_0(D, \gamma, \varepsilon).$$

Example 49. Let A be a ring with even involution (4), and let C be concentrated in degree k with $C_k = A^r$. For odd $k = 2j+1$,

$$\widehat{Q}^0(C^{0-*}) = 0$$

and there is only one chain ε-bundle $\gamma = 0$ over C, with

$$Q_n(C, \gamma) = Q_n(C) = \begin{cases} \bigoplus\limits_r \widehat{H}^0(\mathbb{Z}_2; A) & \text{if } n \geqslant 4j+2,\ n \equiv 0 (\text{mod}\, 2), \\ 0 & \text{otherwise.} \end{cases}$$

For even $k = 2j$,

$$\widehat{Q}^0(C^{0-*}) = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A),$$

a chain ε-bundle $\gamma \in \widehat{Q}^0(C^{0-*})$ is represented by a diagonal matrix

$$\gamma = X = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A),$$

with $\overline{x}_i = x_i \in A$, and there is defined an exact sequence

$$\widehat{Q}^{4j+1}(C) = 0 \to Q_{4j}(C, \gamma) \to Q^{4j}(C) \xrightarrow{J_\gamma} \widehat{Q}^{4j}(C) \to Q_{4j-1}(C, \gamma) \to Q^{4j-1}(C) = 0,$$

with

$$J_\gamma : Q^{4j}(C) = \mathrm{Sym}_r(A) \to \widehat{Q}^{4j}(C) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)};\ M \mapsto M - MXM,$$

so that

$$Q_n(C,\gamma) = \begin{cases} \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) & \text{if } n \geqslant 4j+1, \\ & \quad \text{and } n \equiv 1 (\text{mod } 2), \\ \{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\} & \text{if } n = 4j, \\ M_r(A)/\{M - MXM - (N + N^t) \mid M \in \mathrm{Sym}_r(A),\ \ N \in M_r(A)\} & \text{if } n = 4j-1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Proposition 38(ii) gives an exact sequence

$$0 \to \bigoplus_{i=1}^{r} Q_{4j}(B, x_i) \to Q_{4j}(C,\gamma) \to \bigoplus_{r(r-1)/2} A \to \bigoplus_{i=1}^{r} Q_{4j-1}(B, x_i) \to Q_{4j-1}(C,\gamma) \to 0$$

with B concentrated in degree $2j$ with $B_{2j} = A$.

2.4. The relative twisted quadratic Q-groups

Let $(f,\chi) : (C,\gamma) \to (D,\delta)$ be a map of chain ε-bundles, and let (ϕ,θ) be an n-dimensional ε-symmetric structure on (C,γ), so that $\chi \in (\widehat{W}^{\%}C)_1$, $\phi \in (W^{\%}C)_n$ and $\theta \in (\widehat{W}^{\%}C)_{n+1}$. Composing the chain map $\phi_0 : C^{n-*} \to C$ with f, we get an induced map

$$(\widehat{f\phi_0})^{\%} : \widehat{W}^{\%}C^{n-*} \to \widehat{W}^{\%}D.$$

The morphisms of twisted quadratic Q-groups

$$(f,\chi)_{\%} : Q_n(C,\gamma,\varepsilon) \to Q_n(D,\delta,\varepsilon);\ \ (\phi,\theta) \mapsto (f^{\%}(\phi), \widehat{f}^{\%}(\theta) + (\widehat{f\phi_0})^{\%}(S^n\chi))$$

are induced by a simplicial map of simplicial abelian groups. The relative homotopy groups are the *relative twisted ε-quadratic Q-groups* $Q_n(f,\chi,\varepsilon)$, designed to fit into a long exact sequence

$$\cdots \longrightarrow Q_n(C,\gamma,\varepsilon) \xrightarrow{(f,\chi)_{\%}} Q_n(D,\delta,\varepsilon) \longrightarrow Q_n(f,\chi,\varepsilon) \longrightarrow Q_{n-1}(C,\gamma,\varepsilon) \longrightarrow \cdots .$$

Proposition 50. *For any chain ε-bundle map $(f,\chi):(C,\gamma)\to(D,\delta)$ the various Q-groups fit into a commutative diagram with exact rows and columns*

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(C,\epsilon) \arrow[r, "H_\gamma"] \arrow[d, "\widehat{f}^{\%}"] & Q_n(C,\gamma,\epsilon) \arrow[r, "N_\gamma"] \arrow[d, "(f,\chi)_{\%}"] & Q^n(C,\epsilon) \arrow[r, "J_\gamma"] \arrow[d, "f^{\%}"] & \widehat{Q}^n(C,\epsilon) \arrow[r] \arrow[d, "\widehat{f}^{\%}"] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(D,\epsilon) \arrow[r, "H_\delta"] \arrow[d] & Q_n(D,\delta,\epsilon) \arrow[r, "N_\gamma"] \arrow[d] & Q^n(D,\epsilon) \arrow[r, "J_\delta"] \arrow[d] & \widehat{Q}^n(D,\epsilon) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(f) \arrow[r, "H_\chi"] \arrow[d] & Q_n(f,\chi,\epsilon) \arrow[r, "N_\chi"] \arrow[d] & Q^n(f,\epsilon) \arrow[r, "J_\chi"] \arrow[d] & \widehat{Q}^n(f,\epsilon) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^n(C,\epsilon) \arrow[r, "H_\gamma"] \arrow[d] & Q_{n-1}(C,\gamma,\epsilon) \arrow[r, "N_\delta"] \arrow[d] & Q^{n-1}(C,\epsilon) \arrow[r, "J_\gamma"] \arrow[d] & \widehat{Q}^{n-1}(C,\epsilon) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Proof. These are the exact sequences of the homotopy groups of the simplicial abelian groups in the commutative diagram of fibration sequences

$$
\begin{tikzcd}
K(J_\gamma) \arrow[r] \arrow[d] & K(W^{\%}C) \arrow[r, "J_\gamma"] \arrow[d, "f^{\%}"] & K(\widehat{W}^{\%}C) \arrow[d, "\widehat{f}^{\%}"] \\
K(J_\delta) \arrow[r] \arrow[d] & K(W^{\%}D) \arrow[r, "J_\delta"] \arrow[d] & K(\widehat{W}^{\%}D) \arrow[d] \\
K(J_\chi) \arrow[r] & K(W^{\%}\mathcal{C}(f)) \arrow[r, "J_\chi"] & K(\widehat{W}^{\%}\mathcal{C}(f))
\end{tikzcd}
$$

with

$$\pi_n(K(J_\chi)) = Q_n(f,\chi,\varepsilon). \qquad \square$$

There is also a twisted ε-quadratic Q-group version of the algebraic Thom constructions (12, 18, 24):

Proposition 51. *Let $(f,\chi):(C,0)\to(D,\delta)$ be a chain ε-bundle map, and let $(B,\beta)=\mathcal{C}(f,\chi)$ be the cone chain ε-bundle* (34). *The relative twisted ε-quadratic Q-groups*

$Q_*(f, \chi, \varepsilon)$ are related to the (absolute) twisted ε-quadratic Q-groups $Q_*(B, \beta, \varepsilon)$ by a commutative braid of exact sequences

$$\begin{tikzcd}
\widehat{Q}^{n+1}(B,\epsilon) \arrow[dr] \arrow[rr, bend left] & & Q_n(B,\beta,\epsilon) \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(B\otimes_A C) \\
& Q_n(f,\chi,\epsilon) \arrow[ur, "t"] \arrow[dr] & & Q^n(B,\epsilon) \arrow[ur] \arrow[dr] & \\
H_n(B\otimes_A C) \arrow[ur, "F"] \arrow[rr, bend right, "F"'] & & Q^n(f,\epsilon) \arrow[ur, "t"] \arrow[rr, bend right] & & \widehat{Q}^n(B,\epsilon)
\end{tikzcd}$$

involving the exact sequence of 18,

$$\cdots \to H_n(B\otimes_A C) \xrightarrow{\ F\ } Q^n(f,\varepsilon) \xrightarrow{\ t\ } Q^n(B,\varepsilon) \to H_{n-1}(B\otimes_A C) \to \cdots .$$

Proof. The identity

$$\widehat{f}^{*\%}(\delta) = d\chi \in (\widehat{W}C^{0-*})_0$$

determines a homotopy $\rightsquigarrow$ in the square

$$\begin{tikzcd}
K(W^{\%}C) \arrow[r, "J"] \arrow[d, "f^{\%}"] \arrow[dr, squiggly] & K(\widehat{W}^{\%}C) \arrow[d, "\widehat{f}^{\%}"] \\
K(W^{\%}D) \arrow[r, "J_\delta"] & K(\widehat{W}^{\%}D)
\end{tikzcd}$$

(with $J = J_0$) and hence maps of the mapping fibres

$$J_\chi : K(\mathcal{C}(f^{\%})) \to K(\mathcal{C}(\widehat{f}^{\%})),\ \ (f,\chi)_{\%} : K(J) \to K(J_\delta).$$

The map J_χ is related to $J_\beta : K(W^{\%}B) \to K(\widehat{W}^{\%}B)$ by a homotopy commutative diagram

$$\begin{tikzcd}
K(\mathcal{C}(f^{\%})) \arrow[r, "J_\chi"] \arrow[d, "t"] \arrow[dr, squiggly] & K(\mathcal{C}(\widehat{f}^{\%})) \arrow[d, "\widehat{t}", "\simeq"'] \\
K(W^{\%}B) \arrow[r, "J_\beta"] & K(\widehat{W}^{\%}B)
\end{tikzcd}$$

with $\widehat{t} : K(\mathcal{C}(\widehat{f}^{\%})) \simeq K(\widehat{W}^{\%}B)$ a simplicial homotopy equivalence inducing the algebraic Thom isomorphisms $\widehat{t} : \widehat{Q}^*(f,\varepsilon) \cong \widehat{Q}^*(B,\varepsilon)$ of Proposition 12, and t :

$K(\mathcal{C}(f^{\%})) \to K(W^{\%}B)$ a simplicial map inducing the algebraic Thom maps $t : Q^*(f,\varepsilon) \to Q^*(B,\varepsilon)$ of Proposition 18, with mapping fibre $K(t) \simeq K(B \otimes_A C)$. The braid in the statement is the commutative braid of homotopy groups induced by the homotopy commutative braid of fibrations

$$\begin{tikzcd}[column sep=small]
 & & K(J_\beta) \arrow[dr] & & \\
 & K(J_\chi) \arrow[ur] \arrow[rr, squiggly] \arrow[dr] & & K(W^{\%}B) \arrow[dr, "J_\beta"] & \\
K(B \otimes_A C) \arrow[ur, "F"] \arrow[rr, bend right, "F"'] & & K(\mathcal{C}(f^{\%})) \arrow[ur, "t"] \arrow[rr, bend right, "\widehat{t}J_\chi"'] & & K(\widehat{W}^{\%}B)
\end{tikzcd}$$

□

Proposition 52. *Let (C,γ) be a chain ε-bundle over a f.g. projective A-module chain complex which is concentrated in degrees $k, k+1$,*

$$C : \cdots \to 0 \to C_{k+1} \xrightarrow{\quad d \quad} C_k \to 0 \to \cdots\ ,$$

so that (C,γ) can be taken (up to equivalence) to be the cone $\mathcal{C}(d,\chi)$ of a chain ε-bundle map $(d,\chi) : (C_{k+1}, 0) \to (C_k, \delta)$ (36), regarding C_k, C_{k+1} as chain complexes concentrated in degree k. The relative twisted ε-quadratic Q-groups $Q_*(d,\chi,\varepsilon)$ and the absolute twisted ε-quadratic Q-groups $Q_*(C,\gamma,\varepsilon)$ are given as follows:

(i) *For $n \neq 2k-1, 2k, 2k+1, 2k+2$*

$$Q_n(C,\gamma,\varepsilon) = Q_n(d,\chi,\varepsilon) = Q_n(C,\varepsilon) = \begin{cases} \widehat{Q}^{n+1}(C,\varepsilon) & \text{if } n \geqslant 2k+3, \\ 0 & \text{if } n \leqslant 2k-2, \end{cases}$$

with

$$\widehat{Q}^{n+1}(C,\varepsilon) = \frac{\{(\phi,\theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi = (-1)^{n+k}\varepsilon\phi^*, d\phi d^* = \theta + (-1)^{n+k}\varepsilon\theta^*\}}{\{(\sigma + (-1)^{n+k}\varepsilon\sigma^*, d\sigma d^* + \tau + (-1)^{n+k+1}\varepsilon\tau^* \mid (\sigma,\tau) \in S(C^{k+1}) \oplus S(C^k)\}}$$

as given by Proposition 13.

(ii) *For $n = 2k-1, 2k, 2k+1, 2k+2$ the relative twisted ε-quadratic Q-groups are given by*

$$Q_n(d,\chi,\varepsilon)$$
$$= \begin{cases} \dfrac{\{(\phi,\theta) \in S(C^{k+1}) \oplus S(C^k) \mid \phi = (-1)^k\varepsilon\phi^*, d\phi d^* = \theta + (-1)^k\varepsilon\theta^*\}}{\{(\sigma + (-1)^k\varepsilon\sigma^*, d\sigma d^* + \tau + (-1)^{k+1}\varepsilon\tau^* \mid (\sigma,\tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+2, \\[2ex] \dfrac{\{(\psi,\eta) \in S(C^{k+1}) \oplus S(C^k) \mid (d,\chi)_{\%}(\psi) = (0, \eta + (-1)^{k+1}\varepsilon\eta^*)\}}{\{(\sigma + (-1)^{k+1}\varepsilon\sigma^*, d\sigma d^* + \tau + (-1)^k\varepsilon\tau^*) \mid (\sigma,\tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+1, \\[2ex] \operatorname{coker}((d,\chi)_{\%} : Q_{2k}(C_{k+1},\varepsilon) \to Q_{2k}(C_k,\delta,\varepsilon)) & \text{if } n = 2k, \\ Q_{2k-1}(C_k,\delta,\varepsilon) & \text{if } n = 2k-1, \end{cases}$$

with

$$Q_{2k}(C_k,\delta,\varepsilon)=\frac{\{(\phi,\theta)\in S(C^k)\oplus S(C^k)\mid \phi=(-1)^k\varepsilon\phi^*,\ \phi-\phi\delta\phi^*=\theta+(-1)^k\varepsilon\theta^*\}}{\{(0,\eta+(-1)^{k+1}\varepsilon\eta^*)\mid \eta\in S(C^k)\}},$$

$$Q_{2k-1}(C_k,\delta,\varepsilon)=\frac{\{\sigma\in S(C^k)\mid \sigma=(-1)^k\varepsilon\sigma^*\}}{\{\phi-\phi\delta\phi^*-(\theta+(-1)^k\varepsilon\theta^*)\mid \phi=(-1)^k\varepsilon\phi^*,\ \theta\in S(C^k)\}},$$

$$(d,\chi)_\% : Q_{2k}(C_{k+1},\varepsilon)=H_0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\varepsilon)\to Q_{2k}(C_k,\delta,\varepsilon);$$

$$\psi\mapsto(d(\psi+(-1)^k\varepsilon\psi^*)d^*,\ d\psi d^*-d(\psi+(-1)^k\varepsilon\psi^*)\chi(\psi^*+(-1)^k\varepsilon\psi)d^*).$$

The absolute twisted quadratic Q-groups are such that

$$Q_{2k-1}(C,\gamma,\varepsilon)=Q_{2k-1}(d,\chi,\varepsilon)=Q_{2k-1}(C_k,\delta,\varepsilon)$$

and there is defined an exact sequence

$$\begin{aligned}0\to\ & Q_{2k+2}(d,\chi,\varepsilon)\xrightarrow{t} Q_{2k+2}(C,\gamma,\varepsilon)\\ \to\ & H_{k+1}(C)\otimes_A C_{k+1}\xrightarrow{F} Q_{2k+1}(d,\chi,\varepsilon)\xrightarrow{t} Q_{2k+1}(C,\gamma,\varepsilon)\\ \to\ & H_k(C)\otimes_A C_{k+1}\xrightarrow{F} Q_{2k}(d,\chi,\varepsilon)\xrightarrow{t} Q_{2k}(C,\gamma,\varepsilon)\to 0,\end{aligned}$$

with

$$\begin{aligned}&F: H_k(C)\otimes_A C_{k+1}=\mathrm{coker}(d^*:\mathrm{Hom}_A(C^{k+1},C_{k+1})\to\mathrm{Hom}_A(C^k,C_{k+1}))\\ &\to Q_{2k}(d,\chi);\ \ \lambda\mapsto(\lambda d^*+(-1)^k\varepsilon d\lambda^*-d\lambda^*\delta\lambda d^*,\\ &\lambda d^*-\lambda\chi\lambda^*-d\lambda^*\delta\lambda\chi\lambda^*\delta\lambda d^*-d\lambda^*\delta(\lambda d^*+(-1)^k\varepsilon d\lambda^*)\\ &-(\lambda d^*+(-1)^k\varepsilon d\lambda^*)\delta d\lambda^*\delta\lambda d^*).\end{aligned}$$

Proof. The absolute and relative twisted ε-quadratic Q-groups are related by the exact sequence of 51

$$\cdots\to Q_n(d,\chi,\varepsilon)\xrightarrow{t} Q_n(C,\gamma,\varepsilon)\to H_{n-k-1}(C)\otimes_A C_{k+1}\xrightarrow{F} Q_{n-1}(d,\chi,\varepsilon)\to\cdots .$$

The twisted ε-quadratic Q-groups of $(C_{k+1}, 0)$ are given by Proposition 22

$$Q_n(C_{k+1}, 0, \varepsilon) = Q_n(C_{k+1}, \varepsilon) = H_{n-2k}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon)$$
$$= \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & \text{if } n \geqslant 2k+1, \\ H_0(\mathbb{Z}_2; S(C^{k+1}), (-1)^k T_\varepsilon) & \text{if } n = 2k, \\ 0 & \text{if } n \leqslant 2k-1. \end{cases}$$

The twisted ε-quadratic Q-groups of (C_k, δ) are given by Proposition 47

$$Q_n(C_k, \delta, \varepsilon)$$
$$= \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2; S(C^k), (-1)^k T_\varepsilon) & \text{if } n \geqslant 2k+1, \\ \dfrac{\{(\phi, \theta) \in S(C^k) \oplus S(C^k) \mid \phi = (-1)^k \varepsilon \phi^*, \phi - \phi\delta\phi^* = \theta + (-1)^k \varepsilon \theta^*\}}{\{(0, \eta + (-1)^{k+1} \varepsilon \eta^*) \mid \eta \in S(C^k)\}} & \text{if } n = 2k, \\ \dfrac{\{\sigma \in S(C^k) \mid \sigma = (-1)^k \varepsilon \sigma^*\}}{\{\phi - \phi\delta\phi^* - (\theta + (-1)^k \varepsilon \theta^*) \mid \phi = (-1)^k \varepsilon \phi^*, \theta \in S(C^k)\}} & \text{if } n = 2k-1, \\ 0 & \text{if } n \leqslant 2k-2. \end{cases}$$

The twisted ε-quadratic Q-groups of (d, χ) fit into the exact sequence

$$\cdots \longrightarrow Q_n(C_{k+1}, \varepsilon) \xrightarrow{(d,\chi)_\%} Q_n(C_k, \delta, \varepsilon) \longrightarrow Q_n(d, \chi, \varepsilon) \longrightarrow Q_{n-1}(C_{k+1}, \varepsilon) \longrightarrow \cdots$$

giving the expressions in the statements of (i) and (ii). □

2.5. The computation of $Q_*(C(X), \gamma(X))$

In this section, we compute the twisted quadratic Q-groups $Q_*(C(X), \gamma(X))$ of the following chain bundles over an even commutative ring A.

Definition 53. For $X \in \mathrm{Sym}_r(A)$ let

$$(C(X), \gamma(X)) = \mathcal{C}(d, \chi)$$

be the cone of the chain bundle map over A,

$$(d, \chi) : (C(X)_1, 0) \to (C(X)_0, \delta)$$

defined by

$$d = 2 : C(X)_1 = A^r \to C(X)_0 = A^r,$$
$$\delta = X : C(X)_0 = A^r \to C(X)^0 = A^r,$$
$$\chi = 2X : C(X)_1 = A^r \to C(X)^1 = A^r.$$

By Proposition 36 every chain bundle (C,γ) with $C_1 = A^r \xrightarrow{2} C_0 = A^r$ is of the form $(C(X),\gamma(X))$ for some $X = (x_{ij}) \in \mathrm{Sym}_r(A)$, with the equivalence class given by

$$\gamma = \gamma(X) = X = (x_{11}, x_{22}, \ldots, x_{rr})$$
$$\in \widehat{Q}^0(C(X)^{-*}) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} = \bigoplus_r \widehat{H}^0(\mathbb{Z}_2; A) \ (14).$$

The 0th Wu class of $(C(X),\gamma(X))$ is the A-module morphism

$$\widehat{v}_0(\gamma(X)) : H_0(C(X)) = (A_2)^r \to \widehat{H}^0(\mathbb{Z}_2; A);$$
$$a = (a_1, a_2, \ldots, a_r) \mapsto aXa^t = \sum_{i=1}^r a_i x_{ij} a_j \ = \sum_{i=1}^r (a_i)^2 x_{ii}.$$

In Theorem 60 below the universal chain bundle (B^A, β^A) of a commutative even ring A with $\widehat{H}^0(\mathbb{Z}_2; A)$ a f.g. free A_2-module will be constructed from $(C(X),\gamma(X))$ for a diagonal $X \in \mathrm{Sym}_r(A)$ with $\widehat{v}_0(\gamma(X))$ an isomorphism, and the twisted quadratic Q-groups $Q_*(B^A,\beta^A)$ will be computed using the following computation of $Q_*(C(X),\gamma(X))$ (which holds for arbitrary X).

Theorem 54. *Let A be an even commutative ring, and let $X \in \mathrm{Sym}_r(A)$.*
(i) *The twisted quadratic Q-groups of $(C(X),\gamma(X))$ are given by*

$$Q_n(C(X),\gamma(X))$$
$$= \begin{cases} 0 & \text{if } n \leqslant -2, \\[1ex] \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{M - MXM \mid M \in \mathrm{Sym}_r(A)\}} & \text{if } n = -1, \\[1ex] \dfrac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N+N^t) - 4N^tXN \mid N \in M_r(A)\}} & \text{if } n = 0, \\[1ex] \dfrac{\{N \in M_r(A) \mid N + N^t - 2N^tXN \in 2\mathrm{Quad}_r(A)\}}{2M_r(A)} \oplus \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} & \text{if } n = 1, \\[1ex] \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} & \text{if } n \geqslant 2. \end{cases}$$

(ii) *The boundary maps* $\partial : Q_n(C(X), \gamma(X)) \to L_{n-1}(A)$ *are given by*

$$\partial : Q_{-1}(C(X), \gamma(X)) \to L_{-2}(A);\quad M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix}\right),$$

$$\partial : Q_0(C(X), \gamma(X)) \to L_{-1}(A);\quad M \mapsto (H_-(A^r); A^r, \mathrm{im}\begin{pmatrix} 1-XM \\ M \end{pmatrix} : A^r \to A^r \oplus (A^r)^*)),$$

$$\partial : Q_1(C(X), \gamma(X)) \to L_0(A);\quad (N, P) \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^t X N) & 1-2NX \\ 0 & -2X \end{pmatrix}\right).$$

(iii) *The twisted quadratic Q-groups of the chain bundles*

$$(B(i), \beta(i)) = (C(X), \gamma(X))_{*+2i} \quad (i \in \mathbb{Z})$$

are just the twisted quadratic Q-groups of $(C(X), \gamma(X))$ *with a dimension shift*

$$Q_n(B(i), \beta(i)) = Q_{n-4i}(C(X), \gamma(X)).$$

Proof. (i) Proposition 52(i) and Example 14(ii) give

$$Q_n(C(X), \gamma(X)) = \begin{cases} 0 & \text{if } n \leqslant -2, \\ \widehat{Q}^{n+1}(C(X)) = \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} & \text{if } n \geqslant 3. \end{cases}$$

For $-1 \leqslant n \leqslant 2$ Examples 14, 20, 49 and Proposition 52(ii) show that the commutative diagram with exact rows and columns

$$\begin{tikzcd}
Q^2(d) \arrow[r, "J_\chi"] \arrow[d] & \widehat{Q}^2(d) \arrow[r, "H_\chi"] \arrow[d] & Q_1(d,\chi) \arrow[r, "N_\chi"] \arrow[d] & Q^1(d) \arrow[r, "J_\chi"] \arrow[d] & \widehat{Q}^1(d) \arrow[d] \\
Q^1(C(X)_1) \arrow[r, "J"] \arrow[d, "d^\%"] & \widehat{Q}^1(C(X)_1) \arrow[r, "H"] \arrow[d, "\widehat{d}^\%"] & Q_0(C(X)_1) \arrow[r, "1+T"] \arrow[d, "(d,\chi)_\%"] & Q^0(C(X)_1) \arrow[r, "J"] \arrow[d, "d^\%"] & \widehat{Q}^0(C(X)_1) \arrow[d, "\widehat{d}^\%"] \\
Q^1(C(X)_0) \arrow[r, "J_\delta"] \arrow[d] & \widehat{Q}^1(C(X)_0) \arrow[r, "H_\delta"] \arrow[d] & Q_0(C(X)_0, \delta) \arrow[r, "N_\delta"] \arrow[d] & Q^0(C(X)_0) \arrow[r, "J_\delta"] \arrow[d] & \widehat{Q}^0(C(X)_0) \arrow[d] \\
Q^1(d) \arrow[r, "J_\chi"] \arrow[d] & \widehat{Q}^1(d) \arrow[r, "H_\chi"] \arrow[d] & Q_0(d,\chi) \arrow[r, "N_\chi"] \arrow[d] & Q^0(d) \arrow[r, "J_\chi"] \arrow[d] & \widehat{Q}^0(d) \arrow[d] \\
Q^0(C(X)_1) \arrow[r, "J"] & \widehat{Q}^0(C(X)_1) \arrow[r, "H"] & Q_{-1}(C(X)_1) \arrow[r, "1+T"] & Q^{-1}(C(X)_1) \arrow[r, "J"] & \widehat{Q}^{-1}(C(X)_1)
\end{tikzcd}$$

is given by

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[r, "1"] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[d, "1"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{Quad}_r(A) \arrow[r] \arrow[d, "4"] & \mathrm{Sym}_r(A) \arrow[r] \arrow[d, "4"] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[d, "0"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Q_0(C(X)_0, \delta) \arrow[r] \arrow[d] & \mathrm{Sym}_r(A) \arrow[r, "J_\delta"] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[d, "1"] \\
0 \arrow[r] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[r, "4"] \arrow[d, "1"] & Q_0(d, \chi) \arrow[r] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} \arrow[r] \arrow[d] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[d] \\
\mathrm{Sym}_r(A) \arrow[r] & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

with

$$
\begin{aligned}
& J_\delta : \mathrm{Sym}_r(A) \to \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}; \quad M \mapsto M - MXM, \\
& Q_0(C(X)_0, \delta) = \ker(J_\delta) = \{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}, \\
& Q_0(d, \chi) = \mathrm{coker}((d, \chi)_\% : Q_0(C(X)_1) \to Q_0(C(X)_0, \delta)) \\
& \qquad\qquad = \frac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A)}, \\
& N_\chi : Q_0(d, \chi) \to Q^0(d) = \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)}; \quad M \mapsto M.
\end{aligned}
$$

Furthermore, the commutative braid of exact sequences

$$
\begin{tikzcd}[column sep=tiny]
& Q^1(d) \arrow[dr] \arrow[rr, bend left] & & \widehat{Q}^1(C(X)) \arrow[dr] \arrow[rr, bend left] & & Q_0(C(X), \gamma(X)) \arrow[dr] \arrow[rr, bend left] & & H_{-1}(C(X) \otimes_A C(X)_1) \\
Q_1(d, \chi) \arrow[ur] \arrow[dr] & & Q^1(C(X)) \arrow[ur, "J_{\gamma(X)}"] \arrow[dr] & & Q_0(d, \chi) \arrow[ur] \arrow[dr] & & Q^0(C(X)) \arrow[ur] \arrow[dr, "J_{\gamma(X)}"] & \\
& Q_1(C(X), \gamma(X)) \arrow[ur] \arrow[rr, bend right] & & H_0(C(X) \otimes_A C(X)_1) \arrow[ur, "F"] \arrow[rr, bend right] & & Q^0(d) \arrow[ur] \arrow[rr, bend right] & & \widehat{Q}^0(C(X))
\end{tikzcd}
$$

is given by

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dr] \arrow[rr, bend left] & & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[dr] \arrow[rr, bend left] & & Q_0(C(X),\gamma(X)) \arrow[dr] \arrow[rr, bend left] & & 0 \\
\dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \arrow[ur] \arrow[dr] & & Q^1(C(X)) \arrow[ur, "J_{\gamma(X)}"] \arrow[dr] & & Q_0(d,\chi) \arrow[ur] \arrow[dr, "N_\chi"] & & \dfrac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \arrow[ur] \arrow[dr, "J_{\gamma(X)}"] & \\
 & Q_1(C(X),\gamma(X)) \arrow[ur] \arrow[rr, bend right] & & \dfrac{M_r(A)}{2M_r(A)} \arrow[ur, "F"] \arrow[rr, bend right] & & \dfrac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} \arrow[ur] \arrow[rr, bend right] & & \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}
\end{tikzcd}$$

with

$$\begin{aligned}
&\frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \cong Q^0(C(X));\ \ M \mapsto \phi \ \text{(where } \phi_0 = M : C^0 \to C(X)_0\text{)},\\
&J_{\gamma(X)} : \frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \to \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)};\ \ M \mapsto M - MXM,\\
&F : H_0(C(X) \otimes_A C(X)_1) = \frac{M_r(A)}{2M_r(A)} \to Q_0(d,\chi);\ \ N \mapsto 2(N + N^t) - 4N^t XN,\\
&Q^1(C(X)) = \ker(N_\chi F : H_0(C(X) \otimes_A C(X)_1) \to Q^0(d))\\
&\qquad\qquad\quad = \frac{\{N \in M_r(A) \mid N + N^t \in 2\mathrm{Sym}_r(A)\}}{2M_r(A)}\\
&\qquad\qquad\quad\ \ \text{(where } \phi \in Q^1(C(X)) \text{ corresponds to } N = \phi_0 \in M_r(A)\text{)},\\
&J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X)) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)};\ \ N \mapsto \frac{1}{2}(N + N^t) - N^t XN.
\end{aligned}$$

It follows that

$$\begin{aligned}
Q_0(C(X),\gamma(X)) &= \mathrm{coker}\left(F : \frac{M_r(A)}{2M_r(A)} \to Q_0(d,\chi)\right)\\
&= \frac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N + N^t) - 4N^t XN \mid N \in M_r(A)\}},\\
Q_{-1}(C(X),\gamma(X)) &= Q_{-1}(d,\chi)\\
&= \mathrm{coker}(J_{\gamma(X)} : Q^0(C(X)) \to \widehat{Q}^0(C(X)))\\
&= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{M - MXM \mid M \in \mathrm{Sym}_r(A)\}},
\end{aligned}$$

with

$$\widehat{Q}^1(C(X)) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \to Q_0(C(X),\gamma(X));\ \ M \mapsto 4M,$$
$$\widehat{Q}^0(C(X)) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \to Q_{-1}(C(X),\gamma(X));\ \ M \mapsto M.$$

Also

$$\begin{aligned}
&(d,\chi)_\% = 0 : Q_2(d,\chi) = Q_1(C(X)_1) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \\
&\qquad\qquad\qquad\qquad \to Q_1(C(X)_0,\delta) = \widehat{Q}^2(C(X)_0) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}, \\
&Q^1(C(X)) = \ker(N_\chi F : H_0(C(X)\otimes_A C(X)_1) \to Q^0(d)) \\
&\qquad\qquad = \frac{\{N \in M_r(A) \mid N + N^t \in 2\mathrm{Sym}_r(A)\}}{2M_r(A)}, \\
&J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X)) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)};\ \ N \mapsto \frac{1}{2}(N+N^t) - N^tXN, \\
&Q_2(C(X),\gamma(X)) = Q_2(d,\chi) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}.
\end{aligned}$$

From the definition, an element $(\phi,\theta) \in Q_1(C(X),\gamma(X))$ is represented by a collection of A-module morphisms

$$\begin{aligned}
&\phi_0 : C(X)^0 \to C(X)_1,\ \ \widetilde{\phi}_0 : C(X)^1 \to C(X)_0,\ \ \phi_1 : C(X)^1 \to C(X)_0, \\
&\theta_0 : C(X)^1 \to C(X)_1,\ \ \theta_{-1} : C(X)^0 \to C(X)_1,\ \ \widetilde{\theta}_{-1} : C(X)^1 \to C(X)_0, \\
&\theta_{-2} : C(X)^0 \to C(X)_0
\end{aligned}$$

such that

$$\begin{aligned}
&d\phi_0 + \widetilde{\phi}_0 d^* = 0 : C^0 \to C_1, \\
&\phi_0 - \widetilde{\phi}_0^* + \phi_1 d^* = 0 : C(X)^0 \to C(X)_1, \\
&\widetilde{\phi}_0 - \phi_0^* - d\phi_1 = 0 : C(X)^1 \to C(X)_0, \\
&\phi_1 - \phi_1^* = 0 : C(X)^1 \to C(X)_1, \\
&\phi_0 - \phi_0\gamma(X)_{-1}\widetilde{\phi}_0^* = -\theta_0 d^* - \theta_{-1} - \widetilde{\theta}_{-1}^* : C(X)^0 \to C(X)_1, \\
&\widetilde{\phi}_0 - \widetilde{\phi}_0\widetilde{\gamma}(X)_{-1}\phi_0^* = d\theta_0 - \theta_{-1}^* - \widetilde{\theta}_{-1} : C(X)^1 \to C(X)_0, \\
&\phi_1 - \phi_0\gamma(X)_0\phi_0^* = \theta_0 + \theta_0^* : C(X)^1 \to C(X)_1, \\
&-\widetilde{\phi}_0\gamma(X)_{-2}\widetilde{\phi}_0^* = d\theta_{-1} + \widetilde{\theta}_{-1}d^* + \theta_{-2} - \theta_{-2}^* : C(X)^0 \to C(X)_0,
\end{aligned}$$

where

$$\gamma(X)_0 = X,\ \ \gamma(X)_{-1} = 0,\ \ \widetilde{\gamma}(X)_{-1} = -2X,\ \ \gamma(X)_{-2} = 0.$$

The maps in the exact sequence

$$\begin{aligned}
&0 \to \widehat{Q}^2(C(X)) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \to Q_1(C(X), \gamma(X)) \\
&\to \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))) \\
&= \frac{\{N \in M_r(A) \mid N + N^t - 2N^t X N \in 2\mathrm{Quad}_r(A)\}}{2M_r(A)} \to 0
\end{aligned}$$

are defined by

$$\begin{aligned}
&Q_1(C(X), \gamma(X)) \to \ker(J_{\gamma(X)});\ \ (\phi, \theta) \mapsto N = \phi_0, \\
&\widehat{Q}^2(C(X)) \to Q_1(C(X), \gamma(X));\ \ \theta_{-2} \mapsto (0, \theta)\ \ \ (\theta_0 = 0,\ \ \theta_{-1} = 0,\ \ \widetilde{\theta}_{-1} = 0),
\end{aligned}$$

with $\widehat{Q}^2(C(X)) \to Q_1(C(X), \gamma(X))$ split by

$$Q_1(C(X), \gamma(X)) \to \widehat{Q}^2(C(X));\ \ (\phi, \theta) \mapsto \theta_{-2}.$$

(ii) The expressions for $\partial : Q_n(C(X), \gamma(X)) \to L_{n-1}(A)$ are given by the boundary construction of Proposition 43 and its expression in terms of forms and formations (44, 45). The form in the case $n = -1$ (resp. the formation in the case $n = 0$) is given by 45 (resp. 44) applied to the n-dimensional symmetric structure $(\phi, \theta) \in Q_n(C(X), \gamma(X))$ corresponding to $M \in \mathrm{Sym}_r(A)$. For $n = 1$ the boundary of the one-dimensional symmetric structure $(\phi, \theta) \in Q_1(C(X), \gamma(X))$ corresponding to $N \in M_r(A)$ with

$$N + N^t \in 2\,\mathrm{Sym}_r(A),\ \ \tfrac{1}{2}(N + N^t) - N^t X N \in \mathrm{Quad}_r(A)$$

is a zero-dimensional quadratic Poincaré complex (C, ψ) with

$$C = \mathcal{C}(N : C(X)^{1-*} \to C(X))_{*+1}.$$

The instant surgery obstruction (45) is the nonsingular quadratic form

$$\begin{aligned} & I(C,\psi) \\ & = \left(\operatorname{coker}\left(\begin{pmatrix} -2 \\ N^t \\ 1+2XN^t \end{pmatrix} : A^r \to A^r \oplus A^r \oplus A^r\right),\right. \\ & \qquad \left.\begin{pmatrix} \frac{1}{4}(N+N^t-2NXN^t) & 1 & N \\ 0 & -2X & 2 \\ 0 & 0 & 0 \end{pmatrix}\right), \end{aligned}$$

such that there is defined an isomorphism

$$\begin{pmatrix} 1 & -4X & 2 \\ N^t & 1-2N^tX & N^t \end{pmatrix} : I(C,\psi) \to \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N+N^t-2N^tXN) & 1-2NX \\ 0 & -2X \end{pmatrix}\right).$$

(iii) *The even multiple skew-suspension isomorphisms of the symmetric Q-groups*

$$\overline{S}^{2i} : Q^{n-4i}(C(X)_{*+2i}) \xrightarrow{\cong} Q^n(C(X));\ \ \{\phi_s \mid s \geqslant 0\} \mapsto \{\phi_s \mid s \geqslant 0\} \quad (i \in \mathbb{Z})$$

are defined also for the hyperquadratic, quadratic and twisted quadratic Q-groups. $\square$

2.6. The universal chain bundle

For any A-module chain complexes B, C the additive group $H_0(\mathrm{Hom}_A(C,B))$ consists of the chain homotopy classes of A-module chain maps $f : C \to B$. For a chain ε-bundle (B,β) there is thus defined a morphism

$$H_0(\mathrm{Hom}_A(C,B)) \to \widehat{Q}^0(C^{0-*},\varepsilon);\ \ (f : C \to B) \mapsto \widehat{f^*}(\beta).$$

Proposition 55 (*Weiss [21]*). (i) *For every ring with involution A and $\varepsilon = \pm 1$ there exists a universal chain ε-bundle $(B^{A,\varepsilon}, \beta^{A,\varepsilon})$ over A such that for any finite f.g. projective A-module chain complex C the morphism*

$$H_0(\mathrm{Hom}_A(C, B^{A,\varepsilon})) \to \widehat{Q}^0(C^{0-*},\varepsilon);\ \ (f : C \to B^{A,\varepsilon}) \mapsto \widehat{f^*}(\beta^{A,\varepsilon})$$

is an isomorphism. Thus every chain ε-bundle (C,γ) is classified by a chain ε-bundle map

$$(f,\chi) : (C,\gamma) \to (B^{A,\varepsilon}, \beta^{A,\varepsilon}).$$

(ii) *The universal chain* ε*-bundle* $(B^{A,\varepsilon},\beta^{A,\varepsilon})$ *is characterized* (*uniquely up to equivalence*) *by the property that its Wu classes are A-module isomorphisms*

$$\widehat{v}_k(\beta^{A,\varepsilon}) : H_k(B^{A,\varepsilon}) \xrightarrow{\cong} \widehat{H}^k(\mathbb{Z}_2; A,\varepsilon) \quad (k\in\mathbb{Z}).$$

(iii) *An n-dimensional* (ε*-symmetric*, ε*-quadratic*) *Poincaré pair over A has a canonical universal* ε*-bundle* $(B^{A,\varepsilon},\beta^{A,\varepsilon})$*-structure.*

(iv) *The* 4*-periodic* $(B^{A,\varepsilon},\beta^{A,\varepsilon})$*-structure L-groups are the* 4*-periodic versions of the* ε*-symmetric and* ε*-hyperquadratic L-groups of A*:

$$L\langle B^{A,\varepsilon},\beta^{A,\varepsilon}\rangle^{n+4*}(A,\varepsilon) = L^{n+4*}(A,\varepsilon),$$

$$\widehat{L}\langle B^{A,\varepsilon},\beta^{A,\varepsilon}\rangle^{n+4*}(A,\varepsilon) = \widehat{L}^{n+4*}(A,\varepsilon).$$

(v) *The twisted* ε*-quadratic Q-groups of* $(B^{A,\varepsilon},\beta^{A,\varepsilon})$ *fit into an exact sequence*

$$\cdots \to L_n(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L^{n+4*}(A,\varepsilon) \to Q_n(B^{A,\varepsilon},\beta^{A,\varepsilon},\varepsilon) \xrightarrow{\partial} L_{n-1}(A,\varepsilon) \to \cdots,$$

with

$$\partial : Q_n(B^{A,\varepsilon},\beta^{A,\varepsilon},\varepsilon) \to L_{n-1}(A,\varepsilon);\ \ (\phi,\theta)\mapsto (C,\psi)$$

given by the construction of Proposition 42(ii), *with*

$$C = \mathcal{C}(\phi_0 : (B^{A,\varepsilon})^{n-*} \to B^{A,\varepsilon})_{*+1} \ \textit{etc.}$$

For $\varepsilon = 1$ write

$$(B^{A,1},\beta^{A,1}) = (B^A,\beta^A)$$

and note that

$$(B^{A,-1},\beta^{A,-1}) = (B^A,\beta^A)_{*-1}.$$

In general, the chain A-modules $B^{A,\varepsilon}$ are not finitely generated, although $B^{A,\varepsilon}$ is a direct limit of f.g. free A-module chain complexes. In our applications the involution on A will satisfy the following conditions:

Proposition 56 (*Connolly and Ranicki [10, Section 2.6]*). *Let A be a ring with an even involution such that $\widehat{H}^0(\mathbb{Z}_2; A)$ has a one-dimensional f.g. projective A-module resolution*

$$0 \to C_1 \xrightarrow{d} C_0 \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \to 0.$$

Let $(C, \gamma) = \mathcal{C}(d, \chi)$ be the cone of a chain bundle map $(d, \chi) : (C_1, 0) \to (C_0, \delta)$ with

$$\widehat{v}_0(\delta) = x : C_0 \to \widehat{H}^0(\mathbb{Z}_2; A)$$

and set

$$(B^A(i), \beta^A(i)) = (C, \gamma)_{*+2i} \quad (i \in \mathbb{Z}).$$

(i) *The chain bundle over A*

$$(B^A, \beta^A) = \bigoplus_i (B^A(i), \beta^A(i))$$

is universal.

(ii) *The twisted quadratic Q-groups of (B^A, β^A) are given by*

$$Q_n(B^A, \beta^A) = \begin{cases} Q_0(C, \gamma) & \text{if } n \equiv 0 (\operatorname{mod} 4), \\ \ker(J_\gamma : Q^1(C) \to \widehat{Q}^1(C)) & \text{if } n \equiv 1 (\operatorname{mod} 4), \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4), \\ Q_{-1}(C, \gamma) & \text{if } n \equiv 3 (\operatorname{mod} 4). \end{cases}$$

The inclusion $(B^A(2j), \beta^A(2j)) \to (B^A, \beta^A)$ is a chain bundle map which induces isomorphisms

$$Q_n(B^A, \beta^A) \cong \begin{cases} Q_n(B^A(2j), \beta^A(2j)) & \text{if } n = 4j, 4j-1, \\ \ker(J_{\beta^A(2j)} : Q^n(B^A(2j)) \to \widehat{Q}^n(B^A(2j))) & \text{if } n = 4j+1. \end{cases}$$

Proof. (i) The Wu classes of the chain bundle $(C, \gamma)_{*+2i}$ are isomorphisms

$$\widehat{v}_k(\gamma) : H_k(C_{*+2i}) \xrightarrow{\cong} \widehat{H}^k(\mathbb{Z}_2; A)$$

for $k = 2i, 2i+1$.

(ii) See [10] for the detailed analysis of the exact sequence of 38(ii)

$$\cdots \to \sum_{i=-\infty}^{\infty} Q_n(B^A(i), \beta^A(i)) \to Q_n(B^A, \beta^A) \to \sum_{i<j} H_n(B^A(i) \otimes_A B^A(j)) \to \sum_{i=-\infty}^{\infty} Q_{n-1}(B^A(i), \beta^A(i)) \to \cdots . \qquad \square$$

As in the introduction:

Definition 57. A ring with involution A is *r-even* for some $r \geqslant 1$ if

(i) A is commutative, with the identity involution,
(ii) $2 \in A$ is a nonzero divisor,
(iii) $\widehat{H}^0(\mathbb{Z}_2; A)$ is a f.g. free A_2-module of rank r with a basis $\{x_1 = 1, x_2, \ldots, x_r\}$.

Example 58. $\mathbb{Z}$ is 1-even.

Proposition 59. *If A is 1-even the polynomial extension $A[x]$ is 2-even, with $A[x]_2 = A_2[x]$ and $\{1, x\}$ an $A_2[x]$-module basis of $\widehat{H}^0(\mathbb{Z}_2; A[x])$.*

Proof. For any $a = \sum_{i=0}^{\infty} a_i x^i \in A[x]$:

$$\begin{aligned} a^2 &= \sum_{i=0}^{\infty} (a_i)^2 x^{2i} + 2 \sum_{0 \leqslant i < j < \infty} a_i a_j x^{i+j} \\ &= \sum_{i=0}^{\infty} a_i x^{2i} \in A_2[x]. \end{aligned}$$

The $A_2[x]$-module morphism

$$A_2[x] \oplus A_2[x] \to \widehat{H}^0(\mathbb{Z}_2; A[x]); \; (p, q) \mapsto p^2 + q^2 x$$

is thus an isomorphism, with inverse

$$\widehat{H}^0(\mathbb{Z}_2; A[x]) \xrightarrow{\cong} A_2[x] \oplus A_2[x]; \; a = \sum_{i=0}^{\infty} a_i x^i \mapsto \left(\sum_{j=0}^{\infty} a_{2j} x^j, \sum_{j=0}^{\infty} a_{2j+1} x^j \right). \qquad \square$$

Proposition 59 is the special case $k = 1$ of a general result: if A is 1-even and $t_1, t_2, \ldots, t_k$ are commuting indeterminates over A then the polynomial ring

$A[t_1, t_2, \ldots, t_k]$ is 2^k-even with

$$\{x_1 = 1, x_2, x_3, \ldots, x_{2^k}\} = \{(t_1)^{i_1}(t_2)^{i_2}\cdots(t_k)^{i_k} \mid i_j = 0 \text{ or } 1, 1 \leqslant j \leqslant k\}$$

an $A_2[t_1, t_2, \ldots, t_k]$-module basis of $\widehat{H}^0(\mathbb{Z}_2; A[t_1, t_2, \ldots, t_k])$.

We can now prove Theorem C.

Theorem 60. *Let A be an r-even ring with involution.*

(i) *The A-module morphism*

$$x : A^r \to \widehat{H}^0(\mathbb{Z}_2; A);\ \ (a_1, a_2, \ldots, a_r) \mapsto \sum_{i=1}^{r}(a_i)^2 x_i$$

fits into a one-dimensional f.g. free A-module resolution of $\widehat{H}^0(\mathbb{Z}_2; A)$,

$$0 \to C_1 = A^r \xrightarrow{\ 2\ } C_0 = A^r \xrightarrow{\ x\ } \widehat{H}^0(\mathbb{Z}_2; A) \to 0.$$

The symmetric and hyperquadratic L-groups of A are 4-periodic

$$L^n(A) = L^{n+4}(A),\ \ \widehat{L}^n(A) = \widehat{L}^{n+4}(A).$$

(ii) *Let $(C(X), \gamma(X))$ be the chain bundle over A given by the construction of* (53) *for*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A),$$

with $C(X) = \mathcal{C}(2 : A^r \to A^r)$. The chain bundle over A defined by

$$(B^A, \beta^A) = \bigoplus_i (C(X), \gamma(X))_{*+2i} = \bigoplus_i (B^A(i), \beta^A(i))$$

is universal. The hyperquadratic L-groups of A are given by

$$\widehat{L}^n(A) = Q_n(B^A, \beta^A)$$
$$= \begin{cases} Q_0(C(X),\gamma(X)) = \dfrac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N+N^t) - N^tXN \mid N \in M_r(A)\}} & \text{if } n=0,\\ \mathrm{im}(N_{\gamma(X)} : Q_1(C(X),\gamma(X)) \to Q^1(C(X))) = \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))) & \\ \quad = \dfrac{\{N \in M_r(A) \mid N + N^t \in 2\mathrm{Sym}_r(A),\ \frac{1}{2}(N+N^t) - N^tXN \in \mathrm{Quad}_r(A)\}}{2M_r(A)} & \text{if } n=1,\\ 0 & \text{if } n=2,\\ Q_{-1}(C(X),\gamma(X)) = \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{L - LXL \mid L \in \mathrm{Sym}_r(A)\}} & \text{if } n=3, \end{cases}$$

with

$$\partial : \widehat{L}^0(A) \to L_{-1}(A);\ \ M \mapsto \left(H_-(A^r); A^r, \mathrm{im}\left(\begin{pmatrix} 1-XM \\ M \end{pmatrix} : A^r \to A^r \oplus (A^r)^*\right)\right),$$
$$\partial : \widehat{L}^1(A) \to L_0(A);\ \ N \mapsto \left(A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N+N^t-2N^tXN) & 1-2NX \\ 0 & -2X \end{pmatrix}\right),$$
$$\partial : \widehat{L}^3(A) \to L_2(A);\ \ M \mapsto \left(A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix}\right).$$

Proof. Combine Proposition 30, Theorem 54 and Proposition 56, noting that the direct summand

$$\widehat{Q}^2(C(X)) = \mathrm{Sym}_r(A)/\mathrm{Quad}_r(A) \subseteq Q_1(C(X),\gamma(X))$$

is precisely the image of $H_2(C(X) \otimes C(X)_{*+2}) = \widehat{Q}^2(C(X))$ under the first map in the exact sequence

$$\begin{aligned} &H_2(C(X) \otimes_A C(X)_{*+2}) \to Q_1(C(X),\gamma(X)) \oplus Q_1(C(X)_{*+2},\gamma(X)_{*+2})\\ &\quad \to Q_1(C(X) \oplus C(X)_{*+2}, \gamma(X) \oplus \gamma(X)_{*+2}) \to H_1(C(X) \otimes_A C(X)_{*+2}) = 0 \end{aligned}$$

of Proposition 38(ii), with $Q_1(C(X)_{*+2},\gamma(X)_{*+2}) = 0$, so that

$$Q_1(C(X) \oplus C(X)_{*+2}, \gamma(X) \oplus \gamma(X)_{*+2}) = \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))). \quad \square$$

We can now prove Theorem A.

Corollary 61. *Let A be a* 1*-even ring with* $\psi^2 = 1$.

(i) *The universal chain bundle* (B^A, β^A) *over* A *is given by*

$$B^A : \cdots \longrightarrow B^A_{2k+2} = A \xrightarrow{0} B^A_{2k+1} = A \xrightarrow{2} B^A_{2k} = A \xrightarrow{0} B^A_{2k-1} = A \longrightarrow \cdots ,$$

$$(\beta^A)_{-4k} = 1 : B^A_{2k} = A \to (B^A)^{2k} = A \quad (k \in \mathbb{Z}).$$

(ii) *The hyperquadratic L-groups of* A *are given by*

$$\widehat{L}^n(A) = Q_n(B^A, \beta^A) = \begin{cases} A_8 & \text{if } n \equiv 0 (\operatorname{mod} 4), \\ A_2 & \text{if } n \equiv 1, 3 (\operatorname{mod} 4), \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4), \end{cases}$$

with

$$\partial : \widehat{L}^0(A) = A_8 \to L_{-1}(A);\ a \mapsto \left(H_-(A); A, \operatorname{im}\left(\left(\begin{matrix} 1-a \\ a \end{matrix} \right) : A \to A \oplus A \right) \right),$$

$$\partial : \widehat{L}^1(A) = A_2 \to L_0(A);\ a \mapsto \left(A \oplus A, \left(\begin{matrix} a(1-a)/2 & 1-2a \\ 0 & -2 \end{matrix} \right) \right),$$

$$\partial : \widehat{L}^3(A) = A_2 \to L_2(A);\ a \mapsto \left(A \oplus A, \left(\begin{matrix} a & 1 \\ 0 & 1 \end{matrix} \right) \right).$$

(iii) *The map* $L^0(A) \to \widehat{L}^0(A)$ *sends the Witt class* $(K, \lambda) \in L^0(A)$ *of a nonsingular symmetric form* (K, λ) *over* A *to*

$$[K, \lambda] = \lambda(v, v) \in \widehat{L}^0(A) = A_8$$

for any $v \in K$ *such that*

$$\lambda(x, x) = \lambda(x, v) \in A_2 \quad (x \in K).$$

Proof. (i)+(ii) The A-module morphism

$$\widehat{v}_0(\beta^A) : H_0(B^A) = A_2 \to \widehat{H}^0(\mathbb{Z}_2; A);\ a \mapsto a^2$$

is an isomorphism. Apply Theorem 60 with $r = 1$, $x_1 = 1$.

(ii) The computation of $\widehat{L}^*(A) = Q_*(B^A, \beta^A)$ is given by Theorem 60, using the fact that $a - a^2 \in 2A$ $(a \in A)$ for a 1-even A with $\psi^2 = 1$. The explicit descriptions of ∂ are special cases of the formulae in Theorem 54(ii).

(iii) As in Example 48 regard (K, λ) as a zero-dimensional symmetric Poincaré complex (D, ϕ) with

$$\phi_0 = \varepsilon\lambda^{-1} : D^0 = K \to D^0 = K^*.$$

The Spivak normal chain bundle $\gamma = \lambda^{-1} \in \widehat{Q}^0(D^{0-*})$ is classified by the chain bundle map $(v, 0) : (D, \gamma) \to (B^A, \beta^A)$ with

$$g : D_0 = K^* \to \widehat{H}^0(\mathbb{Z}_2; A);\ \ x \mapsto \lambda^{-1}(x, x) = x(v).$$

The algebraic normal invariant $(\phi, 0) \in Q_0(D, \gamma)$ has image

$$g_{\%}(\phi, 0) = \lambda(v, v) \in Q_0(B^A, \beta^A) = A_8. \qquad \square$$

Example 62. For $R = \mathbb{Z}$,

$$\widehat{L}^n(\mathbb{Z}) = Q_n(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) = \begin{cases} \mathbb{Z}_8 & \text{if } n \equiv 0 (\operatorname{mod} 4), \\ \mathbb{Z}_2 & \text{if } n \equiv 1, 3 (\operatorname{mod} 4), \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4). \end{cases}$$

as recalled (from [15]) in the Introduction.

3. The generalized Arf invariant for forms

A nonsingular ε-quadratic form (K, ψ) over A corresponds to a zero-dimensional ε-quadratic Poincaré complex over A. The zero-dimensional ε-quadratic L-group $L_0(A, \varepsilon)$ is the Witt group of nonsingular ε-quadratic forms, and similarly for $L^0(A, \varepsilon)$ and ε-symmetric forms. In this section we define the ‘generalized Arf invariant’

$$(K, \psi; L) \in Q_1(B^{A,\varepsilon}, \beta^{A,\varepsilon}) = \widehat{L}^{4*+1}(A, \varepsilon)$$

for a nonsingular ε-quadratic form (K, ψ) over A with a lagrangian L for the ε-symmetric form $(K, \psi + \varepsilon\psi^*)$, so that

$$\begin{aligned} (K, \psi) &= \partial(K, \psi; L) \in \ker(1 + T : L_0(A, \varepsilon) \to L^{4*}(A, \varepsilon)) \\ &= \operatorname{im}(\partial : Q_1(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \to L_0(A, \varepsilon)). \end{aligned}$$

3.1. Forms and formations

Given a f.g. projective A-module K and the inclusion $j : L \to K$ of a direct summand, let $f : C \to D$ be the chain map defined by

$$\begin{aligned} &C : \cdots \to 0 \to C_k = K^* \to 0 \to \cdots\ , \\ &D : \cdots \to 0 \to D_k = L^* \to 0 \to \cdots\ , \\ &f = j^* : C_k = K^* \to D_k = L^*. \end{aligned}$$

The symmetric Q-group

$$Q^{2k}(C) = H^0(\mathbb{Z}_2; S(K), (-1)^k T) = \{\phi \in S(K) \mid \phi^* = (-1)^k \phi\}$$

is the additive group of $(-1)^k$-symmetric pairings on K, and

$$f^{\%} = S(j) : Q^{2k}(C) \to Q^{2k}(D);\ \ \phi \mapsto f\phi f^* = j^*\phi j = \phi|_L$$

sends such a pairing to its restriction to L. A $2k$-dimensional symmetric (Poincaré) complex $(C, \phi \in Q^{2k}(C))$ is the same as a (nonsingular) $(-1)^k$-symmetric form (K, ϕ). The relative symmetric Q-group of f:

$$\begin{aligned} Q^{2k+1}(f) &= \ker(f^{\%} : Q^{2k}(C) \to Q^{2k}(D)) \\ &= \{\phi \in S(K) \mid \phi^* = (-1)^k \phi \in S(K), \phi|_L = 0 \in S(L)\}, \end{aligned}$$

consists of the $(-1)^k$-symmetric pairings on K which restrict to 0 on L. The submodule $L \subset K$ is a lagrangian for (K, ϕ) if and only if ϕ restricts to 0 on L and

$$L^{\perp} = \{x \in K \mid \phi(x)(L) = \{0\} \subset A\} = L,$$

if and only if $(f : C \to D, (0, \phi) \in Q^{2k+1}(f))$ defines a $(2k+1)$-dimensional symmetric Poincaré pair, with an exact sequence

$$0 \longrightarrow D^k = L \xrightarrow{f^*=j} C^k = K \xrightarrow{f\phi=j^*\phi} D_k = L^* \longrightarrow 0.$$

Similarly for the quadratic case, with

$$\begin{aligned} &Q_{2k}(C) = H_0(\mathbb{Z}_2; S(K), (-1)^k T), \\ &Q_{2k+1}(f) = \frac{\{(\psi, \chi) \in S(K) \oplus S(L) \mid f^*\psi f = \chi + (-1)^{k+1}\chi^* \in S(L)\}}{\{(\theta + (-1)^{k+1}\theta^*, f\theta f^* + \nu + (-1)^k \nu^*) \mid \theta \in S(K), \nu \in S(L)\}}. \end{aligned}$$

A quadratic structure $\psi \in Q_{2k}(C)$ determines and is determined by the pair (λ, μ) with $\lambda = \psi + (-1)^k \psi^* \in Q^{2k}(C)$ and

$$\mu : K \to H_0(\mathbb{Z}_2; A, (-1)^k);\ \ x \mapsto \psi(x)(x).$$

A $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ is a nonsingular $(-1)^k$-quadratic form (K, ψ) together with a lagrangian $L \subset K$ for the nonsingular $(-1)^k$-symmetric form $(K, \psi + (-1)^k \psi^*)$.

Lemma 63. *Let (K, ψ) be a nonsingular $(-1)^k$-quadratic form over A, and let $L \subset K$ be a lagrangian for $(K, \psi + (-1)^k\psi^*)$. There exists a direct complement for $L \subset K$ which is also a lagrangian for $(K, \psi + (-1)^k\psi^*)$.*

Proof. Choosing a direct complement $L' \subset K$ to $L \subset K$ write

$$\psi = \begin{pmatrix} \mu & \lambda \\ 0 & \nu' \end{pmatrix} : K = L \oplus L' \to K^* = L^* \oplus (L')^*$$

with $\lambda : L' \to L^*$ an isomorphism and

$$\mu + (-1)^k\mu^* = 0 : L \to L^*.$$

In general $\nu' + (-1)^k(\nu')^* \neq 0 : L^* \to L$, but if the direct complement L' is replaced by

$$L'' = \{(-(\lambda^{-1})^*(\nu')^*(x), x) \in L \oplus L' \,|\, x \in L'\} \subset K$$

and the isomorphism

$$\lambda'' : L'' \to L^*;\ \ (-(\lambda^{-1})^*(\nu')^*(x), x) \mapsto \lambda(x)$$

is used as an identification then

$$\psi = \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} : K = L \oplus L^* \to K^* = L^* \oplus L,$$

with $\nu = (\nu')^*\mu\nu' : L^* \to L$ such that

$$s\nu + (-1)^k\nu^* = 0 : L^* \to L.$$

Thus $L'' = L^* \subset K$ is a direct complement for L which is a lagrangian for $(K, \psi + (-1)^k\psi^*)$, with

$$\psi + (-1)^k\psi^* = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} : K = L \oplus L^* \to K^* = L^* \oplus L. \qquad \square$$

A lagrangian L for the $(-1)^k$-symmetrization $(K, \psi + (-1)^k\psi^*)$ is a lagrangian for the $(-1)^k$-quadratic form (K, ψ) if and only if $\psi|_L = \mu$ is a $(-1)^{k+1}$-symmetrization, i.e.

$$\mu = \theta + (-)^{k+1}\theta^* : L \to L^*$$

for some $\theta \in S(L)$, in which case the inclusion $j : (L, 0) \to (K, \psi)$ extends to an isomorphism of $(-1)^k$-quadratic forms

$$\begin{pmatrix} 1 & -v^* \\ 0 & 1 \end{pmatrix} : H_{(-1)^k}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \xrightarrow{\cong} (K, \psi),$$

with $v = \psi|_{L^*}$. The $2k$-dimensional quadratic L-group $L_{2k}(A)$ is the Witt group of stable isomorphism classes of nonsingular $(-1)^k$-quadratic forms over A, such that

$$\begin{aligned} (K, \psi) = (K', \psi') \in L_{2k}(A) \text{ if and only if there exists an isomorphism} \\ (K, \psi) \oplus H_{(-1)^k}(L) \cong (K', \psi') \oplus H_{(-1)^k}(L'). \end{aligned}$$

Proposition 64. *Given a $(-1)^k$-quadratic form (L, μ) over A such that*

$$\mu + (-1)^k \mu^* = 0 : L \to L^*,$$

let (B, β) be the chain bundle over A given by

$$\begin{aligned} &B : \cdots \to 0 \to B_{k+1} = L \to 0 \to \cdots , \\ &\beta = \mu \in \widehat{Q}^0(B^{0-*}) = \mathrm{Hom}_A(L, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) = \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1}T). \end{aligned}$$

(i) *The $(2k+1)$-dimensional twisted quadratic Q-group of (B, β):*

$$\begin{aligned} Q_{2k+1}(B, \beta) &= \frac{\{v \in S(L^*) \mid v + (-1)^k v^* = 0\}}{\{\phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) \mid \phi^* = (-1)^{k+1}\phi, \theta \in S(L^*)\}} \\ &= \mathrm{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T) \to \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T)) \end{aligned}$$

classifies nonsingular $(-1)^k$-quadratic forms (K, ψ) over A for which there exists a lagrangian L for $(K, \psi + (-1)^k\psi^*)$ such that

$$\begin{aligned} \psi|_L &= \mu \in \mathrm{im}(\widehat{H}^1(\mathbb{Z}_2; S(L), (-1)^k T) \to H_0(\mathbb{Z}_2; S(L), (-1)^k T)) \\ &= \ker(1 + (-1)^k T : H_0(\mathbb{Z}_2; S(L), (-1)^k T) \to H^0(\mathbb{Z}_2; S(L), (-1)^k T)). \end{aligned}$$

Specifically, for any $(-1)^k$-quadratic form (L^*, v) such that

$$v + (-1)^k v^* = 0 : L^* \to L,$$

the nonsingular $(-1)^k$*-quadratic form* (K, ψ) *defined by*

$$\psi = \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} : K = L \oplus L^* \to K^* = L^* \oplus L$$

is such that L *is a lagrangian of* $(K, \psi + (-1)^k \psi^*)$, *and*

$$\partial : Q_{2k+1}(B, \beta) \to L_{2k}(A); \;\; \nu \mapsto (K, \psi).$$

(ii) *The algebraic normal invariant of a* $(2k+1)$*-dimensional* (*symmetric*, *quadratic*) *Poincaré pair* $(f : C \to D, (\delta\phi, \psi))$ *concentrated in degree* k *with*

$$\begin{aligned} & C_k = K^*, \;\; D_k = L^*, \\ & f\psi_0 f^* = \mu \in \ker(1 + (-1)^k T : H_0(\mathbb{Z}_2; S(L), (-1)^k T) \to H^0(\mathbb{Z}_2; S(L), (-1)^k T)) \end{aligned}$$

is given by

$$(\phi, \theta) = \nu \in Q_{2k+1}(\mathcal{C}(f), \gamma) = Q_{2k+1}(B, \beta),$$

with

$$\widehat{v}_{k+1}(\gamma) = \widehat{v}_{k+1}(\beta) : L = H_{k+1}(f) = H_{k+1}(B) \to \widehat{H}^{k+1}(\mathbb{Z}_2; A); \;\; x \mapsto \mu(x)(x)$$

and $\nu = \psi|_{L^*}$ *the restriction of* ψ *to any lagrangian* $L^* \subset K$ *of* $(K, \psi + (-1)^k \psi^*)$ *complementary to* L.

Proof. (i) Given $(-1)^{k+1}$-symmetric forms (L^*, ν), (L^*, ϕ) and $\theta \in S(L^*)$ replacing ν by

$$\nu' = \nu + \phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) : L^* \to L$$

results in a $(-1)^k$-quadratic form (K, ψ') such that there is defined an isomorphism

$$\begin{pmatrix} 1 & \phi^* \\ 0 & 1 \end{pmatrix} : (K, \psi') \to (K, \psi)$$

which is the identity on L.

(ii) This is the translation of Proposition 42(iii) into the language of forms and lagrangians. $\square$

More generally:

Proposition 65. *Given $(-1)^k$-quadratic forms (L,μ), (L^*,ν) over A such that*

$$\mu+(-1)^k\mu^*=0:L\to L^*,\quad \nu+(-1)^k\nu^*=0:L^*\to L$$

define a nonsingular $(-1)^k$-quadratic form

$$(K,\psi)=\left(L\oplus L^*,\begin{pmatrix}\mu & 1\\ 0 & \nu\end{pmatrix}\right),$$

such that L and L^* are complementary lagrangians of the nonsingular $(-1)^k$-symmetric form

$$(K,\psi+(-1)^k\psi^*)=\left(L\oplus L^*,\begin{pmatrix}0 & 1\\ (-1)^k & 0\end{pmatrix}\right)$$

and let $(f:C\to D,(\delta\phi,\psi))$ be the $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair concentrated in degree k defined by

$$f=\begin{pmatrix}1 & 0\end{pmatrix}:C_k=K^*=L^*\oplus L\to D_k=L^*,\ \ \delta\phi=0,$$

with $\mathcal{C}(f)\simeq L_{*-k-1}$.

(i) *The Spivak normal bundle of $(f:C\to D,(\delta\phi,\psi))$ is given by*

$$\gamma=\mu\in\widehat{Q}^0(\mathcal{C}(f)^{0-*})=\widehat{H}^0(\mathbb{Z}_2;S(L),(-1)^{k+1}T)$$

and

$$\begin{aligned}Q_{2k+1}(\mathcal{C}(f),\gamma)&=\frac{\{\lambda\in S(L^*)\mid\lambda+(-1)^k\lambda^*=0\}}{\{\phi-\phi\mu\phi^*-(\theta+(-1)^{k+1}\theta^*)\mid\phi^*=(-1)^{k+1}\phi,\theta\in S(L^*)\}}\\&=\operatorname{coker}(J_\mu:H^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T)\\&\quad\to\widehat{H}^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T)).\end{aligned}$$

The algebraic normal invariant of $(f:C\to D,(\delta\phi,\psi))$ is

$$(\phi,\theta)=\nu\in Q_{2k+1}(\mathcal{C}(f),\gamma).$$

(ii) *Let (B,β) be a chain bundle concentrated in degree $k+1$*

$$\begin{aligned}&B:\cdots\to 0\to B_{k+1}\to 0\to\cdots\ ,\\&\beta\in\widehat{Q}^0(B^{0-*})=\operatorname{Hom}_A(B_{k+1},\widehat{H}^{k+1}(\mathbb{Z}_2;A))=\widehat{H}^0(\mathbb{Z}_2;S(B_{k+1}),(-1)^{k+1}T),\end{aligned}$$

so that

$$\begin{aligned} Q_{2k+1}(B,\beta) &= \frac{\{\lambda \in S(B^{k+1}) \mid \lambda + (-1)^k\lambda^* = 0\}}{\{\phi - \phi\beta\phi^* - (\theta + (-1)^{k+1}\theta^*) \mid \phi^* = (-1)^{k+1}\phi, \theta \in S(B^{k+1})\}} \\ &= \mathrm{coker}(J_\beta : H^0(\mathbb{Z}_2; S(B^{k+1}), (-1)^{k+1}T) \\ &\quad \to \widehat{H}^0(\mathbb{Z}_2; S(B^{k+1}), (-1)^{k+1}T)). \end{aligned}$$

A (B,β)*-structure on* $(f : C \to D, (\delta\phi, \psi))$ *is given by a chain bundle map* (g,χ) : $(\mathcal{C}(f),\gamma) \to (B,\beta)$, *corresponding to an A-module morphism* $g : L \to B_{k+1}$ *such that*

$$g^*\beta g = \mu \in \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1}T),$$

with

$$(g,\chi)_\% : Q_{2k+1}(\mathcal{C}(f),\gamma) \to Q_{2k+1}(B,\beta);\ \ \lambda \mapsto g\lambda g^*.$$

The 4*-periodic* (B,β)*-structure cobordism class is thus given by*

$$\begin{aligned} (K,\psi;L) = (f : C \to D, (\delta\phi,\psi)) &= (g,\chi)_\%(\phi,\theta) = g\nu g^* \\ &\in \widehat{L}\langle B,\beta\rangle^{4*+2k+1}(A) = Q_{2k+1}(B,\beta), \end{aligned}$$

with

$$\begin{aligned} (K,\psi) &= \left(B_{k+1} \oplus B^{k+1}, \begin{pmatrix} \beta & 1 \\ 0 & g\nu g^* \end{pmatrix}\right) \\ &\in \mathrm{im}(\partial : Q_{2k+1}(B,\beta) \to L_{2k}(A)) = \ker(L_{2k}(A) \to L\langle B,\beta\rangle^{4*+2k}(A)). \end{aligned}$$

3.2. The generalized Arf invariant

Definition 66. The *generalized Arf invariant* of a nonsingular $(-1)^k$-quadratic form (K,ψ) over A together with a lagrangian $L \subset K$ for the $(-1)^k$-symmetric form $(K,\psi + (-1)^k\psi^*)$ is the image

$$(K,\psi;L) = (g,\chi)_\%(\phi,\theta) \in \widehat{L}^{4*+2k+1}(A) = Q_{2k+1}(B^A,\beta^A)$$

of the algebraic normal invariant $(\phi,\theta) \in Q_{2k+1}(\mathcal{C}(f),\gamma)$ (43) of the corresponding $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi,\psi) \in Q^{2k+1}_{2k+1}(f))$

$$\begin{aligned} (\phi,\theta) &= \nu \in Q_{2k+1}(\mathcal{C}(f),\gamma) \\ &= \mathrm{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T) \to \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T)) \end{aligned}$$

under the morphism $(g,\chi)_\%$ induced by the classifying chain bundle map $(g,\chi):(\mathcal{C}(f),\gamma)\to(B^A,\beta^A)$. As in 64 $v=\psi|_{L^*}$ is the restriction of ψ to a lagrangian $L^*\subset K$ of $(K,\psi+(-1)^k\psi^*)$ complementary to L.

A nonsingular $(-1)^k$-symmetric formation $(K,\phi;L,L')$ is a nonsingular $(-1)^k$-symmetric form (K,ϕ) together with two lagrangians L, L'. This type of formation is essentially the same as a $(2k+1)$-dimensional symmetric Poincaré complex concentrated in degrees $k,k+1$, and represents an element of $L^{4*+2k+1}(A)$.

Proposition 67. (i) *The generalized Arf invariant is such that*

$$(K,\psi;L)=0\in Q_{2k+1}(B^A,\beta^A)=\widehat{L}^{4*+2k+1}(A)$$

if and only if there exists an isomorphism of $(-1)^k$*-quadratic forms*

$$(K,\psi)\oplus H_{(-1)^k}(L')\cong H_{(-1)^k}(L'')$$

such that

$$((K,\psi+(-1)^k\psi^*)\oplus(1+T)H_{(-1)^k}(L');L\oplus L',L'')=0\in L^{4*+2k+1}(A).$$

(ii) *If* (K,ψ) *is a nonsingular* $(-1)^k$*-quadratic form over* A *and* $L,L'\subset K$ *are lagrangians for* $(K,\psi+(-1)^k\psi^*)$ *then*

$$\begin{aligned}(K,\psi;L)-(K,\psi;L')&=(K,\psi+(-1)^k\psi^*;L,L')\\&\in \operatorname{im}(L^{4*+2k+1}(A)\to\widehat{L}^{4*+2k+1}(A))=\ker(\widehat{L}^{4*+2k+1}(A)\to L_{2k}(A)).\end{aligned}$$

Proof. This is the translation of the isomorphism $Q_{2k+1}(B^A,\beta^A)\cong\widehat{L}^{4*+2k+1}(A)$ given by 46 into the language of forms and formations. $\square$

Example 68. Let A be a field, so that each $\widehat{H}^n(\mathbb{Z}_2;A)$ is a free A-module, and the universal chain bundle over A can be taken to be

$$B^A=\widehat{H}^*(\mathbb{Z}_2;A):\cdots\longrightarrow B^A_n=\widehat{H}^n(\mathbb{Z}_2;A)\xrightarrow{0}B^A_{n-1}=\widehat{H}^{n-1}(\mathbb{Z}_2;A)\xrightarrow{0}\cdots.$$

If A is a perfect field of characteristic 2 with the identity involution squaring defines an A-module isomorphism

$$A\xrightarrow{\cong}\widehat{H}^n(\mathbb{Z}_2;A);\quad a\mapsto a^2.$$

Every nonsingular $(-1)^k$-quadratic form over A is isomorphic to one of the type

$$(K, \psi) = \left(L \oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \right),$$

with $L = A^\ell$ f.g. free and

$$\mu = (-1)^{k+1}\mu^* : L \to L^*, \ \ \nu = (-1)^{k+1}\nu^* : L^* \to L.$$

For $j = 1, 2, \ldots, \ell$ let

$$\begin{aligned} e_j &= (0, \ldots, 0, 1, 0, \ldots, 0) \in L, \ \ g_j = \mu(e_j)(e_j) \in A, \\ e_j^* &= (0, \ldots, 0, 1, 0, \ldots, 0) \in L^*, \ \ h_j = \nu(e_j^*)(e_j^*) \in A. \end{aligned}$$

The generalized Arf invariant in this case was identified in [18, §11] with the original invariant of Arf [1]

$$(K, \psi; L) = \sum_{j=1}^{\ell} g_j h_j \in Q_{2k+1}(B^A, \beta^A) = A/\{c + c^2 \mid c \in A\}.$$

For $k = 0$ we have:

Proposition 69. *Suppose that the involution on A is even. If* (K, ψ) *is a nonsingular quadratic form over A and L is a lagrangian of* $(K, \psi + \psi^*)$ *then L is a lagrangian of* (K, ψ)*, the Witt class is*

$$(K, \psi) = 0 \in L_0(A),$$

the algebraic normal invariant is

$$(\phi, \theta) = 0 \in Q_1(\mathcal{C}(f), \gamma) = 0$$

and the generalized Arf invariant is

$$(K, \psi; L) = (g, \chi)_\%(\phi, \theta) = 0 \in \widehat{L}^{4*+1}(A) = Q_1(B^A, \beta^A).$$

Proof. By hypothesis $\widehat{H}^1(\mathbb{Z}_2; A) = 0$, and $L = A^\ell$, so that by Proposition 64(i)

$$Q_1(\mathcal{C}(f), \gamma) = \widehat{H}^0(\mathbb{Z}_2; S(L^*), -T) = \bigoplus_{\ell} \widehat{H}^1(\mathbb{Z}_2; A) = 0. \qquad \square$$

For $k = 1$ we have:

Theorem 70. *Let A be an r-even ring with A_2-module basis $\{x_1 = 1, x_2, \ldots, x_r\} \subset \widehat{H}^0(\mathbb{Z}_2; A)$, and let*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)$$

so that by Theorem 60

$$Q_3(B^A, \beta^A) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{L - LXL \mid L \in \mathrm{Sym}_r(A)\}}.$$

(i) *Given $M \in \mathrm{Sym}_r(A)$ define the nonsingular (-1)-quadratic form over A*

$$(K_M, \psi_M) = \left(A^r \oplus (A^r)^*, \begin{pmatrix} X & 1 \\ 0 & M \end{pmatrix}\right)$$

such that $L_M = A^r \subset K_M$ is a lagrangian of $(K_M, \psi_M - \psi_M^*)$. The function

$$Q_3(B^A, \beta^A) \to \widehat{L}^{4*+3}(A); \;\; M \mapsto (K_M, \psi_M; L_M)$$

is an isomorphism, with inverse given by the generalized Arf invariant.

(ii) *Let (K, ψ) be a nonsingular (-1)-quadratic form over A of the type*

$$(K, \psi) = \left(L \oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix}\right),$$

with

$$\mu - \mu^* = 0 : L \to L^*, \;\; \nu - \nu^* = 0 : L^* \to L$$

and let $g : L \to A^r$, $h : L^* \to A^r$ be A-module morphisms such that

$$\mu = g^* X g \in \widehat{H}^0(\mathbb{Z}_2; S(L), T), \;\; \nu = h^* X h \in \widehat{H}^0(\mathbb{Z}_2; S(L^*), T).$$

The generalized Arf invariant of $(K, \psi; L)$ is

$$(K, \psi; L) = g \nu g^* = g h^* X h g^* \in Q_3(B^A, \beta^A).$$

If $L = A^\ell$ *then*

$$g = (g_{ij}) : L = A^\ell \to A^r,\ \ h = (h_{ij}) : L^* = A^\ell \to A^r,$$

with the coefficients $g_{ij}, h_{ij} \in A$ *such that*

$$\mu(e_j)(e_j) = \sum_{i=1}^{r} (g_{ij})^2 x_i,\ \ \nu(e_j^*)(e_j^*) = \sum_{i=1}^{r} (h_{ij})^2 x_i \in \widehat{H}^0(\mathbb{Z}_2; A)$$
$$(e_j = (0, \dots, 0, 1, 0, \dots, 0) \in L = A^\ell,\ \ e_j^* = (0, \dots, 0, 1, 0, \dots, 0) \in L^* = A^\ell)$$

and

$$(K, \psi; L) = gh^*Xhg^* = \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 \\ 0 & 0 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_r \end{pmatrix} \in Q_3(B^A, \beta^A),$$

with

$$c_i = \sum_{k=1}^{r} \left(\sum_{j=1}^{\ell} g_{ij} h_{kj} \right)^2 x_k \in \widehat{H}^0(\mathbb{Z}_2; A).$$

(iii) *For any* $M = (m_{ij}) \in \mathrm{Sym}_r(A)$ *let* $h = (h_{ij}) \in M_r(A)$ *be such that*

$$m_{jj} = \sum_{i=1}^{r} (h_{ij})^2 x_i \in \widehat{H}^0(\mathbb{Z}_2; A)\ \ (1 \leqslant j \leqslant r),$$

so that

$$M = \begin{pmatrix} m_{11} & 0 & 0 & \dots & 0 \\ 0 & m_{22} & 0 & \dots & 0 \\ 0 & 0 & m_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m_{rr} \end{pmatrix} = h^*Xh \in \widehat{H}^0(\mathbb{Z}_2; M_r(A), T) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}$$

and the generalized Arf invariant of the triple $(K_M, \psi_M; L_M)$ *in* (i) *is*

$$(K_M, \psi_M; L_M) = h^* X h = M \in Q_3(B^A, \beta^A)$$

(*with* $g = (\delta_{ij})$ *here*).

Proof. (i) The isomorphism $Q_3(B^A, \beta^A) \to \widehat{L}^3(A); M \mapsto (K_M, \psi_M; L_M)$ is given by Proposition 46.

(ii) As in Definition 66 let $(\phi, \theta) \in Q_3(\mathcal{C}(f), \gamma)$ be the algebraic normal invariant of the three-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ concentrated in degree 1, with

$$f = \begin{pmatrix} 1 & 0 \end{pmatrix} : C_1 = K^* = L^* \oplus L \to D_1 = L^*, \;\; \delta\phi = 0.$$

The A-module morphism

$$\widehat{v}_2(\gamma) : H_2(\mathcal{C}(f)) = H^1(D) = L \to \widehat{H}^0(\mathbb{Z}_2; A); \;\; y \mapsto \mu(y)(y)$$

is induced by the A-module chain map

$$g : \mathcal{C}(f) \simeq L_{*-2} \to B^A(1)$$

and

$$(g, 0) : (\mathcal{C}(f), \gamma) \to (B^A(1), \beta^A(1)) \to (B^A, \beta^A)$$

is a classifying chain bundle map. The induced morphism

$$\begin{aligned}(g,0)_\% : Q_3(\mathcal{C}(f), \gamma) &= \operatorname{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), T) \to \widehat{H}^0(\mathbb{Z}_2; S(L^*), T)) \\ \to Q_3(B^A, \beta^A) &= \operatorname{coker}(J_X : H^0(\mathbb{Z}_2; M_r(A), T) \to \widehat{H}^0(\mathbb{Z}_2; M_r(A), T)); \;\; \sigma \mapsto g\sigma g^*\end{aligned}$$

sends the algebraic normal invariant

$$(\phi, \theta) = v = h^* X h \in Q_3(\mathcal{C}(f), \gamma)$$

to the generalized Arf invariant

$$(g, 0)_\%(\phi, \theta) = g h^* X h g^* \in Q_3(B^A, \beta^A).$$

(iii) By construction. $\square$

In particular, the generalized Arf invariant for $A = \mathbb{Z}_2$ is just the classical Arf invariant.

4. The generalized Arf invariant for linking forms

An ε-quadratic formation $(Q, \psi; F, G)$ over A corresponds to a one-dimensional ε-quadratic Poincaré complex. The one-dimensional ε-quadratic L-group $L_1(A, \varepsilon)$ is the Witt group of ε-quadratic formations, or equivalently the cobordism group of one-dimensional ε-quadratic Poincaré complexes over A. We could define a generalized Arf invariant $\alpha \in Q_2(B^A, \beta^A, \varepsilon)$ for any formation with a null-cobordism of the one-dimensional ε-symmetric Poincaré complex, so that

$$\begin{aligned}(Q, \psi; F, G) &= \partial(\alpha) \in \ker(1 + T_\varepsilon : L_1(A, \varepsilon) \to L^{4*+1}(A, \varepsilon)) \\ &= \operatorname{im}(\partial : Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) \to L_1(A, \varepsilon)).\end{aligned}$$

However, we do not need quite such a generalized Arf invariant here. For our application to UNil, it suffices to work with a localization $S^{-1}A$ of A and to only consider a formation $(Q, \psi; F, G)$ such that

$$F \cap G = \{0\}, \ \ S^{-1}(Q/(F + G)) = 0,$$

which corresponds to a $(-\varepsilon)$-quadratic linking form (T, λ, μ) over (A, S) with

$$T = Q/(F + G), \ \ \lambda : T \times T \to S^{-1}A/A.$$

Given a lagrangian $U \subset T$ for the $(-\varepsilon)$-symmetric linking form (T, λ) we define in this section a 'linking Arf invariant'

$$(T, \lambda, \mu; U) \in Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}, \varepsilon) = \widehat{L}^{4*+2}(A, \varepsilon)$$

such that

$$\begin{aligned}(Q, \psi; F, G) &= \partial(T, \lambda, \mu; U) \in \ker(1 + T_\varepsilon : L_1(A, \varepsilon) \to L^{4*+1}(A, \varepsilon)) \\ &= \operatorname{im}(\partial : Q_2(B^{A,\varepsilon}, \beta^{A,\varepsilon}) \to L_1(A, \varepsilon)).\end{aligned}$$

4.1. Linking forms and formations

Given a ring with involution A and a multiplicative subset $S \subset A$ of central nonzero divisors such that $\overline{S} = S$ let $S^{-1}A$ be the localized ring with involution obtained from A by inverting S. We refer to [16] for the localization exact sequences in ε-symmetric and ε-quadratic algebraic L-theory

$$\begin{aligned}&\cdots \to L^n(A, \varepsilon) \to L^n_I(S^{-1}A, \varepsilon) \to L^n(A, S, \varepsilon) \to L^{n-1}(A, \varepsilon) \to \cdots \ , \\ &\cdots \to L_n(A, \varepsilon) \to L^I_n(S^{-1}A, \varepsilon) \to L_n(A, S, \varepsilon) \to L_{n-1}(A, \varepsilon) \to \cdots \ .\end{aligned}$$

with $I = \operatorname{im}(\widetilde{K}_0(A) \to \widetilde{K}_0(S^{-1}A))$, $L^n(A, S, \varepsilon)$ the cobordism group of $(n-1)$-dimensional ε-symmetric Poincaré complexes (C, ϕ) over A such that $H_*(S^{-1}C) = 0$, and similarly for $L_n(A, S, \varepsilon)$. An (A, S)-*module* is an A-module T with a one-dimensional f.g. projective A-module resolution

$$0 \longrightarrow P \xrightarrow{d} Q \longrightarrow T \longrightarrow 0$$

such that $S^{-1}d : S^{-1}P \to S^{-1}Q$ is an $S^{-1}A$-module isomorphism. In particular,

$$S^{-1}T = 0.$$

The *dual* (A, S)-*module* is defined by

$$\begin{aligned} T\widehat{\ } &= \operatorname{Ext}^1_A(T, A) = \operatorname{Hom}_A(T, S^{-1}A/A) \\ &= \operatorname{coker}(d^* : Q^* \to P^*), \end{aligned}$$

with

$$A \times T\widehat{\ } \to T\widehat{\ };\ \ (a, f) \mapsto (x \mapsto f(x)\overline{a}).$$

For any (A, S)-modules T, U there is defined a duality isomorphism

$$\operatorname{Hom}_A(T, U) \to \operatorname{Hom}_A(U\widehat{\ }, T\widehat{\ });\ \ f \mapsto f\widehat{\ },$$

with

$$f\widehat{\ } : U\widehat{\ } \to T\widehat{\ };\ \ g \mapsto (x \mapsto g(f(x))).$$

An element $\lambda \in \operatorname{Hom}_A(T, T\widehat{\ })$ can be regarded as a sesquilinear linking pairing

$$\lambda : T \times T \to S^{-1}A/A;\ \ (x, y) \mapsto \lambda(x, y) = \lambda(x)(y),$$

with

$$\begin{aligned} &\lambda(x, ay + bz) = a\lambda(x, y) + b\lambda(x, z), \\ &\lambda(ay + bz, x) = \lambda(y, x)\overline{a} + \lambda(z, x)\overline{b}, \\ &\lambda\widehat{\ }(x, y) = \overline{\lambda(y, x)} \in S^{-1}A/A \quad (a, b \in A,\ x, y, z \in T). \end{aligned}$$

Definition 71. Let $\varepsilon = \pm 1$.

(i) An *ε-symmetric linking form over* (A, S) (T, λ) is an (A, S)-module T together with $\lambda \in \mathrm{Hom}_A(T, T^{\widehat{}})$ such that $\lambda^{\widehat{}} = \varepsilon\lambda$, so that

$$\overline{\lambda(x, y)} = \varepsilon\lambda(y, x) \in S^{-1}A/A \quad (x, y \in T).$$

The linking form is *nonsingular* if $\lambda : T \to T^{\widehat{}}$ is an isomorphism. A *lagrangian* for (T, λ) is an (A, S)-submodule $U \subset T$ such that the sequence

$$0 \longrightarrow U \xrightarrow{j} T \xrightarrow{j^{\widehat{}}\lambda} U^{\widehat{}} \longrightarrow 0$$

is exact with $j \in \mathrm{Hom}_A(U, T)$ the inclusion. Thus λ restricts to 0 on U and

$$U^{\perp} = \{x \in T \mid \lambda(x)(U) = \{0\} \subset S^{-1}A/A\} = U.$$

(ii) A *(nonsingular) ε-quadratic linking form over* (A, S) (T, λ, μ) is a (nonsingular) ε-symmetric linking form (T, λ) together with a function

$$\mu : T \to Q_{\varepsilon}(A, S) = \frac{\{b \in S^{-1}A \mid \varepsilon\overline{b} = b\}}{\{a + \varepsilon\overline{a} \mid a \in A\}}$$

such that

$$\begin{aligned}
&\mu(ax) = a\mu(x)\overline{a},\\
&\mu(x + y) = \mu(x) + \mu(y) + \lambda(x, y) + \lambda(y, x) \in Q_{\varepsilon}(A, S),\\
&\mu(x) = \lambda(x, x) \in \mathrm{im}(Q_{\varepsilon}(A, S) \to S^{-1}A/A) \quad (x, y \in T, a \in A).
\end{aligned}$$

A *lagrangian* $U \subset T$ for (T, λ, μ) is a lagrangian for (T, λ) such that $\mu|_U = 0$.

We refer to [16, 3.5] for the development of the theory of ε-symmetric and ε-quadratic linking formations over (A, S).

From now on, we shall only be concerned with A, S which satisfy:

Hypothesis 72. A, S are such that

$$\widehat{H}^*(\mathbb{Z}_2; S^{-1}A) = 0.$$

Example 73. Hypothesis 72 is satisfied if $1/2 \in S^{-1}A$, e.g. if A is even and

$$S = (2)^{\infty} = \{2^i \mid i \geqslant 0\} \subset A, \;\; S^{-1}A = A[1/2].$$

Proposition 74. (i) *For $n = 2$ (resp. 1) the relative group $L^n(A, S, \varepsilon)$ in the ε-symmetric L-theory localization exact sequence*

$$\cdots \to L^n(A, \varepsilon) \to L^n_I(S^{-1}A, \varepsilon) \to L^n(A, S, \varepsilon) \to L^{n-1}(A, \varepsilon) \to \cdots$$

is the Witt group of nonsingular $(-\varepsilon)$-symmetric linking forms (resp. ε-symmetric linking formations) over (A, S), with $I = \mathrm{im}(\widetilde{K}_0(A) \to \widetilde{K}_0(S^{-1}A))$. The skew-suspension maps

$$\overline{S} : L^n(A, S, \varepsilon) \to L^{n+2}(A, S, -\varepsilon) \ (n \geqslant 1)$$

are isomorphisms if and only if the skew-suspension maps

$$\overline{S} : L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon) \ (n \geqslant 0)$$

are isomorphisms.

(ii) *The relative group $L_n(A, S, \varepsilon)$ for $n = 2k$ (resp. $2k + 1$) in the ε-quadratic L-theory localization exact sequence*

$$\cdots \to L_n(A, \varepsilon) \to L^I_n(S^{-1}A, \varepsilon) \to L_n(A, S, \varepsilon) \to L_{n-1}(A, \varepsilon) \to \cdots$$

is the Witt group of nonsingular $(-1)^k\varepsilon$-quadratic linking forms (resp. formations) over (A, S).

(iii) *The 4-periodic ε-symmetric and ε-quadratic localization exact sequences interleave in a commutative braid of exact sequences*

$$\begin{tikzcd}
Q_{n+1}(B^A, \beta^A, \epsilon) \arrow[dr, "\partial^S"] \arrow[rr, bend left, "\partial"] & & L_n(A, \epsilon) \arrow[dr] \arrow[rr, bend left] & & L^I_n(S^{-1}A, \epsilon) \\
& L_{n+1}(A, S, \epsilon) \arrow[ur] \arrow[dr] & & L^{n+4*}(A, \epsilon) \arrow[ur] \arrow[dr] & \\
L^I_{n+1}(S^{-1}A, \epsilon) \arrow[ur] \arrow[rr, bend right] & & L^{n+4*+1}(A, S, \epsilon) \arrow[ur] \arrow[rr, bend right] & & Q_n(B^A, \beta^A, \epsilon)
\end{tikzcd}$$

Proof. (i)+(ii) See [16, §3].

(iii) For A, S satisfying Hypothesis 72 the ε-symmetrization maps for the L-groups of $S^{-1}A$ are isomorphisms

$$1 + T_\varepsilon : L^I_n(S^{-1}A, \varepsilon) \xrightarrow{\cong} L^n_I(S^{-1}A, \varepsilon). \qquad \square$$

Definition 75. (i) An *ε-quadratic S-formation* $(Q, \psi; F, G)$ over A is an ε-quadratic formation such that

$$S^{-1}F \oplus S^{-1}G = S^{-1}Q,$$

or equivalently such that $Q/(F+G)$ is an (A, S)-module.

(ii) A *stable isomorphism* of ε-quadratic S-formations over A

$$[f] : (Q_1, \psi_1; F_1, G_1) \to (Q_2, \psi_2; F_2, G_2)$$

is an isomorphism of the type

$$f : (Q_1, \psi_1; F_1, G_1) \oplus (N_1, \nu_1; H_1, K_1) \to (Q_2, \psi_2; F_2, G_2) \oplus (N_2, \nu_2; H_2, K_2),$$

with $N_1 = H_1 \oplus K_1,\ N_2 = H_2 \oplus K_2$.

Proposition 76. (i) *A* $(-\varepsilon)$*-quadratic S-formation* $(Q, \psi; F, G)$ *over A determines a nonsingular ε-quadratic linking form* (T, λ, μ) *over* (A, S), *with*

$$\begin{aligned}
&T = Q/(F+G),\\
&\lambda : T \times T \to S^{-1}A/A;\ \ (x, y) \mapsto (\psi - \varepsilon\psi^*)(x)(z)/s,\\
&\mu : T \to Q_\varepsilon(A, S);\ \ y \mapsto (\psi - \varepsilon\psi^*)(x)(z)/s - \psi(y)(y)\\
&(x, y \in Q,\ \ z \in G,\ \ s \in S,\ \ sy - z \in F).
\end{aligned}$$

(ii) *The isomorphism classes of nonsingular ε-quadratic linking forms over A are in one–one correspondence with the stable isomorphism classes of* $(-\varepsilon)$*-quadratic S-formations over A.*

Proof. See Proposition 3.4.3 of [16]. □

For any $S^{-1}A$-contractible f.g. projective A-module chain complexes concentrated in degrees $k, k+1$

$$\begin{aligned}
&C : \cdots \to 0 \to C_{k+1} \to C_k \to 0 \to \cdots\ ,\\
&D : \cdots \to 0 \to D_{k+1} \to D_k \to 0 \to \cdots
\end{aligned}$$

there are natural identifications

$$\begin{aligned}
&H^{k+1}(C) = H_k(C)\widehat{\ },\ \ H_k(C) = H^{k+1}(C)\widehat{\ },\\
&H^{k+1}(D) = H_k(D)\widehat{\ },\ \ H_k(D) = H^{k+1}(D)\widehat{\ },\\
&H_0(\mathrm{Hom}_A(C, D)) = \mathrm{Hom}_A(H_k(C), H_k(D)) = \mathrm{Tor}_1^A(H^{k+1}(C), H_k(D)),\\
&H_1(\mathrm{Hom}_A(C, D)) = H^{k+1}(C) \otimes_A H_k(D) = \mathrm{Ext}_A^1(H_k(C), H_k(D)),\\
&H_{2k}(C \otimes_A D) = H_k(C) \otimes_A H_k(D) = \mathrm{Ext}_A^1(H^{k+1}(C), H_k(D)),\\
&H_{2k+1}(C \otimes_A D) = \mathrm{Hom}_A(H^{k+1}(C), H_k(D)) = \mathrm{Tor}_1^A(H_k(C), H_k(D)).
\end{aligned}$$

In particular, an element $\lambda \in H_{2k+1}(C \otimes_A D)$ is a sesquilinear linking pairing

$$\lambda : H^{k+1}(C) \times H^{k+1}(D) \to S^{-1}A/A.$$

An element $\phi \in H_{2k}(C \otimes_A D)$ is a chain homotopy class of chain maps $\phi : C^{2k-*} \to D$, classifying the extension

$$0 \to H_k(D) \to H_k(\phi) \to H^{k+1}(C) \to 0.$$

Proposition 77. *Given an (A, S)-module T let*

$$B : \cdots \to 0 \to B_{k+1} \xrightarrow{\ d\ } B_k \to 0 \to \cdots$$

be a f.g. projective A-module chain complex concentrated in degrees $k, k+1$ such that $H^{k+1}(B) = T$, $H^k(B) = 0$, so that $H_k(B) = T\widehat{\ }$, $H_{k+1}(B) = 0$. The Q-groups in the exact sequence

$$Q^{2k+2}(B) = 0 \longrightarrow \widehat{Q}^{2k+2}(B) \xrightarrow{\ H\ } Q_{2k+1}(B) \xrightarrow{\ 1+T\ } Q^{2k+1}(B) \xrightarrow{\ J\ } \widehat{Q}^{2k+1}(B)$$

have the following interpretation in terms of T.

(i) *The symmetric Q-group*

$$Q^{2k+1}(B) = H^0(\mathbb{Z}_2; \mathrm{Hom}_A(T, T\widehat{\ }), (-1)^{k+1})$$

is the additive group of $(-1)^{k+1}$-symmetric linking pairings λ on T, with $\phi \in Q^{2k+1}(B)$ corresponding to

$$\lambda : T \times T \to S^{-1}A/A;\ \ (x, y) \mapsto \phi_0(d^*)^{-1}(x)(y)\quad (x, y \in B^{k+1}).$$

(ii) *The quadratic Q-group*

$$Q_{2k+1}(B) = \frac{\{(\psi_0, \psi_1) \in \mathrm{Hom}_A(B^k, B_{k+1}) \oplus S(B^k) \,|\, d\psi_0 = \psi_1 + (-1)^{k+1}\psi_1^* \in S(B^k)\}}{\{((\chi_0 + (-1)^{k+1}\chi_0^*)d^*, d\chi_0 d^* + \chi_1 + (-1)^k\chi_1^*) \,|\, (\chi_0, \chi_1) \in S(B^{k+1}) \oplus S(B^k)\}}$$

is the additive group of $(-1)^{k+1}$*-quadratic linking structures* (λ, μ) *on* T*. The element* $\psi = (\psi_0, \psi_1) \in Q_{2k+1}(B)$ *corresponds to*

$$\lambda : T \times T \to S^{-1}A/A;\ \ (x, y) \mapsto \psi_0(d^*)^{-1}(x)(y) \quad (x, y \in B^{k+1}),$$
$$\mu : T \to Q_{(-1)^{k+1}}(A, S);\ \ x \mapsto \psi_0(d^*)^{-1}(x)(x).$$

(iii) *The hyperquadratic Q-groups of* B

$$\widehat{Q}^n(B) = H_n(\widehat{d}^{\%} : \widehat{W}^{\%}B_{k+1} \to \widehat{W}^{\%}B_k)$$

are such that

$$\widehat{Q}^{2k}(B) = \frac{\{(\delta, \chi) \in S(B^{k+1}) \oplus S(B^k) \,|\, \delta^* = (-1)^{k+1}\delta,\ d\delta d^* = \chi + (-1)^{k+1}\chi^*\}}{\{(\mu + (-1)^{k+1}\mu^*, d\mu d^* + \nu + (-1)^k\nu^*) \,|\, (\mu, \nu) \in S(B^{k+1}) \oplus S(B^k)\}},$$

$$\widehat{Q}^{2k+1}(B) = \frac{\{(\delta, \chi) \in S(B^{k+1}) \oplus S(B^k) \,|\, \delta^* = (-1)^k\delta,\ d\delta d^* = \chi + (-1)^k\chi^*\}}{\{(\mu + (-1)^k\mu^*, d\mu d^* + \nu + (-1)^{k+1}\nu^*) \,|\, (\mu, \nu) \in S(B^{k+1}) \oplus S(B^k)\}},$$

with universal coefficient exact sequences

$$0 \to T\widehat{\ } \otimes_A \widehat{H}^k(\mathbb{Z}_2; A) \to \widehat{Q}^{2k}(B) \xrightarrow{\widehat{v}_{k+1}} \mathrm{Hom}_A(T, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) \to 0,$$

$$0 \to T\widehat{\ } \otimes_A \widehat{H}^{k+1}(\mathbb{Z}_2; A) \to \widehat{Q}^{2k+1}(B) \xrightarrow{\widehat{v}_k} \mathrm{Hom}_A(T, \widehat{H}^k(\mathbb{Z}_2; A)) \to 0.$$

Let $f : C \to D$ be a chain map of $S^{-1}A$-contractible A-module chain complexes concentrated in degrees $k, k+1$, inducing the A-module morphism

$$f^* = j : U = H^{k+1}(D) \to T = H^{k+1}(C).$$

By Proposition 77(i) a $(2k+1)$-dimensional symmetric Poincaré complex (C, ϕ) is essentially the same as a nonsingular $(-1)^{k+1}$-symmetric linking form (T, λ), and a $(2k+2)$-dimensional symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ is essentially the same as a lagrangian U for (T, λ), with $j = f^* : U \to T$ the inclusion. Similarly, a $(2k+1)$-dimensional quadratic Poincaré complex (C, ψ) is essentially the same as a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ), and a $(2k+2)$-dimensional quadratic Poincaré pair $(f : C \to D, (\delta\psi, \psi))$ is essentially the same as a lagrangian $U \subset T$ for (T, λ, μ). A $(2k+2)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ is a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ) together with a lagrangian $U \subset T$ for the nonsingular $(-1)^{k+1}$-symmetric linking form (T, λ).

Proposition 78. *Let U be an (A, S)-module together with an A-module morphism $\mu_1 : U \to \widehat{H}^{k+1}(\mathbb{Z}_2; A)$, defining a $(-1)^{k+1}$-quadratic linking form (U, λ_1, μ_1) over (A, S) with $\lambda_1 = 0$.*

(i) *There exists a map of chain bundles $(d, \chi) : (B_{k+2}, 0) \to (B_{k+1}, \delta)$ concentrated in degree $k+1$ such that the cone chain bundle $(B, \beta) = \mathcal{C}(d, \chi)$ has*

$$\begin{aligned}
&H_{k+1}(B) = U,\ \ H^{k+2}(B) = U^{\wedge},\ \ H_{k+2}(B) = H^{k+1}(B) = 0,\\
&\beta = [\delta] = \mu_1 \in \widehat{Q}^0(B^{0-*}) = \mathrm{Hom}_A(U, \widehat{H}^{k+1}(\mathbb{Z}_2; A)).
\end{aligned}$$

(ii) *The $(2k+2)$-dimensional twisted quadratic Q-group of (B, β) as in* (i)

$$\begin{aligned}
&Q_{2k+2}(B, \beta)\\
&\quad = \frac{\{(\phi, \theta) \in S(B^{k+1}) \oplus S(B^{k+1}) \mid \phi^* = (-1)^{k+1}\phi,\ \phi - \phi\delta\phi^* = \theta + (-1)^{k+1}\theta^*\}}{\{(d, \chi)_{\%}(v) + (0, \eta + (-1)^k\eta^*) \mid v \in S(B^{k+2}),\ \eta \in S(B^{k+1})\}}\\
&((d, \chi)_{\%}(v) = (d(v + (-1)^{k+1}v^*)d^*,\ dvd^* - d(v + (-1)^{k+1}v^*)\chi(v^* + (-1)^{k+1}v)d^*))
\end{aligned}$$

is the additive group of isomorphism classes of extensions of U to a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ) over (A, S) such that $U \subset T$ is a lagrangian of the $(-1)^{k+1}$-symmetric linking form (T, λ) and

$$\beta = \mu|_U : H_{k+1}(B) = U \to \widehat{H}^{k+1}(\mathbb{Z}_2; A) = \ker(Q_{(-1)^{k+1}}(A, S) \to S^{-1}A/A).$$

(iii) *An element $(\phi, \theta) \in Q_{2k+2}(B, \beta)$ is the algebraic normal invariant* (43) *of the $(2k+2)$-dimensional* (*symmetric, quadratic*) *Poincaré pair* $(f : C \to D, (\delta\phi, \psi) \in Q^{2k+2}_{2k+2}(f))$ *with*

$$\begin{aligned}
&d_C = \begin{pmatrix} d & \phi \\ 0 & d^* \end{pmatrix} : C_{k+1} = B_{k+2} \oplus B^{k+1} \to C_k = B_{k+1} \oplus B^{k+2},\\
&f = \text{projection} : C \to D = B^{2k+2-*}
\end{aligned}$$

constructed as in Proposition 42(ii), *corresponding to the quadruple $(T, \lambda, \mu; U)$ given by*

$$j = f^* : U = H^{k+1}(D) = H_{k+1}(B) \to T = H^{k+1}(C).$$

The A-module extension

$$0 \to U \to T \to U\widehat{\ } \to 0$$

is classified by

$$[\phi] \in H_{2k+2}(B \otimes_A B) = U \otimes_A U = \mathrm{Ext}^1_A(U\widehat{\ }, U).$$

(iv) *The* $(-1)^{k+1}$*-quadratic linking form* (T, λ, μ) *in* (iii) *corresponds to the* $(-1)^k$*-quadratic S-formation* $(Q, \psi; F, G)$ *with*

$$\begin{aligned}
&(Q,\psi) = H_{(-1)^k}(F), \ \ F = B_{k+2} \oplus B^{k+1}, \\
&G = \mathrm{im}\left(\begin{pmatrix} 1 & 0 \\ -\delta d & 1-\delta\phi \\ 0 & (-1)^{k+1}d^* \\ d & \phi \end{pmatrix} : B_{k+2} \oplus B^{k+1} \to B_{k+2} \oplus B^{k+1} \oplus B^{k+2} \oplus B_{k+1}\right) \\
&\quad \subset F \oplus F^*
\end{aligned}$$

such that

$$F \cap G = \{0\}, \ \ Q/(F+G) = H^{k+1}(C) = T.$$

The inclusion $U \to T$ *is resolved by*

$$\begin{tikzcd}
0 \arrow[r] & B_{k+2} \arrow[r, "d"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & B_{k+1} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{k+2} \oplus B^{k+1} \arrow[r, "\begin{pmatrix} 0 & (-)^{k+1}d^* \\ d & \phi \end{pmatrix}"] & B^{k+2} \oplus B_{k+1} \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

(v) *If the involution on A is even and* $k = -1$ *then*

$$Q_0(B,\beta) = \frac{\{\phi \in \mathrm{Sym}(B^0) \mid \phi - \phi\delta\phi \in \mathrm{Quad}(B^0)\}}{\{d\sigma d^* \mid \sigma \in \mathrm{Quad}(B^1)\}}.$$

An extension of $U = \operatorname{coker}(d : B_1 \to B_0)$ *to a nonsingular quadratic linking form* (T, λ, μ) *over* (A, S) *with* $\mu|_U = \mu_1$ *and* $U \subset T$ *a lagrangian of* (T, λ) *is classified by* $\phi \in Q_0(B, \beta)$ *such that* $\lambda : T \to T\widehat{\ }$ *is resolved by*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & B_1 \oplus B^0 \arrow[r, "\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ -\delta d & 1-\delta\phi \end{pmatrix}"'] & B^1 \oplus B_0 \arrow[r] \arrow[d, "\begin{pmatrix} 1 & -d^*\delta \\ 0 & 1-\phi\delta \end{pmatrix}"] & T \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & B_1 \oplus B^0 \arrow[r, "\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}"] & B^1 \oplus B_0 \arrow[r] & T\widehat{\ } \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{aligned}
& T = \operatorname{coker}\left(\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix} : B_1 \oplus B^0 \to B^1 \oplus B_0\right), \\
& \lambda : T \times T \to S^{-1}A/A; \\
& ((x_1, x_0), (y_1, y_0)) \mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(y_1) + d^{-1}(x_1)(y_0) + (d^*)^{-1}(x_0)(y_1), \\
& \mu : T \to Q_{+1}(A, S); \\
& (x_1, x_0) \mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(x_1) + d^{-1}(x_1)(x_0) + (d^*)^{-1}(x_0)(x_1) - \delta(x_0)(x_0), \\
& (x_0, y_0 \in B_0,\ \ x_1, y_1 \in B^1).
\end{aligned}$$

4.2. The linking Arf invariant

Definition 79. The *linking Arf invariant* of a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ) over (A, S) together with a lagrangian $U \subset T$ for (T, λ) is the image

$$(T, \lambda, \mu; U) = (g, \chi)_\%(\phi, \theta) \in \widehat{L}^{4*+2k+2}(A) = Q_{2k+2}(B^A, \beta^A)$$

of the algebraic normal invariant $(\phi, \theta) \in Q_{2k+2}(\mathcal{C}(f), \gamma)$ (43) of the corresponding $(2k+2)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi) \in Q^{2k+2}_{2k+2}(f))$ concentrated in degrees $k, k+1$ with

$$f^* = j : H^{k+1}(D) = U \to H^{k+1}(C) = T$$

and $(g, \chi)_\%$ induced by the classifying chain bundle map $(g, \chi) : (\mathcal{C}(f), \gamma) \to (B^A, \beta^A)$.

The chain bundle $(\mathcal{C}(f), \gamma)$ in 79 is (up to equivalence) of the type (B, β) considered in Proposition 78(i): the algebraic normal invariant $(\phi, \theta) \in Q_{2k+2}(B, \beta)$ classifies the extension of (U, β) to a lagrangian of a $(-1)^{k+1}$-symmetric linking form (T, λ) with a $(-1)^{k+1}$-quadratic function μ on T such that $\mu|_U = \beta$. The linking Arf invariant $(T, \lambda, \mu; U) \in Q_{2k+2}(B^A, \beta^A)$ gives the Witt class of $(T, \lambda, \mu; U)$. The boundary map

$$\partial : Q_{2k+2}(B^A, \beta^A) \to L_{2k+1}(A);\ \ (T, \lambda, \mu; U) \mapsto (Q, \psi; F, G)$$

sends the linking Arf invariant to the Witt class of the $(-1)^k$-quadratic formation $(Q, \psi; F, G)$ constructed in 78(iv).

Theorem 80. *Let A be an r-even ring with A_2-module basis $\{x_1 = 1, x_2, \ldots, x_r\} \subset \widehat{H}^0(\mathbb{Z}_2; A)$, and let*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A),$$

so that by Theorem 60

$$Q_{2k}(B^A, \beta^A) = \begin{cases} \dfrac{\{M \in \mathrm{Sym}_r(A) \mid M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N + N^t) - N^t XN \mid N \in M_r(A)\}} & \text{if } k = 0, \\ 0 & \text{if } k = 1. \end{cases}$$

(i) *Let*

$$S = (2)^\infty \subset A,$$

so that

$$S^{-1}A = A[1/2]$$

and $\widehat{H}^0(\mathbb{Z}_2; A)$ is an (A, S)-module. The hyperquadratic L-group $\widehat{L}^0(A)$ fits into the exact sequence

$$\cdots \to L^1(A, S) \to \widehat{L}^0(A) \to L_0(A, S) \to L^0(A, S) \to \cdots\ .$$

The linking Arf invariant of a nonsingular quadratic linking form (T, λ, μ) over (A, S) with a lagrangian $U \subset T$ for (T, λ) is the Witt class

$$(T, \lambda, \mu; U) \in Q_0(B^A, \beta^A) = \widehat{L}^{4*}(A).$$

(ii) *Given* $M \in \mathrm{Sym}_r(A)$ *such that* $M - MXM \in \mathrm{Quad}_r(A)$ *let* (T_M, λ_M, μ_M) *be the nonsingular quadratic linking form over* (A, S) *corresponding to the* (-1)*-quadratic* S*-formation over* A (76)

$$(Q_M, \psi_M; F_M, G_M) = \left(H_-(A^{2r}); A^{2r}, \mathrm{im}\left(\begin{pmatrix} \begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix} \\ \begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix} \end{pmatrix} : A^{2r} \to A^{2r} \oplus A^{2r} \right) \right)$$

and let

$$U_M = (A_2)^r \subset T_M = Q_M/(F_M + G_M) = \mathrm{coker}(G_M \to F_M^*)$$

be the lagrangian for the nonsingular symmetric linking form (T_M, λ_M) *over* (A, S) *with the inclusion* $U_M \to T_M$ *resolved by*

$$\begin{tikzcd}
0 \arrow[r] & A^r \arrow[r, "2I"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & A^r \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & U_M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^r \oplus A^r \arrow[r, "\begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix}"] & A^r \oplus A^r \arrow[r] & T_M \arrow[r] & 0
\end{tikzcd}$$

The function

$$Q_0(B^A, \beta^A) \to \widehat{L}^{4*}(A); \ \ M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

is an isomorphism, with inverse given by the linking Arf invariant.

(iii) *Let* (T, λ, μ) *be a nonsingular quadratic linking form over* (A, S) *together with a lagrangian* $U \subset T$ *for* (T, λ). *For any f.g. projective* A*-module resolution of* U

$$0 \to B_1 \xrightarrow{d} B_0 \to U \to 0$$

let

$$\delta \in \mathrm{Sym}(B_0), \ \ \phi \in \mathrm{Sym}(B^0), \ \ \beta = [\delta] = \mu|_U \in \widehat{Q}^0(B^{0-*})$$
$$= \mathrm{Hom}_A(U, \widehat{H}^0(\mathbb{Z}_2; A))$$

be as in Proposition 78(i) and (v), *so that*

$$d^*\delta d \in \mathrm{Quad}(B_1),\ \ \phi - \phi\delta\phi \in \mathrm{Quad}(B^0)$$

and

$$\phi \in Q_0(B,\beta) = \frac{\ker(J_\delta : \mathrm{Sym}(B^0) \to \mathrm{Sym}(B^0)/\mathrm{Quad}(B^0))}{\mathrm{im}((d^*)^\% : \mathrm{Quad}(B^1) \to \mathrm{Sym}(B^0))}$$

classifies $(T,\lambda,\mu;U)$. *Lift* $\beta : U \to \widehat{H}^0(\mathbb{Z}_2;A)$ *to an A-module morphism* $g : B_0 \to A^r$ *such that*

$$gd(B_1) \subseteq 2A^r,\ \ \delta = g^*Xg \in \widehat{H}^0(\mathbb{Z}_2; S(B^0),T) = \mathrm{Sym}(B^0)/\mathrm{Quad}(B^0).$$

The linking Arf invariant is

$$(T,\lambda,\mu;U) = g\phi g^* \in Q_0(B^A,\beta^A).$$

(iv) *For any* $M = (m_{ij}) \in \mathrm{Sym}_r(A)$ *with* $m_{ij} \in 2A$

$$M - MXM = 2(M/2 - 2(M/2)X(M/2)) \in \mathrm{Quad}_r(A)$$

and so M represents an element $M \in Q_0(B^A,\beta^A)$. *The invertible matrix*

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} \in M_{2r}(A)$$

is such that

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & 2I \\ 2I & vM \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix}\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix}$$

so that $(Q_M,\psi_M;F_M,G_M)$ *is isomorphic to the* (-1)*-quadratic S-formation*

$$\begin{aligned}&(Q'_M,\psi'_M;F'_M,G'_M)\\ &\quad = \left(H_-(A^{2r}); A^{2r}, \mathrm{im}\left(\begin{pmatrix} \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix} \\ \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} \end{pmatrix} : A^{2r} \to A^{2r} \oplus A^{2r}\right)\right),\end{aligned}$$

corresponding to the nonsingular quadratic linking form over (A, S)

$$(T'_M, \lambda'_M, \mu'_M) = \left((A_2)^r \oplus (A_2)^r, \begin{pmatrix} -M/4 & I/2 \\ I/2 & 0 \end{pmatrix}, \begin{pmatrix} -M/4 \\ -X \end{pmatrix}\right),$$

with $2T'_M = 0$, *and* $U'_M = 0 \oplus (A_2)^r \subset T'_M$ *a lagrangian for the symmetric linking form* (T'_M, λ'_M). *The linking Arf invariant of* $(T'_M, \lambda'_M, \mu'_M; U'_M)$ *is*

$$(T'_M, \lambda'_M, \mu'_M; U'_M) = M \in Q_0(B^A, \beta^A).$$

Proof. (i) $\widehat{H}^0(\mathbb{Z}_2; A)$ has an $S^{-1}A$-contractible f.g. free A-module resolution

$$0 \longrightarrow A^r \xrightarrow{2} A^r \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2; A) \longrightarrow 0.$$

The exact sequence for $\widehat{L}^0(A)$ is given by the exact sequence of Proposition 74(iii)

$$\cdots \to L^{4*+1}(A, S) \to Q_0(B^A, \beta^A) \to L_0(A, S) \to L^{4*}(A, S) \to \cdots$$

and the isomorphism $Q_0(B^A, \beta^A) \cong \widehat{L}^{4*}(A)$.

(ii) The isomorphism

$$Q_0(B^A, \beta^A) \to \widehat{L}^{4*}(A); \;\; M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

is given by Proposition 46.

(iii) Combine (ii) and Proposition 78.

(iv) By construction. □

5. Application to UNil

5.1. Background

The topological context for the unitary nilpotent L-groups UNil_* is the following. Let N^n be a closed connected manifold together with a decomposition into n-dimensional connected submanifolds $N_-, N_+ \subset N$ such that

$$N = N_- \cup N_+$$

and

$$N_\cap = N_- \cap N_+ = \partial N_- = \partial N_+ \subset N$$

is a connected $(n-1)$-manifold with $\pi_1(N_\cap) \to \pi_1(N_\pm)$ injective. Then

$$\pi_1(N) = \pi_1(N_-) *_{\pi_1(N_\cap)} \pi_1(N_+),$$

with $\pi_1(N_\pm) \to \pi_1(N)$ injective. Let M be an n-manifold. A homotopy equivalence $f : M \to N$ is called *splittable along* $N_\cap$ if it is homotopic to a map f', transverse regular to $N_\cap$ (whence $f'^{-1}(N_\cap)$ is an $(n-1)$-dimensional submanifold of M), and whose restriction $f'^{-1}(N_\cap) \to N_\cap$, and a fortiori also $f'^{-1}(N_\pm) \to N_\pm$, is a homotopy equivalence.

We ask the following question: given a simple homotopy equivalence $f : M \to N$, when is M h-cobordant to a manifold M' such that the induced homotopy equivalence $f' : M' \to N$ is splittable along $N_\cap$? The answer is given by Cappell [5,6]: the problem has a positive solution if and only if a Whitehead torsion obstruction

$$\overline{\Phi}(\tau(f)) \in \widehat{H}^n(\mathbb{Z}_2; \ker(\widetilde{K}_0(A) \to \widetilde{K}_0(B_+) \oplus \widetilde{K}_0(B_-)))$$

(which is 0 if f is simple) and an algebraic L-theory obstruction

$$\chi^h(f) \in \mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

vanish, where

$$A = \mathbb{Z}[\pi_1(N_\cap)], \;\; B_\pm = \mathbb{Z}[\pi_1(N_\pm)], \;\; \mathcal{N}_\pm = B_\pm - A.$$

The groups $\mathrm{UNil}_*(A; \mathcal{N}_-, \mathcal{N}_+)$ are 4-periodic and 2-primary, and vanish if the inclusions $\pi_1(N_\cap) \hookrightarrow \pi_1(N_\pm)$ are square root closed. The groups $\mathrm{UNil}_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ arising from the expression of the infinite dihedral group as a free product

$$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$$

are of particular interest. Cappell [3] showed that

$$\mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}_2 - \{1\}], \mathbb{Z}[\mathbb{Z}_2 - \{1\}])$$

contains $(\mathbb{Z}_2)^\infty$, and deduced that there is a manifold homotopy equivalent to the connected sum $\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ which does not have a compatible connected sum decomposition. With

$$B = \mathbb{Z}[\pi_1(N)] = B_1 *_A B_2,$$

the map

$$\mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow L_{n+1}(B)$$

given by sending the splitting obstruction $\chi^h(f)$ to the surgery obstruction of an $(n+1)$-dimensional normal map between f and a split homotopy equivalence, is a split monomorphism, and

$$L_{n+1}(B) = L^K_{n+1}(A \to B_+ \cup B_-) \oplus \mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

with $K = \ker(\widetilde{K}_0(A) \to \widetilde{K}_0(B_+) \oplus \widetilde{K}_0(B_-))$. Farrell [11] established a factorization of this map as

$$\mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow \mathrm{UNil}_{n+1}(B; B, B) \longrightarrow L_{n+1}(B).$$

Thus the groups $\mathrm{UNil}_n(A; A, A)$ for any ring A with involution acquire special importance, and we shall use the usual abbreviation

$$\mathrm{UNil}_n(A) = \mathrm{UNil}_n(A; A, A).$$

Cappell [3–5] proved that $\mathrm{UNil}_{4k}(\mathbb{Z}) = 0$ and that $\mathrm{UNil}_{4k+2}(\mathbb{Z})$ is infinitely generated. Farrell [11] showed that for any ring A, $4\mathrm{UNil}_*(A) = 0$. Connolly and Koźniewski [9] obtained $\mathrm{UNil}_{4k+2}(\mathbb{Z}) = \bigoplus_1^\infty \mathbb{Z}_2$.

For any ring with involution A let NL_* denote the L-theoretic analogues of the nilpotent K-groups

$$NK_*(A) = \ker(K_*(A[x]) \to K_*(A)),$$

that is

$$NL_*(A) = \ker(L_*(A[x]) \to L_*(A)),$$

where $A[x] \to A$ is the augmentation map $x \mapsto 0$. Ranicki [16, 7.6] used the geometric interpretation of $\mathrm{UNil}_*(A)$ to identify $NL_*(A) = \mathrm{UNil}_*(A)$ in the case when $A = \mathbb{Z}[\pi]$ is the integral group ring of a finitely presented group π. The following was obtained by pure algebra:

Proposition 81 (*Connolly and Ranicki [10]*). *For any ring with involution A*

$$\mathrm{UNil}_*(A) \cong NL_*(A).$$

It was further shown in [10] that $\mathrm{UNil}_1(\mathbb{Z}) = 0$ and $\mathrm{UNil}_3(\mathbb{Z})$ was computed up to extensions, thus showing it to be infinitely generated.

Connolly and Davis [8] related $\mathrm{UNil}_3(\mathbb{Z})$ to quadratic linking forms over $\mathbb{Z}[x]$ and computed the Grothendieck group of the latter. By Proposition 81

$$\mathrm{UNil}_3(\mathbb{Z}) \cong \ker(L_3(\mathbb{Z}[x]) \to L_3(\mathbb{Z})) = L_3(\mathbb{Z}[x])$$

using the classical fact $L_3(\mathbb{Z}) = 0$. From a diagram chase one gets

$$L_3(\mathbb{Z}[x]) \cong \ker(L_0(\mathbb{Z}[x], (2)^\infty) \to L_0(\mathbb{Z}, (2)^\infty)).$$

By definition, $L_0(\mathbb{Z}[x], (2)^\infty)$ is the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$, with $2^n T = 0$ for some $n \geqslant 1$. Let $\mathcal{L}(\mathbb{Z}[x], 2)$ be a similar Witt group, the difference being that the underlying module T is required to satisfy $2T = 0$. The main results of [8] are

$$L_0(\mathbb{Z}[x], (2)^\infty) \cong \mathcal{L}(\mathbb{Z}[x], 2)$$

and

$$\mathcal{L}(\mathbb{Z}[x], 2) \cong \frac{x\mathbb{Z}_4[x]}{\{2(p^2 - p) \mid p \in x\mathbb{Z}_4[x]\}} \oplus \mathbb{Z}_2[x].$$

By definition, a ring A is *one-dimensional* if it is hereditary and noetherian, or equivalently if every submodule of a f.g. projective A-module is f.g. projective. In particular, a Dedekind ring A is one-dimensional. The symmetric and hyperquadratic L-groups of a one-dimensional A are 4-periodic

$$L^n(A) = L^{n+4}(A), \ \widehat{L}^n(A) = \widehat{L}^{n+4}(A).$$

Proposition 82 (*Connolly and Ranicki [10]*). *For any one-dimensional ring A with involution*

$$Q_{n+1}(B^{A[x]}, \beta^{A[x]}) = Q_{n+1}(B^A, \beta^A) \oplus \mathrm{UNil}_n(A) \quad (n \in \mathbb{Z}).$$

Proof. For any ring with involution A the inclusion $A \to A[x]$ and the augmentation $A[x] \to A; x \mapsto 0$ determine a functorial splitting of the exact sequence

$$\cdots \to L_n(A[x]) \to L^n(A[x]) \to \widehat{L}^n(A[x]) \to L_{n-1}(A[x]) \to \cdots$$

as a direct sum of the exact sequences

$$\cdots \to L_n(A) \to L^n(A) \to \widehat{L}^n(A) \to L_{n-1}(A) \to \cdots \ ,$$
$$\cdots \to NL_n(A) \to NL^n(A) \to N\widehat{L}^n(A) \to NL_{n-1}(A) \to \cdots \ .$$

with $\widehat{L}^{n+4*}(A) = Q_n(B^A, \beta^A)$. It is proved in [10] that for a one-dimensional A

$$L^n(A[x]) = L^n(A), \ NL^n(A) = 0, \ N\widehat{L}^{n+1}(A) = NL_n(A) = \mathrm{UNil}_n(A). \quad \square$$

Example 83. Proposition 82 applies to $A = \mathbb{Z}$, so that

$$Q_{n+1}(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = Q_{n+1}(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \oplus \mathrm{UNil}_n(\mathbb{Z})$$

with $Q_*(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) = \widehat{L}^*(\mathbb{Z})$ as given by Example 62.

5.2. The computation of $Q_*(B^{A[x]}, \beta^{A[x]})$ for 1-even A with $\psi^2 = 1$

We shall now compute the groups

$$\widehat{L}^n(A[x]) = Q_n(B^{A[x]}, \beta^{A[x]}) \quad (n(\mathrm{mod}\, 4))$$

for a 1-even ring A with $\psi^2 = 1$. The special case $A = \mathbb{Z}$ computes

$$\widehat{L}^n(\mathbb{Z}[x]) = Q_n(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \widehat{L}^n(\mathbb{Z}) \oplus \mathrm{UNil}_{n-1}(\mathbb{Z}).$$

Proposition 84. *The universal chain bundle over $A[x]$ is given by*

$$(B^{A[x]}, \beta^{A[x]}) = \bigoplus_{i=-\infty}^{\infty} (C(X), \gamma(X))_{*+2i},$$

with $(C(X), \gamma(X))$ the chain bundle over $A[x]$ given by the construction of (53) *for*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \mathrm{Sym}_2(A[x]).$$

The twisted quadratic Q-groups of $(B^{A[x]}, \beta^{A[x]})$ are

$$Q_n(B^{A[x]}, \beta^{A[x]}) = \begin{cases} Q_0(C(X), \gamma(X)) = \dfrac{\{M \in \mathrm{Sym}_2(A[x]) \mid M - MXM \in \mathrm{Quad}_2(A[x])\}}{4\mathrm{Quad}_2(A[x]) + \{2(N + N^t) - N^t XN \mid N \in M_2(A[x])\}} & \text{if } n = 0, \\ \mathrm{im}(N_{\gamma(X)} : Q_1(C(X), \gamma(X)) \to Q^1(C(X))) = \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))) & \\ = \dfrac{\{N \in M_2(A[x]) \mid N + N^t \in 2\mathrm{Sym}_2(A[x]), \frac{1}{2}(N + N^t) - N^t XN \in \mathrm{Quad}_2(A[x])\}}{2M_2(A[x])} & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ Q_{-1}(C(X), \gamma(X)) = \dfrac{\mathrm{Sym}_2(A[x])}{\mathrm{Quad}_2(A[x]) + \{L - LXL \mid L \in \mathrm{Sym}_2(A[x])\}} & \text{if } n = 3. \end{cases}$$

Proof. A special case of Theorem 60, noting that by Proposition 59 $A[x]$ is 2-even, with $\{1, x\}$ an $A_2[x]$-module basis for $\widehat{H}^0(\mathbb{Z}_2; A[x])$. $\square$

Our strategy for computing $Q_*(B^{A[x]}, \beta^{A[x]})$ will be to first compute $Q_*(C(1), \gamma(1))$, $Q_*(C(x), \gamma(x))$ and then to compute $Q_*(C(X), \gamma(X))$ for

$$(C(X), \gamma(X)) = (C(1), \gamma(1)) \oplus (C(x), \gamma(x))$$

using the exact sequence given by Proposition 38(ii)

$$\cdots \to H_{n+1}(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\overline{\partial}} Q_n(C(1), \gamma(1)) \oplus Q_n(C(x), \gamma(x))$$
$$\to Q_n(C(X), \gamma(X)) \to H_n(C(1) \otimes_{A[x]} C(x)) \to \cdots .$$

The connecting maps ∂ have components

$$\begin{aligned}
\partial(1) : H_{n+1}(C(1) \otimes_{A[x]} C(x)) &\to \widehat{Q}^{n+1}(C(1)) \to Q_n(C(1), \gamma(1)) \\
(f(1) : C(x)^{n+1-*} \to C(1)) &\mapsto (0, \widehat{f(1)}^{\%}(S^{n+1}\gamma(x))), \\
\partial(x) : H_{n+1}(C(1) \otimes_{A[x]} C(x)) &\to \widehat{Q}^{n+1}(C(x)) \to Q_n(C(x), \gamma(x)) \\
(f(x) : C(1)^{n+1-*} \to C(x)) &\mapsto (0, \widehat{f(x)}^{\%}(S^{n+1}\gamma(1))).
\end{aligned}$$

Proposition 85. (i) *The twisted quadratic Q-groups*

$$Q_n(C(1), \gamma(1)) = \begin{cases} \dfrac{A[x]}{2A[x] + \{a - a^2 \mid a \in A[x]\}} & \text{if } n = -1, \\ \dfrac{\{a \in A[x] \mid a - a^2 \in 2A[x]\}}{8A[x] + \{4b - 4b^2 \mid b \in A[x]\}} & \text{if } n = 0, \\ \dfrac{\{a \in A[x] \mid a - a^2 \in 2A[x]\}}{2A[x]} & \text{if } n = 1, \end{cases}$$

(*as given by Theorem* 54) *are such that*

$$Q_n(C(1), \gamma(1)) \cong \begin{cases} A_2[x] & \text{if } n = -1, \\ A_8 \oplus A_4[x] \oplus A_2[x] & \text{if } n = 0, \\ A_2 & \text{if } n = 1, \end{cases}$$

with isomorphisms

$$\begin{aligned}
&f_{-1}(1) : Q_{-1}(C(1), \gamma(1)) \to A_2[x];\ \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 + \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}\right) x^{i+1},\\
&f_0(1) : Q_0(C(1), \gamma(1)) \to A_8 \oplus A_4[x] \oplus A_2[x];\\
&\qquad\qquad \sum_{i=0}^{\infty} a_i x^i \mapsto \left(a_0, \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}/2\right) x^i, \sum_{k=0}^{\infty} (a_{2k+2}/2) x^k\right),\\
&f_1(1) : Q_1(C(1), \gamma(1)) \to A_2;\ a = \sum_{i=0}^{\infty} a_i x^i \mapsto a_0.
\end{aligned}$$

The connecting map components $\partial(1)$ *are given by*

$$\begin{aligned}
&\partial(1) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to Q_0(C(1), \gamma(1));\ c \mapsto (0, 2c, 0),\\
&\partial(1) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to Q_{-1}(C(1), \gamma(1));\ c \mapsto cx.
\end{aligned}$$

(ii) *The twisted quadratic Q-groups*

$$Q_n(C(x), \gamma(x)) = \begin{cases} \dfrac{A[x]}{2A[x] + \{a - a^2 x \mid a \in A[x]\}} & \text{if } n = -1, \\ \dfrac{\{a \in A[x] \mid a - a^2 x \in 2A[x]\}}{8A[x] + \{4b - 4b^2 x \mid b \in A[x]\}} & \text{if } n = 0, \\ \dfrac{\{a \in A[x] \mid a - a^2 x \in 2A[x]\}}{2A[x]} & \text{if } n = 1 \end{cases}$$

(*as given by Theorem* 54) *are such that*

$$Q_n(C(x), \gamma(x)) \cong \begin{cases} A_2[x] & \text{if } n = -1, \\ A_4[x] \oplus A_2[x] & \text{if } n = 0, \\ 0 & \text{if } n = 1, \end{cases}$$

with isomorphisms

$$\begin{aligned}
&f_{-1}(x) : Q_{-1}(C(x), \gamma(x)) \to A_2[x];\ a = \sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j-1}\right) x^i,\\
&f_0(x) : Q_0(C(x), \gamma(x)) \to A_4[x] \oplus A_2[x];\\
&\qquad\qquad \sum_{i=0}^{\infty} a_i x^i \mapsto \left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j-1}/2\right) x^i, \sum_{k=0}^{\infty} (a_{2k+1}/2) x^k\right).
\end{aligned}$$

The connecting map components $\partial(x)$ are given by

$$\partial(x) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to Q_0(C(x), \gamma(x)); \;\; c \mapsto (2c, 0),$$
$$\partial(x) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to Q_{-1}(C(x), \gamma(x)); \;\; c \mapsto c.$$

Proof. (i) We start with $Q_1(C(1), \gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that $a(x) - a(x)^2 \in 2A[x]$ if and only if

$$a_{2i+1}, \;\; a_{2i+2} - (a_{i+1})^2 \in 2A \quad (i \geqslant 0),$$

if and only if $a_k \in 2A$ for all $k \geqslant 1$, so that $f_1(1)$ is an isomorphism.

Next, we consider $Q_{-1}(C(1), \gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a(x) \in 2A[x] + \{b(x) - b(x)^2 \mid b(x) \in A[x]\}$$

if and only if there exist $b_1, b_2, \ldots \in A$ such that

$$a_0 = 0, \;\; a_1 = b_1, \;\; a_2 = b_2 - b_1, \;\; a_3 = b_3, \;\; a_4 = b_4 - b_2, \;\; \ldots \in A_2,$$

if and only if

$$a_0 = \sum_{j=0}^{\infty} a_{(2i+1)2^j} = 0 \in A_2 \quad (i \geqslant 0)$$

(with $b_{(2i+1)2^j} = \sum_{k=0}^{j} a_{(2i+1)2^k} \in A_2$ for any $i, j \geqslant 0$). Thus $f_{-1}(1)$ is well-defined and injective. The morphism $f_{-1}(1)$ is surjective, since

$$\sum_{i=0}^{\infty} c_i x^i = f_{-1}(1)\left(c_0 + \sum_{i=0}^{\infty} c_{i+1} x^{2i+1}\right) \in A_2[x] \quad (c_i \in A).$$

The map $\widehat{Q}^1(C(1)) \to Q_0(C(1), \gamma(1))$ is given by

$$\widehat{Q}^1(C(1)) = A_2[x] \to Q_0(C(1), \gamma(1)) = A_8 \oplus A_4[x] \oplus A_2[x],$$
$$a = \sum_{i=0}^{\infty} a_i x^i \mapsto \left(4a_0, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j}\right) x^i, 0\right).$$

If $a = c^2x$ for $c = \sum\limits_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\left(4a_0, \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j}\right)x^i\right) = (0, 2c) \in A_8 \oplus A_4[x],$$

so that the composite

$$\begin{aligned}\partial(1) : H_1(C(1) \otimes_{A[x]} C(x)) \;&=\; A_2[x] \to \widehat{Q}^1(C(1)) \to Q_0(C(1), \gamma(1)) \\ &=\; A_8 \oplus A_4[x] \oplus \mathbb{Z}_2[x]\end{aligned}$$

is given by $c \mapsto (0, 2c, 0)$.

Next, we consider $Q_0(C(1), \gamma(1))$. A polynomial $a(x) \in A \oplus 2xA[x]$ is such that

$$a(x) \in 8A[x] + \{4(b(x) - b(x)^2) \mid b(x) \in A[x]\},$$

if and only if there exist $b_1, b_2, \ldots \in A$ such that

$$a_0 = 0,\;\; a_1 = 4b_1,\;\; a_2 = 4(b_2 - b_1),\;\; a_3 = 4b_3,\;\; a_4 = 4(b_4 - b_2), \cdots \in A_8,$$

if and only if

$$\begin{aligned}&a_1 = a_2 = a_3 = a_4 = \cdots = 0 \in A_4, \\ &a_0 = \sum_{j=0}^{\infty} a_{(2i+1)2^j} = 0 \in A_8 \quad (i \geqslant 0).\end{aligned}$$

Thus $f_0(1)$ is well-defined and injective. The morphism $f_0(1)$ is surjective, since

$$\begin{aligned}\left(a_0, \sum_{i=0}^{\infty} b_i x^i, \sum_{i=0}^{\infty} c_i x^i\right) \;&=\; f_0(1)\left(a_0 + 2\sum_{i=0}^{\infty} b_i x^{2i+1} + 2\sum_{i=0}^{\infty} c_i x^{2i+2}\right) \\ &\in\; A_8 \oplus A_4[x] \oplus A_2[x] \quad (a_0, b_i, c_i \in A).\end{aligned}$$

The map $\widehat{Q}^0(C(1)) \to Q_{-1}(C(1), \gamma(1))$ is given by

$$\begin{aligned}&\widehat{Q}^0(C(1)) = A_2[x] \to Q_{-1}(C(1), \gamma(1)) = A_2[x], \\ &\qquad a = \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 + \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}\right)x^{i+1}.\end{aligned}$$

If $a = c^2x$ for $c = \sum\limits_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$a_0 + \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}\right)x^{i+1} = cx \in A_2[x],$$

so that the composite

$$\partial(1) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to \widehat{Q}^0(C(1)) \to Q_{-1}(C(1), \gamma(1)) = A_2[x]$$

is given by $c \mapsto cx$.

(ii) We start with $Q_1(C(x), \gamma(x))$. For any polynomial $a = \sum\limits_{i=0}^{\infty} a_i x^i \in A[x]$

$$a - a^2x = \sum_{i=0}^{\infty} a_i x^i - \sum_{i=0}^{\infty} a_i x^{2i+1} \in A_2[x].$$

Now $a - a^2x \in 2A[x]$ if and only if the coefficients $a_0, a_1, \ldots \in A$ are such that

$$a_0 = a_1 - a_0 = a_2 = a_3 - a_1 = \cdots = 0 \in A_2,$$

if and only if

$$a_0 = a_1 = a_2 = a_3 = \cdots = 0 \in A_2.$$

It follows that $Q_1(C(x), \gamma(x)) = 0$.

Next, $Q_{-1}(C(x), \gamma(x))$. A polynomial $a = \sum\limits_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a \in 2A[x] + \{b - b^2x \mid v \in A[x]\},$$

if and only if there exist $b_0, b_1, \ldots \in A$ such that

$$a_0 = b_0, \quad a_1 = b_1 - b_0, \quad a_2 = b_2, \quad a_3 = b_3 - b_1, \quad a_4 = b_4, \quad \ldots \quad \in A_2,$$

if and only if

$$\sum_{j=0}^{\infty} a_{(2i+1)2^j-1} = 0 \in A_2 \quad (i \geqslant 0).$$

Thus $f_{-1}(x)$ is well-defined and injective. The morphism $f_{-1}(x)$ is surjective, since

$$\begin{aligned}\sum_{i=0}^{\infty} c_i x^i &= f_{-1}(x)\left(\sum_{i=0}^{\infty} c_i x^{2i}\right)\\ &\in A_2[x] \quad (c_i \in A).\end{aligned}$$

The map $\widehat{Q}^0(C(x)) \to Q_{-1}(C(x), \gamma(x))$ is given by

$$\begin{aligned}\widehat{Q}^0(C(x)) = A_2[x] &\to Q_{-1}(C(x), \gamma(x)) = A_2[x];\\ b = \sum_{i=0}^{\infty} b_i x^i &\mapsto \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1}\right) x^i.\end{aligned}$$

If $b = c^2$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1}\right) x^i = c \in A_2[x],$$

so that the composite

$$\partial(x) : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to \widehat{Q}^0(C(x)) \to Q_{-1}(C(x), \gamma(x)) = A_2[x]$$

is just the identity $c \mapsto c$.

Next, $Q_0(C(x), \gamma(x))$. For any $a \in A[x]$

$$a \in 8A[x] + \{4(b - b^2 x) \mid b \in A[x]\},$$

if and only there exist $b_0, b_1, \ldots \in A$ such that

$$a_0 = 4b_0,\ a_1 = 4(b_1 - b_0),\ a_2 = 4b_2,\ a_3 = 4(b_3 - b_1),\ \cdots \in A_8,$$

if and only if

$$\begin{aligned}&a_0 = a_1 = a_2 = a_3 = \cdots = 0 \in A_4,\\ &\sum_{j=0}^{\infty} a_{(2i+1)2^j-1} = 0 \in A_8 \quad (i \geqslant 0).\end{aligned}$$

Thus $f_0(x)$ is well-defined and injective. The morphism $f_0(x)$ is surjective, since

$$\left(\sum_{i=0}^{\infty} c_i x^i, \sum_{i=0}^{\infty} d_i x^i\right) = f_0(x)\left(\sum_{i=0}^{\infty} 2c_i x^{2i} + \sum_{i=0}^{\infty} 2d_i x^{2i+1}\right)$$
$$\in A_4[x] \oplus A_2[x] \quad (c_i, d_i \in A).$$

The map $\widehat{Q}^1(C(x)) \to Q_0(C(x), \gamma(x))$ is given by

$$\widehat{Q}^1(C(x)) = A_2[x] \to Q_0(C(x), \gamma(x)) = A_4[x] \oplus A_2[x];$$
$$b = \sum_{i=0}^{\infty} b_i x^i \mapsto \left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} 2b_{(2i+1)2^j-1}\right) x^i, 0\right).$$

If $b = c^2$ for $c = \sum_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j} x^i\right) = 2c \in A_4[x],$$

so that the composite

$$\partial(x) : H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to \widehat{Q}^1(C(x)) \to Q_0(C(x), \gamma(x))$$
$$= A_4[x] \oplus A_2[x]$$

is given by $c \mapsto (2c, 0)$.

We can now prove Theorem B:

Theorem 86. *The hyperquadratic L-groups of $A[x]$ for a 1-even A with $\psi^2 = 1$ are given by*

$$\widehat{L}^n(A[x]) = Q_n(B^{A[x]}, \beta^{A[x]}) = \begin{cases} A_8 \oplus A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 0 (\operatorname{mod} 4),\\ A_2 & \text{if } n \equiv 1 (\operatorname{mod} 4),\\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4),\\ A_2[x] & \text{if } n \equiv 3 (\operatorname{mod} 4).\end{cases}$$

(i) *For $n = 0$*

$$Q_0(B^{A[x]}, \beta^{A[x]}) = \frac{\{M \in \mathrm{Sym}_2(A[x]) \mid M - MXM \in \mathrm{Quad}_2(A[x])\}}{4\mathrm{Quad}_2(A[x]) + \{2(N + N^t) - 4N^t XN \mid N \in M_2(A[x])\}}.$$

An element $M \in Q_0(B^{A[x]}, \beta^{A[x]})$ *is represented by a matrix*

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}_2(A[x]) \quad \left(a = \sum_{i=0}^{\infty} a_i x^i, c = \sum_{i=0}^{\infty} c_i x^i \in A[x] \right),$$

with $a - a_0, b, c \in 2A[x]$. *The isomorphism*

$$Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} \widehat{L}^0(A[x]) = \widehat{L}^1(A[x], (2)^\infty); \;\; M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

sends M *to the Witt class of the nonsingular quadratic linking form* (T_M, λ_M, μ_M) *over* $(A[x], (2)^\infty)$ *with a lagrangian* $U_M \subset T_M$ *for* (T_M, λ_M) *corresponding to the* (-1)*-quadratic* $(2)^\infty$*-formation over* $A[x]$

$$\partial(M) = \left(H_-(A[x]^4); A[x]^4, \mathrm{im}\left(\begin{pmatrix} \begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix} \\ \begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix} \end{pmatrix} : A[x]^4 \to A[x]^4 \oplus A[x]^4 \right) \right)$$

(80), *with*

$$\partial : Q_0(B^{A[x]}, \beta^{A[x]}) = \widehat{L}^0(A[x]) \to L_{-1}(A[x]); \;\; M \mapsto \partial(M).$$

The inverse isomorphism is defined by the linking Arf invariant (79). *Writing*

$$2\Delta : A_2[x] \to A_4[x] \oplus A_4[x]; \;\; d \mapsto (2d, 2d),$$

there are defined isomorphisms

$$Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} A_8 \oplus \mathrm{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x];$$

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \;\; (c' = c - b^2)$$

$$\mapsto \left(a_0, \left[\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}/2 \right) x^i, \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} c'_{(2i+1)2^j - 1}/2 \right) x^i \right], \sum_{k=0}^{\infty} (a_{2k+2}/2) x^k, \right.$$
$$\left. \sum_{k=0}^{\infty} (c'_{2k+1}/2) x^k \right),$$

$$\mathrm{coker}(2\Delta) \xrightarrow{\cong} A_4[x] \oplus A_2[x]; \;\; [d, e] \mapsto (d - e, d).$$

In particular $M \in Q_0(B^{A[x]}, \beta^{A[x]})$ *can be represented by a diagonal matrix* $\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix}$.

(ii) *For* $n=1$:

$$Q_1(B^{A[x]},\beta^{A[x]}) \\ = \frac{\{N\in M_2(A[x])\,|\,N+N^t\in 2\mathrm{Sym}_2(A[x]),\ \frac{1}{2}(N+N^t)-N^tXN\in \mathrm{Quad}_2(A[x])\}}{2M_2(A[x])}$$

and there is defined an isomorphism

$$Q_1(B^{A[x]},\beta^{A[x]}) \xrightarrow{\cong} Q_1(B^A,\beta^A)=A_2;\quad N=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto a_0,$$

with

$$\begin{aligned}&\partial: Q_1(B^{A[x]},\beta^{A[x]})=\widehat{L}^1(A[x])=A_2\to L_0(A[x]);\\ &a_0\mapsto A[x]\otimes_A\left(A\oplus A,\begin{pmatrix} a_0(a_0-1)/2 & 1-2a_0\\ 0 & -2\end{pmatrix}\right).\end{aligned}$$

(iii) *For* $n=2$:

$$Q_2(B^{A[x]},\beta^{A[x]})=0.$$

(iv) *For* $n=3$:

$$Q_3(B^{A[x]},\beta^{A[x]})=\frac{\mathrm{Sym}_2(A[x])}{\mathrm{Quad}_2(A[x])+\{M-MXM\,|\,M\in\mathrm{Sym}_2(A[x])\}}.$$

There is defined an isomorphism

$$\begin{aligned}&Q_3(B^{A[x]},\beta^{A[x]}) \xrightarrow{\cong} A_2[x];\\ &M=\begin{pmatrix} a & b\\ b & c\end{pmatrix}=\begin{pmatrix} a' & 0\\ 0 & c'\end{pmatrix}\mapsto d_0+\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}d_{(2i+1)2^j}\right)x^{i+1}\\ &(a'=a-b^2x,\ \ c'=c-b^2\in A[x],\ \ d=a'+c'x=a+cx\in A_2[x]).\end{aligned}$$

The isomorphism

$$Q_3(B^{A[x]},\beta^{A[x]}) \xrightarrow{\cong} \widehat{L}^3(A[x]);\quad M=\begin{pmatrix} a & b\\ b & c\end{pmatrix}\mapsto (K_M,\psi_M;L_M)$$

sends M *to the Witt class of the nonsingular* (-1)*-quadratic form over* $A[x]$

$$(K_M, \psi_M) = \left(A[x]^2 \oplus A[x]^2, \begin{pmatrix} X & 1 \\ 0 & M \end{pmatrix} \right),$$

with a lagrangian $L_M = A[x]^2 \oplus 0 \subset K_M$ *for* $(K_M, \psi_M - \psi_M^*)$ (70), *and*

$$\partial : Q_3(B^{A[x]}, \beta^{A[x]}) = \widehat{L}^3(A[x]) \to L_2(A[x]); \;\; M \mapsto (K_M, \psi_M).$$

In particular $M \in Q_3(B^{A[x]}, \beta^{A[x]})$ *can be represented by a diagonal matrix* $\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix}$. *The inverse isomorphism is defined by the generalized Arf invariant* (66).

Proof. Proposition 84 expresses $Q_n(B^{A[x]}, \beta^{A[x]})$ in terms of 2×2 matrices. We deal with the four cases separately.

(i) Let $n = 0$. Proposition 85 gives an exact sequence

$$0 \to H_1(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)) \to Q_0(C(X), \gamma(X)) \to 0$$

with

$$\begin{aligned} & H_1(C(1) \otimes_{A[x]} C(x)) = A_2[x] \\ & \to Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)) = (A_8 \oplus A_4[x] \oplus A_2[x]) \oplus (A_4[x] \oplus A_2[x]); \\ & \qquad x \mapsto ((0, 2c, 0), (2c, 0), \end{aligned}$$

so that there is defined an isomorphism

$$\mathrm{coker}(\partial) \xrightarrow{\cong} A_8 \oplus \mathrm{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x]; \;\; (s, t, u, v, w) \mapsto (s, [t, v], u, w).$$

We shall define an isomorphism $Q_0(C(X), \gamma(X)) \cong \mathrm{coker}(\partial)$ by constructing a splitting map

$$Q_0(C(X), \gamma(X)) \to Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x)).$$

An element in $Q_0(C(X), \gamma(X))$ is represented by a symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}_2(A[x]),$$

such that

$$M - MXM = \begin{pmatrix} a - a^2 - b^2x & b - ab - bcx \\ b - ab - bcx & c - b^2 - c^2x \end{pmatrix} \in \mathrm{Quad}_2(A[x]),$$

so that

$$a - a^2 - b^2x,\ \ c - b^2 - c^2x \in 2A[x].$$

Given $a = \sum\limits_{i=0}^{\infty} a_i x^i \in A[x]$ let

$$d = \max\{i \geqslant 0 \,|\, a_i \notin 2A\}\ \ (= 0 \text{ if } a \in 2A[x])$$

so that $a \in A_2[x]$ has degree $d \geqslant 0$,

$$(a_d)^2 = a_d \neq 0 \in A_2$$

and $a - a^2 \in A_2[x]$ has degree $2d$. Thus if $b \neq 0 \in A_2[x]$ the degree of $a - a^2 = b^2x \in A_2[x]$ is both even and odd, so $b \in 2A[x]$ and hence also $a - a^2, c - c^2x \in 2A[x]$. It follows from $a(1-a) = 0 \in A_2[x]$ that $a = 0$ or $1 \in A_2[x]$, so $a - a_0 \in 2A[x]$. Similarly, it follows from $c(1 - cx) = 0 \in A_2[x]$ that $c = 0 \in A_2[x]$, so $c \in 2A[x]$. The matrices defined by

$$N = \begin{pmatrix} 0 & -b/2 \\ 0 & 0 \end{pmatrix} \in M_2(A[x]), \quad M' = \begin{pmatrix} a & 0 \\ 0 & c - b^2 \end{pmatrix} \in \mathrm{Sym}_2(A[x])$$

are such that

$$M + 2(N + N^t) - 4N^tXN = M' \in \mathrm{Sym}_2(A[x])$$

and so $M = M' \in Q_0(C(X), \gamma(X))$. The explicit splitting map is given by

$$Q_0(C(X), \gamma(X)) \to Q_0(C(1), \gamma(1)) \oplus Q_0(C(x), \gamma(x));\ \ M = M' \mapsto (a, c - b^2).$$

The isomorphism

$$Q_0(C(X), \gamma(X)) \xrightarrow{\ \cong\ } \mathrm{coker}(\partial);\ \ M \mapsto (a, c - b^2)$$

may now be composed with the isomorphisms given in the proof of Proposition 85(i)

$$Q_0(C(1),\gamma(1)) \xrightarrow{\cong} A_8 \oplus A_4[x] \oplus A_2[x];$$
$$\sum_{i=0}^{\infty} d_i x^i \mapsto \left(d_0, \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} d_{(2i+1)2^j}/2\right)x^i, \sum_{k=0}^{\infty}(d_{2k+2}/2)x^k\right),$$
$$Q_0(C(x),\gamma(x)) \xrightarrow{\cong} A_4[x] \oplus A_2[x];$$
$$\sum_{i=0}^{\infty} e_i x^i \mapsto \left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} e_{(2i+1)2^j-1}/2\right)x^i, \sum_{k=0}^{\infty}(e_{2k+1}/2)x^k\right).$$

(ii) Let $n = 1$. If $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A[x])$ represents an element $N \in Q_1(B^{A[x]}, \beta^{A[x]})$

$$N + N^t = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \in 2\mathrm{Sym}_2(A[x]),$$
$$\frac{1}{2}(N+N^t) - N^t X N = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} - \begin{pmatrix} a^2+c^2x & ab+cdx \\ ab+cdx & b^2+d^2x \end{pmatrix} \in \mathrm{Quad}_2(A[x])$$

then

$$b+c,\ a-a^2-c^2x,\ d-b^2-d^2x \in 2A[x].$$

If $d \notin 2A[x]$ then the degree of $d - d^2x = b^2 \in A_2[x]$ is both even and odd, so that $d \in 2A[x]$ and hence $b, c \in 2A[x]$. Thus $a - a^2 \in 2A[x]$ and so (as above) $a - a_0 \in 2A[x]$. It follows that

$$Q_1(B^{A[x]}, \beta^{A[x]}) = Q_1(B^A, \beta^A) = A_2.$$

(iii) Let $n = 2$. $Q_2(B^{A[x]}, \beta^{A[x]}) = 0$ by 85.

(iv) Let $n = 3$. Proposition 85 gives an exact sequence

$$0 \to H_0(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_{-1}(C(1),\gamma(1)) \oplus Q_{-1}(C(x),\gamma(x)) \to Q_3(C(X),\gamma(X)) \to 0$$

with

$$\partial : H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to$$
$$Q_{-1}(C(1),\gamma(1)) \oplus Q_{-1}(C(x),\gamma(x)) = A_2[x] \oplus A_2[x];\ \ c \mapsto (cx, c),$$

so that there is defined an isomorphism

$$\mathrm{coker}(\partial) \xrightarrow{\cong} A_2[x];\ \ (a,b) \mapsto a+bx.$$

We shall define an isomorphism $Q_3(C(X),\gamma(X)) \cong \mathrm{coker}(\partial)$ by constructing a splitting map

$$Q_3(C(X),\gamma(X)) \to Q_{-1}(C(1),\gamma(1)) \oplus Q_{-1}(C(x),\gamma(x)).$$

For any $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}_2(A[x])$ the matrices

$$L = \begin{pmatrix} 0 & -b \\ -b & 0 \end{pmatrix},\ \ M' = \begin{pmatrix} a-b^2x & 0 \\ 0 & c-b^2 \end{pmatrix} \in \mathrm{Sym}_2(A[x])$$

are such that

$$M' = M + L - LXL \in \mathrm{Sym}_2(A[x])$$

so $M = M' \in Q_3(C(X),\gamma(X))$. The explicit splitting map is given by

$$Q_3(C(X),\gamma(X)) \to Q_{-1}(C(1),\gamma(1)) \oplus Q_{-1}(C(x),\gamma(x));$$
$$M = M' \mapsto (a-b^2x, c-b^2).$$

The isomorphism

$$Q_3(C(X),\gamma(X)) \xrightarrow{\cong} Q_{-1}(C(1),\gamma(1));\ \ M \mapsto (a-b^2x)+(c-b^2)x = a+cx$$

may now be composed with the isomorphism given in the proof of Proposition 85(ii)

$$Q_{-1}(C(1),\gamma(1)) \xrightarrow{\cong} A_2[x];\ \ d = \sum_{i=0}^{\infty} d_i x^i \mapsto d_0 + \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} d_{(2i+1)2^j}\right) x^{i+1}. \qquad \square$$

Remark 87. (i) Substituting the computation of $Q_*(B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]})$ given by Theorem 86 in the formula

$$Q_{n+1}(B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]}) = Q_{n+1}(B^{\mathbb{Z}},\beta^{\mathbb{Z}}) \oplus \mathrm{UNil}_n(\mathbb{Z})$$

recovers the computations

$$\mathrm{UNil}_n(\mathbb{Z}) = NL_n(\mathbb{Z}) = \begin{cases} 0 & \text{if } n \equiv 0, 1 (\mathrm{mod}\, 4), \\ \mathbb{Z}_2[x] & \text{if } n \equiv 2 (\mathrm{mod}\, 4), \\ \mathbb{Z}_4[x] \oplus \mathbb{Z}_2[x]^3 & \text{if } n \equiv 3 (\mathrm{mod}\, 4). \end{cases}$$

of Connolly and Ranicki [10] and Connolly and Davis [8].

(ii) The twisted quadratic Q-group

$$Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \mathbb{Z}_8 \oplus L_{-1}(\mathbb{Z}[x]) = \mathbb{Z}_8 \oplus \mathrm{UNil}_3(\mathbb{Z})$$

fits into a commutative braid of exact sequences

$$\begin{tikzcd}[column sep=tiny]
0 \arrow[dr] \arrow[rr, bend left] & & Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) \arrow[dr] \arrow[rr, bend left] & & L_{-1}(\mathbb{Z}[x]) \arrow[dr] \arrow[rr, bend left] & & L^{-1}(\mathbb{Z}[1/2][x]) = 0 \\
& L^0(\mathbb{Z}[x]) = \mathbb{Z} \arrow[ur] \arrow[dr] & & L_0(\mathbb{Z}[x], (2)^\infty) \arrow[ur] \arrow[dr] & & L^{-1}(\mathbb{Z}[x]) = 0 \arrow[ur] \arrow[dr] & \\
L_0(\mathbb{Z}[x]) = \mathbb{Z} \arrow[ur, "8"] \arrow[rr, bend right] & & L^0(\mathbb{Z}[1/2][x]) = \mathbb{Z} \oplus \mathbb{Z}_2 \arrow[ur] \arrow[rr, bend right] & & L^0(\mathbb{Z}[x], (2)^\infty) = \mathbb{Z}_2 \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

with $L_0(\mathbb{Z}[x], (2)^\infty)$ (resp. $L^0(\mathbb{Z}[x], (2)^\infty)$) the Witt group of nonsingular quadratic (resp. symmetric) linking forms over $(\mathbb{Z}[x], (2)^\infty)$, and

$$L^0(\mathbb{Z}[x], (2)^\infty) \xrightarrow{\cong} \mathbb{Z}_2; \ \ (T, \lambda) \mapsto n \ \text{ if } \ |\mathbb{Z} \otimes_{\mathbb{Z}[x]} T| = 2^n.$$

The twisted quadratic Q-group $Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ is thus the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$ with $|\mathbb{Z} \otimes_{\mathbb{Z}[x]} T| = 4^m$ for some $m \geqslant 0$. $Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ can also be regarded as the Witt group of nonsingular quadratic linking forms (T, λ, μ) over $(\mathbb{Z}[x], (2)^\infty)$ together with a lagrangian $U \subset T$ for the symmetric linking form (T, λ). The isomorphism class of any such quadruple $(T, \lambda, \mu; U)$ is an element $\phi \in Q_0(B, \beta)$. The chain bundle β is classified by a chain bundle map

$$(f, \chi) : (B, \beta) \to (B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$$

and the Witt class is given by the linking Arf invariant

$$(T, \lambda, \mu; U) = (f, \chi)_\%(\phi) \in Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) = \mathbb{Z}_8 \oplus \mathbb{Z}_4[x] \oplus \mathbb{Z}_2[x]^3.$$

(iii) Here is an explicit procedure obtaining the generalized linking Arf invariant

$$(T, \lambda, \mu; U) \in Q_0(B^{A[x]}, \beta^{A[x]}) = A_8 \oplus A_4[x] \oplus A_2[x]^3$$

for a nonsingular quadratic linking form (T, λ, μ) over $(A[x], (2)^\infty)$ together with a lagrangian $U \subset T$ for the symmetric linking form (T, λ) such that $[U] = 0 \in \widetilde{K}_0(A[x])$, for any 1-even ring A with $\psi^2 = 1$.

Use a set of $A[x]$-module generators $\{g_1, g_2, \dots, g_u\} \subset U$ to obtain a f.g. free $A[x]$-module resolution

$$0 \to B_1 \xrightarrow{\ d\ } B_0 = A[x]^u \xrightarrow{(g_1, g_2, \dots, g_u)} U \to 0.$$

Let $(p_i, q_i) \in A_2[x] \oplus A_2[x]$ be the unique elements such that

$$\mu(g_i) = (p_i)^2 + x(q_i)^2 \in \widehat{H}^0(\mathbb{Z}_2; A[x]) = A_2[x] \quad (1 \leqslant i \leqslant u)$$

and use arbitrary lifts $(p_i, q_i) \in A[x] \oplus A[x]$ to define

$$\begin{aligned} &b_i = (p_i)^2 + x(q_i)^2 \in A[x], \\ &p = (p_1, p_2, \dots, p_u),\ \ q = (q_1, q_2, \dots, q_u) \in A[x]^u. \end{aligned}$$

The diagonal symmetric form on B_0,

$$\beta = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_u \end{pmatrix} \in \mathrm{Sym}(B_0)$$

is such that

$$d^*\beta d \in \mathrm{Quad}(B_1) \subset \mathrm{Sym}(B_1)$$

and represents the chain bundle

$$\beta = \mu|_U \in \widehat{Q}^0(B^{-*}) = \mathrm{Hom}_A(U, \widehat{H}^0(\mathbb{Z}_2; A[x])).$$

The $A[x]$-module morphisms

$$f_0 = \begin{pmatrix} p \\ q \end{pmatrix} : B_0 = A[x]^u \to B_0^{A[x]} = A[x] \oplus A[x];\ \ (a_1, a_2, \dots, a_u) \mapsto \sum_{i=1}^{u} a_i(p_i, q_i),$$

$$f_1 : B_1 = A[x]^u \to B_1^{A[x]} = A[x] \oplus A[x];\ \ a = (a_1, a_2, \dots, a_u) \mapsto \frac{f_0 d(a)}{2}$$

define a chain bundle map

$$(f,0):(B,\beta)\to(B^{A[x]},\beta^{A[x]}),$$

with

$$\beta_0^{A[x]}=\begin{pmatrix}1&0\\0&x\end{pmatrix}:B_0^{A[x]}=A[x]\oplus A[x]\to(B_0^{A[x]})^*=A[x]\oplus A[x].$$

The $(2)^\infty$-torsion dual of U has f.g. free $A[x]$-module resolution

$$0\to B^0=A[x]^u\xrightarrow{d^*}B^1\to U^\wedge\to 0.$$

Lift a set of $A[x]$-module generators $\{h_1,h_2,\dots,h_u\}\subset U^\wedge$ to obtain a basis for B^1, and hence an identification $B^1=A[x]^u$. Also, lift these generators to elements $\{h_1,h_2,\dots,h_u\}\subset T$, so that $\{g_1,g_2,\dots,g_u,h_1,h_2,\dots,h_u\}\subset T$ is a set of $A[x]$-module generators such that

$$d^{-1}=(\lambda(g_i,h_j))\in\frac{\mathrm{Hom}_{A[1/2][x]}(B_0[1/2],B_1[1/2])}{\mathrm{Hom}_{A[x]}(B_0,B_1)}.$$

Lift the symmetric $u\times u$ matrix $(\lambda(h_i,h_j))$ with entries in $A[1/2][x]/A[x]$ to a symmetric form on the f.g. free $A[1/2][x]$-module $B^1[1/2]=A[1/2][x]^u$:

$$\Lambda=(\lambda_{ij})\in\mathrm{Sym}(B^1[1/2])$$

such that $\lambda_{ii}\in A[1/2][x]$ has image $\mu(h_i)\in A[1/2][x]/2A[x]$. Let $\phi=(\phi_{ij})$ be the symmetric form on $B^0=A[x]^u$ defined by

$$\phi=d\Lambda d^*\in\mathrm{Sym}(B^0)\subset\mathrm{Sym}(B^0[1/2]).$$

Then T has a f.g. free $A[x]$-module resolution

$$0\to B_1\oplus B^0\xrightarrow{\begin{pmatrix}0&d^*\\d&\phi\end{pmatrix}}B^1\oplus B_0\xrightarrow{(g_1,\dots,g_u,h_1,\dots,h_u)}T\to 0$$

and

$$\phi_{ii}-\sum_{j=1}^{u}(\phi_{ij})^2b_j\in 2A[x].$$

The symmetric form on $(B_0^{A[x]})^* = A[x] \oplus A[x]$ defined by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = f_0 \phi f_0^* = \begin{pmatrix} \phi(p,p) & \phi(p,q) \\ \phi(q,p) & \phi(q,q) \end{pmatrix} \in \mathrm{Sym}((B_0^{A[x]})^*)$$
$$(p = (p_1, p_2, \ldots, p_u), q = (q_1, q_2, \ldots, q_u) \in B^0 = A[x]^u)$$

is of the type considered in the proof of Theorem 86(i), with

$$a - a^2 = b^2 x,\ \ c - c^2 x = b^2 \in A_2[x],\ \ b \in 2A[x].$$

The Witt class is

$$\begin{aligned}(T, \lambda, \mu; U) &= (f, 0)_\%(\phi) \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \in Q_0(B^{A[x]}, \beta^{A[x]}) \quad (c' = c - b^2),\end{aligned}$$

with isomorphisms

$$Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} A_8 \oplus \mathrm{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x];$$
$$\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \mapsto \left(a_0, \left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{(2i+1)2^j}/2\right)x^i, \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} c'_{(2i+1)2^j-1}/2\right)x^i\right],\right.$$
$$\left.\sum_{k=0}^{\infty}(a_{2k+2}/2)x^k, \sum_{k=0}^{\infty}(c'_{2k+1}/2)x^k\right),$$

$$\mathrm{coker}(2\Delta) \xrightarrow{\cong} A_4[x] \oplus A_2[x];\ \ [m, n] \mapsto (m - n, m),$$

where

$$2\Delta : A_2[x] \to A_4[x] \oplus A_4[x];\ \ m \mapsto (2m, 2m)$$

as in Theorem 86, and

$$Q_0(B^{A[x]}, \beta^{A[x]}) = A_8 \oplus A_4[x] \oplus A_2[x]^3.$$

For Dedekind A the splitting formula of [10] gives

$$\mathrm{UNil}_3(A) \cong Q_0(B^{A[x]}, \beta^{A[x]})/A_8 \cong A_4[x] \oplus A_2[x]^3.$$

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