THE ALGEBRAIC THEORY OF SURGERY II. APPLICATIONS TO TOPOLOGY

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[Received 4 May 1978]

Introduction

The theory of algebraic surgery on chain complexes with an abstract Poincaré duality developed in Part I (Ranicki [**22**]) is applied here to the study of geometric surgery on manifolds.

In § 2 {respectively § 4, § 9} below we shall associate an n-dimensional symmetric Poincaré {respectively quadratic Poincaré, hyperquadratic} complex over $\mathbf{Z}[\pi_1(X)]$ $\sigma^*(X)$ {respectively $\sigma_*(f,b)$, $\hat{\sigma}^*(p)$} to an n-dimensional geometric Poincaré complex X {an n-dimensional normal map $(f\colon M \to X, b\colon \nu_M \to \nu_X)$, a stable spherical fibration $p\colon X \to BG$ over an n-dimensional CW complex X} such that

$$(1+T)\sigma_*(f,b) \oplus \sigma^*(X) = \sigma^*(M),$$
$$J\sigma^*(X) = \hat{\sigma}^*(\nu_X).$$

The quadratic signature $\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$ will be identified in § 7 with the Wall surgery obstruction. The Mishchenko symmetric signature invariant $\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)])$ appears in the product formula for surgery obstructions obtained in § 8,

$$\begin{aligned}\sigma_*(f \times g\colon M \times N \to X \times Y,\ b \times c\colon \nu_M \times \nu_N \to \nu_X \times \nu_Y) \\ = \sigma_*(f,b) \otimes \sigma_*(g,c) + \sigma^*(X) \otimes \sigma_*(g,c) + \sigma_*(f,b) \otimes \sigma^*(Y) \\ \in L_{m+n}(\mathbf{Z}[\pi_1(X \times Y)]).\end{aligned}$$

In § 9 there is obtained a formula describing the effect on the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$ of a change in the bundle map $b\colon \nu_M \to \nu_X$. It turns out that the surgery obstruction of a $([\frac{1}{2}n]-1)$-connected n-dimensional normal map (f,b) is independent of b for $n \neq 2, 3, 6, 7, 14, 15$.

Part II contains the following sections:

§ 1. The chain constructions;
§ 2. Geometric Poincaré complexes;
§ 3. Equivariant S-duality;
§ 4. Normal maps;
§ 5. Intersections and self-intersections;
§ 6. Geometric Poincaré cobordism;

§7. Geometric surgery;
§8. Products;
§9. Wu classes.

1. The chain constructions

We develop two chain level constructions on topological spaces, which we shall use in §2 to obtain algebraic Poincaré complexes from geometric Poincaré complexes. The 'symmetric construction' associates to the singular chain complex $C(X)$ of a topological space X a natural chain homotopy class of chain maps

$$\varphi_X: C(X) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X)\otimes_{\mathbf{Z}} C(X)),$$

inducing abelian group morphisms in homology

$$\varphi_X: H_n(X) \to H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X)\otimes_{\mathbf{Z}} C(X))) = Q^n(C(X)).$$

In fact, φ_X is the Alexander–Whitney diagonal approximation underlying the construction of the Steenrod squares, which may be recovered from $\varphi_X: H_n(X) \to Q^n(C(X))$ by applying the symmetric Wu class operations v_r of §I.1. The 'quadratic construction' associates to a stable map of spaces $F: \Sigma^p X \to \Sigma^p Y$ (p large) a natural chain homotopy class of chain maps

$$\psi_F: C(X) \to W \otimes_{\mathbf{Z}[\mathbf{Z}_2]}(C(Y)\otimes_{\mathbf{Z}} C(Y)),$$

inducing abelian group morphisms in homology

$$\psi_F: H_n(X) \to H_n(W\otimes_{\mathbf{Z}[\mathbf{Z}_2]}(C(Y)\otimes_{\mathbf{Z}} C(Y))) = Q_n(C(Y)).$$

The two constructions are related to each other by

$$f^{\%}\varphi_X - \varphi_Y f_* = (1+T)\psi_F: H_n(X) \to Q^n(C(Y)),$$

where $f: C(X) \to C(Y)$ is any of the chain maps in the chain homotopy class of the composite

$$C(X) \xrightarrow{\Sigma^p} \Omega^p C(\Sigma^p X) \xrightarrow{F} \Omega^p C(\Sigma^p Y) \xrightarrow{(\Sigma^p)^{-1}} C(Y).$$

If f is induced by a geometric map, that is if $F = \Sigma^p F_0$ for some $F_0: X \to Y$, then

$$\psi_F = 0: H_n(X) \to Q_n(C(Y)),$$

$$f^{\%}\varphi_X - \varphi_Y f_* = 0: H_n(X) \to Q^n(C(Y)).$$

Thus the quadratic construction ψ_F is a chain level desuspension obstruction, and measures the failure of f to respect the symmetric constructions φ_X, φ_Y. The effect of applying the quadratic Wu class operations v^r to $\psi_F: H_n(X) \to Q_n(C(Y))$ can be expressed in terms of the functional Steenrod squares.

Actually, we shall develop the symmetric and quadratic constructions in the context of spaces with a discrete group action and equivariant maps, in order to deal with the action of the group of covering translations π on the singular chain complex $C(\tilde{X})$ of a covering $\tilde{X}$ of a space X.

Let R be a commutative ring with 1.

Write the singular R-module chain complex functor as

$$C(\ ;R)\colon (\text{topological spaces}) \to (R\text{-module chain complexes});\quad X \mapsto C(X;R),$$

and denote the homology and cohomology R-modules by

$$H_*(C(X;R)) = H_*(X;R),\quad H^*(C(X;R)) = H^*(X;R).$$

As usual, for $R = \mathbf{Z}$ we write

$$C(X;\mathbf{Z}) = C(X),\quad H_*(X;\mathbf{Z}) = H_*(X),\quad H^*(X;\mathbf{Z}) = H^*(X).$$

(Thus $C(X;R) = R \otimes_{\mathbf{Z}} C(X)$.)

PROPOSITION 1.1. (i) *There exists a functorial diagonal chain map*

$$\Delta\colon C(\ ;R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(\ ;R)\otimes_R C(\ ;R));$$

that is for each topological space X there is given an R-module chain map

$$\Delta_X\colon C(X;R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X;R)\otimes_R C(X;R)),$$

such that for any map of spaces $f\colon X \to Y$ there is defined a commutative diagram of R-module chain complexes

$$\begin{array}{ccc} C(X;R) & \xrightarrow{\Delta_X} & \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X;R)\otimes_R C(X;R)) \\ \downarrow f & & \downarrow f^{\%} \\ C(Y;R) & \xrightarrow{\Delta_Y} & \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(Y;R)\otimes_R C(Y;R)) \end{array}$$

with $T \in \mathbf{Z}_2$ acting on $C(X;R)\otimes_R C(X;R)$ by

$$T\colon C_p(X;R)\otimes_R C_q(X;R) \to C_q(X;R)\otimes_R C_p(X;R);\ x\otimes y \mapsto (-)^{pq} y \otimes x.$$

(ii) *Any two such functorial diagonal chain maps Δ, Δ' are related by a functorial chain homotopy*

$$\Gamma\colon \Delta \simeq \Delta'\colon C(\ ;R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(\ ;R)\otimes_R C(\ ;R)).$$

Any two such chain homotopies are related by a functorial homotopy.

Proof. The proof is by standard acyclic model theory.

We recall the construction from Δ_X of the squaring operations introduced by Steenrod [**26**].

The *Steenrod squares* of a topological space X are the $\mathbf{Z}_2$-module morphisms

$$Sq^i\colon H^r(X;\mathbf{Z}_2) \to H^{r+i}(X;\mathbf{Z}_2);$$

$$(c\colon C_r(X;\mathbf{Z}_2) \to \mathbf{Z}_2) \mapsto (Sq^i(c) = (c\otimes c)\Delta_X(-)(1_{r-i})\colon C_{r+i}(X;\mathbf{Z}_2) \to \mathbf{Z}_2)$$

with

$$1_{r-i} \in W_{r-i} = \begin{cases}\mathbf{Z}[\mathbf{Z}_2] & (r \geqslant i)\\ 0 & (r < i)\end{cases}$$

the generator, and Δ_X any of the diagonal chain approximations given by Proposition 1.1(i) for $R = \mathbf{Z}_2$.

The *functional Steenrod squares* of a map of spaces $f\colon X \to Y$ are the $\mathbf{Z}_2$-module morphisms

$$Sq^i_f\colon \ker\left(\begin{pmatrix}f^*\\ Sq^i\end{pmatrix}\colon H^r(Y;\mathbf{Z}_2) \to H^r(X;\mathbf{Z}_2)\oplus H^{r+i}(Y;\mathbf{Z}_2)\right)$$

$$\to \operatorname{coker}((Sq^i\ f^*)\colon H^{r-1}(X;\mathbf{Z}_2)\oplus H^{r+i-1}(Y;\mathbf{Z}_2) \to H^{r+i-1}(X;\mathbf{Z}_2));$$

$$(c\colon C_r(Y;\mathbf{Z}_2) \to \mathbf{Z}_2) \mapsto (Sq^i_f(c)$$

$$= (g\otimes g)(\Delta_X(-)(1_{r-i-1}) + (1\otimes d_X)\Delta_X(-)(1_{r-i})) + hf\colon C_{r+i-1}(X;\mathbf{Z}_2) \to \mathbf{Z}_2)$$

$$(cf = gd_X\colon C_r(X;\mathbf{Z}_2) \to \mathbf{Z}_2,\ g\colon C_{r-1}(X;\mathbf{Z}_2) \to \mathbf{Z}_2,$$

$$(c\otimes c)\Delta_Y(-)(1_{r-i}) = hd_Y\colon C_{r+i}(Y;\mathbf{Z}_2) \to \mathbf{Z}_2,\ h\colon C_{r+i-1}(Y;\mathbf{Z}_2) \to \mathbf{Z}_2).$$

Let π be a group, and let $R[\pi]$ be the group ring with elements formal sums $\sum_{g\in\pi} n_g g$ $(n_g \in R)$ such that only a finite number of the coefficients n_g is not 0.

Given a group morphism $w\colon \pi \to \mathbf{Z}_2 = \{\pm 1\}$ define the *w-twisted* involution on $R[\pi]$

$$^-\colon R[\pi] \to R[\pi];\ \sum_{g\in\pi} n_g g \mapsto \sum_{g\in\pi} w(g) n_g g^{-1}.$$

The *untwisted* involution on $R[\pi]$ is the w-twisted involution in the case where $w = 1$ (so that $\bar{g} = g^{-1}$ $(g \in \pi)$).

We refer to § 1 of Part I for our conventions and definitions regarding modules over a ring with involution.

Given an $R[\pi]$-module M let ${}^wM$ denote the $R[\pi]$-module defined by the additive group of M, with $R[\pi]$ acting by

$$R[\pi]\times {}^wM \to {}^wM;\ (\sum_{g\in\pi} n_g g, x) \mapsto \sum_{g\in\pi} n_g w(g)(gx).$$

The right $R[\pi]$-module $({}^wM)^t$ (respectively the dual $R[\pi]$-module $({}^wM)^*$) defined with respect to the untwisted involution on $R[\pi]$ is the same as the right $R[\pi]$-module M^t (respectively the dual $R[\pi]$-module M^*) defined with respect to the w-twisted involution on $R[\pi]$.

A *π-action* on a topological space X is a continuous function

$$\pi \times X \to X;\ (g, x) \mapsto gx$$

(with the discrete topology on π), such that

$$g(hx) = (gh)x,$$
$$1x = x \quad (x \in X,\ g, h \in \pi).$$

The singular chain complex $C(X; R)$ is an $R[\pi]$-module chain complex, and there are defined *homology* and *cohomology* $R[\pi]$-modules

$$H_*(X; R) = H_*(C(X; R)), \quad H^*(X; R) = H^*(C(X; R))$$

using the untwisted dual $R[\pi]$-module structure on

$$C(X; R)^* = \mathrm{Hom}_{R[\pi]}(C(X; R), R[\pi]).$$

For $R = \mathbf{Z}$ write

$$H_*(X; \mathbf{Z}) = H_*(X), \quad H^*(X; \mathbf{Z}) = H^*(X).$$

(Warning: if π is infinite $H^*(X)$ is not the singular cohomology of X.) Define also the homology and cohomology R-modules of a space with π-action X taking coefficients in an $R[\pi]$-module M,

$$H^{\pi}_*(X; M) = H_*(C(X; M)), \quad H^*_{\pi}(X; M) = H^*(C(X; M)),$$

using the R-module chain complex

$$C(X; M) = M^t \otimes_{R[\pi]} C(X; R).$$

Given a group morphism $w: \pi \to \mathbf{Z}_2$ there are natural identifications of $R[\pi]$-modules

$${}^wH_*(X; R) = H_*({}^wC(X; R)), \quad {}^wH^*(X; R) = H^*({}^wC(X; R)),$$

making use of the natural isomorphism of $R[\pi]$-module chain complexes

$${}^w\mathrm{Hom}_{R[\pi]}(C(X; R), R[\pi]) \to \mathrm{Hom}_{R[\pi]}({}^wC(X; R), R[\pi]);$$
$$f \mapsto (x \mapsto \sum_{g \in \pi} w(g) n_g g) \quad (f(x) = \sum_{g \in \pi} n_g g \in R[\pi],\ n_g \in R).$$

Given a pointed topological space X define the quotient R-module chain complex

$$\dot{C}(X; R) = C(X; R)/C(\mathrm{pt.}; R).$$

Write the reduced homology and cohomology R-modules as

$$\dot{H}_*(X; R) = H_*(\dot{C}(X; R)), \quad \dot{H}^*(X; R) = H^*(\dot{C}(X; R)).$$

For $R = \mathbf{Z}$ we shall write

$$\dot{C}(X; \mathbf{Z}) = \dot{C}(X), \quad \dot{H}_*(X; \mathbf{Z}) = \dot{H}_*(X), \quad \dot{H}^*(X; \mathbf{Z}) = \dot{H}^*(X).$$

Given a functorial diagonal chain map

$$\Delta_X\colon C(X;\,R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X;\,R)\otimes_R C(X;\,R)),$$

there is induced a diagonal chain map

$$\begin{aligned}\dot{\Delta}_X\colon \dot{C}(X;\,R) &= C(X;\,R)/C(\mathrm{pt.};\,R)\\ &\xrightarrow{[\Delta_X]} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X;\,R)\otimes_R C(X;\,R))/\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(\mathrm{pt.};\,R)\\ &\qquad\qquad\qquad\qquad\qquad\otimes_R C(\mathrm{pt.};\,R))\\ &\xrightarrow{[(\mathrm{pr.})^{\%}]} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X;\,R)\otimes_R \dot{C}(X;\,R))\end{aligned}$$

with pr.: $C(X;\,R) \to \dot{C}(X;\,R)$ the projection. Then $\dot{\Delta}_X$ is functorial on the category of pointed spaces and basepoint-preserving maps.

Given an (unpointed) space X define a pointed space by adjoining a point

$$X_+ = X \cup \{\mathrm{pt.}\}.$$

We shall identify

$$\dot{C}(X_+;\,R) = C(X;\,R),\quad \dot{H}_*(X_+;\,R) = H_*(X;\,R),$$
$$\dot{H}^*(X_+;\,R) = H^*(X;\,R),\quad \dot{\Delta}_{X_+} = \Delta_X.$$

A *π-space* is a pointed space X with a π-action

$$\pi \times X \to X;\quad (g, x) \mapsto gx,$$

such that $g(\mathrm{pt.}) = \mathrm{pt.} \in X$ $(g \in \pi)$. The induced $R[\pi]$-action on $C(X;\,R)$ preserves $C(\mathrm{pt.};\,R) \subseteq C(X;\,R)$, so that there is induced an $R[\pi]$-action on $\dot{C}(X;\,R)$. Also, there are defined reduced homology and cohomology $R[\pi]$-modules

$$\dot{H}_*(X;\,R) = H_*(\dot{C}(X;\,R)),\quad \dot{H}^*(X;\,R) = H^*(\dot{C}(X;\,R)).$$

The reduced diagonal chain map

$$\dot{\Delta}_X\colon \dot{C}(X;\,R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X;\,R)\otimes_R \dot{C}(X;\,R))$$

is an $R[\pi]$-module chain map, with the diagonal π-action on

$$\dot{C}(X;\,R)\otimes_R \dot{C}(X;\,R).$$

A *π-map* of π-spaces is a map of spaces

$$f\colon X \to Y,$$

such that

$$f(\mathrm{pt.}) = \mathrm{pt.},\quad f(gx) = gf(x) \in Y\quad (x \in X,\ g \in \pi).$$

There are induced $R[\pi]$-module maps

$$f\colon \dot{C}(X;\,R) \to \dot{C}(Y;\,R),\qquad f_*\colon \dot{H}_*(X;\,R) \to \dot{H}_*(Y;\,R),$$
$$f^*\colon \dot{H}^*(Y;\,R) \to \dot{H}^*(X;\,R),$$

and

$$f^{\%}\dot{\Delta}_X = \dot{\Delta}_Y f\colon \dot{C}(X\,;R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(Y\,;R)\otimes_R \dot{C}(Y\,;R)).$$

We have the following *symmetric construction* $\dot{\varphi}_X$.

PROPOSITION 1.2. *Let π be a group, $w\colon \pi \to \mathbf{Z}_2$ a group morphism, R a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution. Regard R as an $R[\pi]$-module by*

$$R[\pi]\times R \to R\,;\ (\sum_{g\in\pi} n_g g, r) \mapsto (\sum_{g\in\pi} n_g)r.$$

Given a π-space X there are defined in a natural way R-module morphisms

$$\dot{\varphi}_X\colon \dot{H}^{\pi}_n(X\,;{}^wR) \to Q^n(\dot{C}(X\,;R)) \quad (n \geqslant 0)$$

such that

(i) *for each $x \in \dot{H}^{\pi}_n(X\,;{}^wR)$,*

$$\dot{\varphi}_X(x)_0\backslash - \; = x\cap - \colon {}^w\dot{H}^r(X\,;R) \to \dot{H}_{n-r}(X\,;R),$$

(ii) *for each π-map of π-spaces $f\colon X \to Y$ there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc}
\dot{H}^{\pi}_n(X\,;{}^wR) & \xrightarrow{\ \dot{\varphi}_X\ } & Q^n(\dot{C}(X\,;R)) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f^{\%}} \\
\dot{H}^{\pi}_n(Y\,;{}^wR) & \xrightarrow{\ \dot{\varphi}_Y\ } & Q^n(\dot{C}(Y\,;R))
\end{array}$$

(iii) *for each morphism $h\colon R \to S$ of commutative rings there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc}
\dot{H}^{\pi}_n(X\,;{}^wR) & \xrightarrow{\ \dot{\varphi}_X\ } & Q^n(\dot{C}(X\,;R)) \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} \\
\dot{H}^{\pi}_n(X\,;{}^wS) & \xrightarrow{\ \varphi_X\ } & Q^n(\dot{C}(X\,;S))
\end{array}$$

in which the vertical maps are the change of rings $h\colon R[\pi] \to S[\pi]$.

Proof. Applying $R^t\otimes_{R[\pi]} -$ to a functorial diagonal $R[\pi]$-module chain map

$$\dot{\Delta}_X\colon \dot{C}(X\,;R) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X\,;R)\otimes_R \dot{C}(X\,;R)),$$

we obtain a functorial $\mathbf{Z}$-module chain map

$$\begin{aligned}\dot{\Delta}_X\colon R^t\otimes_{R[\pi]}\dot{C}(X\,;R) \to{}& R^t\otimes_{R[\pi]}\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X\,;R)\otimes_R \dot{C}(X\,;R)) \\ ={}& \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X\,;R)^t\otimes_{R[\pi]}\dot{C}(X\,;R)),\end{aligned}$$

inducing the required $\mathbf{Z}$-module maps in homology

$$\dot{\varphi}_X = (\dot{\Delta}_X)_* : H_n(R^t \otimes_{R[\pi]} \dot{C}(X;R)) = \dot{H}^\pi_n(X; {}^wR)$$
$$\to H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X;R)^t \otimes_{R[\pi]} \dot{C}(X;R))) = Q^n(\dot{C}(X;R)).$$

Applying the symmetric construction to the π-space X_+ obtained from a space with π-action X by adjoining a base point we obtain an absolute symmetric construction

$$\varphi_X = \dot{\varphi}_{X_+} : H^\pi_n(X; {}^wR) = \dot{H}^\pi_n(X_+; {}^wR) \to Q^n(C(X;R)) = Q^n(\dot{C}(X_+;R)).$$

Applying the symmetric Wu class operations v_r of § I.1 to the symmetric construction for $\pi = \{1\}$, $R = \mathbf{Z}_2$ we obtain the Steenrod squares:

PROPOSITION 1.3. *Let X be a $\{1\}$-space. The composite $\mathbf{Z}_2$-module morphism*

$$\dot{H}_n(X;\mathbf{Z}_2) \xrightarrow{\dot{\varphi}_X} Q^n(\dot{C}(X;\mathbf{Z}_2)) \xrightarrow{v_r} \mathrm{Hom}_{\mathbf{Z}_2}(\dot{H}^{n-r}(X;\mathbf{Z}_2), H^{n-2r}(\mathbf{Z}_2;\mathbf{Z}_2))$$
$$= \begin{cases} \mathrm{Hom}_{\mathbf{Z}_2}(\dot{H}^{n-r}(X;\mathbf{Z}_2), \mathbf{Z}_2) & \text{if } n \geqslant 2r, \\ 0 & \text{if } n < 2r, \end{cases}$$

is given by

$$v_r(\dot{\varphi}_X(x))(y) = \langle Sq^r(y), x\rangle \in \mathbf{Z}_2 \quad (x \in \dot{H}_n(X;\mathbf{Z}_2),\ y \in \dot{H}^{n-r}(X;\mathbf{Z}_2)),$$

with $\langle\ ,\ \rangle$ the Kronecker product.

Let $f, f' : C \to D$ be chain maps of A-module chain complexes (for any ring A), and let $g, g' : f \simeq f' : C \to D$ be chain homotopies. A *homotopy* of chain homotopies

$$h : g \simeq g' : f \simeq f' : C \to D$$

is a collection of A-module morphisms $\{h \in \mathrm{Hom}_A(C_r, D_{r+2}) \mid r \in \mathbf{Z}\}$ such that

$$g' - g = d_D h - h d_C : C_r \to D_{r+1}.$$

The *suspension* of a π-space X is the reduced suspension π-space

$$\Sigma X = S^1 \wedge X = (S^1 \times X)/(S^1 \times \text{pt.} \cup \text{pt.} \times X),$$

with the trivial π-action on S^1. Acyclic models give functorial $\mathbf{Z}[\pi]$-module chain equivalences on the category of π-spaces and π-maps,

$$\Sigma_X : S\dot{C}(X) \to \dot{C}(\Sigma X), \quad \Sigma_X{}^{-1} : \dot{C}(\Sigma X) \to S\dot{C}(X),$$

and functorial $\mathbf{Z}[\pi]$-module chain homotopies,

$$h_X : \Sigma_X(\Sigma_X{}^{-1}) \simeq 1 : \dot{C}(\Sigma X) \to \dot{C}(\Sigma X),$$
$$h_X{}^{-1} : (\Sigma_X{}^{-1})\Sigma_X \simeq 1 : S\dot{C}(X) \to S\dot{C}(X),$$

such that $\Sigma_X, \Sigma_X{}^{-1}$ are unique up to functorial chain homotopy, and

h_X, h_X^{-1} are unique up to functorial homotopy of chain homotopies. (This follows from the standard proof of the Eilenberg–Zilber theorem, which, in particular, gives inverse chain equivalences $\dot{C}(S^1)\otimes_{\mathbf{Z}}\dot{C}(X) \rightleftarrows \dot{C}(S^1 \wedge X)$, since $S\mathbf{Z} \subset \dot{C}(S^1)$, $S\dot{C}(X) = S\mathbf{Z}\otimes_{\mathbf{Z}}\dot{C}(X) \subset \dot{C}(S^1)\otimes_{\mathbf{Z}}\dot{C}(X)$ are chain homotopy deformation retracts.) Let

$$\dot{\Delta}_X : \dot{C}(X) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W\dot{C}(X)\otimes_{\mathbf{Z}}\dot{C}(X))$$

be a functorial diagonal $\mathbf{Z}[\pi]$-module chain map, as before, and let

$$S : S\,\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X)\otimes_{\mathbf{Z}}\dot{C}(X)) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S\dot{C}(X)\otimes_{\mathbf{Z}}S\dot{C}(X))$$

be the algebraic suspension chain map of §I.1. Acyclic models also give a functorial $\mathbf{Z}[\pi]$-module chain homotopy

$$\Gamma_X : \dot{\Delta}_{\Sigma X}\Sigma_X \simeq \Sigma_X^{\%} S\dot{\Delta}_X : S\dot{C}(X) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(\Sigma X)\otimes_{\mathbf{Z}}\dot{C}(\Sigma X))$$

in the diagram

$$\begin{array}{ccc}
S\dot{C}(X) & \xrightarrow{\quad\Sigma_X\quad} & \dot{C}(\Sigma X) \\
{\scriptstyle S\dot{\Delta}_X}\big\downarrow & \searrow^{\Gamma_X} & \big\downarrow{\scriptstyle \dot{\Delta}_{\Sigma X}} \\
\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S\dot{C}(X)\otimes_{\mathbf{Z}}S\dot{C}(X)) & \xrightarrow{\Sigma_X^{\%}} & \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(\Sigma X)\otimes_{\mathbf{Z}}\dot{C}(\Sigma X))
\end{array}$$

(This is the chain level relation implying that the Steenrod squares commute with suspensions in cohomology. The chain map $S\dot{\Delta}_X$ can also be expressed as the composite

$$\begin{aligned}
S\dot{\Delta}_X : S\mathbf{Z}\otimes_{\mathbf{Z}}\dot{C}(X) &\xrightarrow{\Delta_S\otimes\dot{\Delta}_X} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S\mathbf{Z}\otimes_{\mathbf{Z}}S\mathbf{Z})\otimes_{\mathbf{Z}}\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X)\\
&\qquad\qquad\qquad\qquad\otimes_{\mathbf{Z}}\dot{C}(X))\\
&\xrightarrow{\Delta_W^*} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, (S\mathbf{Z}\otimes_{\mathbf{Z}}\dot{C}(X))\otimes_{\mathbf{Z}}(S\mathbf{Z}\otimes_{\mathbf{Z}}\dot{C}(X))),
\end{aligned}$$

where $\Delta_S : S\mathbf{Z} \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S\mathbf{Z}\otimes_{\mathbf{Z}}S\mathbf{Z})$ is the restriction of

$$\dot{\Delta}_{S^1} : \dot{C}(S^1) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(S^1)\otimes_{\mathbf{Z}}\dot{C}(S^1))$$

defined by

$$\Delta_S : (S\mathbf{Z})_1 = \mathbf{Z} \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W_1, (S\mathbf{Z}\otimes_{\mathbf{Z}}S\mathbf{Z})_2) = \mathbf{Z};\quad 1 \mapsto 1,$$

and $\Delta_W : W \to W\otimes_{\mathbf{Z}} W$ is the diagonal $\mathbf{Z}[\mathbf{Z}_2]$-module chain map defined by

$$\Delta_W : W_s = \mathbf{Z}[\mathbf{Z}_2] \to (W\otimes_{\mathbf{Z}} W)_s = \sum_{r=0}^{s} W_r\otimes_{\mathbf{Z}} W_{s-r};\quad 1_s \mapsto \sum_{r=0}^{s} 1_r \otimes T^r_{s-r} \qquad (s \geqslant 0),$$

exactly as in §I.8. More generally, for any pointed spaces X, Y there are defined a chain equivalence $\dot{C}(X\wedge Y) \simeq \dot{C}(X)\otimes_{\mathbf{Z}}\dot{C}(Y)$ and a chain homotopy $\dot{\Delta}_{X\wedge Y} \simeq (\dot{\Delta}_X\otimes\dot{\Delta}_Y)\Delta_W^*$, cf. the proof of Proposition 8.1 below.)

Applying $R\otimes_{\mathbf{Z}}-$ we have the same types of chain maps and chain homotopies for any coefficient ring R.

Thus the algebraic and geometric suspension operations correspond to each other under the symmetric construction:

PROPOSITION 1.4. *For any π-space X, commutative ring R, and group morphism $w\colon \pi \to \mathbf{Z}_2$ there is defined a commutative diagram of R-modules*

$$\begin{array}{ccccc} \dot{H}^{\pi}_n(X;{}^wR) & \xrightarrow{\dot{\varphi}_X} & Q^n(\dot{C}(X;R)) & \xrightarrow{S} & Q^{n+1}(S\dot{C}(X;R)) \\ \Sigma_X \wr \downarrow & & & & \downarrow \wr \Sigma^{\%}_X \\ \dot{H}^{\pi}_{n+1}(\Sigma X;{}^wR) & & \xrightarrow{\dot{\varphi}_{\Sigma X}} & & Q^{n+1}(\dot{C}(\Sigma X;R)) \end{array}$$

Proof. The underlying chain maps are chain homotopic.

A π-homotopy of π-maps $f_0, f_1\colon X \to Y$ is a π-map

$$H\colon X\wedge I_+ \to Y$$

which restricts to f_i on $X\wedge\{i\}$ $(i = 0,1)$, with the trivial π-action on $I = [0,1]$. The functoriality of the usual proof of the homotopy invariance of singular homology ensures that H induces an $R[\pi]$-module chain homotopy (for any ring R)

$$H\colon f_0 \simeq f_1\colon \dot{C}(X;R) \to \dot{C}(Y;R).$$

We have the following *quadratic construction* ψ_F.

PROPOSITION 1.5. *Let π be a group, let $w\colon \pi \to \mathbf{Z}_2$ be a group morphism, let R be a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution.*

Given π-spaces X, Y and a π-map $F\colon \Sigma^p X \to \Sigma^p Y$ $(p \geqslant 0)$ there are defined in a natural way R-module morphisms

$$\psi_F\colon \dot{H}^{\pi}_n(X;{}^wR) \to Q^{[0,p-1]}_n(\dot{C}(Y;R)) \quad (n \geqslant 0)$$

such that:

(i) *ψ_F depends only on the π-homotopy class of F, and $\psi_{\Sigma F}$ is given by*

$$\psi_{\Sigma F}\colon \dot{H}^{\pi}_n(X;{}^wR) \xrightarrow{\psi_F} Q^{[0,p-1]}_n(\dot{C}(Y;R)) \longrightarrow Q^{[0,p]}_n(\dot{C}(Y;R)),$$

with $\Sigma F\colon \Sigma^{p+1}X \to \Sigma^{p+1}Y$ the suspension of F. Passing to the suspension limit $\varinjlim_k \Sigma^k F$ there are defined R-module morphisms

$$\psi_F\colon \dot{H}^{\pi}_n(X;{}^wR) \to \varinjlim_k Q^{[0,p+k-1]}_n(\dot{C}(Y;R)) = Q_n(\dot{C}(Y;R)),$$

depending only on the stable π-homotopy class of F; if $p = 0$ then $\psi_F = 0$ (unstably);

(ii) $(1+T)\psi_F = \dot{\phi}_Y f_* - f^{\%}\dot{\phi}_X : \dot{H}^{\pi}_n(X; {}^wR) \to Q^n(\dot{C}(Y; R))$,

with f the composite $R[\pi]$-module chain map

$$f: \dot{C}(X; R) \xrightarrow{\Sigma^p_X} \Omega^p\dot{C}(\Sigma^p X; R) \xrightarrow{F} \Omega^p\dot{C}(\Sigma^p Y; R) \xrightarrow{\Sigma^{-p}_Y} \dot{C}(Y; R);$$

(iii) *if $G: \Sigma^p Y \to \Sigma^p Z$ is another π-map between p-fold suspensions of π-spaces Y, Z, and $g = (\Sigma^{-p}_Z)G\Sigma^p_Y : \dot{C}(Y; R) \to \dot{C}(Z; R)$, then*

$$\psi_{GF} = g_{\%}\psi_F + \psi_G f_* : \dot{H}^{\pi}_n(X; {}^wR) \to Q^{[0,p-1]}_n(\dot{C}(Z; R));$$

(iv) *if $j: R \to S$ is a morphism of commutative rings then there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc} \dot{H}^{\pi}_n(X; {}^wR) & \xrightarrow{\psi_F} & Q^{[0,p-1]}_n(\dot{C}(Y; R)) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ \dot{H}^{\pi}_n(X; {}^wS) & \xrightarrow{\psi_F} & Q^{[0,p-1]}_n(\dot{C}(Y; S)) \end{array}$$

in which the vertical maps are the change of rings for $j: R[\pi] \to S[\pi]$.

Proof. (We consider only the case where $R = \mathbf{Z}$. To obtain the general case apply $R \otimes_{\mathbf{Z}} -$ on the chain level.)

Iterating the previous constructions p times there are defined functorial $\mathbf{Z}[\pi]$-module chain equivalences on the category of π-spaces and π-maps

$$\Sigma^p_X : S^p\dot{C}(X) \to \dot{C}(\Sigma^p X), \quad \Sigma^{-p}_X : \dot{C}(\Sigma^p X) \to S^p\dot{C}(X),$$

and a functorial $\mathbf{Z}[\pi]$-module chain homotopy

$$h^p_X : \Sigma^p_X\Sigma^{-p}_X \simeq 1 : \dot{C}(\Sigma^p X) \to \dot{C}(\Sigma^p X).$$

Also, applying $\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} -$ we see that there are defined a functorial diagonal $\mathbf{Z}$-module chain map

$$\dot{\Delta}_X : \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X))$$

and a functorial $\mathbf{Z}$-module chain homotopy

$$\Gamma^p_X : \dot{\Delta}_{\Sigma^p X}\Sigma^p_X \simeq \Sigma^{p\%}_X S^p\dot{\Delta}_X :$$
$$\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X) \to \Omega^p\,\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(\Sigma^p X)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(\Sigma^p X)),$$

where S^p is the p-fold algebraic suspension chain map

$$S^p : \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(X)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X)) \to \Omega^p\,\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S^p\dot{C}(X)^t \otimes_{\mathbf{Z}[\pi]} S^p\dot{C}(X))$$

and ΩC denotes the desuspension of a chain complex C, $\Omega C_r = C_{r+1}$, $d_{\Omega C} = d_C$.

Given a π-map $F\colon \Sigma^p X \to \Sigma^p Y$ define a $\mathbf{Z}$-module chain map

$$\psi_F\colon \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X) \to \Omega C(S^p)$$

to the desuspension $\Omega C(S^p)$ of the algebraic mapping cone $C(S^p)$ of the p-fold algebraic suspension chain map

$$S^p\colon \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \Omega^p \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \Omega^p \dot{C}(\Sigma^p Y)) \to \Omega^p \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(\Sigma^p Y)),$$

by

$$\psi_F = \begin{pmatrix} \Sigma_Y^{p\%}\dot{\Delta}_Y \Sigma_Y^{-p} F \Sigma_X^p - F^{\%}\Sigma_X^{p\%}\dot{\Delta}_X \\ F^{\%}\Gamma_X^p - \Gamma_Y^p \Sigma_Y^{-p} F \Sigma_X^p - \dot{\Delta}_{\Sigma^p Y} h_Y^p F \Sigma_X^p \end{pmatrix}\colon$$

$$\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X)_n \to \Omega C(S^p)_n = \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \Omega^p \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \Omega^p \dot{C}(\Sigma^p Y))_n \oplus \Omega^p \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(\Sigma^p Y))_{n+1}.$$

The composition of $\psi_F\colon \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X) \to \Omega C(S^p)$ with the $\mathbf{Z}$-module chain equivalence given by Proposition I.1.3,

$$\Omega C(S^p) \to W[0, p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (\Omega^p \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \Omega^p \dot{C}(\Sigma^p Y)),$$

and the $\mathbf{Z}$-module chain equivalence induced by $\Sigma_Y^{-p}\colon \Omega^p \dot{C}(\Sigma^p Y) \to \dot{C}(Y)$,

$$(\Sigma_Y^{-p})_{\%}\colon W[0, p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (\Omega^p \dot{C}(\Sigma^p Y)^t \otimes_{\mathbf{Z}[\pi]} \Omega^p \dot{C}(\Sigma^p Y)) \to W[0, p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (\dot{C}(Y)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(Y)),$$

is a $\mathbf{Z}$-module chain map

$$\psi_F\colon \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X) \to W[0, p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (\dot{C}(Y)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(Y))$$

inducing the quadratic construction in homology

$$\psi_F\colon H_n(\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(X)) = \dot{H}_n^{\pi}(X; {}^w\mathbf{Z}) \to H_n(W[0, p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (\dot{C}(Y)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(Y))) = Q_n^{[0,p-1]}(\dot{C}(Y)).$$

For π-maps of the type $F\colon \Sigma^p(X_+) \to \Sigma^p(Y_+)$ (for some spaces with π-action X, Y) we have an absolute quadratic construction

$$\psi_F = \psi_F\colon H_n(X; {}^wR) = \dot{H}_n(X_+; {}^wR) \to Q_n^{[0,p-1]}(C(Y; R)) = Q_n^{[0,p-1]}(\dot{C}(Y_+; R)).$$

REMARK.† As noted in the Introduction to Part I there is an alternative expression for the quadratic construction $\psi_F\colon \dot{H}_n^{\pi}(X; {}^wR) \to Q_n(\dot{C}(Y; R))$ of a stable π-map $F\colon \Sigma^\infty X \to \Sigma^\infty Y$, using the adjoint π-map

$$\mathrm{adj}(F)\colon X \to \Omega^\infty \Sigma^\infty Y$$

and the canonical group completion π-map

$$\Big(\coprod_{k \geqslant 1} E\Sigma_k \times_{\Sigma_k} \prod_k Y\Big)/\sim\ \to \Omega^\infty \Sigma^\infty Y$$

† See note added in proof (p. 279).

(where $\sim$ is the equivalence relation given by the inclusions $\Sigma_k \subset \Sigma_{k+1}$ and the base point of Y), as the composite

$$\begin{aligned}\psi_F\colon \dot{H}^\pi_n(X;\, {}^wR) &\xrightarrow{(\mathrm{adj}\,F)_*} \dot{H}^\pi_n(\Omega^\infty\Sigma^\infty Y;\, {}^wR)\\ &= \Big(\bigoplus_{k=1}^{\infty} \dot{H}^\pi_n(E\Sigma_k \ltimes_{\Sigma_k} (\textstyle\bigwedge_k Y)/\pi;\, {}^wR)\Big) \otimes_{R[\mathbf{N}]} R[\mathbf{Z}]\\ &\xrightarrow{\text{projection}} \dot{H}^\pi_n(E\Sigma_2 \ltimes_{\Sigma_2} (Y \wedge_\pi Y);\, {}^wR) = Q_n(\dot{C}(Y;\, R)).\end{aligned}$$

The unstable quadratic construction $\psi_F\colon \dot{H}^\pi_n(X;\, {}^wR) \to Q^{[0,p-1]}_n(\dot{C}(Y;\, R))$ has a similar description, using the approximation theorem for $\Omega^p\Sigma^p Y$, and the adjoint π-map $\mathrm{adj}(F)\colon X \to \Omega^p\Sigma^p Y$ of a π-map $F\colon \Sigma^p X \to \Sigma^p Y$.

The result of applying the quadratic Wu class operation v^r of §I.1 to the quadratic construction ψ_F for $\pi = \{1\}$, $R = \mathbf{Z}_2$ can be expressed in terms of the functional Steenrod squares:

PROPOSITION 1.6. *Let X, Y be $\{1\}$-spaces, and let $F\colon \Sigma^p X \to \Sigma^p Y$ be a $\{1\}$-map, inducing the chain map $f\colon \dot{C}(X;\, \mathbf{Z}_2) \to \dot{C}(Y;\, \mathbf{Z}_2)$. The composite*

$$\begin{aligned}\dot{H}_n(X;\, \mathbf{Z}_2) &\xrightarrow{\psi_F} Q^{[0,p-1]}_n(\dot{C}(Y;\, \mathbf{Z}_2))\\ &\xrightarrow{v^r} \mathrm{Hom}_{\mathbf{Z}_2}(\dot{H}^{n-r}(Y;\, \mathbf{Z}_2), Q^{[0,p-1]}_n(S^{n-r}\mathbf{Z}_2))\\ &= \begin{cases}\mathrm{Hom}_{\mathbf{Z}_2}(\dot{H}^{n-r}(Y;\, \mathbf{Z}_2), \mathbf{Z}_2) & \text{if } n \leqslant 2r \leqslant n+p-1,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

is given by

$$v^r(\psi_F(x))(y) = \langle Sq^{r+1}_h(\Sigma^p\iota), \Sigma^p_X(x)\rangle \in \mathbf{Z}_2,$$

where $x \in \dot{H}_n(X;\, \mathbf{Z}_2)$, $\iota = \text{generator} \in \dot{H}^{n-r}(K(\mathbf{Z}_2, n-r);\, \mathbf{Z}_2) = \mathbf{Z}_2$,

$$y \in \dot{H}^{n-r}(Y;\, \mathbf{Z}_2) = [Y, K(\mathbf{Z}_2, n-r)],$$

$$f^*y \in \dot{H}^{n-r}(X;\, \mathbf{Z}_2) = [X, K(\mathbf{Z}_2, n-r)],$$

$$h = (\Sigma^p y)F - \Sigma^p(f^*y) \in [\Sigma^p X, \Sigma^p K(\mathbf{Z}_2, n-r)],$$

satisfying the sum formula

$$\begin{aligned}&v^r(\psi_F(x))(y_1+y_2) - v^r(\psi_F(x))(y_1) - v^r(\psi_F(x))(y_2)\\ &\quad= \begin{cases}\langle f^*(y_1 \cup y_2) - (f^*y_1 \cup f^*y_2), x\rangle \in \mathbf{Z}_2 & \text{if } n = 2r,\\ 0 & \text{otherwise,}\end{cases}\\ &\qquad (x \in \dot{H}_n(X;\, \mathbf{Z}_2),\ y_1, y_2 \in \dot{H}^{n-r}(Y;\, \mathbf{Z}_2)).\end{aligned}$$

2. Geometric Poincaré complexes

Given an oriented covering $\tilde{X}$ of an n-dimensional geometric Poincaré complex X with group π of covering translations we shall use the symmetric construction of § 1 to define an n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi]$

$$\sigma^*(X) = (C(\tilde{X}), \varphi \in Q^n(C(\tilde{X}))).$$

A degree 1 map of n-dimensional geometric Poincaré complexes $f\colon M \to X$ has a kernel n-dimensional symmetric Poincaré complex $\sigma^*(f)$ such that

$$\sigma^*(M) = \sigma^*(f) \oplus \sigma^*(X).$$

Given also a stable π-map $F\colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$ inducing the Umkehr $f^!\colon C(\tilde{X}) \to C(\tilde{M})$ we shall use the quadratic construction of § 1 to define an n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi]\sigma_*(f, F)$ such that

$$(1+T)\sigma_*(f, F) = \sigma^*(f).$$

In § 4 we shall show how such geometric Umkehr maps F may be obtained for normal maps $(f\colon M \to X, b\colon \nu_M \to \nu_X)$, and in §§ 5 and 7 we shall relate the resulting quadratic Poincaré complex $\sigma_*(f, b) = \sigma_*(f, F)$ to the surgery obstruction.

An *n-dimensional geometric Poincaré complex* X (as defined by Wall [**29**]) is a finitely dominated CW complex X together with an *orientation* group morphism

$$w(X)\colon \pi_1(X) \to \mathbf{Z}_2$$

and a *fundamental class*

$$[X] \in H_n^{\pi_1(X)}(\tilde{X};\, {}^{w(X)}\mathbf{Z}),$$

such that the cap products

$$[X] \cap -\colon {}^{w(X)}H^r(\tilde{X}) \to H_{n-r}(\tilde{X}) \quad (0 \leqslant r \leqslant n)$$

are $\mathbf{Z}[\pi_1(X)]$-module isomorphisms, with $\tilde{X}$ the universal cover of X and $\pi_1(X)$ acting on the left of $\tilde{X}$ as the group of covering translations. The singular chain complex $C(\tilde{X})$ is an n-dimensional $\mathbf{Z}[\pi_1(X)]$-module chain complex, and the Poincaré duality isomorphisms are induced by a $\mathbf{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap -\colon {}^{w(X)}C(\tilde{X})^{n-*} \to C(\tilde{X}).$$

(For finite X and $w(X) = 1$ such a geometric Poincaré complex X is a P-space of formal dimension n in the sense of Spivak [**25**], since applying $\mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Z})$ we obtain a $\mathbf{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap -\colon \mathrm{Hom}_{\mathbf{Z}}(C(\tilde{X}), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{Z}}(C(\tilde{X})^{n-*}, \mathbf{Z}) = C^{LF}(\tilde{X})_{n-*}$$

inducing Poincaré duality isomorphisms

$$[X] \cap -\colon H^*(\tilde{X}) \to H^{LF}_{n-*}(\tilde{X})$$

between the singular cohomology groups of $\tilde{X}$ and the homology groups of $\tilde{X}$ defined by locally finite infinite chains, with $[\tilde{X}] \in H_n^{LF}(\tilde{X})$ the transfer of the fundamental class $[X] \in H_n(X)$.)

Let X be an n-dimensional geometric Poincaré complex. If $\tilde{X}$ is a (not necessarily connected) covering of X with group of covering translations π and $\tilde{\tilde{X}}$ is the universal covering of X, the natural projection

$$\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[\pi_1(X)]} C(\tilde{\tilde{X}}) \to C(\tilde{X})$$

is a chain equivalence of n-dimensional $\mathbf{Z}[\pi]$-module chain complexes, with $\mathbf{Z}[\pi_1(X)] \to \mathbf{Z}[\pi]$ the group ring morphism defined by the characteristic map $\pi_1(X) \to \pi$. The covering $\tilde{X}$ of X is *oriented with data* (π, w) if π is equipped with a group morphism $w: \pi \to \mathbf{Z}_2$ such that the orientation map $w(X)$ factors as

$$w(X): \pi_1(X) \longrightarrow \pi \xrightarrow{w} \mathbf{Z}_2.$$

In particular, the universal cover $\tilde{\tilde{X}}$ is oriented with data $(\pi_1(X), w(X))$. If $\tilde{X}$ is oriented with data (π, w) applying $\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} -$ to the above $\mathbf{Z}[\pi]$-module chain equivalence we obtain a $\mathbf{Z}$-module chain equivalence

$$\mathbf{Z}^{t(X)} \otimes_{\mathbf{Z}[\pi_1(X)]} C(\tilde{\tilde{X}}) = \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} (\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[\pi_1(X)]} C(\tilde{\tilde{X}})) \to \mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} C(\tilde{X}),$$

where $t(X)$ {respectively t} refers to the $w(X)$ {w}-twisted involution on $\mathbf{Z}[\pi_1(X)]$ {$\mathbf{Z}[\pi]$}, so that there is a fundamental class

$$[X] \in H_n^{\pi}(\tilde{X}; {}^{w}\mathbf{Z}) = H_n(\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} C(\tilde{X}))$$

for $\tilde{X}$. Applying $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[\pi_1(X)]} -$ to the $\mathbf{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap - : {}^{w(X)}C(\tilde{\tilde{X}})^{n-*} \to C(\tilde{\tilde{X}}),$$

we obtain a $\mathbf{Z}[\pi]$-module chain equivalence

$$[X] \cap - : {}^{w}C(\tilde{X})^{n-*} \to C(\tilde{X}).$$

Thus a geometric Poincaré complex satisfies Poincaré duality with respect to any oriented cover $\tilde{X}$.

The symmetric construction of Proposition 1.2 associates a symmetric Poincaré complex to every oriented covering of a geometric Poincaré complex, by a chain homotopy invariant version of the procedure of Mishchenko [**18**].

PROPOSITION 2.1. *Given an n-dimensional geometric Poincaré complex X and an oriented cover $\tilde{X}$ with data (π, w) there is defined in a natural way an n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi]$ with the w-twisted involution*

$$\sigma^*(\tilde{X}) = (C(\tilde{X}), \varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X}))).$$

If $\tilde{\tilde{X}}$ is the universal cover of X then

$$\sigma^*(\tilde{X}) = \mathbf{Z}[\pi]\otimes_{\mathbf{Z}[\pi_1(X)]}\sigma^*(\tilde{\tilde{X}})$$

up to homotopy equivalence.

Proof. Evaluating $\varphi_{\tilde{X}}\colon H_n^\pi(\tilde{X};{}^w\mathbf{Z}) \to Q^n(C(\tilde{X}))$ on the fundamental class $[X] \in H_n^\pi(\tilde{X};{}^w\mathbf{Z})$ we see that there is obtained a $\mathbf{Z}_2$-hypercohomology class $\varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X}))$ such that slant product with

$$\varphi_{\tilde{X}}[X]_0 \in H_n(C(\tilde{X})^t\otimes_{\mathbf{Z}[\pi]}C(\tilde{X}))$$

defines the Poincaré duality $\mathbf{Z}[\pi]$-module isomorphisms

$$\varphi_{\tilde{X}}[X]_0\backslash - = [X]\cap - : {}^wH^*(\tilde{X}) \to H_{n-*}(\tilde{X})$$

(cf. Proposition 1.2(i)). Also, there is defined a commutative diagram

$$\begin{array}{ccc} H_n^{\pi_1(X)}(\tilde{\tilde{X}};{}^{w(X)}\mathbf{Z}) & \xrightarrow{\varphi_{\tilde{\tilde{X}}}} & Q^n(C(\tilde{\tilde{X}})) \\ \big\downarrow & & \big\downarrow \\ H_n^\pi(\tilde{X};{}^w\mathbf{Z}) & \xrightarrow{\varphi_{\tilde{X}}} & Q^n(C(\tilde{X})) \end{array}$$

in which the vertical maps are the change of rings $\mathbf{Z}[\pi_1(X)] \to \mathbf{Z}[\pi]$.

We shall normally write $\sigma^*(\tilde{X})$ as $\sigma^*(X)$.

A *map* of geometric Poincaré complexes (not necessarily of the same dimension)

$$f\colon M \to X$$

is a map of the underlying spaces which preserves the orientation maps, that is such that $w(M)$ factors as

$$w(M)\colon \pi_1(M) \xrightarrow{f} \pi_1(X) \xrightarrow{w(X)} \mathbf{Z}_2.$$

If $\tilde{X}$ is an oriented cover of X with data (π, w) then the pullback $\tilde{M}$ is an oriented cover of M with data (π, w).

Let $f\colon M \to X$ be a map of n-dimensional geometric Poincaré complexes, and let $\tilde{X}$ be a (not necessarily connected) cover of X with group of covering translations π and induced cover $\tilde{M}$ of M. Define the *Umkehr* $\mathbf{Z}[\pi]$-module chain map

$$f^!\colon C(\tilde{X}) \to C(\tilde{M})$$

(up to non-canonical $\mathbf{Z}[\pi]$-module chain homotopy) by applying $\mathbf{Z}[\pi]\otimes_{\mathbf{Z}[\pi_1(X)]}-$ to the composite $\mathbf{Z}[\pi_1(X)]$-module chain map

$$f^!\colon C(\tilde{\tilde{X}}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\tilde{\tilde{X}})^{n-*} \xrightarrow{\tilde{\tilde{f}}^*} {}^{w(X)}C(\tilde{\tilde{M}})^{n-*} \xrightarrow{[M]\cap -} C(\tilde{\tilde{M}})$$

with $\tilde{X}$ the universal cover of X and $\tilde{M}$ the induced oriented cover of M. If $\tilde{X}$ is an oriented cover of X with data (π, w) then the Umkehr factors as

$$f^!: C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} {}^wC(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} {}^wC(\tilde{M})^{n-*} \xrightarrow{[M]\cap -} C(\tilde{M}).$$

A map of n-dimensional geometric Poincaré complexes $f: M \to X$ is of *degree* 1 if it preserves the fundamental classes, that is if

$$f_*[M] = [X] \in H_n^\pi(\tilde{X}; {}^w\mathbf{Z})$$

for any oriented cover $\tilde{X}$ of X with data (π, w). The induced chain map $\tilde{f}: C(\tilde{M}) \to C(\tilde{X})$ defines a map of n-dimensional symmetric Poincaré complexes over $\mathbf{Z}[\pi]$

$$\tilde{f}: \sigma^*(\tilde{M}) \to \sigma^*(\tilde{X}),$$

which is a homotopy equivalence if $f: M \to X$ is a homotopy equivalence of spaces. Conversely, if $f: M \to X$ is a degree 1 map inducing an isomorphism $f: \pi_1(M) \to \pi_1(X)$ and a homotopy equivalence

$$\tilde{f}: \sigma^*(\tilde{M}) \to \sigma^*(\tilde{X}),$$

with $\tilde{M}, \tilde{X}$ the universal covers then $f: M \to X$ is a homotopy equivalence, by Whitehead's theorem.

PROPOSITION 2.2. *Let $f: M \to X$ be a degree* 1 *map of n-dimensional geometric Poincaré complexes. Let $\tilde{X}$ be a cover of X with group of covering translations π and induced cover $\tilde{M}$ of M. Then the* Umkehr $\mathbf{Z}[\pi]$-module chain map

$$f^!: C(\tilde{X}) \to C(\tilde{M})$$

is a chain homotopy right inverse for $\tilde{f}: C(\tilde{M}) \to C(\tilde{X})$, that is

$$\tilde{f}f^! \simeq 1: C(\tilde{X}) \to C(\tilde{X}).$$

The inclusion in the algebraic mapping cone $e: C(\tilde{M}) \to C(f^!)$ is such that

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix}: C(\tilde{M}) \to C(f^!) \oplus C(\tilde{X})$$

defines a chain equivalence of n-dimensional $\mathbf{Z}[\pi]$-module chain complexes. If $\tilde{X}$ is an oriented cover of X with data (π, w) the symmetric kernel *of f,*

$$\sigma^*(f) = (C(f^!), e^\%(\varphi_{\tilde{M}}[M]) \in Q^n(C(f^!))),$$

is an n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi]$ with the w-twisted involution, and there is defined a homotopy equivalence of such complexes

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix}: \sigma^*(M) \to \sigma^*(f) \oplus \sigma^*(X).$$

Proof. To obtain $\tilde{f}f^! \simeq 1$ apply $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[\pi_1(X)]} -$ to the $\mathbf{Z}[\pi_1(X)]$-module chain homotopy commutative diagram

$$\begin{array}{ccc}
{}^{w(X)}C(\tilde{\tilde{X}})^{n-*} & \xrightarrow{\tilde{\tilde{f}}^*} & {}^{w(X)}C(\tilde{\tilde{M}})^{n-*} \\
{\scriptstyle [X]\cap -}\downarrow & & \downarrow{\scriptstyle [M]\cap -} \\
C(\tilde{\tilde{X}}) & \xleftarrow{\tilde{\tilde{f}}} & C(\tilde{\tilde{M}})
\end{array}$$

with $\tilde{\tilde{X}}$ the universal cover of X and $\tilde{\tilde{M}}$ the induced cover of M. To show that

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix}^{\%} : Q^n(C(\tilde{M})) \to Q^n(C(f^!) \oplus C(\tilde{X})) = Q^n(C(f^!)) \oplus Q^n(C(\tilde{X})) \\ \oplus H_n(C(f^!)^t \otimes_{\mathbf{Z}[\pi]} C(\tilde{X}))$$

sends $\varphi_{\tilde{M}}[M]$ to $e^{\%}(\varphi_{\tilde{M}}[M]) \oplus \varphi_{\tilde{X}}[X] \oplus 0$ (using the decomposition of Proposition I.1.4(i)) consider the chain homotopy commutative diagram

$$\begin{array}{ccccc}
{}^wC(f^!)^{n-*} & \xrightarrow{e^*} & {}^wC(\tilde{M})^{n-*} & \xrightarrow{\tilde{f}^{!*}} & {}^wC(\tilde{X})^{n-*} \\
 & & {\scriptstyle [X]\cap -}\downarrow & & \downarrow{\scriptstyle [M]\cap -} \\
 & & C(\tilde{M}) & \xrightarrow{\tilde{f}} & C(\tilde{X})
\end{array}$$

which gives $\tilde{f}([M] \cap -)e^* \simeq 0 \colon {}^wC(f^!)^{n-*} \to C(\tilde{X})$, and so

$$(e^t \otimes_{\mathbf{Z}[\pi]} \tilde{f})\varphi_{\tilde{M}}[M]_0 = 0 \in H_n(C(f^!)^t \otimes_{\mathbf{Z}[\pi]} C(\tilde{X})).$$

Define the *homology* {*cohomology*} *kernel* $\mathbf{Z}[\pi]$-modules of a degree 1 map of n-dimensional geometric Poincaré complexes $f\colon M \to X$ with respect to a covering $\tilde{X}$ of X with group of covering translations π

$$\begin{cases} K_*(M) = H_*(C(f^!)) \\ K^*(M) = H^*(C(f^!)) \end{cases}$$

using any w-twisted involution on $\mathbf{Z}[\pi]$ to define the dual $\mathbf{Z}[\pi]$-module structure on $C(f^!)^*$. Proposition 2.2 gives natural direct sum decompositions

$$\begin{cases} H_*(\tilde{M}) = K_*(M) \oplus H_*(\tilde{X}), \\ {}^wH^*(\tilde{M}) = K^*(M) \oplus {}^wH^*(\tilde{X}). \end{cases}$$

If $\tilde{X}$ is oriented with data (π, w) the symmetric kernel $\sigma^*(f)$ gives Poincaré duality in the kernel modules

$$K^*(M) = K_{n-*}(M).$$

A *geometric Umkehr map* for a degree 1 map $f\colon M \to X$ and a cover $\tilde{X}$ is a π-map

$$F\colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+ \quad (p \geqslant 0),$$

inducing the Umkehr $f^!$ on chain level, that is such that there exists a $\mathbf{Z}[\pi]$-module chain homotopy

$$(\Sigma^p_{\tilde{M}})^{-1} F (\Sigma^p_{\tilde{X}}) \simeq f^!\colon C(\tilde{X}) \to C(\tilde{M}).$$

PROPOSITION 2.3. *Given a degree 1 map of n-dimensional geometric Poincaré complexes $f\colon M \to X$ and a geometric Umkehr map*

$$F\colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$$

with respect to an oriented cover $\tilde{X}$ of X with data (π, w), there is defined in a natural way an n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi]$ with the w-twisted involution, the quadratic kernel *of (f, F),*

$$\sigma_*(f, F) = (C(f^!), e_\% \psi_F[X] \in Q_n(C(f^!)))$$

depending only on the stable π-homotopy class of F, such that

$$(1+T)\sigma_*(f, F) = \sigma^*(f).$$

Proof. The absolute version of the quadratic construction of Proposition 1.5

$$\psi_F\colon H^\pi_n(\tilde{X}; {}^w\mathbf{Z}) \to Q^{[0,p-1]}_n(C(\tilde{M}))$$

is such that

$$\varphi_{\tilde{M}} f^!_* - f^{!\%} \varphi_{\tilde{X}} = (1+T)\psi_F\colon H^\pi_n(\tilde{X}; {}^w\mathbf{Z}) \to Q^n(C(\tilde{M})).$$

Let $e\colon C(\tilde{M}) \to C(f^!)$ be the inclusion, so that

$$\begin{aligned}(1+T)e_\% \psi_F[X] &= e^\%(1+T)\psi_F[X] \\ &= e^\% \varphi_{\tilde{M}} f^!_*[X] - e^\% f^{!\%} \varphi_{\tilde{X}}[X] = e^\% \varphi_{\tilde{M}}[M] \in Q^n(C(\tilde{M})).\end{aligned}$$

Here, as elsewhere, we let $e_\% \psi_F[X]$ stand both for an element of $Q^{[0,p-1]}_n(C(f^!))$ and for its image in $Q_n(C(f^!))$.

The symmetric {quadratic} kernels $\sigma^*(f)$ $\{\sigma_*(f, F)\}$ of a degree 1 map $f\colon M \to X$ {with Umkehr $F\colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$} associated to the various oriented covers of X are all induced from the kernel associated to the universal cover $\tilde{X}$.

In § 7 below we shall show how to obtain the surgery obstruction of a normal map $(f, b)\colon M \to X$ from the quadratic kernel $\sigma_*(f, F)$, using the given normal bundle map $b\colon \nu_M \to \nu_X$ and the equivariant S-duality of § 3 to produce a geometric Umkehr map $F\colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$ for the universal cover $\tilde{X}$. In favourable circumstances it is possible to obtain F directly from (f, b) without the S-duality machinery. For example, if $f\colon M \to X$ is a degree 1 map of manifolds which is covered by a map $b: \nu_M \to \nu_X$ of stable normal bundles then f can be approximated by a framed embedding $M \times D^p \subset \text{interior}(X \times D^p)$ (p large) which lifts to an embedding of covers $\tilde{M} \times D^p \subset \tilde{X} \times D^p$ for any cover $\tilde{X}$ of X, giving F by the Pontrjagin–Thom construction

$$F: \Sigma^p \tilde{X}_+ = \tilde{X} \times D^p / \tilde{X} \times S^{p-1} \xrightarrow{\text{collapse}} \tilde{X} \times D^p / \overline{\tilde{X} \times D^p - \tilde{M} \times D^p}$$
$$= \tilde{M} \times D^p / \tilde{M} \times S^{p-1} = \Sigma^p \tilde{M}_+.$$

The case where $p = 1$ is of interest in codimension 2 surgery.

The mod 2 reduction of the quadratic kernel construction gives the $\mathbf{Z}_2$-valued quadratic form used by Browder [3, Chapter III, § 4] to define the Arf invariant. An *n-dimensional geometric* $\mathbf{Z}_2$*-Poincaré complex* is a finitely dominated CW complex X together with a mod 2 fundamental class $[X] \in H_n(X; \mathbf{Z}_2)$ defining mod 2 Poincaré duality isomorphisms

$$[X] \cap - : H^*(X; \mathbf{Z}_2) \to H_{n-*}(X; \mathbf{Z}_2).$$

PROPOSITION 2.4. (i) *Given an n-dimensional geometric* $\mathbf{Z}_2$*-Poincaré complex* X *there is defined in a natural way an n-dimensional symmetric Poincaré complex over* $\mathbf{Z}_2$

$$\sigma^*(X) = (C(X; \mathbf{Z}_2), \varphi_X[X] \in Q^n(C(X; \mathbf{Z}_2)))$$

such that the symmetric Wu classes of $\sigma^*(X)$ *are just the Wu classes of* X

$$v_r(\varphi_X[X]) = v_r(X) \in \text{Hom}_{\mathbf{Z}_2}(H^{n-r}(X; \mathbf{Z}_2), \mathbf{Z}_2) = H^r(X; \mathbf{Z}_2),$$

as characterized by

$$Sq^r(y) = \langle v_r(X) \cup y, [X] \rangle \in \mathbf{Z}_2 \quad (y \in H^{n-r}(X; \mathbf{Z}_2)).$$

(ii) *Given a degree* 1 (mod 2) *map* $f: M \to X$ *of n-dimensional geometric* $\mathbf{Z}_2$*-Poincaré complexes and a* $\{1\}$*-map* $F: \Sigma^p X_+ \to \Sigma^p M_+$ *inducing the* mod 2 *Umkehr* $f^!: C(X; \mathbf{Z}_2) \to C(M; \mathbf{Z}_2)$ *there is defined in a natural way an n-dimensional quadratic Poincaré complex over* $\mathbf{Z}_2$,

$$\sigma_*(f, F) = (C(f^!), e_\% \psi_F[X] \in Q_n(C(f^!))),$$

such that

$$\sigma^*(M) = (1+T)\sigma_*(f, F) \oplus \sigma^*(X)$$

up to homotopy equivalence. The quadratic Wu classes of $\sigma_*(f, F)$,

$$v^r = v^r(e_\% \psi_F[X]): K^{n-r}(M; \mathbf{Z}_2) \to \begin{cases} \mathbf{Z}_2 & \text{if } n \leqslant 2r \leqslant n+p-1, \\ 0 & \text{otherwise}, \end{cases}$$

can be expressed in terms of functional Steenrod squares

$$v^r(y) = \langle Sq^{r+1}_{(\Sigma^p y)F}(\Sigma^p(\iota)), \Sigma^p[X]\rangle \in \mathbf{Z}_2 \quad (n \leqslant 2r),$$

$$(y \in K^{n-r}(M;\mathbf{Z}_2) \subseteq H^{n-r}(M;\mathbf{Z}_2) = [M_+, K(\mathbf{Z}_2, n-r)],$$

$$\iota \in H^{n-r}(K(\mathbf{Z}_2, n-r);\mathbf{Z}_2) = \mathbf{Z}_2),$$

and are such that

$$v^r(y_1+y_2) - v^r(y_1) - v^r(y_2) = \begin{cases} \langle y_1 \cup y_2, [M]\rangle \in \mathbf{Z}_2 & \text{if } n = 2r, \\ 0 \in \mathbf{Z}_2 & \text{if } n < 2r. \end{cases}$$

Proof. Apply Propositions 1.3 and 1.6.

The kernel constructions behave as follows under composition.

PROPOSITION 2.5. *Let X, Y, Z be n-dimensional geometric Poincaré complexes. The composite of degree 1 {geometric Umkehr} maps $f\colon X \to Y$, $g\colon Y \to Z$ $\{F\colon \Sigma^p\tilde{Y}_+ \to \Sigma^p\tilde{X}_+,\ G\colon \Sigma^p\tilde{Z}_+ \to \Sigma^p\tilde{Y}_+\}$ is a degree 1 {geometric Umkehr} map $gf\colon X \to Z$ $\{FG\colon \Sigma^p\tilde{Z}_+ \to \Sigma^p\tilde{X}_+\}$, with symmetric {quadratic} kernel*

$$\begin{cases} \sigma^*(gf) = \sigma^*(f) \oplus \sigma^*(g), \\ \sigma_*(gf, FG) = \sigma_*(f, F) \oplus \sigma_*(g, (\Sigma^p\tilde{f}_+)FG) \end{cases}$$

up to homotopy equivalence, with $\tilde{X}$, $\tilde{Y}$ the oriented covers of X, Y induced from an oriented cover $\tilde{Z}$ of Z.

Proof. Write the inclusions in the algebraic mapping cones as

$$e_f\colon C(\tilde{X}) \to C(f^!),$$
$$e_g\colon C(\tilde{Y}) \to C(g^!),$$
$$e_{gf}\colon C(\tilde{Z}) \to C((gf)^!),$$

and note that $(gf)^! = f^!g^!\colon C(\tilde{Z}) \to C(\tilde{Y}) \to C(\tilde{X})$. The stable composite of the chain equivalences,

$$\begin{pmatrix} e_f \\ \tilde{f} \end{pmatrix}\colon C(\tilde{X}) \to C(f^!) \oplus C(\tilde{Y}),$$

$$\begin{pmatrix} e_g \\ \tilde{g} \end{pmatrix}\colon C(\tilde{Y}) \to C(g^!) \oplus C(\tilde{Z}),$$

is a chain equivalence

$$\begin{pmatrix} e_f \\ e_g\tilde{f} \\ \tilde{g}\tilde{f} \end{pmatrix}\colon C(\tilde{X}) \to C(f^!) \oplus C(g^!) \oplus C(\tilde{Z}),$$

allowing us to identify

$$e_{gf} = \begin{pmatrix} e_f \\ e_g\tilde{f} \end{pmatrix} : C(\tilde{X}) \to C(f^!g^!) = C(f^!) \oplus C(g^!).$$

Using the direct sum decomposition of Proposition I.1.4(i) we have

$$e_{gf}^{\%}\varphi_{\tilde{X}}[X] = (e_f^{\%}\varphi_{\tilde{X}}[X], e_g^{\%}\tilde{f}^{\%}\varphi_{\tilde{X}}[X], (e_f^t \otimes e_g\tilde{f})\varphi_{\tilde{X}}[X]_0)$$
$$\in Q^n(C(f^!) \oplus C(g^!)) = Q^n(C(f^!)) \oplus Q^n(C(g^!)) \oplus H_n(C(f^!)^t \otimes_{\mathbf{Z}[\pi]} C(g^!)).$$

Now $f^{\%}\varphi_{\tilde{X}}[X] = \varphi_{\tilde{Y}}f_*[X] = \varphi_{\tilde{Y}}[Y] \in Q^n(C(\tilde{Y}))$, and

$$(e_f^t \otimes e_g\tilde{f})(\varphi_{\tilde{X}}[X])_0 = 0 \in H_n(C(f^!)^t \otimes_{\mathbf{Z}[\pi]} C(g^!))$$

since there is defined a $\mathbf{Z}[\pi]$-module chain homotopy commutative diagram

$$\begin{array}{ccccc} {}^wC(f^!)^{n-*} & \xrightarrow{e_f^*} & {}^wC(\tilde{X})^{n-*} & \xrightarrow{f^{!*}} & {}^wC(\tilde{Y})^{n-*} & & \\ & & \Big\downarrow \varphi_{\tilde{X}}[X]_0 = [X]\cap - & & \Big\downarrow [Y]\cap - & & \\ & & C(\tilde{X}) & \xrightarrow{\tilde{f}} & C(\tilde{Y}) & \xrightarrow{e_g} & C(g^!) \end{array}$$

with $f^{!*}e_f^* \simeq 0$. Thus

$$e_{gf}^{\%}\varphi_{\tilde{X}}[X] = e_f^{\%}\varphi_{\tilde{X}}[X] \oplus e_g^{\%}\varphi_{\tilde{Y}}[Y] \in Q^n(C((gf)^!)) = Q^n(C(f^!) \oplus C(g^!)),$$

and so

$$\sigma^*(gf) = \sigma^*(f) \oplus \sigma^*(g).$$

(The formula $\sigma^*(X) = \sigma^*(f) \oplus \sigma^*(Y)$ is the special case $Z = \emptyset$.)

In the quadratic case we have

$$\sigma_*(gf, FG) = (C((gf)^!), e_{gf\%}\psi_{FG}[Z] \in Q_n(C((gf)^!)))$$

with $\psi_{FG} = \psi_F g_*^! + f_{\%}^!\psi_G$: $H_n^{\pi}(\tilde{Z}; {}^w\mathbf{Z}) \to Q_n(C(\tilde{X}))$ by the sum formula of Proposition 1.5(iii). Working as above, we have

$$\begin{aligned} e_{gf\%}\psi_{FG}[Z] &= (e_{f\%}\psi_{FG}[Z], e_{g\%}\tilde{f}_{\%}\psi_{FG}[Z], (e_f^t \otimes e_g\tilde{f})((1+T)\psi_{FG}[Z])_0) \\ &= (e_{f\%}\psi_F[Y], e_{g\%}(\tilde{f}_{\%}\psi_F g_*^! + \psi_G)[Z], 0) \\ &= e_{f\%}\psi_F[Y] \oplus e_{g\%}\psi_{(\Sigma^p\tilde{f}_+)FG}[Z] \\ &\in Q_n(C((gf)^!)) = Q_n(C(f^!) \oplus C(g^!)) \end{aligned}$$

so that

$$\sigma_*(gf, FG) = \sigma_*(f, F) \oplus \sigma_*(g, (\Sigma^p\tilde{f}_+)FG).$$

A degree 1 map of n-dimensional geometric Poincaré complexes $f: M \to X$ is *k-connected* with respect to some covering $\tilde{X}$ of X if $K_r(M) = 0$ for $r \leqslant k$. Recalling the definition of skew-suspension $\bar{S}$ in § I.1 we have:

PROPOSITION 2.6. *The symmetric {quadratic} kernel $\sigma^*(f)$ $\{\sigma_*(f, F)\}$ of an $(r-1)$-connected degree 1 map of n-dimensional geometric Poincaré* *complexes* $f: M \to X$ *{with geometric Umkehr* π*-map* F*} with respect to an oriented covering* $\tilde{X}$ *of* X *with data* (π, w) *is the r-fold skew-suspension of an* $(n-2r)$*-dimensional* $(-)^r$*-symmetric* $\{(-)^r$*-quadratic*$\}$ *Poincaré complex over* $\mathbf{Z}[\pi]$ $\sigma^r(f)$ $\{\sigma_r(f,F)\}$, *with*

$$\begin{cases} \bar{S}\sigma^r(f) = \sigma^{r-1}(f), \quad \sigma^0(f) = \sigma^*(f), \\ \bar{S}\sigma_r(f,F) = \sigma_{r-1}(f,F), \quad \sigma_0(f,F) = \sigma_*(f,F), \quad (1+T_{(-)^r})\sigma_r(f,F) = \sigma^r(f). \end{cases}$$

In § 5 below we shall identify the quadratic kernel $\sigma_i(f,F)$ associated to an $(i-1)$-connected $2i$ $\{2i+1\}$-dimensional normal map $(f,b): M \to X$ with the surgery obstruction kernel obtained in § 5 {§ 6} of Wall [**30**], using the one-to-one correspondence between 0-dimensional {1-dimensional} $(-)^i$ quadratic Poincaré complexes and non-singular $(-)^i$ quadratic forms {formations} of Proposition I.2.1 {I.2.5}.

3. Equivariant S-duality

The S-duality between M_+ and the Thom space $T(\nu_M)$ of the normal bundle ν_M of an embedding $M^n \subset S^{n+p}$ (p large) of a compact manifold M was first established by Milnor and Spanier [**17**]. This was then generalized by Atiyah [**2**], and extended to geometric Poincaré complexes by Spivak [**25**] and Wall [**29**]. In particular, if $f: M \to X$ is a degree 1 map of geometric Poincaré complexes which is covered by a map of Spivak normal fibrations $b: \nu_M \to \nu_X$ then the S-dual of $T(b): T(\nu_M) \to T(\nu_X)$ is a geometric Umkehr map $F: \Sigma^p X_+ \to \Sigma^p M_+$, and this was used by Browder [**3**] to obtain the surgery obstruction in the simply-connected case $\pi_1(X) = \{1\}$. We shall now develop an equivariant S-duality theory for π-spaces with a special type of π-equivariant cell structure ('$CW\pi$-complexes') in order to obtain a geometric Umkehr π-map

$$F: \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+ \quad (p \text{ large})$$

for any covering $\tilde{X}$ of X with group of covering translations π, giving the non-simply-connected surgery obstruction by means of the quadratic construction ψ_F of § 1.

Given π-spaces X, Y let $[X, Y]_\pi$ be the pointed set of π-homotopy classes of π-maps $f: X \to Y$. Regarding the loop space $\Omega X = (X, \text{pt.})^{(S^1, \text{pt.})}$ as a π-space using the trivial π-action on S^1 we have that

$$[\Sigma^p X, Y]_\pi = [X, \Omega^p Y]_\pi$$

is a group for $p \geqslant 1$, abelian for $p \geqslant 2$. Define the abelian group of *$S\pi$-maps* between π-spaces X, Y to be the direct limit

$$\{X, Y\}_\pi = \varinjlim_p [\Sigma^p X, \Sigma^p Y]_\pi$$

of the suspension sequence

$$[X, Y]_\pi \xrightarrow{\Sigma} [\Sigma X, \Sigma Y]_\pi \xrightarrow{\Sigma} [\Sigma^2 X, \Sigma^2 Y]_\pi \xrightarrow{\Sigma} [\Sigma^3 X, \Sigma^3 Y]_\pi \longrightarrow \dots .$$

For $\pi = \{1\}$ we write $[X, Y]_{\{1\}} = [X, Y], \{X, Y\}_{\{1\}} = \{X, Y\}$ as usual.

The *mapping cone* of a π-map $f: X \to Y$ is the π-space

$$C_f = Y \cup_f X \wedge I.$$

The cofibration sequence of π-spaces and π-maps

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow C_{\Sigma f} \longrightarrow \dots$$

induces the following π-equivariant analogue of the Puppe exact sequence.

PROPOSITION 3.1. *For any π-map $f: X \to Y$ and π-space Z there is defined in a natural way an exact sequence*

$$[X,Z]_\pi \xleftarrow{f} [Y,Z]_\pi \longleftarrow [C_f,Z]_\pi \longleftarrow [\Sigma X,Z]_\pi$$
$$\xleftarrow{\Sigma f} [\Sigma Y,Z]_\pi \longleftarrow \dots .$$

For any $\{1\}$-space K regard $\bigvee_\pi K$ as a π-space by permutation of the summands. Note that for any π-space X,

$$[\bigvee_\pi K, X]_\pi = [K, X], \quad \{\bigvee_\pi K, X\}_\pi = \{K, X\}.$$

Define the π-space obtained from a π-space X by *attaching an r-dimensional π-cell* to be the disjoint union {identification space}

$$X' = \begin{cases} X \cup \pi \times D^0 & \text{if } r = 0, \\ X \cup_{\pi \times f} \pi \times D^r, \text{ for some map } f: S^{r-1} \to X & \text{if } r \geqslant 1. \end{cases}$$

The π-cell is *pointed* if the attaching map $f: S^{r-1} \to X$ $(r \geqslant 1)$ preserves basepoints, in which case f extends to a π-map $f': \bigvee_\pi S^{r-1} \to X$ such that

$$X' = C_{f'} = X \cup_{f'} \bigvee_\pi D^r.$$

If X is a path-connected π-space then any map $f: S^{r-1} \to X$ $(r \geqslant 1)$ is homotopic to a basepoint-preserving map $f_0: S^{r-1} \to X$ extending to a π-map $f_0': \bigvee_\pi S^{r-1} \to X$, and $X' = X \cup_{\pi \times f} \pi \times D^r$ is π-homotopic to the mapping cone π-space $C_{f_0'} = X \cup_{f_0'} \bigvee_\pi D^r$.

A *$CW\pi$-complex* X is a π-space which is a based CW complex obtained from the base 0-cell by successively attaching π-cells of non-decreasing dimension. A $CW\pi$-complex is thus an ordinary CW complex with a cellular π-action which preserves the base 0-cell and which acts freely by permutation on the other cells. The suspension of a $CW\pi$-complex X is a $CW\pi$-complex ΣX, with one r-dimensional π-cell for each $(r-1)$-dimensional π-cell of X $(r \geqslant 0)$.

$CW\pi$-complexes arise as follows.

PROPOSITION 3.2. *If $(\tilde{X}, \tilde{Y})$ is a covering of a CW pair (X, Y) with group π of covering translations, then $\tilde{X}/\tilde{Y}$ is a CW π-complex with one π-cell for each cell of $X - Y$. If $Y = \emptyset$ interpret the quotient as $\tilde{X}/\emptyset = \tilde{X}_+$.*

A $CW\pi$-complex is *pointed* if it involves only pointed π-cells. A $CW\pi$-complex with no 0-dimensional π-cells (for example, a suspension) is π-homotopic to a pointed $CW\pi$-complex.

A $CW\pi$-complex X is *finite* if it involves only a finite number of π-cells. A $CW\pi$-complex X is *finitely-dominated* if there exist a finite $CW\pi$-complex K and π-maps $f: X \to K$, $g: K \to X$ such that $gf = 1 \in [X, X]_\pi$, and it is *n-dimensional* if $\dot{H}^r(X) = 0$ for $r > n$, in which case $\dot{C}(X)$ is an n-dimensional $\mathbf{Z}[\pi]$-module chain complex.

We have the following analogue of the Freudenthal suspension theorem.

PROPOSITION 3.3. *Let X be an n-dimensional finite pointed $CW\pi$-complex, and let Y be a π-space of the homotopy type of a CW complex. Then the suspension map*

$$\Sigma: [\Sigma^p X, \Sigma^p Y]_\pi \to [\Sigma^{p+1} X, \Sigma^{p+1} Y]_\pi$$

is an isomorphism for $p \geqslant n+1$, and

$$\{X, Y\}_\pi = [\Sigma^{n+1} X, \Sigma^{n+1} Y]_\pi .$$

Proof. The proof is by induction on the number of pointed π-cells in X. The result is trivial for $n = 0$; assume it is true for X, and let $X' = X \cup_f \bigvee_\pi D^n$ for some π-map $f: \bigvee_\pi S^{n-1} \to X$ $(n \geqslant 1)$. There is defined a commutative diagram of abelian groups and morphisms

$$\begin{array}{ccccccccc}
[\Sigma^{p+1} X, \Sigma^p Y]_\pi & \longrightarrow & [\bigvee_\pi S^{n+p}, \Sigma^p Y]_\pi & \longrightarrow & [\Sigma^p X', \Sigma^p Y]_\pi & \longrightarrow & [\Sigma^p X, \Sigma^p Y]_\pi & \longrightarrow & [\bigvee_\pi S^{n+p-1}, \Sigma^p Y]_\pi \\
\downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\
[\Sigma^{p+2} X, \Sigma^{p+1} Y]_\pi & \longrightarrow & [\bigvee_\pi S^{n+p+1}, \Sigma^{p+1} Y]_\pi & \longrightarrow & [\Sigma^{p+1} X', \Sigma^{p+1} Y]_\pi & \longrightarrow & [\Sigma^{p+1} X, \Sigma^{p+1} Y]_\pi & \longrightarrow & [\bigvee_\pi S^{n+p}, \Sigma^{p+1} Y]_\pi
\end{array}$$

in which the rows are exact (Proposition 3.1). The suspension maps involving X are isomorphisms for $p \geqslant n+1$ by the inductive hypothesis. Since Y is of the homotopy type of a CW complex $\Sigma^p Y$ is $(p-1)$-connected and

$$\Sigma\colon [\bigvee_{\pi} S^{n+p}, \Sigma^p Y]_\pi = [S^{n+p}, \Sigma^p Y] \to [S^{n+p+1}, \Sigma^{p+1} Y]$$

is an isomorphism for $p \geqslant n+1$ by the ordinary Freudenthal suspension theorem. Application of the 5-lemma gives the induction step.

Given π-spaces X, Y define the $\{1\}$-space

$$X \wedge_\pi Y = (X \wedge Y)/\pi$$

to be the space of orbits of the diagonal π-action

$$\pi \times X \wedge Y \to X \wedge Y;\ (g, x \wedge y) \mapsto gx \wedge gy.$$

Note that for any π-space X and $\{1\}$-space K

$$X \wedge_\pi (\bigvee_{\pi} K) = X \wedge K.$$

A *π-spectrum* $\underline{Z}$ is a sequence of π-spaces Z_p $(p \geqslant 0)$ and π-maps $\xi_p\colon \Sigma Z_p \to Z_{p+1}$ $(p \geqslant 0)$. Given a π-space X define the abelian group

$$\{X, \underline{Z}\}_\pi = \varinjlim_{p} [\Sigma^p X, Z_p]_\pi$$

to be the direct limit of the sequence

$$[X, Z_0]_\pi \xrightarrow{\Sigma} [\Sigma X, \Sigma Z_0]_\pi \xrightarrow{\xi_0} [\Sigma X, Z_1]_\pi \xrightarrow{\Sigma} [\Sigma^2 X, \Sigma Z_1]_\pi \xrightarrow{\xi_1} \cdots.$$

In particular, for $\xi_p = \mathrm{id.}\colon \Sigma Z_p = \Sigma^{p+1} Z_0 \to Z_{p+1} = \Sigma^{p+1} Z_0$ we have

$$\{X, \underline{Z}\}_\pi = \{X, Z_0\}_\pi.$$

Given a π-space X and a π-spectrum $\underline{Z}$ let $X \wedge_\pi \underline{Z}$ be the $\{1\}$-spectrum defined by

$$(X \wedge_\pi \underline{Z})_p = X \wedge_\pi Z_p, \quad 1 \wedge \xi_p\colon \Sigma(X \wedge_\pi Z_p) = X \wedge_\pi \Sigma Z_p \to X \wedge_\pi Z_{p+1}.$$

PROPOSITION 3.4. *Given a $\{1\}$-space W, a π-map $f\colon X \to Y$ and a π-spectrum $\underline{Z}$ there are defined exact sequences of abelian groups*

$$\{W, X \wedge_\pi \underline{Z}\} \to \{W, Y \wedge_\pi \underline{Z}\} \to \{W, C_f \wedge_\pi \underline{Z}\} \to \{W, \Sigma X \wedge_\pi \underline{Z}\} \to \cdots$$
$$\{X, \underline{Z}\}_\pi \leftarrow \{Y, \underline{Z}\}_\pi \leftarrow \{C_f, \underline{Z}\}_\pi \leftarrow \{\Sigma X, \underline{Z}\}_\pi \leftarrow \cdots.$$

Proof. The first sequence is just the Puppe sequence associated to the (co)fibration sequence of $\{1\}$-spectra

$$X \wedge_\pi \underline{Z} \xrightarrow{f \wedge 1} Y \wedge_\pi \underline{Z} \longrightarrow C_f \wedge_\pi \underline{Z} \longrightarrow X \wedge_\pi \underline{Z} \longrightarrow \cdots.$$

The exactness of the other sequence may be established as in the case where $\pi = \{1\}$ (Puppe sequence again) by insisting on π-maps and π-homotopies.

Given π-spaces X, Y and a $\{1\}$-map

$$\alpha : S^N \to X \wedge_\pi Y$$

for some $N \geqslant 0$ define slant products for any π-spectrum $\underline{Z}$,

$$\alpha \backslash - : \{X, \underline{Z}\}_\pi \longrightarrow \{S^N, \underline{Z} \wedge_\pi Y\};$$

$$(f : \Sigma^p X \longrightarrow Z_p) \longmapsto (S^{N+p} \xrightarrow{\Sigma^p \alpha} \Sigma^p X \wedge_\pi Y \xrightarrow{f \wedge 1} Z_p \wedge_\pi Y),$$

$$\alpha \backslash - : \{Y, \underline{Z}\}_\pi \longrightarrow \{S^N, X \wedge_\pi \underline{Z}\};$$

$$(g : \Sigma^p Y \longrightarrow Z_p) \longmapsto (S^{N+p} \xrightarrow{\Sigma^p \alpha} X \wedge_\pi \Sigma^p Y \xrightarrow{1 \wedge g} X \wedge_\pi Z^p).$$

Call $\alpha : S^N \to X \wedge_\pi Y$ an *$S\pi$-duality map* if these slant products are isomorphisms for every π-spectrum $\underline{Z}$, in which case the suspensions

$$\Sigma\alpha : S^{N+1} \to \Sigma(X \wedge_\pi Y) = \Sigma X \wedge_\pi Y, \quad \Sigma\alpha : S^{N+1} \to \Sigma(X \wedge_\pi Y) = X \wedge_\pi \Sigma Y$$

are also $S\pi$-duality maps. For $\pi = \{1\}$ this is classical Spanier–Whitehead S-duality.

Given $S\pi$-duality maps

$$\alpha : S^N \to X \wedge_\pi Y, \quad \alpha' : S^N \to X' \wedge_\pi Y',$$

define the *$S\pi$-dual* of an $S\pi$-map $f \in \{X, X'\}_\pi$ to be the $S\pi$-map $g \in \{Y', Y\}_\pi$ to which f is sent by the composite isomorphism

$$\{X, X'\}_\pi \xrightarrow{\alpha \backslash} \{S^N, X' \wedge_\pi Y\} \xrightarrow{(\alpha' \backslash)^{-1}} \{Y', Y\}_\pi.$$

In particular, if $X = X'$ the $S\pi$-duals of $1 \in \{X, X\}_\pi$ are an inverse pair of $S\pi$-homotopy equivalences $g \in \{Y, Y'\}_\pi$, $g' \in \{Y', Y\}_\pi$.

PROPOSITION 3.5. *Every finite $CW\pi$-complex X admits an $S\pi$-duality map*

$$\alpha : S^N \to X \wedge_\pi Y$$

with Y a finite $CW\pi$-complex.

Proof. Suspending if necessary we may assume that X is a pointed $CW\pi$-complex. Our construction of an $S\pi$-dual is by induction on the pointed π-cells: given an $S\pi$-duality map $\alpha : S^N \to X \wedge_\pi Y$ between finite pointed $CW\pi$-complexes X, Y and a π-map $f : \bigvee_\pi S^{r-1} \to X$ we shall construct an $S\pi$-duality map $\alpha' : S^{N'} \to X' \wedge_\pi Y'$ for $X' = X \cup_f \bigvee_\pi D^r$.

Let $m = \max(\text{dimension}(Y) + 1, 2r - 1 - N)$. Replacing $\alpha : S^N \to X \wedge_\pi Y$ by $\Sigma^m \alpha : S^{N+m} \to X \wedge_\pi \Sigma^m Y$ we have that $N - r + 1 \geqslant 0$ and

$$\{\bigvee_\pi S^{r-1}, X\}_\pi = [\bigvee_\pi S^N, \Sigma^{N-r+1} X]_\pi, \quad \{Y, \bigvee_\pi S^{N-r+1}\}_\pi = [Y, \bigvee_\pi S^{N-r+1}]_\pi$$

(by Proposition 3.3). Define an $S\pi$-duality map

$$\beta: S^N \to (\bigvee_\pi S^{r-1}) \wedge_\pi (\bigvee_\pi S^{N-r+1}) = \bigvee_\pi S^N$$

by sending S^N to the summand labelled by $1 \in \pi$. Let $g: Y \to \bigvee_\pi S^{N-r+1}$ be a π-map representing the $S\pi$-dual of $f \in \{\bigvee_\pi S^{r-1}, X\}_\pi$, and let $Y' = C_g$ be the mapping cone π-space. Denote the cofibration sequences by

$$\bigvee_\pi S^{r-1} \xrightarrow{f} X \xrightarrow{e} X' \xrightarrow{d} \bigvee_\pi S^r,$$

$$Y \xrightarrow{g} \bigvee_\pi S^{N-r+1} \xrightarrow{h} Y' \xrightarrow{k} \Sigma Y.$$

The diagram of $\{1\}$-spaces and $\{1\}$-maps

$$\begin{array}{ccccc} S^N & \xrightarrow{\alpha} & X \wedge_\pi Y & \xrightarrow{e \wedge 1} & X' \wedge_\pi Y \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1 \wedge g} & & \downarrow{\scriptstyle 1 \wedge g} \\ (\bigvee_\pi S^{r-1}) \wedge_\pi (\bigvee_\pi S^{N-r+1}) & \xrightarrow{f \wedge 1} & X \wedge_\pi (\bigvee_\pi S^{N-r+1}) & \xrightarrow{e \wedge 1} & X' \wedge_\pi (\bigvee_\pi S^{N-r+1}) \end{array}$$

is homotopy commutative, with the bottom row null-homotopic. It is thus possible to define a $\{1\}$-map $j: D^{N+1} \to X' \wedge_\pi (\bigvee_\pi S^{N-r+1})$ such that the diagram

$$\begin{array}{ccc} S^N & \xrightarrow{(e \wedge 1)\alpha} & X' \wedge_\pi Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1 \wedge g} \\ D^{N+1} & \xrightarrow{j} & X' \wedge_\pi (\bigvee_\pi S^{N-r+1}) \end{array}$$

is actually commutative, with $i: S^N \to D^{N+1}$ the inclusion. The induced $\{1\}$-map of mapping cones

$$\alpha': C_i = S^{N+1} \to C_{1 \wedge g} = X' \wedge_\pi Y'$$

is such that both the squares in the diagram of $\{1\}$-spaces and $\{1\}$-maps

$$\begin{array}{ccccc} X \wedge_\pi \Sigma Y & \xleftarrow{\Sigma\alpha} & S^{N+1} & \xrightarrow{\Sigma\beta} & (\bigvee_\pi S^r) \wedge_\pi (\bigvee_\pi S^{N-r+1}) \\ \downarrow{\scriptstyle e \wedge 1} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle 1 \wedge h} \\ X' \wedge_\pi \Sigma Y & \xleftarrow{1 \wedge k} & X' \wedge_\pi Y' & \xrightarrow{d \wedge 1} & (\bigvee_\pi S^r) \wedge_\pi Y' \end{array}$$

are homotopy commutative. There is thus defined a commutative diagram of abelian groups and morphisms

$$\begin{array}{ccccc}
\{\Sigma X,\underline{Z}\}_\pi & \xrightarrow{\Sigma f} & \{\bigvee_\pi S^r,\underline{Z}\}_\pi & \xrightarrow{d} & \{X',\underline{Z}\}_\pi \\
\downarrow{\scriptstyle (\Sigma\alpha)\backslash} & & \downarrow{\scriptstyle (\Sigma\beta)\backslash} & & \downarrow{\scriptstyle \alpha'\backslash} \\
\{S^{N+1},\underline{Z}\wedge_\pi Y\} & \xrightarrow{g} & \{S^{N+1},\underline{Z}\wedge_\pi(\bigvee_\pi S^{N-r+1})\} & \xrightarrow{h} & \{S^{N+1},\underline{Z}\wedge_\pi Y'\}
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{e} & \{X,\underline{Z}\}_\pi & \xrightarrow{f} & \{\bigvee_\pi S^{r-1},\underline{Z}\}_\pi \\
& \downarrow{\scriptstyle \Sigma\alpha\backslash} & & \downarrow{\scriptstyle \Sigma\beta\backslash} \\
\xrightarrow{k} & \{S^{N+1},\underline{Z}\wedge_\pi\Sigma Y\} & \xrightarrow{\Sigma g} & \{S^{N+1},\underline{Z}\wedge_\pi(\bigvee_\pi S^{N-r+2})\}
\end{array}$$

for any π-spectrum $\underline{Z}$, with exact rows (Proposition 3.4). Applying the 5-lemma we have that the middle column is an isomorphism, and similarly for the other type of slant product. Therefore $\alpha'\colon S^{N+1}\to X'\wedge_\pi Y'$ is an $S\pi$-duality map.

We can use $S\pi$-duality to prove an equivariant analogue of Whitehead's theorem.

PROPOSITION 3.6. *A π-map of finite $CW\pi$-complexes $f\colon X\to Y$ induces isomorphisms in homology if and only if $\Sigma^p f\colon \Sigma^p X\to\Sigma^p Y$ is a π-homotopy equivalence for some $p\geqslant 0$.*

Proof. Let $f\colon X\to Y$ induce isomorphisms in homology. Applying the ordinary Whitehead theorem we have that $\Sigma f\colon\Sigma X\to\Sigma Y$ is a homotopy equivalence, and hence that

$$f\colon\{S^N,(\bigvee_\pi S^r)\wedge_\pi X\}\to\{S^N,(\bigvee_\pi S^r)\wedge_\pi Y\}$$

is an isomorphism for all $N,r\geqslant 0$. This gives the induction step in proving that

$$f\colon\{S^N, W\wedge_\pi X\}\to\{S^N, W\wedge_\pi Y\}$$

is an isomorphism for every finite $CW\pi$-complex W. Given an $S\pi$-duality map $\alpha\colon S^N\to W\wedge_\pi Y$ (by Proposition 3.5) we thus have isomorphisms

$$\{Y,X\}_\pi\xrightarrow{\alpha\backslash}\{S^N, W\wedge_\pi X\}\xrightarrow{f}\{S^N, W\wedge_\pi Y\}\xrightarrow{(\alpha\backslash)^{-1}}\{Y,Y\}_\pi.$$

The element $g\in\{Y,X\}_\pi$ corresponding to $1\in\{Y,Y\}_\pi$ is represented by a

π-map $g\colon \Sigma^p Y \to \Sigma^p X$ for some $p \geqslant 0$ (by Proposition 3.2) which is a π-homotopy inverse for $\Sigma^p f\colon \Sigma^p X \to \Sigma^p Y$.

For any π-spaces X, Y the natural projection defines a $\mathbf{Z}$-module chain map

$$\mathbf{Z}^t \otimes_{\mathbf{Z}[\pi]} (\dot{C}(X) \otimes_{\mathbf{Z}} \dot{C}(Y)) = \dot{C}(X)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(Y) \to \dot{C}(X \wedge_\pi Y),$$

where t refers to the untwisted involution on $\mathbf{Z}[\pi]$. If X, Y are finitely-dominated $CW\pi$-complexes this is a chain equivalence (consider the reduced cellular chain complexes) and the chain level slant product

$$(\dot{C}(X)^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(Y)) \otimes_{\mathbf{Z}} \dot{C}(X)^* \to \dot{C}(Y);\ (x \otimes y) \otimes f \mapsto \overline{f(x)}y$$

induces a slant product in homology

$$\backslash : \dot{H}_N(X \wedge_\pi Y) \otimes_{\mathbf{Z}} \dot{H}^r(X) \to \dot{H}_{N-r}(Y).$$

The $S\pi$-duality map $\alpha\colon S^N \to X \wedge_\pi Y$ constructed in Proposition 3.5 is such that $\alpha_*[S^N]\backslash - : \dot{C}(X)^{N-*} \to \dot{C}(Y)$ is a $\mathbf{Z}[\pi]$-module chain equivalence, since the π-cellular structure was constructed as the dual of that of X. We shall show (in Proposition 3.8 below) that this property characterizes $S\pi$-duality maps for finite $CW\pi$-complexes, generalizing the case $\pi = \{1\}$ of ordinary S-duality.

Define the *r-dimensional Eilenberg–MacLane π-spectrum* $\underline{K\pi(\mathbf{Z}, r)}$ by

$$K\pi(\mathbf{Z}, r)_p = \bigvee_\pi K(\mathbf{Z}, p+r), \quad \xi_p = \bigvee_\pi \eta_p\colon \Sigma K\pi(\mathbf{Z}, r)_p = \bigvee_\pi \Sigma K(\mathbf{Z}, p+r)$$
$$\to K\pi(\mathbf{Z}, r)_{p+1} = \bigvee_\pi K(\mathbf{Z}, p+r+1) \quad (r \geqslant 0),$$

with $\eta_p\colon \Sigma K(\mathbf{Z}, p+r) \to K(\mathbf{Z}, p+r+1)$ the standard map. For $\pi = \{1\}$ this is the usual Eilenberg–MacLane spectrum $\underline{K(\mathbf{Z}, r)}$.

PROPOSITION 3.7. *If X is a finite $CW\pi$-complex X then*

$$\dot{H}_r(X) = \{S^r, X \wedge_\pi \underline{K\pi(\mathbf{Z}, 0)}\}, \quad \dot{H}^r(X) = \{X, \underline{K\pi(\mathbf{Z}, r)}\}_\pi \quad (r \geqslant 0).$$

Proof. For any $CW\pi$-complex X we have

$$\{S^r, X \wedge_\pi \underline{K\pi(\mathbf{Z}, 0)}\} = \{S^r, X \wedge \underline{K(\mathbf{Z}, 0)}\} = \dot{H}_r(X) \quad (r \geqslant 0)$$

by the usual identification of integral homology with $\underline{K(\mathbf{Z}, 0)}$-homology. Also, there is defined a natural $\mathbf{Z}[\pi]$-module morphism

$$\iota\colon \{X, \underline{K\pi(\mathbf{Z}, r)}\}_\pi \to \dot{H}^r(X);\quad (f\colon \Sigma^p X \to \bigvee_\pi K(\mathbf{Z}, p+r)) \mapsto f^*(1)$$

with $f^*\colon \dot{H}^{p+r}(\bigvee_\pi K(\mathbf{Z}, p+r)) = \mathbf{Z}[\pi] \to \dot{H}^{p+r}(\Sigma^p X) = \dot{H}^r(X)$. If X is finite we have an $S\pi$-duality $\alpha\colon S^N \to X \wedge_\pi Y$ (Proposition 3.5) and ι can be identified with the $S\pi$-duality isomorphism

$$\alpha\backslash - : \{X, \underline{K\pi(\mathbf{Z}, r)}\}_\pi \to \{S^N, \underline{K\pi(\mathbf{Z}, r)} \wedge_\pi Y\} = \dot{H}_{N-r}(Y) = \dot{H}^r(X).$$

If X, Y are finite $CW\pi$-complexes and $\alpha: S^N \to X \wedge_\pi Y$ is a $\{1\}$-map then the identification of Proposition 3.7 carries the chain level slant product

$$\alpha_*[S^N]\backslash - : \dot{H}^r(X) \to \dot{H}_{N-r}(Y) \quad (\alpha_*[S^N] \in \dot{H}_N(X \wedge_\pi Y)),$$

to the geometric slant product

$$\alpha\backslash - : \{X, \underline{K\pi(\mathbf{Z}, r)}\}_\pi \to \{S^N, \underline{K\pi(\mathbf{Z}, r)} \wedge_\pi Y\} = \{S^{N-r}, \underline{K\pi(\mathbf{Z}, 0)} \wedge_\pi Y\},$$

defined previously.

Proposition 3.8. *Let X, Y be finite $CW\pi$-complexes. A $\{1\}$-map $\alpha: S^N \to X \wedge_\pi Y$ is an $S\pi$-duality map if and only if the chain level slant product*

$$\alpha_*[S^N]\backslash - : \dot{C}(X)^{N-*} \to \dot{C}(Y)$$

is a $\mathbf{Z}[\pi]$-module chain equivalence.

Proof. If $\alpha: S^N \to X \wedge_\pi Y$ is an $S\pi$-duality map then the chain level slant product with $\alpha_*[S^N] \in \dot{H}_N(X \wedge_\pi Y)$ induces the $S\pi$-duality isomorphisms

$$\dot{H}^r(X) = \{X, \underline{K\pi(\mathbf{Z}, r)}\}_\pi \to \{S^N, \underline{K\pi(\mathbf{Z}, r)} \wedge_\pi Y\}$$
$$= \{S^{N-r}, \underline{K\pi(\mathbf{Z}, 0)} \wedge_\pi Y\} = \dot{H}_{N-r}(Y).$$

Conversely, suppose given a $\{1\}$-map $\alpha: S^N \to X \wedge_\pi Y$ such that $\alpha_*[S^N]\backslash - : \dot{H}^*(X) \to \dot{H}_{N-*}(Y)$ is an isomorphism. Let $\alpha': S^N \to X \wedge_\pi Y'$ be the $S\pi$-duality map constructed for X in Proposition 3.5 for N sufficiently large, and let $f \in \{Y', Y\}_\pi$ correspond to $\alpha \in \{S^N, X \wedge_\pi Y\}$ under the $S\pi$-duality isomorphism

$$\alpha'\backslash - : \{Y', Y\}_\pi \to \{S^N, X \wedge_\pi Y\}.$$

Now $f \in \{Y', Y\}_\pi$ induces isomorphisms in homology

$$f = \alpha_*[S^N]\backslash - : \dot{H}_*(Y') = \dot{H}^{N-*}(X) \to \dot{H}_*(Y).$$

Applying Proposition 3.6 we have that $f \in \{Y', Y\}$ is an $S\pi$-homotopy equivalence, and hence that $\alpha: S^N \to X \wedge_\pi Y$ is an $S\pi$-duality map.

With a little more effort Propositions 3.5–3.8 can be made to apply also for finitely-dominated $CW\pi$-complexes.

$S\pi$-duality maps arise as follows.

Proposition 3.9. *Let E be a compact N-dimensional submanifold of S^N with non-empty boundary ∂E and let $\tilde{E}$ be a covering space of E with group of covering translations π, with $\widetilde{\partial E} \subset \tilde{E}$ covering ∂E. Then the* *composite* $\{1\}$*-map*

$$\alpha\colon S^N \xrightarrow{\text{collapse}} S^N/\overline{S^N - E} = E/\partial E = (\tilde{E}/\widetilde{\partial E})/\pi \xrightarrow{\text{diagonal}} \tilde{E}_+ \wedge_\pi (\tilde{E}/\widetilde{\partial E})$$

is an $S\pi$*-duality map.*

Proof. The diagonal map is obtained from

$$\Delta\colon \tilde{E}/\widetilde{\partial E} \to \tilde{E}_+ \wedge \tilde{E}/\widetilde{\partial E} = (\tilde{E} \times \tilde{E})/(\tilde{E} \times \widetilde{\partial E});\ x \mapsto (x, x)$$

by quotienting out the π-action. Now $\tilde{E}_+$ and $\tilde{E}/\widetilde{\partial E}$ are finite $CW\pi$-complexes by Proposition 3.2, and $\alpha\colon S^N \to \tilde{E}_+ \wedge_\pi \tilde{E}/\widetilde{\partial E}$ is an $S\pi$-duality map by Proposition 3.8 since

$$\alpha_*[S^N]/- = [E] \cap - :\ \dot{H}^r(\tilde{E}/\widetilde{\partial E}) = H^r(\tilde{E}, \widetilde{\partial E}) \to \dot{H}_{N-r}(\tilde{E}_+) = H_{N-r}(\tilde{E})$$

defines the Poincaré–Lefschetz duality isomorphisms of $(E, \partial E)$, with $[E] \in H_N(E, \partial E)$ the fundamental class.

Given a fibration $F \longrightarrow E \xrightarrow{p} B$ and a covering $\tilde{B}$ of the base space B with group π of covering translations define the *Thom* π*-space* to be the mapping cone π-space of the induced π-map $\tilde{p}_+\colon \tilde{E}_+ \to \tilde{B}_+$,

$$T\pi(p) = \tilde{B}_+ \cup_{\tilde{p}_+} \tilde{E}_+ \wedge I = \tilde{B} \cup_{\tilde{p}\times 0} \tilde{E} \times I/\tilde{E} \times 1.$$

The quotient $\{1\}$-space $T\pi(p)/\pi = T(p)$ is the usual Thom $\{1\}$-space of p, and if $\tilde{B} = \pi \times B$ is the trivial covering then

$$T\pi(p) = \bigvee_\pi T(p).$$

If $p\colon E \to B$ is a cellular map of CW complexes then $T\pi(p)$ is a $CW\pi$-complex by Proposition 3.2.

Fibre homotopy equivalence classes of $(k-1)$-spherical fibrations

$$S^{k-1} \longrightarrow E \xrightarrow{p} X$$

over a CW complex X are in a natural one–one correspondence with the homotopy classes of maps $p\colon X \to BG(k)$, for the appropriate classifying space $BG(k)$. Given such a fibration we shall say that a covering $\tilde{X}$ of X is *oriented with respect to* p if the group of covering translations π is equipped with a group morphism $w\colon \pi \to \mathbf{Z}_2$ such that the first Stiefel–Whitney class $w_1(p) \in H^1(X; \mathbf{Z}_2) = \mathrm{Hom}(\pi_1(X), \mathbf{Z}_2)$ factors as

$$w_1(p)\colon \pi_1(X) \longrightarrow \pi \xrightarrow{w} \mathbf{Z}_2$$

with $\pi_1(X) \to \pi$ the characteristic map, and the pair (π, w) is the *data* of the covering. A covering $\tilde{X}$ of X can be oriented with respect to $p\colon X \to BG(k)$ if and only if the pullback $\tilde{p}\colon \tilde{X} \longrightarrow X \xrightarrow{p} BG(k)$ is an orientable $(k-1)$-spherical fibration, but the choice of w is not unique. If $f\colon M \to X$ is a map of CW complexes then the pullback cover $\tilde{M}$ of M is oriented with respect to the pullback fibration

$$f^*(p)\colon M \xrightarrow{f} X \xrightarrow{p} BG(k),$$

with the same data (π, w).

A covering $\tilde{X}$ of a geometric Poincaré complex X is oriented with data (π, w) in the sense of § 2 if and only if it is oriented in the above sense with respect to the Spivak normal fibration $\nu_X\colon X \to BG$ with data (π, w) (cf. Proposition 4.1).

Spherical fibrations are characterized by the following equivariant generalization of the Thom isomorphism theorem.

PROPOSITION 3.10. *Let* $F \longrightarrow E \xrightarrow{p} B$ *be a fibration of* CW *complexes, with* E, B *finitely dominated. If* $F = S^{k-1}$ *(up to homotopy equivalence) and* $\tilde{B}$ *is an oriented covering of* B *with data* (π, w) *then there exists an element* $U_p \in \dot{H}^k_\pi(T\pi(p);\, {}^w\mathbf{Z})$*, the* Thom class *of* p*, such that the cap product*

$$U_p \cap - \colon {}^w\dot{C}(T\pi(p)) \to S^kC(\tilde{B})$$

is a chain equivalence of finite-dimensional $\mathbf{Z}[\pi]$*-module chain complexes, the* Thom equivalence. *Conversely, if* F *is simply-connected and there exists an element* $U_p \in \dot{H}^k_{\pi_1(B)}(T\pi_1(B)(p);\, {}^w\mathbf{Z})$ $(k \geqslant 3)$ *for the Thom* $\pi_1(B)$*-space with respect to the universal cover* $\tilde{B}$ *of* B*, for some group morphism* $w\colon \pi_1(B) \to \mathbf{Z}_2$*, such that*

$$U_p \cap - \colon {}^w\dot{C}(T\pi_1(B)) \to S^kC(\tilde{B})$$

is a $\mathbf{Z}[\pi_1(B)]$*-module chain equivalence then* F *is a homotopy* S^{k-1} *and* $w = w_1(p)\colon \pi_1(B) \to \mathbf{Z}_2$ *is the first Stiefel–Whitney class of* p.

Proof. The proof is by the spectral sequence argument of Browder [**3**, Lemma I.4.3] applied to the pullback $F \longrightarrow \tilde{E} \xrightarrow{\tilde{p}} \tilde{B}$ of p to the universal cover $\tilde{B}$ of B.

We can now state the analogue of Proposition 4.4 of Spivak [**25**] appropriate to geometric Poincaré complexes in the sense of Wall [**29**] (cf. Browder [**4**]).

PROPOSITION 3.11. *Let* $X \subset S^N$ *be a finite subcomplex with a closed regular neighbourhood* E*, and let* F *be the homotopy-theoretic fibre of the inclusion* $p\colon \partial E \to E$*. Then* X *is an* n*-dimensional geometric Poincaré complex if and only if* F *is a homotopy* S^{N-n-1} $(N \geqslant n+3)$.

Proof. The inclusion $X \hookrightarrow E$ is a (simple) homotopy equivalence, so we can identify $\pi_1(X) = \pi_1(E) = \pi$, $\tilde{X} = \tilde{E}$, $T\pi(p) = \tilde{E}/\widetilde{\partial E}$ with $\tilde{X}, \tilde{E}$ the universal covers. Proposition 3.9 gives an $S\pi$-duality map

$$\alpha: S^N \to \tilde{X}_+ \wedge_\pi T\pi(p)$$

such that there is defined a commutative diagram

$$\begin{array}{ccc} H_n^\pi(\tilde{X}; {}^w\mathbf{Z}) \otimes_{\mathbf{Z}} {}^wH^r(\tilde{X}) & \xrightarrow{\quad \cap \quad} & H_{n-r}(\tilde{X}) \\ \Big\downarrow {\scriptstyle (\alpha_*[S^N]\backslash -)^{-1} \otimes (\alpha_*[S^N]\backslash -)} & & \Big\downarrow {\scriptstyle \mathrm{id.}} \\ \dot{H}_\pi^{N-n}(T\pi(p); {}^w\mathbf{Z}) \otimes_{\mathbf{Z}} {}^w\dot{H}_{N-r}(T\pi(p)) & \xrightarrow{\quad \cap \quad} & H_{n-r}(\tilde{X}) \end{array}$$

for any group morphism $w: \pi \to \mathbf{Z}_2$. Comparison of the definition of a geometric Poincaré complex (as in § 2) with the criterion of Proposition 3.10 gives the required correspondence.

4. Normal maps

Given a degree 1 map of n-dimensional geometric Poincaré complexes $f: M \to X$ and a covering map of the Spivak stable normal fibrations $b: \nu_M \to \nu_X$ we shall apply the equivariant S-duality of § 3 to obtain a geometric Umkehr map $F: \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$, and hence by the quadratic kernel construction of § 2 an n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi]$,

$$\sigma_*(f,b) = \sigma_*(f,F).$$

In § 7 we shall identify the quadratic Poincaré cobordism class

$$\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$$

with the Wall surgery obstruction.

An *n-dimensional normal space* (X, ν_X, ρ_X) is an n-dimensional finitely-dominated CW complex X together with a $(k-1)$-spherical fibration $\nu_X: X \to BG(k)$ and an element $\rho_X \in \pi_{n+k}(T(\nu_X))$. (This concept is due to Quinn [**21**].) Given a covering $\tilde{X}$ of X with group of covering translations π define the *fundamental map* of (X, ν_X, ρ_X) to be the composite $\{1\}$-map

$$\alpha_X: S^{n+k} \xrightarrow{\rho_X} T(\nu_X) = X/E = (\tilde{X}/\tilde{E})/\pi \xrightarrow{\Delta} \tilde{X}_+ \wedge_\pi \tilde{X}/\tilde{E} = \tilde{X}_+ \wedge_\pi T\pi(\nu_X),$$

with $\tilde{E}$ the induced covering of the total space E of ν_X, and Δ the diagonal map. If $\tilde{X}$ is oriented with data (π, w) with respect to ν_X define the *fundamental class* to be the twisted homology class

$$[X] = U_{\nu_X} \cap h(\rho_X) \in H_n^\pi(\tilde{X}; {}^w\mathbf{Z}),$$

with $U_{\nu_X} \in \dot{H}^k_\pi(T\pi(\nu_X); {}^w\mathbf{Z})$ the Thom class of ν_X and

$$h: \pi_{n+k}(T(\nu_X)) \to \dot{H}_{n+k}(T(\nu_X))$$

the Hurewicz map. The fundamental map is related to the fundamental class by a $\mathbf{Z}[\pi]$-module chain homotopy commutative diagram

$$\begin{array}{ccc} {}^wC(\tilde{X})^{n-*} & \xrightarrow{[X]\cap -} & C(\tilde{X}) \\ {\scriptstyle U_{\nu_X}\cup -}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id.}} \\ \dot{C}(T\pi(\nu_X))^{n+k-*} & \xrightarrow{\alpha_*[S^{n+k}]\backslash -} & C(\tilde{X}) \end{array}$$

in which the cup product with U_{ν_X} is a chain equivalence (a variant of the Thom equivalence of Proposition 3.10).

A *normal map* of n-dimensional normal spaces

$$(f, b): (M, \nu_M, \rho_M) \to (X, \nu_X, \rho_X)$$

consists of a map $f: M \to X$ of the underlying spaces together with a stable fibre homotopy class of stable fibre maps $b: \nu_M \to \nu_X$ over f such that

$$T(b)\rho_M = \rho_X \in \pi_{n+k}(T(\nu_X))$$

for sufficiently large k.

An *equivalence* of normal structures (ν_X, ρ_X), (ν'_X, ρ'_X) on a space X is a normal map of the type

$$(1, b): (X, \nu_X, \rho_X) \to (X, \nu'_X, \rho'_X).$$

PROPOSITION 4.1. *An n-dimensional geometric Poincaré complex X admits a normal structure (ν_X, ρ_X) with $w_1(\nu_X) = w(X)$ and the same fundamental class $[X] \in H^\pi_n(\tilde{X}; {}^w\mathbf{Z})$ such that the fundamental map*

$$\alpha_X: S^{n+k} \to \tilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

defines an $S\pi$-duality for every covering $\tilde{X}$ of X, with π the group of covering translations. Any two such normal structures (ν_X, ρ_X), (ν'_X, ρ'_X) are related by a unique equivalence $(1, b): (X, \nu_X, \rho_X) \to (X, \nu'_X, \rho'_X)$. Conversely, if X is a finitely-dominated CW complex with a normal structure (ν_X, ρ_X) such that the fundamental map

$$\alpha_X: S^{n+k} \to \tilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

with respect to the universal cover $\tilde{X}$ ($\pi = \pi_1(X)$) defines an $S\pi$-duality map then X is an n-dimensional geometric Poincaré complex with $w(X) = w_1(\nu_X)$ and the same fundamental class $[X] \in H^\pi_n(\tilde{X}; {}^w\mathbf{Z})$.

Proof. If X is finite there exists an embedding $X \subset S^N$ for $N \geqslant 2$(geometric dimension of X)$+1$ by general position, with closed regular neighbourhood E say. If $\tilde{X}$ is any covering of X with group of covering translations π then Proposition 3.9 gives an $S\pi$-duality map

$$\alpha_X \colon S^N \xrightarrow{\rho_X = \text{collapse}} E/\partial E \xrightarrow{\quad\Delta\quad} \tilde{X}_+ \wedge_\pi \tilde{E}/\widetilde{\partial E}.$$

Let F be the homotopy-theoretic fibre of the inclusion $\partial E \subset E$, so that there is defined a fibration

$$F \longrightarrow \partial E \xrightarrow{\nu_X} X$$

with $T\pi(\nu_X) = \tilde{E}/\widetilde{\partial E}$. If X is an n-dimensional geometric Poincaré complex then $F \simeq S^{N-n-1}$, by Proposition 3.11, and (ν_X, ρ_X) defines a normal structure with $S\pi$-duality. If X is not finite use the trick of Wall [**29**, § 3] of crossing with S^1 to reduce to the finite case. The uniqueness clause is as in [**29**, Corollary 3.6] (see also Theorem I.4.19 of Browder [**3**]). Conversely, given a normal structure with $S\pi$-duality for the universal cover we can obtain Poincaré duality by combining the $S\pi$-duality criterion of Proposition 3.8 with the Thom isomorphism of Proposition 3.10.

Thus an n-dimensional geometric Poincaré complex X carries a canonical equivalence class of normal structures (ν_X, ρ_X) with $S\pi$-duality. We shall call this the *Spivak normal class*, calling any such ν_X a *Spivak normal fibration* of X. *A normalization* of X is a choice of normal structure (ν_X, ρ_X) in the Spivak normal class.

We are now in a position to apply $S\pi$-duality to obtain geometric Umkehr maps of the type considered in § 2 for degree 1 maps of geometric Poincaré complexes which preserve Spivak normal structures.

PROPOSITION 4.2. *Given a degree* 1 *normal map of normalized n-dimensional geometric Poincaré complexes*

$$(f, b) \colon (M, \nu_M, \rho_M) \to (X, \nu_X, \rho_X)$$

and a cover $\tilde{X}$ of X with group of covering translations π there is induced a π-map of Thom π-spaces $T\pi(b) \colon T\pi(\nu_M) \to T\pi(\nu_X)$ such that the $S\pi$-dual of $T\pi(b)$ with respect to the fundamental $S\pi$-duality maps

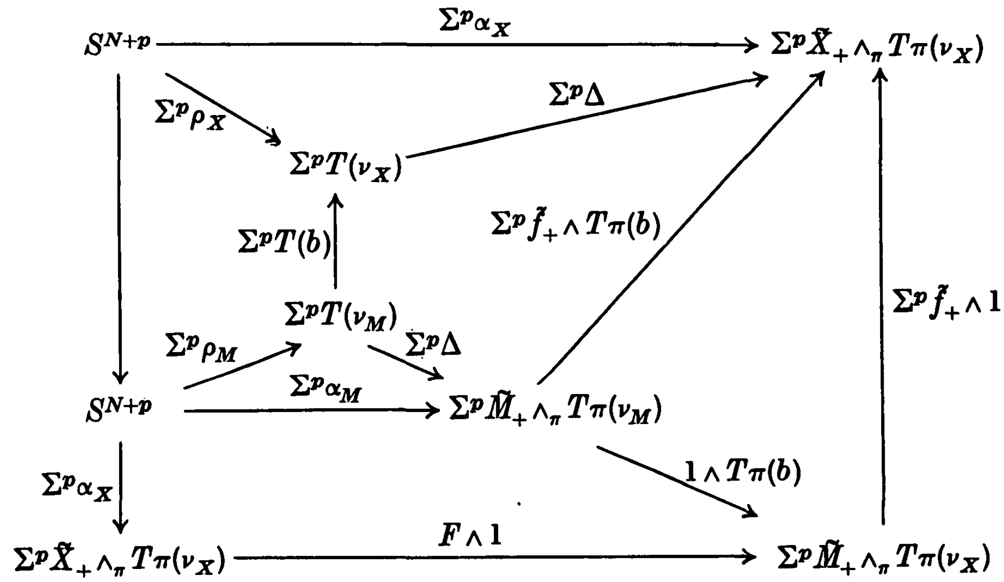
$$\alpha_M \colon S^N \to \tilde{M}_+ \wedge_\pi T\pi(\nu_M), \quad \alpha_X \colon S^N \to \tilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

is an $S\pi$-homotopy class $F \in \{\tilde{X}_+, \tilde{M}_+\}_\pi$ of geometric Umkehr maps $F \colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$ such that $(\Sigma^p \tilde{f}_+)F \simeq 1 \colon \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{X}_+$ up to stable π-homotopy.

Proof. The $S\pi$-duality is defined by the composite

$$\{T\pi(\nu_M), T\pi(\nu_X)\}_\pi \xrightarrow{(\alpha_M\backslash -)^{-1}} \{S^N, T\pi(\nu_X)\wedge_\pi \tilde{M}_+\} \xrightarrow{(\alpha_X\backslash -)} \{\tilde{X}_+, \tilde{M}_+\}_\pi .$$

Working round the stable homotopy commutative diagram



we have that

$$((\Sigma^p\tilde{f}_+)F\wedge 1)(\Sigma^p\alpha_X) \simeq (\Sigma^p\alpha_X) : S^{N+p} \to \Sigma^p\tilde{X}_+\wedge_\pi T\pi(\nu_X).$$

Since $\Sigma^p\alpha_X$ is also an $S\pi$-duality map it follows that

$$(\Sigma^p\tilde{f}_+)F \simeq 1 : \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{X}_+$$

for p large enough. The diagram also shows that F induces the Umkehr $f^! : C(\tilde{X}) \to C(\tilde{M})$ on the chain level, identifying the Poincaré duality chain equivalences with the appropriate Thom equivalences.

Define the *quadratic kernel* of a normal map of normalized n-dimensional geometric Poincaré complexes

$$(f,b) : (M, \nu_M, \rho_M) \to (X, \nu_X, \rho_X)$$

with respect to an oriented cover $\tilde{X}$ of X with data (π, w) to be the n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi]$ with the w-twisted involution

$$\sigma_*(f,b) = \sigma_*(f,F) = (C(f^!), e_\%\psi_F[X] \in Q_n(C(f^!))),$$

using the quadratic kernel construction of Proposition 2.3 with any of the geometric Umkehr maps $F : \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{M}_+$ such that $(\Sigma^p\tilde{f}_+)F \simeq 1$ provided by Proposition 4.2. All such quadratic kernels are induced from that associated to the universal covering $\tilde{X}$ of X with data $(\pi_1(X), w(X))$. We have the sum formula:

PROPOSITION 4.3. *The quadratic kernel of the composite*

$$(gf, cb)\colon (X, \nu_X, \rho_X) \xrightarrow{(f,b)} (Y, \nu_Y, \rho_Y) \xrightarrow{(g,c)} (Z, \nu_Z, \rho_Z)$$

of normal maps of normalized n-dimensional geometric Poincaré complexes is

$$\sigma_*(gf, cb) = \sigma_*(f, b) \oplus \sigma_*(g, c)$$

up to homotopy equivalence.

Proof. This is immediate from Proposition 2.5, since $(\Sigma^p \tilde{f}_+)F \simeq 1$.

The difference $\psi' - \psi \in \ker((1+T)\colon Q_n(C(f^!)) \to Q^n(C(f^!)))$ of the hyperhomology classes appearing in the quadratic kernels $\sigma_*(f,b) = (C(f^!), \psi)$, $\sigma_*(f,b') = (C(f^!), \psi')$ of normal maps

$$(f,b)\colon (M, \nu_M, \rho_M) \to (X, \nu_X, \rho_X), \quad (f,b')\colon (M, \nu_M, \rho_M) \to (X, \nu_X, \rho'_X)$$

such that $b' = bc\colon \nu_M \to \nu_X$ for some automorphism $c\colon \nu_M \to \nu_M$ will be expressed in terms of c in Proposition 9.10 below.

A *normal bundle map*

$$(f,b)\colon M \to X$$

is a degree 1 map $f\colon M \to X$ from an n-dimensional smooth manifold M to an n-dimensional geometric Poincaré complex X together with a bundle map $b\colon \nu_M \to \nu_X$ from the normal bundle $\nu_M\colon M \to BO(k)$ for some embedding $M \subset S^{n+k}$ $(k \gg n)$ to some bundle $\nu_X\colon X \to BO(k)$. This is the definition of normal map due to Browder [**3**] (with M compact and X finite). The *quadratic kernel* of such a normal bundle map with respect to an oriented cover $\tilde{X}$ of X is the quadratic kernel

$$\sigma_*(f,b) = \sigma_*(f, Jb) = (C(f^!), e_\%(\psi_F[X]) \in Q_n(C(f^!)))$$

of the normal map of normalized n-dimensional geometric Poincaré complexes

$$(f, Jb)\colon (M, J\nu_M, \rho_M) \to (X, J\nu_X, \rho_X),$$

obtained by passing to the associated spherical fibrations $J\nu_M\colon M \to BG(k)$, $J\nu_X\colon X \to BG(k)$, with

$$\rho_M\colon S^{n+k} \xrightarrow{\text{collapse}} T(\nu_M), \quad \rho_X = T(b)\rho_M\colon S^{n+k} \longrightarrow T(\nu_X).$$

The surgery obstruction of a $2q$-dimensional normal bundle map $(f,b)\colon M \to X$ such that $\pi_1(X) = \{1\}$ is $\frac{1}{8}$(signature) {the Arf invariant} of the non-singular quadratic form over $\mathbf{Z}$ $\{\mathbf{Z}_2\}$ defined on $K^q(M)$ $\{K^q(M; \mathbf{Z}_2)\}$ by $\sigma_*(f,b)$ $\{\mathbf{Z}_2 \otimes \sigma_*(f,b)\}$ if $q \equiv 0$ $\{q \equiv 1\}$ (mod 2) (cf. Propositions 2.4(i), I.7.1, and I.7.2).

A normal map in the sense of Wall [**30**]

$$(f, B): M \to X$$

is a degree 1 map $f: M \to X$ from an n-dimensional smooth manifold M to an n-dimensional geometric Poincaré complex X together with a bundle isomorphism $B: \varepsilon_M \to \tau_M \oplus f^*\nu_X$, with $\tau_M: M \to BO(n)$ the tangent bundle of M, $\nu_X: X \to BO(k)$ some bundle over X, and $\varepsilon_M = 0: M \to BO(n+k)$ the trivial $(n+k)$-plane bundle (with M compact and X finite). Choosing an embedding $M \subset S^N$ ($N \gg n$) with normal bundle $\nu_M: M \to BO(N-n)$ we have a stable inverse ν_M for τ_M and a bundle map over f

$$\begin{aligned} b: \nu_M \oplus \varepsilon_M &\xrightarrow{1 \oplus B} \nu_M \oplus (\tau_M \oplus f^*\nu_X) \\ &= (\nu_M \oplus \tau_M) \oplus f^*\nu_X \longrightarrow f^*\varepsilon_X \oplus f^*\nu_X \longrightarrow \varepsilon_X \oplus \nu_X \end{aligned}$$

with $\varepsilon_X = 0: X \to BO(N)$. The quadratic kernel $\sigma_*(f, b)$ of the normal bundle map $(f, b): M \to X$ does not depend on the choice of ν_M: for if ν_M, ν'_M are two such then there exists a bundle isomorphism $c: \nu'_M \to \nu_M$ such that $b' = bc: \nu'_M \to \nu_X$ and $T(c)(\rho'_M) = \rho_M \in \pi_N(T(\nu_M))$ (by the uniqueness of embeddings $M \subset S^N$ for $N \gg n$), so that applying the sum formula of Proposition 4.3 to the composite normal map

$$(f, b'): (M, \nu'_M, \rho'_M) \xrightarrow{(1,c)} (M, \nu_M, \rho_M) \xrightarrow{(f,b)} (X, \nu_X, \rho_X),$$

we have that up to homotopy equivalence

$$\sigma_*(f, b') = \sigma_*(f, b) \oplus \sigma_*(1, c) = \sigma_*(f, b).$$

Conversely, a normal bundle map $(f, b): M \to X$ determines a normal map in the sense of Wall [**30**] $(f, B): M \to X$ with

$$B: \varepsilon_M = \tau_M \oplus \nu_M \xrightarrow{1 \oplus b} \tau_M \oplus f^*\nu_X.$$

From now on we shall not distinguish between the two formulations of normal bundle maps.

5. Intersections and self-intersections

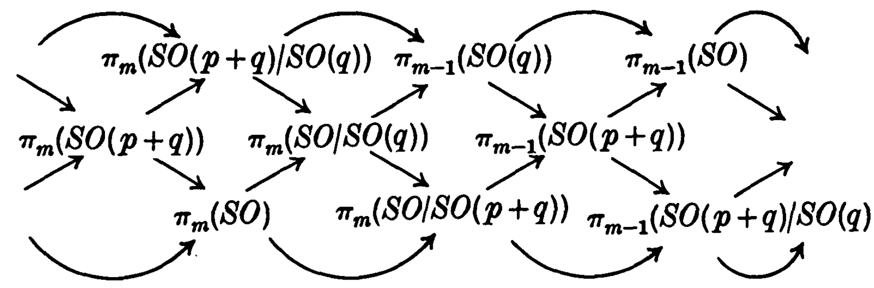
We have used the quadratic construction ψ of §1 to define in §4 the quadratic kernel $\sigma_*(f, b) = \sigma_*(f, F)$ of a normal bundle map $(f, b): M \to X$, using the equivariant S-duality of §3 to obtain the geometric Umkehr map $F: \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+$. We shall now describe the self-intersections of an immersion $S^r \to M^n$ in terms of the quadratic construction ψ, allowing us to identify the quadratic kernel $\sigma_*(f, b)$ for a highly-connected f with the geometrically defined surgery obstruction kernel of Wall [**30**, §§5, 6]. (See the note added in proof (p. 279) for the generalization to arbitrary immersions of manifolds.)

PROPOSITION 5.1. *For any $p, q \geqslant 0$ there is defined in a natural way a morphism of commutative braids of exact sequences of abelian groups*

$$j: \pi \to Q$$

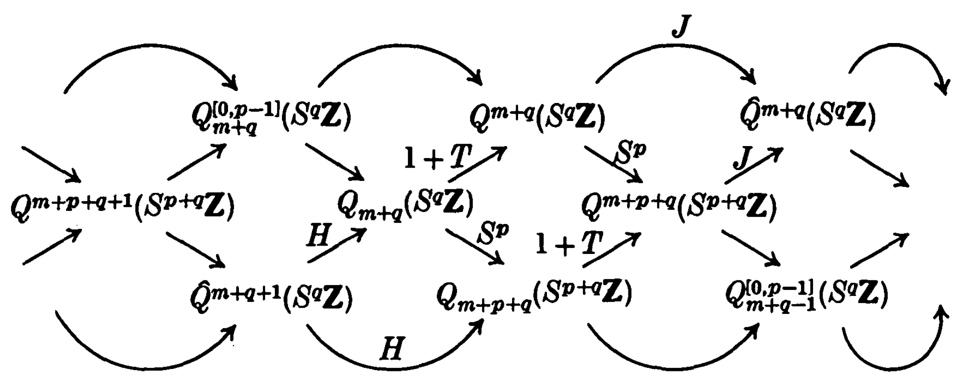
from

π:



to

Q:



Proof. Q is a particular case of the braid of Proposition I.1.3.

Define as follows abelian group morphisms

$$j: \pi_m(BSO(q)) = \pi_{m-1}(SO(q)) \to Q^{m+q}(S^q\mathbf{Z}) = H^{q-m}(\mathbf{Z}_2; \mathbf{Z}, (-)^q),$$

$$j: \pi_{m+1}(BSO(p+q), BSO(q)) = \pi_m(SO(p+q)/SO(q))$$
$$\to Q^{[0,p-1]}_{m+q}(S^q\mathbf{Z}) \quad (= H_{m-q}(\mathbf{Z}_2; \mathbf{Z}, (-)^q) \text{ if } m-q < p-1 \neq 0).$$

Given an oriented q-plane bundle $\alpha: S^m \to BSO(q)$ over S^m apply Lemma 1 of Milnor [15] to identify the Thom space $T(\alpha)$ with the mapping cone of $J(\alpha) \in \pi_{m+q-1}(S^q)$

$$T(\alpha) = S^q \cup_{J(\alpha)} e^{m+q}.$$

Applying the symmetric construction $\dot{\varphi}$ and the symmetric Wu class v_m

$$\mathbf{Z} = \dot{H}_{m+q}(T(\alpha)) \xrightarrow{\dot{\varphi}_{T(\alpha)}} Q^{m+q}(\dot{C}(T(\alpha))) \xrightarrow{v_m} \mathrm{Hom}_{\mathbf{Z}}(\dot{H}^q(T(\alpha)), Q^{m+q}(S^q\mathbf{Z}))$$

set

$$j(\alpha) = v_m(\dot{\varphi}_{T(\alpha)}(1))(1) \in Q^{m+q}(S^q\mathbf{Z}) \quad (\dot{H}^q(T(\alpha)) = \mathbf{Z}).$$

(By Propositions 1.2(i) (if $m = q$) and 1.3 (if $m \neq q$) $j(\alpha)$ can be expressed in terms of the cup product structure of $\dot{H}^*(T(\alpha))$ and the action of the Steenrod algebra on $\dot{H}^*(T(\alpha); \mathbf{Z}_2)$. In § 9 below we shall use this to identify $j(\alpha)$ with the Hopf invariant of $J(\alpha)$.) Furthermore, given a null-homotopy $\beta: D^{m+1} \to BSO(p+q)$ of $\alpha \oplus \varepsilon^p: S^m \to BSO(p+q)$ there is defined an isomorphism $\beta: \varepsilon^{p+q} \to \alpha \oplus \varepsilon^p$ of oriented $(p+q)$-plane bundles over S^m, inducing a homotopy equivalence of Thom spaces

$$T(\beta): T(\varepsilon^{p+q}) = S^{p+q} \vee S^{m+p+q} \to T(\alpha \oplus \varepsilon^p) = \Sigma^p T(\alpha).$$

The composite

$$I(\beta): \Sigma^p(S^{m+q}) = S^{m+p+q} \xrightarrow{\text{inclusion}} S^{p+q} \vee S^{m+p+q} \xrightarrow{T(\beta)} \Sigma^p T(\alpha)$$

represents the generator $1 \in \dot{H}_{m+p+q}(\Sigma^p T(\alpha)) = \mathbf{Z}$. Applying the quadratic construction ψ and the quadratic Wu class v^m

$$\mathbf{Z} = \dot{H}_{m+q}(S^{m+q}) \xrightarrow{\psi_{I(\beta)}} Q_{m+q}^{[0,p-1]}(\dot{C}(T(\alpha))) \xrightarrow{v^m} \mathrm{Hom}_{\mathbf{Z}}(\dot{H}^q(T(\alpha)), Q_{m+q}^{[0,p-1]}(S^q\mathbf{Z}))$$

set

$$j(\alpha, \beta) = v^m(\psi_{I(\beta)}(1))(1) \in Q_{m+q}^{[0,p-1]}(S^q\mathbf{Z}).$$

Applying the symmetric Wu class operation v_m to the relation

$$\dot{\varphi}_{T(\alpha)}(1) - ((\Sigma^p)^{-1}I(\beta)\Sigma^p)^{\%}\dot{\varphi}_{S^{m+q}}[S^{m+q}] = (1+T)\psi_{I(\beta)}(1) \in Q^{m+q}(\dot{C}(T(\alpha)))$$

given by Proposition 1.5(ii) we have that

$$j(\alpha) = (1+T)j(\alpha, \beta) \in Q^{m+q}(S^q\mathbf{Z}).$$

The remaining morphisms $j: \pi \to Q$ are obtained from these by passing to the suspension limits in both the geometry and the algebra (cf. Proposition 1.4).

(It is possible to factorize the map of braids $j: \pi \to Q$ as $j: \pi \to \Pi \to Q$, with Π defined exactly as π but using SG instead of SO. In particular, for $m = q = 2k$, j factorizes as

$$j: \pi_{2k}(BSO(2k)) \xrightarrow{J} \pi_{4k-1}(S^{2k}) \xrightarrow{\text{Hopf invariant}} Q^{4k}(S^{2k}\mathbf{Z}) = \mathbf{Z},$$

and is just the function assigning the Euler number $\chi(\alpha) \in \mathbf{Z}$ to the oriented $2k$-plane bundle over S^{2k} classified by $\alpha: S^{2k} \to BSO(2k)$. For $m \leqslant q$, $j: \pi_m(SO/SO(q)) \to Q_{m+q}(S^q\mathbf{Z})$ is an isomorphism.)

Let M^n be an n-manifold, which for the sake of simplicity we take to be compact, smooth, and closed. Let $(\pi, w) = (\pi_1(M), w(M))$, and give the group ring $\mathbf{Z}[\pi]$ the w-twisted involution. Let $S_r(M)$ $(r \geqslant 2)$ be the $\mathbf{Z}[\pi]$-module of regular homotopy classes of oriented immersions $g: S^r \to M$ with a preferred lift $\tilde{g}: \tilde{S}^r = \pi \times S^r \to \tilde{M}$ to the universal cover $\tilde{M}$ of M, where addition is by connected sum and π acts by changing lifts. Given such an immersion g define a $\mathbf{Z}[\pi]$-module chain map

$g^!\colon C(\tilde{M}) \to \dot{C}(T\pi(\nu_g))$ by

$$g^!\colon C(\tilde{M}) \xrightarrow[\sim]{([M]\cap -)^{-1}} C(\tilde{M})^{n-*} \xrightarrow{\tilde{g}^*} C(\tilde{S}^r)^{n-*} \xrightarrow[\sim]{([S^r]\cap -)} S^{n-r}C(\tilde{S}^r) \xrightarrow[\sim]{(U_{\nu_g}\cap -)} \dot{C}(T\pi(\nu_g)),$$

where $T\pi(\nu_g) = \bigvee_\pi T(\nu_g)$ is the Thom π-space of the normal bundle $\nu_g\colon S^r \to BSO(n-r)$, and $U_{\nu_g} \in \dot{H}^{n-r}(T(\nu_g))$ is the Thom class of ν_g.

The *symmetric self-intersection* of an immersion $g\colon S^r \to M^n$ is the $\mathbf{Z}_2$-cohomology class

$$\lambda(g) = v_r(\varphi_{\tilde{M}}[M])(x) \in H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^{n-r})$$

obtained by evaluating the composite

$$H_n^\pi(\tilde{M}; {}^w\mathbf{Z}) \xrightarrow{\varphi_{\tilde{M}}} Q^n(C(\tilde{M})) \xrightarrow{v_r} \mathrm{Hom}_{\mathbf{Z}[\pi]}({}^wH^{n-r}(\tilde{M}), H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^{n-r})),$$

with $x = g^{!*}(U_{\nu_g}) \in {}^wH^{n-r}(\tilde{M})$ the Poincaré dual of $\tilde{g}_*[S^r] \in H_r(\tilde{M})$. For a fixed M the class $\lambda(g)$ depends only on $\tilde{g}_*[S^r] \in H_r(\tilde{M})$. In the case where $n \mp 2r$, $\lambda(g)$ can be identified with the evaluation $\lambda(g,g)$ of the geometric intersection pairing

$$\lambda\colon S_r(M) \times S_r(M) \to \mathbf{Z}[\pi_1(M)].$$

By Propositions 1.3 and 2.4(i) the mod 2 reduction of $\lambda(g)$ (for $n \geqslant 2r$) can be expressed as

$$\lambda(g) = \langle Sq^r(x), [M]\rangle = \langle v_r(M) \cup x, [M]\rangle \in \mathbf{Z}_2 \qquad (x = g^{!*}(U_{\nu_g}) \in H^{n-r}(M; \mathbf{Z}_2)).$$

Given an immersion $g\colon S^r \to M^n$ and a non-negative integer

$$p > 2r - n + 1,$$

it is possible to deform the immersion $g \times 1\colon S^r \to M^n \times D^p$ by a regular homotopy to an embedding $g'\colon S^r \hookrightarrow \mathrm{interior}(M^n \times D^p)$ with normal bundle

$$\nu_{g'} = \nu_g \oplus \varepsilon^p\colon S^r \to BSO(n-r+p).$$

Let E be a closed tubular neighbourhood of $g'(S^r)$ in $M^n \times D^p$, with induced cover $\tilde{E} = \pi \times E \subset \tilde{M} \times D^p$. The π-map

$$G\colon \Sigma^p\tilde{M}_+ = \tilde{M} \times D^p/\tilde{M} \times S^{p-1} \xrightarrow{\text{collapse}} \tilde{M} \times D^p/\overline{\tilde{M} \times D^p - \tilde{E}} = \tilde{E}/\partial\tilde{E} = T\pi(\nu_{g'}) = \Sigma^p T\pi(\nu_g)$$

induces $g^!\colon C(\tilde{M}) \to \dot{C}(T\pi(\nu_g))$ on the chain level.

The *quadratic self-intersection* of an immersion $g\colon S^r \to M^n$ is the $\mathbf{Z}_2$-homology class

$$\mu(g) = -v^r(\psi_G[M])(U_{\nu_g}) \in H_{2r-n}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)], (-)^{n-r})$$

obtained by evaluating the composite

$$H_n^{\pi}(\tilde{M};\, {}^w\mathbf{Z}) \xrightarrow{\psi_G} Q_n(\dot{C}(T\pi(\nu_g))) \xrightarrow{v^r} \mathrm{Hom}_{\mathbf{Z}[\pi]}({}^w\dot{H}^{n-r}(T\pi(\nu_g)), H_{2r-n}(\mathbf{Z}_2;\, \mathbf{Z}[\pi], (-)^{n-r})).$$

In the case where $n = 2r$, $\mu(g)$ will be identified with the geometric self-intersection of g (in Proposition 5.2 below). By Proposition 1.6, the mod 2 reduction of $\mu(g)$ can be expressed as

$$\mu(g) = \langle Sq_h^{r+1}(\Sigma^p\iota), \Sigma^p[M]\rangle \in \mathbf{Z}_2$$

$$(h = (\Sigma^p U_{\nu_g})G - \Sigma^p(x) \in [\Sigma^p M_+, \Sigma^p K(\mathbf{Z}_2, n-r)],$$
$$\iota = \text{generator} \in \dot{H}^{n-r}(K(\mathbf{Z}_2, n-r);\, \mathbf{Z}_2) = \mathbf{Z}_2).$$

PROPOSITION 5.2. *The symmetric and quadratic self-intersections define functions*

$$\lambda\colon S_r(M^n) \to H^{n-2r}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)], (-)^{n-r});\ (g\colon S^r \to M^n) \mapsto \lambda(g),$$
$$\mu\colon S_r(M^n) \to H_{2r-n}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)], (-)^{n-r});\ (g\colon S^r \to M^n) \mapsto \mu(g),$$

such that

(i) $\lambda(ag) = a\lambda(g)\bar{a},\quad \mu(ag) = a\mu(g)\bar{a}\quad (a \in \mathbf{Z}[\pi_1(M)],\ g \in S_r(M^n)),$

(ii) $\lambda(g) = (j(\nu_g), 0) + (1+T)\mu(g) \in H^{n-2r}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)], (-)^{n-r})$
$= H^{n-2r}(\mathbf{Z}_2;\, \mathbf{Z}, (-)^{n-r}) \oplus H^{n-2r}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)]/\mathbf{Z}, (-)^{n-r}),$

(iii) $\mu(g_1+g_2) - \mu(g_1) - \mu(g_2) = \begin{cases} [\lambda(g_1, g_2)] & \text{if } n = 2r \\ 0 & \text{otherwise} \end{cases}\quad (g_1, g_2 \in S_r(M^n)),$

(iv) *if the class $g \in S_r(M^n)$ contains an embedding then $\mu(g) = 0$, and if it contains a framed embedding then also $\lambda(g) = 0$,*

(v) *if $n = 2r \geqslant 6$ and $\mu(g) = 0$ then the class $g \in S_r(M^{2r})$ contains an embedding.*

Proof. (i) By construction.

(ii) Apply the symmetric Wu class v_r to the relation given by Proposition 1.5(i),

$$g^{!\%}\varphi_{\tilde{M}}[M] = \dot{\varphi}_{T\pi(\nu_g)} g^!_*[M] - (1+T)\psi_G[M] \in Q^n(\dot{C}(T\pi(\nu_g))).$$

(iii) The quadratic self-intersection $\mu(g_1+g_2)$ of the connected sum g_1+g_2 of immersions $g_1, g_2\colon S^r \to M^n$ is given by

$$\mu(g_1+g_2) = -v^r(\psi_G[M])(U_{\nu_{g_1}} \oplus U_{\nu_{g_2}}) \in H_{2r-n}(\mathbf{Z}_2;\, \mathbf{Z}[\pi_1(M)], (-)^{n-r}),$$

with

$$G = G_1 \vee G_2 \colon \Sigma^p \tilde{M}_+ \to \Sigma^p(T\pi(\nu_{g_1}) \vee T\pi(\nu_{g_2})) = \Sigma^p T\pi(\nu_{g_1}) \vee \Sigma^p T\pi(\nu_{g_2}).$$

There is a natural identification of $\mathbf{Z}[\pi]$-module chain complexes

$$\dot{C}(T\pi(\nu_{g_1}) \vee T\pi(\nu_{g_2})) = \dot{C}(T\pi(\nu_{g_1})) \oplus \dot{C}(T\pi(\nu_{g_2})),$$

and Proposition I.1.4(i) allows us to express ψ_G as

$$\psi_G = \begin{pmatrix} \psi_{G_1} \\ \psi_{G_2} \\ -(g_1^! \otimes g_2^!)\Delta_0 \end{pmatrix} : H_n^\pi(\tilde{M}; {}^w\mathbf{Z}) \to Q_n(\dot{C}(T\pi(\nu_{g_1})) \oplus \dot{C}(T\pi(\nu_{g_2})))$$

$$= Q_n(\dot{C}(T\pi(\nu_{g_1}))) \oplus Q_n(\dot{C}(T\pi(\nu_{g_2}))) \oplus H_n(\dot{C}(T\pi(\nu_{g_1}))^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(T\pi(\nu_{g_2}))),$$

with $(g_1^! \otimes g_2^!)\Delta_0$ the composite

$$(g_1^! \otimes g_2^!)\Delta_0 \colon H_n^\pi(\tilde{M}; {}^w\mathbf{Z}) \xrightarrow{\Delta_0} H_n(C(\tilde{M})^t \otimes_{\mathbf{Z}[\pi]} C(\tilde{M}))$$

$$\xrightarrow{g_1^! \otimes g_2^!} H_n(\dot{C}(T\pi(\nu_{g_1}))^t \otimes_{\mathbf{Z}[\pi]} \dot{C}(T\pi(\nu_{g_2}))).$$

Now apply the rth quadratic Wu class v^r to the identity

$$\psi_G[M] = (\psi_{G_1}[M], \psi_{G_2}[M], -(g_1^! \otimes g_2^!)\Delta_0[M]) \in Q_n(\dot{C}(T\pi(\nu_{g_1}) \vee T\pi(\nu_{g_2}))).$$

(iv) By definition ψ_G is a composite

$$\psi_G \colon H_n^\pi(\tilde{M}; {}^w\mathbf{Z}) \to Q_n^{[0,p-1]}(\dot{C}(T\pi(\nu_g))) \to Q_n(\dot{C}(T\pi(\nu_g))),$$

and the middle group is 0 if $p = 0$.

(v) Let $\hat{\mu}(g) \in H_0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^r)$ be the geometric self-intersection of an immersion $g \colon S^r \to M^{2r}$ $(r \geqslant 2)$, as defined by Wall in [**30**, Theorem 5.2]. It was proved there that

$$\hat{\mu}(g_1+g_2) - \hat{\mu}(g_1) - \hat{\mu}(g_2) = [\lambda(g_1, g_2)] \in H_0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^r),$$

and that, for $r \geqslant 3$, $\hat{\mu}(g)$ is the sole obstruction to deforming g to an embedding.

We shall prove that $\mu(g) = \hat{\mu}(g)$ (for $r \geqslant 3$) by a generalization of the trick used by Browder in the proof of Theorem IV.4.1 of [**3**].† Lift $\hat{\mu}(g)$ to some element $a \in \mathbf{Z}[\pi_1(M)]$, and let $g' \colon S^r \to M'^{2r} = M^{2r} \# (S^r \times S^r)$ be an immersion representing the homology class

$$\tilde{g}'_*[S^r] = (0, -a, 1) \in H_r(\tilde{M}') = H_r(\tilde{M}) \oplus \mathbf{Z}[\pi_1(M)] \oplus \mathbf{Z}[\pi_1(M)].$$

The immersion $g \# 0 \colon S^r \to M'^{2r}$ represents the homology class

$$(\tilde{g} \# 0)_*[S^r] = (\tilde{g}_*[S^r], 0, 0) \in H_r(\tilde{M}') = H_r(\tilde{M}) \oplus \mathbf{Z}[\pi_1(M)] \oplus \mathbf{Z}[\pi_1(M)].$$

Define an immersion

$$g'' = (g \# 0) + g' \colon S^r \to M'^{2r},$$

† See note added in proof on p. 279.

and apply the sum formulae for μ and $\hat{\mu}$ to obtain

$$\mu(g'') = \mu(g \# 0) + \mu(g') = \mu(g) - [a] = \mu(g) - \hat{\mu}(g) \in H_0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^r),$$

$$\hat{\mu}(g'') = \hat{\mu}(g \# 0) + \hat{\mu}(g') = \hat{\mu}(g) - [a] = 0 \in H_0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^r).$$

Thus g'' can be deformed to an embedding, and $\mu(g'') = 0$ by (iv).

The relation of Proposition 5.2(ii) for $n = 2r$,

$$\lambda(g) = (j(\nu_g), 0) + (1+T)\mu(g) \in H^0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(M)], (-)^r)$$

is precisely the relation of Theorem 5.2(iii) of Wall [**30**], with

$$j(\nu_g) = \chi(\nu_g) \in H^0(\mathbf{Z}_2; \mathbf{Z}, (-)^r)$$

the Euler number of $\nu_g \in \pi_r(BSO(r))$.

PROPOSITION 5.3. *Let $f\colon M \to X$ $\{(f,b)\colon M \to X\}$ be a degree 1 $\{$normal bundle$\}$ map from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X. Let $g\colon S^r \to M$ be an immersion with an oriented normal bundle $\nu_g\colon S^r \to BSO(n-r)$ and a null-homotopy $h\colon D^{r+1} \to X$ of $fg\colon S^r \to X$ $\{$and let $\nu_h\colon D^{r+1} \to BSO$ be the stable trivialization of $\nu_g\colon S^r \to BSO(n-r)$ determined by $b\colon \nu_M \to \nu_X\}$. The rth symmetric $\{$quadratic$\}$ Wu class of the symmetric $\{$quadratic$\}$ kernel,*

$$\begin{cases} \sigma^*(f) = (C(f^!), \varphi = e^{\%}\varphi_{\tilde{M}}[M] \in Q^n(C(f^!))), \\ \sigma_*(f,b) = (C(f^!), \psi = e_{\%}\psi_F[X] \in Q_n(C(f^!))), \end{cases}$$

$$\begin{cases} v_r(\varphi)\colon H^{n-r}(C(f^!)) = K^{n-r}(M) \to H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^{n-r}), \\ v^r(\psi)\colon H^{n-r}(C(f^!)) = K^{n-r}(M) \to H_{2r-n}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^{n-r}), \end{cases}$$

sends the Poincaré dual $x \in K^{n-r}(M)$ of the Hurewicz image in

$$H_{r+1}(\tilde{f}) = K_r(M)$$

of $(h,g) \in \pi_{r+1}(f) = \pi_{r+1}(\tilde{f})$ to

$$\begin{cases} v_r(\varphi)(x) = \lambda(g) = (j(\nu_g), 0) + (1+T)\mu(g) \in H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^{n-r}), \\ v^r(\psi)(x) = (j(\nu_h, \nu_g), 0) + \mu(g) \in H_{2r-n}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^{n-r}). \end{cases}$$

Proof. The expression for $v_r(\varphi)(x)$ is immediate from Proposition 5.2(i), so only the normal bundle case need be considered. The commutative diagram of maps of spaces

$$\begin{array}{ccc} S^r & \xrightarrow{\;g\;} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ D^{r+1} & \xrightarrow{\;h\;} & X \end{array}$$

is covered by a commutative diagram of bundle maps

$$\begin{array}{ccc} g^*\nu_M & \longrightarrow & \nu_M \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ h^*\nu_X & \longrightarrow & \nu_X \end{array}$$

with $h^*\nu_X$ a trivial bundle and $c\colon g^*\nu_M \to h^*\nu_X$ giving rise to

$$\nu_h\colon D^{r+1} \to BSO.$$

There is induced a commutative diagram of Thom π-spaces and π-maps ($\pi = \pi_1(X)$)

$$\begin{array}{ccc} T\pi(g^*\nu_M) & \longrightarrow & T\pi(\nu_M) \\ {\scriptstyle T\pi(c)}\big\downarrow & & \big\downarrow{\scriptstyle T\pi(b)} \\ T\pi(h^*\nu_X) & \longrightarrow & T\pi(\nu_X) \end{array}$$

whose $S\pi$-dual is a π-homotopy commutative diagram of π-maps

$$\begin{array}{ccc} \Sigma^p T\pi(\nu_g) & \xleftarrow{\ G\ } & \Sigma^p \tilde{M}_+ \\ {\scriptstyle I(\nu_h)}\big\uparrow & & \big\uparrow{\scriptstyle F} \\ \Sigma^p(\bigvee_\pi S^n) & \xleftarrow{\ H\ } & \Sigma^p \tilde{X}_+ \end{array}$$

for $p \geqslant 0$ sufficiently large. Applying the sum formula for the quadratic construction of Proposition 1.5(iii) we have

$$\begin{aligned} g^!_{\%}\psi_F[X] + \psi_G[M] &= \psi_{GF}[X] = \psi_{I(\nu_h)H}[X] \\ &= \psi_{I(\nu_h)}(1) + I(\nu_h)_*\psi_H[X] \in Q_n(\dot{C}(T\pi(\nu_g))). \end{aligned}$$

The disc theorem for geometric Poincaré complexes (Wall [**29**, Theorem 2.4]) provides a homotopy equivalence

$$X \to Y \cup_k e^n$$

with Y a homologically $(n-1)$-dimensional complex and $k\colon S^{n-1} \to Y$ some map. Passing to the universal covers, adjoining basepoints, and collapsing $\tilde{Y}$ we obtain an unstable π-map

$$H\colon \tilde{X}_+ \to (\tilde{Y} \cup_{\tilde{k}} (\pi \times e^n))_+ \to \bigvee_\pi S^n,$$

representing the $S\pi$-dual of $T\pi(h^*\nu_X) \to T\pi(\nu_X)$, so that

$$\psi_H = 0\colon H^\pi_n(\tilde{X}; {}^w\mathbf{Z}) \to Q_n(\dot{C}(\bigvee_\pi S^n)) \quad (w = w(X)).$$

Applying the rth quadratic Wu class v^r to the $\mathbf{Z}_2$-hyperhomology class

$$g^!_{\%}\psi_F[X] = \psi_{I(\nu_h)}(1) - \psi_G[M] \in Q_n(\dot{C}(T\pi(\nu_g)))$$

we obtain the desired expression for $v^r(\psi)(x)$.

At this point it is instructive to compare the approaches taken by Wall [**30**] and Browder [**3**] to the problem of performing framed surgery on an element $\alpha \in \pi_{r+1}(f)$ for some n-dimensional normal bundle map (f, b): $M \to X$. Theorem 1.1 of [**30**] establishes that for $r \leqslant n-2$ every $\alpha \in \pi_{r+1}(f)$ determines a regular homotopy class of framed immersions $g: S^r \to M$ together with a prescribed null-homotopy $h: D^{r+1} \to X$ of $fg: S^r \to X$, such that $(\nu_h, \nu_g) = 0 \in \pi_{r+1}(BSO, BSO(n-r))$. Surgery on α is possible if and only if this class contains an embedding, so that on the chain level the surgery obstruction is

$$v^r(\psi)(x) = \mu(g) \in H_{2r-n}(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^{n-r}).$$

On the other hand, Theorem IV.1.6 of [**3**] assumes that $\alpha \in \pi_{r+1}(f)$ is already represented by an embedding $g: S^r \hookrightarrow M$ with a null-homotopy $h: D^{r+1} \to X$ of $fg: S^r \to X$, so that $\mu(g) = 0$. Surgery on α is possible if and only if $(\nu_h, \nu_g) = 0 \in \pi_{r+1}(BSO, BSO(n-r))$ $(= \pi_r(SO/SO(n-r)) = \pi_r(V_{k,n-r})$, k large), so that the chain level surgery obstruction is $v^r(\psi)(x) = (j(\nu_h, \nu_g), 0)$. In Proposition I.4.6(i) we interpreted the ε-quadratic Wu class $v^r(\psi)(x) \in H_{2r-n}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon)$ associated to an abstract n-dimensional ε-quadratic Poincaré complex over A, $(C, \psi \in Q_n(C, \varepsilon))$, as the obstruction to performing algebraic surgery on $x \in H_r(C)$. (Algebraic surgery will be related to geometric surgery in §7 below.)

Given an $(i-1)$-connected $2i$-dimensional $\{(2i+1)$-dimensional$\}$ normal bundle map for $i \geqslant 3$ $\{i \geqslant 2\}$, $(f, b): M \to X$, let

$$\theta(f,b) = \begin{cases} (K_i(M), \lambda, \mu) \\ (H_{(-)^i}(K_{i+1}(U, \partial U)); K_{i+1}(U, \partial U), K_{i+1}(M_0, \partial U)) \end{cases}$$

be the non-singular $(-)^i$ quadratic form {formation} over $\mathbf{Z}[\pi_1(X)]$ with the $w(X)$-twisted involution obtained by Wall in §5 {§6} of [**30**] as the surgery obstruction kernel, using geometrically defined intersection and self-intersection forms. The odd-dimensional terminology involves the union U of disjoint framed embeddings $S^i \times D^{i+1} \subset M$ such that the images $f(S^i \times D^{i+1}) \subset X$ are contractible, and such that the corresponding elements of $K_i(M)$ are a set of generators, with $M_0 = \overline{M \backslash U} \subset M$. The quadratic kernel $\sigma_*(f, b)$ is the i-fold skew-suspension of a 0-dimensional {1-dimensional} $(-)^i$ quadratic Poincaré complex over $\mathbf{Z}[\pi_1(X)]$,

$$\sigma_*(f,b) = \bar{S}^i \sigma_i(f,b)$$

(as in Proposition 2.6), and $\sigma_i(f,b)$ can be regarded as a non-singular $(-)^i$ quadratic form {formation} by Proposition I.2.1 {I.2.5}.

PROPOSITION 5.4. *The surgery obstruction kernel of a highly-connected n-dimensional normal bundle map $(f,b)\colon M \to X$ agrees with the quadratic kernel defined using a geometric Umkehr map $F \in \{\tilde{X}_+, \tilde{M}_+\}_\pi$ $(\pi = \pi_1(X))$*

$$\theta(f,b) = \sigma_i(f,b) \quad (n = 2i \text{ or } 2i+1 \geqslant 5).$$

Proof. Consider first the case where $n = 2i$. Now $C(f^!)$ is given up to chain equivalence by

$$C(f^!)\colon \ldots \to 0 \to K_i(M) \to 0 \to \ldots,$$

and the quadratic kernel is given by

$$\begin{aligned}\sigma_*(f,b) = (C(f^!), \psi = e_\%\psi_F[X] \in Q_{2i}(C(f^!)) \\ = \operatorname{coker}(1-T_{(-)^i}\colon \operatorname{Hom}_{\mathbf{Z}[\pi]}(K_i(M), K_i(M)^*) \\ \to \operatorname{Hom}_{\mathbf{Z}[\pi]}(K_i(M), K_i(M)^*))),\end{aligned}$$

identifying $K_i(M) = K^i(M)$ by Poincaré duality. By [**30**, Theorem 1.1], every element $x \in K_i(M)$ is represented by a framed immersion $g\colon S^i \to M^{2i}$ together with a null-homotopy $h\colon D^{i+1} \to X$ of $fg\colon S^i \to X$, and Propositions 5.2 and 5.3 allow the identification

$$\psi(x)(x) = \mu(g) \in H_0(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)], (-)^i).$$

Thus $\theta(f,b) = \sigma_i(f,b)$ if $n = 2i$.

In the case where $n = 2i+1$ we have that up to chain equivalence

$$C(f^!)\colon \ldots \to 0 \to K_{i+1}(M,U) \to K_i(U) \to 0 \to \ldots,$$

so that $\sigma_i(f,b)$ is a non-singular $(-)^i$ quadratic formation over $\mathbf{Z}[\pi_1(X)]$

$$\sigma_i(f,b) = (H_{(-)^i}(K_i(U)^*);\ K_i(U)^*, K_{i+1}(M,U)).$$

Identifying $K_{i+1}(U,\partial U) = K^i(U) = K_i(U)^*$ by Poincaré duality and the universal coefficient theorem we can write the inclusion of the lagrangian

$$K_{i+1}(M,U) \to K_i(U)^* \oplus K_i(U)$$

as the map

$$K_{i+1}(M,U) = K_{i+1}(M_0,\partial U) \to K_i(\partial U) = K_{i+1}(U,\partial U) \oplus K_{i+1}(U,\partial U)^*$$

appearing in the definition of $\theta(f,b)$. Thus $\theta(f,b) = \sigma_i(f,b)$ if $n = 2i+1$.

6. Geometric Poincaré cobordism

We now relativize all of the results of §§1–4, in order to construct algebraic Poincaré pairs from geometric Poincaré pairs. Given an $(n+1)$-dimensional geometric Poincaré pair $(X,\partial X)$ we define an $(n+1)$-dimensional symmetric Poincaré pair $\sigma^*(X,\partial X)$ with boundary $\sigma^*(\partial X)$, and given a degree 1 {normal} map of $(n+1)$-dimensional geometric Poincaré pairs $(f,\partial f)\colon (M,\partial M)\to (X,\partial X)$ $\{((f,\partial f),(b,\partial b))\colon (M,\partial M)\to (X,\partial X)\}$ we define an $(n+1)$-dimensional symmetric {quadratic} Poincaré pair $\sigma^*(f,\partial f)$ $\{\sigma_*((f,\partial f),(b,\partial b))\}$ with boundary $\sigma^*(\partial f)$ $\{\sigma_*(\partial f,\partial b)\}$.

The *relative symmetric construction* φ_f defined below is a relative version of the absolute symmetric construction φ_X of Proposition 1.2.

PROPOSITION 6.1. *Let π be a group, and give $\mathbf{Z}[\pi]$ the w-twisted involution for some group morphism $w\colon \pi\to \mathbf{Z}_2$. Given a π-map of π-spaces*

$$f\colon X\to Y$$

there are defined in a natural way abelian group morphisms

$$\varphi_f\colon H^\pi_{n+1}(f;\,{}^w\mathbf{Z})\to Q^{n+1}(f\colon \dot{C}(X)\to \dot{C}(Y))\quad (n\in \mathbf{Z})$$

such that

(i) *for each $z\in H^\pi_{n+1}(f;\,{}^w\mathbf{Z})$,*

$$\varphi_f(z)_0\backslash - = z\cap -\colon {}^wH^r(f)\to \dot{H}_{n+1-r}(Y),$$

(ii) *there is defined a morphism of long exact sequences*

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \dot{H}^\pi_{n+1}(Y;\,{}^w\mathbf{Z}) & \longrightarrow & H^\pi_{n+1}(f;\,{}^w\mathbf{Z}) & \longrightarrow & \dot{H}^\pi_n(X;\,{}^w\mathbf{Z}) & \\
& \downarrow{\scriptstyle \dot{\varphi}_Y} & & \downarrow{\scriptstyle \varphi_f} & & \downarrow{\scriptstyle \dot{\varphi}_X} & \\
\cdots \longrightarrow & Q^{n+1}(C(Y)) & \longrightarrow & Q^{n+1}(f) & \longrightarrow & Q^n(\dot{C}(X)) &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{f_*} & \dot{H}^\pi_n(Y;\,{}^w\mathbf{Z}) & \longrightarrow & H^\pi_n(f;\,{}^w\mathbf{Z}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \dot{\varphi}_Y} & & \downarrow{\scriptstyle \varphi_f} & \\
\xrightarrow{f^\%} & Q^n(\dot{C}(Y)) & \longrightarrow & Q^n(f) & \longrightarrow \cdots
\end{array}$$

Proof. Choosing a functorial diagonal approximation Δ we have a commutative diagram of abelian group chain complexes and chain maps

$$\begin{array}{ccc}
\mathbf{Z}^t\otimes_{\mathbf{Z}[\pi]}\dot{C}(X) & \xrightarrow{\quad 1\otimes f\quad} & \mathbf{Z}^t\otimes_{\mathbf{Z}[\pi]}\dot{C}(Y) \\
{\scriptstyle \dot{\varphi}_X = 1\otimes\dot{\Delta}_X}\downarrow & & \downarrow{\scriptstyle 1\otimes\dot{\Delta}_Y = \dot{\varphi}_Y} \\
\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\dot{C}(X)^t\otimes_{\mathbf{Z}[\pi]}\dot{C}(X)) & \xrightarrow{\ f^\%\ } & \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\dot{C}(Y)^t\otimes_{\mathbf{Z}[\pi]}\dot{C}(Y))
\end{array}$$

with corresponding chain map of algebraic mapping cones

$$\varphi_f\colon C(1\otimes f)\to C(f^\%).$$

The relative symmetric construction is given by the maps induced in the homology groups

$$\varphi_f\colon H_{n+1}(C(1\otimes f)) = H^{\pi}_{n+1}(f;\ {}^w\mathbf{Z}) \to H_{n+1}(C(f^{\%})) = Q^{n+1}(f).$$

Given a pair of π-spaces (X, Y) we shall write the relative symmetric construction for the inclusion $i\colon Y \hookrightarrow X$ as

$$\varphi_{X,Y} = \varphi_i\colon H^{\pi}_{n+1}(X, Y;\ {}^w\mathbf{Z}) = H^{\pi}_{n+1}(i;\ {}^w\mathbf{Z}) \to Q^{n+1}(i\colon \dot{C}(Y) \to \dot{C}(X)).$$

An $(n+1)$-*dimensional geometric Poincaré pair* $(X, \partial X)$ is a CW pair of finitely-dominated CW complexes, such that ∂X is an n-dimensional geometric Poincaré complex, together with a group morphism $w(X)\colon \pi_1(X) \to \mathbf{Z}_2$ such that $w(\partial X)$ factors as

$$w(\partial X)\colon \pi_1(\partial X) \longrightarrow \pi_1(X) \xrightarrow{w(X)} \mathbf{Z}_2$$

and with a relative homology class $[X] \in H^{\pi_1(X)}_{n+1}(\tilde{X}, \widetilde{\partial X};\ {}^{w(X)}\mathbf{Z})$ such that the cap products

$$[X]\cap - :\ {}^{w(X)}H^r(\tilde{X}, \widetilde{\partial X}) \to H_{n+1-r}(\tilde{X}) \quad (0 \leqslant r \leqslant n+1)$$

are $\mathbf{Z}[\pi_1(X)]$-module isomorphisms (Poincaré–Lefschetz duality) and

$$\partial_*[X] = [\partial X] \in H^{\pi_1(X)}_n(\widetilde{\partial X};\ {}^{w(X)}\mathbf{Z}),$$

with $\tilde{X}$ the universal cover of X and $\widetilde{\partial X}$ the induced cover of ∂X.

The relative symmetric construction of Proposition 6.1 gives a relative version of the construction of $\sigma^*(X)$ in Proposition 2.1.

PROPOSITION 6.2. *Given an $(n+1)$-dimensional geometric Poincaré pair $(X, \partial X)$ and an oriented cover $\tilde{X}$ of X with data (π, w) and induced cover $\widetilde{\partial X}$ of ∂X there is defined in a natural way an $(n+1)$-dimensional symmetric Poincaré pair over $\mathbf{Z}[\pi]$ with the w-twisted involution*

$$\sigma^*(X, \partial X) = (i_{\tilde{X}}\colon C(\widetilde{\partial X}) \to C(\tilde{X}), \varphi_{\tilde{X},\widetilde{\partial X}}[X] \in Q^{n+1}(i_{\tilde{X}}))$$

with boundary $\sigma^*(\partial X) = (C(\widetilde{\partial X}), \varphi_{\widetilde{\partial X}}[\partial X] \in Q^n(C(\widetilde{\partial X})))$, where $i_{\tilde{X}}$ is the inclusion.

Define the *symmetric signature* of an n-dimensional geometric Poincaré complex X with respect to an oriented cover $\tilde{X}$ of X with data (π, w) to be the symmetric Poincaré cobordism class

$$\sigma^*(X) \in L^n(\mathbf{Z}[\pi])$$

with $\sigma^*(X) = (C(\tilde{X}),\ \varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X})))$ the n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi]$ with the w-twisted involution constructed in Proposition 2.1. The symmetric signature $\sigma^*(X) \in L^n(\mathbf{Z}[\pi])$ is induced via the change of ring maps $\mathbf{Z}[\pi_1(X)] \to \mathbf{Z}[\pi]$ from the universal symmetric signature $\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)])$ associated to the universal cover of X.

The symmetric signature invariant $\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)])$ was introduced by Mishchenko [**18**].

Given a space K and a group morphism $w: \pi_1(K) \to \mathbf{Z}_2$ let $\Omega_n^P(K, w)$ be the group of geometric Poincaré bordism classes of maps $f: X \to K$ from n-dimensional geometric Poincaré complexes X such that the orientation map factors as

$$w(X): \pi_1(X) \xrightarrow{f} \pi_1(K) \xrightarrow{w} \mathbf{Z}_2,$$

that is, such that the cover $\tilde{X}$ of X induced from the universal cover $\tilde{K}$ of K is oriented with data $(\pi_1(K), w)$.

PROPOSITION 6.3. *The symmetric signature defines abelian group morphisms*

$$\sigma^*: \Omega_n^P(K, w) \to L^n(\mathbf{Z}[\pi_1(K)]);\ (f: X \to K) \mapsto \sigma^*(X) \quad (n \geqslant 0).$$

Proof. If $(g;\, f, f'): (Y;\, X, X') \to K$ is an $(n+1)$-dimensional geometric Poincaré bordism then the construction of Proposition 6.2 defines an $(n+1)$-dimensional symmetric Poincaré cobordism over $\mathbf{Z}[\pi_1(K)]$

$$\sigma^*(Y;\, X, X')$$

from $\sigma^*(X)$ to $\sigma^*(X')$.

As a special case of the geometric Poincaré bordism invariance of the symmetric signature we have homotopy invariance: if $f: X \to X'$ is a homotopy equivalence of n-dimensional geometric Poincaré complexes then

$$\sigma^*(X) = \sigma^*(X') \in L^n(\mathbf{Z}[\pi_1(X)]).$$

It follows from the computation of $L^*(\mathbf{Z})$ (Proposition I.7.2) that the simply-connected symmetric signature map

$$\sigma^*: \Omega_n^P(\text{pt.}) \to L^n(\mathbf{Z}) = \begin{cases} \mathbf{Z} \\ \mathbf{Z}_2 \\ 0 \\ 0 \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

sends an oriented $4k$-dimensional $\{(4k+1)$-dimensional$\}$ geometric Poincaré complex X $(w(X) = 1)$ to

$$\begin{cases} \sigma^*(X^{4k}) = (\text{signature of } X) \equiv (\text{signature of the Poincaré duality} \\ \qquad\qquad \text{intersection form } (H^{2k}(X;\, \mathbf{R}), \varphi_X[X]_0)) \in L^{4k}(\mathbf{Z}) = \mathbf{Z}, \\ \sigma^*(X^{4k+1}) = (\text{deRham invariant of } X) \equiv (\text{deRham invariant of the} \\ \qquad\qquad \text{Seifert linking form } (H^{2k+1}(X;\, \mathbf{Q}/\mathbf{Z}), \varphi_X[X]_0)) \\ \qquad\qquad = (\dim_{\mathbf{Z}_2} H^{2k+1}(X;\, \mathbf{Z}_2)) \in L^{4k+1}(\mathbf{Z}) = \mathbf{Z}_2. \end{cases}$$

The *relative quadratic construction* $\psi_{F,G}$ described below is the relative version of the absolute quadratic construction ψ_F of Proposition 1.5.

PROPOSITION 6.4. *Let π be a group, and give $\mathbf{Z}[\pi]$ the w-twisted involution for some group morphism $w\colon \pi \to \mathbf{Z}_2$. Given π-spaces X, X', Y, Y' and π-maps $f\colon X \to Y$, $f'\colon X' \to Y'$, $F\colon \Sigma^p X \to \Sigma^p X'$, $G\colon \Sigma^p Y \to \Sigma^p Y'$ for some $p \geqslant 0$ such that the diagram*

$$\begin{array}{ccc} \Sigma^p X & \xrightarrow{\Sigma^p f} & \Sigma^p Y \\ {\scriptstyle F}\big\downarrow & h & \big\downarrow{\scriptstyle G} \\ \Sigma^p X' & \xrightarrow{\Sigma^p f'} & \Sigma^p Y' \end{array}$$

commutes there are defined in a natural way abelian group morphisms

$$\psi_{F,G}\colon H^{\pi}_{n+1}(f;\, {}^{w}\mathbf{Z}) \to Q_{n+1}(f'\colon \dot{C}(X') \to \dot{C}(Y')) \quad (n \in \mathbf{Z})$$

such that

(i) $h^{\%}\varphi_f - \varphi_{f'}h_* = (1+T)\psi_{F,G}\colon H^{\pi}_{n+1}(f;\, {}^{w}\mathbf{Z}) \to Q^{n+1}(f')$, *with* $h_*\colon H^{\pi}_{n+1}(f;\, {}^{w}\mathbf{Z}) \to H^{\pi}_{n+1}(f';\, {}^{w}\mathbf{Z})$, $h^{\%}\colon Q^{n+1}(f) \to Q^{n+1}(f')$ *the induced maps,*

(ii) *there is defined a map of long exact sequences*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{\pi}_{n+1}(f;\, {}^{w}\mathbf{Z}) & \longrightarrow & \dot{H}^{\pi}_{n}(X;\, {}^{w}\mathbf{Z}) & \xrightarrow{f_*} & \dot{H}^{\pi}_{n}(Y;\, {}^{w}\mathbf{Z}) & \longrightarrow & H^{\pi}_{n}(f;\, {}^{w}\mathbf{Z}) & \longrightarrow \cdots \\ & {\scriptstyle \psi_{F,G}}\big\downarrow & & {\scriptstyle \psi_F}\big\downarrow & & {\scriptstyle \psi_G}\big\downarrow & & {\scriptstyle \psi_{F,G}}\big\downarrow & \\ \cdots \longrightarrow & Q_{n+1}(f') & \longrightarrow & Q_n(\dot{C}(X')) & \xrightarrow{f'_{\%}} & Q_n(\dot{C}(Y')) & \longrightarrow & Q_n(f') & \longrightarrow \cdots \end{array}$$

(iii) $\psi_{F,G}$ *factorizes through* $Q^{[0,p-1]}_{n+1}(f')$

$$\psi_{F,G}\colon H^{\pi}_{n+1}(f;\, {}^{w}\mathbf{Z}) \to Q^{[0,p-1]}_{n+1}(f') \to Q^{[0,\infty]}_{n+1}(f') \equiv Q_{n+1}(f').$$

If $p = 0$ then $\psi_{F,G} = 0$.

Given a degree 1 map of $(n+1)$-dimensional geometric Poincaré pairs

$$(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$$

and a covering $\tilde{X}$ of X with group of covering translations π define $\mathbf{Z}[\pi]$-module *Umkehr* chain maps

$$\begin{cases} f^!: C(\tilde{X}) \to C(\tilde{M}) \\ \partial f^!: C(\widetilde{\partial X}) \to C(\widetilde{\partial M}) \end{cases}$$

by applying $\mathbf{Z}[\pi]\otimes_{\mathbf{Z}[\pi_1(X)]}-$ to the $\mathbf{Z}[\pi_1(X)]$-module Umkehr chain maps

$$\begin{cases} f^!: C(\tilde{\tilde{X}}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\tilde{\tilde{X}}, \widetilde{\widetilde{\partial X}})^{n+1-*} \\ \qquad \xrightarrow{\tilde{\tilde{f}}^*} {}^{w(X)}C(\tilde{\tilde{M}}, \widetilde{\widetilde{\partial M}})^{n+1-*} \xrightarrow{[M]\cap -} C(\tilde{\tilde{M}}) \\ \partial f^!: C(\widetilde{\widetilde{\partial X}}) \xrightarrow{([\partial X]\cap -)^{-1}} {}^{w(X)}C(\widetilde{\widetilde{\partial X}})^{n-*} \\ \qquad \xrightarrow{\widetilde{\widetilde{\partial f}}^*} {}^{w(X)}C(\widetilde{\widetilde{\partial M}})^{n-*} \xrightarrow{[\partial M]\cap -} C(\widetilde{\widetilde{\partial M}}) \end{cases}$$

with $\tilde{\tilde{X}}$ the universal cover of X and $\tilde{\tilde{M}}$, $\widetilde{\widetilde{\partial M}}$, $\widetilde{\widetilde{\partial X}}$ the induced covers of M, ∂M, ∂X. There is defined a chain homotopy commutative diagram

$$\begin{array}{ccc} C(\widetilde{\partial X}) & \xrightarrow{i_{\tilde{X}}} & C(\tilde{X}) \\ {\scriptstyle \partial f^!}\downarrow & & \downarrow{\scriptstyle f^!} \\ C(\widetilde{\partial M}) & \xrightarrow{i_{\tilde{M}}} & C(\tilde{M}) \end{array}$$

with $i_{\tilde{M}}$, $i_{\tilde{X}}$ the inclusions, so that there is induced a $\mathbf{Z}[\pi]$-module chain map in the algebraic mapping cones

$$i_{\tilde{f}}: C(\partial f^!) \to C(f^!).$$

A *geometric Umkehr map* for $(f, \partial f)$ is a π-map of pairs of π-spaces

$$(F, \partial F): (\Sigma^p \tilde{X}_+, \Sigma^p(\widetilde{\partial X})_+) \to (\Sigma^p \tilde{M}_+, \Sigma^p(\widetilde{\partial M})_+)$$

for some $p \geqslant 0$, which induces the Umkehr $(f^!, \partial f^!)$ on the chain level.

The relative symmetric {quadratic} construction of Proposition 6.1 {Proposition 6.4} can be used to obtain a relative analogue of the symmetric {quadratic} kernel $\sigma^*(f)$ $\{\sigma_*(f, F)\}$ of Proposition 2.2 {Proposition 2.3} as follows.

PROPOSITION 6.5. *Given a degree* 1 *map of* $(n+1)$*-dimensional geometric Poincaré pairs*

$$(f, \partial f): (M, \partial M) \to (X, \partial X)$$

and or oriented cover $\tilde{X}$ *of* X *with data* (π, w) *there is defined in a natural way a* symmetric kernel $(n+1)$*-dimensional symmetric Poincaré pair over*

$\mathbf{Z}[\pi]$ *with the w-twisted involution*

$$\sigma^*(f,\partial f) = (i_{\bar{f}}\colon C(\partial f^!) \to C(f^!), e^{\%}\varphi_{\widetilde{M},\widetilde{\partial M}}[M] \in Q^{n+1}(i_f))$$

with boundary $\sigma^*(\partial f)$, *and such that*

$$\sigma^*(M,\partial M) = \sigma^*(f,\partial f) \oplus \sigma^*(X,\partial X)$$

up to homotopy equivalence of pairs. Given also a geometric Umkehr map

$$(F,\partial F)\colon (\Sigma^p\tilde{X}_+, \Sigma^p\widetilde{\partial X}_+) \to (\Sigma^p\tilde{M}_+, \Sigma^p\widetilde{\partial M}_+)$$

there is defined in a natural way a quadratic kernel $(n+1)$-*dimensional quadratic Poincaré pair over* $\mathbf{Z}[\pi]$

$$\sigma_*(f,\partial f;\, F,\partial F) = (i_{\bar{f}}\colon C(\partial f^!) \to C(f^!), e_{\%}\psi_{F,\partial F}[X] \in Q_{n+1}(i_{\bar{f}}))$$

with boundary $\sigma_*(\partial f,\partial F)$, *and such that*

$$(1+T)\sigma_*(f,\partial f;\, F,\partial F) = \sigma^*(f,\partial f).$$

Next, we outline the relative version of the equivariant S-duality theory of § 3 required to obtain geometric Umkehr maps for normal bundle maps of pairs. A *π-pair* (X, Y) is a pair of π-spaces, $Y \subset X$, in which case the suspension $\Sigma(X, Y) = (\Sigma X, \Sigma Y)$ is also a π-pair. Given π-pairs (X, Y), (A, B) let $\{X, Y; A, B\}_\pi$ be the abelian group of stable relative π-homotopy classes of π-maps of π-pairs $(f,g)\colon \Sigma^p(X, Y) \to \Sigma^p(A,B)$ $(p \geqslant 0)$. The π-pairs (X, Y), (X^*, Y^*) are *relatively $S\pi$-dual* if there is given a $\{1\}$-map of pairs

$$(\alpha,\beta)\colon (D^N, S^{N-1}) \to (X\wedge_\pi X^*,\ Y\wedge_\pi Y^*)$$

such that for every π-spectrum of pairs $(\underline{A},\underline{B})$ the slant products

$$\begin{aligned}
\backslash\colon \{X, Y;\, \underline{A},\underline{B}\}_\pi &\to \{D^N, S^{N-1};\, \underline{A}\wedge_\pi X^*,\ \underline{B}\wedge_\pi Y^*\};\\
&((f,g)\colon (\Sigma^p X, \Sigma^p Y) \to (A_p, B_p))\\
&\quad\mapsto (((f\wedge 1)\Sigma^p\alpha, (g\wedge 1)\Sigma^p\beta)\colon\\
&\qquad\qquad (D^{N+p}, S^{N+p-1}) \to (A_p\wedge_\pi X^*,\ B_p\wedge_\pi Y^*)),\\
\backslash\colon \{X^*, Y^*;\, \underline{A},\underline{B}\}_\pi &\to \{D^N, S^{N-1};\, X\wedge_\pi \underline{A},\ Y\wedge_\pi \underline{B}\};\\
&((f^*,g^*)\colon (\Sigma^p X^*, \Sigma^p Y^*) \to (A_p, B_p))\\
&\quad\mapsto (((1\wedge f^*)\Sigma^p\alpha, (1\wedge g^*)\Sigma^p\beta)\colon\\
&\qquad\qquad (D^{N+p}, S^{N+p-1}) \to (X\wedge_\pi A_p,\ Y\wedge_\pi B_p))
\end{aligned}$$

are isomorphisms, and such that the $\{1\}$-map $\beta\colon S^{N-1} \to Y\wedge_\pi Y^*$ is an absolute $S\pi$-duality map. It then follows that there are defined absolute $S\pi$-duality maps

$$\alpha/\beta\colon S^N \to (X/Y)\wedge_\pi X^*, \quad \alpha/\beta\colon S^N \to X\wedge_\pi (X^*/Y^*).$$

An n-dimensional geometric Poincaré pair $(X, \partial X)$ can be embedded in (D^{n+k}, S^{n+k-1}) (k large) with $X \cap S^{n+k-1} = \partial X \subset S^{n+k-1}$, such that there exists a closed regular neighbourhood E of X in D^{n+k} with

$$E' = E \cap S^{n+k-1} \subset \partial E$$

a closed regular neighbourhood of ∂X in S^{n+k-1}. The inclusions

$$\overline{\partial E \backslash E'} \hookrightarrow E, \quad \partial E' \hookrightarrow E'$$

define $(k-1)$-spherical fibrations

$$S^{k-1} \longrightarrow \overline{\partial E \backslash E'} \xrightarrow{\nu_X} E = X,$$

$$S^{k-1} \longrightarrow \partial E' \xrightarrow{\nu_{\partial X}} E' = \partial X,$$

such that $\nu_{\partial X}$ is the restriction of ν_X to ∂X

$$\nu_{\partial X} \colon \partial X \hookrightarrow X \xrightarrow{\nu_X} BG(k).$$

The collapsing map of $\{1\}$-pairs

$$\begin{aligned}(\rho_X, \rho_{\partial X}) \colon (D^{n+k}, S^{n+k-1}) &\to (D^{n+k}/\overline{D^{n+k} - E}, S^{n+k-1}/\overline{S^{n+k-1} - E'}) \\ &= (E/\overline{\partial E \backslash E'}, E'/\partial E') = (T(\nu_X), T(\nu_{\partial X}))\end{aligned}$$

can be used to define a relative $S\pi$-duality map

$$\begin{aligned}(\alpha_X, \alpha_{\partial X}) \colon (D^{n+k}, S^{n+k-1}) &\xrightarrow{(\rho_X, \rho_{\partial X})} (T(\nu_X), T(\nu_{\partial X})) \\ &\xrightarrow{\Delta} (\tilde{X}_+ \wedge_\pi T\pi(\nu_X), \widetilde{\partial X}_+ \wedge_\pi T\pi(\nu_{\partial X}))\end{aligned}$$

between the π-pairs $(\tilde{X}_+, \widetilde{\partial X}_+)$ and $(T\pi(\nu_X), T\pi(\nu_{\partial X}))$ for any covering $\tilde{X}$ of X with group of covering translations π. Given n-dimensional geometric Poincaré pairs $(M, \partial M)$, $(X, \partial X)$ and any coverings $\tilde{M}$, $\tilde{X}$ with the same group of covering translations π we thus have relative $S\pi$-duality isomorphisms

$$\begin{aligned}&\{T\pi(\nu_M), T\pi(\nu_{\partial M}); T\pi(\nu_X), T\pi(\nu_{\partial X})\}_\pi \\ &\qquad \to \{D^{n+k}, S^{n+k-1}; \tilde{M}_+ \wedge_\pi T\pi(\nu_X), \widetilde{\partial M}_+ \wedge_\pi T\pi(\nu_{\partial X})\} \\ &\qquad \to \{\tilde{X}_+, \widetilde{\partial X}_+; \tilde{M}_+, \widetilde{\partial M}_+\}_\pi.\end{aligned}$$

Thus given a normal bundle map of pairs

$$(f, \partial f; b, \partial b) \colon (M, \partial M) \to (X, \partial X)$$

and an oriented covering $\tilde{X}$ of X with data (π, w) the $S\pi$-dual of the π-map of π-pairs

$$(T\pi(b), T\pi(\partial b)) \colon (T\pi(\nu_M), T\pi(\nu_{\partial M})) \to (T\pi(\nu_X), T\pi(\nu_{\partial X}))$$

is the relative $S\pi$-homotopy class of a relative geometric Umkehr map

$$(F, \partial F)\colon (\Sigma^p \tilde{X}_+, \Sigma^p \widetilde{\partial X}_+) \to (\Sigma^p \tilde{M}_+, \Sigma^p \widetilde{\partial M}_+).$$

The construction of Proposition 6.5 now gives a *quadratic kernel* n-dimensional quadratic Poincaré pair over $\mathbf{Z}[\pi]$

$$\sigma_*(f, \partial f; b, \partial b) = \sigma_*(f, \partial f; F, \partial F)$$

with boundary $\sigma_*(f, b)$.

Define the *symmetric* {*quadratic*} *signature*

$$\sigma^*(f) \in L^n(\mathbf{Z}[\pi_1(X)]) \quad \{\sigma_*(f, b) \in L_n(\mathbf{Z}[\pi_1(X)])\}$$

of a degree 1 {normal} map $f\colon M \to X$ $\{(f, b)\colon M \to X\}$ of n-dimensional geometric Poincaré complexes to be the symmetric {quadratic} Poincaré cobordism class of the symmetric {quadratic} kernel

$$\begin{cases} \sigma^*(f) = (C(f^!), e^{\%}\varphi_{\tilde{M}}[M] \in Q^n(C(f^!))) \\ \sigma^*(f, b) = (C(f^!), e_{\%}\psi_F[X] \in Q_n(C(f^!))) \end{cases}$$

defined in Proposition 2.2 {Proposition 4.3}.

A *degree* 1 {*normal*} *bordism* between n-dimensional degree 1 {normal} maps

$$\begin{cases} f\colon M \to X,\ f'\colon M' \to X \\ (f, b)\colon M \to X,\ (f', b')\colon M' \to X \end{cases}$$

is a degree 1 {normal} map of $(n+1)$-dimensional geometric Poincaré cobordisms

$$\begin{cases} (g;\ f, f')\colon (N;\ M, M') \to (X \times I;\ X \times \{0\}, X \times \{1\}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (I = [0,1]) \\ ((g;\ f,\ f'), (c;\ b, b'))\colon (N;\ M, M') \to (X \times I;\ X \times \{0\}, X \times \{1\}). \end{cases}$$

PROPOSITION 6.6. (i) *The symmetric* {*quadratic*} *signature*

$$\sigma^*(f) \in L^n(\mathbf{Z}[\pi_1(X)]) \quad \{\sigma_*(f, b) \in L_n(\mathbf{Z}[\pi_1(X)])\}$$

of an n-dimensional degree 1 {*normal*} *map* $f\colon M \to X$ $\{(f, b)\colon M \to X\}$ *is a degree* 1 {*normal*} *bordism invariant such that*

$$\begin{cases} \sigma^*(f) = \sigma^*(M) - \sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)]), \\ (1+T)\sigma_*(f, b) = \sigma^*(f) \in L^n(\mathbf{Z}[\pi_1(X)]). \end{cases}$$

(ii) *The symmetric* {*quadratic*} *signature of the composite* $gf\colon M \to Y$ $\{(gf, cb)\colon M \to Y\}$ *of n-dimensional degree* 1 {*normal*} *maps*

$$f\colon M \to X,\ g\colon X \to Y \quad \{(f, b)\colon M \to X,\ (g, c)\colon X \to Y\}$$

is the sum

$$\begin{cases} \sigma^*(gf) = \sigma^*(f)+\sigma^*(g) \in L^n(\mathbf{Z}[\pi_1(Y)]), \\ \sigma_*(gf, cb) = \sigma_*(f,b)+\sigma_*(g,c) \in L_n(\mathbf{Z}[\pi_1(Y)]). \end{cases}$$

Proof. The symmetric {quadratic} kernel

$$\sigma^*(g;\, f, f') \quad \{\sigma_*((g;\, f, f'),\, (c;\, b, b'))\}$$

of a degree 1 {normal} bordism $(g;\, f, f')$ $\{((g; f, f'),\, (c;\, b, b'))\}$ is a symmetric {quadratic} Poincaré cobordism between the symmetric {quadratic} kernels $\sigma^*(f), \sigma^*(f')$ $\{\sigma_*(f,b),\ \sigma_*(f',b')\}$. Proposition 2.2 {Proposition 2.3} gives that $\sigma^*(f)\oplus\sigma^*(X) = \sigma^*(M)$ $\{(1+T)\sigma_*(f,b) = \sigma^*(f)\}$, and Proposition 2.5 {Proposition 4.3} that

$$\sigma^*(gf) = \sigma^*(f)\oplus\sigma^*(g) \quad \{\sigma_*(gf, cb) = \sigma_*(f,b)\oplus\sigma_*(g,c)\},$$

up to homotopy equivalence. By Proposition I.3.2 homotopy equivalent symmetric {quadratic} Poincaré complexes are cobordant.

In Proposition 7.1 we shall identify the quadratic signature

$$\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$$

of an n-dimensional normal bundle map $(f,b)\colon M \to X$ with the surgery obstruction $\theta(f,b) \in L_n(\pi_1(X), w(X))$ obtained by Wall [**30**] using geometric intersection and self-intersection forms. We have already related the two constructions in Proposition 5.4, and the normal bordism invariance of the quadratic signature (Proposition 6.6(i)) ensures that there is defined a morphism of abelian groups

$$L_n(\pi_1(X), w(X)) \to L_n(\mathbf{Z}[\pi_1(X)]);\quad \theta(f,b) \mapsto \sigma_*(f,b).$$

In §7 below we shall identify this geometrically defined map with the algebraically defined isomorphism of §I.5.

In view of the above, the quadratic signature sum formula of Proposition 6.6(ii) may be considered as a homotopy-theoretic version of the sum formulae of Wall [**30**, §17H] and Theorem 7.0 of Jones [**6**].

7. Geometric surgery

The original work of Milnor [**16**] and Wallace [**31**] {Kervaire and Milnor [**8**]} developed oriented {framed} surgery as a method for killing the homotopy groups of an oriented {framed} smooth manifold M. The framed surgery technique was generalized to surgery on a normal bundle map $(f,b)\colon M \to X$ from a compact manifold M to a finite geometric Poincaré complex X (previously $X = S^n$) by Browder [**3**], Novikov [**20**] (for $\pi_1(X) = \{1\}$), and Wall [**30**] (for any $\pi_1(X)$). The manifold M may be taken to be smooth, PL, topological (Kirby and Siebenmann [**9**]) and even a homology manifold (Maunder [14]). There are also versions for paracompact M and infinite X (Taylor [27], Maumary [13]). Various authors, Levitt [11], Jones [6], Quinn [21], Lannes, Latour and Morlet [10], went on to consider framed surgery on normal maps of geometric Poincaré complexes. In all cases the obstruction to making a normal map a homotopy equivalence by a sequence of framed surgeries is an element $\theta(f,b)$ of the group $L_n(\pi_1(X),w(X))$ of Wall [30], or of one of the closely related variants described in §I.9. We shall now identify the surgery obstruction

$$\theta(f,b)\in L_n(\pi_1(X),w(X))$$

with the quadratic signature

$$\sigma_*(f,b)\in L_n(\mathbf{Z}[\pi_1(X)]).$$

Also, we shall show that the chain level effect of an oriented {framed} geometric surgery on a degree 1 {normal} map $f\colon M\to X$ $\{(f,b)\colon M\to X\}$ is an elementary symmetric {quadratic} surgery on the symmetric {quadratic} kernel $\sigma^*(f)$ $\{\sigma_*(f,b)\}$ (as defined in §I.4). For the purpose of exposition we shall consider only smooth surgery in geometry, and only the projective symmetric {quadratic} L-groups $L^n(A)$ $\{L_n(A)\}$ in algebra.

PROPOSITION 7.1. *The quadratic signature of an n-dimensional normal bundle map $(f,b)\colon M\to X$ is the Wall surgery obstruction*

$$\sigma_*(f,b)=\theta(f,b)\in L_n(\mathbf{Z}[\pi_1(X)])=L_n(\pi_1(X),w(X))\quad(n\geqslant 5).$$

Proof. Let $n=2i$ or $2i+1$, and write $(f',b')\colon M'\to X$ for the normal bordant $(i-1)$-connected n-dimensional normal bundle map obtained from $(f,b)\colon M\to X$ by framed surgery below the middle dimension, as in Theorem 1.2 of Wall [30]. Propositions 4.2 and 2.6 give an $(n-2i)$-dimensional $(-)^i$ quadratic Poincaré complex over $\mathbf{Z}[\pi_1(X)]$, $\sigma_i(f',b')$, such that

$$\bar{S}^i\sigma_i(f',b')=\sigma_*(f',b').$$

It follows from the normal bordism invariance of the quadratic signature (Proposition 6.6(i)) that

$$\sigma_*(f',b')=\sigma_*(f,b)\in L_n(\mathbf{Z}[\pi_1(X)]).$$

Proposition 5.4 identifies the $(-)^i$ quadratic Poincaré cobordism class

$$\sigma_i(f',b')\in L_{n-2i}(\mathbf{Z}[\pi_1(X)],(-)^i)$$

with the framed surgery obstruction obtained by Wall in [30, §§5, 6],

$$\theta(f,b)=\theta(f',b')\in L_n(\pi_1(X),w(X)),$$

under the identifications

$$L_{n-2i}(\mathbf{Z}[\pi_1(X)], (-)^i) = L_n(\pi_1(X), w(X))$$

of Propositions I.5.1 and I.5.2. The i-fold skew-suspension isomorphism of Proposition I.4.3

$$\bar{S}^i : L_{n-2i}(\mathbf{Z}[\pi_1(X)], (-)^i) \to L_n(\mathbf{Z}[\pi_1(X)])$$

is therefore such that

$$\bar{S}^i\theta(f, b) = \bar{S}^i\theta(f', b') = \bar{S}^i\sigma_i(f', b') = \sigma_*(f', b') = \sigma_*(f, b) \in L_n(\mathbf{Z}[\pi_1(X)]).$$

As yet, there is no geometric interpretation of the symmetric signature $\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)])$ of an n-dimensional geometric Poincaré complex X. At any rate, the symmetric signature appears in the product formula for surgery obstructions (Proposition 8.1(ii) below).

Given an n-dimensional normal bundle map of pairs

$$((f, \partial f), (b, \partial b)) : (M, \partial M) \to (X, \partial X)$$

such that $\partial f : \partial M \to \partial X$ is a homotopy equivalence we have that the quadratic kernel $\sigma_*((f, \partial f), (b, \partial b))$ is an n-dimensional quadratic Poincaré pair over $\mathbf{Z}[\pi_1(X)]$ with contractible boundary $(n-1)$-dimensional quadratic Poincaré complex $\sigma_*(\partial f, \partial b)$. The homotopy equivalence classes of n-dimensional quadratic Poincaré pairs with contractible boundary are in a natural one–one correspondence with the homotopy equivalence classes of n-dimensional quadratic Poincaré complexes (by Proposition I.3.4(i)). We thus obtain a quadratic signature

$$\sigma_*((f, \partial f), (b, \partial b)) \in L_n(\mathbf{Z}[\pi_1(X)]),$$

which we can identify with the obstruction obtained by Wall [**30**] (for $n \geqslant 5$) to making $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ a homotopy equivalence by a sequence of framed surgeries on the interior of M, keeping $\partial f : \partial M \to \partial X$ fixed.

The identification of Proposition 7.1 can be interpreted as an instant surgery obstruction, solving Problem 5 of Shaneson [**24**]. Given a $2i$-dimensional $\{(2i+1)$-dimensional$\}$ normal bundle map $(f, b) : M \to X$ we can write down a non-singular $(-)^i$-quadratic form $\{$formation$\}$ over $\mathbf{Z}[\pi_1(X)]$ representing the surgery obstruction $\sigma_*(f, b) \in L_{2i}(\mathbf{Z}[\pi_1(X)])$ $\{\sigma_*(f, b) \in L_{2i+1}(\mathbf{Z}[\pi_1(X)])\}$, without preliminary surgeries below the middle dimension.

PROPOSITION 7.2. *Let $(f, b) : M \to X$ be an n-dimensional normal bundle map, with quadratic kernel*

$$\sigma_*(f, b) = (C(f^!), e_{\%}\psi_F[X]) = (C, \psi \in Q_n(C)).$$

Then the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$ *is the class of the non-singular* $(-)^i$ *quadratic form {formation} over* $\mathbf{Z}[\pi_1(X)]$,

$$\Omega^i\sigma_*(f,b) = \begin{cases} \left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T)\psi_0 & d \end{pmatrix}: C^{i-1}\oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right) \\ \left(H_{(-)^i}(C_{i+1});\ C_{i+1}, \operatorname{im}\left(\begin{pmatrix} (1+T)\psi_0 & d \\ d^* & 0 \end{pmatrix}: C^i \oplus C_{i+2} \to C_{i+1}\oplus C^{i+1}\right)\right) \end{cases}$$

$$\in L_n(\pi_1(X), w(X)) \quad \textit{if } n = \begin{cases} 2i, \\ 2i+1. \end{cases}$$

Proof. This is just the explicit inverse

$$\Omega^i\colon L_n(\mathbf{Z}[\pi_1(X)]) \to L_{n-2i}(\mathbf{Z}[\pi_1(X)], (-)^i)$$

to the i-fold skew-suspension isomorphism $\bar{S}^i$ of Proposition I.4.3. (It is required that all the chain modules C_r appearing in the above formulae be finitely generated projective $\mathbf{Z}[\pi_1(X)]$-modules.)

We shall now show that a geometric surgery induces an elementary algebraic surgery on the chain level (as defined in § I.4). Let us recall the elements of geometric surgery.

An *elementary oriented {framed} surgery of type* $(r, n-r-1)$ $(0 \leqslant r \leqslant n-1)$ on a degree 1 {normal bundle} map $f\colon M \to X$ $\{(f,b)\colon M \to X\}$ from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X is determined by the following data:

(i) an embedding $g\colon S^r \hookrightarrow M$ with an oriented normal bundle $\nu_g\colon S^r \to BSO(n-r)$;

(ii) a null-homotopy $\bar{\nu}_g\colon D^{r+1} \to BSO(n-r)$ of $\nu_g\colon S^r \to BSO(n-r)$, that is an embedding $\bar{g}\colon S^r \times D^{n-r} \hookrightarrow M$ extending g;

(iii) a null-homotopy $h\colon D^{r+1} \to X$ of $fg\colon S^r \to X$;

and in the framed case also

(iv) a relative null-homotopy $(\bar{\nu}_h, \bar{\nu}_g)\colon (D^{r+1}, S^r)\wedge I \to (BSO, BSO(n-r))$ extending $\bar{\nu}_g$, of the map of pairs

$$(\nu_h, \nu_g)\colon (D^{r+1}, S^r) \to (BSO, BSO(n-r)),$$

with $\nu_h\colon D^{r+1} \to BSO$ the null-homotopy of the classifying map for the stable normal bundle $\nu_g\colon S^r \to BSO(n-r) \to BSO$ determined by $b\colon \nu_M \to \nu_X$ and $h\colon D^{r+1} \to X$.

The surgery replaces f $\{(f,b)\}$ by the n-dimensional degree 1 {normal bundle} map $f'\colon M'\to X$ $\{(f',b')\colon M'\to X\}$ appearing in the $(n+1)$-dimensional degree 1 {normal bundle} map of cobordisms

$$\begin{cases} (e;\ f,f')\colon (N;\ M,M')\to(X\times I;\ X\times 0,\ X\times 1),\\ ((e;\ f,f'),(a;\ b,b'))\colon (N;\ M,M')\to(X\times I;\ X\times 0,X\times 1)\end{cases}$$

defined by

$$N = M\times I\cup_{\bar{g}\times 1} D^{r+1}\times D^{n-r},\quad M' = \overline{M\backslash\bar{g}(S^r\times D^{n-r})}\cup D^{r+1}\times S^{n-r-1},$$

using $h\colon D^{r+1}\to X$ to extend $f\colon M\to X\times 0$ to a map of pairs

$$(e,f')\colon (N,M')\to(X\times I,X\times 1),$$

and in the framed case using $\tilde{\nu}_h$ to extend $b\colon \nu_M\to\nu_{X\times 0}$ to a bundle map of pairs $(a,b')\colon(\nu_N,\nu_{M'})\to(\nu_{X\times I},\nu_{X\times 1})$. The surgery is said to *kill*

$$(h,g)\in\pi_{r+1}(f).$$

PROPOSITION 7.3. *Let*

$$f\colon M\to X,\ f'\colon M'\to X\quad \{(f,b)\colon M\to X,\ (f',b')\colon M'\to X\}$$

be n-dimensional degree 1 *{normal bundle} maps such that* f' $\{(f',b')\}$ *is obtained from* f $\{(f,b)\}$ *by an elementary oriented {framed} surgery of type* $(r,n-r-1)$ *killing* $(h,g)\in\pi_{r+1}(f)$. *Then the symmetric {quadratic} kernel* $\sigma^*(f')$ $\{\sigma_*(f',b')\}$ *is obtained from* $\sigma^*(f)$ $\{\sigma_*(f,b)\}$ *by an elementary symmetric {quadratic} surgery of type* $(r,n-r-1)$ *killing the image of* $(h,g)\in\pi_{r+1}(f)$ *under the Hurewicz map*

$$\pi_{r+1}(f)=\pi_{r+1}(\tilde{f})\to H_{r+1}(\tilde{f})=K_r(M).$$

Proof. Let

$$\begin{cases} (e;\ f,f')\colon (N;\ M,M')\to(X\times I;\ X\times 0,X\times 1)\\ ((e;\ f,f'),(a;\ b,b'))\colon (N;\ M,M')\to(X\times I;\ X\times 0,X\times 1)\end{cases}$$

be the associated degree 1 {normal bundle} bordism, with symmetric {quadratic} kernel $(n+1)$-dimensional symmetric {quadratic} Poincaré pair over $\mathbf{Z}[\pi_1(X)]$

$$\begin{cases} \sigma^*(e;\ f,f')=((i\ \ i')\colon C(f^!)\oplus C(f'^!)\to C(e^!),(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((i\ \ i'))),\\ \sigma_*((e;\ f,f'),(a;\ b,b'))\\ \qquad\qquad =((i\ \ i')\colon C(f^!)\oplus C(f'^!)\to C(e^!),(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((i\ \ i'))).\end{cases}$$

Let $k\colon C(e^!)\to C(i')$ be the inclusion of $C(e^!)$ in the algebraic mapping cone of $i'\colon C(f'^!)\to C(e^!)$, which is such that $C(i')=S^{n-r}\mathbf{Z}[\pi_1(X)]$ up to chain equivalence. Use the chain homotopy commutative diagram of

$\mathbf{Z}[\pi_1(X)]$-module chain maps (with $j\colon ki(1\ 0) \simeq k(i\ i')$ any chain homotopy)

$$\begin{array}{ccc} C(f^!)\oplus C(f'^!) & \xrightarrow{(i\ i')} & C(e^!) \\ {\scriptstyle (1\ 0)}\big\downarrow & \searrow{\scriptstyle j} & \big\downarrow{\scriptstyle k} \\ C(f^!) & \xrightarrow{\bar{\imath} = ki} & C(i') = S^{n-r}\mathbf{Z}[\pi_1(X)] \end{array}$$

to define a relative $\mathbf{Z}_2$-hypercohomology $\{\mathbf{Z}_2$-hyperhomology$\}$ class

$$\begin{cases} (\overline{\delta\varphi},\varphi) = ((1\quad 0),k;j)^{\%}(\delta\varphi,\varphi\oplus-\varphi') \in Q^{n+1}(\bar{\imath}), \\ (\overline{\delta\psi},\psi) = ((1\quad 0),k;j)_{\%}(\delta\psi,\psi\oplus-\psi') \in Q_{n+1}(\bar{\imath}). \end{cases}$$

The symmetric $\{$quadratic$\}$ kernel

$$\begin{cases} \sigma^*(f') = (C(f'^!),\varphi' \in Q^n(C(f'^!))), \\ \sigma_*(f',b') = (C(f'^!),\psi' \in Q_n(C(f'^!))) \end{cases}$$

is obtained from $\sigma^*(f) = (C(f^!),\varphi)$ $\{\sigma_*(f,b) = (C(f^!),\psi)\}$ by an elementary symmetric $\{$quadratic$\}$ surgery on the $(n+1)$-dimensional symmetric $\{$quadratic$\}$ pair

$$\begin{cases} (\bar{\imath}\colon C(f^!) \to S^{n-r}\mathbf{Z}[\pi_1(X)], (\overline{\delta\varphi},\varphi) \in Q^{n+1}(\bar{\imath})), \\ (\bar{\imath}\colon C(f^!) \to S^{n-r}\mathbf{Z}[\pi_1(X)], (\overline{\delta\psi},\psi) \in Q_{n+1}(\bar{\imath})). \end{cases}$$

We have the following partial converse to Proposition 7.3.

PROPOSITION 7.4. *Let $(f,b)\colon M \to X$ be an n-dimensional normal bundle map with quadratic kernel $\sigma_*(f,b) = (C(f^!),\psi = e_{\%}\psi_F[X] \in Q_n(C(f^!)))$. If $f\colon M \to X$ is $(r-1)$-connected $(2r \leqslant n)$ and $n \geqslant 5$ it is possible to kill $x \in \pi_{r+1}(f) = K_r(M)$ by an elementary framed surgery if and only if it can be killed by an elementary quadratic surgery on $\sigma_*(f,b)$, that is if and only if*

$$v^r(\psi)(x) = 0 \in \hat{H}_{2r-n}(\mathbf{Z}_2;\ \mathbf{Z}[\pi_1(X)],(-)^{n-r}) \quad (= 0 \text{ if } 2r < n).$$

For the sake of completeness we shall now describe the effect on an elementary geometric surgery of a change in the framing of the embedded sphere. We recall that the investigation of such changes played an important part in the original proof that there is no surgery obstruction in the odd-dimensional simply-connected case (Kervaire and Milnor [**8**, §6], Browder [**3**, Chapter IV.3]).

Let $f\colon M \to X$ $\{(f,b)\colon M \to X\}$ be a degree 1 $\{$normal bundle$\}$ map from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X. Let $g\colon S^r \hookrightarrow M$ be an embedding with a null-homotopy $h\colon D^{r+1} \to X$ of $fg\colon S^r \to X$ on which it is possible to perform oriented {framed} surgery, and let

$$f'\colon M'\to X,\, f''\colon M''\to X \quad \{(f',b')\colon M'\to X,\ (f'',b'')\colon M''\to X\}$$

be the degree 1 {normal bundle} maps obtained from f $\{(f,b)\}$ by oriented {framed} surgery using two different extensions $\bar{g},\bar{\bar{g}}\colon S^r\times D^{n-r}\hookrightarrow M$ of g {and two different relative null-homotopies $(\bar{\nu}_h,\bar{\nu}_g)$, $(\bar{\bar{\nu}}_h,\bar{\bar{\nu}}_g)$ of

$$(\nu_h,\nu_g)\colon (D^{r+1},S^r)\to (BSO,BSO(n-r))\}.$$

The differences are measured by elements

$$\alpha\in\pi_r(SO(n-r)) \quad \{\beta\in\pi_{r+1}(SO/SO(n-r))\}.$$

The symmetric {quadratic} kernels $\sigma^*(f')$, $\sigma^*(f'')$ $\{\sigma_*(f',b'),\sigma_*(f'',b'')\}$ are obtained from the symmetric {quadratic} kernel

$$\begin{cases} \sigma^*(f) = (C(f^!),\varphi = e^{\%}\varphi_{\widetilde{M}}[M]\in Q^n(C(f^!))) \\ \sigma_*(f,b) = (C(f^!),\psi = e_{\%}\psi_F[X]\in Q_n(C(f^!))) \end{cases}$$

by elementary symmetric {quadratic} surgeries of type $(r,n-r-1)$ on the $(n+1)$-dimensional symmetric {quadratic} pairs

$$\begin{cases} (\bar{\imath}\colon C(f^!)\to S^{n-r}\mathbf{Z}[\pi_1(X)],\ (\overline{\delta\varphi},\varphi)\in Q^{n+1}(\bar{\imath})), \\ \qquad\qquad (\bar{\bar{\imath}}\colon C(f^!)\to S^{n-r}\mathbf{Z}[\pi_1(X)],\ (\overline{\overline{\delta\varphi}},\varphi)\in Q^{n+1}(\bar{\bar{\imath}})) \\ (\bar{\imath}\colon C(f^!)\to S^{n-r}\mathbf{Z}[\pi_1(X)],\ (\overline{\delta\psi},\psi)\in Q_{n+1}(\bar{\imath})), \\ \qquad\qquad (\bar{\bar{\imath}}\colon C(f^!)\to S^{n-r}\mathbf{Z}[\pi_1(X)],\ (\overline{\overline{\delta\psi}},\psi)\in Q_{n+1}(\bar{\bar{\imath}})) \end{cases}$$

defined in the proof of Proposition 7.3, with $\bar{\imath}=\bar{\bar{\imath}}\colon C(f^!)\to S^{n-r}\mathbf{Z}[\pi_1(X)]$. Now Proposition I.3.1 gives exact sequences

$$\begin{cases} Q^{n+1}(S^{n-r}\mathbf{Z}[\pi]) \xrightarrow{\ \gamma\ } Q^{n+1}(\bar{\imath}) \xrightarrow{\ \partial\ } Q^n(C(f^!)) \\ Q_{n+1}(S^{n-r}\mathbf{Z}[\pi]) \xrightarrow{\ \gamma\ } Q_{n+1}(\bar{\imath}) \xrightarrow{\ \partial\ } Q_n(C(f^!)) \end{cases} \quad (\pi=\pi_1(X)),$$

so that

$$\begin{cases} (\overline{\overline{\delta\varphi}},\varphi)-(\overline{\delta\varphi},\varphi)\in\ker(\partial)=\mathrm{im}(\gamma\colon Q^{n+1}(S^{n-r}\mathbf{Z}[\pi])\to Q^{n+1}(\bar{\imath})), \\ (\overline{\overline{\delta\psi}},\psi)-(\overline{\delta\psi},\psi)\in\ker(\partial)=\mathrm{im}(\gamma\colon Q_{n+1}(S^{n-r}\mathbf{Z}[\pi])\to Q_{n+1}(\bar{\imath})). \end{cases}$$

Next, recall from Proposition 5.1 the morphism

$$\begin{cases} j\colon \pi_r(SO(n-r))\to Q^{n+1}(S^{n-r}\mathbf{Z}), \\ j\colon \pi_{r+1}(SO/SO(n-r))\to Q_{n+1}(S^{n-r}\mathbf{Z}). \end{cases}$$

PROPOSITION 7.5. *The algebraic effect on an elementary oriented {framed} surgery of type* $(r,n-r-1)$ *of a change of framing by* $\alpha\in\pi_r(SO(n-r))$

$\{\beta \in \pi_{r+1}(SO/SO(n-r))\}$ *is given by*

$$\begin{cases} (\overline{\overline{\delta\varphi}},\varphi)-(\overline{\delta\varphi},\varphi) = \gamma(j(\alpha),0) \in Q^{n+1}(\bar{\imath}), \\ (\overline{\overline{\delta\psi}},\psi)-(\overline{\delta\psi},\psi) = \gamma(j(\beta),0) \in Q_{n+1}(\bar{\imath}), \end{cases}$$

with

$$\begin{cases} (j(\alpha),0) \in Q^{n+1}(S^{n-r}\mathbf{Z}) \oplus H^{n-2r-1}(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z},(-)^{n-r}) = Q^{n+1}(S^{n-r}\mathbf{Z}[\pi]), \\ (j(\beta),0) \in Q_{n+1}(S^{n-r}\mathbf{Z}) \oplus H_{2r-n+1}(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z},(-)^{n-r}) = Q_{n+1}(S^{n-r}\mathbf{Z}[\pi]). \end{cases}$$

8. Products

We shall now apply the L-theoretic product operations of §I.8 to obtain product formulae for the symmetric signatures of geometric Poincaré complexes, and for the quadratic signatures (surgery obstructions) of normal maps.

PROPOSITION 8.1. (i) *The symmetric signature of the cartesian product* $X \times Y$ *of geometric Poincaré complexes is*

$$\sigma^*(X \times Y) = \sigma^*(X) \otimes \sigma^*(Y) \in L^{m+n}(\mathbf{Z}[\pi_1(X \times Y)]),$$

where $m = \dim X$, $n = \dim Y$.

(ii) *The symmetric* {*quadratic*} *signature of the cartesian product* $f \times g\colon M \times N \to X \times Y$ $\{(f\times g, b \times c)\colon M \times N \to X \times Y\}$ *of degree* 1 {*normal*} *maps* $f\colon M \to X$, $g\colon N \to Y$ $\{(f,b)\colon M \to X,\ (g,c)\colon N \to Y\}$ *of geometric Poincaré complexes is*

$$\begin{cases} \sigma^*(f\times g) = \sigma^*(f)\otimes\sigma^*(g) + \sigma^*(X)\otimes\sigma^*(g) \\ \qquad\qquad + \sigma^*(f)\otimes\sigma^*(Y) \in L^{m+n}(\mathbf{Z}[\pi_1(X\times Y)]), \\ \sigma_*(f\times g, b\times c) = \sigma_*(f,b)\otimes\sigma_*(g,c) + \sigma^*(X)\otimes\sigma_*(g,c) \\ \qquad\qquad + \sigma_*(f,b)\otimes\sigma^*(Y) \in L_{m+n}(\mathbf{Z}[\pi_1(X\times Y)]). \end{cases}$$

Proof. (i) Choose a functorial diagonal chain approximation Δ. The standard acyclic model proof of the Eilenberg–Zilber theorem gives a functorial chain equivalence on the category (topological spaces) × (topological spaces)

$$h_{X,Y}\colon C(X\times Y) \to C(X)\otimes_{\mathbf{Z}} C(Y)$$

and the acyclic model argument underlying the Cartan product formula for the Steenrod squares gives a functorial chain homotopy

$$k_{X,Y}\colon \Delta^*(\Delta_X \otimes \Delta_Y)h_{X,Y} \simeq h^{\%}_{X,Y}\Delta_{X\times Y}$$

in the diagram

$$\begin{array}{ccc}
C(X\times Y) & \xrightarrow{\quad h_{X,Y}\quad} & C(X)\otimes_{\mathbf{Z}} C(Y) \\
 & \searrow{\scriptstyle k_{X,Y}} & \big\downarrow{\scriptstyle \Delta_X\otimes\Delta_Y} \\
\big\downarrow{\scriptstyle \Delta_{X\times Y}} & & \begin{array}{c}\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X)\otimes_{\mathbf{Z}} C(X)) \\ \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(Y)\otimes_{\mathbf{Z}} C(Y))\end{array} \\
 & & \big\downarrow{\scriptstyle \Delta^*} \\
\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C(X\times Y)\otimes_{\mathbf{Z}} C(X\times Y)) & \xrightarrow{h^{\%}_{X,Y}} & \begin{array}{c}\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, (C(X)\otimes_{\mathbf{Z}} C(Y)) \\ \otimes_{\mathbf{Z}} (C(X)\otimes_{\mathbf{Z}} C(Y)))\end{array}
\end{array}$$

with $\Delta\colon W \to W\otimes_{\mathbf{Z}} W$ an algebraic diagonal approximation for $W = C(S^\infty)$. The product of an m-dimensional geometric Poincaré complex X and an n-dimensional geometric Poincaré complex Y is an $(m+n)$-dimensional geometric Poincaré complex $X\times Y$, with orientation map

$$w(X\times Y) = w(X)\times w(Y)\colon \pi_1(X\times Y) = \pi_1(X)\times\pi_1(Y) \to \mathbf{Z}_2$$

and fundamental class

$$\begin{aligned}[X\times Y] = [X]\otimes[Y] &\in H^{\pi_1(X\times Y)}_{m+n}(\tilde{X}\times\tilde{Y};\, {}^{w(X\times Y)}\mathbf{Z}) \\ &= H^{\pi_1(X)}_{m}(\tilde{X};\, {}^{w(X)}\mathbf{Z})\otimes_{\mathbf{Z}} H^{\pi_1(Y)}_{n}(\tilde{Y};\, {}^{w(Y)}\mathbf{Z}),\end{aligned}$$

where $\tilde{X}$, $\tilde{Y}$ are the universal covers of X, Y. It now follows from the chain homotopy invariance of the Q-groups that there is defined a homotopy equivalence of $(m+n)$-dimensional symmetric Poincaré complexes over $\mathbf{Z}[\pi_1(X\times Y)] = \mathbf{Z}[\pi_1(X)]\otimes_{\mathbf{Z}}\mathbf{Z}[\pi_1(Y)]$,

$$\begin{aligned}h_{\tilde{X},\tilde{Y}}\colon \sigma^*(X\times Y) &= (C(\tilde{X}\times\tilde{Y}), \varphi_{\tilde{X}\times\tilde{Y}}[X\times Y]\in Q^{m+n}(C(\tilde{X}\times\tilde{Y}))) \\ &\to \sigma^*(X)\otimes\sigma^*(Y) \\ &= (C(\tilde{X})\otimes_{\mathbf{Z}} C(\tilde{Y}), \varphi_{\tilde{X}}[X]\otimes\varphi_{\tilde{Y}}[Y]\in Q^{m+n}(C(\tilde{X})\otimes_{\mathbf{Z}} C(\tilde{Y}))),\end{aligned}$$

and the homotopy invariance of symmetric Poincaré cobordism gives

$$\sigma^*(X\times Y) = \sigma^*(X)\otimes\sigma^*(Y)\in L^{m+n}(\mathbf{Z}[\pi_1(X\times Y)]).$$

(ii) Consider first the special case of the product $f\times 1\colon M\times N\to X\times N$ $\{(f\times 1, b\times 1)\colon M\times N\to X\times N\}$ of an m-dimensional degree 1 {normal} map $f\colon M\to X$ $\{(f,b)\colon M\to X\}$ with an n-dimensional geometric Poincaré complex N. Given an Umkehr chain map for $f\colon M\to X$

$$f^!\colon C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\tilde{X})^{m-*} \xrightarrow{\;f^*\;} {}^{w(X)}C(\tilde{M})^{m-*} \xrightarrow{[M]\cap -} C(\tilde{M})$$

there is defined an Umkehr chain map for $f\times 1: M\times N\to X\times N$

$$(f\times 1)^!: C(\tilde{X}\times\tilde{N}) \xrightarrow{h_{\tilde{X},\tilde{N}}} C(\tilde{X})\otimes_{\mathbf{Z}} C(\tilde{N}) \xrightarrow{f^!\otimes 1} C(\tilde{M})\otimes_{\mathbf{Z}} C(\tilde{N}) \xrightarrow{h^{-1}_{\tilde{M},\tilde{N}}} C(\tilde{M}\times\tilde{N}).$$

There are also defined a chain equivalence

$$h'_{\tilde{M},\tilde{N}}: C((f\times 1)^!)\to C(f^!\otimes 1) = C(f^!)\otimes_{\mathbf{Z}} C(\tilde{N})$$

and a chain homotopy commutative diagram

$$\begin{array}{ccc} C(\tilde{M}\times\tilde{N}) & \xrightarrow{e_{\tilde{f}\times 1}} & C((f\times 1)^!) \\ \downarrow h_{\tilde{M},\tilde{N}} & & \downarrow h'_{\tilde{M},\tilde{N}} \\ C(\tilde{M})\otimes_{\mathbf{Z}} C(\tilde{N}) & \xrightarrow{e_{\tilde{f}}\otimes 1} & C(f^!)\otimes_{\mathbf{Z}} C(\tilde{N}) \end{array}$$

with $e_{\tilde{f}}$, $e_{\tilde{f}\times 1}$ the inclusions. The homotopy equivalence of $(m+n)$-dimensional symmetric Poincaré complexes over $\mathbf{Z}[\pi_1(X\times N)]$

$$h'_{\tilde{M},\tilde{N}}: \sigma^*(f\times 1) = (C((f\times 1)^!), e^{\%}_{\tilde{f}\times 1}\varphi_{\tilde{M}\times\tilde{N}}[M\times N]) \to \sigma^*(f)\otimes\sigma^*(N) = (C(f^!)\otimes_{\mathbf{Z}} C(\tilde{N}), e^{\%}_{\tilde{f}}\varphi_{\tilde{M}}[M]\otimes\varphi_{\tilde{N}}[N])$$

implies that

$$\sigma^*(f\times 1) = \sigma^*(f)\otimes\sigma^*(N)\in L^{m+n}(\mathbf{Z}[\pi_1(X\times N)]).$$

Furthermore, if $F: \Sigma^p\tilde{X}_+\to\Sigma^p\tilde{M}_+$ is a geometric Umkehr map for $(f,b): M\to X$ then

$$F\wedge 1: \Sigma^p(\tilde{X}\times\tilde{N})_+ = \Sigma^p\tilde{X}_+\wedge\tilde{N}_+\to\Sigma^p(\tilde{M}\times\tilde{N})_+ = \Sigma^p\tilde{M}_+\wedge\tilde{N}_+$$

is a geometric Umkehr map for $(f\times 1, b\times 1): M\times N\to X\times N$, with quadratic construction

$$\psi_{F\wedge 1}: H^{\pi_1(X\times N)}_{m+n}(\tilde{X}\times\tilde{N};\, {}^{w(X\times N)}\mathbf{Z})\to Q_{m+n}(C(\tilde{M}\times\tilde{N}))$$

such that

$$h_{\tilde{M},\tilde{N}\%}\psi_{F\wedge 1}[X\times N] = \psi_F[X]\otimes\varphi_{\tilde{N}}[N]\in Q_{m+n}(C(\tilde{M})\otimes_{\mathbf{Z}} C(\tilde{N})).$$

The homotopy equivalence of $(m+n)$-dimensional quadratic Poincaré complexes over $\mathbf{Z}[\pi_1(X\times N)]$,

$$h'_{\tilde{M},\tilde{N}}: \sigma_*(f\times 1, b\times 1) = (C((f\times 1)^!), e_{\tilde{f}\times 1\%}\psi_{F\wedge 1}[X\times N]) \to \sigma_*(f,b)\otimes\sigma^*(N) = (C(f^!)\otimes_{\mathbf{Z}} C(\tilde{N}), e_{\tilde{f}\%}\psi_F[X]\otimes\varphi_{\tilde{N}}[N]),$$

implies that

$$\sigma_*(f\times 1, b\times 1) = \sigma_*(f,b)\otimes\sigma^*(N)\in L_{m+n}(\mathbf{Z}[\pi_1(X\times N)]).$$

In the general case express the product degree 1 {normal} map as the composite

$$\begin{cases} f\times g\colon M\times N \xrightarrow{f\times 1} X\times N \xrightarrow{1\times g} X\times Y, \\ (f\times g, b\times c)\colon M\times N \xrightarrow{(f\times 1, b\times 1)} X\times N \xrightarrow{(1\times g, 1\times c)} X\times Y, \end{cases}$$

and apply the sum formula of Proposition 2.5 {Proposition 4.3} to obtain that

$$\begin{cases} \sigma^*(f\times g) = \sigma^*(f\times 1)\oplus \sigma^*(1\times g) \\ \qquad = \sigma^*(f)\otimes\sigma^*(N)\oplus\sigma^*(X)\otimes\sigma^*(g), \\ \sigma_*(f\times g, b\times c) = \sigma_*(f\times 1, b\times 1)\oplus\sigma_*(1\times g, 1\times c) \\ \qquad = \sigma_*(f,b)\otimes\sigma^*(N)\oplus\sigma^*(X)\otimes\sigma_*(g,c) \end{cases}$$

up to homotopy equivalence. Now $\sigma^*(N) = \sigma^*(g)+\sigma^*(Y)\in L^n(\mathbf{Z}[\pi_1(Y)])$ (by Proposition 2.2), so that

$$\begin{cases} \sigma^*(f\times g) = \sigma^*(f)\otimes\sigma^*(g)+\sigma^*(X)\otimes\sigma^*(g) \\ \qquad +\sigma^*(f)\otimes\sigma^*(Y)\in L^{m+n}(\mathbf{Z}[\pi_1(X\times Y)]), \\ \sigma_*(f\times g, b\times c) = \sigma_*(f,b)\otimes\sigma_*(g,c)+\sigma^*(X)\otimes\sigma_*(g,c) \\ \qquad +\sigma_*(f,b)\otimes\sigma^*(Y)\in L_{m+n}(\mathbf{Z}[\pi_1(X\times Y)]). \end{cases}$$

The product formula for symmetric signatures of Propositions 8.1(i) is a generalization of the classical product formula for the signature.

The product formula for surgery obstructions (quadratic signatures) of Proposition 8.1(ii) is a common generalization of the product formulae of Sullivan (for $\pi_1(X) = \{1\}$, $\pi_1(Y) = \{1\}$, proved by Browder in [**3**, Chapter III]), Williamson [**32**], Shaneson [**23**], and Morgan [**19**] (all for $\pi_1(X) = \{1\}$, $f = 1\colon M\to X = M$).

PROPOSITION 8.2. *The periodicity isomorphism in the ε-quadratic L-groups is defined by the product with* $\sigma^*(\mathbf{C}P^2)\in L^4(\mathbf{Z})$,

$$\bar{S}^2 = \sigma^*(\mathbf{C}P^2)\otimes - : L_n(A,\varepsilon)\to L_{n+4}(A,\varepsilon) \quad (\mathbf{Z}\otimes_{\mathbf{Z}} A = A,\ n\in\mathbf{Z}).$$

Proof. Removing the fundamental class $[\mathbf{C}P^2]\in H_4(\mathbf{C}P^2)$ by symmetric surgery represent $\sigma^*(\mathbf{C}P^2)\in L^4(\mathbf{Z})$ by the 4-dimensional symmetric Poincaré complex over $\mathbf{Z}$, $(C, \varphi\in Q^4(C))$, defined by

$$C_r = \begin{cases} \mathbf{Z} & \text{if } r = 2 \\ 0 & \text{if } r\neq 2, \end{cases} \qquad \varphi_0 = 1\colon C^2\to C_2.$$

The algebraic 4-periodicity in the ε-quadratic L-groups is thus seen to correspond to the geometrically defined 4-periodicity of surgery obstructions

$$L_n(\pi) \to L_{n+4}(\pi);$$
$$\sigma_*((f,b)\colon M \to X) \mapsto \sigma_*((f\times 1, b\times 1)\colon M\times \mathbf{C}P^2 \to X\times \mathbf{C}P^2)$$

of Wall [**30**, Theorem 9.9].

9. Wu classes

We shall now use the equivariant S-duality of § 3 to describe the extent to which the n-dimensional symmetric Poincaré complex

$$\sigma^*(X) = (C(\tilde{X}), \varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X})))$$

of an n-dimensional geometric Poincaré complex X reflects the properties of the Spivak stable normal fibration $\nu_X\colon X \to BG$. For any stable spherical fibration $p\colon X \to BG$ over any n-dimensional CW complex X we shall construct an n-dimensional hyperquadratic complex over $\mathbf{Z}[\pi_1(X)]$

$$\hat{\sigma}^*(p) = (C(\tilde{X})^{n-*}, \theta_p \in \hat{Q}^n(C(\tilde{X})^{n-*})).$$

The hyperquadratic Wu classes $\hat{v}_r(\theta_p)\colon H_r(\tilde{X}) \to \hat{H}^r(\mathbf{Z}_2; \mathbf{Z}[\pi_1(X)])$ of $\hat{\sigma}^*(p)$ are equivariant analogues of the Wu classes $v_r(p) \in H^r(X; \mathbf{Z}_2)$. In particular, for the Spivak normal fibration $\nu_X\colon X \to BG$ of a geometric Poincaré complex X there is a natural identification

$$J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$$

(up to homotopy equivalence), giving rise to an equivariant analogue of the classical Wu formula

$$v_r(X) = v_r(\nu_X) \in H^r(X; \mathbf{Z}_2)$$

relating the diagonal structure of $C(X; \mathbf{Z}_2)$ to ν_X. The quadratic structure in the kernel $\sigma_*(f,b)$ of a normal map $(f\colon M \to X, b\colon \nu_M \to \nu_X)$ expresses the vanishing of the equivariant Wu classes of ν_M on

$$K_*(M) = \ker(\tilde{f}_*\colon H_*(\tilde{M}) \to H_*(\tilde{X})).$$

We shall also develop equivariant analogues of the suspended Wu classes $(\sigma v)_r(h) \in H^{r-1}(X; \mathbf{Z}_2)$ of the automorphisms $h\colon p \to p$ of a stable spherical fibration $p\colon X \to BG$ (over any CW complex X). We shall use them to describe the effect on the quadratic kernel $\sigma_*(f,b)$ of a normal map $(f,b)\colon M \to X$ of a change in the bundle map $b\colon \nu_M \to \nu_X$.

To the symmetric and quadratic constructions of § 1

$$\varphi_X\colon H_*(X) \to Q^*(C(X)), \quad \psi_F\colon H_*(X) \to Q_*(C(Y)) \quad (F \in \{X, Y\})$$

we now add the 'hyperquadratic construction'

$$\theta_X\colon H^*(X) \to \hat{Q}^*(C(X)^*).$$

This is defined to be the composite

$$\theta_X\colon H^*(X) = H_*(Y) \xrightarrow{\varphi_Y} Q^*(C(Y)) = Q^*(C(X)^*) \xrightarrow{J} \hat{Q}^*(C(X)^*)$$

for any S- (or $S\pi$-)dual Y of X. For example, if $\nu_X\colon X \to BG(k)$ is a Spivak normal fibration of an n-dimensional geometric Poincaré complex X, and $\tilde{X}$ is an oriented covering of X with data (π, w), then the $S\pi$-duality between $\tilde{X}_+$ and $T\pi(\nu_X)$ obtained in Proposition 4.1 expresses the stable symmetric construction $J\varphi_{\tilde{X}}$ on the fundamental class $[X] \in H_n^\pi(\tilde{X}; {}^w\mathbf{Z})$ in terms of the hyperquadratic construction $\theta_{T\pi(\nu_X)}$ on the Thom class $U_{\nu_X} \in \dot{H}_\pi^k(T\pi(\nu_X); {}^w\mathbf{Z})$.

Given a group π and an $S\pi$-duality map $\alpha\colon S^N \to X \wedge_\pi Y$ between finitely-dominated $CW\pi$-complexes X, Y there is defined a chain equivalence of finite-dimensional $R[\pi]$-module chain complexes

$$\chi_\alpha = (\alpha_*[S^N]\backslash -)^{-1}\colon \dot{C}(Y; R) \to \dot{C}(X; R)^{N-*}$$

for any commutative coefficient ring R, which is obtained by applying $R[\pi]\otimes_{\mathbf{Z}[\pi]}-$ to the $\mathbf{Z}[\pi]$-module chain equivalence $\chi_\alpha\colon \dot{C}(Y) \to \dot{C}(X)^{N-*}$ given by Proposition 3.8. Given a group morphism $w\colon \pi \to \mathbf{Z}_2$ endow $R[\pi]$ with the w-twisted involution, and define an $R[\pi]$-module chain map

$$\begin{aligned}
\theta_{X,\alpha}\colon & \operatorname{Hom}_{R[\pi]}(\dot{C}(X; R), {}^wR) \\
& \xrightarrow{\chi_\alpha^*} \operatorname{Hom}_{R[\pi]}(\dot{C}(Y; R)^{N-*}, {}^wR) = R^t \otimes_{R[\pi]} \dot{C}(Y; R)_{N-*} \\
& \xrightarrow{\dot{\varphi}_Y = 1 \otimes \dot{\Delta}_Y} \operatorname{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \dot{C}(Y; R)^t \otimes_{R[\pi]} \dot{C}(Y; R))_{N-*} \\
& \xrightarrow{\chi_\alpha^\%} \operatorname{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, (\dot{C}(X; R)^{N-*})^t \otimes_{R[\pi]} \dot{C}(X; R)^{N-*})_{N-*}.
\end{aligned}$$

This induces R-module morphisms in homology

$$\theta_{X,\alpha}\colon \dot{H}_\pi^k(X; {}^wR) \to Q^{N-k}(\dot{C}(X; R)^{N-*})$$

such that there is defined a commutative diagram of R-module morphisms

$$\begin{array}{ccc}
\dot{H}_n^\pi(Y; {}^wR) & \xrightarrow{\dot{\varphi}_Y} & Q^n(\dot{C}(Y; R)) \\
\Big\downarrow{\scriptstyle \chi_\alpha} & & \Big\downarrow{\scriptstyle \chi_\alpha^\%} \\
\dot{H}_\pi^{N-n}(X; {}^wR) & \xrightarrow{\theta_{X,\alpha}} & Q^n(\dot{C}(X; R)^{N-*})
\end{array}$$

with $\dot{\varphi}_Y$ the symmetric construction of Proposition 1.2. Thus for the $S\pi$-duality map $\alpha_X: S^{n+k} \to \tilde{X}_+ \wedge_\pi T\pi(\nu_X)$ associated to a normalized n-dimensional geometric Poincaré complex

$$(X, \nu_X: X \to BG(k), \rho_X \in \pi_{n+k}(T(\nu_X)))$$

and an oriented covering $\tilde{X}$ with data (π, w) we have a commutative diagram

$$\begin{array}{ccc} H_n^\pi(\tilde{X};\, {}^wR) & \xrightarrow{\quad \varphi_{\tilde{X}} \quad} & Q^n(C(\tilde{X};\, R)) \\ \Big\downarrow{\scriptstyle \chi_{\alpha_X}*} & & \Big\downarrow{\scriptstyle \chi^{\%}_{\alpha_X}} \\ \dot{H}^k_\pi(T\pi(\nu_X);\, {}^wR) & \xrightarrow{\theta_{T\pi(\nu_X),\alpha_X}} & Q^n(\dot{C}(T\pi(\nu_X);\, R)^{n+k-*}) \end{array}$$

using the untwisted dual $R[\pi]$-module structure in $\dot{C}(T\pi(\nu_X);\, R)^{n+k-*}$. Evaluating on the fundamental class $[X] \in H_n^\pi(\tilde{X};\, {}^wR)$ and using the isomorphism $\chi^{\%}_{\alpha_X}$ as an identification we can write

$$\varphi_{\tilde{X}}[X] = \theta_{T\pi(\nu_X),\alpha_X}(U_{\nu_X}) \in Q^n(C(\tilde{X};\, R)),$$

with $U_{\nu_X} \in \dot{H}^k_\pi(T\pi(\nu_X);\, {}^wR)$ the Thom class of ν_X. (We are using only the orientability of X with coefficients in R here.)

We shall now show that for a fixed finitely-dominated $CW\pi$-complex X the composite

$$\dot{H}^k_\pi(X;\, {}^wR) \xrightarrow{\theta_{X,\alpha}} Q^{N-k}(\dot{C}(X;\, R)^{N-*}) \xrightarrow{\ J\ } \hat{Q}^{-k}(\dot{C}(X;\, R)^{-*})$$

is independent of the $S\pi$-duality map $\alpha: S^N \to X \wedge_\pi Y$, with J as in Proposition I.1.2. We have the following *hyperquadratic construction.*

PROPOSITION 9.1. *Let π be a group, $w: \pi \to \mathbf{Z}_2$ a group morphism, R a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution.*

Given a finitely-dominated $CW\pi$-complex X there are defined in a natural way R-module morphisms

$$\theta_X: \dot{H}^k_\pi(X;\, {}^wR) \to \hat{Q}^{-k}(\dot{C}(X;\, R)^{-*}) \quad (k \geqslant 0)$$

with the untwisted dual $R[\pi]$-module structure on $\dot{C}(X;\, R)^{-*}$, such that

(i) *if $\alpha: S^N \to X \wedge_\pi Y$ is an $S\pi$-duality map there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc} & \overset{\theta_{X,\alpha}}{\nearrow} & Q^{N-k}(\dot{C}(X;\, R)^{N-*}) \\ \dot{H}^k_\pi(X;\, {}^wR) & & \Big\downarrow{\scriptstyle J} \\ & \underset{\theta_X}{\searrow} & \hat{Q}^{-k}(\dot{C}(X;\, R)^{-*}) \end{array}$$

(ii) *if $f: X \to Y$ is a π-map of finitely-dominated $CW\pi$-complexes then there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc} \dot{H}^k_\pi(Y;{}^wR) & \xrightarrow{\theta_Y} & \hat{Q}^{-k}(\dot{C}(Y;R)^{-*}) \\ \Big\downarrow{f^*} & & \Big\downarrow{\hat{f}^{*\%}} \\ \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\theta_X} & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \end{array}$$

(iii) *the construction is invariant under suspension, in that there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc} \dot{H}^{k+1}_\pi(\Sigma X;{}^wR) & \xrightarrow{\theta_{\Sigma X}} & \hat{Q}^{-k-1}(\dot{C}(\Sigma X;R)^{-*}) \\ \Big\downarrow{\Sigma^*_X} & & \Big\downarrow{\hat{\Sigma}^{*\%}_X} \\ \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\theta_X} & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \end{array}$$

in which the vertical maps are the suspension isomorphisms,

(iv) *if $h: R \to S$ is a morphism of commutative rings, there is defined a commutative diagram of R-modules*

$$\begin{array}{ccc} \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\theta_X} & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \\ \Big\downarrow{h} & & \Big\downarrow{h} \\ \dot{H}^k_\pi(X;{}^wS) & \xrightarrow{\theta_X} & \hat{Q}^{-k}(\dot{C}(X;S)^{-*}) \end{array}$$

in which the vertical maps are the change of rings $h: R[\pi] \to S[\pi]$.

Proof. If $\alpha: S^N \to X \wedge_\pi Y$ is an $S\pi$-duality map then so is

$$\Sigma\alpha: S^{N+1} \to X \wedge_\pi \Sigma Y,$$

and Proposition 1.4 gives a commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\theta_{X,\alpha}} & \hat{Q}^{N-k}(\dot{C}(X;R)^{N-*}) \\ \dot{H}^k_\pi(X;{}^wR) & & \Big\downarrow{S} \\ & \xrightarrow{\theta_{X,\Sigma\alpha}} & \hat{Q}^{N-k+1}(\dot{C}(X;R)^{N+1-*}) \end{array}$$

If $\alpha': S^N \to X \wedge_\pi Y'$ is another $S\pi$-duality map let $F \in \{Y, Y'\}_\pi$ be the image of $1 \in \{X, X\}_\pi$ under the $S\pi$-duality isomorphism

$$\{X, X\}_\pi \xrightarrow{(\alpha'\backslash -)} \{S^N, Y' \wedge_\pi X\} \xrightarrow{(\alpha\backslash -)^{-1}} \{Y, Y'\}_\pi.$$

Applying the quadratic construction of Proposition 1.5 we obtain

$$\psi_F : \dot{H}^\pi_{N-k}(Y; {}^wR) \to Q_{N-k}(\dot{C}(Y'; R))$$

such that

$$F^\%\dot\varphi_Y - \dot\varphi_{Y'}F_* = (1+T)\psi_F : \dot{H}^\pi_{N-k}(Y; {}^wR) \to Q^{N-k}(\dot{C}(Y'; R)).$$

The composite

$$Q_{N-k}(\dot{C}(Y'; R)) \xrightarrow{1+T} Q^{N-k}(\dot{C}(Y'; R)) \xrightarrow{J} \hat{Q}^{N-k}(\dot{C}(Y'; R))$$

is 0 (Proposition I.1.2), so that there is defined a commutative diagram

$$\begin{array}{ccccccc}
\dot{H}^k_\pi(X; {}^wR) & & & \xrightarrow{J\theta_{X,\alpha}} & & & \hat{Q}^{-k}(\dot{C}(X; R)^{-*}) \\
 & \searrow^{\chi_{\alpha*}} & & & & \nearrow^{\hat{\chi}^\%_\alpha} & \\
\Big\downarrow 1 & & \dot{H}^\pi_{N-k}(Y; {}^wR) \xrightarrow{\dot\varphi_Y} & Q^{N-k}(\dot{C}(Y; R)) & \xrightarrow{J} \hat{Q}^{N-k}(\dot{C}(Y; R)) & & \Big\downarrow 1 \\
 & & \Big\downarrow F_* & & \Big\downarrow \hat{F}^\% & & \\
 & & \dot{H}^\pi_{N-k}(Y'; {}^wR) \xrightarrow{\dot\varphi_{Y'}} & Q^{N-k}(\dot{C}(Y'; R)) & \xrightarrow{J} \hat{Q}^{N-k}(\dot{C}(Y'; R)) & & \\
 & \nearrow^{\chi_{\alpha'*}} & & & & \searrow^{\hat{\chi}^\%_{\alpha'}} & \\
\dot{H}^k_\pi(X; {}^wR) & & & \xrightarrow{J\theta_{X,\alpha'}} & & & \hat{Q}^{-k}(\dot{C}(X; R)^{-*})
\end{array}$$

Thus

$$J\theta_{X,\alpha} = J\theta_{X,\alpha'} : \dot{H}^k_\pi(X; {}^wR) \to \hat{Q}^{-k}(\dot{C}(X; R)^{-*})$$

is independent of the $S\pi$-duality maps involved, and may be written as θ_X.

Applying the hyperquadratic Wu class operations $\hat{v}_r$ of §I.1 to the hyperquadratic construction for $\pi = \{1\}$, $R = \mathbf{Z}_2$, we recover the duals of the Steenrod squares.

PROPOSITION 9.2. *Let X be a finitely-dominated CW complex. The composite*

$$\dot{H}^k(X; \mathbf{Z}_2) \xrightarrow{\theta_X} \hat{Q}^{-k}(\dot{C}(X; \mathbf{Z}_2)^{-*}) \xrightarrow{\hat{v}_r} \mathrm{Hom}_{\mathbf{Z}_2}(\dot{H}_{k+r}(X; \mathbf{Z}_2), \mathbf{Z}_2)$$

is given by

$$\hat{v}_r(\theta_X(x))(y) = \langle \chi(Sq^r)(x), y\rangle \in \mathbf{Z}_2 \quad (x \in \dot{H}^k(X; \mathbf{Z}_2), y \in \dot{H}_{k+r}(X; \mathbf{Z}_2))$$

with $\chi(Sq^r)$ the image of Sq^r under the canonical anti-automorphism χ of the mod 2 *Steenrod algebra, as characterized by*

$$\sum_{i+j=r} \chi(Sq^i)Sq^j = \begin{cases} Sq^0 & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases}$$

Proof. Apply Proposition 1.3 to an S-dual Y of X, and use the result of Thom [**28**] that Steenrod squares in Y correspond to the duals of the Steenrod squares in X.

We shall say that a space X is *n-dimensional* if it is a finitely-dominated CW complex and the universal cover $\tilde{X}$ is such that $H^r(\tilde{X}) = 0$ for $r > n$, in which case $C(\tilde{X})$ is an n-dimensional $\mathbf{Z}[\pi_1(X)]$-module chain complex. In particular, an n-dimensional geometric Poincaré complex is an n-dimensional space.

The hyperquadratic construction associates a hyperquadratic complex to every oriented covering of the base space of a stable spherical fibration over a finite-dimensional space.

PROPOSITION 9.3. *Given a stable spherical fibration $p\colon X \to BG$ over an n-dimensional space X and an oriented covering $\tilde{X}$ with data (π, w), and given also a commutative ring R, there is defined in a natural way an n-dimensional hyperquadratic complex over $R[\pi]$ with the w-twisted involution, the* Wu complex *of p,*

$$\hat{\sigma}^*(p) = ({}^wC(\tilde{X}\,;\,R)^{n-*}, \theta_{T\pi(p)}(U_p) \in \hat{Q}^n({}^wC(\tilde{X}\,;\,R)^{n-*}))$$

depending only on the stable fibre homotopy class of p.

The hyperquadratic Wu classes of $\sigma^*(p)$ are the Wu classes *of p, $R[\pi]$-module morphisms*

$$v_r(p) = \hat{v}_r(\theta_{T\pi(p)}(U_p))\colon H_r(\tilde{X}\,;\,R) \to \hat{H}^r(\mathbf{Z}_2\,;\,R[\pi]) \quad (r \geqslant 0)$$

such that

(i) *the 0th Wu class is the augmentation map*

$$v_0(p)\colon H_0(\tilde{X}\,;\,R) \to \hat{H}^0(\mathbf{Z}_2\,;\,R[\pi])\,;$$

$$\sum_{g\in\pi} n_g gx \mapsto \sum_{g\in\pi} n_g \in R/2R = \hat{H}^0(\mathbf{Z}_2\,;\,R) \subseteq \hat{H}^0(\mathbf{Z}_2\,;\,R[\pi])$$

with $x \in H_0(\tilde{X}\,;\,R)$ the geometric $R[\pi]$-module generator defined by any path-component of $\tilde{X}$,

(ii) *if $f\colon M \to X$ is a map of n-dimensional spaces with induced cover $\tilde{M}$ and pullback fibration $f^*p\colon M \xrightarrow{\;f\;} X \xrightarrow{\;p\;} BG$ then there is defined a map of Wu complexes*

$$\tilde{f}^*\colon \hat{\sigma}^*(p) \to \hat{\sigma}^*(f^*p)$$

and the Wu classes are such that there is defined a commutative diagram

$$\begin{array}{ccc} H_r(\tilde{M}\,;\,R) & \xrightarrow{\quad \tilde{f}_* \quad} & H_r(\tilde{X}\,;\,R) \\ & {\scriptstyle v_r(f^*p)}\searrow \quad \swarrow{\scriptstyle v_r(p)} & \\ & \hat{H}^r(\mathbf{Z}_2\,;\,R[\pi]) & \end{array}$$

(iii) *if* $p\colon X \to BG$ *is stably fibre homotopy trivial then*

$$v_r(p) = 0\colon H_r(\tilde{X}; R) \to \hat{H}^r(\mathbf{Z}_2; R[\pi]) \quad (r > 0),$$

(iv) $\hat{\sigma}^*(p)$ *is induced via* $R[\pi]\otimes_{R[\pi_1(X)]}-$ *from the Wu complex* $\hat{\sigma}^*(\tilde{\tilde{p}})$ *associated to the universal cover* $\tilde{\tilde{X}}$, *and there is defined a commutative diagram*

$$\begin{array}{ccc} H_r(\tilde{\tilde{X}}; R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; R[\pi_1(X)]) \\ \downarrow & & \downarrow \\ H_r(\tilde{X}; R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; R[\pi]) \end{array}$$

in which the vertical maps are the change of rings $R[\pi_1(X)] \to R[\pi]$, *with* $\pi_1(X) \to \pi$ *the characteristic map of the covering,*

(v) *if* $h\colon R \to S$ *is a morphism of commutative rings there is defined a commutative diagram*

$$\begin{array}{ccc} H_r(\tilde{X}; R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; R[\pi]) \\ h\downarrow & & \downarrow h \\ H_r(\tilde{X}; S) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; S[\pi]) \end{array}$$

in which the vertical maps are the change of rings $h\colon R[\pi] \to S[\pi]$.

Proof. Choose a representative $(k-1)$-spherical fibration $p\colon X \to BG(k)$, evaluate the hyperquadratic construction

$$\theta_{T\pi(p)}\colon \dot{H}^k_\pi(T\pi(p); {}^wR) \to \hat{Q}^{-k}(\dot{C}(T\pi(p); R)^{-*}) = \hat{Q}^n(\dot{C}(T\pi(p); R)^{n+k-*})$$

on the Thom class $U_p \in \dot{H}^k_\pi(T\pi(p); {}^wR)$, and use the Thom equivalence

$$U_p \cap - : \dot{C}(T\pi(p); R) \to {}^wS^kC(\tilde{X}; R)$$

to obtain an element $\theta_{T\pi(p)}(U_p) \in \hat{Q}^n({}^wC(\tilde{X}; R)^{n-*})$.

To prove (iii), that $v_r(0) = 0$ $(r > 0)$, let Y be a skeleton of $K(\pi, 1)$ of dimension greater than r containing the image of the classifying map $f\colon X \to K(\pi, 1)$ of the covering $\tilde{X}$ (assuming that X is finite, in the first instance), and apply the naturality property (ii) to obtain a commutative diagram

$$\begin{array}{ccc} H_r(\tilde{X}; R) & \xrightarrow{\tilde{g}_*} & H_r(\tilde{Y}; R) = 0 \\ & {\scriptstyle v_r(g^*0) = v_r(0)}\searrow \quad \swarrow{\scriptstyle v_r(0)} & \\ & \hat{H}^r(\mathbf{Z}_2; R[\pi]) & \end{array} \qquad (g = f| : X \to Y)$$

The mod 2 *Stiefel–Whitney classes* $w_*(p) \in H^*(X; \mathbf{Z}_2)$ of a spherical fibration $p: X \to BG(k)$ are characterized by the property

$$U_p \cap w_j(p) = Sq^j(U_p) \in \dot{H}^{j+k}(T(p); \mathbf{Z}_2)$$

(Thom [28]), which may be expressed in terms of the symmetric construction and the symmetric Wu classes as

$$w_j(p): H_j(X; \mathbf{Z}_2) \xrightarrow{(U_p \cap -)^{-1}} \dot{H}_{j+k}(T(p); \mathbf{Z}_2) \xrightarrow{\dot{\varphi}_{T(p)}} Q^{j+k}(\dot{C}(T(p); \mathbf{Z}_2))$$
$$\xrightarrow{v_j} \operatorname{Hom}_{\mathbf{Z}_2}(\dot{H}^k(T(p); \mathbf{Z}_2), H^{k-j}(\mathbf{Z}_2; \mathbf{Z}_2)) = \begin{cases} \mathbf{Z}_2 & \text{if } j \leqslant k \\ 0 & \text{if } j > k \end{cases}$$

(cf. Proposition 1.3), with $U_p \in \dot{H}^k(T(p); \mathbf{Z}_2)$ the mod 2 Thom class.

The mod 2 *Wu classes* $v_*(p) \in H^*(X; \mathbf{Z}_2)$ of a stable spherical fibration $p: X \to BG$ over a finite-dimensional space X are defined by

$$v_r(p) = \sum_{i+j=r} \chi(Sq^i) w_j(-p) \in H^r(X; \mathbf{Z}_2) \quad (r \geqslant 0)$$

with $-p: X \to BG$ any stable inverse for p. The mod 2 Wu classes are characterized by the property

$$v_r(p)(U_p \cap z) = \langle \chi(Sq^r)(U_p), z \rangle \in \mathbf{Z}_2 \quad (z \in \dot{H}_{r+k}(T(p); \mathbf{Z}_2)).$$

PROPOSITION 9.4. *The* mod 2 *reductions of the Wu classes of a stable spherical fibration* $p: X \to BG$ *over a finite-dimensional space* X *with respect to an oriented cover* $\tilde{X}$ *of* X *with data* (π, w) *agree with the* mod 2 *Wu classes, that is there are defined commutative diagrams*

$$\begin{array}{ccc} H_r(\tilde{X}) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; \mathbf{Z}[\pi]) \\ \downarrow & & \downarrow \\ H_r(X; \mathbf{Z}_2) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbf{Z}_2; \mathbf{Z}_2) = \mathbf{Z}_2 \end{array}$$

in which the vertical maps are the change of rings

$$\mathbf{Z}[\pi] \to \mathbf{Z}_2; \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g.$$

Proof. Applying Proposition 9.2 we can express the mod 2 Wu classes of p in terms of the hyperquadratic construction by evaluating the composite

$$\dot{H}^k(T(p); \mathbf{Z}_2) \xrightarrow{\theta_{T(p)}} \hat{Q}^{-k}(\dot{C}(T(p); \mathbf{Z}_2)) \xrightarrow{v_r} \operatorname{Hom}_{\mathbf{Z}_2}(\dot{H}_{k+r}(T(p); \mathbf{Z}_2), \mathbf{Z}_2)$$
$$= \operatorname{Hom}_{\mathbf{Z}_2}(H_r(X; \mathbf{Z}_2), \mathbf{Z}_2) = H^r(X; \mathbf{Z}_2)$$

on the mod 2 Thom class $U_p \in \dot{H}^k(T(p); \mathbf{Z}_2)$, so that

$$v_r(p) = \hat{v}_r \theta_{T(p)}(U_p) \in H^r(X; \mathbf{Z}_2).$$

Define the *Hopf invariant* function,

$$H: \pi_{m+n}(S^m) \to H^{m-n-1}(\mathbf{Z}_2; \mathbf{Z}, (-)^m) = \begin{cases} \mathbf{Z} & \text{if } m = n+1,\ n \equiv 1 \pmod 2, \\ \mathbf{Z}_2 & \text{if } m > n+1,\ n \equiv 1 \pmod 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$(f: S^{m+n} \to S^m) \mapsto H(f),$$

by applying the symmetric construction to the mapping cone

$$X = S^m \cup_f e^{m+n+1},$$

with

$$\mathbf{Z} = \dot{H}_{m+n+1}(X) \xrightarrow{\dot{\varphi}_X} Q^{m+n+1}(\dot{C}(X))$$

$$\xrightarrow{v_{n+1}} \mathrm{Hom}_{\mathbf{Z}}(\dot{H}^m(X), H^{m-n-1}(\mathbf{Z}_2, \mathbf{Z}, (-)^m));$$

$$1 \mapsto v_{n+1}(\dot{\varphi}_X(1)) = H(f) \quad (\dot{H}^m(X) = \mathbf{Z}).$$

Alternatively, apply the quadratic construction to

$$f: \Sigma^m(S^n) = S^{m+n} \to \Sigma^m(S^0) = S^m,$$

$$\psi_f: \dot{H}_n(S^n) = \mathbf{Z} \to Q_n^{[0,m-1]}(\dot{C}(S^0))$$

$$= H^{m-n-1}(\mathbf{Z}_2; \mathbf{Z}, (-)^m);\ 1 \mapsto \psi_f(1) = H(f) \quad (n > 0).$$

Both these ways agree with the construction of the Hopf invariant due to Steenrod [**26**], by Propositions 1.2(i), 1.3 and 1.6. The morphism j defined in § 5 is the composite

$$j: \pi_n(SO(m)) \xrightarrow{J} \pi_n(SG(m)) = \pi_{m+n}(S^m) \xrightarrow{H} Q^{m+n+1}(S^m\mathbf{Z}).$$

The diagram

$$\begin{array}{ccc} \pi_{m+n}(S^m) & \xrightarrow{H} & H^{m-n-1}(\mathbf{Z}_2; \mathbf{Z}, (-)^m) \\ \Sigma \downarrow & & \downarrow S\ (= \text{id. if } m > n+1) \\ \pi_{m+n+1}(S^{m+1}) & \xrightarrow{H} & H^{m-n}(\mathbf{Z}_2; \mathbf{Z}, (-)^{m+1}) \end{array}$$

commutes, so that it is possible to define the *stable Hopf invariant*

$$\hat{H}: \pi_n^S = \varinjlim_m \pi_{m+n}(S^m) \to \hat{H}^{n+1}(\mathbf{Z}_2; \mathbf{Z}) = \begin{cases} \mathbf{Z}_2 & \text{if } n \equiv 1 \pmod 2, \\ 0 & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

PROPOSITION 9.5. *The Wu classes of a stable spherical fibration* $p\colon S^m \to BG$ *over* S^m $(m \geqslant 2)$ *are given by*

$$v_0(p)\colon H_0(S^m) = \mathbf{Z} \to \hat{H}^0(\mathbf{Z}_2; \mathbf{Z}) = \mathbf{Z}_2;\ z \mapsto z^2 \equiv z \pmod 2,$$

$$v_m(p)\colon H_m(S^m) = \mathbf{Z} \to \hat{H}^m(\mathbf{Z}_2; \mathbf{Z}) = \begin{cases} \mathbf{Z}_2 & \text{if } m \equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 1 \pmod 2; \end{cases}$$

$z \mapsto z^2$ (stable Hopf invariant of $p \in \pi_m(BG) = \pi^S_{m-1}$).

Proof. Choosing a representative $(k-1)$-spherical fibration

$$p\colon S^m \to BG(k) \quad (k \gg m)$$

we have that the Thom space $T(p)$ is the mapping cone of

$$p \in \pi_m(BG(k)) = \pi_{m+k-1}(S^k), \quad T(p) = S^k \cup_p e^{k+m}.$$

Now

$$v_m(p) = \sum_{i+j=m} \chi(Sq^i)w_j(-p) = w_m(-p) = w_m(p) \in H^m(S^m; \mathbf{Z}_2) = \mathbf{Z}_2,$$

and $w_m(p) = \hat{H}(p) \in \mathbf{Z}_2$ by construction.

The *rth Wu class* of an n-dimensional geometric Poincaré complex X with respect to an oriented cover $\tilde{X}$ of X with data (π, w) is the rth symmetric Wu class of the associated n-dimensional symmetric Poincaré complex $\sigma^*(X) = (C(\tilde{X}), \varphi_{\tilde{X}}[X])$ over $\mathbf{Z}[\pi]$, the $\mathbf{Z}[\pi]$-module morphism

$$v_r(X) = v_r(\varphi_{\tilde{X}}[X])\colon H_r(\tilde{X}) \to H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^{n-r}) \quad (r \geqslant 0).$$

The mod 2 *Wu classes* $v_r(X) \in H^r(X; \mathbf{Z}_2)$ of X are characterized by

$$v_r(X)([X] \cap y) = \langle Sq^r(y), [X]\rangle \in \mathbf{Z}_2 \quad (y \in H^{n-r}(X; \mathbf{Z}_2), [X] \in H_n(X; \mathbf{Z}_2))$$

and Proposition 1.3 gives commutative diagrams relating the two types of Wu class

$$\begin{array}{ccc} H_r(\tilde{X}) & \xrightarrow{v_r(X)} & H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^{n-r}) \\ \downarrow & & \downarrow \\ H_r(X; \mathbf{Z}_2) & \xrightarrow{v_r(X)} & H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}_2, (-)^{n-r}) = \begin{cases} \mathbf{Z}_2 & \text{if } 2r \leqslant n \\ 0 & \text{if } 2r > n, \end{cases} \end{array}$$

in which the vertical maps are the change of rings

$$\mathbf{Z}[\pi] \to \mathbf{Z}_2;\ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g.$$

The *reduced Wu classes* of X are defined by passing to the reduced (Tate) cohomology groups

$$\hat{v}_r(X)\colon H_r(\tilde{X}) \xrightarrow{v_r(X)} H^{n-2r}(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^{n-r}) \xrightarrow{\text{reduction}} \hat{H}^r(\mathbf{Z}_2; \mathbf{Z}[\pi]).$$

Note that $\hat{v}_r(X) = v_r(X)$ for $n \neq 2r$.

PROPOSITION 9.6. *If X is an n-dimensional geometric Poincaré complex and $\tilde{X}$ is an oriented covering with data (π, w) then the Poincaré duality chain equivalence*

$$[X] \cap - : {}^wC(\tilde{X})^{n-*} \to C(\tilde{X})$$

defines a homotopy equivalence of n-dimensional hyperquadratic complexes over $\mathbf{Z}[\pi]$ with the w-twisted involution

$$[X] \cap - : \hat{\sigma}^*(\nu_X) = ({}^wC(\tilde{X})^{n-*}, \theta_{T\pi(\nu_X)}(U_{\nu_X}) \in \hat{Q}^n({}^wC(\tilde{X})^{n-*}))$$
$$\to J\sigma^*(X) = (C(\tilde{X}), J\varphi_{\tilde{X}}[X] \in \hat{Q}^n(C(\tilde{X}))).$$

In particular, the reduced Wu classes of X are just the Wu classes of the Spivak normal fibration $\nu_X\colon X \to BG$,

$$\hat{v}_r(X) = v_r(\nu_X)\colon H_r(\tilde{X}) \to \hat{H}^r(\mathbf{Z}_2; \mathbf{Z}[\pi]) \quad (r \geqslant 0).$$

Proof. We have already obtained the identity

$$\varphi_{\tilde{X}}[X] = \theta_{T\pi(\nu_X), \alpha_X}(U_{\nu_X}) \in Q^n(C(\tilde{X}))$$

(just before Proposition 9.1). Now apply the J-homomorphism of passing to the suspension limit to remove the dependence on the choice of $S\pi$-duality α_X.

The identities $J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$, $\hat{v}_r(X) = v_r(\nu_X)$ may be considered as equivariant generalizations of the formulae of Wu [**33**] and Thom [**28**] relating the mod 2 Wu classes of a manifold X to the mod 2 Stiefel–Whitney classes of the tangent bundle τ_X, since $v_r(X) = v_r(\nu_X) \in H^r(X; \mathbf{Z}_2)$ can be written as

$$v_r(X) = \sum_{i+j=r} \chi(Sq^i) w_j(\tau_X) \in H^r(X; \mathbf{Z}_2),$$

or equivalently

$$w_r(\tau_X) = \sum_{i+j=r} Sq^i v_j(X) \in H^r(X; \mathbf{Z}_2).$$

If $T \in \mathbf{Z}_2$ acts on a group ring $\mathbf{Z}[\pi]$ by the w-twisted involution, for some group morphism $w\colon \pi \to \mathbf{Z}_2$, then the direct sum decomposition of $\mathbf{Z}[\mathbf{Z}_2]$-modules

$$\mathbf{Z}[\pi] = \mathbf{Z} \oplus \mathbf{Z}[\pi]/\mathbf{Z}$$

gives rise to a direct sum decomposition of $\mathbf{Z}_2$-cohomology {$\mathbf{Z}_2$-homology, Tate $\mathbf{Z}_2$-cohomology} groups,

$$\begin{cases} H^r(\mathbf{Z}_2;\, \mathbf{Z}[\pi], \varepsilon) = H^r(\mathbf{Z}_2;\, \mathbf{Z}, \varepsilon) \oplus H^r(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z}, \varepsilon), \\ H_r(\mathbf{Z}_2;\, \mathbf{Z}[\pi], \varepsilon) = H_r(\mathbf{Z}_2;\, \mathbf{Z}, \varepsilon) \oplus H_r(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z}, \varepsilon), \\ \hat{H}^r(\mathbf{Z}_2;\, \mathbf{Z}[\pi], \varepsilon) = \hat{H}^r(\mathbf{Z}_2;\, \mathbf{Z}, \varepsilon) \oplus \hat{H}^r(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z}, \varepsilon), \end{cases}$$

with $\varepsilon = \pm 1 \in \mathbf{Z}$. We shall call elements of these groups *regular* if they have a decomposition of the type $(?, 0)$. The Wu classes of an orientable spherical fibration $p: X \to BG(k)$ with respect to the trivial cover $\tilde{X} = \pi \times X$ take regular values,

$$v_m(p):\, H_m(\tilde{X}) = \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} H_m(X) \to \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}[\pi]);\ 1 \otimes x \mapsto (v_m(p)(x), 0).$$

A map of geometric Poincaré complexes $f: M \to X$ such that $\dim M = m \leqslant \dim X = n$ *represents* the homology class $x \in H_m(\tilde{X})$ if $\tilde{X}$ is an oriented cover of X with data (π, w) such that the composite $\pi_1(M) \xrightarrow{f} \pi_1(X) \longrightarrow \pi$ is trivial, so that $\tilde{M} = \pi \times M$ is the trivial cover of M and M is oriented (since $w(M): \pi_1(M) \xrightarrow{f} \pi_1(X) \longrightarrow \pi \xrightarrow{w} \mathbf{Z}_2$ is trivial), and if the induced $\mathbf{Z}[\pi]$-module morphism

$$\tilde{f}:\, H_m(\tilde{M}) = \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} H_m(M) = \mathbf{Z}[\pi] \to H_m(\tilde{X})$$

sends the generator to $\tilde{f}(1 \otimes [M]) = x \in H_m(\tilde{X})$ for some lift $\tilde{f}: \tilde{M} \to \tilde{X}$. The lift is non-unique, all such lifts being given by $g\tilde{f} = \tilde{f}g: \tilde{M} \to \tilde{X}$ $(g \in \pi)$, so that if $x \in H_m(\tilde{X})$ is representable then so is $gx \in H_m(\tilde{X})$ $(g \in \pi)$. (Note that for $\pi = \{1\}$ we have the result of Levitt **[11]** that every homology class $x \in H_m(X)$ is representable in this sense, for any CW complex X.) The homology classes $x \in H_m(\tilde{X})$ which are represented by maps $f: S^m \to X$ are *spherical*.

PROPOSITION 9.7. *The mth reduced Wu class of a geometric Poincaré complex X with respect to an oriented cover $\tilde{X}$ of X with data (π, w)*

$$\hat{v}_m(X):\, H_m(\tilde{X}) \to \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}[\pi])$$

takes regular values on representable homology classes. If $x \in H_m(\tilde{X})$ is represented by $f: M \to X$ then

$$\begin{aligned} \hat{v}_m(X)(x) &= (v_m(f^*\nu_X)([M]), 0) \in \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}[\pi]) \\ &= \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}) \oplus \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}[\pi]/\mathbf{Z}), \end{aligned}$$

and if $M = S^m$ then

$$\hat{v}_m(X)(x) = (\text{stable Hopf invariant of } f^*\nu_X \in \pi_m(BG) = \pi^S_{m-1}, 0) \in \hat{H}^m(\mathbf{Z}_2;\, \mathbf{Z}[\pi]).$$

Proof. Combining Propositions 9.3(ii) and 9.6 we have

$$\begin{aligned}\hat{v}_m(X)(\tilde{f}(1\otimes[M])) &= v_m(\nu_X)(\tilde{f}(1\otimes[M]))\\ &= v_m(f^*\nu_X)(1\otimes[M])\\ &= 1\otimes v_m(f^*\nu_X)([M]) \in \hat{H}^m(\mathbf{Z}_2;\ \mathbf{Z}[\pi]).\end{aligned}$$

For spherical homology classes apply Proposition 9.5 to identify the Wu class with the stable Hopf invariant.

The result of Proposition 9.7 restricts the $\pm$symmetric forms and formations occurring as the symmetric kernels of highly-connected degree 1 maps of geometric Poincaré complexes to be even, except in dimensions related to Hopf invariant 1. (See §I.2 for the definition of 'even'.)

PROPOSITION 9.8. *Let $f\colon M\to X$ be an $(i-1)$-connected degree 1 map of n-dimensional geometric Poincaré complexes ($2i\leqslant n$), with symmetric kernel $\sigma^*(f) = (C,\varphi\in Q^n(C))$. The reduced ith symmetric Wu class*

$$\hat{v}_i(\varphi)\colon H^{n-i}(C) = K_i(M)\to \hat{H}^i(\mathbf{Z}_2;\ \mathbf{Z}[\pi_1(X)])$$

is such that $\hat{v}_i(\varphi)=0$ for $i\neq 2,4,8$, in which case $\sigma^*(f) = \bar{S}^i\sigma^i(f)$ is the i-fold skew-suspension of an $(n-2i)$-dimensional even $(-)^i$-symmetric Poincaré complex $\sigma^i(f)$, and

$$\sigma^*(f)\in \operatorname{im}(\bar{S}^{i+1}\colon L^{n-2(i+1)}(\mathbf{Z}[\pi_1(X)],(-)^{i+1})\to L^n(\mathbf{Z}[\pi_1(X)]))$$

(killing $K_i(M)$ by symmetric surgery if $2(i+1)\leqslant n$); for $i=2,4,8$, $\hat{v}_i(\varphi)$ only takes regular values in $\hat{H}^i(\mathbf{Z}_2;\ \mathbf{Z}[\pi_1(X)])$.

Proof. By the Hurewicz theorem $K_i(M)=\pi_{i+1}(f)$, so that every homology class $x\in K_i(M)\subseteq H_i(\tilde{M})$ is spherical, corresponding to a relative homotopy class

$$x = ((h,g)\colon (D^{i+1},S^i)\to (X,M))\in \pi_{i+1}(f) = K_i(M).$$

By Proposition 9.7,

$$\hat{v}_i(\varphi)(x) = (\hat{H}(g^*\nu_M),0)\in \hat{H}^i(\mathbf{Z}_2;\ \mathbf{Z}[\pi_1(X)]).$$

Now apply the result of Adams [**1**] that the stable Hopf invariant map $\hat{H}\colon \pi_i(BG)=\pi^S_{i-1}\to \hat{H}^i(\mathbf{Z}_2;\ \mathbf{Z})$ is 0 for $i\neq 2,4,8$.

(Note that even if $2(i+1)\leqslant n$ and $\hat{v}_i(\varphi)=0$ it may not be possible to kill $K_i(M)$ by geometric Poincaré surgery, that is obtain a degree 1 geometric Poincaré bordism to an i-connected degree 1 map $f'\colon M'\to X$,

$$(e;f,f')\colon (N;\ M,M')\to (X\times I;\ X\times 0, X\times 1),$$

since this would require every element $x=(h,g)\in K_i(M)$ to be such that $g^*\nu_M = 0\in \pi_i(BG)=\pi^S_{i-1}$, and we are only given that

$$\hat{H}(g^*\nu_M) = 0\in \hat{H}^i(\mathbf{Z}_2;\ \mathbf{Z}).)$$

In the case where $n = 2i$ $\{n = 2i+1\}$ Proposition 9.8 states that for $i \neq 2, 4, 8$ the symmetric kernel $\sigma^i(f)$ must correspond to a non-singular $(-)^i$-symmetric form {formation} which is even (under the correspondence of Proposition I.2.1 {I.2.3}). By contrast, the realization Theorem 5.8 {6.5} of Wall [**30**] shows that for $n \geqslant 5$ every non-singular $(-)^i$-quadratic form {formation} over $\mathbf{Z}[\pi]$ is the quadratic kernel $\sigma_i(f, b)$ of an $(i-1)$-connected n-dimensional normal map $(f, b)\colon M \to X$, with $\pi = \pi_1(X)$ any finitely-presented group.

For $i = 2, 4, 8$ let M be the $(i-1)$-connected $2i$-dimensional geometric Poincaré complex defined by the complex projective plane $\mathbf{C}P^2 = S^2 \cup_\eta e^4$, the quaternion projective plane $\mathbf{H}P^2 = S^4 \cup_\eta e^8$ and the Cayley projective plane $\mathbf{O}P^2 = S^8 \cup_\eta e^{16}$ respectively, with $\eta \in \pi_{2i-1}(S^i)$ the Hopf invariant 1 elements. The symmetric kernels of the associated degree 1 maps $f\colon M \to S^{2i}$ are all given by the non-singular symmetric form over $\mathbf{Z}$,

$$\sigma^i(f) = (\mathbf{Z}, 1),$$

with non-trivial reduced Wu class. Further, crossing with S^1 gives $(i-1)$-connected $(2i+1)$-dimensional degree 1 maps

$$f \times 1\colon M \times S^1 \to S^{2i} \times S^1$$

such that the symmetric kernels are all given by the non-singular symmetric formation over the Laurent extension $\mathbf{Z}[z, z^{-1}]$ $(\bar{z} = z^{-1})$

$$\sigma^i(f \times 1) = \sigma^i(f) \otimes \sigma^*(S^1) = \left(\mathbf{Z}[z, z^{-1}] \oplus \mathbf{Z}[z, z^{-1}]^*, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{Z}[z, z^{-1}],\right.$$
$$\left. \operatorname{im}\left(\begin{pmatrix} 1 \\ z-1 \end{pmatrix}\colon \mathbf{Z}[z, z^{-1}] \to \mathbf{Z}[z, z^{-1}] \oplus \mathbf{Z}[z, z^{-1}]^*\right)\right)$$

with non-trivial reduced Wu class. We can also construct simply-connected odd-dimensional examples, as follows. For each of $i = 2, 4, 8$ let N be the $(i-1)$-connected $(2i+1)$-dimensional manifold obtained from $M \times I$ by glueing the ends together using the conjugation map $M \to M$, and killing π_1 by oriented surgery. The symmetric kernel $\sigma^i(g)$ of the associated $(i-1)$-connected degree 1 map $g\colon N \to S^{2i+1}$ is given by the non-singular symmetric formation over $\mathbf{Z}$ of deRham invariant 1

$$\sigma^i(g) = \left(\mathbf{Z} \oplus \mathbf{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{Z}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\mathbf{Z}\right)$$

with non-trivial reduced Wu class, representing the generator

$$\sigma^i(g) = 1 \in L^{2i+1}(\mathbf{Z}) = \mathbf{Z}_2.$$

The stable automorphisms $h\colon p \to p$ over $1\colon X \to X$ of a stable spherical fibration $p\colon X \to BG$ over a finite-dimensional space X are classified by homotopy classes of maps $h\colon X \to G = \varinjlim_m G(m)$, or equivalently by the relative homotopy classes of maps $h\colon (X \times D^1, X \times S^0) \to BG$ such that $h|_{X\times S^0} = p \cup p\colon X \times S^0 \to BG$. Define the *rth suspended* mod 2 *Wu class* of h, $(\sigma v_r)(h) \in H^{r-1}(X; \mathbf{Z}_2)$, to be the image of the universal rth mod 2 Wu class $v_r \in H^r(BG; \mathbf{Z}_2)$ under the composite

$$H^r(BG; \mathbf{Z}_2) \xrightarrow{\;h^*\;} H^r(X \times D^1, X \times S^0; \mathbf{Z}_2) \xrightarrow{\;(\text{proj.})^{*-1}\;} \dot{H}^r(\Sigma X_+; \mathbf{Z}_2) \xrightarrow{\;\Sigma^*\;} H^{r-1}(X; \mathbf{Z}_2).$$

In terms of the mod 2 Stiefel–Whitney classes this is just

$$(\sigma v_r)(h) = \sum_{i+j=r} \chi(Sq^i)(\Sigma^* w_j(-h)) \in H^{r-1}(X; \mathbf{Z}_2)$$
$$(w_j(-h) \in H^j(X \times D^1, X \times S^0; \mathbf{Z}_2) = \dot{H}^j(\Sigma X_+; \mathbf{Z}_2)).$$

PROPOSITION 9.9. *Let $h\colon p \to p$ be a stable automorphism over $1\colon X \to X$ of a stable spherical fibration $p\colon X \to BG$ over an n-dimensional space X. Let $\tilde{X}$ be an oriented cover of X with data (π, w), and let R be a commutative ring. Then there is defined in a natural way an $(n+1)$-dimensional hyperquadratic complex over $R[\pi]$ with the w-twisted involution, the* suspended Wu complex *of (p,h)*

$$\hat{\sigma}^*(p,h) = ({}^wC(\tilde{X}; R)^{n-*}, \theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\tilde{X}; R)^{n-*})),$$

depending only on the homotopy class of $h\colon X \to G$, such that

(i) *if $\alpha\colon S^{n+k} \to Y \wedge_\pi T\pi(p)$ is an $S\pi$-duality map for some finite-dimensional $CW\pi$-complex Y, and $H \in \{Y, Y\}_\pi$ is the $S\pi$-dual of $T\pi(h) \in \{T\pi(p), T\pi(p)\}_\pi$ then the $\mathbf{Z}[\pi]$-module chain equivalence*

$$j\colon {}^wC(\tilde{X}; R)^{n-*} \xrightarrow{\;U_p \cap -\;} \dot{C}(T\pi(p); R)^{n+k-*} \xrightarrow{\;(\alpha[S^{n+k}]\backslash -)\;} \dot{C}(Y; R)$$

sends $H\theta_{p,h} \in Q_n({}^wC(\tilde{X}; R)^{n-*})$ (with H as in Proposition I.1.2) *to $\psi_H(\alpha[S^{n+k}]\backslash U_p) \in Q_n(\dot{C}(Y; R))$,*

$$\psi_H(\alpha[S^{n+k}]\backslash U_p) = j_\% H\theta_{p,h} \in Q_n(\dot{C}(Y; R)),$$

(ii) *if $f\colon M \to X$ is a map of n-dimensional spaces with induced cover $\tilde{M}$ there is defined a map of the suspended Wu complexes*

$$\tilde{f}^*\colon \hat{\sigma}^*(p,h) \to \hat{\sigma}^*(f^*p, f^*h),$$

(iii) $\theta_{p,gh} = \theta_{p,g} + \theta_{p,h}, \theta_{p,1} = 0,$

(iv) *for $R = \mathbf{Z}_2$, the* mod 2 *reduction of the rth hyperquadratic Wu class of $\theta_{p,h}$ is the rth suspended* mod 2 *Wu class of h*

$$\hat{v}_r(\theta_{p,h}) = (\sigma v_r)(h) \in \operatorname{Hom}_{\mathbf{Z}_2}(H_{r-1}(X; \mathbf{Z}_2), \mathbf{Z}_2) = H^{r-1}(X; \mathbf{Z}_2).$$

For $X = S^n$,

$$\hat{v}_{n+1}(\theta_{p,h}) = (\textit{stable Hopf invariant of } h \in \pi_n(G) = \pi_n^S) \in H^n(S^n; \mathbf{Z}_2) = \mathbf{Z}_2.$$

Proof. The relative version of the Wu complex construction of Proposition 9.3 applied to $h: (X \times D^1, X \times S^0) \to BG$ gives a relative Tate $\mathbf{Z}_2$-hypercohomology class

$$\theta_{T\pi(h)}(U_h) \in \hat{Q}^{n+1}(i = (\text{inclusion})^*: {}^wC(\tilde{X} \times D^1; R)^{n-*} \to {}^wC(\tilde{X} \times S^0; R)^{n-*}).$$

The inclusion ${}^wC(\tilde{X} \times S^0; R)^{n-*} \to C(i) = {}^wC(\tilde{X}; R)^{n-*}$ sends $\theta_{T\pi(h)}(U_h)$ to the required element $\theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\tilde{X}; R)^{n-*})$.

The hyperquadratic Wu classes of the suspended Wu complex $\hat{\sigma}^*(p,h) = ({}^wC(\tilde{X}; R)^{n-*}, \theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\tilde{X}; R)^{n-*}))$ are the *suspended Wu classes* of an automorphism $h: p \to p$ of $p: X \to BG$, $R[\pi]$-module morphisms

$$(\sigma v_r)(h) = \hat{v}_r(\theta_{p,h}): H_{r-1}(\tilde{X}; R) \to \hat{H}^r(\mathbf{Z}_2; R[\pi]) \quad (r \geqslant 1).$$

We have already related these classes with the suspended mod 2 Wu classes, in Proposition 9.9(iv) above. Note that the quadratic Wu classes of $H\theta_{p,h} \in Q_n({}^wC(\tilde{X}; R)^{n-*})$ are given by

$$v^r(H\theta_{p,h}) = H(\sigma v_{r+1})(h): H_r(\tilde{X}; R) \to H_{2r-n}(\mathbf{Z}_2; R[\pi], (-)^{n-r}) \quad (r \geqslant 0)$$

with $H: \hat{Q}^{n+1}({}^wC(\tilde{X}; R)^{n-*}) \to Q_n({}^wC(\tilde{X}; R)^{n-*})$ as defined in Proposition I.1.2, and $H: \hat{H}^{r+1}(\mathbf{Z}_2; R[\pi]) \to H_{2r-n}(\mathbf{Z}_2; R[\pi], (-)^{n-r})$ the natural map.

PROPOSITION 9.10. *Let* $(f,b): (M, \nu_M, \rho_M) \to (X, \nu_X, \rho_X)$ *be a normal map of normalized n-dimensional geometric Poincaré complexes with quadratic kernel* $\sigma_*(f,b) = (C(f^!), \psi = e_{\%}\psi_F[X] \in Q_n(C(f^!)))$. *Given an automorphism* $c: \nu_M \to \nu_M$ *of* $\nu_M: M \to BG(k)$ *define normal maps*

$$(f,b'): (M, \nu_M, \rho_M) \to (X, \nu_X, \rho'_X), \quad (f,b''): (M, \nu_M, \rho'_M) \to (X, \nu_X, \rho'_X)$$

by

$$\rho'_M = T(c)\rho_M \in \pi_{n+k}(T(\nu_M)), \quad \rho'_X = T(b)\rho'_M \in \pi_{n+k}(T(\nu_X)),$$
$$b' = bc, \quad b'' = b: \nu_M \to \nu_X.$$

Then the quadratic kernels of (f,b'), (f,b'') *are given by*

$$\sigma_*(f,b') = \sigma_*(f,b'') = (C(f^!), \psi' = \psi + H\theta \in Q_n(C(f^!)))$$

with $\theta = \hat{e}^{\%}\theta_{\nu_M,c} \in \hat{Q}^{n+1}(C(f^!))$, *and the quadratic Wu classes are such that*

$$v^r(\psi') - v^r(\psi) = H(\sigma v_{r+1})(c): K_r(M) \to H_{2r-n}(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^{n-r}) \quad (r \geqslant 0).$$

Proof. Let $(1,d): (X, \nu_X, \rho'_X) \to (X, \nu_X, \rho_X)$ be the canonical equivalence of Spivak normal structures given by Proposition 4.1. The fundamental

$S\pi$-duality maps

$$\alpha_X = \Delta\rho_X,\ \alpha'_X = \Delta\rho'_X : S^{n+k} \to \tilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

are such that there is defined a homotopy commutative diagram

$$\begin{array}{ccccc}
S^{n+k} & & \xrightarrow{\quad\alpha'_X\quad} & & \tilde{X}_+\wedge_\pi T\pi(\nu_X) \\
 & \searrow^{\rho'_X} & & \nearrow^{\Delta} & \\
\Big\downarrow 1 & & T(\nu_X) & & \Big\downarrow 1\wedge T\pi(d) \\
 & & \Big\downarrow T(d) & & \\
 & & T(\nu_X) & & \\
 & \nearrow_{\rho_X} & & \searrow_{\Delta} & \\
S^{n+k} & & \xrightarrow{\quad\alpha_X\quad} & & \tilde{X}_+\wedge_\pi T\pi(\nu_X)
\end{array}$$

Let $H: \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{X}_+$ be a π-map which is $S\pi$-dual to

$$T\pi(d): T\pi(\nu_X) \to T\pi(\nu_X)$$

with respect to α'_X, so that there is defined a homotopy commutative diagram

$$\begin{array}{ccc}
S^{n+k+p} & \xrightarrow{\Sigma^p\alpha'_X} & \Sigma^p\tilde{X}_+\wedge_\pi T\pi(\nu_X) \\
\Big\downarrow \Sigma^p\alpha'_X & \searrow^{\Sigma^p\alpha_X} & \Big\downarrow H\wedge 1 \\
\Sigma^p\tilde{X}_+\wedge_\pi T\pi(\nu_X) & \xrightarrow{1\wedge T\pi(d)} & \Sigma^p\tilde{X}_+\wedge_\pi T\pi(\nu_X)
\end{array}$$

Working as in the proof of Theorem 3.5 of Wall [**29**] we can take H to be

$$H: \Sigma^p\tilde{X}_+ = \tilde{X}_+\wedge S^p \to \tilde{X}_+\wedge S^p;\ \tilde{x}\wedge s \mapsto \tilde{x}\wedge d(x)(s)$$

with $d: X \to G(p)$ a classifying map for $d: \nu_X \to \nu_X$, and similarly for a π-map $G: \Sigma^p\tilde{M}_+ \to \Sigma^p\tilde{M}_+$, $S\pi$-dual to $T\pi(c): T\pi(\nu_M) \to T\pi(\nu_M)$ with respect to the fundamental $S\pi$-duality map

$$\alpha_M = \Delta\rho_M: S^{n+k} \to \tilde{M}_+\wedge_\pi T\pi(\nu_M).$$

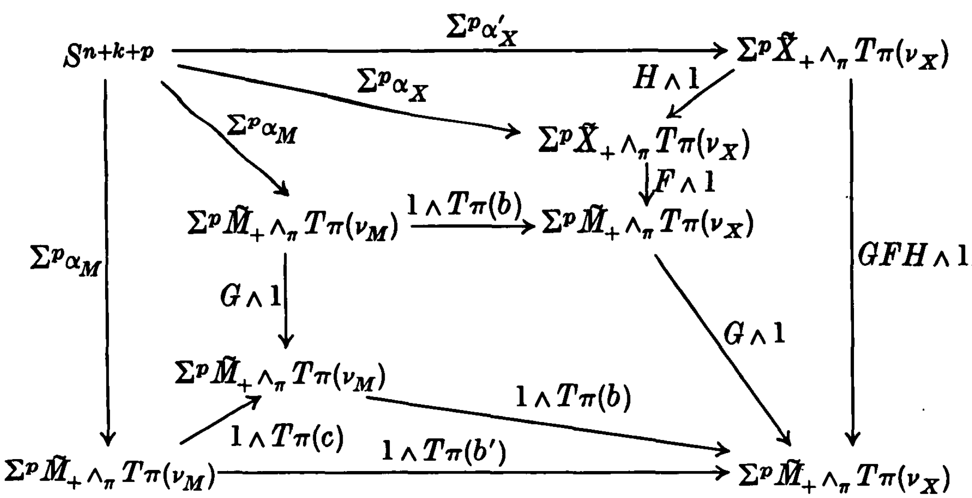
By the definition of quadratic kernel we have that

$$\begin{cases} \sigma_*(f,b) = (C(f^!), e_\%\psi_F[X] \in Q_n(C(f^!))), \\ \sigma_*(f,b') = (C(f^!), e_\%\psi_{F'}[X] \in Q_n(C(f^!))), \end{cases}$$

with $F: \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{M}_+$ $\{F': \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{M}_+\}$ a π-map $S\pi$-dual to

$$T\pi(b): T\pi(\nu_M) \to T\pi(\nu_X) \quad \{T\pi(b'): T\pi(\nu_M) \to T\pi(\nu_X)\}$$

with respect to α_M, α_X $\{\alpha_M, \alpha'_X\}$. Considering the homotopy commutative diagram



we can identify

$$F' = GFH\colon \Sigma^p\tilde{X}_+ \to \Sigma^p\tilde{M}_+.$$

Applying the sum formula for the quadratic construction of Proposition 1.5(iii) we obtain

$$\psi_{F'} = \psi_G f^!_* + \psi_F + f^!_\%\psi_H\colon H^\pi_n(\tilde{X};\, {}^w\mathbf{Z}) \to Q_n(C(\tilde{M})),$$

so that

$$e_\%\psi_{F'}[X] - e_\%\psi_F[X] = e_\%\psi_G[M] \in Q_n(C(f^!)).$$

Further, applying the construction of Proposition 9.9(i) to the fundamental $S\pi$-duality map $\alpha_M\colon S^{n+k} \to \tilde{M}_+ \wedge_\pi T\pi(\nu_M)$ we can identify

$$\psi_G[M] = H\theta_{\nu_M,c} \in Q_n(C(\tilde{M}))$$

with $\theta_{\nu_M,c} \in \hat{Q}^{n+1}({}^wC(\tilde{M})^{n-*}) = \hat{Q}^{n+1}(C(\tilde{M}))$, and so

$$e_\%\psi_{F'}[X] - e_\%\psi_F[X] = H(\hat{e}^\%\theta_{\nu_M,c}) \in Q_n(C(f^!)).$$

Applying the quadratic kernel sum formula of Proposition 4.3 to the composition of normal maps

$$(f, b')\colon (M, \nu_M, \rho_M) \xrightarrow{(1,c)} (M, \nu_M, \rho'_M) \xrightarrow{(f,b'')} (X, \nu_X, \rho'_X),$$

we have that up to homotopy equivalence

$$\sigma_*(f, b') = \sigma_*(f, b'') \oplus \sigma_*(1, c) = \sigma_*(f, b'').$$

The mod 2 reduction of the quadratic Wu class identity of Proposition 9.10 in the case where $n = 2r \equiv 2 \pmod 4$, $\pi = \{1\}$ is the formula for the twisting of the Arf form due to Brown [5].

PROPOSITION 9.11. (i) *Let* $(f,b)\colon M\to X$, $(f,b')\colon M\to X$ *be normal bundle maps with* $b' = bc\colon \nu_M \to \nu_X$ *for some stable bundle automorphism* $c\colon \nu_M\to\nu_M$ *classified by* $c\colon M\to SO$. *The quadratic kernels,*

$$\sigma_*(f,b) = (C,\psi) = (C(f^!), e_{\%}\psi_F[X]\in Q_n(C(f^!))),$$
$$\sigma_*(f,b') = (C,\psi') = (C(f^!), e_{\%}\psi_{F'}[X]\in Q_n(C(f^!))),$$

are such that if $x\in K_r(M) = H_{r+1}(\tilde f)$ *is the Hurewicz image of*

$$(h,g)\in \pi_{r+1}(f) = \pi_{r+1}(\tilde f)$$

with $g\colon S^r\to M$ *an immersion and* $h\colon D^{r+1}\to X$ *a null-homotopy of* $fg\colon S^r\to X$ *then*

$$\begin{aligned} v^r(\psi')(x) - v^r(\psi)(x) &= (Hj(cg),0)\in Q_n(S^{n-r}\mathbf{Z}[\pi]) \\ &= Q_n(S^{n-r}\mathbf{Z})\oplus H_{2r-n}(\mathbf{Z}_2;\mathbf{Z}[\pi]/\mathbf{Z},(-)^{n-r}) \quad (\pi=\pi_1(X)) \end{aligned}$$

with $Hj\colon \pi_r(SO)\xrightarrow{j}\hat Q^{n+1}(S^{n-r}\mathbf{Z})\xrightarrow{H} Q_n(S^{n-r}\mathbf{Z})$.

(ii) *The surgery obstruction* $\sigma_*(f,b)\in L_n(\mathbf{Z}[\pi_1(X)])$ *of an* $(i-1)$*-connected* n*-dimensional normal bundle map* $(f,b)\colon M\to X$ *for* $n=2i$ *or* $2i+1$ *is independent of the bundle map* $b\colon \nu_M\to\nu_X$ *for* $i\neq 1,3,7$.

Proof. (i) The universal cover $\tilde X$ of X induces the trivial cover $\tilde S^r = \pi\times S^r$ of S^r, so that applying Propositions 9.10 and 9.9(iv) we have

$$v^r(\psi')(x) - v^r(\psi)(x) = (H\sigma v_{r+1}(c)(x),0) = (Hj(cg),0)\in Q_n(S^{n-r}\mathbf{Z}[\pi]),$$

since j is the composite

$$j\colon \pi_r(SO)\xrightarrow{J}\pi_r(SG) = \pi_r^S \xrightarrow{\text{stable Hopf invariant}} \hat H^{r+1}(\mathbf{Z}_2;\mathbf{Z})$$

by construction.

(ii) Let $\psi,\psi'\in Q_n(C(f^!))$ be the $\mathbf{Z}_2$-hyperhomology classes appearing in the quadratic kernels $\sigma_*(f,b) = (C(f^!),\psi)$, $\sigma_*(f,b') = (C(f^!),\psi')$ of $(i-1)$-connected n-dimensional normal bundle maps

$$(f,b)\colon M\to X,\quad (f,b')\colon M\to X$$

for $n=2i$ or $2i+1$, with $b'=bc\colon \nu_M\to\nu_X$ for some automorphism $c\colon \nu_M\to\nu_M$ classified by $c\colon M\to SO$. By the Hurewicz theorem every element $x\in K_i(M) = \pi_{i+1}(f)$ is represented by an immersion $g\colon S^i\to M$ together with a null-homotopy $h\colon D^{i+1}\to X$ of $fg\colon S^i\to X$, so that by (i)

$$v^i(\psi')(x) - v^i(\psi)(x) = (Hj(cg),0)\in Q_n(S^{n-i}\mathbf{Z}[\pi]).$$

Now

$$j(cg) = (\text{stable Hopf invariant of } J(cg)\in\pi_i(SG) = \pi_i^S) = 0\in \hat H^{i+1}(\mathbf{Z}_2;\mathbf{Z})$$

for $i\neq 1,3,7$ by the result of Adams [1], so that the $(-)^i$ quadratic forms {formations} associated to $\sigma_*(f,b)$, $\sigma_*(f,b')$ are isomorphic by Proposition I.2.1 {Proposition I.2.5}.

For $X = S^{2i}$, $i \equiv 1 \pmod 2$ Proposition 9.11(ii) is the familiar result that the Arf invariant of an $(i-1)$-connected framed $2i$-manifold is independent of the framing for $i \neq 1, 3, 7$. Indeed, the original definition by Kervaire in [**7**] of the Arf invariant of such a manifold was independent of the choice of framing. For $i = 1, 3, 7$ there exists an $(i-1)$-connected $2i$-dimensional normal bundle map

$$(f, b): S^i \times S^i \to S^{2i}$$

involving an exotic framing b of $S^i \times S^i$, with Arf invariant

$$\sigma_*(f, b) = 1 \in L_{2i}(\mathbf{Z}) = \mathbf{Z}_2.$$

Moreover, crossing with S^1 gives an $(i-1)$-connected $(2i+1)$-dimensional normal bundle map

$$(f \times 1, b \times 1): S^i \times S^i \times S^1 \to S^{2i} \times S^1$$

involving an exotic framing $b \times 1$ of $S^i \times S^i \times S^1$, with surgery obstruction

$$\sigma_*(f \times 1, b \times 1) = 1 \in L_{2i+1}(\mathbf{Z}[\mathbf{Z}]) = \mathbf{Z}_2.$$

Proposition 9.11(ii) has the following consequence: for $n \neq 2, 3, 6, 7, 14, 15$ the bundle map b of an n-dimensional normal bundle map $(f, b): M \to X$ determines the sequence of framed surgeries below the middle dimension needed to obtain a normal bordant $([\frac{1}{2}n]-1)$-connected normal bundle map $(f', b'): M' \to X$, but the surgery obstruction

$$\sigma_*(f, b) = \sigma_*(f', b') \in L_n(\mathbf{Z}[\pi_1(X)])$$

is independent of the bundle map b'. For example, let $(f, b): S^1 \times S^1 \to S^2$ be a normal bundle map of Arf invariant 1, and let

$$(f', b'): M^4 \to S^2 \times S^1 \times S^1$$

be the 1-connected 4-dimensional normal bundle map obtained from $(f \times 1, b \times 1): S^1 \times S^1 \times S^1 \times S^1 \to S^2 \times S^1 \times S^1$ by two framed surgeries on $\pi_2(f \times 1) = \mathbf{Z}[\mathbf{Z}^2] \oplus \mathbf{Z}[\mathbf{Z}^2]$. The surgery obstruction

$$\sigma_*(f \times 1, b \times 1) = \sigma_*(f', b') = (0, 0, 0, 1) \in L_4(\mathbf{Z}[\mathbf{Z}^2]) = \mathbf{Z} \oplus 0 \oplus 0 \oplus \mathbf{Z}_2$$

is independent of b', and in fact can be expressed as the Witt class of the non-singular even symmetric form over $\mathbf{Z}[\mathbf{Z}^2]$ associated to the degree 1 map

$$f': M^4 \to S^2 \times S^1 \times S^1,$$

$$\sigma^1(f') = (0, 0, 0, 1) \in L\langle v_0 \rangle^+(\mathbf{Z}[\mathbf{Z}^2]) = L_4(\mathbf{Z}[\mathbf{Z}^2]).$$

Added in proof. The recent paper of Koschorke and Sanderson [**34**] interprets the approximation theorem $\Omega^\infty \Sigma^\infty X = (\coprod_{k \geqslant 1} E\Sigma_k \times_{\Sigma_k} (\prod_k X))/\sim$ for a connected pointed space X (with $\sim$ the equivalence relation given by

$\Sigma_k \subset \Sigma_{k+1}$ and the base point) in terms of immersion theory. This interpretation can be used to give a direct proof of the identification in Proposition 5.2 of the quadratic self-intersection $\mu(f)$ of an immersion $f: S^r \to M^{2r}$ defined by the quadratic construction ψ of § 1 with the geometric self-intersection defined in § 5 of Wall [**30**]. We shall give only a sketch of the argument here, leaving the details to a later occasion.

An oriented immersion of smooth manifolds $f: M^m \to N^n$ $(m \leqslant n)$ with normal bundle $\nu_f: M \to BSO(n-m)$ can be approximated by an embedding $f' = f \times g: M \hookrightarrow N \times \mathbf{R}^p$ (p large). The Pontrjagin–Thom construction applied to f' by collapsing the complement of a tubular neighbourhood of $f'(M)$ in $N \times \mathbf{R}^p$ gives a stable map $F: \Sigma^\infty N_+ \to \Sigma^\infty T(\nu_f)$ inducing the Umkehr chain map

$$f^!: C(N) = C(N)^{n-*} \xrightarrow{f^*} C(M)^{n-*} = S^{n-m}C(M) = \dot{C}(T(\nu_f)).$$

Assuming that f is in general position we have that the k-tuple point set of f,

$$S_k(f) = \{(x_1, x_2, \ldots, x_k) \in \prod_k M \mid f(x_i) = f(x_j) \in N,\ x_i \neq x_j \text{ for } i \neq j\}/\Sigma_k,$$

is an $(n - k(n-m))$-dimensional manifold, with an immersion

$$f_k: S_k(f) \to N;\ [x_1, x_2, \ldots, x_k] \mapsto f(x_1) \quad (k \geqslant 1,\ S_1(f) = M,\ f_1 = f).$$

Let Σ_k act on the contractible space

$$E\Sigma_k = \{(t_1, t_2, \ldots, t_k) \in \prod_k \mathbf{R}^\infty \mid t_i \neq t_j \text{ for } i \neq j\}$$

by permutation of components, as usual, and define a map

$$F_k: S_k(f) \to E\Sigma_k \times_{\Sigma_k} (\prod_k M);$$

$$[x_1, x_2, \ldots, x_k] \mapsto [(g(x_1), g(x_2), \ldots, g(x_k)), (x_1, x_2, \ldots, x_k)] \quad (\mathbf{R}^p \subset \mathbf{R}^\infty, k \geqslant 1).$$

The maps appearing in the approximation theorem

$$a_k: E\Sigma_k \times_{\Sigma_k} (\prod_k X) \to \Omega^\infty\Sigma^\infty X \quad (k \geqslant 1)$$

are also defined by the Pontrjagin–Thom construction: given

$$b = [(t_1, t_2, \ldots, t_k), (x_1, x_2, \ldots, x_k)] \in E\Sigma_k \times_{\Sigma_k} (\prod_k M),$$

let $t: S^q \to \bigvee_{i=1}^k S^q$ be the map obtained by collapsing the complement of a tubular neighbourhood of $\{t_1, t_2, \ldots, t_k\} \subset \mathbf{R}^q$ (q large), define base-point-preserving maps

$$x_i: S^q \to \Sigma^q X = S^q \wedge X;\ s \mapsto s \wedge x_i \quad (1 \leqslant i \leqslant k),$$

and define $a_k(b) \in \Omega^\infty\Sigma^\infty X = \underrightarrow{\mathrm{Lim}}_q \Omega^q\Sigma^q X$ by

$$a_k(b): S^q \xrightarrow{t} \bigvee_{i=1}^{k} S^q \xrightarrow{\bigvee_i x_i} \Sigma^q X.$$

Thus there is defined a commutative diagram

$$\begin{array}{ccc}
\coprod_{k\geq 1} S_k(f) & \xrightarrow{\coprod_k F_k} & \coprod_{k\geq 1} E\Sigma_k \times_{\Sigma_k} (\prod_k M) \\
\Big\downarrow \coprod_k f_k & & \Big\downarrow \coprod_k 1 \times (\prod_k z) \\
N_+ & \xrightarrow{\mathrm{adj}(F)} & \Omega^\infty\Sigma^\infty T(\nu_f) = (\coprod_{k\geq 1} E\Sigma_k \times_{\Sigma_k} (\prod_k T(\nu_f)))/\sim
\end{array}$$

with $z: M \hookrightarrow T(\nu_f)$ the inclusion given by the zero section of ν_f. Now $\Omega^\infty\Sigma^\infty T(\nu_f) = (\coprod_{k\geq 1} E\Sigma_k \times_{\Sigma_k}(\prod_k T(\nu_f)))/\sim$ is the stratified Thom space of the extended power bundles

$$e_k(\nu_f): E\Sigma_k \times_{\Sigma_k} (\prod_k M) \to BO(k(n-m))$$

$(k \geqslant 1, T(e_k(\nu_f)) = E\Sigma_k \ltimes_{\Sigma_k} (\bigwedge_k T(\nu_f)))$ and $\mathrm{adj}(F): N_+ \to \Omega^\infty\Sigma^\infty T(\nu_f)$ is transverse regular at the stratified zero section

$$\coprod_{k\geq 1} E\Sigma_k \times_{\Sigma_k} (\prod_k M) \hookrightarrow \Omega^\infty\Sigma^\infty T(\nu_f),$$

with inverse image

$$S(f) \equiv \mathrm{im}(\coprod_k f_k: \coprod_{k\geq 1} S_k(f) \to N) = \{y \in N \mid f^{-1}(y) \neq \emptyset\}$$

stratified by $\mathrm{im}(f_k: S_k(f) \to N) = \{y \in N \mid |f^{-1}(y)| \geqslant k\}$, and

$$\nu_{f_k}: S_k(f) \xrightarrow{F_k} E\Sigma_k \times_{\Sigma_k} (\prod_k M) \xrightarrow{e_k(\nu_f)} BO(k(n-m)) \quad (k \geqslant 1).$$

It follows that the map

$$\begin{aligned}
\mathrm{adj}(F)_*: H_n(N) = \dot{H}_n(N_+) \to \dot{H}_n(\Omega^\infty\Sigma^\infty T(\nu_f)) \\
= \bigoplus_{k\geq 1} \dot{H}_n(T(e_k(\nu_f))) \\
= \bigoplus_{k\geq 1} H^w_{n-k(n-m)}(E\Sigma_k \times_{\Sigma_k} (\prod_k M))
\end{aligned}$$

sends the fundamental class $[N] \in H_n(N)$ (defined using $w(N)$-twisted homology) to

$$\mathrm{adj}(F)_*[N] = \bigoplus_{k\geq 1} F_k[S_k(f)] \in \bigoplus_{k\geq 1} H^w_{n-k(n-m)}(E\Sigma_k \times_{\Sigma_k} (\prod_k M)),$$

where $[S_k(f)] \in H^w_{n-k(n-m)}(S_k(f))$ is the fundamental class of $S_k(f)$ and w

refers to $w(N)w_1(e_k(\nu_f))$-twisted homology, with

$$w_1(e_k(\nu_f))\colon \pi_1(E\Sigma_k \times_{\Sigma_k} (\prod_k M)) = \Sigma_k \wr \pi_1(M) \longrightarrow \Sigma_k \xrightarrow{(n-m)\mathrm{sgn}} \mathbf{Z}_2.$$

Similar considerations apply to the multiple point manifolds $S_*(\tilde{f})$ of a π-equivariant immersion $\tilde{f}\colon \tilde{M} \to \tilde{N}$ lifting f, with $\tilde{N}$ a covering of N with group of covering translations π, taking into account the diagonal π-action on $S_*(\tilde{f})$. In particular, for an immersion $f\colon S^r \to N^{2r}$ $(r \geqslant 2)$ and the universal cover $\tilde{N}$ of N we have $\tilde{S}^r = \pi \times S^r$ $(\pi = \pi_1(N))$ and a stable π-map $F\colon \Sigma^\infty \tilde{N}_+ \to \Sigma^\infty T\pi(\nu_f)$ such that the quadratic component of $\mathrm{adj}(F)_*[N]$ is

$$\psi(\mathrm{adj}(F)_*[N]) = F_2[S_2(\tilde{f})/\pi]$$
$$\in H_0^w(E\Sigma_2 \times_{\Sigma_2} (\tilde{S}^r \times_\pi \tilde{S}^r)) = Q_0(C(\tilde{S}^r), (-)^r) = H_0(\mathbf{Z}_2; \mathbf{Z}[\pi], (-)^r),$$

with the $w(N)$-twisted involution on $\mathbf{Z}[\pi]$. The left-hand side of the equation is the quadratic self-intersection $\mu(f)$ defined in § 5 above, while the right-hand side is the geometric self-intersection $\mu(f)$ defined in § 5 of Wall [**30**].

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