The algebraic theory of surgery

by Andrew Ranicki

Introduction

An algebraic theory of surgery on chain complexes with an abstract Poincaré duality should be a "simple and satisfactory algebraic version of the whole setup" to quote §17G of C.T.C.Wall's "Surgery on compact manifolds" ([5]), which includes a first approximation to such a theory. A second and a third attempt are due to A.S.Mishchenko [1],[2]. These are sufficient to obtain the symmetric part of the surgery obstruction, and so determine it modulo 2-torsion. The theory presented here obtains the quadratic structure as well, thus giving all of the surgery obstruction. Our theory of surgery is homotopy invariant in geometry and chain homotopy invariant in algebra. It applies also to codimension 2 surgery obstructions, such as arise in knot theory.

An n-dimensional algebraic Poincaré complex over a ring A with involution $\bar{}:A\longrightarrow A;a\longmapsto\bar{a}$ is an A-module chain complex C with an n-dimensional Poincaré duality

$$H^*(C) = H_{n-*}(C) .$$

We shall use n-dimensional algebraic Poincaré complexes to define functors

$$\begin{cases} L^n \\ L_n \end{cases} : (\text{rings with involution}) \longrightarrow (\text{abelian groups}) \qquad (n\in\mathbb{Z})$$

such that $\begin{cases} L^0(A) \\ L_0(A) \end{cases}$ is the Witt group of non-singular $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ forms over A.

The quadratic L-groups $L_n(A)$ $(n\in\mathbb{Z})$ will turn out to be the surgery obstruction groups of Wall [5], with a 4-periodicity

$$L_n(A) = L_{n+4}(A) \quad (n\in\mathbb{Z}) .$$

The higher symmetric L-groups $L^n(A)$ $(n\geqslant 0)$ are the "algebraic Poincaré bordism" groups $\Omega_n(A)$ of Mishchenko [2], and are not 4-periodic in general. The lower symmetric L-groups are such that

$$L^n(A) = L_n(A) \quad (n\leqslant -3) .$$

The symmetric L-groups differ from the quadratic L-groups in 2-torsion only, with $L^n(A) = L_n(A)$ $(n\in\mathbb{Z})$ if 2 is invertible in A.

The $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ L-groups $\begin{cases} L^n(A) \\ L_n(A) \end{cases}$ $(n\in\mathbb{Z})$ are to the $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Witt group $\begin{cases} L^0(A) \\ L_0(A) \end{cases}$ what the algebraic K-groups $K_n(A)$ $(n\in\mathbb{Z})$ are to the projective class group $K_0(A)$. In order to justify this assertion we shall exhibit

i) a change of rings exact sequence

$$\begin{cases} \dots \longrightarrow L^n(A) \xrightarrow{f} L^n(B) \longrightarrow L^n(f) \longrightarrow L^{n-1}(A) \xrightarrow{f} L^{n-1}(B) \longrightarrow \dots \\ \dots \longrightarrow L_n(A) \xrightarrow{f} L_n(B) \longrightarrow L_n(f) \longrightarrow L_{n-1}(A) \xrightarrow{f} L_{n-1}(B) \longrightarrow \dots \end{cases} \quad (n\in\mathbb{Z}) ,$$

ii) products

$$\begin{cases} \otimes : L^m(A)\otimes_{\mathbb{Z}}L^n(B) \longrightarrow L^{m+n}(A\otimes_{\mathbb{Z}}B) \\ \otimes : L_m(A)\otimes_{\mathbb{Z}}L_n(B) \longrightarrow L_{m+n}(A\otimes_{\mathbb{Z}}B) \end{cases} \quad (m,n\in\mathbb{Z}) ,$$

iii) a localization exact sequence

$$\begin{cases} \dots \longrightarrow L^n(A) \longrightarrow L^n(S^{-1}A) \longrightarrow L^n(A,S) \longrightarrow L^{n-1}(A) \longrightarrow L^{n-1}(S^{-1}A) \longrightarrow \dots \\ \dots \longrightarrow L_n(A) \longrightarrow L_n(S^{-1}A) \longrightarrow L_n(A,S) \longrightarrow L_{n-1}(A) \longrightarrow L_{n-1}(S^{-1}A) \longrightarrow \dots \end{cases} \quad (n\in\mathbb{Z})$$

involving the $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ L-groups $\begin{cases} L^n(A,S) \\ L_n(A,S) \end{cases}$ of algebraic Poincaré complexes over A which become contractible over a localization $S^{-1}A$,

iv) simplicial spectra $\begin{cases} \mathcal{L}^*(A) \\ \mathcal{L}_*(A) \end{cases}$ such that

$$\begin{cases} \pi_n(\mathcal{L}^*(A)) = L^n(A) \\ \pi_n(\mathcal{L}_*(A)) = L_n(A) \end{cases} \quad (n\in\mathbb{Z}) .$$

Furthermore, in certain cases we shall have

v) a Künneth formula

$$\begin{cases} L^n(A[z,z^{-1}]) = L^n(A)\oplus L^{n-1}(A) \\ L_n(A[z,z^{-1}]) = L_n(A)\oplus L_{n-1}(A) \end{cases} \quad (\bar{z} = z^{-1}) ,$$

vi) a split exact sequence

$$\begin{cases} 0 \longrightarrow L^n(A) \longrightarrow L^n(A[x])\oplus L^n(A[x^{-1}]) \longrightarrow L^n(A[x,x^{-1}]) \longrightarrow L^n(A) \longrightarrow 0 \\ 0 \longrightarrow L_n(A) \longrightarrow L_n(A[x])\oplus L_n(A[x^{-1}]) \longrightarrow L_n(A[x,x^{-1}]) \longrightarrow L_n(A) \longrightarrow 0 \end{cases} \quad (\bar{x} = x) ,$$

vii) a Mayer-Vietoris exact sequence for a cartesian square $\begin{matrix} A \to B \\ \downarrow \quad \downarrow \\ B' \to A' \end{matrix}$,

$$\begin{cases} \dots \longrightarrow L^n(A) \longrightarrow L^n(B)\oplus L^n(B') \longrightarrow L^n(A') \longrightarrow L^{n-1}(A) \longrightarrow L^{n-1}(B)\oplus L^{n-1}(B') \longrightarrow \dots \\ \dots \longrightarrow L_n(A) \longrightarrow L_n(B)\oplus L_n(B') \longrightarrow L_n(A') \longrightarrow L_{n-1}(A) \longrightarrow L_{n-1}(B)\oplus L_{n-1}(B') \longrightarrow \dots \end{cases} .$$

An "algebraic Poincaré complex" in the sense of Mishchenko [2] is a chain complex of f.g. projective modules over a ring with involution A

$$C : C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \cdots \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

together with a collection of A-module morphisms

$$\varphi_s : C^{n-r+s} \longrightarrow C_r \quad (s \geqslant 0)$$

such that

$$d\varphi_s + (-)^r \varphi_s d^* + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T\varphi_{s-1}) = 0 : C^{n-r+s-1} \longrightarrow C_r \quad (\varphi_{-1} = 0)$$

and such that the chain map

$$\varphi_0 : C^{n-*} \longrightarrow C$$

is a chain equivalence, inducing abstract Poincaré duality isomorphisms

$$\varphi_0 : H^{n-r}(C) \longrightarrow H_r(C) \quad ,$$

where C^{n-*} is the chain complex of dual A-modules $C^r = C_r^* = \mathrm{Hom}_A(C_r, A)$ defined by

$$(C^{n-*})_r = C^{n-r} \quad , \quad d_{C^{n-*}} = (-)^r d^* : C^{n-r} \longrightarrow C^{n-r+1}$$

and T is the duality involution

$$T : \mathrm{Hom}_A(C^p, C_q) \longrightarrow \mathrm{Hom}_A(C^q, C_p) \ ; \ \varphi \longmapsto (-)^{pq}\varphi^* \qquad (C_p^{**} = C_p) \ .$$

The motivation for this definition comes from the symmetry properties of the chain equivalence

$$\varphi_0 = [X] \cap - : C(X)^{n-*} \longrightarrow C(X) \qquad ([X] = 1 \in H_n(X) = \mathbb{Z})$$

inducing the Poincaré duality isomorphisms

$$\varphi_0 = [X] \cap - : H^{n-r}(X) \longrightarrow H_r(X)$$

of a compact oriented n-dimensional manifold X, with φ_1 a chain homotopy between φ_0 and $T\varphi_0$, φ_2 a higher chain homotopy between φ_1 and $T\varphi_1$, and so on. The "algebraic Poincaré bordism" groups $\Omega_n(A)$ of Mishchenko [2] (which we denote by $L^n(A)$) are the groups of equivalence classes of such n-dimensional symmetric Poincaré complexes over A (as we shall call them) under a cobordism relation given by abstract Poincaré-Lefschetz duality.

An n-dimensional geometric Poincaré complex X determines in a natural way an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi_1(X)]$

$$\sigma^*(X) = (C(\widetilde{X}), \varphi_{\widetilde{X}})$$

with $\widetilde{X}$ the universal cover of X. The "higher signature" of Mishchenko [2] is the symmetric Poincaré cobordism class

$$\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)]) \ .$$

This is a geometric Poincaré bordism invariant which is a $\pi_1(X)$-equivariant generalization of the signature.

Given a degree 1 map of n-dimensional geometric Poincaré complexes

$$f : M \longrightarrow X$$

there is defined a kernel n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi_1(X)]$

$$\sigma^*(f) = (C(f^!), \varphi_f)$$

such that up to chain equivalence preserving the symmetric structure

$$\sigma^*(M) = \sigma^*(f) \oplus \sigma^*(X) \ ,$$

where $C(f^!)$ is the algebraic mapping cone of the Umkehr chain map

$$f^! : C(\widetilde{X}) \xrightarrow{([X]\cap -)^{-1}} C(\widetilde{X})^{n-*} \xrightarrow{\widetilde{f}^*} C(\widetilde{M})^{n-*} \xrightarrow{[M]\cap -} C(\widetilde{M})$$

with $\widetilde{M}$ the covering space of M induced by f from the universal cover $\widetilde{X}$ of X, and with $\sigma^*(M)$ defined using $\widetilde{M}$ here.

A normal map of geometric Poincaré complexes

$$(f : M \longrightarrow X, b : \nu_M \longrightarrow \nu_X)$$

is a degree 1 map f together with a covering map b of Spivak normal fibrations. The fundamental construction of this paper refines the symmetric structure φ_f of the kernel $\sigma^*(f) = (C(f^!), \varphi_f)$ of a normal map (f,b) to a quadratic structure ψ_b depending only on the fibre homotopy class of b (Proposition 2.9). Among others, we shall prove the following two results:

i) The surgery obstruction of a normal map (f,b) is the equivalence class

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

of the quadratic kernel $\sigma_*(f,b) = (C(f^!), \psi_b)$ in the cobordism group of such n-dimensional quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X)]$ (Proposition 8.1). In Proposition 8.2 we shall explicitly construct the corresponding element of the surgery obstruction group $L_n(\pi_1(X))$ of Wall [5] . This does not require geometric surgery below the middle dimension, and is thus an "instant surgery obstruction".

ii) The surgery obstruction of the product of normal maps $(f\times g:M\times N \longrightarrow X\times Y, b\times c:\nu_M\times\nu_N \longrightarrow \nu_X\times\nu_Y)$ is given by the formula

$$\sigma_*(f\times g,b\times c) = \sigma_*(f,b)\otimes\sigma_*(g,c) + \sigma^*(X)\otimes\sigma_*(g,c) + \sigma_*(f,b)\otimes\sigma^*(Y) \in L_{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) .$$

involving the "higher signatures" of Mishchenko [2] (Proposition 11.2). Prior to such applications it is necessary to develop a theory of abstract quadratic structures appropriate for the range of $b \longmapsto \psi_b$.

In the first version of the present theory of quadratic structures an n-dimensional quadratic Poincaré complex over a ring with involution A was defined to be a chain complex of f.g. projective A-modules

$$C : C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

together with a collection of A-module morphisms

$$\psi_s : C^{n-r-s} \longrightarrow C_r \quad (s \geqslant 0)$$

such that

$$d\psi_s + (-)^r\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}) = 0 : C^{n-r-s-1} \longrightarrow C_r$$

and such that the chain map

$$(1+T)\psi_0 : C^{n-*} \longrightarrow C$$

is a chain equivalence. This definition suggested itself on the pragmatic grounds that the cobordism groups $L_n(A)$ of n-dimensional quadratic Poincaré complexes over A (C,ψ) could be shown to be 4-periodic

$$L_n(A) = L_{n+4}(A) \quad (n \geqslant 0) ,$$

coinciding for a group ring $A = \mathbb{Z}[\pi]$ with the surgery obstruction groups of Wall [5]

$$L_n(\mathbb{Z}[\pi]) = L_n(\pi) \quad (n(\mathrm{mod}\ 4)) .$$

There remained the problem of exhibiting such a quadratic structure ψ on the chain complex kernel $C(f^!)$ of a normal map $(f,b):M \longrightarrow X$. Graeme Segal pointed out that for the chain complex $C = C(X)$ of a topological space X such collections

$\psi = \{\psi_s | s \geqslant 0\}$ (for $A = \mathbb{Z}$) are the cycles of classes in the homology groups of the 'quadratic construction' $(S^\infty \times X \times X)/\mathbb{Z}_2$

$$H_n((S^\infty \times X \times X)/\mathbb{Z}_2) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C(X)\otimes C(X)))$$

with the generator $T\in\mathbb{Z}_2$ acting by the antipodal map on S^∞ and by the transposition $T:(x,y) \longmapsto (y,x)$ on $X\times X$, where

$$W = C(S^\infty) : \ldots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

and $C(X)\otimes_{\mathbb{Z}}C(X)$ is identified with $\mathrm{Hom}_{\mathbb{Z}}(C(X)^*,C(X))$ in the natural way. This led to a formulation of the quadratic theory in which a quadratic structure on an A-module chain complex C is defined to be a class $\psi\in Q_n(C)$ in the $\mathbb{Z}_2$-hyperhomology group

$$Q_n(C) = H_n(\mathbb{Z}_2,C\otimes_A C) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C\otimes_A C))$$

with $T\in\mathbb{Z}_2$ acting on $C\otimes_A C$ by the transposition

$$T : C_p\otimes_A C_q \longrightarrow C_q\otimes_A C_p \ ; \ x\otimes y \longmapsto (-)^{pq}y\otimes x ,$$

which corresponds to the duality involution T on $\mathrm{Hom}_A(C^*,C)$ under the natural identification $\mathrm{Hom}_A(C^*,C) = C\otimes_A C$ for f.g. projective C. In turn, this led to a reformulation of the symmetric theory in which the symmetric structures $\{\varphi_s | s \geqslant 0\}$ of Mishchenko [2] are viewed as the cycles of classes $\varphi\in Q^n(C)$ in the $\mathbb{Z}_2$-hypercohomology group

$$Q^n(C) = H^n(\mathbb{Z}_2,C\otimes_A C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C\otimes_A C))$$

Transfer defines a map from quadratic structures to symmetric structures

$$(1+T) : Q_n(C) \longrightarrow Q^n(C) \ ; \ \psi \longmapsto (1+T)\psi \quad , \quad ((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & s = 0 \\ 0 & s > 0 . \end{cases}$$

This reduces the problem to the lifting of the $\mathbb{Z}_2$-hypercohomology class $\varphi_f \in Q^n(C(f^!))$ appearing in the symmetric kernel $\sigma^*(f) = (C(f^!),\varphi_f)$ of a normal map $(f:M \longrightarrow X, b:\nu_M \longrightarrow \nu_X)$ to a $\mathbb{Z}_2$-hyperhomology class $\psi_b \in Q_n(C(f^!))$ such that

$$(1+T)\psi_b = \varphi_f \in Q^n(C(f^!)) ,$$

allowing the definition of a quadratic kernel

$$\sigma_*(f,b) = (C(f^!),\psi_b)$$

containing the surgery obstruction. Ib Madsen suggested using a more refined version of the construction of the Arf invariant in §4 of Chapter III of Browder [2] which involved the S-dual of the induced map of Thom spaces $T(b):T(\nu_M)\longrightarrow T(\nu_X)$, a stable map $F:\Sigma^p X_+\longrightarrow\Sigma^p M_+$ (p large) inducing the Umkehr $f^!:C(X)\longrightarrow C(M)$ on the chain level. (The adjoint $F:X_+\longrightarrow\Omega^\infty\Sigma^\infty M_+$ sends the fundamental class $[X]\in H_n(X)$ to an element $F[X]\in H_n(\Omega^\infty\Sigma^\infty M_+)$, which contains $Q_n(C(M)) = H_n((S^\infty\times M\times M)/\mathbb{Z}_2)$ as a direct summand). This led to the observation that an abstract $\mathbb{Z}_2$-hypercohomology class $\varphi\in Q^n(C)$ lies in $\mathrm{im}((1+T):Q_n(C)\longrightarrow Q^n(C))$ if and only if

$$S^p\varphi = 0\in Q^{n+p}(S^pC) \quad \text{(p large)} ,$$

where SC is the suspension of C ($SC_r = C_{r-1}$) and

$$S : Q^n(C)\longrightarrow Q^{n+1}(SC) \; ; \; \varphi\longmapsto S\varphi \; , \quad (S\varphi)_s = \begin{cases}\varphi_{s-1} & s\geqslant 1\\ 0 & s=0\end{cases}$$

(Proposition 1.2). The $\mathbb{Z}_2$-hypercohomology class $\varphi_f\in Q^n(C(f^!))$ is of this type if there exists a $\pi_1(X)$-equivariant map $F:\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+$ (p large) inducing the Umkehr $f^!:C(\widetilde{X})\longrightarrow C(\widetilde{M})$ on the chain level, in which case it is possible to obtain a specific $\mathbb{Z}_2$-hyperhomology class $\psi_F\in Q_n(C(f^!))$ depending only on the stable $\pi_1(X)$-equivariant homotopy class F, and such that

$$(1+T)\psi_F = \varphi_f\in Q^n(C(f^!))$$

(Proposition 2.9). A normal map $(f:M\longrightarrow X, b:\nu_M\longrightarrow\nu_X)$ gives rise to such an equivariant geometric Umkehr map $F:\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+$ via an equivariant S-duality theory (Proposition 3.13), and $\psi_b = \psi_F\in Q_n(C(f^!))$ defines the kernel n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$

$$\sigma_*(f,b) = (C(f^!),\psi_b) \; .$$

For highly-connected f, with M a manifold and b a map of bundles, this agrees with the surgery obstruction kernel obtained by Wall [5] using geometric intersection and self-intersection forms.

The localization exact sequence of §13 will be used to prove that the natural maps

$$\begin{cases} L^n(\mathbb{Z}[\pi])\longrightarrow L^n(\mathbb{R}[\pi]) \\ L_n(\mathbb{Z}[\pi])\longrightarrow L_n(\mathbb{R}[\pi])\end{cases} \quad (n\in\mathbb{Z})$$

are isomorphisms modulo 8-torsion, for any group π (Proposition 13.24).

§1. Quadratic algebra

Let A be an associative ring (with 1) and let

$$\bar{\ } : A \longrightarrow A \; ; \; a \longmapsto \bar{a}$$

be an involution on A, with

$$\overline{(a+b)} = \bar{a} + \bar{b} \;,\; \overline{(ab)} = \bar{b}.\bar{a} \;,\; \bar{\bar{a}} = a \;,\; \bar{1} = 1 \qquad (a,b \in A) \;.$$

Given a left A-module M let M^t be the right A-module defined by the additive group of M with A acting by

$$M^t \times A \longrightarrow M^t \;;\; (x,a) \longmapsto \bar{a}x \;.$$

Except where a right action is specified "A-module" will refer to a left A-action.

The dual of an A-module M is the A-module

$$M^* = \mathrm{Hom}_A(M,A)$$

with A acting by

$$A \times M^* \longrightarrow M^* \;;\; (a,f) \longmapsto (x \longmapsto f(x).\bar{a}) \;.$$

The dual of an A-module morphism $f \in \mathrm{Hom}_A(M,N)$ is the A-module morphism

$$f^* : N^* \longrightarrow M^* ; g \longmapsto (x \longmapsto g(f(x))) \;.$$

If M is f.g. projective then so is M^*, and there is defined a natural A-module isomorphism

$$M \longrightarrow M^{**} \;;\; x \longmapsto (f \longmapsto \overline{f(x)})$$

which we shall use as an identification.

The homology and cohomology A-modules of an A-module chain complex

$$C : \ldots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \longrightarrow \ldots \qquad (d^2 = 0,\; r \in \mathbb{Z})$$

are defined by

$$H_r(C) = \ker(d : C_r \longrightarrow C_{r-1})/\mathrm{im}(d : C_{r+1} \longrightarrow C_r)$$
$$H^r(C) = \ker(d^* : C^r \longrightarrow C^{r+1})/\,\mathrm{im}(d^* : C^{r-1} \longrightarrow C^r) \qquad (C^r = C_r^*).$$

An A-module chain complex C is n-dimensional if it is chain equivalent to a f.g. projective A-module chain complex of the type

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \longrightarrow 0 \qquad \ldots \;.$$

A finite ~~dimensional~~ f.g. projective chain complex C is n-dimensional if and only if $H_r(C) = 0$ for $r < 0$ and $H^r(C) = 0$ for $r > n$.

Given A-module chain complexes C,D let $C^t \otimes_A D$, $\mathrm{Hom}_A(C,D)$ be the abelian group chain complexes defined by

$$(C^t \otimes_A D)_n = \sum_{p+q=n} C_p^t \otimes_A D_q \;,\; d_{C^t \otimes_A D}(x \otimes y) = x \otimes d_D(y) + (-)^q d_C(x) \otimes y$$

$$\mathrm{Hom}_A(C,D)_n = \sum_{q-p=n} \mathrm{Hom}_A(C_p,D_q) \;,\; d_{\mathrm{Hom}_A(C,D)}(f) = d_D f + (-)^q f d_C \;.$$

The slant chain map

$$\backslash : C^t \otimes_A D \longrightarrow \mathrm{Hom}_A(C^{-*},D) \;;\; x \otimes y \longmapsto (f \longmapsto \overline{f(x)}y)$$

is an isomorphism if C is f.g. projective, where C^{-*} is the A-module chain complex defined by $(C^{-*})_r = C^{-r}$, $d_{C^{-*}} = d_C^*$.

Given a central unit $\varepsilon \in A$ such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$ (e.g. $\varepsilon = \pm 1 \in A$) and an A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on $C^t \otimes_A C$ by the ε-transposition involution

$$T_\varepsilon : C_p^t \otimes_A C_q \longrightarrow C_q^t \otimes_A C_p \;;\; x \otimes y \longmapsto (-)^{pq} y \otimes \varepsilon x \;,$$

and define the Q-groups

$$\begin{cases} Q^n_{[i,j]}(C,\varepsilon) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j], C^t \otimes_A C)) \\ Q_n^{[i,j]}(C,\varepsilon) = H_n(W[i,j] \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes_A C)) \end{cases} \qquad (-\infty \leqslant i \leqslant j \leqslant \infty \;,\; n \geqslant 0)$$

with $W[i,j]$ the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex given by

$$W[i,j]_r = \begin{cases} \mathbb{Z}[\mathbb{Z}_2] & i \leqslant r \leqslant j \\ 0 & \text{otherwise} \end{cases} \;,\quad d_{W[i,j]} = 1 + (-)^r T : W[i,j]_r \longrightarrow W[i,j]_{r-1} \;.$$

An element $\begin{cases} \varphi \in Q^n_{[i,j]}(C,\varepsilon) \\ \psi \in Q_n^{[i,j]}(C,\varepsilon) \end{cases}$ is represented by a collection of chains

$$\begin{cases} \varphi = \{\varphi_s \in (C^t \otimes_A C)_{n+s} \,|\, i \leqslant s \leqslant j\} \\ \psi = \{\psi_s \in (C^t \otimes_A C)_{n-s} \,|\, i \leqslant s \leqslant j\} \end{cases}$$

such that

$$\begin{cases} d_{C^t \otimes_A C}\varphi_s + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon \varphi_{s-1}) = 0 \in (C^t \otimes_A C)_{n+s-1} & (i \leqslant s \leqslant j,\; \varphi_{i-1} = 0) \\ d_{C^t \otimes_A C}\psi_s + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\varepsilon \psi_{s+1}) = 0 \in (C^t \otimes_A C)_{n-s-1} & (i \leqslant s \leqslant j,\; \psi_{j+1} = 0) \;. \end{cases}$$

The notation is heavily redundant, allowing identifications

$$Q^n_{[i,j]}(C,\varepsilon) = Q^{n+k}_{[i-k,j-k]}(C,(-)^k \varepsilon) = Q_{n+k}^{[k-j,k-i]}(C,(-)^{k+1}\varepsilon) \qquad (k \in \mathbb{Z}) \;.$$

For $i = j$ we have $Q^n_{[i,i]}(C,\varepsilon) = H_{n+i}(C^t \otimes_A C)$, $Q_n^{[i,i]}(C,\varepsilon) = H_{n-i}(C^t \otimes_A C)$.

$$A\begin{cases}\underline{\text{chain map}}\\ \underline{\text{chain homotopy}}\\ \underline{\text{homotopy}}\end{cases}\text{of } A\text{-module chain}\begin{cases}\text{complexes } f:C\longrightarrow D\\ \text{maps } g:f\simeq f':C\longrightarrow D\\ \text{homotopies } h:g\simeq g':f\simeq f':C\longrightarrow D\end{cases}$$

$$\text{is a collection of } A\text{-module morphisms}\begin{cases}\{f\in \mathrm{Hom}_A(C_r,D_r)\,|\,r\in\mathbb{Z}\}\\ \{g\in \mathrm{Hom}_A(C_r,D_{r+1})\,|\,r\in\mathbb{Z}\}\\ \{h\in \mathrm{Hom}_A(C_r,D_{r+2})\,|\,r\in\mathbb{Z}\}\end{cases}\text{such that}$$

$$\begin{cases}d_Df-fd_C=0:C_r\longrightarrow D_{r-1}\\ f'-f=d_Dg+gd_C:C_r\longrightarrow D_r \qquad (r\in\mathbb{Z})\\ g'-g=d_Dh-hd_C:C_r\longrightarrow D_{r+1}\end{cases}.$$

A chain map which admits a chain homotopy inverse is a <u>chain equivalence</u>. We recall that a chain map $f:C\longrightarrow D$ of finite-dimensional chain complexes is a chain equivalence if and only if it induces isomorphisms $f_*:H_*(C)\longrightarrow H_*(D)$ in homology (or alternatively in cohomology $f^*:H^*(D)\longrightarrow H^*(C)$).

$$\text{The isomorphism type of the groups}\begin{cases}Q^n_{[i,j]}(C,\varepsilon)\\ Q_n^{[i,j]}(C,\varepsilon)\end{cases}\text{depends only on}$$

the chain homotopy type of C, despite the quadratic nature of the construction used in their definition.

$$\underline{\text{Proposition 1.1}}\text{ i) An } A\text{-module}\begin{cases}\text{chain map } f:C\longrightarrow D\\ \text{chain homotopy } g:f\simeq f':C\longrightarrow D\\ \text{homotopy } h:g\simeq g':f\simeq f':C\longrightarrow D\end{cases}\text{induces a}$$

$$\mathbb{Z}\text{-module}\begin{cases}\text{chain map}\\ \text{chain homotopy}\\ \text{homotopy}\end{cases}$$

$$\begin{cases}\begin{cases}f^{\%}\\ g^{\%}:f^{\%}\simeq f'^{\%}\\ h^{\%}:g^{\%}\simeq g'^{\%}:f^{\%}\simeq f'^{\%}\end{cases} : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],C^t\otimes_A C)\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],D^t\otimes_A D)\\ \begin{cases}f_{\%}\\ g_{\%}:f_{\%}\simeq f'_{\%}\\ h_{\%}:g_{\%}\simeq g'_{\%}:f_{\%}\simeq f'_{\%}\end{cases} : W[i,j]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C)\longrightarrow W[i,j]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(D^t\otimes_A D)\end{cases}$$

$$\text{ii) If } C \text{ is } m\text{-dimensional then}\begin{cases}Q^n_{[i,j]}(C,\varepsilon)=0\\ Q_n^{[i,j]}(C,\varepsilon)=0\end{cases}\text{if}\begin{cases}n+i>2m \text{ or } n+j<0\\ n-j>2m \text{ or } n-i<0\end{cases}.$$

iii) For $-\infty\leq i\leq j\leq k\leq\infty$ there is defined a long exact sequence of abelian groups

$$\begin{cases}\dots\longrightarrow Q^n_{[j+1,k]}(C,\varepsilon)\longrightarrow Q^n_{[i,k]}(C,\varepsilon)\longrightarrow Q^n_{[i,j]}(C,\varepsilon)\longrightarrow Q^{n-1}_{[j+1,k]}(C,\varepsilon)\longrightarrow\dots\\ \dots\longrightarrow Q_n^{[i,j]}(C,\varepsilon)\longrightarrow Q_n^{[i,k]}(C,\varepsilon)\longrightarrow Q_n^{[j+1,k]}(C,\varepsilon)\longrightarrow Q_{n-1}^{[i,j]}(C,\varepsilon)\longrightarrow\dots\end{cases}.$$

<u>Proof</u> i) Given $\varphi\in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],C^t\otimes_A C)_n$ set

$$\begin{cases}f^{\%}(\varphi)_s=(f^t\otimes_A f)\varphi_s\in(D^t\otimes_A D)_{n+s}\\ g^{\%}(\varphi)_s=(f^t\otimes_A g+(-)^q g^t\otimes_A f')\varphi_s+(-)^{q+1}(g^t\otimes_A g)\varphi_{s-1}\\ \qquad\qquad\in(D^t\otimes_A D)_{n+s+1}=\sum\limits_{q=-\infty}^{\infty}D^t_{n-q+s+1}\otimes_A D_q\\ h^{\%}(\varphi)_s=(f^t\otimes_A h+(-)^q h^t\otimes_A f')\varphi_s+(-)^{q+1}(g^t\otimes_A h+(-)\quad h^t\otimes_A g')\varphi_{s-1}\\ \qquad\qquad+(-)\quad(h^t\otimes_A h)\varphi_{s-2}\in(D^t\otimes_A D)_{n+s+2}=\sum\limits_{q=\infty}^{\infty}D^t_{n-q+s+2}\otimes_A D_q\\ \qquad\qquad(i\leq s\leq j,\ \varphi_{i-1}=0\ ,\ \varphi_{i-2}=0)\ .\end{cases}$$

The lower case is the same, after identifying

$$S(W[i,j]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C))=\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-j-1,-i-1],C^t\otimes_A C)\quad(i\leq j).$$

ii) By the chain homotopy invariance of i) it may be assumed that C is such that $C_r=0$ for $r<0$ or $r>m$.

iii) Given intervals $[i,j],[i',j']$ such that $i\leq i'\leq j\leq j'$ define a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$W[i,j]\longrightarrow W[i',j']$$

by the identity in the overlap dimensions $[i,j]\cap[i',j']=[i',j]$, so that

$$\text{there are induced}\begin{cases}\text{contravariantly}\\ \text{covariantly}\end{cases}\text{abelian group morphisms}$$

$$\begin{cases}Q^n_{[i',j']}(C,\varepsilon)\longrightarrow Q^n_{[i,j]}(C,\varepsilon)\\ Q_n^{[i,j]}(C,\varepsilon)\longrightarrow Q_n^{[i',j']}(C,\varepsilon)\end{cases}.$$

For $-\infty\leq i\leq j\leq k\leq\infty$ there is defined a split short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$0\longrightarrow W[i,j]\longrightarrow W[i,k]\longrightarrow W[j+1,k]\longrightarrow 0\ .$$

$$\text{Now apply}\begin{cases}\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(-,C^t\otimes_A C)\\ -\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C)\end{cases}\text{and consider the homology long exact sequence.}$$

[]

(The chain homotopy invariance of the Q-groups is basic to this paper - it is further clarified by the following discussion. Define the $\underline{\mathbb{Z}_2\text{-isovariant}}$ category with objects $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes and morphisms $f:C\longrightarrow D$ $\mathbb{Z}_2$-hypercohomology classes $f\in H^0(\mathbb{Z}_2;\mathrm{Hom}_{\mathbb{Z}}(C,D)) = H_0(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_{\mathbb{Z}}(C,D)))$, with $T\in\mathbb{Z}_2$ acting on $\mathrm{Hom}_{\mathbb{Z}}(C,D)$ by

$$T:\mathrm{Hom}_{\mathbb{Z}}(C,D)\longrightarrow\mathrm{Hom}_{\mathbb{Z}}(C,D)\ ;\ g\longmapsto T_D g T_C\ .$$

A $\mathbb{Z}_2$-isovariant morphism $f:C\longrightarrow D$ is thus an equivalence class of collections $\{f_s\in\mathrm{Hom}_{\mathbb{Z}}(C_r,D_{r+s})\,|\,r\in\mathbb{Z},s\geqslant 0\}$ such that

$$d_Df_s+(-)^{s-1}f_sd_C+(-)^{s-1}(f_{s-1}+(-)^sT_Df_{s-1}T_C)=0:C_r\longrightarrow D_{r+s-1}\quad (f_{-1}=0)$$

corresponding to a $\mathbb{Z}$-module chain map $f_0:C\to D$ together with a $\mathbb{Z}$-module chain homotopy $f_1:f_0\simeq T_Df_0T_C:C\longrightarrow D$ and the higher chain homotopies $f_2,f_3,\ldots$. The diagonal $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$\Delta: W[i,j]\longrightarrow W\otimes_{\mathbb{Z}}W[i,j]\ ;\ 1_s\longmapsto\sum_{r=0}^{s-i}1_r\otimes T^r1_{s-r}\qquad (i\leqslant s\leqslant j)$$

can be used to define products

$$H_0(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_{\mathbb{Z}}(C,D))\otimes_{\mathbb{Z}}H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],C))\longrightarrow H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],D))\ ,$$

so that a $\mathbb{Z}_2$-isovariant morphism $f:C\longrightarrow D$ induces in a natural way abelian group morphisms

$$f^{\%}:H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],C))\longrightarrow H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],D))\ .$$

An A-module chain map $f:C\longrightarrow D$ of A-module chain complexes induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f^t\otimes_Af:C^t\otimes_AC\longrightarrow D^t\otimes_AD$ of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes, but an A-module chain homotopy $g:f\simeq f':C\longrightarrow D$ does not induce a $\mathbb{Z}[\mathbb{Z}_2]$-module chain homotopy $f^t\otimes_Af\simeq f'^t\otimes_Af':C^t\otimes_AC\longrightarrow D^t\otimes_AD$, only the $\mathbb{Z}_2$-isovariant chain homotopy used to prove Proposition 1.1 i). Note that for any topological space X there is a natural $\mathbb{Z}_2$-isovariant equivalence $C(X\times X)\longrightarrow C(X)\otimes_{\mathbb{Z}}C(X)$, as used in the construction of the Steenrod squares in singular cohomology (cf. §2 below)).

Define the $\left\{\begin{array}{l}\underline{\mathbb{Z}_2\text{-hypercohomology}}\\ \underline{\mathbb{Z}_2\text{-hyperhomology}}\\ \underline{\text{Tate }\mathbb{Z}_2\text{-hypercohomology}}\end{array}\right.$ groups of the action of $T\in\mathbb{Z}_2$ on $C^t\otimes_AC$ by T_ε

$$\left\{\begin{array}{l}Q^n(C,\varepsilon)=Q^n_{[0,\infty]}(C,\varepsilon)=H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C^t\otimes_AC))\\ Q_n(C,\varepsilon)=Q^{[0,\infty]}_n(C,\varepsilon)=H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_AC))\\ \hat{Q}^n(C,\varepsilon)=Q^n_{[-\infty,\infty]}(C,\varepsilon)=H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W},C^t\otimes_AC))\end{array}\right.$$

with $W=W[0,\infty]$ a free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$ and $\hat{W}=W[-\infty,\infty]$ a complete resolution for $\mathbb{Z}_2$ (cf. Chapters XII, XVII of Cartan and Eilenberg [1]). The groups $\hat{Q}^n(C,\varepsilon)$ are of exponent 2. For n-dimensional C we have

$$\left\{\begin{array}{l}Q^n(C,\varepsilon)=Q^n_{[0,n+1]}(C,\varepsilon)\\ Q_n(C,\varepsilon)=Q^{[0,n+1]}_n(C,\varepsilon)\\ \hat{Q}^n(C,\varepsilon)=Q^n_{[-n-1,n+1]}(C,\varepsilon)\end{array}\right.$$

by Proposition 1.1.

__Proposition 1.2__ There is defined in a natural way an exact sequence of abelian groups for any A-module chain complex C

$$\ldots\longrightarrow\hat{Q}^{n+1}(C,\varepsilon)\xrightarrow{H}Q_n(C,\varepsilon)\xrightarrow{1+T_\varepsilon}Q^n(C,\varepsilon)\xrightarrow{J}\hat{Q}^n(C,\varepsilon)\xrightarrow{H}Q_{n-1}(C,\varepsilon)\longrightarrow\ldots$$

with

$$H:\hat{Q}^{n+1}(C,\varepsilon)\longrightarrow Q_n(C,\varepsilon)\ ;\ \theta\longmapsto\{(H\theta)_s=\theta_{-s-1}\,|\,s\geqslant 0\}$$

$$1+T_\varepsilon:Q_n(C,\varepsilon)\longrightarrow Q^n(C,\varepsilon)\ ;\ \psi\longmapsto\left\{((1+T_\varepsilon)\psi)_s=\left\{\begin{array}{ll}(1+T_\varepsilon)\psi_0 & s=0\\ 0 & s\geqslant 1\end{array}\right.\right\}$$

$$J:Q^n(C,\varepsilon)\longrightarrow\hat{Q}^n(C,\varepsilon)\ ;\ \varphi\longmapsto\left\{(J\varphi)_s=\left\{\begin{array}{ll}\varphi_s & s\geqslant 0\\ 0 & s<0\end{array}\right.\right\}.$$

__Proof:__ This is just a special case of the exact sequence of Proposition 1.1 iii), with Q_*, $i=-\infty$, $j=-1$, $k=\infty$. []

(Proposition 1.2 is related to the EHP sequence, cf. Proposition 4.1 below).

An $\underline{n\text{-dimensional}}$ $\left\{\begin{array}{l}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\\ \underline{\varepsilon\text{-hyperquadratic}}\end{array}\right.$ $\underline{\text{complex over }A}$ $\left\{\begin{array}{l}(C,\varphi)\\ (C,\psi)\\ (C,\theta)\end{array}\right.$ is

an n-dimensional A-module chain complex C together with an element $\left\{\begin{array}{l}\varphi\in Q^n(C,\varepsilon)\\ \psi\in Q_n(C,\varepsilon)\\ \theta\in\hat{Q}^n(C,\varepsilon)\end{array}\right.$.

$$\text{The}\begin{cases}\underline{\varepsilon\text{-hyperquadratization}}\\ \underline{\varepsilon\text{-symmetrization}}\\ \underline{\varepsilon\text{-quadratization}}\end{cases}\text{ of an n-dimensional }\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \varepsilon\text{-hyperquadratic}\end{cases}\text{ complex over }A$$

$$\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\\ (C,\theta\in\hat{Q}^n(C,\varepsilon))\end{cases}\text{ such that }\begin{cases}-\\ -\\ H^n(C)=0\end{cases}\text{ is the }\begin{cases}n-\\ n-\\ (n-1)-\end{cases}\text{ dimensional }\begin{cases}\varepsilon\text{-hyperquadratic}\\ \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$$

complex

$$\begin{cases}J(C,\varphi) = (C,J\varphi\in\hat{Q}^n(C,\varepsilon))\\ (1+T_\varepsilon)(C,\psi) = (C,(1+T_\varepsilon)\psi\in Q^n(C,\varepsilon))\\ H(C,\theta) = (C,H\theta\in Q_{n-1}(C,\varepsilon))\ .\end{cases}$$

If $1/2\in A$ the ε-symmetrization map $(1+T_\varepsilon):Q_n(C,\varepsilon)\longrightarrow Q^n(C,\varepsilon)$ is an isomorphism (the inverse being given by $Q^n(C,\varepsilon)\longrightarrow Q_n(C,\varepsilon);\varphi\longmapsto\psi=\{\psi_s=\begin{cases}\frac{1}{2}(1+T_\varepsilon)\varphi_0 & s=0\\ 0 & s=1\end{cases}\})$

and $\hat{Q}^n(C,\varepsilon)=0$, so that there is no difference between $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-hyperquadratic}\end{cases}$ and $\begin{cases}\varepsilon\text{-quadratic}\\ \text{chain}\end{cases}$ complexes over A. The case $1/2\in A$ has been treated in Mishchenko [1].

An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ is <u>Poincaré</u>

if $\begin{cases}\varphi_0\in H_n(C^t\otimes_A C)\\ (1+T_\varepsilon)\psi_0\in H_n(C^t\otimes_A C)\end{cases}$ determines A-module isomorphisms

$$\begin{cases}\varphi_0 : H^r(C)\longrightarrow H_{n-r}(C)\\ (1+T_\varepsilon)\psi_0 : H^r(C)\longrightarrow H_{n-r}(C)\end{cases}\qquad(0\leqslant r\leqslant n)$$

via the slant product

$$\backslash : H^r(C)\otimes_{\mathbb{Z}}H_n(C^t\otimes_A C)\longrightarrow H_{n-r}(C)\ ;\ f\otimes(x\otimes y)\longmapsto\overline{f(x)}y\ .$$

The ε-symmetrization of an n-dimensional ε-quadratic Poincaré complex (C,ψ) is an n-dimensional ε-symmetric Poincaré complex $(1+T_\varepsilon)(C,\psi)$.

A <u>map</u> of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \varepsilon\text{-hyperquadratic}\end{cases}$ complexes over A

$$\begin{cases}f : (C,\varphi)\longrightarrow(C',\varphi')\\ f : (C,\psi)\longrightarrow(C',\psi')\\ f : (C,\theta)\longrightarrow(C',\theta')\end{cases}$$

is an A-module chain map $f:C\longrightarrow C'$ such that

$$\begin{cases}f^{\%}(\varphi) = \varphi'\in Q^n(C',\varepsilon)\\ f_{\%}(\psi) = \psi'\in Q_n(C',\varepsilon)\\ \hat{f}^{\%}(\theta) = \theta'\in\hat{Q}^n(C',\varepsilon)\ .\end{cases}$$

The map is a <u>homotopy equivalence</u> if $f:C\longrightarrow C'$ is a chain equivalence. Homotopy equivalence is an equivalence relation.

The homotopy theory of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes for $n=0$ (resp. 1) is the isomorphism (resp. stable isomorphism) theory of $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms (resp. formations), as elaborated in Propositions 1.5 - 1.9 below.

For $\varepsilon=1\in A$ we shall contract the terminology by writing

$$\begin{cases}Q^n_{[i,j]}(C,1) = Q^n_{[i,j]}(C)\\ Q_n^{[i,j]}(C,1) = Q_n^{[i,j]}(C)\end{cases}\qquad\begin{cases}Q^n(C,1) = Q^n(C)\\ Q_n(C,1) = Q_n(C)\\ \hat{Q}^n(C,1) = \hat{Q}^n(C)\ ,\end{cases}$$

and calling $\begin{cases}1\text{-symmetric}\\ 1\text{-quadratic}\\ 1\text{-hyperquadratic}\end{cases}$ complexes $\begin{cases}\text{symmetric}\\ \text{quadratic}\\ \text{hyperquadratic}\end{cases}$. As indicated in the Introduction our notion of "symmetric Poincaré complex" is a chain homotopy invariant version of the "algebraic Poincaré complex" due to Mishchenko [2]..

In $\begin{cases}\S2\\ \S3\\ \S18\end{cases}$ below we shall show that $\begin{cases}\text{an n-dimensional geometric Poincaré complex } X\\ \text{a normal map of n-dimensional geometric Poincaré complexes } (f,b):(M,\nu_M)\longrightarrow(X,\nu_X)\\ \text{a spherical fibration } p:X\longrightarrow BG \text{ over an n-dimensional CW complex } X\end{cases}$ determines in a natural way

an n-dimensional $\begin{cases}\text{symmetric Poincaré}\\ \text{quadratic Poincaré}\\ \text{hyperquadratic}\end{cases}$ complex $\begin{cases}\sigma^*(X)\\ \sigma_*(f,b)\\ \hat{\sigma}^*(p)\end{cases}$ over the fundamental group ring $\mathbb{Z}[\pi_1(X)]$ with the orientation-twisted involution, such that

$$\begin{cases}(1+T)\sigma_*(f,b)\oplus\sigma^*(X) = \sigma^*(M)\\ J\sigma^*(X) = \hat{\sigma}^*(\nu_X),\ \text{if } \nu_X:X\longrightarrow BG \text{ is the Spivak normal fibration of a geometric Poincaré complex } X.\end{cases}$$

The surgery obstruction of a normal map (f,b) may be obtained from $\sigma_*(f,b)$.

For f.g. projective A-module chain complexes C the slant map isomorphism $C^t\otimes_A C \longrightarrow \mathrm{Hom}_A(C^*,C)$ allows us to represent elements $\begin{cases}\varphi \in Q^n(C,\varepsilon)\\ \psi \in Q_n(C,\varepsilon)\\ \theta \in \hat{Q}^n(C,\varepsilon)\end{cases}$ by collections of A-module morphisms $\begin{cases}\{\varphi_s \in \mathrm{Hom}_A(C^{n-r+s},C_r)\,|\,r\in\mathbb{Z}, s\geqslant 0\}\\ \{\psi_s \in \mathrm{Hom}_A(C^{n-r-s},C_r)\,|\,r\in\mathbb{Z}, s\geqslant 0\}\\ \{\theta_s \in \mathrm{Hom}_A(C^{n-r+s},C_r)\,|\,r\in\mathbb{Z}, s\in\mathbb{Z}\}\end{cases}$ such that

$$\begin{cases} d_C\varphi_s + (-)^r\varphi_s d_C^* + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon\varphi_{s-1}) = 0 : C^{n-r+s-1} \longrightarrow C_r & (s\geqslant 0, \varphi_{-1}=0)\\ d_C\psi_s + (-)^r\psi_s d_C^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\varepsilon\psi_{s+1}) = 0 : C^{n-r-s-1} \longrightarrow C_r & (s\geqslant 0)\\ d_C\theta_s + (-)^r\theta_s d_C^* + (-)^{n+s-1}(\theta_{s-1} + (-)^s T_\varepsilon\theta_{s-1}) = 0 : C^{n-r+s-1} \longrightarrow C_r & (s\in\mathbb{Z})\ ,\end{cases}$$

where T_ε is the <u>ε-duality</u> involution

$$T_\varepsilon : \mathrm{Hom}_A(C^p,C_q) \longrightarrow \mathrm{Hom}_A(C^q,C_p)\ ;\ \theta \longmapsto (-)^{pq}\varepsilon.\theta^*\ .$$

An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ with C f.g. projective is Poincaré if and only if the chain map

$$\begin{cases}\varphi_0 : C^{n-*} \longrightarrow C\\ (1+T_\varepsilon)\psi_0 : C^{n-*} \longrightarrow C\end{cases}$$

is a chain equivalence.

The <u>suspension</u> of an A-module chain complex C is the A-module chain complex SC obtained by dimension shift -1

$$(SC)_r = C_{r-1}\ ,\ d_{SC} = d_C\ .$$

We shall denote the inverse operation by ΩC ($(\Omega C)_r = C_{r+1}$, $d_{\Omega C} = d_C$), and can identify

$$\begin{cases}H_r(SC) = H_{r-1}(C)\ ,\ H_r(\Omega C) = H_{r+1}(C)\\ H^r(SC) = H^{r-1}(C)\ ,\ H^r(\Omega C) = H^{r+1}(C)\end{cases}\ ,\quad S\Omega C = \Omega SC = C$$

$$\begin{cases}Q^n_{[i,j]}(C,\varepsilon) = Q^{n+1}_{[i+1,j+1]}(SC,\varepsilon) = Q^{n+2}_{[i,j]}(SC,-\varepsilon)\\ Q^{[i,j]}_n(C,\varepsilon) = Q^{[i-1,j-1]}_{n+1}(SC,\varepsilon) = Q^{[i,j]}_{n+2}(SC,-\varepsilon)\ .\end{cases}$$

In particular,

$$\hat{Q}^n(C,\varepsilon) = \hat{Q}^{n+1}(SC,\varepsilon)\ .$$

The <u>skew-suspension</u> of an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ (Poincaré) complex over A $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ is the $(n+2)$-dimensional $\begin{cases}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ (Poincaré) complex over A

$$\begin{cases}\bar{S}(C,\varphi) = (SC,\bar{S}\varphi\in Q^{n+2}(SC,-\varepsilon))\\ \bar{S}(C,\psi) = (SC,\bar{S}\psi\in Q_{n+2}(SC,-\varepsilon))\end{cases}\ ,$$

with $\begin{cases}\bar{S} : Q^n(C,\varepsilon) \longrightarrow Q^{n+2}(SC,-\varepsilon)\\ \bar{S} : Q_n(C,\varepsilon) \longrightarrow Q_{n+2}(SC,-\varepsilon)\end{cases}$ the abelian group isomorphism induced by the isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\bar{S} : C^t\otimes_A C \longrightarrow \Omega^2(SC^t\otimes_A SC)\ ;\ x\otimes y \longmapsto (-)^p x\otimes y \quad (x\in C_p, y\in C_q).$$

Here $T\in\mathbb{Z}_2$ acts by T_ε on $C^t\otimes_A C$ and by $T_{-\varepsilon}$ on $SC^t\otimes_A SC$. An $(n+2)$-dimensional $\begin{cases}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D,\varphi\in Q^{n+2}(D,-\varepsilon))\\ (D,\psi\in Q_{n+2}(D,-\varepsilon))\end{cases}$ is the skew-suspension of an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(\Omega D,\bar{S}^{-1}(\varphi)\in Q^n(\Omega D,\varepsilon))\\ (\Omega D,\bar{S}^{-1}(\psi)\in Q_n(\Omega D,\varepsilon))\end{cases}$ if and only if ΩD is an n-dimensional A-module chain complex, that is if

$$H_0(D) = 0\ ,\ H^{n+2}(D) = 0\ .$$

Thus the study of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ such that $H_r(C) = 0$, $H^r(C) = 0$ for $r < i$ reduces to the study of $(n-2i)$-dimensional $\begin{cases}(-)^i\varepsilon\text{-symmetric}\\ (-)^i\varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes $\begin{cases}(\Omega^i C,\bar{S}^{-i}(\varphi))\\ (\Omega^i C,\bar{S}^{-i}(\psi))\end{cases}$ $(n\geqslant 2i)$.

Given an A-module chain complex C define the <u>suspension</u> chain map

$$\begin{cases} S : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t\otimes_A C) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-1,\infty], C^t\otimes_A C) = \Omega\,\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, SC^t\otimes_A SC)\ ; \\ \qquad \varphi \longmapsto \{(S\varphi)_s = \iota\varphi_{s-1} \mid s \geqslant 0\} \quad (\varphi_{-1} = 0) \\ S : W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C) \longrightarrow W[1,\infty]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C) = \Omega(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(SC^t\otimes_A SC))\ ; \\ \qquad \psi \longmapsto \{(S\psi)_s = \iota\psi_{s+1} \mid s \geqslant 0\} \end{cases}$$

using the natural chain map $\begin{cases} W[-1,\infty] \longrightarrow W[0,\infty] = W \\ W = W[0,\infty] \longrightarrow W[1,\infty] \end{cases}$, so that there are

induced suspension maps in $\begin{cases} \mathbb{Z}_2\text{-hypercohomology} \\ \mathbb{Z}_2\text{-hyperhomology} \end{cases}$

$$\begin{cases} S : Q^n(C,\varepsilon) \longrightarrow Q^{n+1}(SC,\varepsilon) \\ S : Q_n(C,\varepsilon) \longrightarrow Q_{n+1}(SC,\varepsilon)\ . \end{cases}$$

Now $W[-\infty,\infty] = \underset{\overleftarrow{p}}{\mathrm{Lim}}\, W[-p,\infty]$ so that the Tate $\mathbb{Z}_2$-hypercohomology groups are the direct limits

$$\hat{Q}^n(C,\varepsilon) = \underset{\overrightarrow{p}}{\mathrm{Lim}}\, Q^{n+p}(S^pC,\varepsilon)$$

of the directed systems of suspension maps

$$Q^n(C,\varepsilon) \xrightarrow{S} Q^{n+1}(SC,\varepsilon) \xrightarrow{S} Q^{n+2}(S^2C,\varepsilon) \xrightarrow{S} \dots$$

with $J : Q^n(C,\varepsilon) \longrightarrow \hat{Q}^n(C,\varepsilon)$ the natural map. Thus the relation $\ker J = \mathrm{im}(1+T_\varepsilon)$ in the exact sequence of Proposition 1.2 can be interpreted as saying that an n-dimensional ε-symmetric complex $(C,\varphi\in Q^n(C,\varepsilon))$ is the ε-symmetrization $(1+T_\varepsilon)(C,\psi)$ of an n-dimensional ε-quadratic complex $(C,\psi\in Q_n(C,\varepsilon))$ if and only if $S^p\varphi = 0 \in Q^{n+p}(S^pC,\varepsilon)$ for some $p \geqslant 0$. This is the mechanism by which we shall obtain quadratic information in the topological context (in §2).

The exact sequence of Proposition 1.2 admits a generalization:

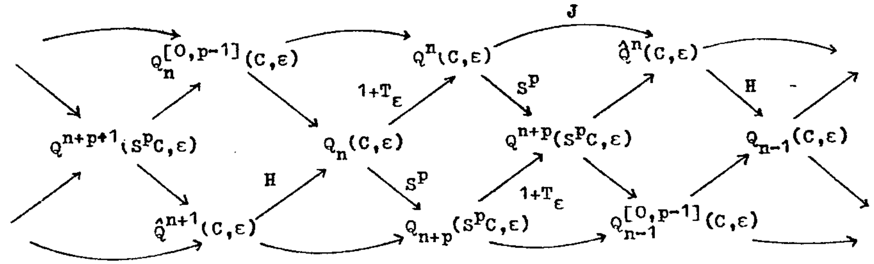
<u>Proposition 1.3</u> Given an A-module chain complex C there is defined in a natural way a chain equivalence of $\mathbb{Z}$-module chain complexes

$$C(S^p) \longrightarrow S(W[0,p-1]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C))$$

for each $p \geqslant 0$, with $C(S^p)$ the algebraic mapping cone of the p-fold suspension chain map

$$S^p : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t\otimes_A C) \longrightarrow \Omega^p\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, S^pC^t\otimes_A S^pC)\ ,$$

and there is also defined a commutative braid of exact sequences of $\mathbb{Z}$-modules



If C is n-dimensional then for $p \geqslant n+1$

$$Q_{n+p}(S^pC,\varepsilon) = 0\ ,\quad Q_n^{[0,p]}(C,\varepsilon) = Q_n(C,\varepsilon)\ ,\quad Q^{n+p}(S^pC,\varepsilon) = \hat{Q}^n(C,\varepsilon)\ ,$$

and the braid collapses to the exact sequence

$$\dots \longrightarrow \hat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H} Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{J} \hat{Q}^n(C,\varepsilon) \xrightarrow{H} Q_{n-1}(C,\varepsilon) \rightarrow \dots\ .$$

<u>Proof</u>: Applying $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(-, C^t\otimes_A C)$ to the chain equivalence of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$SW[-p,-1] \longrightarrow C(W[-p,\infty] \longrightarrow W[0,\infty])$$

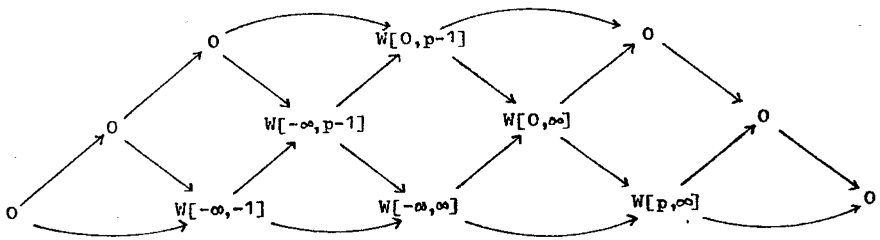
arising from the split short exact sequence

$$0 \longrightarrow W[-p,-1] \longrightarrow W[-p,\infty] \longrightarrow W[0,\infty] \longrightarrow 0$$

we have a chain equivalence of $\mathbb{Z}$-module chain complexes

$$C(S^p) = C(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[0,\infty], C^t\otimes_A C) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-p,\infty], C^t\otimes_A C))$$
$$\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(SW[-p,-1], C^t\otimes_A C) = S(W[0,p-1]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C))\ .$$

To obtain the braid apply $-\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C)$ to the commutative diagram of split



and consider the associated long exact sequences of homology groups (which are all special cases of those of Proposition 1.1 iii)).

[]

Given A-module chain complexes C,C' there are defined <u>direct sum</u> operations

$$\begin{cases} \oplus : Q^n_{[i,j]}(C,\varepsilon)\oplus Q^n_{[i,j]}(C',\varepsilon) \longrightarrow Q^n_{[i,j]}(C\oplus C',\varepsilon) \;;\; (\varphi,\varphi')\longmapsto \varphi\oplus\varphi' \\ \oplus : Q_n^{[i,j]}(C,\varepsilon)\oplus Q_n^{[i,j]}(C',\varepsilon) \longrightarrow Q_n^{[i,j]}(C\oplus C',\varepsilon) \;;\; (\psi,\psi')\longmapsto \psi\oplus\psi' \;. \end{cases}$$

The direct sum of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ (Poincaré) complexes over A $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon)),\ (C',\varphi'\in Q^n(C',\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon)),\ (C',\psi'\in Q_n(C',\varepsilon))\end{cases}$ is an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ (Poincaré) complex over A

$$\begin{cases} (C,\varphi)\oplus(C',\varphi') = (C\oplus C',\varphi\oplus\varphi'\in Q^n(C\oplus C',\varepsilon)) \\ (C,\psi)\oplus(C',\psi') = (C\oplus C',\psi\oplus\psi'\in Q_n(C\oplus C',\varepsilon)) \;. \end{cases}$$

The $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\\ \text{Tate }\mathbb{Z}_2\text{-hypercohomology}\end{cases}$ groups behave as follows under the direct sum operation.

<u>Proposition 1.4</u> i) Given A-module chain complexes C,D there are natural direct sum decompositions of abelian groups

$$\begin{cases} Q^n(C\oplus D,\varepsilon) = Q^n(C,\varepsilon)\oplus Q^n(D,\varepsilon)\oplus H_n(C^t\otimes_A D) \\ Q_n(C\oplus D,\varepsilon) = Q_n(C,\varepsilon)\oplus Q_n(D,\varepsilon)\oplus H_n(C^t\otimes_A D) \\ \hat{Q}^n(C\oplus D,\varepsilon) = \hat{Q}^n(C,\varepsilon)\oplus\hat{Q}^n(D,\varepsilon) \;. \end{cases}$$

The ε-symmetrization $(1+T_\varepsilon):Q_n(C\oplus D,\varepsilon)\longrightarrow Q^n(C\oplus D,\varepsilon)$ is an isomorphism on $H_n(C^t\otimes_A D)$.

ii) Given A-module chain maps $f,g:C\longrightarrow D$ there are defined factorizations

$$\begin{cases} (f+g)^\% - f^\% - g^\% : Q^n(C,\varepsilon)\longrightarrow H_n(C^t\otimes_A C)\xrightarrow{f^t\otimes_A g} H_n(D^t\otimes_A D)\longrightarrow Q^n(D,\varepsilon) \\ (f+g)_\% - f_\% - g_\% : Q_n(C,\varepsilon)\longrightarrow H_n(C^t\otimes_A C)\xrightarrow{f^t\otimes_A g} H_n(D^t\otimes_A D)\longrightarrow Q_n(D,\varepsilon) \\ \widehat{(f+g)}^\% - \hat{f}^\% - \hat{g}^\% = 0 : \hat{Q}^n(C,\varepsilon)\longrightarrow\hat{Q}^n(D,\varepsilon) \end{cases}$$

with

$$Q^n(C,\varepsilon)\longrightarrow H_n(C^t\otimes_A C);\varphi\longmapsto\varphi_0\ ,\ H_n(D^t\otimes_A D)\longrightarrow Q^n(D,\varepsilon);\theta\longmapsto\{\varphi_s = \begin{cases}(1+T_\varepsilon)\theta & s=0\\ 0 & s\geq 1\end{cases}\}$$

$$Q_n(C,\varepsilon)\longrightarrow H_n(C^t\otimes_A C);\psi\longmapsto(1+T_\varepsilon)\psi_0\ ,\ H_n(D^t\otimes_A D)\longrightarrow Q_n(D,\varepsilon);\theta\longmapsto\{\psi_s = \begin{cases}\theta & s=0\\ 0 & s\geq 1.\end{cases}\}$$

<u>Proof</u>: i) Applying $H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],-))$ to the direct sum decomposition of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$(C\oplus D)^t\otimes_A(C\oplus D) = (C^t\otimes_A C)\oplus(D^t\otimes_A D)\oplus((C^t\otimes_A D)\oplus(D^t\otimes_A C))$$

we have a direct sum decomposition of abelian groups

$$Q^n_{[i,j]}(C\oplus D,\varepsilon) = Q^n_{[i,j]}(C,\varepsilon)\oplus Q^n_{[i,j]}(D,\varepsilon)\oplus H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],(C^t\otimes_A D)\oplus(D^t\otimes_A C)))$$

with $T\in\mathbb{Z}_2$ acting on $(C^t\otimes_A D)\oplus(D^t\otimes_A C)$ by

$$T_\varepsilon : C^t_p\otimes_A D_q\oplus D^t_r\otimes_A C_s\longrightarrow C^t_s\otimes_A D_r\oplus D^t_q\otimes_A C_p \;;\; (u\otimes v,x\otimes y)\longmapsto((-)^{rs}y\otimes\varepsilon x,(-)^{pq}v\otimes\varepsilon u)$$

By direct computation

$$\begin{cases} H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[0,\infty],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = H_n(C^t\otimes_A D) \\ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-\infty,-1],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = H_n(C^t\otimes_A D) \\ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[-\infty,\infty],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = 0 \;. \end{cases}$$

ii) Substitute the decomposition $Q^n(C\oplus C,\varepsilon) = Q^n(C,\varepsilon)\oplus Q^n(C,\varepsilon)\oplus H_n(C^t\otimes_A C)$ in

$$(f+g)^\% : Q^n(C,\varepsilon)\xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}^\%} Q^n(C\oplus C,\varepsilon)\xrightarrow{(f\ g)^\%} Q^n(D,\varepsilon)$$

and similarly for the other two cases.

[]

We can associate Wu classes to algebraic Poincaré complexes by analogy with the Wu classes of geometric Poincaré complexes. We shall relate algebraic Wu classes to geometric Wu classes in §18 below.

Let $T \in \mathbb{Z}_2$ act on A by

$$T_\varepsilon : A \longrightarrow A \; ; \; a \longmapsto \varepsilon\bar{a}$$

and define the $\begin{cases} \mathbb{Z}_2\text{-cohomology} \\ \mathbb{Z}_2\text{-homology} \\ \text{Tate } \mathbb{Z}_2\text{-cohomology} \end{cases}$ groups

$$\begin{cases} H^r(\mathbb{Z}_2;A,\varepsilon) = \begin{cases} \ker(1-T_\varepsilon : A \longrightarrow A) & r = 0 \\ \hat{H}^r(\mathbb{Z}_2;A,\varepsilon) & r \geqslant 1 \\ 0 & r < 0 \end{cases} \\ H_r(\mathbb{Z}_2;A,\varepsilon) = \begin{cases} \operatorname{coker}(1-T_\varepsilon : A \longrightarrow A) & r = 0 \\ \hat{H}^{r+1}(\mathbb{Z}_2;A,\varepsilon) & r \geqslant 1 \\ 0 & r < 0 \end{cases} \\ \hat{H}^r(\mathbb{Z}_2;A,\varepsilon) = \ker(1-(-)^r T_\varepsilon : A \longrightarrow A)/\operatorname{im}(1+(-)^r T_\varepsilon : A \longrightarrow A) \qquad r \in \mathbb{Z} \; . \end{cases}$$

The function

$$A \times \hat{H}^r(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^r(\mathbb{Z}_2;A,\varepsilon) \; ; \; (a,x) \longmapsto ax\bar{a}$$

defines an A-module structure on $\hat{H}^r(\mathbb{Z}_2;A,\varepsilon)$ (which is of exponent 2, and vanishes if $1/2 \in A$). The functions

$$\begin{cases} A \times H^0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow H^0(\mathbb{Z}_2;A,\varepsilon) \; ; \; (a,x) \longmapsto ax\bar{a} \\ A \times H_0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow H_0(\mathbb{Z}_2;A,\varepsilon) \; ; \; (a,x) \longmapsto ax\bar{a} \end{cases}$$

are not linear in A, and so do not define A-module structures. Nevertheless, we shall write $\begin{cases} \operatorname{Hom}_A(M,H^0(\mathbb{Z}_2;A,\varepsilon)) \\ \operatorname{Hom}_A(M,H_0(\mathbb{Z}_2;A,\varepsilon)) \end{cases}$ for the abelian group of functions $\begin{cases} f : M \longrightarrow H^0(\mathbb{Z}_2;A,\varepsilon) \\ f : M \longrightarrow H_0(\mathbb{Z}_2;A,\varepsilon) \end{cases}$ defined on an A-module M such that

$$f(ax) = af(x)\bar{a} \in \begin{cases} H^0(\mathbb{Z}_2;A,\varepsilon) \\ H_0(\mathbb{Z}_2;A,\varepsilon) \end{cases} \qquad (a \in A, x \in M) \; ,$$

calling such functions "A-module morphisms". Write

$$\begin{cases} H^r(\mathbb{Z}_2;A,1) = H^r(\mathbb{Z}_2;A) \\ H_r(\mathbb{Z}_2;A,1) = H_r(\mathbb{Z}_2;A) \\ \hat{H}^r(\mathbb{Z}_2;A,1) = \hat{H}^r(\mathbb{Z}_2;A) \end{cases} \; .$$

The cohomology classes $f \in H^m(C)$ of an A-module chain complex C may be regarded as the chain homotopy classes of A-module chain maps

$$f : C \longrightarrow S^m A \; ,$$

where $S^m A$ is the A-module chain complex defined by

$$(S^m A)_r = \begin{cases} A & \text{if } r = m \\ 0 & \text{otherwise} \end{cases} \; .$$

The induced abelian group morphisms

$$\begin{cases} f^{\%} : Q^n_{[i,j]}(C,\varepsilon) \longrightarrow Q^n_{[i,j]}(S^m A,\varepsilon) \; ; \; \varphi \longmapsto (f^t \otimes_A f)\varphi_{2m-n} & (\varphi_{2m-n} \in C^t_m \otimes_A C_m) \\ f_{\%} : Q_n^{[i,j]}(C,\varepsilon) \longrightarrow Q_n^{[i,j]}(S^m A,\varepsilon) \; ; \; \psi \longmapsto (f^t \otimes_A f)\psi_{n-2m} & (\psi_{n-2m} \in C^t_m \otimes_A C_m) \end{cases}$$

depend only on $f \in H^m(C)$, on account of the chain homotopy invariance of Proposition 1.1 i). Using the $\mathbb{Z}[\mathbb{Z}_2]$-module isomorphism

$$A \longrightarrow A^t \otimes_A A \; ; \; a \longmapsto 1 \otimes a$$

as an identification we have that these morphisms take values in

$$\begin{cases} Q^n_{[i,j]}(S^m A,\varepsilon) = \begin{cases} H^{2m-n-i}(\mathbb{Z}_2;A,(-)^{m-i}\varepsilon) & \text{if } 2m-n < j \neq i \\ H_0(\mathbb{Z}_2;A,(-)^{n-m+1}\varepsilon) & \text{if } 2m-n = j \neq i \\ A & \text{if } 2m-n = j = i \\ 0 & \text{otherwise} \end{cases} \\ Q_n^{[i,j]}(S^m A,\varepsilon) = \begin{cases} H_{n-2m-i}(\mathbb{Z}_2;A,(-)^{m-i}\varepsilon) & \text{if } n-2m < j \neq i \\ H^0(\mathbb{Z}_2;A,(-)^{n-m+1}\varepsilon) & \text{if } n-2m = j \neq i \\ A & \text{if } n-2m = j = i \\ 0 & \text{otherwise} \end{cases} \end{cases} \; .$$

Define the <u>rth</u> $\begin{cases} \underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}} \\ \underline{\varepsilon\text{-hyperquadratic}} \end{cases}$ <u>Wu class</u> of an element $\begin{cases} \varphi \in Q^n(C,\varepsilon) \\ \psi \in Q_n(C,\varepsilon) \\ \theta \in \hat{Q}^n(C,\varepsilon) \end{cases}$ for some A-module chain complex C to be the A-module morphism

$$\begin{cases} v_r = v_r(\varphi) : H^{n-r}(C) \longrightarrow Q^n(S^{n-r}A,\varepsilon) = H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon) \; ; \; f \longmapsto (f^t \otimes_A f)\varphi_{n-2r} \\ v^r = v^r(\psi) : H^{n-r}(C) \longrightarrow Q_n(S^{n-r}A,\varepsilon) = H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon) \; ; \; f \longmapsto (f^t \otimes_A f)\psi_{2r-n} \\ \hat{v}_r = \hat{v}_r(\theta) : H^{n-r}(C) \longrightarrow \hat{Q}^n(S^{n-r}A,\varepsilon) = \hat{H}^r(\mathbb{Z}_2;A,\varepsilon) \; ; \; f \longmapsto (f^t \otimes_A f)\theta_{n-2r} \end{cases}$$

$$(f : C_{n-r} \longrightarrow A, \; \varphi_{n-2r}, \psi_{2r-n}, \theta_{n-2r} \in C^t_{n-r} \otimes_A C_{n-r}) \; .$$

Note that $\begin{cases} v_r = 0 \\ v^r = 0 \end{cases}$ for $\begin{cases} 2r > n \\ 2r < n \end{cases}$, and that the Wu classes satisfy the addition formulae (cf. Proposition 1.4 ii))

$$\begin{cases} v_r(\varphi)(f+g) - v_r(\varphi)(f) - v_r(\varphi)(g) = \begin{cases} (1+T_{(-)^r\varepsilon})(f^t\otimes_A g)\varphi_0 \in H^0(\mathbb{Z}_2;A,(-)^r\varepsilon) \\ 0\in \hat{H}^0(\mathbb{Z}_2;A,(-)^r\varepsilon) \end{cases} \text{if } \begin{cases} n=2r \\ n>2r \end{cases} \\ v^r(\psi)(f+g) - v^r(\psi)(f) - v^r(\psi)(g) = \begin{cases} (f^t\otimes_A g)(1+T_\varepsilon)\psi_0 \in H_0(\mathbb{Z}_2;A,(-)^r\varepsilon) \\ 0\in \hat{H}^0(\mathbb{Z}_2;A,(-)^{r+1}\varepsilon) \end{cases} \text{if } \begin{cases} n=2r \\ n<2r \end{cases} \\ \hat{v}_r(\theta)(f+g) - \hat{v}_r(\theta)(f) - \hat{v}_r(\theta)(g) = 0\in \hat{H}^0(\mathbb{Z}_2;A,(-)^r\varepsilon) \quad (r\in\mathbb{Z}) . \end{cases}$$

The Wu classes are compatible with all the maps appearing in the braid of Proposition 1.3. In particular, the Wu classes commute with the suspension maps:

$$\begin{array}{ccc} Q^n(C,\varepsilon) & \xrightarrow{v_r} & \mathrm{Hom}_A(H^{n-r}(C),H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \\ S\downarrow & & \downarrow S(=\text{id. if } n>2r) \\ Q^{n+1}(SC,\varepsilon) & \xrightarrow{v_r} & \mathrm{Hom}_A(H^{n-r+1}(SC),H^{n-2r+1}(\mathbb{Z}_2;A,(-)^{n-r+1}\varepsilon)) \end{array}$$

$$\begin{array}{ccc} Q_n(C,\varepsilon) & \xrightarrow{v^r} & \mathrm{Hom}_A(H^{n-r}(C),H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \\ S\downarrow & & \downarrow S(=\text{id. if } n<2r) \\ Q_{n+1}(SC,\varepsilon) & \xrightarrow{v^r} & \mathrm{Hom}_A(H^{n-r+1}(SC),H_{2r-n+1}(\mathbb{Z}_2;A,(-)^{n-r+1}\varepsilon)) \end{array} ,$$

and also with the skew-suspension isomorphisms:

$$\begin{array}{ccc} Q^n(C,\varepsilon) & \xrightarrow{v_r} & \mathrm{Hom}_A(H^{n-r}(C),H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \\ \bar{S}\downarrow & & \downarrow \text{id.} \\ Q^{n+2}(SC,-\varepsilon) & \xrightarrow{v_{r+1}} & \mathrm{Hom}_A(H^{n-r+1}(SC),H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \end{array}$$

$$\begin{array}{ccc} Q_n(C,\varepsilon) & \xrightarrow{v^r} & \mathrm{Hom}_A(H^{n-r}(C),H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \\ \bar{S}\downarrow & & \downarrow \text{id.} \\ Q_{n+2}(SC,-\varepsilon) & \xrightarrow{v^{r+1}} & \mathrm{Hom}_A(H^{n-r+1}(SC),H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \end{array} .$$

The composite

$$Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{v_r} \mathrm{Hom}_A(H^{n-r}(C),H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon))$$

is 0 for $n\neq 2r$. For $n=2r$ there is defined a commutative diagram

$$\begin{array}{ccc} Q_{2r}(C,\varepsilon) & \xrightarrow{1+T_\varepsilon} & Q^{2r}(C,\varepsilon) \\ v^r\downarrow & & \downarrow v_r \\ \mathrm{Hom}_A(H^r(C),H_0(\mathbb{Z}_2;A,(-)^r\varepsilon)) & \xrightarrow{1+T_{(-)^r\varepsilon}} & \mathrm{Hom}_A(H^r(C),H^0(\mathbb{Z}_2;A,(-)^r\varepsilon)) \end{array} .$$

There is also defined a commutative diagram

$$\begin{array}{ccc} \hat{Q}^{n+1}(C,\varepsilon) & \xrightarrow{\hat{v}_{r+1}} & \mathrm{Hom}_A(H^{n-r}(C),\hat{H}^{r+1}(\mathbb{Z}_2;A,\varepsilon)) \\ H\downarrow & & \downarrow H\ (=\text{id. if } 2r>n) \\ Q_n(C,\varepsilon) & \xrightarrow{v^r} & \mathrm{Hom}_A(H^{n-r}(C),H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)) \end{array} .$$

Given a morphism of rings with 1

$$f:A\longrightarrow B$$

(such that $f(1_A)=1_B$) regard B as a (B,A)-bimodule by

$$B\times B\times A\longrightarrow B\ ;\ (b,x,a)\longmapsto b.x.f(a)\ ,$$

so that an A-module M induces a B-module $B\otimes_A M$, with $B\otimes_A A=B$. If f is a morphism of rings with involution, with

$$\overline{f(a)} = f(\bar{a})\in B \qquad (a\in A)$$

then for any f.g. projective A-module M there is defined a natural isomorphism of f.g. projective B-modules

$$B\otimes_A M^* \longrightarrow (B\otimes_A M)^*\ ;\ b\otimes g\longmapsto (c\otimes m\longmapsto c.fg(m).\bar{b})\ .$$

For any A-module chain complex C there are defined natural $\mathbb{Z}$-module chain maps

$$C\longrightarrow B\otimes_A C\ ;\ x\longmapsto 1\otimes x$$

$$C^*\longrightarrow (B\otimes_A C)^*\ ;\ g\longmapsto (b\otimes x\longmapsto b.fg(x))$$

inducing the <u>change of rings</u> maps in $\begin{cases}\text{homology}\\ \text{cohomology}\end{cases}$

$$\begin{cases} f:H_*(C)\longrightarrow H_*(B\otimes_A C) \\ f:H^*(C)\longrightarrow H^*(B\otimes_A C)\ . \end{cases}$$

If $\varepsilon\in A$ is a central unit such that $\bar{\varepsilon}=\varepsilon^{-1}\in A$ and such that $\eta=f(\varepsilon)\in B$ is central (necessarily such that $\bar{\eta}=\eta^{-1}\in B$) then there are also induced change of rings maps

$$\begin{cases} f:Q^n_{[i,j]}(C,\varepsilon)\longrightarrow Q^n_{[i,j]}(B\otimes_A C,\eta) \\ f:Q_n^{[i,j]}(C,\varepsilon)\longrightarrow Q_n^{[i,j]}(B\otimes_A C,\eta)\ . \end{cases}$$

An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ (Poincaré) complex over A $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ induces an n-dimensional $\begin{cases}\eta\text{-symmetric}\\ \eta\text{-quadratic}\end{cases}$ (Poincaré) complex over B $\begin{cases}B\otimes_A(C,\varphi) = (B\otimes_A C,1\otimes\varphi)\\ B\otimes_A(C,\psi) = (B\otimes_A C,1\otimes\psi)\ .\end{cases}$

The Wu classes remain invariant under change of rings, in the sense that the following diagram commutes

$$\begin{array}{ccc} H^{n-r}(C) & \xrightarrow{f} & H^{n-r}(B\otimes_A C) \\ v_r(\varphi)\downarrow & & \downarrow v_r(1\otimes\varphi) \\ H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon) & \xrightarrow{f} & H^{n-2r}(\mathbb{Z}_2;B,(-)^{n-r}\eta) \end{array} \qquad (\varphi\in Q^n(C,\varepsilon))$$

and similarly for $Q_n,\hat{Q}^n$.

We now specialise to the study of algebraic Poincaré complexes in dimensions 0,1.

An $\begin{cases}\underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}}\end{cases}$ $\underline{\text{form over } A}$ $\begin{cases}(M,\varphi) \\ (M,\psi)\end{cases}$ is a f.g. projective A-module M together with an element $\begin{cases}\varphi \in Q^{\varepsilon}(M) = \ker(1-T_{\varepsilon}:\mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*)) \\ \psi \in Q_{\varepsilon}(M) = \mathrm{coker}(1-T_{\varepsilon}:\mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*))\end{cases}$,

where T_{ε} is the ε-duality involution

$$T_{\varepsilon} : \mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*) \ ; \ \varphi \longmapsto (\varepsilon\varphi^* : x \longmapsto (y \longmapsto \varepsilon.\overline{\varphi(y)(x)})),$$

and it is non-singular if $\begin{cases}\varphi \in \mathrm{Hom}_A(M,M^*) \\ (1+T_{\varepsilon})\psi \in \mathrm{Hom}_A(M,M^*)\end{cases}$ is an isomorphism.

A morphism (resp. isomorphism) of such forms

$$\begin{cases}f : (M,\varphi) \longrightarrow (M',\varphi') \\ f : (M,\psi) \longrightarrow (M',\psi')\end{cases}$$

is an A-module morphism (resp. isomorphism) $f \in \mathrm{Hom}_A(M,M')$ such that

$$\begin{cases}f^*\varphi' f = \varphi \in Q^{\varepsilon}(M) \\ f^*\psi' f = \psi \in Q_{\varepsilon}(M) \ .\end{cases}$$

The above definition of an ε-quadratic form over an arbitrary ring with involution A is a generalization due to Wall [6] of the definition due to Tits [1] for division rings A, which itself goes back to the work of Klingenberg and Witt [1] on the invariant of Arf [1] for $A = \mathbb{Z}_2$. As explained in Wall [6] this definition is equivalent (up to isomorphism) to that given in §5 of Wall [5], as a triple

$$(\text{f.g. projective A-module } M, \lambda \in \mathrm{Hom}_A(M,M^*), \mu : M \longrightarrow H_0(\mathbb{Z}_2;A,\varepsilon))$$

such that

$$\lambda(x)(x) = \mu(x) + \varepsilon\overline{\mu(x)} \in H^0(\mathbb{Z}_2;A,\varepsilon)$$
$$\lambda(x)(y) = \mu(x+y) - \mu(x) - \mu(y) \in H_0(\mathbb{Z}_2;A,\varepsilon)$$
$$\mu(ax) = a\mu(x)\bar{a} \in H_0(\mathbb{Z}_2;A,\varepsilon) \qquad (x,y \in M,\ a \in A)\ ,$$

with the transformation $(M,\psi) \longmapsto (M,\lambda,\mu)$ given by

$$\lambda(x)(y) = \psi(x)(y) + \varepsilon\overline{\psi(y)(x)} \in A \ ,\ \mu(x) = \psi(x)(x) \in H_0(\mathbb{Z}_2;A,\varepsilon) \quad (x,y \in M).$$

This definition of ε-quadratic form is also equivalent to, but not quite the same as, the definition of Ranicki [1] - cf. the discussion in §12.

Proposition 1.5 The homotopy equivalence classes of 0-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ (Poincaré) complexes over A are in a natural one-one correspondence with the isomorphism classes of (non-singular) $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms over A.

Proof: The ε-duality involution

$$T_{\varepsilon} : \mathrm{Hom}_A(M,M^*) \longrightarrow \mathrm{Hom}_A(M,M^*)$$

agrees with the ε-transposition involution

$$T_{\varepsilon} : M^* {}^t\!\otimes_A M^* \longrightarrow M^* {}^t\!\otimes_A M^* \ ; \ f \otimes g \longmapsto g \otimes \varepsilon f$$

under the slant map isomorphism

$$\backslash : M^* {}^t\!\otimes_A M^* \longrightarrow \mathrm{Hom}_A(M,M^*) \ ; \ f \otimes g \longmapsto (x \longmapsto (y \longmapsto g(y).\overline{f(x)}))$$

for any f.g. projective A-module M. Thus for any 0-dimensional A-module chain complex C we can identify

$$\begin{cases}Q^0(C,\varepsilon) = Q^{\varepsilon}(H^0(C)) \\ Q_0(C,\varepsilon) = Q_{\varepsilon}(H^0(C)) \ .\end{cases}$$

An $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form $\begin{cases}(M,\varphi) \\ (M,\psi)\end{cases}$ can be thus considered as $\begin{cases}\text{a fixed point} \\ \text{an orbit space}\end{cases}$ of the ε-duality involution T_{ε} on $\mathrm{Hom}_A(M,M^*)$, corresponding to a $\begin{cases}\mathbb{Z}_2\text{-cohomology} \\ \mathbb{Z}_2\text{-homology}\end{cases}$ class $\begin{cases}\varphi \in H^0(\mathbb{Z}_2;\mathrm{Hom}_A(M,M^*)) = Q^{\varepsilon}(M) \\ \psi \in H_0(\mathbb{Z}_2;\mathrm{Hom}_A(M,M^*)) = Q_{\varepsilon}(M)\end{cases}$, and defines a 0-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi \in Q^0(C,\varepsilon)) \\ (C,\psi \in Q_0(C,\varepsilon))\end{cases}$ with $C_0 = M^*, C_r = 0\ (r \neq 0)$.

Note that the quadratic functions associated to the forms are just the Wu classes associated to the algebraic Poincaré complexes

$$\begin{cases}\lambda = v_0(\varphi) : M = H^0(C) \longrightarrow H^0(\mathbb{Z}_2;A,\varepsilon) \ ; \ x \longmapsto \varphi(x)(x) \\ \mu = v^0(\psi) : M = H^0(C) \longrightarrow H_0(\mathbb{Z}_2;A,\varepsilon) \ ; \ x \longmapsto \psi(x)(x) \ .\end{cases}$$

[]

Given $\begin{cases}\text{an } \varepsilon\text{-symmetric form over } A\ (M,\varphi) \\ \text{a f.g. projective } A\text{-module } L\end{cases}$ define the $\begin{cases}\underline{\text{metabolic}} \\ \underline{\text{hyperbolic}}\end{cases}$ non-singular $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form over A

$$\begin{cases} H^{\varepsilon}(M,\varphi) = (M^*\oplus M, \begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix} \in Q^{\varepsilon}(M^*\oplus M)) \\ H_{\varepsilon}(L) = (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_{\varepsilon}(L\oplus L^*)) \ . \end{cases}$$

We shall write $H^{\varepsilon}(L^*,0)$ as $H^{\varepsilon}(L)$, so that

$$H^{\varepsilon}(L) = H^{\varepsilon}(L^*,0) = (1+T_{\varepsilon})H_{\varepsilon}(L) \ .$$

Given an $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form $\begin{cases}(M,\varphi\in Q^{\varepsilon}(M)) \\ (M,\psi\in Q_{\varepsilon}(M))\end{cases}$ define the <u>annihilator</u> of a submodule L of M to be

$$\begin{cases} L^{\perp} = \ker(j^*\varphi : M \longrightarrow L^*) \\ L^{\perp} = \ker(j^*(\psi+\varepsilon\psi^*) : M \longrightarrow L^*) \ , \end{cases}$$

with $j\in \mathrm{Hom}_A(L,M)$ the inclusion. A <u>sublagrangian</u> of $\begin{cases}(M,\varphi\in Q^{\varepsilon}(M)) \\ (M,\psi\in Q_{\varepsilon}(M))\end{cases}$ is a direct summand L of M such that $\begin{cases} j^*\varphi \in \mathrm{Hom}_A(M,L^*) \\ j^*(\psi+\varepsilon\psi^*)\in \mathrm{Hom}_A(M,L^*)\end{cases}$ is onto and $\begin{cases} j^*\varphi j = 0 \in Q^{\varepsilon}(L) \\ j^*\psi j = 0 \in Q_{\varepsilon}(L)\end{cases}$, so that $L \subseteq L^{\perp}$ and $L^{\perp}$ is a direct summand of M. A <u>lagrangian</u> is a sublagrangian L such that $L^{\perp} = L$, i.e. such that the sequence

$$\begin{cases} 0 \longrightarrow L \xrightarrow{\ j\ } M \xrightarrow{\ j^*\varphi\ } L^* \longrightarrow 0 \\ 0 \longrightarrow L \xrightarrow{\ j\ } M \xrightarrow{\ j^*(\psi+\varepsilon\psi^*)\ } L^* \longrightarrow 0 \end{cases}$$

is exact. For example, L is a lagrangian of $\begin{cases} H^{\varepsilon}(L^*,\varphi) \\ H_{\varepsilon}(L)\end{cases}$ and in fact an $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form is isomorphic to a $\begin{cases}\text{metabolic} \\ \text{hyperbolic}\end{cases}$ form if and only if it admits a lagrangian:

<u>Proposition 1.6</u> The inclusion of a sublagrangian L in an $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form $\begin{cases}(M,\varphi\in Q^{\varepsilon}(M)) \\ (M,\psi\in Q_{\varepsilon}(M))\end{cases}$ defines a morphism

$$\begin{cases} j : (L,0) \longrightarrow (M,\varphi) \\ j : (L,0) \longrightarrow (M,\psi) \end{cases}$$

which can be extended to an isomorphism

$$\begin{cases} f : H^{\varepsilon}(L^*,\theta)\oplus(L^{\perp}/L,\varphi^{\perp}/\varphi) \longrightarrow (M,\varphi) \\ f : H_{\varepsilon}(L)\oplus(L^{\perp}/L,\varphi^{\perp}/\varphi) \longrightarrow (M,\varphi) \ . \end{cases}$$

If $\varphi\in \mathrm{im}((1+T_{\varepsilon}):Q_{\varepsilon}(M)\longrightarrow Q^{\varepsilon}(M))$ then $j:(L,0)\longrightarrow(M,\varphi)$ extends to an isomorphism

$$f : H^{\varepsilon}(L)\oplus(L^{\perp}/L,\varphi^{\perp}/\varphi) \longrightarrow (M,\varphi) \ .$$

<u>Proof</u>: A morphism of $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms

$$\begin{cases} g : (N,\nu) \longrightarrow (N',\nu') \\ g : (N,\chi) \longrightarrow (N',\chi') \end{cases}$$

with $\begin{cases}(N,\nu) \\ (N,\chi)\end{cases}$ non-singular extends to an isomorphism

$$\begin{cases} (g\ g^{\perp}) : (N,\nu)\oplus(N^{\perp},\nu^{\perp}) \longrightarrow (N',\nu') \\ (g\ g^{\perp}) : (N,\chi)\oplus(N^{\perp},\chi^{\perp}) \longrightarrow (N',\chi') \end{cases}$$

with $g^{\perp}\in \mathrm{Hom}_A(N^{\perp},N')$ the inclusion of $\begin{cases} N^{\perp} = \ker(g^*\nu' : N' \longrightarrow N^*) \\ N^{\perp} = \ker(g^*(\chi'+\varepsilon\chi'^*) : N' \longrightarrow N^*)\end{cases}$ and

$\begin{cases} \nu^{\perp} = (g^{\perp})^*\nu' g^{\perp} \in Q^{\varepsilon}(N^{\perp}) \\ \chi^{\perp} = (g^{\perp})^*\chi' g^{\perp} \in Q_{\varepsilon}(N^{\perp})\end{cases}$, since the exact sequence

$$\begin{cases} 0 \longrightarrow N^{\perp} \xrightarrow{\ g^{\perp}\ } N' \xrightarrow{\ g^*\nu'\ } N^* \longrightarrow 0 \\ 0 \longrightarrow N^{\perp} \xrightarrow{\ g^{\perp}\ } N' \xrightarrow{\ g^*(\chi'+\varepsilon\chi'^*)\ } N^* \longrightarrow 0 \end{cases}$$

is split by $\begin{cases} g\nu^{-1} \in \mathrm{Hom}_A(N^*,N') \\ g(\chi+\varepsilon\chi^*)^{-1} \in \mathrm{Hom}_A(N^*,N')\end{cases}$.

The inclusion of a sublagrangian L in an $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form $\begin{cases}(M,\phi\in Q^{\varepsilon}(M)) \\ (M,\psi\in Q_{\varepsilon}(M))\end{cases}$ extends to a morphism

$$\begin{cases} g = (j \quad k) : H^{\varepsilon}(L^*,k^*\phi k) = (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & k^*\phi k\end{pmatrix}) \longrightarrow (M,\phi) \\ g = (j \quad (k-\bar{\varepsilon}jk^*\psi k)) : H_{\varepsilon}(L) = (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}) \longrightarrow (M,\psi) \end{cases},$$

with $k\in \mathrm{Hom}_A(L^*,M)$ any A-module morphism such that $\begin{cases} j^*\phi k = 1\in \mathrm{Hom}_A(L^*,L^*) \\ j^*(\psi+\varepsilon\psi^*)k = 1 \in \mathrm{Hom}_A(L^*,L^*)\end{cases}$.

Now $\begin{cases} H^{\varepsilon}(L^*,k^*\phi k) \\ H_{\varepsilon}(L)\end{cases}$ is non-singular, so that g extends to an isomorphism

$$\begin{cases} f = (g \quad h) : H^{\varepsilon}(L^*,k^*\phi k)\oplus(L^{\perp}/L,\phi^{\perp}/\phi) \longrightarrow (M,\phi) \\ f = (g \quad h) : H_{\varepsilon}(L)\oplus(L^{\perp}/L,\psi^{\perp}/\psi) \longrightarrow (M,\psi) \end{cases}.$$

If $\phi\in \mathrm{im}((1+T_{\varepsilon}):Q_{\varepsilon}(M)\longrightarrow Q^{\varepsilon}(M))$, say $\phi = \psi+\varepsilon\psi^*$ $(\psi\in Q_{\varepsilon}(M))$, then there is defined an isomorphism of metabolic forms

$$\begin{pmatrix} 1 & -\bar{\varepsilon}k^*\psi k \\ 0 & 1\end{pmatrix} : H^{\varepsilon}(L) = (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0\end{pmatrix}) \longrightarrow H^{\varepsilon}(L^*,k^*\phi k) = (L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & k^*\phi k\end{pmatrix})$$

which is the identity on L, so that the inclusion $j:(L,0)\longrightarrow(M,\phi)$ also extends to an isomorphism

$$f' = (g' \quad h) : H^{\varepsilon}(L)\oplus(L^{\perp}/L,\phi^{\perp}/\phi) \longrightarrow (M,\phi)$$

(with $g' = (j \quad (k-\bar{\varepsilon}jk^*\psi k)) : L\oplus L^* \longrightarrow M$).

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An $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ formation over A $(M,\phi;F,G)$ is a non-singular $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form over A (M,ϕ) together with a lagrangian F and a sublagrangian G. A formation $(M,\phi;F,G)$ is non-singular if G is a lagrangian. An isomorphism of formations

$$f : (M,\phi;F,G) \longrightarrow (M',\phi';F',G')$$

is an isomorphism of forms $f:(M,\phi)\longrightarrow(M',\phi')$ such that $f(F) = F'$, $f(G) = G'$. A stable isomorphism of formations

$$[f] : (M,\phi;F,G) \longrightarrow (M',\phi';F',G')$$

is an isomorphism of the type

$$f : (M,\phi;F,G)\oplus(N,\psi;H,K) \longrightarrow (M',\phi';F',G')\oplus(N',\psi';H',K')$$

with $(N,\psi;H,K)$, $(N',\psi';H',K')$ non-singular formations such that $N = H\oplus K$, $N' = H'\oplus K'$.

Formations first appeared (as "pairs of subkernels") in the work of Wall [1] on the classification of linking forms (for $A = \mathbb{Z}$) - the relationship of formations to linking forms is discussed in §13 below. The odd-dimensional surgery obstruction of §6 of Wall [5] was obtained as an equivalence class of the matrix of an automorphism of a hyperbolic $\pm$quadratic form (alias "kernel") $\alpha: H_{\pm}(L)\longrightarrow H_{\pm}(L)$ with L free. The work of Novikov [2] made apparent that only the structure of the non-singular $\pm$quadratic formation $(H_{\pm}(L);L,\alpha(L))$ was relevant. Moreover, the proper surgery obstructions (Maumary [1], see also §12 below) in the odd dimensions are given by equivalence classes of non-singular $\pm$quadratic formations $(M,\psi;F,G)$ with f.g. projective lagrangians F,G for which there may be no automorphism $\alpha: (M,\psi)\longrightarrow(M,\psi)$ such that $\alpha(F) = G$. Thus formations cater for a wider range of surgery obstructions than automorphisms of hyperbolic forms. The algebraic properties of $\pm$quadratic formations were studied in Ranicki [1].

A 1-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\phi\in Q^{1}(C,\varepsilon)) \\ (C,\psi\in Q_{1}(C,\varepsilon))\end{cases}$ is connected if $\begin{cases} H_0(\phi_0:C^{1-*}\longrightarrow C) = 0 \\ H_0((1+T_{\varepsilon})\psi_0:C^{1-*}\longrightarrow C) = 0\end{cases}$. In particular, Poincaré complexes are connected.

Proposition 1.7 The homotopy equivalence classes of connected 1-dimensional ε-symmetric complexes over A are in a natural one-one correspondence with the stable isomorphism classes of ε-symmetric formations over A. Poincaré complexes correspond to non-singular formations.

Proof: Let C be a 1-dimensional f.g. projective A-module chain complex

$$C : \dots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \dots .$$

A cycle $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))_1$ representing a $\mathbb{Z}_2$-hypercohomology class $\varphi \in Q^1(C,\varepsilon)$ is defined by A-module morphisms

$$\varphi_0 : C^0 \longrightarrow C_1 , \ \tilde{\varphi}_0 : C^1 \longrightarrow C_0 , \ \varphi_1 : C^1 \longrightarrow C_1$$

such that

$$d\varphi_0 + \tilde{\varphi}_0 d^* = 0 : C^0 \longrightarrow C_0 , \ d\varphi_1 - \tilde{\varphi}_0 + \varepsilon\varphi_0^* = 0 : C^1 \longrightarrow C_0 , \ \varphi_1 - \varepsilon\varphi_1^* = 0 : C^1 \longrightarrow C_1 .$$

The algebraic mapping cone $C(\varphi_0 : C^{1-*} \longrightarrow C)$ can be expressed as

$$C(\varphi_0) : 0 \longrightarrow C^0 \xrightarrow{\begin{pmatrix} \bar{\varepsilon}\varphi_0 \\ d^* \end{pmatrix}} C_1 \oplus C^1 \xrightarrow{(\varepsilon d \ \ \tilde{\varphi}_0) = (\varepsilon\varphi_0^* \ \ d)\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi_1 \end{pmatrix}} C_0 \longrightarrow 0 .$$

If (C,φ) is connected we thus have an ε-symmetric formation

$$(H^{\varepsilon}(C^1,\varphi_1); C_1, C^0) = (C_1 \oplus C^1, \begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi_1 \end{pmatrix}); C_1, \mathrm{im}(\begin{pmatrix} \bar{\varepsilon}\varphi_0 \\ d^* \end{pmatrix} : C^0 \longrightarrow C_1 \oplus C^1)) ,$$

which is non-singular if (and only if) (C,φ) is Poincaré.

Given a homotopy equivalence of f.g. projective connected 1-dimensional ε-symmetric complexes over A

$$f : (C,\varphi) \longrightarrow (C',\varphi')$$

defined by a chain equivalence

$$\begin{array}{ccccccccc} C : & \dots \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{d} & C_0 & \longrightarrow 0 \longrightarrow & \dots \\ f \downarrow & & & & f \downarrow & & \downarrow \tilde{f} & & \\ C' : & \dots \longrightarrow & 0 & \longrightarrow & C_1' & \xrightarrow{d'} & C_0' & \longrightarrow 0 \longrightarrow & \dots \end{array}$$

we have that the algebraic mapping cone

$$C(f) : \dots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\begin{pmatrix} f \\ d \end{pmatrix}} C_1' \oplus C_0 \xrightarrow{(d' \ -\tilde{f})} C_0' \longrightarrow 0 \longrightarrow \dots$$

is chain contractible. Choosing a contraction of C(f)

$$\dots \longrightarrow 0 \longrightarrow C_0' \xrightarrow{\begin{pmatrix} g \\ e \end{pmatrix}} C_1' \oplus C_0 \xrightarrow{(f' \ g')} C_1 \longrightarrow 0 \longrightarrow \dots$$

there are defined inverse A-module isomorphisms

$$C_1 \oplus C_0' \ \underset{\begin{pmatrix} f' & g' \\ d' & e' \end{pmatrix}}{\overset{\begin{pmatrix} f & g \\ d & e \end{pmatrix}}{\rightleftarrows}} \ C_1' \oplus C_0$$

where $e' = -\tilde{f} \in \mathrm{Hom}_A(C_0, C_0')$. Choosing representative cycles $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))_1$, $\varphi' \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C'^*, C'))_1$ we have that

$$f^{\%}(\varphi) - \varphi' = d\chi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C'^*, C'))_1$$

for some chain $\chi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))_2$, which is represented by an A-module morphism $\chi_0 \in \mathrm{Hom}_A(C'^1, C_1')$ such that

$$f\varphi_0\tilde{f}^* - \varphi_0' = -\chi_0 d'^* : C'^0 \longrightarrow C_1'$$
$$\tilde{f}\tilde{\varphi}_0 f^* - \tilde{\varphi}_0' = d'\chi_0 : C'^1 \longrightarrow C_0'$$
$$f\varphi_1 f^* - \varphi_1' = \chi_0 + \varepsilon\chi_0^* : C'^1 \longrightarrow C_1' .$$

The A-module isomorphism

$$h = \begin{bmatrix} f & g & \bar{\varepsilon}(\chi_0 f'^* - f\varphi_0 g'^*) & \bar{\varepsilon}\varphi_0' \\ d & e & \bar{\varepsilon}\tilde{\varphi}_0 & 0 \\ 0 & 0 & f'^* & d'^* \\ 0 & 0 & g'^* & e'^* \end{bmatrix} : C_1 \oplus C_0' \oplus C^1 \oplus C'^0 \longrightarrow C_1' \oplus C_0 \oplus C'^1 \oplus C^0$$

defines an isomorphism of ε-symmetric forms

$$h : H^{\varepsilon}(C^1 \oplus C'^0, \varphi_1 \oplus 0) \longrightarrow H^{\varepsilon}(C'^1 \oplus C^0, \varphi_1' \oplus 0)$$

such that

$$h(C_1 \oplus C_0') = C_1' \oplus C_0 ,$$

and such that there is defined a commutative diagram

$$\begin{array}{ccc} C^0 \oplus C'^0 & \xrightarrow{k = \begin{pmatrix} -e^* & 1 \\ 1 - e'^* e^* & e'^* \end{pmatrix}} & C'^0 \oplus C^0 \\ \begin{pmatrix} \bar{\varepsilon}\varphi_0 & 0 \\ 0 & 0 \\ d^* & 0 \\ 0 & 1 \end{pmatrix} \Big\downarrow & & \Big\downarrow \begin{pmatrix} \bar{\varepsilon}\varphi_0' & 0 \\ 0 & 0 \\ d'^* & 0 \\ 0 & 1 \end{pmatrix} \\ C_1 \oplus C_0' \oplus C^1 \oplus C'^0 & \xrightarrow{h} & C_1' \oplus C_0 \oplus C'^1 \oplus C^0 \end{array}$$

with $k \in \mathrm{Hom}_A(C^0 \oplus C'^0, C'^0 \oplus C^0)$ an isomorphism. Thus there is defined an isomorphism of ε-symmetric formations

$$h : (H^{\varepsilon}(C^1,\varphi_1);C_1,C^0)\oplus(H^{\varepsilon}(C'_0);C'_0,C'^0) \longrightarrow (H^{\varepsilon}(C'^1,\varphi'_1);C'_1,C'^0)\oplus(H^{\varepsilon}(C_0);C_0,C^0)$$

and hence a stable isomorphism

$$[h] : (H^{\varepsilon}(C^1,\varphi_1);C_1,C^0) \longrightarrow (H^{\varepsilon}(C'^1,\varphi'_1);C'_1,C'^0)$$

of the ε-symmetric formations associated to (C,φ) and (C',φ').

Every ε-symmetric formation is isomorphic to one of the type $(H^{\varepsilon}(F^*,\lambda);F,G)$ (by Proposition 1.6) and so determines a 1-dimensional ε-symmetric Poincaré complexes $(C,\varphi\in Q^1(C,\varepsilon))$ as follows. Write the inclusion of G in $F\oplus F^*$ as $\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : G \longrightarrow F\oplus F^*$, and let

$$C_1 = F\ ,\ C_0 = G^*\ ,\ d = \mu^* \in \mathrm{Hom}_A(F,G^*)\ ,\ C_r = 0\ (r \neq 0,1),$$

$$\varphi_0 = \varepsilon\gamma \in \mathrm{Hom}_A(G,F)\ ,\ \tilde{\varphi}_0 = \gamma^* + \mu^*\lambda \in \mathrm{Hom}_A(F^*,G^*)\ ,\ \varphi_1 = \lambda \in \mathrm{Hom}_A(F^*,F)\ .$$

Given a stable isomorphism of ε-symmetric formations

$$[h] : (H^{\varepsilon}(F^*,\lambda);F,G) \longrightarrow (H^{\varepsilon}(F'^*,\lambda');F',G')$$

let

$$h : (H^{\varepsilon}(F^*,\lambda);F,G)\oplus(H^{\varepsilon}(P);P,P^*) \longrightarrow (H^{\varepsilon}(F'^*,\lambda');F',G')\oplus(H^{\varepsilon}(P');P',P'^*)$$

and write the restrictions of h to the (sub)lagrangians as

$$\alpha = \begin{pmatrix} a & a_1 \\ a_2 & a_3 \end{pmatrix} : F\oplus P \longrightarrow F'\oplus P'\ ,\ \beta = \begin{pmatrix} b & b_1 \\ b_2 & b_3 \end{pmatrix} : G\oplus P^* \longrightarrow G'\oplus P'^*\ ,$$

denoting the inverse isomorphisms by

$$\alpha^{-1} = \begin{pmatrix} a' & a'_1 \\ a'_2 & a'_3 \end{pmatrix} : F'\oplus P' \longrightarrow F\oplus P\ ,\ \beta^{-1} = \begin{pmatrix} b' & b'_1 \\ b'_2 & b'_3 \end{pmatrix} : G'\oplus P'^* \longrightarrow G\oplus P^*\ .$$

Then there are defined inverse A-module isomorphisms

$$F\oplus G'^* \underset{\begin{pmatrix} a' & a'_1b'^*_1 \\ \mu'^* & -b'^* \end{pmatrix}}{\overset{\begin{pmatrix} a & a_1b^*_1 \\ \mu^* & -b^* \end{pmatrix}}{\rightleftarrows}} F'\oplus G^*$$

so that the chain map $f:C \longrightarrow C'$ defined by

$$\begin{array}{ccccccccc} C: & \dots \longrightarrow & 0 & \longrightarrow & F & \xrightarrow{\mu^*} & G^* & \longrightarrow 0 \longrightarrow & \dots \\ f\downarrow & & & & \downarrow a & & \downarrow b'^* & & \\ C': & \dots \longrightarrow & 0 & \longrightarrow & F' & \xrightarrow{\mu'^*} & G'^* & \longrightarrow 0 \longrightarrow & \dots \end{array}$$

is a chain equivalence, defining a homotopy equivalence

$$f : (C,\varphi) \longrightarrow (C',\varphi')$$

of the connected 1-dimensional ε-symmetric complexes associated to $(H^{\varepsilon}(F^*,\lambda);F,G)$ and $(H^{\varepsilon}(F'^*,\lambda');F',G')$.

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Given a 1-dimensional ε-quadratic complex over A $(C,\psi\in Q_1(C,\varepsilon))$ with C a f.g. projective A-module chain complex

$$C : \dots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \dots$$

we have that the $\mathbb{Z}_2$-hyperhomology class $\psi\in Q_1(C,\varepsilon)$ is represented by A-module morphisms

$$\psi_0 : C^0 \longrightarrow C_1\ ,\ \tilde{\psi}_0 : C^1 \longrightarrow C_0\ ,\ \psi_1 : C^0 \longrightarrow C_0$$

such that

$$d\psi_0 + \tilde{\psi}_0 d^* + \psi_1 - \varepsilon\psi^*_1 = 0 : C^0 \longrightarrow C_0\ .$$

The algebraic mapping cone $C((1+T_{\varepsilon})\psi_0 : C^{1-*} \longrightarrow C)$ can be expressed as

$$0 \longrightarrow C^0 \xrightarrow{\begin{pmatrix} \bar{\varepsilon}\psi_0+\tilde{\psi}^*_0 \\ d^* \end{pmatrix}} C_1\oplus C^1 \xrightarrow{(\varepsilon d\ \ (\tilde{\psi}_0+\varepsilon\psi^*_0)) = ((\bar{\varepsilon}\psi_0+\tilde{\psi}^*_0)^*\ \ d)\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}} C_0 \longrightarrow 0\ .$$

Thus if (C,ψ) is connected there is defined an ε-quadratic formation

$$(H_{\varepsilon}(C_1);C_1,C^0) = \left(C_1\oplus C^1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};C_1, \mathrm{im}\left(\begin{pmatrix} \bar{\varepsilon}\psi_0+\tilde{\psi}^*_0 \\ d^* \end{pmatrix} : C^0 \longrightarrow C_1\oplus C^1\right)\right)\ ,$$

which is non-singular if and only if (C,ψ) is Poincaré. The formation $(H_{\varepsilon}(C_1);C_1,C^0)$ does not involve ψ_1. In Proposition 1.8 below we shall show that homotopy equivalence classes of connected 1-dimensional ε-quadratic complexes correspond to the stable isomorphism classes of ε-quadratic formations together with the extra structure afforded by ψ_1.

Given a f.g. projective A-module M and a direct summand L define the abelian group

$$Q_\varepsilon(M,L) = \frac{\{(\psi,\theta)\in \mathrm{Hom}_A(M,M^*)\oplus \mathrm{Hom}_A(L,L^*)\,|\, j^*\psi j = \theta - \varepsilon\theta^*\}}{\{(\chi-\varepsilon\chi^*, j^*\chi j+\nu+\varepsilon\nu^*)\,|\,(\chi,\nu)\in \mathrm{Hom}_A(M,M^*)\oplus\mathrm{Hom}_A(L,L^*)\}}\ ,$$

so that there is defined an exact sequence

$$Q_\varepsilon(M,L)\xrightarrow{\partial} Q_\varepsilon(M)\xrightarrow{j_\%} Q_\varepsilon(L)$$

with $j\in\mathrm{Hom}_A(L,M)$ the inclusion and

$$\partial : Q_\varepsilon(M,L)\longrightarrow Q_\varepsilon(M)\ ;\ (\psi,\theta)\longmapsto\psi\quad ,\quad j_\% : Q_\varepsilon(M)\longrightarrow Q_\varepsilon(L)\ ;\ \psi\longmapsto j^*\psi j\ .$$

A hessian for a sublagrangian L of an ε-quadratic form $(M,\psi\in Q_\varepsilon(M))$ is a choice of lift of $\psi\in Q_\varepsilon(M)$ to an element $(\psi,\theta)\in Q_\varepsilon(M,L)$ such that $\partial(\psi,\theta) = \psi\in Q_\varepsilon(M)$. Every sublagrangian admits hessians, since $j^*\psi j = 0\in Q_\varepsilon(L)$, but they are not unique. A connected 1-dimensional ε-quadratic complex $(C,\psi\in Q_1(C,\varepsilon))$ (as above) determines an ε-quadratic formation $(H_\varepsilon(C_1);C_1,C^0)$ together with a hessian $\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},-(\psi_1+d\psi_0)\right)\in Q_\varepsilon(C_1\oplus C^1,C^0)$ for C^0 in $H_\varepsilon(C_1)$.

A split ε-quadratic formation over A $(F,G) = (F,(\begin{pmatrix}\gamma\\\mu\end{pmatrix},\theta)G)$ is an ε-quadratic formation over A $(H_\varepsilon(F);F,\mathrm{im}(\begin{pmatrix}\gamma\\\mu\end{pmatrix}:G\to F\oplus F^*))$ (with $\begin{pmatrix}\gamma\\\mu\end{pmatrix}:G\longrightarrow F\oplus F^*$ the inclusion) together with a (-ε)-quadratic form $\theta\in Q_{-\varepsilon}(G)$ such that

$$\gamma^*\mu = \theta-\varepsilon\theta^*\in\mathrm{Hom}_A(G,G^*)\ .$$

This determines a hessian $\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\theta\right)\in Q_\varepsilon(F\oplus F^*,G)$ for G in $H_\varepsilon(F)$, since $\gamma^*\mu\in\mathrm{Hom}_A(G,G^*)$ is the composite

$$\gamma^*\mu : G\xrightarrow{\begin{pmatrix}\gamma\\\mu\end{pmatrix}} F\oplus F^*\xrightarrow{\begin{pmatrix}0&1\\0&0\end{pmatrix}} F^*\oplus F\xrightarrow{(\gamma^*\ \mu^*)} G^*\ .$$

(Warning: our use of "split" accords with the usage of Sharpe [2], but not that of Wall [9]).

An isomorphism of split ε-quadratic formations over A

$$(\alpha,\beta,\psi) : (F,G)\longrightarrow(F',G')$$

is a triple consisting of A-module isomorphisms $\alpha\in\mathrm{Hom}_A(F,F')$, $\beta\in\mathrm{Hom}_A(G,G')$ and a (-ε)-quadratic form $(F^*,\psi\in Q_{-\varepsilon}(F^*))$ such that

$$\alpha\gamma\ +\alpha(\psi-\varepsilon\psi^*)^*\mu = \gamma'\beta : G\longrightarrow F'$$
$$\alpha^{*-1}\mu = \mu'\beta : G\longrightarrow F'^*$$
$$\theta+\mu^*\psi\mu = \beta^*\theta'\beta\in Q_{-\varepsilon}(G)\ .$$

A stable isomorphism of split ε-quadratic formations

$$[\alpha,\beta,\psi] : (F,G)\longrightarrow(F',G')$$

is an isomorphism of the type

$$(\alpha,\beta,\psi) : (F,G)\oplus(P,P^*)\longrightarrow(F',G')\oplus(P',P'^*)$$

for some f.g. projective A-modules P,P' with $(P,P^*) = (P,(\begin{pmatrix}0\\1\end{pmatrix},0)P^*)$.

Proposition 1.8 i) The homotopy equivalence classes of connected 1-dimensional ε-quadratic complexes over A are in a natural one-one correspondence with the stable isomorphism classes of split ε-quadratic formations over A. Poincaré complexes correspond to non-singular formations.

ii) A stable isomorphism of split ε-quadratic formations

$$[\alpha,\beta,\psi] : (F,G)\longrightarrow(F',G')$$

gives rise to a stable isomorphism of the underlying ε-quadratic formations

$$[f] : (H_\varepsilon(F);F,G)\longrightarrow(H_\varepsilon(F');F',G')\ .$$

Conversely, every stable isomorphism of ε-quadratic formations [f] can be lifted (non-uniquely) to a stable isomorphism of split ε-quadratic formations $[\alpha,\beta,\psi]$.

iii) Given connected 1-dimensional ε-quadratic complexes (C,ψ),(C',ψ') and a homotopy equivalence of the ε-symmetrizations

$$f : (C,(1+T_\varepsilon)\psi)\longrightarrow(C',(1+T_\varepsilon)\psi')$$

such that

$$f_\%(\psi)-\psi' = H(\theta)\in Q_1(C',\varepsilon)$$

for some $\theta\in\hat{Q}^2(C',\varepsilon)$ with

$$\hat{v}_1(\theta) = 0 : H^1(C')\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)\ ;\ x\longmapsto\theta_0(x)(x)$$

then there is defined a stable isomorphism between the ε-quadratic formations associated to (C,ψ),(C',ψ').

Proof: i) + iii) Given a connected 1-dimensional ε-quadratic complex $(C,\psi\in Q_1(C,\varepsilon))$ with C a f.g. projective complex of the type

$$C\ :\ \dots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\ d\ } C_0 \longrightarrow 0 \longrightarrow \dots$$

define a split ε-quadratic formation

$$(C_1,C^0) = (C_1,(\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^*\\ d^*\end{pmatrix},-(\psi_1+d\psi_0))C^0)\ .$$

If $\psi' = \psi + H(\theta)\in Q_1(C,\varepsilon)$ for some $\theta\in\hat{Q}^2(C,\varepsilon)$ then θ is represented by A-module morphisms

$$\theta_0 : C^1 \longrightarrow C_1\ ,\ \theta_{-1} : C^0 \longrightarrow C_1\ ,\ \tilde{\theta}_{-1} : C^1 \longrightarrow C_0\ ,\ \theta_{-2} : C^0 \longrightarrow C_0$$

such that

$$\theta_0 + \varepsilon\theta_0^* = 0\ ,\ d\theta_0 - \tilde{\theta}_{-1} - \varepsilon\theta_{-1}^* = 0\ ,\ \theta_0 d^* + \theta_{-1} + \varepsilon\tilde{\theta}_{-1}^* = 0\ ,$$
$$d\theta_{-1} + \tilde{\theta}_{-1}d^* + \theta_{-2} - \varepsilon\theta_{-2}^* = 0\ .$$

Thus

$$\psi_0' = \psi_0 + \theta_{-1}\ :\ C^0 \longrightarrow C_1\ ,\ \tilde{\psi}_0' = \tilde{\psi}_0 + \tilde{\theta}_{-1}\ :\ C^1 \longrightarrow C_0\ ,$$
$$\psi_1' = \psi_1 + \theta_{-2}\ :\ C^0 \longrightarrow C_0\ .$$

If $\hat{v}_1(\theta) = 0$ then $\theta_0 = \chi - \varepsilon\chi^*$ for some $\chi\in \mathrm{Hom}_A(C^1,C_1)$, and there is defined an isomorphism of the ε-quadratic formations associated to $(C,\psi),(C,\psi')$

$$\begin{pmatrix}1 & \theta_0^*\\ 0 & 1\end{pmatrix} : (H_\varepsilon(C_1);C_1,\mathrm{im}(\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^*\\ d^*\end{pmatrix}:C^0 \longrightarrow C_1\oplus C^1))$$
$$\longrightarrow (H_\varepsilon(C_1);C_1,\mathrm{im}(\begin{pmatrix}\bar{\varepsilon}\psi_0'+\tilde{\psi}_0'^*\\ d^*\end{pmatrix}:C^0 \longrightarrow C_1\oplus C^1))\ .$$

Conversely, given a split ε-quadratic formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ define a connected 1-dimensional ε-quadratic complex $(C,\psi\in Q_1(C,\varepsilon))$ by

$$C_1 = F\ ,\ C_0 = G^*\ ,\ d = \mu^*\in \mathrm{Hom}_A(C_1,C_0)\ ,\ C_r = 0\ (r \neq 0,1)$$
$$\psi_0 = \varepsilon\gamma\in \mathrm{Hom}_A(C^0,C_1)\ ,\ \tilde{\psi}_0 = 0\in \mathrm{Hom}_A(C^1,C_0)\ ,\ \psi_1 = -\theta\in \mathrm{Hom}_A(C^0,C_0)$$

for any representative $\theta\in\mathrm{Hom}_A(G,G^*)$ of $\theta\in Q_{-\varepsilon}(G)$.

The verification that homotopy equivalence classes of complexes correspond to stable isomorphism classes of formations proceeds as in the ε-symmetric case (Proposition 1.7).

ii) Given an isomorphism of split ε-quadratic formations over A

$$(\alpha,\beta,\psi)\ :\ (F,G) \longrightarrow (F',G')$$

there is defined a commutative diagram of A-module morphisms

$$\begin{array}{ccc} G & \xrightarrow{\ \beta\ } & G' \\ {\scriptstyle\begin{pmatrix}\gamma\\ \mu\end{pmatrix}}\downarrow & & \downarrow{\scriptstyle\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}} \\ F\oplus F^* & \xrightarrow{\ f\ } & F'\oplus F'^* \end{array}$$

with

$$f = \begin{pmatrix}\alpha & \alpha(\psi-\varepsilon\psi^*)^*\\ 0 & \alpha^{*-1}\end{pmatrix} : F\oplus F^* \longrightarrow F'\oplus F'^*\ ,$$

giving rise to an isomorphism of the underlying ε-quadratic formations

$$f\ :\ (H_\varepsilon(F);F,\mathrm{im}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G \longrightarrow F\oplus F^*)) \longrightarrow (H_\varepsilon(F');F',\mathrm{im}(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}:G' \longrightarrow F'\oplus F'^*))\ .$$

Similarly for stable isomorphisms.

Every ε-quadratic formation is isomorphic to one of the type $(H_\varepsilon(F);F,G)$, by Proposition 1.6, and choosing a hessian $(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\theta)\in Q_{-\varepsilon}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G \longrightarrow F\oplus F^*)$ for G in $H_\varepsilon(F)$ there is defined a split ε-quadratic formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$.

Suppose given a stable isomorphism of ε-quadratic formations

$$[f]\ :\ (H_\varepsilon(F);F,G) \longrightarrow (H_\varepsilon(F');F',G')\ ,$$

as defined by an isomorphism

$$f\ :\ (H_\varepsilon(F);F,G)\oplus(H_\varepsilon(P);P,P^*) \longrightarrow (H_\varepsilon(F');F',G')\oplus(H_\varepsilon(P');P',P'^*)\ .$$

The restrictions of f to the (sub)lagrangians are A-module isomorphisms

$$\alpha = \begin{pmatrix}a & a_1\\ a_2 & a_3\end{pmatrix} : F\oplus P \longrightarrow F'\oplus P'\ ,\ \beta = \begin{pmatrix}b & b_1\\ b_2 & b_3\end{pmatrix} : G\oplus P^* \longrightarrow G'\oplus P'^*\ .$$

Now f can be expressed as

$$f = \begin{pmatrix}\alpha & \alpha(\psi-\varepsilon\psi^*)^*\\ 0 & \alpha^{*-1}\end{pmatrix} : (F\oplus P)\oplus(F^*\oplus P^*) \longrightarrow (F'\oplus P')\oplus(F'^*\oplus P'^*)$$

for some A-module morphism

$$\psi = \begin{pmatrix}s & s_1\\ s_2 & s_3\end{pmatrix} : F^*\oplus P^* \longrightarrow F\oplus P\ ,$$

and there is defined a commutative diagram

$$\begin{array}{ccc} G\oplus P^* & \xrightarrow{\quad\beta\quad} & G'\oplus P'^* \\ \left(\begin{pmatrix}\gamma & 0\\ 0 & 0\end{pmatrix} \atop \begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}\right)\Big\downarrow & & \Big\downarrow\left(\begin{pmatrix}\gamma' & 0\\ 0 & 0\end{pmatrix} \atop \begin{pmatrix}\mu' & 0\\ 0 & 1\end{pmatrix}\right) \\ (F\oplus P)\oplus(F^*\oplus P^*) & \xrightarrow{\quad f\quad} & (F'\oplus P')\oplus(F'^*\oplus P'^*) \end{array} .$$

It follows that

$$\begin{pmatrix}\gamma^*\mu & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\mu^* & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} s-\varepsilon s^* & s_1-\varepsilon s_2^*\\ s_2-\varepsilon s_1^* & s_3-\varepsilon s_3^*\end{pmatrix}\begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} b^* & b_2^*\\ b_1^* & b_3^*\end{pmatrix}\begin{pmatrix}\gamma'^*\mu' & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} b & b_1\\ b_2 & b_3\end{pmatrix} : G\oplus P^* \longrightarrow G^*\oplus P ,$$

and in particular

$$s_3 - \varepsilon s_3^* = b_1^*\gamma'^*\mu' b_1 : P^* \longrightarrow P .$$

Choose a hessian $\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\theta'\right) \in Q_{-\varepsilon}\left(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}: G' \longrightarrow F'\oplus F'^*\right)$ for G' in $H_\varepsilon(F')$,

and define

$$\theta = b^*\theta' b - \mu^* s\mu \in Q_{-\varepsilon}(G) .$$

The isomorphism of split ε-quadratic formations

$$(\alpha,\beta,\tilde{\psi}) = \left(\begin{pmatrix} a & a_1\\ a_2 & a_3\end{pmatrix},\begin{pmatrix} b & b_1\\ b_2 & b_3\end{pmatrix},\begin{pmatrix} s & s_1\\ s_2 & b_1^*\theta' b_1\end{pmatrix}\right)$$

$$: \left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right)\oplus(P,P^*) \longrightarrow \left(F',\left(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta'\right)G'\right)\oplus(P',P'^*)$$

defines a stable isomorphism of split ε-quadratic formations

$$[\alpha,\beta,\psi] : (F,G) \longrightarrow (F',G')$$

covering the stable isomorphism of ε-quadratic formations $[f]$.

[]

The extra structure afforded an ε-quadratic formation by a choice of hessian is required for the uniqueness of glueing of $(-\varepsilon)$-quadratic forms, as defined in §5. The choice of hessian does not affect the cobordism class (= surgery obstruction) of the associated 1-dimensional ε-quadratic Poincaré complex defined in §5 below, but it is needed for the relative L-theory of §10.

§2. Quadratic topology

Write the singular chain complex functor as

$$C(\ ) : (\text{topological spaces}) \longrightarrow (\mathbb{Z}\text{-module chain complexes}) ;\ X \longmapsto C(X).$$

Given a commutative ring R (with 1) there is also defined an R-module chain complex

$$C(X;R) = R\otimes_{\mathbb{Z}} C(X) .$$

We shall write the homology and cohomology R-modules as

$$H_*(C(X;R)) = H_*(X;R) ,\ H^*(C(X;R)) = H^*(X;R)$$

(using the identity involution on R for the dual R-module structure). For $R = \mathbb{Z}$ write $H_*(X;\mathbb{Z}) = H_*(X)$, $H^*(X;\mathbb{Z}) = H^*(X)$. Standard acyclic model theory gives:

Proposition 2.1 i) There exists a functorial diagonal chain map

$$\Delta : C(\ ;R) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\ ;R)\otimes_R C(\ ;R)) ,$$

that is for each space X there is given an R-module chain map

$$\Delta_X : C(X;R) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X;R)\otimes_R C(X;R))$$

such that for each map of spaces $f: X \longrightarrow Y$ there is defined a commutative diagram

$$\begin{array}{ccc} C(X;R) & \xrightarrow{\ \Delta_X\ } & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X;R)\otimes_R C(X;R)) \\ f\Big\downarrow & & \Big\downarrow f^{\%} \\ C(Y;R) & \xrightarrow{\ \Delta_Y\ } & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(Y;R)\otimes_R C(Y;R)) \end{array} ,$$

with $T\in\mathbb{Z}_2$ acting on $C(X;R)\otimes_R C(X;R)$ by

$$T : C_p(X;R)\otimes_R C_q(X;R) \longrightarrow C_q(X;R)\otimes_R C_p(X;R) ;\ x\otimes y \longmapsto (-)^{pq} y\otimes x .$$

ii) Any two such functorial diagonal chain maps Δ,Δ' are related by a functorial chain homotopy

$$\Gamma : \Delta \simeq \Delta' : C(\ ;R) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\ ;R)\otimes_R C(\ ;R))$$

(which is itself unique up to functorial homotopy).

[]

Recall that the Steenrod squares of a space X are defined by

$$Sq^i : H^r(X;\mathbb{Z}_2) \longrightarrow H^{r+i}(X;\mathbb{Z}_2) ;$$

$$(c : C_r(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2) \longmapsto (Sq^i(c) = (c\otimes c)\Delta_X(-)(1_{r-i}) : C_{r+i}(X;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2)$$

with $1_{r-i} \in W_{r-i} = \begin{cases}\mathbb{Z}[\mathbb{Z}_2] & (r \geqslant i)\\ 0 & (r < i)\end{cases}$ the generator, and Δ_X any of the diagonal chain maps given by Proposition 2.1 i) for $R = \mathbb{Z}_2$. The functional Steenrod squares of a map of spaces $f:X \longrightarrow Y$ are defined by

$$Sq^i_f : \ker\left(\begin{pmatrix} f^* \\ Sq^i \end{pmatrix} : H^r(Y;\mathbb{Z}_2) \longrightarrow H^r(X;\mathbb{Z}_2)\oplus H^{r+i}(Y;\mathbb{Z}_2)\right)$$

$$\longrightarrow \operatorname{coker}((Sq^i\ f^*) : H^{r-1}(X;\mathbb{Z}_2)\oplus H^{r+i-1}(Y;\mathbb{Z}_2) \longrightarrow H^{r+i-1}(X;\mathbb{Z}_2))\ ;$$

$$(c:C(Y;\mathbb{Z}_2)_r \longrightarrow \mathbb{Z}_2)$$

$$\longmapsto (Sq^i_f(c) = (g\otimes g)(\Delta_X(-)(1_{r-i-1}) + (1\otimes d_X)\Delta_X(-)(1_{r-i})) + hf : C(X;\mathbb{Z}_2)_{r+i-1} \to \mathbb{Z}_2)$$

$$(cf = gd_X : C(X;\mathbb{Z}_2)_r \longrightarrow \mathbb{Z}_2\ ,\ g : C(X;\mathbb{Z}_2)_{r-1} \longrightarrow \mathbb{Z}_2\ ,$$

$$(c\otimes c)\Delta_Y(-)(1_{r-i}) = hd_Y : C(Y;\mathbb{Z}_2)_{r+i} \longrightarrow \mathbb{Z}_2\ ,\ h : C(Y;\mathbb{Z}_2)_{r+i-1} \longrightarrow \mathbb{Z}_2)$$

(Theorem 16.3 of Steenrod [1]).

Let π be a group, R a commutative ring (with 1).

Given a group morphism $w:\pi \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$ define the <u>w-twisted</u> involution on the group ring $R[\pi]$

$$\bar{\ } : R[\pi] \longrightarrow R[\pi]\ ;\ \sum_{g\in\pi} n_g g \longmapsto \sum_{g\in\pi} w(g) n_g g^{-1} \quad (n_g \in R)\ ,$$

calling it <u>untwisted</u> if $w = 1$ is trivial. Given an $R[\pi]$-module M let ${}^wM$ denote the $R[\pi]$-module defined by the additive group of M with $R[\pi]$ acting by

$$R[\pi] \times {}^wM \longrightarrow {}^wM\ ;\ (\sum_{g\in\pi} n_g g, x) \longmapsto \sum_{g\in\pi} n_g w(g)(gx)\ .$$

Then $({}^wM)^t$ (resp. $({}^wM)^*$) with respect to the untwisted involution on $R[\pi]$ is the same as M^t (resp. M^*) with respect to the w-twisted involution.

A <u>π-action</u> on a topological space X is a continuous function

$$\pi \times X \longrightarrow X\ ;\ (g,x) \longmapsto gx$$

(with the discrete topology on π), such that

$$(gh)x = g(hx)\ ,\ 1x = x \in X \quad (x\in X,\ g,h\in\pi).$$

The <u>homology</u> and <u>π-cohomology</u> $R[\pi]$-modules of X are defined by

$$H_*(X;R) = H_*(C(X;R))\ ,\ H^*_\pi(X;R) = H^*(C(X;R))$$

with the induced $R[\pi]$-module structure on $C(X;R)$ and the untwisted dual $R[\pi]$-module structure on $C(X;R)^{-*} = \operatorname{Hom}_{R[\pi]}(C(X;R),R[\pi])$. For $R = \mathbb{Z}$ write

$$H_*(X;\mathbb{Z}) = H_*(X)\ ,\ H^*_\pi(X;\mathbb{Z}) = H^*_\pi(X)\ .$$

Define also the homology and π-cohomology R-modules of X with coefficients in an $R[\pi]$-module M

$$H^\pi_*(X;M) = H_*(C(X;M))\ ,\ H^*_\pi(X;M) = H^*(C(X;M))$$

with $C(X;M) = M^t\otimes_{R[\pi]} C(X;R)$. In particular, for $M = R[\pi]$

$$H^\pi_*(X;R[\pi]) = H_*(X;R)\ ,\ H^*_\pi(X;R[\pi]) = H^*_\pi(X;R)\ ,$$

If $w:\pi \longrightarrow \mathbb{Z}_2$ is a group morphism as above we have natural identifications of $R[\pi]$-modules

$${}^wH_*(X;R) = H_*({}^wC(X;R))\ ,\ {}^wH^*_\pi(X;R) = H^*({}^wC(X;R))\ ,$$

the latter using the natural $R[\pi]$-module isomorphism

$${}^w\operatorname{Hom}_{R[\pi]}(C(X),R[\pi]) \longrightarrow \operatorname{Hom}_{R[\pi]}({}^wC(X),R[\pi])\ ;\ f \longmapsto (x \longmapsto \sum_{g\in\pi} w(g) n_g g)$$

$$(f(x) = \sum_{g\in\pi} n_g g \in R[\pi],\ n_g \in R)$$

Given a pointed topological space X define the quotient R-module chain complex

$$\dot{C}(X;R) = C(X;R)/C(\text{pt.};R)$$

and write the reduced homology and cohomology R-modules as

$$\dot{H}_*(X;R) = H_*(\dot{C}(X;R))\ ,\ \dot{H}^*(X;R) = H^*(\dot{C}(X;R))\ .$$

For $R = \mathbb{Z}$ write $\dot{C}(X;\mathbb{Z}) = \dot{C}(X)$, $\dot{H}_*(X;\mathbb{Z}) = \dot{H}_*(X)$, $\dot{H}^*(X;\mathbb{Z}) = \dot{H}^*(X)$.

Given a functorial diagonal chain map

$$\Delta_X : C(X;R) \longrightarrow \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X;R)\otimes_R C(X;R))$$

there is induced a diagonal chain map

$$\dot{\Delta}_X : \dot{C}(X;R) = C(X;R)/C(\text{pt.};R)$$

$$\xrightarrow{\ \Delta_X\ } \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X;R)\otimes_R C(X;R))/\operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\text{pt};R)\otimes_R C(\text{pt.};R))$$

$$\xrightarrow{\ (\text{pr.})^{\%}\ } \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \dot{C}(X;R)\otimes_R \dot{C}(X;R))$$

with $\text{pr.}:C(X;R) \longrightarrow \dot{C}(X;R)$ the projection. Then $\dot{\Delta}_X$ is functorial on the category of pointed spaces and point-preserving maps. Given an (unpointed) space X define a pointed space by adjoining a point

$$X_+ = X \cup \{\text{pt.}\}$$

and identify $\dot{C}(X_+;R) = C(X;R)$, $\dot{H}_*(X_+;R) = H_*(X;R)$, $\dot{H}^*(X_+;R) = H^*(X;R)$, $\dot{\Delta}_{X_+} = \Delta_X$.

A <u>π-space</u> is a pointed space X with an action of a group π

$$\pi \times X \longrightarrow X\ ;\ (g,x) \longmapsto gx$$

such that $g(\text{pt.}) = \text{pt.} \in X$ $(g\in\pi)$. The induced $R[\pi]$-action on $C(X;R)$ preserves $C(\text{pt.};R) \subseteq C(X;R)$, so that there is defined an $R[\pi]$-action on $\dot{C}(X;R)$. Also, there are defined reduced homology and π-cohomology $R[\pi]$-modules

$$\dot{H}_*(X;R) = H_*(\dot{C}(X;R))\ ,\ \dot{H}^*_\pi(X;R) = H^*(\dot{C}(X;R))$$

and the reduced diagonal chain maps $\dot{\Delta}_X$ are $R[\pi]$-module maps, with the diagonal π-action on $\dot{C}(X;R)\otimes_R \dot{C}(X;R)$.

A π-map of π-spaces is a map of spaces

$$f : X \longrightarrow Y$$

such that

$$f(pt.) = pt.\ ,\ f(gx) = gf(x) \in Y \quad (x\in X,\ g\in\pi)\ .$$

There are induced $R[\pi]$-module maps $f:\dot{C}(X;R)\longrightarrow\dot{C}(Y;R)$, $f_*:\dot{H}_*(X;R)\longrightarrow\dot{H}_*(Y;R)$, $f^*:\dot{H}^*(Y;R)\longrightarrow\dot{H}^*(X;R)$, and $f^{\%}\dot{\Delta}_X=\dot{\Delta}_Y f:\dot{C}(X;R)\longrightarrow Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(Y;R)\otimes_R\dot{C}(Y;R))$.

We have the following symmetric construction $\dot{\varphi}_X$.

Proposition 2.2 Let π be a group, $w:\pi\longrightarrow\mathbb{Z}_2$ a group morphism, R a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution. Regard R as an $R[\pi]$-module by

$$R[\pi]\times R\longrightarrow R\ ;\ (\sum_{g\in\pi}n_g g,r)\longmapsto(\sum_{g\in\pi}n_g)r\ .$$

Given a π-space X there are defined in a natural way R-module morphisms

$$\dot{\varphi}_X\ :\ \dot{H}^\pi_n(X;{}^wR)\longrightarrow Q^n(\dot{C}(X;R))\qquad(n\geq 0)$$

such that

i) for each $x\in\dot{H}^\pi_n(X;{}^wR)$

$$\dot{\varphi}_X(x)_0\backslash - = x\cap -\ :\ {}^w\dot{H}^r_\pi(X;R)\longrightarrow\dot{H}_{n-r}(X;R)$$

ii) for each π-map of π-spaces $f:X\longrightarrow Y$ there is defined a commutative diagram of R-modules

$$\begin{array}{ccc}\dot{H}^\pi_n(X;{}^wR) & \xrightarrow{\dot{\varphi}_X} & Q^n(\dot{C}(X;R))\\ \downarrow f_* & & \downarrow f^{\%}\\ \dot{H}^\pi_n(Y;{}^wR) & \xrightarrow{\dot{\varphi}_Y} & Q^n(\dot{C}(Y;R))\end{array}$$

iii) for each morphism $h:R\longrightarrow S$ of commutative rings there is defined a commutative diagram of R-modules

$$\begin{array}{ccc}\dot{H}^\pi_n(X;{}^wR) & \xrightarrow{\dot{\varphi}_X} & Q^n(\dot{C}(X;R))\\ \downarrow h & & \downarrow h\\ \dot{H}^\pi_n(X;{}^wS) & \xrightarrow{\dot{\varphi}_X} & Q^n(\dot{C}(X;S))\end{array}$$

in which the vertical maps are the change of rings $h:R[\pi]\longrightarrow S[\pi]$.

Proof: Applying $R^t\otimes_{R[\pi]}-$ to a functorial diagonal $R[\pi]$-module chain map

$$\dot{\Delta}_X\ :\ \dot{C}(X;R)\longrightarrow Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(X;R)\otimes_R\dot{C}(X;R))$$

there is obtained a functorial $\mathbb{Z}$-module chain map

$$\dot{\Delta}_X\ :\ R^t\otimes_{R[\pi]}\dot{C}(X;R)\longrightarrow R^t\otimes_{R[\pi]}Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(X;R)\otimes_R\dot{C}(X;R))$$
$$= Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(X;R)^t\otimes_{R[\pi]}\dot{C}(X;R))\ ,$$

inducing the required $\mathbb{Z}$-module maps in homology

$$\dot{\varphi}_X=(\dot{\Delta}_X)_*\ :\ H_n(R^t\otimes_{R[\pi]}\dot{C}(X;R))=\dot{H}^\pi_n(X;{}^wR)$$
$$\longrightarrow H_n(Hom_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(X;R)^t\otimes_{R[\pi]}\dot{C}(X;R)))=Q^n(\dot{C}(X;R))\ .$$

[]

Applying the symmetric construction to the π-space X_+ obtained from a space with π-action X by adjoining a base point we obtain an absolute symmetric construction

$$\varphi_X=\dot{\varphi}_{X_+}\ :\ H^\pi_n(X;{}^wR)=\dot{H}^\pi_n(X_+;{}^wR)\longrightarrow Q^n(C(X;R))=Q^n(\dot{C}(X_+;R))\ .$$

Applying the symmetric Wu class operations v_r of §1 to the symmetric construction for $\pi=\{1\}$, $R=\mathbb{Z}_2$ we obtain the Steenrod squares:

Proposition 2.3 Let X be a $\{1\}$-space. The composite $\mathbb{Z}_2$-module morphism

$$\dot{H}_n(X;\mathbb{Z}_2)\xrightarrow{\dot{\varphi}_X}Q^n(\dot{C}(X;\mathbb{Z}_2))\xrightarrow{v_r}Hom_{\mathbb{Z}_2}(\dot{H}^{n-r}(X;\mathbb{Z}_2),\hat{H}^{n-2r}(\mathbb{Z}_2;\mathbb{Z}_2))$$
$$=\begin{cases}Hom_{\mathbb{Z}_2}(\dot{H}^{n-r}(X;\mathbb{Z}_2),\mathbb{Z}_2) & \text{if } n\geq 2r\\ 0 & \text{if } n<2r\end{cases}$$

is given by

$$v_r(\dot{\varphi}_X(x))(y)=\langle Sq^r(y),x\rangle\in\mathbb{Z}_2\qquad(x\in\dot{H}_n(X;\mathbb{Z}_2),\ y\in\dot{H}^{n-r}(X;\mathbb{Z}_2))\ ,$$

with $\langle\ ,\ \rangle$ the Kronecker product.

[]

The suspension of a π-space X is the reduced suspension

$$\Sigma X = X\wedge S^1 = X\times S^1/X\times pt.\cup pt.\times S^1$$

with the trivial π-action on S^1. The relative Eilenberg-Zilber theorem gives a functorial $\mathbb{Z}$-module chain equivalence on the category of pointed spaces X

$$E\ :\ \dot{C}(X)\otimes_{\mathbb{Z}}\dot{C}(S^1)\longrightarrow C(X\times S^1,X\times pt.\cup pt.\times S^1)$$

uniquely up to functorial chain homotopy. Evaluating on a fixed cycle $\iota\in\dot{C}_1(S^1)$

generating $\dot{H}_1(S^1) = \mathbb{Z}$ and composing with the chain map induced by the collapsing map $c:(X\times S^1, X\times pt.\cup pt.\times S^1)\longrightarrow(\Sigma X, pt.)$ there is obtained a functorial $\mathbb{Z}$-module chain map

$$\Sigma_X = cE(-\otimes\iota) : S\dot{C}(X)\longrightarrow \dot{C}(\Sigma X)$$

uniquely up to functorial chain homotopy, inducing the suspension isomorphisms

$$\Sigma : \dot{H}_{*-1}(X)\longrightarrow \dot{H}_*(\Sigma X)$$

in homology. Furthermore, if $\Delta_W: W\longrightarrow W\otimes_{\mathbb{Z}} W$ is the diagonal $\mathbb{Z}[\mathbb{Z}_2]$-module chain map defined by

$$\Delta_W : W_s \longrightarrow (W\otimes_{\mathbb Z} W)_s = \sum_{r=0}^{s} W_r\otimes_{\mathbb Z} W_{s-r} \ ; \ 1_s \longmapsto \sum_{r=0}^{s} 1_r\otimes T^r_{s-r} \quad (s\geq 0)$$

and $\dot\Delta_X : \dot C(X)\longrightarrow Hom_{\mathbb Z[\mathbb Z_2]}(W, \dot C(X)\otimes_{\mathbb Z}\dot C(X))$ is one of the functorial $\mathbb{Z}$-module chain maps described above then acyclic models give a functorial $\mathbb{Z}$-module chain homotopy uniquely up to functorial homotopy

$$\gamma_X : \dot\Delta_{X\times S^1} E \simeq E^{\%}\Delta_W^*(\dot\Delta_X\otimes\dot\Delta_{S^1})$$
$$: \dot C(X)\otimes_{\mathbb Z} C(S^1) \longrightarrow Hom_{\mathbb Z[\mathbb Z_2]}(W, C(X\times S^1, X\times pt.\cup pt.\times S^1)\otimes_{\mathbb Z} C(X\times S^1, X\times pt.\cup pt.\times S^1)).$$

Evaluating on $\iota\in \dot C_1(S^1)$ and composing with the chain map $c^\%$ induced by the collapsing map c there is obtained a functorial $\mathbb{Z}$-module chain homotopy uniquely up to functorial homotopy

$$\Gamma_X = c^\%\gamma_X(-\otimes\iota) : \dot\Delta_{\Sigma X}\Sigma_X \simeq \Sigma_X^\% S\dot\Delta_X : S\dot C(X)\longrightarrow Hom_{\mathbb Z[\mathbb Z_2]}(W,\dot C(\Sigma X)\otimes_{\mathbb Z}\dot C(\Sigma X)) ,$$

with S the algebraic suspension chain map of §1

$$S : SHom_{\mathbb Z[\mathbb Z_2]}(W,\dot C(X)\otimes_{\mathbb Z}\dot C(X)) \longrightarrow Hom_{\mathbb Z[\mathbb Z_2]}(W, S\dot C(X)\otimes_{\mathbb Z} S\dot C(X)) .$$

Thus for a π-space X we have functorial $\mathbb{Z}[\pi]$-module chain maps

$$\Sigma_X : S\dot C(X)\longrightarrow \dot C(\Sigma X)$$

inducing isomorphisms in homology, and for each such chain map a functorial $\mathbb{Z}[\pi]$-module chain homotopy

$$\Gamma_X : \dot\Delta_{\Sigma X}\Sigma_X \simeq \Sigma^\%_X S\dot\Delta_X : S\dot C(X)\longrightarrow Hom_{\mathbb Z[\mathbb Z_2]}(W, \dot C(\Sigma X)\otimes_{\mathbb Z}\dot C(\Sigma X)).$$

The chain map is unique up to functorial chain homotopy, and the chain homotopy is unique up to functorial homotopy. Applying $R\otimes_{\mathbb Z}-$ we have the same types of map for any coefficient ring R.

Thus the algebraic and geometric suspension operations correspond to each other under the symmetric construction.

Proposition 2.4 There is defined a commutative diagram of R-modules

$$\begin{array}{ccccc}
\dot H^\pi_n(X; {}^wR) & \xrightarrow{\varphi_X} & Q^n(\dot C(X;R)) & \xrightarrow{S} & Q^{n+1}(S\dot C(X;R)) \\
\downarrow{\scriptstyle \Sigma_X} & & & & \downarrow{\scriptstyle \Sigma^\%_X} \\
\dot H^\pi_{n+1}(\Sigma X;{}^wR) & & \xrightarrow{\varphi_{\Sigma X}} & & Q^{n+1}(\dot C(\Sigma X;R))
\end{array}$$

for any π-space X, commutative ring R, group morphism $w:\pi\longrightarrow\mathbb Z_2$.

Proof: The underlying chain maps are chain homotopic.

[]

A π-space X is n-dimensional if $\dot C(\Sigma^p X)$ is an $(n+p)$-dimensional $\mathbb{Z}[\pi]$-module chain complex for each $p\geq 0$, with $\Sigma^0 X = X$ by convention. For example, if $X=\tilde Y_+$ for some covering $\tilde Y$ of a finite n-dimensional (resp. finitely-dominated) CW complex Y with group of covering translations π then $\Sigma^q X$ is an $(n+q)$-dimensional (resp. finite-dimensional) π-space, $q\geq 0$. If X is a finite-dimensional π-space then $\dot C(\Sigma^pX;R) = R\otimes_{\mathbb Z}\dot C(\Sigma^p X) = R[\pi]\otimes_{\mathbb Z[\pi]}\dot C(\Sigma^pX)$ is a finite-dimensional $R[\pi]$-module chain complex for any coefficient ring R, and the suspension chain maps $\Sigma_X: S\dot C(X;R)\longrightarrow\dot C(\Sigma X;R)$ are $R[\pi]$-module chain equivalences.

A π-homotopy of π-maps $f_0,f_1:X\longrightarrow Y$ is a π-map

$$H : X\wedge I_+\longrightarrow Y \qquad (I=[0,1])$$

with the trivial π-action on I, such that H restricts to f_i on $X\wedge\{i\}_+$ $(i=0,1)$. The functoriality of the usual proof of the homotopy invariance of singular homology ensures that H induces an $R[\pi]$-module chain homotopy

$$H : f_0\simeq f_1 : \dot C(X;R)\longrightarrow\dot C(Y;R) .$$

We have the following quadratic construction, associating to an $S\pi$-map $F:\Sigma^\infty X\longrightarrow\Sigma^\infty Y$ and a group morphism $w:\pi\longrightarrow\mathbb Z_2$ abelian group morphisms

$$\psi_F : \dot H^\pi_n(X;{}^w\mathbb Z)\longrightarrow Q_n(\dot C(Y))$$

in a natural way. We are in effect giving a direct chain level description of the composite

$$X \xrightarrow{\text{adjoint } F} \Omega^\infty\Sigma^\infty Y \xrightarrow[\text{equivalence}]{\text{stable homotopy}} \bigvee_{m=1}^\infty (E\Sigma_m)_+\wedge_{\Sigma_m} Y^{(m)} \xrightarrow{\text{projection}} (E\Sigma_2)_+\wedge_{\Sigma_2} Y^{(2)}$$

where $Y^{(m)} = Y\wedge Y\wedge\ldots\wedge Y$ (m times), $\dot H_n((E\Sigma_2)_+\wedge_{\Sigma_2}(Y\wedge_\pi Y)) = Q_n(\dot C(Y))$ (if $w=1$).

Proposition 2.5 Let π be a group, $w:\pi \longrightarrow \mathbb{Z}_2$ a group morphism, R a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution.

Given finite-dimensional π-spaces X,Y and a π-map $F:\Sigma^p X \longrightarrow \Sigma^p Y$ $(p \geqslant 0)$ there are defined in a natural way R-module morphisms

$$\dot{\psi}_F : \dot{H}^\pi_n(X;{}^wR) \longrightarrow Q_n^{[0,p-1]}(\dot{C}(Y;R)) \qquad (n \geqslant 0)$$

depending only on the π-homotopy class of F, such that

i) $\dot{\varphi}_Y f_* - f^{\%}\dot{\varphi}_X = (1+T)\dot{\psi}_F : \dot{H}^\pi_n(X;{}^wR) \longrightarrow Q^n(\dot{C}(Y;R))$

for any $R[\pi]$-module chain map $f:\dot{C}(X;R) \longrightarrow \dot{C}(Y;R)$ in the chain homotopy class

$$f : \dot{C}(X;R) \xrightarrow{\Sigma^p_X} \Omega^p\dot{C}(\Sigma^pX;R) \xrightarrow{F} \Omega^p\dot{C}(\Sigma^pY;R) \xrightarrow{(\Sigma^p_Y)^{-1}} \dot{C}(Y;R)$$

ii) suspension of F to $\Sigma F:\Sigma^{p+1}X \longrightarrow \Sigma^{p+1}Y$ replaces $\dot{\psi}_F$ by the composite

$$\dot{\psi}_{\Sigma F} : \dot{H}^\pi_n(X;{}^wR) \xrightarrow{\dot{\psi}_F} Q_n^{[0,p-1]}(\dot{C}(Y;R)) \longrightarrow Q_n^{[0,p]}(\dot{C}(Y;R))$$

iii) if $G:\Sigma^pY \longrightarrow \Sigma^pZ$ is another π-map between p-fold suspensions of finite-dimensional π-spaces Y,Z and $g = (\Sigma^p_Z)^{-1}G(\Sigma^p_Y) : \dot{C}(Y;R) \longrightarrow \dot{C}(Z;R)$ then

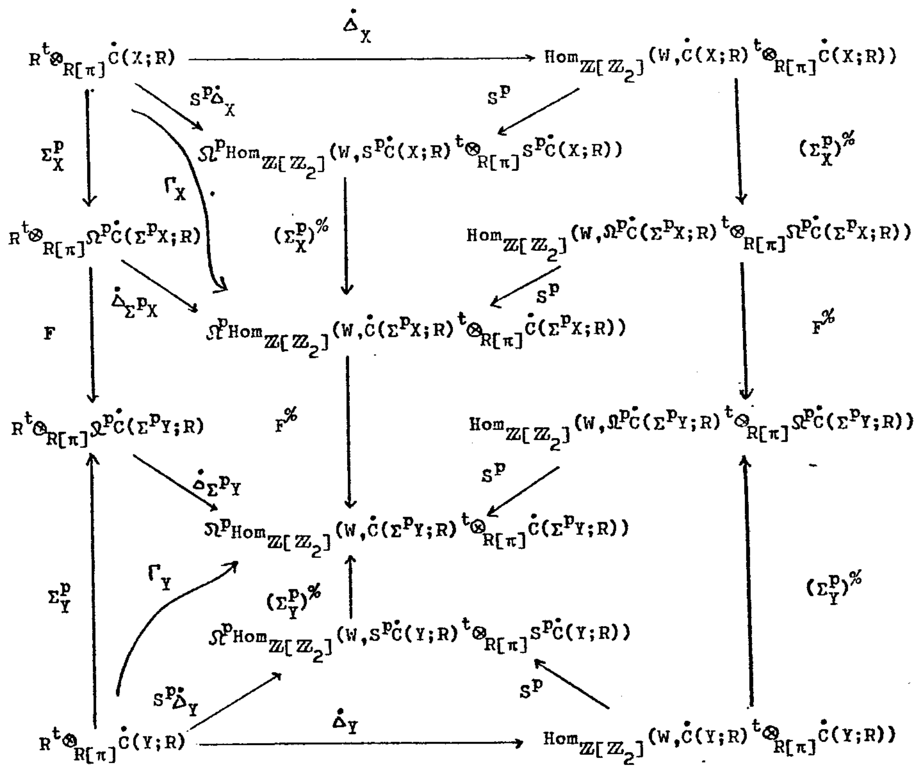
$$\dot{\psi}_{GF} = g_{\%}\dot{\psi}_F + \dot{\psi}_G f_* : \dot{H}^\pi_n(X;{}^wR) \longrightarrow Q_n^{[0,p-1]}(\dot{C}(Z;R))$$

iv) if $h:R \longrightarrow S$ is a morphism of commutative rings then there is defined a commutative diagram of R-modules

$$\begin{array}{ccc} \dot{H}^\pi_n(X;{}^wR) & \xrightarrow{\dot{\psi}_F} & Q_n^{[0,p-1]}(\dot{C}(Y;R)) \\ \downarrow h & & \downarrow h \\ \dot{H}^\pi_n(X;{}^wS) & \xrightarrow{\dot{\psi}_F} & Q_n^{[0,p-1]}(\dot{C}(Y;S)) \end{array}$$

in which the vertical maps are the change of rings $h:R[\pi] \longrightarrow S[\pi]$.

Proof: Make functorial choices of Δ,Σ,Γ and consider the diagram



which commutes except for the parts marked by the chain homotopies Γ_X,Γ_Y. Recall from Proposition 1.3 that there is a natural R-module chain equivalence

$$\begin{aligned} C(S^p) \equiv\ & C(S^p:\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\Omega^p\dot{C}(\Sigma^pY;R)^t\otimes_{R[\pi]}\Omega^p\dot{C}(\Sigma^pY;R)) \\ & \longrightarrow \Omega^p\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(\Sigma^pY;R)^t\otimes_{R[\pi]}\dot{C}(\Sigma^pY;R))) \\ & \longrightarrow S(W[0,p-1]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\Omega^p\dot{C}(\Sigma^pY;R)^t\otimes_{R[\pi]}\Omega^p\dot{C}(\Sigma^pY;R))) \end{aligned}$$

for any π-space Y. Thus for a finite-dimensional π-space Y we can identify

$$H_{n+1}(S^p) = Q_n^{[0,p-1]}(\Omega^p\dot{C}(\Sigma^pY;R)) = Q_n^{[0,p-1]}(\dot{C}(Y;R)) ,$$

using the chain equivalence $\Sigma^p_Y:\dot{C}(Y;R) \longrightarrow \Omega^p\dot{C}(\Sigma^pY;R)$. We shall construct $\dot{\psi}_F$ using a chain homotopy inverse for Σ^p_Y - this cannot be chosen functorially, and we shall need the following result to show that the choice does not matter.

Lemma Let $f:C \longrightarrow D$ be a chain equivalence of finite-dimensional A-module chain complexes, and let $(g,h),(g',h')$ be pairs consisting of a chain map $g:D \longrightarrow C$ and a chain homotopy $h:fg \simeq 1:D \longrightarrow D$. Then there exist a chain homotopy $j:g \simeq g':D \longrightarrow C$ and a homotopy of chain homotopies $k:h \simeq h'+fj:fg \simeq 1:D \longrightarrow D$.

Proof: A chain map of finite-dimensional A-module chain complexes $f:C \longrightarrow D$ is a chain equivalence if and only if the algebraic mapping cone $C(f)$ is contractible. A pair (g,h) defines a chain map $\begin{pmatrix} g \\ h \end{pmatrix}:D \longrightarrow \Omega C(f)$ by

$$\begin{pmatrix} g \\ h \end{pmatrix}: D_r \longrightarrow \Omega C(f)_r = C_r \oplus D_{r+1} .$$

A contraction of $C(f)$ determines a chain homotopy between any two such chain maps

$$\begin{pmatrix} j \\ k \end{pmatrix}: \begin{pmatrix} g \\ h \end{pmatrix} \simeq \begin{pmatrix} g' \\ h' \end{pmatrix} : D \longrightarrow \Omega C(f) .$$

[]

Given a π-map $F:\Sigma^p X \longrightarrow \Sigma^p Y$ with Y a finite-dimensional π-space we have that $\Sigma^p_Y : \dot{C}(Y;R) \longrightarrow \Omega^p \dot{C}(\Sigma^p Y;R)$ is a chain equivalence of finite-dimensional $R[\pi]$-module chain complexes. Choosing a chain homotopy inverse $(\Sigma^p_Y)^{-1} : \Omega^p \dot{C}(\Sigma^p Y;R) \longrightarrow \dot{C}(Y;R)$ and a chain homotopy $h : \Sigma^p_Y(\Sigma^p_Y)^{-1} \simeq 1 : \Omega^p \dot{C}(\Sigma^p Y;R) \longrightarrow \Omega^p \dot{C}(\Sigma^p Y;R)$ define a chain map

$$\dot{\psi}_F : \dot{C}(X;{}^wR) \longrightarrow \Omega C(S^p)$$

by

$$\dot{\psi}_F = \begin{pmatrix} (\Sigma^p_Y)^{\%}\dot{\Delta}_Y(\Sigma^p_Y)^{-1}F\Sigma^p_X - \dot{F}^{\%}(\Sigma^p_X)^{\%}\dot{\Delta}_X \\ F^{\%}\Gamma_X - \Gamma_Y(\Sigma^p_Y)^{-1}F\Sigma^p_X - \dot{\Delta}_{\Sigma^p Y}hF\Sigma^p_X \end{pmatrix}$$

$$: R^t\otimes_{R[\pi]}\dot{C}(X;R)_n \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\Omega^p\dot{C}(\Sigma^p Y;R)^t\otimes_{R[\pi]}\Omega^p\dot{C}(\Sigma^p Y;R))_n$$
$$\oplus\,\Omega^p\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(\Sigma^p Y;R)^t\otimes_{R[\pi]}\dot{C}(\Sigma^p Y;R))_{n+1} .$$

A different choice of $\Delta,\Sigma,\Gamma,((\Sigma^p_Y)^{-1},h)$ affects $\dot{\psi}_F$ only by a chain homotopy, so that the quadratic construction

$$\dot{\psi}_F : \dot{H}^\pi_n(X;{}^wR) \longrightarrow Q_n^{[0,p-1]}(\dot{C}(Y;R))$$

is independent of the choices.

[]

For π-maps of the type $F:\Sigma^p X_+ \longrightarrow \Sigma^p Y_+$ (for some spaces with π-action X,Y) we have an absolute quadratic construction

$$\psi_F = \dot{\psi}_F : H^\pi_n(X;{}^wR) = \dot{H}^\pi_n(X_+;{}^wR) \longrightarrow Q_n^{[0,p-1]}(C(Y;R)) = Q_n^{[0,p-1]}(\dot{C}(Y_+;R)) .$$

The result of applying the quadratic Wu class operation v^r of §1 to the quadratic construction for $\pi = \{1\}$, $R = \mathbb{Z}_2$ can be expressed in terms of the functional Steenrod squares:

Proposition 2.6 Let $F:\Sigma^p X \longrightarrow \Sigma^p Y$ be a $\{1\}$-map, inducing the chain map

$$f : \dot{C}(X) \xrightarrow{\Sigma^p_X} \dot{C}(\Sigma^p X) \xrightarrow{F} \dot{C}(\Sigma^p Y) \xrightarrow{(\Sigma^p_Y)^{-1}} \dot{C}(Y) ,$$

for some finite-dimensional $\{1\}$-spaces X,Y. The composite

$$\dot{H}_n(X;\mathbb{Z}_2) \xrightarrow{\dot{\psi}_F} Q_n^{[0,p-1]}(\dot{C}(Y;\mathbb{Z}_2)) \xrightarrow{v^r} \mathrm{Hom}_{\mathbb{Z}_2}(\dot{H}^{n-r}(Y;\mathbb{Z}_2),Q_n^{[0,p-1]}(S^{n-r}\mathbb{Z}_2))$$
$$= \begin{cases} \mathrm{Hom}_{\mathbb{Z}_2}(\dot{H}^{n-r}(Y;\mathbb{Z}_2),\mathbb{Z}_2) & \text{if } n \leq 2r \leq n+p-1 \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$v^r(\dot{\psi}_F(x))(y) = \langle Sq_h^{r+1}(\Sigma^p\iota),\Sigma^p x\rangle \in \mathbb{Z}_2 \qquad (n \leq 2r)$$

$$(x\in\dot{H}_n(X;\mathbb{Z}_2),\ y\in\dot{H}^{n-r}(Y;\mathbb{Z}_2) = [Y,K(\mathbb{Z}_2,n-r)],\ h = (\Sigma^p y)F - \Sigma^p(f^*y)\in[\Sigma^p X,\Sigma^p K(\mathbb{Z}_2,n-r)]$$
$$\iota = \text{generator} \in \dot{H}^{n-r}(K(\mathbb{Z}_2,n-r);\mathbb{Z}_2) = \mathbb{Z}_2\ ) ,$$

with

$$v^r(\dot{\psi}_F(x))(y_1+y_2) - v^r(\dot{\psi}_F(x))(y_1) - v^r(\dot{\psi}_F(x))(y_2)$$
$$= \begin{cases} \langle f^*(y_1 \cup y_2) - (f^*y_1 \cup f^*y_2),x\rangle \in \mathbb{Z}_2 & n = 2r \\ 0 \in \mathbb{Z}_2 & n < 2r . \end{cases}$$

[]

(It is sometimes useful to consider also the following generalization of the quadratic construction $\dot{\psi}_F$. Given finite-dimensional π-spaces X,Y, a group morphism $w:\pi \longrightarrow \mathbb{Z}_2$ and a π-map $F:X \longrightarrow \Sigma^p Y$ $(p \geq 0)$ there are defined in a natural way abelian group morphisms

$$\dot{\psi}_F : \dot{H}^\pi_{n+p}(X;{}^wR) \longrightarrow Q_n^{[0,p-1]}(C(f))$$

such that

$$(1+T)\dot{\psi}_F : \dot{H}^\pi_{n+p}(X;{}^wR) \xrightarrow{f} \dot{H}^\pi_n(Y;{}^wR) \xrightarrow{\dot{\varphi}_Y} Q^n(\dot{C}(Y;R)) \xrightarrow{e^{\%}} Q^n(C(f))$$

where f is the $R[\pi]$-module chain map

$$f : \Omega^p\dot{C}(X;R) \xrightarrow{F} \Omega^p\dot{C}(Y;R) \xrightarrow{(\Sigma^p_Y)^{-1}} \dot{C}(Y;R)$$

and $e:\dot{C}(Y;R) \longrightarrow C(f)$ is the inclusion.)

An <u>n-dimensional geometric Poincaré complex</u> X in the sense of Wall [4] is a finitely dominated connected CW complex X together with an <u>orientation map</u> group morphism

$$w(X) : \pi_1(X) \longrightarrow \mathbb{Z}_2$$

and a <u>fundamental class</u> $[X] \in H_n^{\pi_1(X)}(\tilde{X};{}^{w(X)}\mathbb{Z})$ such that the cap products

$$[X] \cap - : {}^{w(X)}H^r_{\pi_1(X)}(\tilde{X}) \longrightarrow H_{n-r}(\tilde{X}) \qquad (0 \leqslant r \leqslant n)$$

define $\mathbb{Z}[\pi_1(X)]$-module Poincaré duality isomorphisms, with $\tilde{X}$ the universal cover of X and $\pi_1(X)$ acting on $\tilde{X}$ as the group of covering translations. The singular chain complex $C(\tilde{X})$ is then an n-dimensional $\mathbb{Z}[\pi_1(X)]$-module complex, and the Poincaré duality is induced by a $\mathbb{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap - : {}^{w(X)}C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X}) \ .$$

(For finite X and $w(X) = 1$ such a geometric Poincaré complex X is a P-space of formal dimension n in the sense of Spivak [1], since applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ there is obtained a $\mathbb{Z}[\pi_1(X)]$-module chain equivalence

$$[\tilde{X}] \cap - : \mathrm{Hom}_{\mathbb{Z}}(C(\tilde{X}),\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C(\tilde{X})^{n-*},\mathbb{Z}) \simeq C^{LF}(\tilde{X})_{n-*}$$

inducing Poincaré duality isomorphisms

$$[\tilde{X}] \cap - : H^*(\tilde{X}) \longrightarrow H^{LF}_{n-*}(\tilde{X})$$

between the singular cohomology groups of $\tilde{X}$ and the homology groups of $\tilde{X}$ defined by locally finite infinite chains, with $[\tilde{X}] \in H_n^{LF}(\tilde{X})$ the transfer of the fundamental class $[X] \in H_n(X)$).

Let X be an n-dimensional geometric Poincaré complex. If $\tilde{X}$ is a (not necessarily connected) covering of X with group of covering translations π and $\tilde{\tilde{X}}$ is the universal covering of X the natural projection

$$\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{\tilde{X}}) \longrightarrow C(\tilde{X})$$

is a chain equivalence of n-dimensional $\mathbb{Z}[\pi]$-module chain complexes, with $\mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{Z}[\pi]$ the group ring morphism defined by the characteristic map $\pi_1(X) \longrightarrow \pi$. The covering $\tilde{X}$ of X is <u>oriented with data</u> (π,w) if π is equipped with a group morphism $w : \pi \longrightarrow \mathbb{Z}_2$ such that the orientation map $w(X)$ factors as

$$w(X) : \pi_1(X) \longrightarrow \pi \xrightarrow{\ w\ } \mathbb{Z}_2 \ .$$

In particular, the universal cover $\tilde{\tilde{X}}$ is oriented with data $(\pi_1(X),w(X))$. If $\tilde{X}$ is oriented with data (π,w) applying $\mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} -$ to the above $\mathbb{Z}[\pi]$-module chain equivalence we obtain a $\mathbb{Z}$-module chain equivalence

$$\mathbb{Z}^{t(X)} \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{\tilde{X}}) = \mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} (\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{\tilde{X}})) \longrightarrow \mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} C(\tilde{X})$$

where $\begin{cases} t(X) \\ t \end{cases}$ refers to the $\begin{cases} w(X) \\ w \end{cases}$-twisted involution on $\begin{cases} \mathbb{Z}[\pi_1(X)] \\ \mathbb{Z}[\pi] \end{cases}$, so that there is a fundamental class $[X] \in H_n^{\pi}(\tilde{X};{}^{w}\mathbb{Z}) = H_n(\mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} C(\tilde{X}))$ for $\tilde{X}$. Applying $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1(X)]} -$ to the $\mathbb{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap - : {}^{w(X)}C(\tilde{\tilde{X}})^{n-*} \longrightarrow C(\tilde{\tilde{X}})$$

we obtain a $\mathbb{Z}[\pi]$-module chain equivalence

$$[X] \cap - : {}^{w}C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X}) \ .$$

Thus a geometric Poincaré complex satisfies Poincaré duality with respect to any oriented cover $\tilde{X}$.

The symmetric construction of Proposition 2.2 associates a symmetric Poincaré complex to every oriented covering of a geometric Poincaré complex. As noted in the Introduction this construction is due to Mishchenko [2].

<u>Proposition 2.7</u> Given an n-dimensional geometric Poincaré complex X and an oriented cover $\tilde{X}$ with data (π,w) there is defined in a natural way an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$\sigma^*(\tilde{X}) = (C(\tilde{X}), \varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X}))) .$$

If $\tilde{\tilde{X}}$ is the universal cover of X then

$$\sigma^*(\tilde{X}) = \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi_1(X)]}\sigma^*(\tilde{\tilde{X}})$$

up to homotopy equivalence.

<u>Proof</u>: Evaluating $\varphi_{\tilde{X}}: H_n^{\pi}(\tilde{X};{}^w\mathbb{Z}) \longrightarrow Q^n(C(\tilde{X}))$ on the fundamental class $[X]\in H_n^{\pi}(\tilde{X};{}^w\mathbb{Z})$ there is obtained a $\mathbb{Z}_2$-hypercohomology class $\varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X}))$ such that slant product with $\varphi_{\tilde{X}}[X]_0 \in H_n(C(\tilde{X})^t\otimes_{\mathbb{Z}[\pi]}C(\tilde{X}))$ defines the Poincaré duality $\mathbb{Z}[\pi]$-module isomorphisms

$$\varphi_{\tilde{X}}[X]_0\backslash - = [X]\cap - \; : \; {}^wH^*_{\pi}(\tilde{X}) \longrightarrow H_{n-*}(\tilde{X})$$

(cf. Proposition 2.2 i). Also, there is defined a commutative diagram

$$\begin{array}{ccc} H_n^{\pi_1(X)}(\tilde{\tilde{X}};{}^{w(X)}\mathbb{Z}) & \xrightarrow{\varphi_{\tilde{\tilde{X}}}} & Q^n(C(\tilde{\tilde{X}})) \\ \downarrow & & \downarrow \\ H_n^{\pi}(\tilde{X};{}^w\mathbb{Z}) & \xrightarrow{\varphi_{\tilde{X}}} & Q^n(C(\tilde{X})) \end{array}$$

in which the vertical maps are the change of rings $\mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{Z}[\pi]$.

[]

We shall normally write $\sigma^*(\tilde{X})$ as $\sigma^*(X)$.

A <u>map</u> of geometric Poincaré complexes (not necessarily of the same dimension)

$$f : M \longrightarrow X$$

is a map of the underlying spaces which preserves the orientation maps, that is such that $w(M)$ factors as

$$w(M) : \pi_1(M) \xrightarrow{f} \pi_1(X) \xrightarrow{w(X)} \mathbb{Z}_2 .$$

If $\tilde{X}$ is an oriented cover of X with data (π,w) then the pullback $\tilde{M}$ is an oriented cover of M with data (π,w).

Let $f:M \longrightarrow X$ be a map of geometric Poincaré complexes of dimensions dim M = m, dim X = n and let $\tilde{X}$ be a (not necessarily oriented) cover of X with group of covering translations π and induced cover $\tilde{M}$ of M. Define the <u>Umkehr</u> $\mathbb{Z}[\pi]$-module chain map

$$f^! : C(\tilde{X}) \longrightarrow S^{n-m}C(\tilde{M})$$

(up to non-canonical $\mathbb{Z}[\pi]$-module chain homotopy) by applying $\mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi_1(X)]}-$ to the composite $\mathbb{Z}[\pi_1(X)]$-module chain map

$$f^! : C(\tilde{\tilde{X}}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\tilde{\tilde{X}})^{n-*} \xrightarrow{\tilde{\tilde{f}}^*} {}^{w(X)}C(\tilde{\tilde{M}})^{n-*} \xrightarrow{[M]\cap -} S^{n-m}C(\tilde{\tilde{M}})$$

with $\tilde{\tilde{X}}$ the universal cover of X and $\tilde{\tilde{M}}$ the induced oriented cover of M. If $\tilde{X}$ is an oriented cover of X with data (π,w) then the Umkehr factors as

$$f^! : C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} {}^wC(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} {}^wC(\tilde{M})^{n-*} \xrightarrow{[M]\cap -} S^{n-m}C(\tilde{M}) .$$

A map of n-dimensional geometric Poincaré complexes $f:M \longrightarrow X$ is of <u>degree 1</u> if it preserves the fundamental classes, that is if

$$f_*[M] = [X] \in H_n^{\pi}(\tilde{X};{}^w\mathbb{Z})$$

for any oriented cover $\tilde{X}$ of X with data (π,w). The induced chain map $\tilde{f}:C(\tilde{M}) \longrightarrow C(\tilde{X})$ defines a map of n-dimensional symmetric Poincaré complexes over $\mathbb{Z}[\pi]$

$$f : \sigma^*(M) \longrightarrow \sigma^*(X)$$

which is a homotopy equivalence if $f:M \longrightarrow X$ is a homotopy equivalence of spaces. Conversely, if $f:M \longrightarrow X$ is a degree 1 map inducing an isomorphism $f:\pi_1(M) \longrightarrow \pi_1(X)$ and a homotopy equivalence $\tilde{f}:\sigma^*(\tilde{M}) \longrightarrow \sigma^*(\tilde{X})$ with $\tilde{M},\tilde{X}$ the universal covers then $f:M \longrightarrow X$ is a homotopy equivalence, by Whitehead's theorem.

<u>Proposition 2.8</u> Let $f:M \longrightarrow X$ be a degree 1 map of n-dimensional geometric Poincaré complexes. Let $\tilde{X}$ be a cover of X with group of covering translations π and induced cover $\tilde{M}$ of M. Then the Umkehr $\mathbb{Z}[\pi]$-module chain map

$$f^! : C(\tilde{X}) \longrightarrow C(\tilde{M})$$

is a chain homotopy right inverse for $\tilde{f}:C(\tilde{M}) \longrightarrow C(\tilde{X})$, that is $\tilde{f}f^! \simeq 1 : C(\tilde{X}) \longrightarrow C(\tilde{X})$. The inclusion in the algebraic mapping cone $e:C(\tilde{M}) \longrightarrow C(f^!)$ is such that

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix} : C(\tilde{M}) \longrightarrow C(f^!)\oplus C(\tilde{X})$$

defines a chain equivalence of n-dimensional $\mathbb{Z}[\pi]$-module chain complexes.

If $\widetilde{X}$ is an oriented cover of X with data (π,w) the symmetric kernel of f

$$\sigma^*(f) = (C(f^!), e^{\%}(\varphi_{\widetilde{M}}[M]) \in Q^n(C(f^!)))$$

is an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ with the w-twisted involution, and there is defined a homotopy equivalence of such complexes

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix} : \sigma^*(M) \longrightarrow \sigma^*(f) \oplus \sigma^*(X) \quad .$$

Proof: To obtain $\tilde{f}f^! \simeq 1$ apply $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1(X)]} -$ to the $\mathbb{Z}[\pi_1(X)]$-module chain homotopy commutative diagram

$$\begin{array}{ccc} {}^{w(X)}C(\widetilde{\widetilde{X}})^{n-*} & \xrightarrow{\ \tilde{\tilde{f}}^*\ } & {}^{w(X)}C(\widetilde{\widetilde{M}})^{n-*} \\ {\scriptstyle [X]\cap -}\big\downarrow & & \big\downarrow{\scriptstyle [M]\cap -} \\ C(\widetilde{\widetilde{X}}) & \xleftarrow{\ \tilde{\tilde{f}}\ } & C(\widetilde{\widetilde{M}}) \end{array}$$

with $\widetilde{\widetilde{X}}$ the universal cover of X and $\widetilde{\widetilde{M}}$ the induced cover of M. To show that

$$\begin{pmatrix} e \\ \tilde{f} \end{pmatrix}^{\%} : Q^n(C(\widetilde{M})) \longrightarrow Q^n(C(f^!) \oplus C(\widetilde{X})) = Q^n(C(f^!)) \oplus Q^n(C(\widetilde{X})) \oplus H_n(C(f^!)^t \otimes_{\mathbb{Z}[\pi]} C(\widetilde{X}))$$

sends $\varphi_{\widetilde{M}}[M]$ to $e^{\%}(\varphi_{\widetilde{M}}[M]) \oplus \varphi_{\widetilde{X}}[X] \oplus 0$ (using the decomposition of Proposition 1.4 i)) consider the chain homotopy commutative diagram

$$\begin{array}{ccccc} {}^{w}C(f^!)^{n-*} & \xrightarrow{\ e^*\ } & {}^{w}C(\widetilde{M})^{n-*} & \xrightarrow{\ f^{!*}\ } & {}^{w}C(\widetilde{X})^{n-*} \\ & & {\scriptstyle [M]\cap -}\big\downarrow & & \big\downarrow{\scriptstyle [X]\cap -} \\ & & C(\widetilde{M}) & \xrightarrow{\ \tilde{f}\ } & C(\widetilde{X}) \end{array} ,$$

which gives $\tilde{f}([M]\cap -)e^* \simeq 0 : {}^{w}C(f^!)^{n-*} \longrightarrow C(\widetilde{X})$, and so

$$(e^t \otimes_{\mathbb{Z}[\pi]} \tilde{f})\varphi_{\widetilde{M}}[M]_0 = 0 \in H_n(C(f^!)^t \otimes_{\mathbb{Z}[\pi]} C(\widetilde{X})) \quad .$$

[]

Define the $\left\{\begin{array}{l}\text{homology} \\ \pi\text{-cohomology}\end{array}\right.$ kernel $\mathbb{Z}[\pi]$-modules of a degree 1 map of n-dimensional geometric Poincaré complexes $f:M \longrightarrow X$ with respect to a covering $\widetilde{X}$ of X with group of covering translations π

$$\left\{\begin{array}{l} K_*(M) = H_*(C(f^!)) \\ K^*(M) = H^*(C(f^!)) \end{array}\right.$$

using the untwisted involution on $\mathbb{Z}[\pi]$ to define the dual $\mathbb{Z}[\pi]$-module structure on $C(f^!)^*$. Proposition 2.8 gives natural direct sum decompositions

$$\left\{\begin{array}{l} H_*(M) = K_*(M) \oplus H_*(X) \\ H^*_{\pi}(\widetilde{M}) = K^*(M) \oplus H^*_{\pi}(\widetilde{X}) \end{array}\right. \quad .$$

If $\widetilde{X}$ is oriented with data (π,w) the symmetric kernel $\sigma^*(f)$ gives Poincaré duality in the kernel modules

$${}^{w}K^*(M) = K_{n-*}(M) \quad .$$

A geometric Umkehr map for f is a π-map

$$F : \Sigma^p \widetilde{X}_+ \longrightarrow \Sigma^p \widetilde{M}_+ \qquad (p \geqslant 0)$$

inducing the Umkehr $f^!$ on chain level, that is such that there exists a $\mathbb{Z}[\pi]$-module chain homotopy

$$(\Sigma^p_M)^{-1} F (\Sigma^p_X) \simeq f^! : C(\widetilde{X}) \longrightarrow C(\widetilde{M}) \quad .$$

Proposition 2.9 Given a degree 1 map of n-dimensional geometric Poincaré complexes $f:M \longrightarrow X$ and a geometric Umkehr map $F:\Sigma^p\widetilde{X}_+ \longrightarrow \Sigma^p\widetilde{M}_+$ with respect to an oriented cover $\widetilde{X}$ of X with data (π,w) there is defined in a natural way an n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi]$ with the w-twisted involution, the quadratic kernel of (f,F)

$$\sigma_*(f,F) = (C(f^!), e_{\%}\psi_F[X] \in Q_n(C(f^!)))$$

depending only on the stable π-homotopy class of F, such that

$$(1+T)\sigma_*(f,F) = \sigma^*(f) \quad .$$

Proof: The absolute version of the quadratic construction of Proposition 2.5

$$\psi_F : H^{\pi}_n(\widetilde{X};{}^{w}\mathbb{Z}) \longrightarrow Q_n^{[0,p-1]}(C(\widetilde{M}))$$

is such that

$$\varphi_{\widetilde{M}} f^!_* - f^{!\%}\varphi_{\widetilde{X}} = (1+T)\psi_F : H^{\pi}_n(\widetilde{X};{}^{w}\mathbb{Z}) \longrightarrow Q^n(C(\widetilde{M})) \quad .$$

Let $e:C(\widetilde{M}) \longrightarrow C(f^!)$ be the inclusion, so that

$$\begin{aligned} (1+T)e_{\%}\psi_F[X] &= e^{\%}(1+T)\psi_F[X] \\ &= e^{\%}\varphi_{\widetilde{M}} f^!_*[X] - e^{\%} f^{!\%}\varphi_{\widetilde{X}}[X] = e^{\%}\varphi_{\widetilde{M}}[M] \in Q^n(C(\widetilde{M})) \quad . \end{aligned}$$

Here, as elsewhere, we let $e_{\%}\psi_F[X]$ stand both for an element of $Q_n^{[0,p-1]}(C(f^!))$ and for its image in $Q_n(C(f^!))$.

[]

The quadratic kernels associated to the various oriented covers $\widetilde{X}$ of X are all induced via $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1(X)]} -$ from the quadratic kernel $\sigma_*(f,F)$ associated to the universal cover of X.

In §8 below we shall show how to obtain the surgery obstruction of a normal map $(f,b):M\longrightarrow X$ from the quadratic kernel $\sigma_*(f,F)$, using the given normal bundle map $b:\nu_M\longrightarrow\nu_X$ and the equivariant S-duality of §3 to produce a geometric Umkehr map $F:\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+$ for the universal cover $\widetilde{X}$. In favourable circumstances it is possible to obtain F directly from (f,b) without the S-duality machinery. For example, if $f:M\longrightarrow X$ is a degree 1 map of manifolds which is covered by a map $b:\nu_M\longrightarrow\nu_X$ of stable normal bundles then f can be approximated by a framed embedding $M\times D^p\subset \text{interior}(X\times D^p)$ which lifts to an embedding of covers $\widetilde{M}\times D^p\subset\widetilde{X}\times D^p$ for any cover $\widetilde{X}$ of X, giving F by the Pontrjagin-Thom construction

$$F:\Sigma^p\widetilde{X}_+=\widetilde{X}\times D^p/\widetilde{X}\times S^{p-1}\xrightarrow{\text{collapse}}\widetilde{X}\times D^p/\overline{\widetilde{X}\times D^p-\widetilde{M}\times D^p}=\widetilde{M}\times D^p/\widetilde{M}\times S^{p-1}=\Sigma^p\widetilde{M}_+ .$$

The case $p=1$ is of interest in codimension 2 surgery, see §17.

The mod 2 reduction of the quadratic kernel construction gives the $\mathbb{Z}_2$-valued quadratic form used in §4 of Chapter III of Browder [2] to define the Arf invariant. We recall that a $\underline{\mathbb{Z}_2\text{-Poincaré complex}}$ is a connected space X together with a mod 2 fundamental class $[X]\in H_n(X;\mathbb{Z}_2)$ defining mod 2 Poincaré duality isomorphisms $[X]\cap - : H^*(X;\mathbb{Z}_2)\longrightarrow H_{n-*}(X;\mathbb{Z}_2)$.

Proposition 2.10 i) Given an n-dimensional geometric $\mathbb{Z}_2$-Poincaré complex X there is defined in a natural way an n-dimensional symmetric Poincaré complex over $\mathbb{Z}_2$

$$\sigma^*(X) = (C(X;\mathbb{Z}_2),\varphi_X[X]\in Q^n(C(X;\mathbb{Z}_2)))$$

such that the symmetric Wu classes of $\sigma^*(X)$ are just the Wu classes of X

$$v_r(\varphi_X[X]) = v_r(X)\in \mathrm{Hom}_{\mathbb{Z}_2}(H^{n-r}(X;\mathbb{Z}_2),\mathbb{Z}_2) = H^r(X;\mathbb{Z}_2)$$

as characterized by

$$Sq^r(y) = \langle v_r(X)\cup y,[X]\rangle\in\mathbb{Z}_2 \qquad (y\in H^{n-r}(X;\mathbb{Z}_2)) .$$

ii) Given a degree 1(mod 2) map $f:M\longrightarrow X$ of n-dimensional geometric $\mathbb{Z}_2$-Poincaré complexes and a $\{1\}$-map $F:\Sigma^pX_+\longrightarrow\Sigma^pM_+$ inducing the mod 2 Umkehr $f^!:C(X;\mathbb{Z}_2)\longrightarrow C(M;\mathbb{Z}_2)$ there is defined in a natural way an n-dimensional quadratic Poincaré complex over $\mathbb{Z}_2$

$$\sigma_*(f,F) = (C(f^!),e_\%\psi_F[X]\in Q_n(C(f^!)))$$

such that

$$\sigma^*(M) = (1+T)\sigma_*(f,F)\oplus\sigma^*(X)$$

up to homotopy equivalence. The quadratic Wu classes of $\sigma_*(f,F)$

$$v^r = v^r(e_\%\psi_F[X]) : K^{n-r}(M;\mathbb{Z}_2)\longrightarrow\begin{cases}\mathbb{Z}_2 & \text{if } n\leqslant 2r\leqslant n+p-1\\ 0 & \text{otherwise}\end{cases}$$

are such that

$$v^r(y_1+y_2) - v^r(y_1) - v^r(y_2) = \langle y_1\cup y_2,[M]\rangle \quad (= 0 \text{ if } n\neq 2r) ,$$

and can be expressed in terms of functional Steenrod squares by

$$v^r(y) = \langle Sq^{r+1}_{(\Sigma^p y)F}(\Sigma^p(\iota)),\Sigma^p_X[X]\rangle\in\mathbb{Z}_2 \quad (n\leqslant 2r)$$

$$(y\in K^{n-r}(M;\mathbb{Z}_2)\subseteq H^{n-r}(M;\mathbb{Z}_2)=[M_+,K(\mathbb{Z}_2,n-r)]\ ,\ \iota\neq 0\in H^{n-r}(K(\mathbb{Z}_2,n-r);\mathbb{Z}_2)=\mathbb{Z}_2)$$

Proof: Apply Propositions 2.3, 2.6.

[]

(Similarly, the generalized quadratic construction defined following Proposition 2.6 can be used to obtain the quadratic operation

$$\psi : \ker(F^*:H^q(M;\mathbb{Z}_2) = H^{-q}(\Sigma^{-2q}\underline{M}_+;\mathbb{Z}_2)\rightarrow H^{-q}(\underline{X};\mathbb{Z}_2))\longrightarrow\mathbb{Z}_2$$

defined in §1 of Browder [1] for any $\underline{X}$-orientation $F:\underline{X}\longrightarrow\Sigma^{-2q}\underline{M}_+$ of a 2q-dimensional geometric $\mathbb{Z}_2$- Poincaré complex M, with $\underline{X}$ any Wu(q+1)-cospectrum).

The kernel constructions behave as follows under composition.

Proposition 2.11 The composite of $\begin{cases}\text{degree } 1\\ \text{geometric Umkehr}\end{cases}$ maps

$\begin{cases}f:X\longrightarrow Y,g:Y\longrightarrow Z\\ F:\Sigma^p\widetilde{Y}_+\longrightarrow\Sigma^p\widetilde{X}_+,G:\Sigma^p\widetilde{Z}_+\longrightarrow\Sigma^p\widetilde{Y}_+\end{cases}$ of n-dimensional geometric Poincaré complexes

is another such map $\begin{cases}gf:X\longrightarrow Z\\ FG:\Sigma^p\widetilde{Z}_+\longrightarrow\Sigma^p\widetilde{X}_+\end{cases}$, with $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel

$$\begin{cases}\sigma^*(gf) = \sigma^*(f)\oplus\sigma^*(g)\\ \sigma^*(gf,FG) = \sigma_*(f,F)\oplus\sigma_*(g,(\Sigma^p\widetilde{f}_+)FG)\end{cases}$$

up to homotopy equivalence, taking the oriented covers $\widetilde{X},\widetilde{Y}$ of X,Y induced from an oriented cover $\widetilde{Z}$ of Z.

Proof: Write the inclusions in the algebraic mapping cones as

$$e_f : C(\tilde X) \longrightarrow C(f^!)$$
$$e_g : C(\tilde Y) \longrightarrow C(g^!)$$
$$e_{gf} : C(\tilde Z) \longrightarrow C((gf)^!)\ ,$$

and note that $(gf)^! = f^! g^! : C(\tilde Z) \longrightarrow C(\tilde Y) \longrightarrow C(\tilde X)$. The stable composite of the chain equivalences

$$\begin{pmatrix} e_f \\ \tilde f \end{pmatrix} : C(\tilde X) \longrightarrow C(f^!)\oplus C(\tilde Y)$$
$$\begin{pmatrix} e_g \\ \tilde g \end{pmatrix} : C(\tilde Y) \longrightarrow C(g^!)\oplus C(\tilde Z)$$

is a chain equivalence

$$\begin{pmatrix} e_f \\ e_g\tilde f \\ \tilde g\tilde f \end{pmatrix} : C(\tilde X) \longrightarrow C(f^!)\oplus C(g^!)\oplus C(\tilde Z)$$

allowing us to identify

$$e_{gf} = \begin{pmatrix} e_f \\ e_g\tilde f \end{pmatrix} : C(\tilde X) \longrightarrow C(f^! g^!) = C(f^!)\oplus C(g^!)\ .$$

Using the direct sum decomposition of Proposition 1.4 i) we have

$$e^{\%}_{gf}\varphi_{\tilde X}[X] = (e^{\%}_f\varphi_{\tilde X}[X], e^{\%}_g\tilde f^{\%}\varphi_{\tilde X}[X], (e_f^t\otimes e_g\tilde f)\varphi_{\tilde X}[X]_0)$$
$$\in Q^n(C(f^!)\oplus C(g^!)) = Q^n(C(f^!))\oplus Q^n(C(g^!))\oplus H_n(C(f^!)^t\otimes_{\mathbb Z[\pi]}C(g^!))\ .$$

Now $\tilde f^{\%}\varphi_{\tilde X}[X] = \varphi_{\tilde Y}f_*[X] = \varphi_{\tilde Y}[Y]\in Q^n(C(\tilde Y))$, and

$$(e_f^t\otimes e_g\tilde f)(\varphi_{\tilde X}[X])_0 = 0 \in H_n(C(f^!)^t\otimes_{\mathbb Z[\pi]}C(g^!))$$

since there is defined a $\mathbb Z[\pi]$-module chain homotopy commutative diagram

$$\begin{array}{ccccc} C(f^!)^{n-*} & \xrightarrow{e_f^*} & C(\tilde X)^{n-*} & \xrightarrow{f^{!*}} & C(\tilde Y)^{n-*} & & \\ & & \downarrow{\scriptstyle \varphi_{\tilde X}[X]_0 = [X]\cap -} & & \downarrow{\scriptstyle [Y]\cap -} & & \\ & & C(\tilde X) & \xrightarrow{\tilde f} & C(\tilde Y) & \xrightarrow{e_g} & C(g^!) \end{array}$$

with $f^{!*}e_f^* \simeq 0$. Thus

$$e^{\%}_{gf}\varphi_{\tilde X}[X] = e^{\%}_f\varphi_{\tilde X}[X]\oplus e^{\%}_g\varphi_{\tilde Y}[Y]\in Q^n(C((gf)^!)) = Q^n(C(f^!)\oplus C(g^!))\ ,$$

and so

$$\sigma^*(gf) = \sigma^*(f)\oplus\sigma^*(g)\ .$$

(The formula $\sigma^*(X) = \sigma^*(f)\oplus\sigma^*(Y)$ is the special case $Z = \emptyset$).

$$\sigma_*(gf,FG) = (C((gf)^!), e_{gf\%}\psi_{FG}[Z]\in Q_n(C((gf)^!)))$$

with $\psi_{FG} = \psi_F g^!_* + f^!_{\%}\psi_G : H^{\pi}_n(\tilde Z;{}^w\mathbb Z) \longrightarrow Q_n(C(\tilde X))$ by the sum formula of Proposition 2.5 . Working as above we have

$$\begin{aligned} e_{gf\%}\psi_{FG}[Z] &= (e_{f\%}\psi_{FG}[Z], e_{g\%}\tilde f_{\%}\psi_{FG}[Z], (e_f^t\otimes e_g\tilde f)((1+T)\psi_{FG}[Z])_0) \\ &= (e_{f\%}\psi_F[Y], e_{g\%}(\tilde f_{\%}\psi_F g^!_* + \psi_G)[Z], 0) \\ &= e_{f\%}\psi_F[Y]\oplus e_{g\%}\psi_{(\Sigma^p\tilde f_+)FG}[Z]\in Q_n(C((gf)^!)) = Q_n(C(f^!)\oplus C(g^!)) \end{aligned}$$

so that

$$\sigma_*(gf,FG) = \sigma_*(f,F)\oplus\sigma_*(g,(\Sigma^p\tilde f_+)FG)\ .$$

[]

A degree 1 map of n-dimensional geometric Poincaré complexes $f:M\longrightarrow X$ is <u>k-connected</u> with respect to some covering $\tilde X$ of X if $K_r(M) = 0$ for $r\leqslant k$. Recalling the definition of skew-suspension in §1 we have:

<u>Proposition 2.12</u> The $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ kernel $\left\{\begin{array}{l}\sigma^*(f)\\ \sigma_*(f,F)\end{array}\right.$ of an (r-1)-connected degree 1 map of n-dimensional geometric Poincaré complexes $f:M\longrightarrow X$ with respect to an oriented covering $\tilde X$ of X with data (π,w) is the r-fold skew-suspension of an (n-2r)-dimensional $\left\{\begin{array}{l}(-)^r\text{-symmetric}\\ (-)^r\text{-quadratic}\end{array}\right.$ Poincaré complex over $\mathbb Z[\pi]$ $\left\{\begin{array}{l}\sigma^r(f)\\ \sigma_r(f,F)\end{array}\right.$, with

$$\left\{\begin{array}{l}\bar S\sigma^r(f) = \sigma^{r-1}(f)\ ,\quad \sigma^0(f) = \sigma^*(f)\\ \bar S\sigma_r(f,F) = \sigma_{r-1}(f,F)\ ,\quad \sigma_0(f,F) = \sigma_*(f,F)\ ,\quad (1+T_{(-)^r})\sigma_r(f,F) = \sigma^r(f).\end{array}\right.$$

[]

In §4 below we shall identify the quadratic kernel $\sigma_i(f,F)$ associated to an (i-1)-connected $\left\{\begin{array}{l}2i\\ 2i+1\end{array}\right.$-dimensional normal map $(f,b):M\longrightarrow X$ with the surgery obstruction kernel obtained in $\left\{\begin{array}{l}\S5\\ \S6\end{array}\right.$ of Wall [5], using the one-one correspondence between $\left\{\begin{array}{l}0\text{-}\\ 1\text{-}\end{array}\right.$dimensional $(-)^i$quadratic Poincaré complexes and non-singular $(-)^i$quadratic $\left\{\begin{array}{l}\text{forms}\\ \text{formations}\end{array}\right.$ of Proposition $\left\{\begin{array}{l}1.5\\ 1.8\end{array}\right.$.

§3. Equivariant S-duality

The S-duality between M_+ and the Thom space $T(\nu_M)$ of the normal bundle ν_M of an embedding $M^n \subset S^{n+p}$ (p large) of a compact manifold M was first established by Milnor and Spanier [1]. This was then generalized by Atiyah [1], and extended to geometric Poincaré complexes by Spivak [1] and Wall [4]. In particular, if $f:M \longrightarrow X$ is a degree 1 map of geometric Poincaré complexes which is covered by a map of Spivak normal fibrations $b:\nu_M \longrightarrow \nu_X$ then the S-dual of $T(b):T(\nu_M) \longrightarrow T(\nu_X)$ is a geometric Umkehr map $F:\Sigma^p X_+ \longrightarrow \Sigma^p M_+$, and this was used by Browder [2] to obtain the surgery obstruction in the simply-connected case $\pi_1(X) = \{1\}$. We shall now develop an equivariant S-duality theory for finite-dimensional π-spaces with a π-equivariant cell structure ("CWπ-complexes") in order to obtain a geometric Umkehr π-map $F:\Sigma^p\widetilde{X}_+ \longrightarrow \Sigma^p\widetilde{M}_+$ for any covering $\widetilde{X}$ of X with group of covering translations π, giving the non-simply-connected surgery obstruction by means of the quadratic construction ψ_F.

Given π-spaces X,Y let $[X,Y]_\pi$ be the pointed set of π-homotopy classes of π-maps $f:X \longrightarrow Y$. Regarding the loop space $\Omega X = (X,pt.)^{(S^1,pt.)}$ as a π-space using the trivial π-action on S^1 we have that $[\Sigma^p X,Y]_\pi = [X,\Omega^p Y]_\pi$ is a group for $p \geqslant 1$, abelian for $p \geqslant 2$. Define the abelian group of <u>Sπ-maps</u> between π-spaces X,Y to be the direct limit

$$\{X,Y\}_\pi = \varinjlim_p [\Sigma^p X,\Sigma^p Y]_\pi$$

of the suspension sequence

$$[X,Y]_\pi \xrightarrow{\Sigma} [\Sigma X,\Sigma Y]_\pi \xrightarrow{\Sigma} [\Sigma^2 X,\Sigma^2 Y]_\pi \xrightarrow{\Sigma} [\Sigma^3 X,\Sigma^3 Y]_\pi \longrightarrow \dots \quad .$$

For $\pi = \{1\}$ we write $[X,Y]_{\{1\}} = [X,Y]$, $\{X,Y\}_{\{1\}} = \{X,Y\}$ as usual.

The mapping cone of a π-map $f:X \longrightarrow Y$ is a π-space

$$C_f = Y \cup_f X \wedge I \ .$$

The cofibration sequence of π-spaces and π-maps

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow C_{\Sigma f} \longrightarrow \dots$$

induces the following π-equivariant analogue of the Puppe exact sequence.

<u>Proposition 3.1</u> For any π-map $f:X \longrightarrow Y$ and π-space Z there is defined in a natural way an exact sequence

$$[X,Z]_\pi \xleftarrow{f} [Y,Z]_\pi \longleftarrow [C_f,Z]_\pi \longleftarrow [\Sigma X,Z]_\pi \xleftarrow{\Sigma f} [\Sigma Y,Z]_\pi \longleftarrow \dots \ .$$

[]

For any $\{1\}$-space K regard $\bigvee_\pi K$ as a π-space by permutation of the summands. Note that for any π-space X

$$[\bigvee_\pi K,X]_\pi = [K,X] \quad , \quad \{\bigvee_\pi K,X\}_\pi = \{K,X\} \ .$$

Define the π-space obtained from a π-space X by <u>attaching an r-dimensional π-cell</u> to be the $\begin{cases}\text{disjoint union}\\ \text{identification space}\end{cases}$

$$X' = \begin{cases} X \cup \pi \times D^0 & \\ X \cup_{\pi \times f} \pi \times D^r \ , \text{ for some map } f:S^{r-1} \longrightarrow X \end{cases} \text{ if } \begin{cases} r = 0 \\ r \geqslant 1 \end{cases} .$$

The π-cell is <u>pointed</u> if the attaching map $f:S^{r-1} \longrightarrow X$ $(r \geqslant 1)$ preserves basepoints, in which case it extends to a π-map $f':\bigvee_\pi S^{r-1} \longrightarrow X$ such that

$$X' = C_{f'} = X \cup_{f'} \bigvee_\pi D^r \ .$$

If X is a path-connected π-space then any map $f:S^{r-1} \longrightarrow X$ $(r \geqslant 1)$ is homotopic to a basepoint-preserving map $f_0:S^{r-1} \longrightarrow X$ extending to a π-map $f_0':\bigvee_\pi S^{r-1} \longrightarrow X$, and $X' = X \cup_{\pi \times f} \pi \times D^r$ is π-homotopic to the mapping cone π-space $C_{f_0'} = X \cup_{f_0'} \bigvee_\pi D^r$.

A <u>CWπ-complex</u> X is a π-space which is a based CW complex obtained from the base 0-cell by successively attaching π-cells of non-decreasing dimension. The suspension of a CWπ-complex X is a CWπ-complex ΣX, with one r-dimensional π-cell for each (r-1)-dimensional π-cell of X $(r \geqslant 0)$.

CWπ-complexes arise as follows.

<u>Proposition 3.2</u> If $(\widetilde{X},\widetilde{Y})$ is a covering of a CW pair (X,Y) with group of covering translations π then $\widetilde{X}/\widetilde{Y}$ is a CWπ-complex with one π-cell for each cell of X - Y. If $Y = \emptyset$ interpret the quotient as $\widetilde{X}/\emptyset = \widetilde{X}_+$.

[]

A CWπ-complex is <u>pointed</u> if it involves only pointed π-cells. A CWπ-complex with no 0-dimensional π-cells (e.g. a suspension) is π-homotopic to a pointed CWπ-complex.

A CWπ-complex X is <u>finite</u> if it involves only a finite number of π-cells. A CWπ-complex X is <u>finitely-dominated</u> if there exists a finite CWπ-complex K and π-maps $f:X\longrightarrow K$, $g:K\longrightarrow X$ such that $gf = 1\in [X,X]_\pi$, and it is <u>n-dimensional</u> if $\dot{H}_r(X) = 0$ for $r > n$, in which case $\dot{C}(X)$ is an n-dimensional $\mathbb{Z}[\pi]$-module chain complex.

We have the following analogue of the Freudenthal suspension theorem.

<u>Proposition 3.3</u> Let X be an n-dimensional finite pointed CWπ-complex, and let Y be a π-space of the homotopy type of a CW complex. Then the suspension map

$$\Sigma : [\Sigma^p X, \Sigma^p Y]_\pi \longrightarrow [\Sigma^{p+1}X, \Sigma^{p+1}Y]_\pi$$

is an isomorphism for $p \geqslant n+1$, and

$$\{X,Y\}_\pi = [\Sigma^{n+1}X, \Sigma^{n+1}Y]_\pi \ .$$

<u>Proof</u>: By induction on the number of pointed π-cells in X. Trivial for X = pt.. Assume true for X, and let $X' = X\cup_f \bigvee_\pi D^n$ for some π-map $f:\bigvee_\pi S^{n-1}\longrightarrow X$ ($n\geqslant 1$). There is defined a commutative diagram of abelian groups and morphisms

$$\begin{array}{ccccccccc}
[\Sigma^{p+1}X,\Sigma^pY]_\pi & \to & [\bigvee_\pi S^{n+p},\Sigma^pY]_\pi & \to & [\Sigma^pX',\Sigma^pY]_\pi & \to & [\Sigma^pX,\Sigma^pY]_\pi & \to & [\bigvee_\pi S^{n+p-1},\Sigma^pY]_\pi \\
\Sigma\downarrow & & \Sigma\downarrow & & \Sigma\downarrow & & \Sigma\downarrow & & \downarrow\Sigma \\
[\Sigma^{p+2}X,\Sigma^{p+1}Y]_\pi & \to & [\bigvee_\pi S^{n+p+1},\Sigma^{p+1}Y]_\pi & \to & [\Sigma^{p+1}X',\Sigma^{p+1}Y]_\pi & \to & [\Sigma^{p+1}X,\Sigma^{p+1}Y]_\pi & \to & [\bigvee_\pi S^{n+p},\Sigma^{p+1}Y]_\pi
\end{array}$$

in which the rows are exact (Proposition 3.1). The suspension maps involving X are isomorphisms for $p\geqslant n+1$ by the inductive hypothesis. Since Y is of the homotopy type of a CW complex Σ^pY is $(p-1)$-connected and

$$\Sigma : [\bigvee_\pi S^{n+p},\Sigma^pY]_\pi = [S^{n+p},\Sigma^pY] \longrightarrow [S^{n+p+1},\Sigma^{p+1}Y]$$

is an isomorphism for $p\geqslant n+1$ by the ordinary Freudenthal suspension theorem. Applying the 5-lemma gives the induction step.

[]

Given π-spaces X,Y define the $\{1\}$-space

$$X\wedge_\pi Y = (X\wedge Y)/\pi$$

to be the space of orbits of the diagonal π-action

$$\pi\times X\wedge Y \longrightarrow X\wedge Y \ ; \ (g,x\wedge y)\longmapsto gx\wedge gy \ .$$

Note that for any π-space X and $\{1\}$-space K

$$X\wedge_\pi(\bigvee_\pi K) = X\wedge K \ .$$

A <u>π-spectrum $\underline{Z}$</u> is a sequence of π-spaces Z_p ($p\geqslant 0$) and π-maps $\xi_p:\Sigma Z_p\longrightarrow Z_{p+1}$ ($p\geqslant 0$). Given a π-space X define the abelian group

$$\{X,\underline{Z}\}_\pi = \varinjlim_p [\Sigma^pX, Z_p]_\pi$$

to be the direct limit of the sequence

$$[X,Z_0]_\pi \xrightarrow{\Sigma} [\Sigma X,\Sigma Z_0]_\pi \xrightarrow{\xi_0} [\Sigma X,Z_1]_\pi \xrightarrow{\Sigma} [\Sigma^2X,\Sigma Z_1]_\pi \xrightarrow{\xi_1} \dots \ .$$

In particular, for $\xi_p = \mathrm{id.} : \Sigma Z_p = \Sigma^{p+1}Z_0 \longrightarrow Z_{p+1} = \Sigma^{p+1}Z_0$ we have

$$\{X,\underline{Z}\}_\pi = \{X,Z_0\}_\pi .$$

Given a π-space X and a π-spectrum $\underline{Z}$ let $X\wedge_\pi\underline{Z}$ be the $\{1\}$-spectrum defined by

$$(X\wedge_\pi\underline{Z})_p = X\wedge_\pi Z_p \ , \ 1\wedge\xi_p : \Sigma(X\wedge_\pi Z_p) = X\wedge_\pi\Sigma Z_p \longrightarrow X\wedge_\pi Z_{p+1} \ .$$

<u>Proposition 3.4</u> Given a $\{1\}$-space W, a π-map $f:X\longrightarrow Y$ and a π-spectrum $\underline{Z}$ there are defined exact sequences of abelian groups

$$\{W,X\wedge_\pi\underline{Z}\}\longrightarrow\{W,Y\wedge_\pi\underline{Z}\}\longrightarrow\{W,C_f\wedge_\pi\underline{Z}\}\longrightarrow\{W,\Sigma X\wedge_\pi\underline{Z}\}\longrightarrow\dots$$

$$\{X,\underline{Z}\}_\pi\longleftarrow\{Y,\underline{Z}\}_\pi\longleftarrow\{C_f,\underline{Z}\}_\pi\longleftarrow\{\Sigma X,\underline{Z}\}_\pi\longleftarrow\dots \ .$$

<u>Proof</u>: The first sequence is just the Puppe sequence associated to the (co)fibration sequence of $\{1\}$-spectra

$$X\wedge_\pi\underline{Z} \xrightarrow{f\wedge 1} Y\wedge_\pi\underline{Z} \longrightarrow C_f\wedge_\pi\underline{Z} \longrightarrow X\wedge_\pi\underline{Z} \longrightarrow \dots \ .$$

The exactness of the other sequence may be established as in the case $\pi = \{1\}$ (Puppe sequence again) by insisting on π-maps and π-homotopies.

[]

Given π-spaces X,Y and a $\{1\}$-map

$$\alpha : S^N \longrightarrow X\wedge_\pi Y$$

for some $N\geqslant 0$ define slant products for any π-spectrum $\underline{Z}$

$$\alpha\backslash - : \{X,\underline{Z}\}_\pi\to\{S^N,\underline{Z}\wedge_\pi Y\} ; (f:\Sigma^pX\to Z_p)\longmapsto (S^{N+p}\xrightarrow{\Sigma^p\alpha}\Sigma^pX\wedge_\pi Y\xrightarrow{f\wedge 1} Z_p\wedge_\pi Y)$$

$$\alpha\backslash - : \{Y,\underline{Z}\}_\pi\to\{S^N,X\wedge_\pi\underline{Z}\} ; (g:\Sigma^pY\to Z_p)\longmapsto (S^{N+p}\xrightarrow{\Sigma^p\alpha}X\wedge_\pi\Sigma^pY\xrightarrow{1\wedge g} X\wedge_\pi Z_p) \ .$$

Call $\alpha:S^N\longrightarrow X\wedge_\pi Y$ an <u>Sπ-duality map</u> if these slant products are isomorphisms for every π-spectrum $\underline{Z}$, in which case the suspensions

$$\Sigma\alpha : S^{N+1}\longrightarrow\Sigma(X\wedge_\pi Y) = \Sigma X\wedge_\pi Y \ , \ \Sigma\alpha : S^{N+1}\longrightarrow\Sigma(X\wedge_\pi Y) = X\wedge_\pi\Sigma Y$$

are also Sπ-duality maps. For $\pi = \{1\}$ this is classical Spanier-Whitehead S-duality.

Given Sπ-duality maps

$$\alpha : S^N\longrightarrow X\wedge_\pi Y \ , \ \alpha' : S^N\longrightarrow X'\wedge_\pi Y'$$

define the <u>Sπ-dual</u> of an Sπ-map $f\in\{X,X'\}_\pi$ to be the Sπ-map $g\in\{Y',Y\}_\pi$ to which f is sent by the composite isomorphism

$$\{X,X'\}_\pi \xrightarrow{\alpha\backslash} \{S^N,X'\wedge_\pi Y\} \xrightarrow{(\alpha'\backslash)^{-1}} \{Y',Y\}_\pi .$$

In particular, if X = X' the Sπ-duals of $1\in\{X,X\}_\pi$ are an inverse pair of Sπ-homotopy equivalences $g\in\{Y,Y'\}_\pi$, $g'\in\{Y',Y\}_\pi$.

<u>Proposition 3.5</u> Every finite CWπ-complex X admits an Sπ-duality map

$$\alpha : S^N \longrightarrow X \wedge_\pi Y$$

with Y a finite CWπ-complex.

<u>Proof</u>: Suspending if necessary it may be assumed that X is a pointed CWπ-complex. Our construction of an Sπ-dual is by induction on the pointed π-cells: given an Sπ-duality map $\alpha : S^N \longrightarrow X \wedge_\pi Y$ between finite pointed CWπ-complexes X, Y and a π-map $f : \bigvee_\pi S^{r-1} \longrightarrow X$ we shall construct an Sπ-duality map $\alpha' : S^{N'} \longrightarrow X' \wedge_\pi Y'$ for $X' = X \cup_{f\pi} \bigvee D^r$.

Let $m = \max(\dim(Y) + 1, 2r-1-N)$. Replacing $\alpha : S^N \longrightarrow X \wedge_\pi Y$ by $\Sigma^m \alpha : S^{N+m} \longrightarrow X \wedge_\pi \Sigma^m Y$ we have that $N-r+1 \geqslant 0$ and

$$\{\bigvee_\pi S^{r-1}, X\}_\pi = [\bigvee_\pi S^N, \Sigma^{N-r+1} X]_\pi \quad , \quad \{Y, \bigvee_\pi S^{N-r+1}\}_\pi = [Y, \bigvee_\pi S^{N-r+1}]_\pi$$

(by Proposition 3.3). Define an Sπ-duality map

$$\beta : S^N \longrightarrow (\bigvee_\pi S^{r-1}) \wedge_\pi (\bigvee_\pi S^{N-r+1}) = \bigvee_\pi S^N$$

by sending S^N to the summand labelled by $1 \in \pi$. Let $g : Y \longrightarrow \bigvee_\pi S^{N-r+1}$ be a π-map representing the Sπ-dual of $f \in \{\bigvee_\pi S^{r-1}, X\}_\pi$, and let $Y' = C_g$ be the mapping cone π-space. Denote the cofibration sequences by

$$\bigvee_\pi S^{r-1} \xrightarrow{f} X \xrightarrow{e} X' \xrightarrow{d} \bigvee_\pi S^r$$
$$Y \xrightarrow{g} \bigvee_\pi S^{N-r+1} \xrightarrow{h} Y' \xrightarrow{k} \Sigma Y \ .$$

The diagram of $\{1\}$-spaces and $\{1\}$-maps

$$\begin{array}{ccccc}
S^N & \xrightarrow{\alpha} & X \wedge_\pi Y & \xrightarrow{e \wedge 1} & X' \wedge_\pi Y \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1 \wedge g} & & \downarrow{\scriptstyle 1 \wedge g} \\
(\bigvee_\pi S^{r-1}) \wedge_\pi (\bigvee_\pi S^{N-r+1}) & \xrightarrow{f \wedge 1} & X \wedge_\pi (\bigvee_\pi S^{N-r+1}) & \xrightarrow{e \wedge 1} & X' \wedge_\pi (\bigvee_\pi S^{N-r+1})
\end{array}$$

is homotopy commutative, with the bottom row null-homotopic. It is thus possible to define a $\{1\}$-map $j : D^{N+1} \longrightarrow X' \wedge_\pi (\bigvee_\pi S^{N-r+1})$ such that the diagram

$$\begin{array}{ccc}
S^N & \xrightarrow{(e \wedge 1)\alpha} & X' \wedge_\pi Y \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1 \wedge g} \\
D^{N+1} & \xrightarrow{j} & X' \wedge_\pi (\bigvee_\pi S^{N-r+1})
\end{array}$$

is actually commutative, with $i : S^N \longrightarrow D^{N+1}$ the inclusion. The induced $\{1\}$-map of mapping cones

$$\alpha' : C_i = S^{N+1} \longrightarrow C_{1 \wedge g} = X' \wedge_\pi Y'$$

is such that both the squares in the diagram of $\{1\}$-spaces and $\{1\}$-maps

$$\begin{array}{ccccc}
X \wedge_\pi \Sigma Y & \xleftarrow{\Sigma\alpha} & S^{N+1} & \xrightarrow{\Sigma\beta} & (\bigvee_\pi S^r) \wedge_\pi (\bigvee_\pi S^{N-r+1}) \\
\downarrow{\scriptstyle e \wedge 1} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle 1 \wedge h} \\
X' \wedge_\pi \Sigma Y & \xleftarrow{1 \wedge k} & X' \wedge_\pi Y' & \xrightarrow{d \wedge 1} & (\bigvee_\pi S^r) \wedge_\pi Y'
\end{array}$$

are homotopy commutative. There is thus defined a commutative diagram of abelian groups and morphisms

$$\begin{array}{ccccccccc}
\{\Sigma X, \underline{Z}\}_\pi & \xrightarrow{\Sigma f} & \{\bigvee_\pi S^r, \underline{Z}\}_\pi & \xrightarrow{d} & \{X', \underline{Z}\}_\pi & \xrightarrow{e} & \{X, \underline{Z}\}_\pi & \xrightarrow{f} & \{\bigvee_\pi S^{r-1}, \underline{Z}\}_\pi \\
\downarrow{\scriptstyle (\Sigma\alpha)\backslash} & & \downarrow{\scriptstyle (\Sigma\beta)\backslash} & & \downarrow{\scriptstyle \alpha'\backslash} & & \downarrow{\scriptstyle \Sigma\alpha\backslash} & & \downarrow{\scriptstyle \Sigma\beta\backslash} \\
\{S^{N+1}, \underline{Z} \wedge_\pi Y\} & \xrightarrow{g} & \{S^{N+1}, \underline{Z} \wedge_\pi (\bigvee_\pi S^{N-r+1})\} & \xrightarrow{h} & \{S^{N+1}, \underline{Z} \wedge_\pi Y'\} & \xrightarrow{k} & \{S^{N+1}, \underline{Z} \wedge_\pi \Sigma Y\} & \xrightarrow{\Sigma g} & \{S^{N+1}, \underline{Z} \wedge_\pi (\bigvee_\pi S^{N-r+2})\}
\end{array}$$

for any π-spectrum $\underline{Z}$, with exact rows (Proposition 3.4). Applying the 5-lemma we have that the middle column is an isomorphism, and similarly for the other type of slant product. Therefore $\alpha' : S^{N+1} \longrightarrow X' \wedge_\pi Y'$ is an Sπ-duality map. []

We can use Sπ-duality to prove a π-equivariant analogue of Whitehead's theorem.

<u>Proposition 3.6</u> A π-map of finite CWπ-complexes $f : X \longrightarrow Y$ induces isomorphisms in homology if and only if $\Sigma^p f : \Sigma^p X \longrightarrow \Sigma^p Y$ is a π-homotopy equivalence for some $p \geqslant 0$.

<u>Proof</u>: Let $f : X \longrightarrow Y$ induce isomorphisms in homology. Applying the ordinary Whitehead theorem we have that $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ is a homotopy equivalence, and hence that

$$f : \{S^N, (\bigvee_\pi S^r) \wedge_\pi X\} \longrightarrow \{S^N, (\bigvee_\pi S^r) \wedge_\pi Y\}$$

is an isomorphism for all $N, r \geqslant 0$. This gives the induction step in proving that

$$f : \{S^N, W \wedge_\pi X\} \longrightarrow \{S^N, W \wedge_\pi Y\}$$

is an isomorphism for every finite CWπ-complex W. Given an Sπ-duality map $\alpha : S^N \longrightarrow W \wedge_\pi Y$ (by Proposition 3.5) we thus have isomorphisms

$$\{Y, X\}_\pi \xrightarrow{\alpha\backslash} \{S^N, W \wedge_\pi X\} \xrightarrow{f} \{S^N, W \wedge_\pi Y\} \xrightarrow{(\alpha\backslash)^{-1}} \{Y, Y\}_\pi \ .$$

The element $g \in \{Y, X\}_\pi$ corresponding to $1 \in \{Y, Y\}_\pi$ is represented by a π-map $g : \Sigma^p Y \longrightarrow \Sigma^p X$ for some $p \geqslant 0$ (by Proposition 3.2) which is a π-homotopy inverse for $\Sigma^p f : \Sigma^p X \longrightarrow \Sigma^p Y$. []

For any π-spaces X,Y the natural projection defines a $\mathbb{Z}$-module chain map

$$\mathbb{Z}^t\otimes_{\mathbb{Z}[\pi]}(\dot{C}(X)\otimes_{\mathbb{Z}}\dot{C}(Y)) = \dot{C}(X)^t\otimes_{\mathbb{Z}[\pi]}\dot{C}(Y) \longrightarrow \dot{C}(X\wedge_\pi Y)$$

where t refers to the untwisted involution on $\mathbb{Z}[\pi]$. If X,Y are finitely-dominated CWπ-complexes this is a chain equivalence (- consider the reduced cellular chain complexes) and the chain level slant product

$$(\dot{C}(X)^t\otimes_{\mathbb{Z}[\pi]}\dot{C}(Y))\otimes_{\mathbb{Z}}\dot{C}(X)^* \longrightarrow \dot{C}(Y) \ ; \ (x\otimes y)\otimes f \longmapsto \overline{f(x)}y$$

induces a slant product in homology

$$\backslash : \dot{H}_N(X\wedge_\pi Y)\otimes_{\mathbb{Z}}\dot{H}^r_\pi(X) \longrightarrow \dot{H}_{N-r}(Y) \ .$$

The Sπ-duality map $\alpha: S^N \longrightarrow X\wedge_\pi Y$ constructed in Proposition 2.5 is such that $\alpha_*[S^N]\backslash - : \dot{C}(X)^{N-*} \longrightarrow \dot{C}(Y)$ is a $\mathbb{Z}[\pi]$-module chain equivalence, since the cellular structure was constructed as the dual of that of X. We shall show (in Proposition 3.8 below) that this property characterizes Sπ-duality maps for finite CWπ-complexes, generalizing the case $\pi = \{1\}$ of ordinary S-duality.

Define the <u>r-dimensional Eilenberg-MacLane π-spectrum</u> $\underline{K\pi(\mathbb{Z},r)}$ by

$$K\pi(\mathbb{Z},r)_p = \bigvee_\pi K(\mathbb{Z},p+r) \ , \ \xi_p = \bigvee_\pi \eta_p : \Sigma K\pi(\mathbb{Z},r)_p = \bigvee_\pi \Sigma K(\mathbb{Z},p+r) \longrightarrow K\pi(\mathbb{Z},r)_{p+1} = \bigvee_\pi K(\mathbb{Z},p+r+1) \quad (r\geqslant 0)$$

with $\eta_p : \Sigma K(\mathbb{Z},p+r) \longrightarrow K(\mathbb{Z},p+r+1)$ the standard map. For $\pi = \{1\}$ this is the usual Eilenberg-MacLane spectrum $\underline{K(\mathbb{Z},r)}$.

<u>Proposition 3.7</u> If X is a finite CWπ-complex X then

$$\dot{H}_r(X) = \{S^r, X\wedge_\pi \underline{K\pi(\mathbb{Z},0)}\} \ , \ \dot{H}^r_\pi(X) = \{X, \underline{K\pi(\mathbb{Z},r)}\}_\pi \qquad (r\geqslant 0) \ .$$

<u>Proof</u>: For any CWπ-complex X we have

$$\{S^r, X\wedge_\pi \underline{K\pi(\mathbb{Z},0)}\} = \{S^r, X\wedge \underline{K(\mathbb{Z},0)}\} = \dot{H}_r(X) \qquad (r\geqslant 0)$$

by the usual identification of integral homology with $\underline{K(\mathbb{Z},0)}$-homology. Also, there is defined a natural $\mathbb{Z}[\pi]$-module morphism

$$\iota : \{X, \underline{K\pi(\mathbb{Z},r)}\}_\pi \longrightarrow \dot{H}^r_\pi(X) \ ; \ (f:\Sigma^p X \longrightarrow \bigvee_\pi K(\mathbb{Z},p+r)) \longmapsto f^*(1)$$

with $f^*:\dot{H}^{p+r}_\pi(\bigvee_\pi K(\mathbb{Z},p+r)) = \mathbb{Z}[\pi] \longrightarrow \dot{H}^{p+r}_\pi(\Sigma^p X) = \dot{H}^r_\pi(X)$. If X is finite we have an Sπ-duality $\alpha: S^N \longrightarrow X\wedge_\pi Y$ (Proposition 3.5) and ι can be identified with the Sπ-duality isomorphism

$$\alpha\backslash - : \{X, \underline{K\pi(\mathbb{Z},r)}\}_\pi \longrightarrow \{S^N, \underline{K\pi(\mathbb{Z},r)}\wedge_\pi Y\} = \dot{H}_{N-r}(Y) = \dot{H}^r_\pi(X) \ .$$

[]

If X,Y are finite CWπ-complexes and $\alpha: S^N \longrightarrow X\wedge_\pi Y$ is a $\{1\}$-map then the identification of Proposition 3.7 carries the chain level slant product

$$\alpha_*[S^N]\backslash - : \dot{H}^*_\pi(X) \longrightarrow \dot{H}_{N-*}(Y) \qquad (\alpha_*[S^N] \in \dot{H}_N(X\wedge_\pi Y))$$

to the geometric slant product

$$\alpha\backslash - : \{X, \underline{K\pi(\mathbb{Z},r)}\}_\pi \longrightarrow \{S^N, \underline{K\pi(\mathbb{Z},r)}\wedge_\pi Y\} = \{S^{N-r}, \underline{K\pi(\mathbb{Z},0)}\wedge_\pi Y\}$$

defined previously.

<u>Proposition 3.8</u> Let X,Y be finite CWπ-complexes. A $\{1\}$-map $\alpha: S^N \longrightarrow X\wedge_\pi Y$ is an Sπ-duality map if and only if the chain level slant product

$$\alpha_*[S^N]\backslash - : \dot{C}(X)^{N-*} \longrightarrow \dot{C}(Y)$$

is a $\mathbb{Z}[\pi]$-module chain equivalence.

<u>Proof</u>: If $\alpha: S^N \longrightarrow X\wedge_\pi Y$ is an Sπ-duality map then the chain level slant product with $\alpha_*[S^N] \in \dot{H}_N(X\wedge_\pi Y)$ induces the Sπ-duality isomorphisms

$$\dot{H}^r_\pi(X) = \{X, \underline{K\pi(\mathbb{Z},r)}\}_\pi \longrightarrow \{S^N, \underline{K\pi(\mathbb{Z},r)}\wedge_\pi Y\} = \{S^{N-r}, \underline{K\pi(\mathbb{Z},0)}\wedge_\pi Y\} = \dot{H}_{N-r}(Y) \ .$$

Conversely, suppose given a $\{1\}$-map $\alpha: S^N \longrightarrow X\wedge_\pi Y$ such that $\alpha_*[S^N]\backslash - : \dot{H}^*_\pi(X) \longrightarrow \dot{H}_{N-*}(Y)$ is an isomorphism. Let $\alpha': S^N \longrightarrow X\wedge_\pi Y'$ be the Sπ-duality map constructed for X in Proposition 3.5 for N sufficiently large, and let $f\in\{Y',Y\}_\pi$ correspond to $\alpha\in\{S^N, X\wedge_\pi Y\}$ under the Sπ-duality isomorphism

$$\alpha'\backslash - : \{Y',Y\}_\pi \longrightarrow \{S^N, X\wedge_\pi Y\} \ .$$

Now $f\in\{Y',Y\}_\pi$ induces isomorphisms in homology

$$f = \alpha_*[S^N]\backslash - : \dot{H}_*(Y') = \dot{H}^{N-*}_\pi(X) \longrightarrow \dot{H}_*(Y)$$

Applying Proposition 3.6 we have that $f\in\{Y',Y\}_\pi$ is an Sπ-homotopy equivalence, and hence that $\alpha: S^N \longrightarrow X\wedge_\pi Y$ is an Sπ-duality map.

[]

With a little more effort Propositions 3.5 - 3.8 can be made to apply also for finitely-dominated CWπ-complexes.

$S\pi$-duality maps arise as follows.

Proposition 3.9 Let E be a compact connected N-dimensional submanifold of S^N with non-empty boundary ∂E and let $\widetilde{E}$ be a covering space of E with group of covering translations π, with $\widetilde{\partial E} \subset \widetilde{E}$ covering ∂E. Then the composite $\{1\}$-map

$$\alpha : S^N \xrightarrow{\text{collapse}} S^N/\overline{S^N - E} = E/\partial E = (\widetilde{E}/\widetilde{\partial E})/\pi \xrightarrow{\text{diagonal}} \widetilde{E}_+ \wedge_\pi (\widetilde{E}/\widetilde{\partial E})$$

is an $S\pi$-duality map.

Proof: The diagonal map is obtained from

$$\Delta : \widetilde{E}/\widetilde{\partial E} \longrightarrow \widetilde{E}_+ \wedge \widetilde{E}/\widetilde{\partial E} = (\widetilde{E} \times \widetilde{E})/(\widetilde{E} \times \widetilde{\partial E}) \ ; \ x \longmapsto (x,x)$$

by quotienting out the π-action. Now $\widetilde{E}_+$ and $\widetilde{E}/\widetilde{\partial E}$ are finite $CW\pi$-complexes by Proposition 3.2, and $\alpha : S^N \longrightarrow \widetilde{E}_+ \wedge_\pi \widetilde{E}/\widetilde{\partial E}$ is an $S\pi$-duality map by Proposition 3.8, since

$$\alpha_*[S^N]\backslash - = [E] \cap - : \dot{H}^r_\pi(\widetilde{E}/\widetilde{\partial E}) = H^r_\pi(\widetilde{E},\widetilde{\partial E}) \longrightarrow \dot{H}_{N-r}(\widetilde{E}_+) = H_{N-r}(\widetilde{E})$$

defines the Poincaré-Lefschetz duality isomorphisms of $(E,\partial E)$, with $[E] \in H_N(E,\partial E)$ the fundamental class.

[]

Given a fibration $F \longrightarrow E \xrightarrow{p} B$ and a covering $\widetilde{B}$ of the base space B with group of covering translations π define the Thom π-space to be the mapping cone π-space of the induced π-map $\widetilde{p}_+ : \widetilde{E}_+ \longrightarrow \widetilde{B}_+$

$$T\pi(p) = \widetilde{B}_+ \cup_{\widetilde{p}_+} \widetilde{E}_+ \wedge I = \widetilde{B} \cup_{\widetilde{p} \times 0} \widetilde{E} \times I / \widetilde{E} \times 1 \ .$$

The quotient $\{1\}$-space $T\pi(p)/\pi = T(p)$ is the usual Thom $\{1\}$-space of p, and if $\widetilde{B} = \pi \times B$ is the trivial covering then

$$T\pi(p) = \bigvee_\pi T(p) \ .$$

If $p : E \longrightarrow B$ is a cellular map of connected CW complexes then $T\pi(p)$ is a $CW\pi$-complex by Proposition 3.2.

Fibre homotopy equivalence classes of $(k-1)$-spherical fibrations

$$S^{k-1} \longrightarrow E \xrightarrow{p} X$$

over a connected CW complex X are in a natural one-one correspondence with the homotopy classes of maps $p : X \longrightarrow BG(k)$, for the appropriate classifying space $BG(k)$. Given such a fibration we shall say that a covering $\widetilde{X}$ of X

 is oriented with respect to p if the group of covering translations π is equipped with a group morphism $w : \pi \longrightarrow \mathbb{Z}_2$ such that the first Stiefel-Whitney class $w_1(p) \in H^1(X;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)$ factors as

$$w_1(p) : \pi_1(X) \longrightarrow \pi \xrightarrow{w} \mathbb{Z}_2$$

with $\pi_1(X) \longrightarrow \pi$ the characteristic map, and the pair (π,w) is the data of the covering. A covering $\widetilde{X}$ of X can be oriented with respect to $p : X \longrightarrow BG(k)$ if and only if the pullback $\widetilde{p} : \widetilde{X} \longrightarrow X \xrightarrow{p} BG(k)$ is an orientable $(k-1)$-spherical fibration, but the choice of w is not unique. If $f : M \longrightarrow X$ is a map of connected CW complexes then the pullback cover $\widetilde{M}$ of M is oriented with respect to the pullback fibration $f^*(p) : M \xrightarrow{f} X \xrightarrow{p} BG(k)$, with the same data (π,w).

A covering $\widetilde{X}$ of a geometric Poincaré complex X is oriented with data (π,w) in the sense of §2 if and only if it is oriented in the above sense with respect to the Spivak normal fibration $\nu_X : X \longrightarrow BG$ with data (π,w) (cf. Proposition 3.12 below).

Spherical fibrations are characterized by the following equivariant generalization of the Thom isomorphism theorem.

Proposition 3.10 Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected CW complexes, with E,B finitely dominated. If $F = S^{k-1}$ (up to homotopy equivalence) and $\widetilde{B}$ is an oriented covering of B with data (π,w) then there exists an element $U_p \in \dot{H}^k_\pi(T\pi(p);{}^w\mathbb{Z})$, the Thom class of p, such that the cap product

$$U_p \cap - : {}^w\dot{C}(T\pi(p)) \longrightarrow S^k C(\widetilde{B})$$

is a chain equivalence of finite-dimensional $\mathbb{Z}[\pi]$-module chain complexes, the Thom equivalence. Conversely, if F is simply-connected and there exists an element $U_p \in \dot{H}^k_{\pi_1(B)}(T\pi_1(B)(p);{}^w\mathbb{Z})$ $(k \geqslant 3)$ for the Thom $\pi_1(B)$-space with respect to the universal cover $\widetilde{B}$ of B, for some group morphism $w : \pi_1(B) \longrightarrow \mathbb{Z}_2$ such that

$$U_p \cap - : {}^w\dot{C}(T\pi_1(B)) \longrightarrow S^k C(\widetilde{B})$$

is a $\mathbb{Z}[\pi_1(B)]$-module chain equivalence then F is a homotopy S^{k-1} and $w = w_1(p) : \pi_1(B) \longrightarrow \mathbb{Z}_2$ is the first Stiefel-Whitney class of p.

Proof: By the spectral sequence arguments of Lemma I.4.3 of Browder [2] applied to the pullback $F \longrightarrow \widetilde{E} \xrightarrow{\widetilde{p}} \widetilde{B}$ of p to the universal cover $\widetilde{B}$ of B.

[]

We can now state the analogue of Proposition 4.4 of Spivak [1] appropriate to geometric Poincaré complexes in the sense of Wall [4] (cf. Browder [3]).

Proposition 3.11 Let $X \subset S^N$ be a finite subcomplex with a closed regular neighbourhood E, and let F be the homotopy-theoretic fibre of the inclusion $p:\partial E \longrightarrow E$. Then X is an n-dimensional geometric Poincaré complex if and only if F is a homotopy S^{N-n-1} $(N \geqslant n+3)$.

Proof: The inclusion $X \hookrightarrow E$ is a homotopy equivalence, so we can identify $\pi_1(X) = \pi_1(E) = \pi$, $\tilde{X} = \tilde{E}$, $T\pi(p) = \tilde{E}/\partial\tilde{E}$ with $\tilde{X},\tilde{E}$ the universal covers. Proposition 3.9 gives an $S\pi$-duality map

$$\alpha : S^N \longrightarrow \tilde{X}_+ \wedge_\pi T\pi(p)$$

such that there is defined a commutative diagram

$$\begin{array}{ccc} H^\pi_n(\tilde{X};{}^w\mathbb{Z}) \otimes_{\mathbb{Z}} {}^wH^r_\pi(\tilde{X}) & \xrightarrow{\ \cap\ } & H_{n-r}(\tilde{X}) \\ \Big\downarrow {\scriptstyle (\alpha_*[S^N]\backslash -)^{-1} \otimes (\alpha_*[S^N]\backslash -)} & & \Big\downarrow {\scriptstyle \mathrm{id.}} \\ \dot{H}^{N-n}_\pi(T\pi(p);{}^w\mathbb{Z}) \otimes_{\mathbb{Z}} {}^w\dot{H}_{N-r}(T\pi(p)) & \xrightarrow{\ \cap\ } & H_{n-r}(\tilde{X}) \end{array}$$

for any group morphism $w:\pi \longrightarrow \mathbb{Z}_2$. Comparing the definition of a geometric Poincaré complex (as in §2) with the criterion of Proposition 3.10 gives the required correspondence.

[]

An n-dimensional normal space (X,ν_X,ρ_X) is an n-dimensional finitely-dominated CW complex X together with a $(k-1)$-spherical fibration $\nu_X : X \longrightarrow BG(k)$ and an element $\rho_X \in \pi_{n+k}(T(\nu_X))$. (This concept is due to Quinn [3]). Given a covering $\tilde{X}$ of X with group of covering translations π define the fundamental map of (X,ν_X,ρ_X) to be the composite $\{1\}$-map

$$\alpha_X : S^{n+k} \xrightarrow{\ \rho_X\ } T(\nu_X) = X/E = (\tilde{X}/\tilde{E})/\pi \xrightarrow{\ \Delta\ } \tilde{X}_+ \wedge_\pi \tilde{X}/\tilde{E} = \tilde{X}_+ \wedge_\pi T\pi(\nu_X) ,$$

with $\tilde{E}$ the induced covering of the total space E of ν_X, and Δ the diagonal map. If $\tilde{X}$ is oriented with data (π,w) with respect to ν_X define the fundamental class to be the twisted homology class

$$[X] = U_{\nu_X} \cap h(\rho_X) \in H^\pi_n(\tilde{X};{}^w\mathbb{Z})$$

with $U_{\nu_X} \in \dot{H}^k_\pi(T\pi(\nu_X);{}^w\mathbb{Z})$ the Thom class of ν_X and $h:\pi_{n+k}(T(\nu_X)) \longrightarrow H_{n+k}(T(\nu_X))$ the Hurewicz map. The fundamental map is related to the fundamental class by a $\mathbb{Z}[\pi]$-module chain homotopy commutative diagram

$$\begin{array}{ccc} {}^wC(\tilde{X})^{n-*} & \xrightarrow{\ [X]\cap -\ } & C(\tilde{X}) \\ \Big\downarrow {\scriptstyle U_{\nu_X} \cup -} & & \Big\downarrow {\scriptstyle \mathrm{id.}} \\ \dot{C}(T\pi(\nu_X))^{n+k-*} & \xrightarrow{\ \alpha_*[S^{n+k}]\backslash -\ } & C(\tilde{X}) \end{array}$$

in which the cup product with U_{ν_X} is a chain equivalence (a variant of the Thom equivalence of Proposition 3.10).

A normal map of n-dimensional normal spaces

$$(f,b) : (M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$$

consists of a map $f:M \longrightarrow X$ of the underlying spaces together with a stable fibre homotopy class of stable fibre maps $b:\nu_M \longrightarrow \nu_X$ over f such that

$$T(b)\rho_M = \rho_X \in \pi_{n+k}(T(\nu_X))$$

for sufficiently large k. (This is a normal map in the sense of Quinn [3]). An equivalence of normal structures (ν_X,ρ_X), (ν'_X,ρ'_X) on a space X is a normal map of the type

$$(1,b) : (X,\nu_X,\rho_X) \longrightarrow (X,\nu'_X,\rho'_X) .$$

<u>Proposition 3.12</u> An n-dimensional geometric Poincaré complex X admits a normal structure (ν_X, ρ_X) with $w_1(\nu_X) = w(X)$ and the same fundamental class $[X] \in H_n^{\pi}(\widetilde{X}; {}^w\mathbb{Z})$ such that the fundamental map

$$\alpha_X : S^{n+k} \longrightarrow \widetilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

defines an $S\pi$-duality for every covering $\widetilde{X}$ of X with group of covering translations π. Any two such normal structures (ν_X, ρ_X), (ν_X', ρ_X') are related by a unique equivalence $(1,b) : (X, \nu_X, \rho_X) \longrightarrow (X, \nu_X', \rho_X')$. Conversely, if X is a connected finitely-dominated CW complex with a normal structure $(\nu_X \rho_X)$ such that the fundamental map

$$\alpha_X : S^{n+k} \longrightarrow \widetilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

with respect to the universal cover $\widetilde{X}$ $(\pi = \pi_1(X))$ defines an $S\pi$-duality map then X is an n-dimensional geometric Poincaré complex with $w(X) = w_1(\nu_X)$ and the same fundamental class $[X] \in H_n^{\pi}(\widetilde{X}; {}^w\mathbb{Z})$.

<u>Proof</u>: If X is finite there exists an embedding $X \subseteq S^N$ for $N \geq 2$(geometric dimension of X) + 1 by general position, with closed regular neighbourhood E say. If $\widetilde{X}$ is any covering of X with group of covering translations π then Proposition 3.9 gives an $S\pi$-duality map

$$\alpha_X : S^N \xrightarrow{\rho_X = \text{collapse}} E/\partial E \xrightarrow{\Delta} \widetilde{X}_+ \wedge_\pi \widetilde{E}/\partial\widetilde{E} \ .$$

Let F be the homotopy-theoretic fibre of the inclusion $\partial E \subset E$, so that there is defined a fibration

$$F \longrightarrow \partial E \xrightarrow{\nu_X} X$$

with $T\pi(\nu_X) = \widetilde{E}/\partial\widetilde{E}$. If X is an n-dimensional geometric Poincaré complex then $F \simeq S^{N-n-1}$ by Proposition 3.11 and (ν_X, ρ_X) defines a normal structure with $S\pi$-duality. If X is not finite use the trick of §3 of Wall [4] of crossing with S^1 to reduce to the finite case. The uniqueness clause is as in Corollary 3.6 of Wall [4] (see also Theorem I.4.19 of Browder [2]). Conversely, given a normal structure with $S\pi$-duality for the universal cover we can obtain Poincaré duality by combining the $S\pi$-duality criterion of Proposition 3.8 with the Thom isomorphism of Proposition 3.10.

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Thus an n-dimensional geometric Poincaré complex X carries a canonical equivalence class of normal structures (ν_X, ρ_X) with $S\pi$-duality. We shall call this the <u>Spivak normal class</u>, calling any such ν_X a <u>Spivak normal fibration</u> of X. A <u>normalization</u> of X is a choice of normal structure (ν_X, ρ_X) in the Spivak normal class.

We are now in a position to apply our $S\pi$-duality to obtain geometric Umkehr maps of the type considered in §2 for degree 1 maps of geometric Poincaré complexes which preserve Spivak normal structures.

<u>Proposition 3.13</u> Given a degree 1 normal map of normalized n-dimensional geometric Poincaré complexes

$$(f,b) : (M, \nu_M, \rho_M) \longrightarrow (X, \nu_X, \rho_X)$$

and a cover $\widetilde{X}$ of X with group of covering translations π there is induced a π-map of Thom π-spaces $T\pi(b) : T\pi(\nu_M) \longrightarrow T\pi(\nu_X)$ such that the $S\pi$-dual of $T\pi(b)$ with respect to the fundamental $S\pi$-duality maps

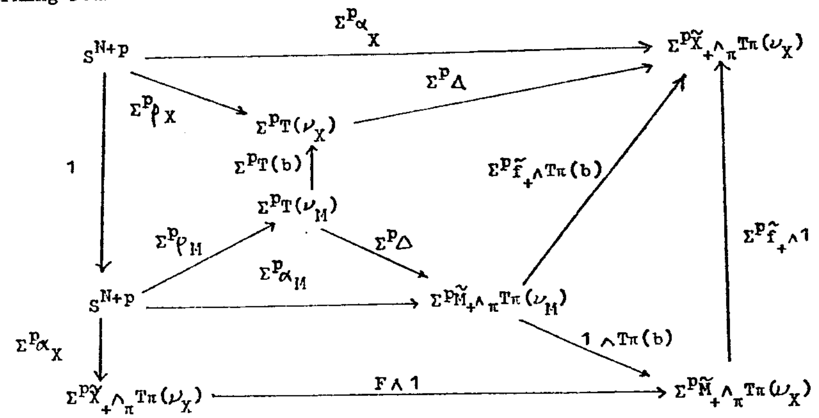
$$\alpha_M : S^N \longrightarrow \widetilde{M}_+ \wedge_\pi T\pi(\nu_M) \ , \ \alpha_X : S^N \longrightarrow \widetilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

is an $S\pi$-homotopy class $F \in \{\widetilde{X}_+, \widetilde{M}_+\}_\pi$ of geometric Umkehr maps $F : \Sigma^p \widetilde{X}_+ \longrightarrow \Sigma^p \widetilde{M}_+$ such that $(\Sigma^p \widetilde{f}_+) F \simeq 1 : \Sigma^p \widetilde{X}_+ \longrightarrow \Sigma^p \widetilde{X}_+$ up to stable π-homotopy.

<u>Proof</u>: The $S\pi$-duality is defined by the composite

$$\{T\pi(\nu_M), T\pi(\nu_X)\}_\pi \xrightarrow{(\alpha_M \backslash -)^{-1}} \{S^N, T\pi(\nu_X) \wedge_\pi \widetilde{M}_+\} \xrightarrow{(\alpha_X \backslash -)} \{\widetilde{X}_+, \widetilde{M}_+\}_\pi \ .$$

Working round the stable homotopy commutative diagram



we have that

$$((\Sigma^p\tilde{f}_+)F\wedge 1)(\Sigma^p\alpha_X)\simeq(\Sigma^p\alpha_X):S^{N+p}\longrightarrow \Sigma^p\tilde{X}_+\wedge_\pi T\pi(\nu_X)\ .$$

Since $\Sigma^p\alpha_X$ is also an $S\pi$-duality map it follows that $\Sigma^p\tilde{f}_+F\simeq 1:\Sigma^p\tilde{X}_+\longrightarrow\Sigma^p\tilde{X}_+$ for p large enough. The diagram also shows that F induces the Umkehr $f^!:C(\tilde{X})\longrightarrow C(\tilde{M})$ on the chain level, identifying the Poincaré duality chain equivalences with the appropriate Thom equivalences.

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Define the quadratic kernel of a normal map of normalized n-dimensional geometric Poincaré complexes

$$(f,b)\ :\ (M,\nu_M,\rho_M)\longrightarrow(X,\nu_X,\rho_X)$$

with respect to an oriented cover $\tilde{X}$ of X with data (π,w) to be the n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$\sigma_*(f,b)\ =\ \sigma_*(f,F)\ =\ (C(f^!),e_\%\psi_F[X]\in Q_n(C(f^!)))$$

using the quadratic kernel construction of Proposition 2.9 with any of the geometric Umkehr maps $F:\Sigma^p\tilde{X}_+\longrightarrow\Sigma^p\tilde{M}_+$ such that $(\Sigma^p\tilde{f}_+)F\simeq 1$ provided by Proposition 3.13. All such quadratic kernels are induced from that associated to the universal covering $\tilde{X}$ of X with data $(\pi_1(X),w(X))$. We have the sum formula:

Proposition 3.14 The quadratic kernel of the composite

$$(gf,cb)\ :\ (X,\nu_X,\rho_X)\xrightarrow{(f,b)}(Y,\nu_Y,\rho_Y)\xrightarrow{(g,c)}(Z,\nu_Z,\rho_Z)$$

of normal maps of normalized n-dimensional geometric Poincaré complexes is

$$\sigma_*(gf,cb)\ =\ \sigma_*(f,b)\oplus\sigma_*(g,c)$$

up to homotopy equivalence.

Proof: This is immediate from Proposition 2.11, since $(\Sigma^p\tilde{f}_+)F\simeq 1$.

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The difference $\psi'-\psi\in\ker((1+T):Q_n(C(f^!))\longrightarrow Q^n(C(f^!)))$ of the hyperhomology classes appearing in the quadratic kernels $\sigma_*(f,b)=(C(f^!),\psi)$, $\sigma_*(f,b')=(C(f^!),\psi')$ of normal maps

$$(f,b):(M,\nu_M,\rho_M)\longrightarrow(X,\nu_X,\rho_X)\ \ ,\ \ (f,b'):(M,\nu_M,\rho_M)\longrightarrow(X,\nu_X,\rho_X')$$

such that $b'=bc:\nu_M\longrightarrow\nu_X$ for some automorphism $c:\nu_M\longrightarrow\nu_M$ will be expressed in terms of c in Proposition 18.10 below.

A normal bundle map

$$(f,b)\ :\ M\longrightarrow X$$

is a degree 1 map $f:M\longrightarrow X$ from an n-dimensional smooth manifold M to an n-dimensional geometric Poincaré complex X together with a bundle map $b:\nu_M\longrightarrow\nu_X$ from the normal bundle $\nu_M:M\longrightarrow BO(k)$ for some embedding $M\subset S^{n+k}$ $(k\gg n)$ to some bundle $\nu_X:X\longrightarrow BO(k)$. This is the definition of normal map due to Browder [2] (with M compact and X finite). The quadratic kernel of such a normal bundle map with respect to an oriented cover $\tilde{X}$ of X is the quadratic kernel

$$\sigma_*(f,b)\ =\ (C(f^!),e_\%(\psi_F[X])\in Q_n(C(f^!)))$$

of the normal map of normalized n-dimensional geometric Poincaré complexes

$$(f,b)\ :\ (M,\nu_M,\rho_M)\longrightarrow(X,\nu_X,\rho_X)$$

obtained by passing to the associated spherical fibrations $\nu_M:M\longrightarrow BG(k)$, $\nu_X:X\longrightarrow BG(k)$ with $\rho_M:S^{n+k}\xrightarrow{\text{collapse}}T(\nu_M)$, $\rho_X=T(b)\rho_M:S^{n+k}\longrightarrow T(\nu_X)$.

The surgery obstruction of a 2q-dimensional normal bundle map $(f,b):M\longrightarrow X$ such that $\pi_1(X)=\{1\}$ is $\begin{cases}\frac{1}{8}\text{ (signature)}\\ \text{the Arf invariant}\end{cases}$ of the quadratic form over $\begin{cases}\mathbb{Z}\\ \mathbb{Z}_2\end{cases}$ defined on $\begin{cases}K^q(M)\\ K^q(M;\mathbb{Z}_2)\end{cases}$ by $\begin{cases}\sigma_*(f,b)\\ \mathbb{Z}_2\otimes_{\mathbb{Z}}\sigma_*(f,b)\end{cases}$ if $q\equiv\begin{cases}0\\1\end{cases}$ (mod 2) (cf. Proposition 2.10 ii)). The relationship of framings of manifolds to quadratic forms was first noted by Milnor [2].

A normal map in the sense of Wall [5]

$$(f,B)\ :\ M\longrightarrow X$$

is a degree 1 map $f:M\longrightarrow X$ from an n-dimensional smooth manifold M to an n-dimensional geometric Poincaré complex X together with a bundle isomorphism $B:\varepsilon_M\longrightarrow\tau_M\oplus f^*\nu_X$ with $\tau_M:M\longrightarrow BO(n)$ the tangent bundle of M, $\nu_X:X\longrightarrow BO(k)$ some bundle over X, and $\varepsilon_M=0:M\longrightarrow BO(n+k)$ the trivial (n+k)-plane bundle, (with M compact and X finite). Choosing an embedding $M\subset S^N$ $(N\gg n)$ with normal bundle $\nu_M:M\longrightarrow BO(N-n)$ we have a stable inverse ν_M for τ_M and a bundle map

$$b\ :\ \nu_M\oplus\varepsilon_M\xrightarrow{1\oplus B}\nu_M\oplus(\tau_M\oplus f^*\nu_X)=(\nu_M\oplus\tau_M)\oplus f^*\nu_X\longrightarrow f^*\varepsilon_X\oplus f^*\nu_X\longrightarrow\varepsilon_X\oplus\nu_X$$

with $\varepsilon_X=0:X\longrightarrow BO(N)$. The quadratic kernel $\sigma_*(f,b)$ of the normal bundle map $(f,b):M \longrightarrow X$ does not depend on the choice of ν_M : for if ν_M, ν'_M are two such then there exists a bundle isomorphism $c:\nu'_M \longrightarrow \nu_M$ such that $b' = bc:\nu'_M \longrightarrow \nu_X$ and $T(c)(\rho'_M) = \rho_M \in \pi_N(T(\nu_M))$ (by the uniqueness of embeddings $M \subset S^N$ for $N \gg n$), so that applying the sum formula of Proposition 3.14 to the composite normal map

$$(f,b') : (M,\nu'_M,\rho'_M) \xrightarrow{(1,c)} (M,\nu_M,\rho_M) \xrightarrow{(f,b)} (X,\nu_X,\rho_X)$$

we have that up to homotopy equivalence

$$\sigma_*(f,b') = \sigma_*(f,b) \oplus \sigma_*(1,c) = \sigma_*(f,b) \ .$$

Conversely, a normal bundle map $(f,b):M \longrightarrow X$ determines a normal map in the sense of Wall [5] $(f,B):M \longrightarrow X$ with

$$B : \epsilon_M = \tau_M \oplus \nu_M \xrightarrow{1 \oplus b} \tau_M \oplus f^*\nu_X \ .$$

From now on we shall not distinguish between the two formulations of normal bundle maps.

§4. Intersections and self-intersections

We have used the quadratic construction Ψ of §2 to define the quadratic kernel $\sigma_*(f,b) = \sigma_*(f,F)$ of a normal bundle map $(f,b):M \longrightarrow X$, using the equivariant S-duality of §3 to obtain a geometric Umkehr map $F:\Sigma^p\tilde{X}_+ \longrightarrow \Sigma^p\tilde{M}_+$. We shall now describe the self-intersections of an immersion $g:S^r \looparrowright M^n$ in terms of the quadratic construction, allowing the identification of $\sigma_*(f,b)$ for highly-connected f with the geometrically defined surgery obstruction kernels of §§5,6 of Wall [5]. (Our methods apply to any immersion of manifolds, as discussed in §18).

Define as follows abelian group morphisms

$$\begin{cases} j : \pi_m(BSO(q)) = \pi_{m-1}(SO(q)) \longrightarrow Q^{m+q}(S^q\mathbb{Z}) = \hat{H}^{q-m}(\mathbb{Z}_2;\mathbb{Z},(-)^q) \\ j : \pi_{m+1}(BSO(p+q),BSO(q)) = \pi_m(SO(p+q)/SO(q)) \\ \qquad \longrightarrow Q^{[0,p-1]}_{m+q}(S^q\mathbb{Z}) (=H_{m-q}(\mathbb{Z}_2;\mathbb{Z},(-)^q) \text{ if } m-q<p-1 \neq 0). \end{cases}$$

Given a q-plane bundle $\alpha:S^m \longrightarrow BSO(q)$ over S^m apply Lemma 1 of Milnor [3] to identify the Thom space $T(\alpha)$ with the mapping cone of $J(\alpha) \in \pi_{m+q-1}(S^q)$

$$T(\alpha) = S^q \cup_{J(\alpha)} e^{m+q} \ .$$

Applying the symmetric construction and the symmetric Wu class

$$\mathbb{Z} = \dot{H}_{m+q}(T(\alpha)) \xrightarrow{\dot{\varphi}_{T(\alpha)}} Q^{m+q}(\dot{C}(T(\alpha))) \xrightarrow{v_m} \mathrm{Hom}_{\mathbb{Z}}(\dot{H}^q(T(\alpha)),Q^{m+q}(S^q\mathbb{Z}))$$

set

$$j(\alpha) = v_m(\dot{\varphi}_{T(\alpha)}(1))(1) \in Q^{m+q}(S^q\mathbb{Z}) \qquad (\dot{H}^q(T(\alpha)) = \mathbb{Z}) \ .$$

Furthermore, given a trivialization $\beta:D^{m+1} \longrightarrow BSO(p+q)$ of $\alpha \oplus \varepsilon^p:S^m \longrightarrow BSO(p+q)$ there is defined a $(p+q)$-plane bundle isomorphism $\beta:\varepsilon^{p+q} \to \alpha \oplus \varepsilon^p$ over $1:S^m \longrightarrow S^m$ inducing a homeomorphism of Thom spaces

$$T(\beta) : T(\varepsilon^{p+q}) = S^{p+q} \vee S^{m+p+q} \longrightarrow T(\alpha \oplus \varepsilon^p) = \Sigma^p T(\alpha) \ .$$

The composite

$$I(\beta) : \Sigma^p(S^{m+q}) = S^{m+p+q} \xrightarrow{\text{inclusion}} S^{p+q} \vee S^{m+p+q} \xrightarrow{T(\beta)} \Sigma^p T(\alpha)$$

represents the generator $1 \in \dot{H}_{m+p+q}(\Sigma^p T(\alpha)) = \mathbb{Z}$. Applying the quadratic construction and the quadratic Wu class

$$\mathbb{Z} = \dot{H}_{m+q}(S^{m+q}) \xrightarrow{\dot{\psi}_{I(\beta)}} Q^{[0,p-1]}_{m+q}(\dot{C}(T(\alpha))) \xrightarrow{v^m} \mathrm{Hom}_{\mathbb{Z}}(\dot{H}^q(T(\alpha)),Q^{[0,p-1]}_{m+q}(S^q\mathbb{Z}))$$

set

$$j(\beta,\alpha) = v^m(\dot{\psi}_{I(\beta)}(1))(1) \in Q^{[0,p-1]}_{m+q}(S^q\mathbb{Z}) \ .$$

Applying the symmetric Wu class operation to the relation

$$\dot{\varphi}_{T(\alpha)}(1) - ((\Sigma^p)^{-1} I(\beta)\Sigma^p)^{\%}\dot{\varphi}_{S^{m+q}}[S^{m+q}] = (1+T)\dot{\psi}_{I(\beta)}(1) \in Q^{m+q}(\dot{C}(T(\alpha)))$$

given by Proposition 2.5 i) we have that

$$j(\alpha) \doteq (1+T)j(\beta,\alpha) \in Q^{m+q}(S^q\mathbb{Z}) .$$

Note that in the case $m = q = 2k$

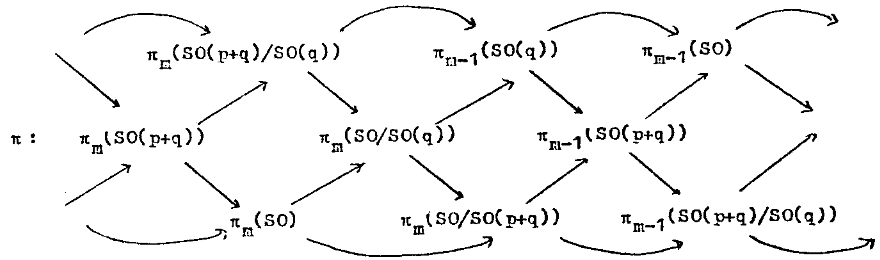
$$j : \pi_{2k}(BSO(2k)) \longrightarrow Q^{4k}(S^{2k}\mathbb{Z}) = \mathbb{Z}$$

is just the Euler number $(= \pi_{2k}(B\dot{S}O(2k)) \xrightarrow{J} \pi_{4k-1}(S^{2k}) \xrightarrow{\text{Hopf invariant}} \mathbb{Z})$.

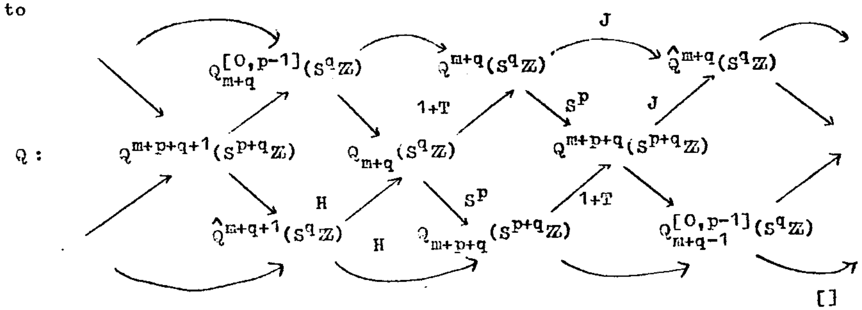
Proposition 4.1 There is defined in a natural way a morphism of commutative braids of exact sequences of abelian groups

$$j : \pi \longrightarrow Q$$

from



to



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(Q is a particular case of the braid of Proposition 1.3. It is possible to factorise j as $j:\pi \xrightarrow{J} \Pi \xrightarrow{H} Q$, with Π defined exactly as π but using SG instead of SO). For example, $j:\pi_m(SO/SO(q)) \longrightarrow Q_{m+q}(S^q\mathbb{Z})$ is an isomorphism for $m \leqslant q$.

Let M^n be an n-manifold, which we shall take to be compact, smooth and closed, for simplicity. As usual, give the group ring $\mathbb{Z}[\pi_1(M)]$ the $w(M)$-twisted involution. Let $S_r(M^n)$ $(r \geqslant 2)$ be the $\mathbb{Z}[\pi_1(M)]$-module of regular homotopy classes of oriented immersions $g:S^r \looparrowright M^n$ with a preferred lift $\tilde{g}:\tilde{S}^r = \pi_1(M)\times S^r \looparrowright \tilde{M}$ to the universal cover $\tilde{M}$ of M^n, where addition is by connected sum. Given such an immersion g define a $\mathbb{Z}[\pi_1(M)]$-module chain map

$$g^! : C(\tilde{M}) \xrightarrow{([M]\cap -)^{-1}} C(\tilde{M})^{n-*} \xrightarrow{\tilde{g}^*} C(\tilde{S}^r)^{n-*} \xrightarrow{([S^r]\cap -)} S^{n-r}C(\tilde{S}^r) \xrightarrow{(U_{\nu_g}\cap -)} \dot{C}(T\pi(\nu_g)) \quad ,$$

where $T\pi(\nu_g) = \bigvee_{\pi} T(\nu_g)$ is the Thom space of the normal bundle $\nu_g:S^r \longrightarrow BSO(n-r)$, $U_{\nu_g} \in \dot{H}^{n-r}(T(\nu_g))$ is the Thom class of ν_g, and $\pi = \pi_1(M)$.

The symmetric self-intersection of an immersion $g:S^r \looparrowright M^n$ is the $\mathbb{Z}_2$-cohomology class

$$\lambda(g) = v_r(\varphi_{\tilde{M}}[M])(x) \in H^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r})$$

obtained by evaluating the composite

$$H_n^{\pi}(\tilde{M};{}^w\mathbb{Z}) \xrightarrow{\varphi_{\tilde{M}}} Q^n(C(\tilde{M})) \xrightarrow{v_r} \mathrm{Hom}_{\mathbb{Z}[\pi]}({}^wH^{n-r}(\tilde{M}),H^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r}))$$

where $x = g^{!*}(U_{\nu_g}) \in {}^wH^{n-r}(\tilde{M})$ is the Poincaré dual of $\tilde{g}_*[S^r] \in H_r(\tilde{M})$, and $w = w(M)$. (Thus for a fixed M^n the class $\lambda(g)$ depends only on $\tilde{g}_*[S^r] \in H_r(\tilde{M})$). In the case $n = 2r$ $\lambda(g)$ can be identified with the evaluation $\lambda(g,g)$ of the geometric intersection pairing

$$\lambda : S_r(M^{2r}) \times S_r(M^{2r}) \longrightarrow \mathbb{Z}[\pi_1(M)] .$$

By Proposition 2.3, the mod 2 reduction of $\lambda(g)$ can be expressed as

$$\lambda(g) = \langle Sq^r(x),[M]\rangle \in \mathbb{Z}_2 \quad (x \in H^{n-r}(M;\mathbb{Z}_2)) .$$

Given an immersion $g:S^r \looparrowright M^n$ and a non-negative integer $p \geq 2r-n+1$ it is possible to deform the immersion $g\times 1:S^r \looparrowright M^n \times D^p$ by a regular homotopy to an embedding $g':S^r \hookrightarrow \text{interior}(M^n \times D^p)$ with normal bundle

$$\nu_{g'} = \nu_g \oplus \varepsilon^p : S^r \longrightarrow BSO(n-r+p) \ .$$

Let E be a closed tubular neighbourhood of $g'(S^r)$ in $M^n \times D^p$, with induced cover $\widetilde{E} = \pi \times E \subset \widetilde{M} \times D^p$. The π-map

$$G : \Sigma^p \widetilde{M}_+ = \widetilde{M}\times D^p/\widetilde{M}\times S^{p-1} \xrightarrow{\text{collapse}} \widetilde{M}\times D^p/\overline{\widetilde{M}\times D^p - \widetilde{E}} = \widetilde{E}/\partial\widetilde{E} = T\pi(\nu_{g'}) = \Sigma^p T\pi(\nu_g)$$

induces $g^!:C(\widetilde{M}) \longrightarrow \dot{C}(T\pi(\nu_g))$ on the chain level.

The <u>quadratic self-intersection</u> of an immersion $g:S^r \looparrowright M^n$ is the $\mathbb{Z}_2$-homology class

$$\mu(g) = -v^r(\dot{\psi}_G[M])(U_{\nu_g}) \in H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r})$$

obtained by evaluating the composite

$$H_n^\pi(\widetilde{M};{}^w\mathbb{Z}) \xrightarrow{\dot{\psi}_G} Q_n(\dot{C}(T\pi(\nu_g))) \xrightarrow{v^r} \text{Hom}_{\mathbb{Z}[\pi]}({}^w\dot{H}^{n-r}(T\pi(\nu_g)),H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r})).$$

In the case $n = 2r$ $\mu(g)$ will be identified with the geometric self-intersection of g (in Proposition 4.2 below). By Proposition 2.6, the mod 2 reduction of $\mu(g)$ can be expressed as

$$\mu(g) = \langle Sq_h^{r+1}(\Sigma^p \iota),\Sigma^p[M]\rangle \in \mathbb{Z}_2$$

$$(h = (\Sigma^p U_{\nu_g})G - \Sigma^p(x) \in [\Sigma^p M_+,\Sigma^p K(\mathbb{Z}_2,n-r)] \ ,$$

$$(\iota = \text{generator} \in \dot{H}^{n-r}(K(\mathbb{Z}_2,n-r);\mathbb{Z}_2) = \mathbb{Z}_2) \ .$$

<u>Proposition 4.2</u> The symmetric and quadratic self-intersections define functions

$$\lambda : S_r(M^n) \longrightarrow H^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r}) \ ; \ (g:S^r \looparrowright M^n) \longmapsto \lambda(g)$$

$$\mu : S_r(M^n) \longrightarrow H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r}) \ ; \ (g:S^r \looparrowright M^n) \longmapsto \mu(g)$$

such that

i) $\lambda(ag) = a\lambda(g)\bar{a}$, $\mu(ag) = a\mu(g)\bar{a}$ $(a \in \mathbb{Z}[\pi_1(M)], g \in S_r(M^n))$

ii) $\lambda(g) = (j(\nu_g),0) + (1+T)\mu(g) \in H^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r})$

$$= H^{n-2r}(\mathbb{Z}_2;\mathbb{Z},(-)^{n-r}) \oplus H^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)]/\mathbb{Z},(-)^{n-r})$$

iii) $\mu(g_1+g_2)-\mu(g_1)-\mu(g_2) = \begin{cases} [\lambda(g_1,g_2)] & \text{if } n = 2r \\ 0 & \text{otherwise} \end{cases}$ $(g_1,g_2 \in S_r(M^n))$

iv) if the class $g \in S_r(M^n)$ contains an embedding then $\mu(g) = 0$, and if it contains a framed embedding then also $\lambda(g) = 0$

v) if $n = 2r \geq 6$ and $\mu(g) = 0$ then the class $g \in S_r(M^{2r})$ contains an embedding.

<u>Proof</u>: i) By construction.

ii) Apply the symmetric Wu class v_r to the relation given by Proposition 2.5 i)

$$g^{!\%}\varphi_{\widetilde{M}}[M] = \dot{\varphi}_{T\pi(\nu_g)} g_*^![M] - (1+T)\dot{\psi}_G[M] \in Q^n(\dot{C}(T\pi(\nu_g))) \ .$$

iii) The quadratic self-intersection $\mu(g_1+g_2)$ of the connected sum g_1+g_2 of immersions $g_1,g_2:S^r \looparrowright M^n$ is given by

$$\mu(g_1+g_2) = -v^r(\dot{\psi}_G[M])(U_{\nu_{g_1}} \oplus U_{\nu_{g_2}}) \in H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^{n-r}) \ ,$$

with

$$G = G_1 \vee G_2 : \Sigma^p \widetilde{M}_+ \longrightarrow \Sigma^p(T\pi(\nu_{g_1}) \vee T\pi(\nu_{g_2})) \ .$$

There is a natural identification of $\mathbb{Z}[\pi]$-module chain complexes

$$\dot{C}(T\pi(\nu_{g_1}) \vee T\pi(\nu_{g_2})) = \dot{C}(T\pi(\nu_{g_1})) \oplus \dot{C}(T\pi(\nu_{g_2})) \ ,$$

and Proposition 1.4 i) allows us to express $\dot{\psi}_G$ as

$$\dot{\Psi}_G = \begin{pmatrix} \dot{\Psi}_{G_1} \\ \dot{\Psi}_{G_2} \\ -(g_1^!\otimes g_2^!)\Delta_0 \end{pmatrix} : H_n^\pi(\tilde{M};{}^w\mathbb{Z}) \longrightarrow Q_n(\dot{C}(T\pi(\nu_{g_1}))\oplus\dot{C}(T\pi(\nu_{g_2})))$$

$$= Q_n(\dot{C}(T\pi(\nu_{g_1})))\oplus Q_n(\dot{C}(T\pi(\nu_{g_2})))\oplus H_n(\dot{C}(T\pi(\nu_{g_1}))^t\otimes_{\mathbb{Z}[\pi]}\dot{C}(T\pi(\nu_{g_2})))$$

with $(g_1^!\otimes g_2^!)\Delta_0$ the composite

$$(g_1^!\otimes g_2^!)\Delta_0 : H_n^\pi(\tilde{M};{}^w\mathbb{Z}) \xrightarrow{\Delta_0} H_n(C(\tilde{M})^t\otimes_{\mathbb{Z}[\pi]}C(\tilde{M}))$$
$$\xrightarrow{g_1^!\otimes g_2^!} H_n(\dot{C}(T\pi(\nu_{g_1}))^t\otimes_{\mathbb{Z}[\pi]}\dot{C}(T\pi(\nu_{g_2}))) .$$

Now apply the rth quadratic Wu class v^r to the identity

$$\dot{\Psi}_G[M] = (\dot{\Psi}_{G_1}[M],\dot{\Psi}_{G_2}[M],-(g_1^!\otimes g_2^!)\Delta_0)\in Q_n(\dot{C}(T\pi(\nu_{g_1})\oplus T\pi(\nu_{g_2}))) .$$

iv) By definition $\dot{\Psi}_G$ is a composite

$$\dot{\Psi}_G : H_n^\pi(\tilde{M};{}^w\mathbb{Z}) \longrightarrow Q_n^{[0,p-1]}(\dot{C}(T\pi(\nu_g))) \longrightarrow Q_n(\dot{C}(T\pi(\nu_g))) ,$$

and the middle group is 0 if $p = 0$.

v) Let $\hat{\mu}(g)\in H_0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^r)$ be the geometric self-intersection of an immersion $g:S^r \looparrowright M^{2r}$ $(r\geqslant 2)$, as defined in Theorem 5.2 of Wall [5]. It was proved there that

$$\hat{\mu}(g_1+g_2) - \hat{\mu}(g_1) - \hat{\mu}(g_2) = [\lambda(g_1,g_2)]\in H_0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^r) ,$$

and that for $r\geqslant 3$ $\hat{\mu}(g)$ is the sole obstruction to deforming g to an embedding.

We shall prove that $\mu(g) = \hat{\mu}(g)$ (for $r\geqslant 3$) by a generalization of the trick used in the proof of Theorem IV.4.1 of Browder [2]. Lift $\hat{\mu}(g)$ to some element $a\in\mathbb{Z}[\pi_1(M)]$, and let $g':S^r\looparrowright M'^{2r} = M^{2r}\#(S^r\times S^r)$ be an immersion representing the homology class

$$\tilde{g}'_*[S^r] = (0,-a,1)\in H_r(\tilde{M}') = H_r(\tilde{M})\oplus\mathbb{Z}[\pi_1(M)]\oplus\mathbb{Z}[\pi_1(M)] .$$

The immersion $g\#0 : S^r\looparrowright M'^{2r}$ represents the homology class

$$(\tilde{g}\#0)_*[S^r] = (\tilde{g}_*[S^r],0,0)\in H_r(\tilde{M}') = H_r(\tilde{M})\oplus\mathbb{Z}[\pi_1(M)]\oplus\mathbb{Z}[\pi_1(M)] .$$

Define an immersion

$$g'' = (g\#0)+g' : S^r\looparrowright M'^{2r} ,$$

and apply the sum formulae for μ and $\hat{\mu}$ to obtain

$$\mu(g'') = \mu(g\#0) + \mu(g') = \mu(g) - [a] = \mu(g) - \hat{\mu}(g)\in H_0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^r)$$
$$\hat{\mu}(g'') = \hat{\mu}(g\#0) + \hat{\mu}(g') = \hat{\mu}(g) - [a] = 0\in H_0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^r) .$$

Thus g" can be deformed to an embedding, and $\mu(g'') = 0$ by iv).

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The relation of Proposition 4.2 ii) for $n = 2r$

$$\lambda(g) = (j(\nu_g),0) + (1+T)\mu(g)\in H^0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(M)],(-)^r)$$

is precisely the relation of Theorem 5.2 iii) of Wall [5], with $j(\nu_g) = \chi(\nu_g)\in H^0(\mathbb{Z}_2;\mathbb{Z},(-)^r)$ the Euler number of $\nu_g\in\pi_r(BSO(r))$.

<u>Proposition 4.3</u> Let $\begin{cases} f:M\longrightarrow X \\ (f,b):M\longrightarrow X \end{cases}$ be a $\begin{cases}\text{degree 1} \\ \text{normal bundle}\end{cases}$ map from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X. Let $g:S^r\looparrowright M$ be an immersion with an oriented normal bundle $\nu_g:S^r\longrightarrow BSO(n-r)$ and a null-homotopy $\begin{cases} h:D^{r+1}\longrightarrow X \text{ of } fg:S^r\longrightarrow X \\ \text{and let } \nu_h:D^{r+1}\longrightarrow BSO \text{ be the stable trivialization} \end{cases}$ $\begin{cases} \\ \text{of } \nu_g:S^r\longrightarrow BSO(n-r) \text{ determined by } b:\nu_M\longrightarrow\nu_X \end{cases}$. The rth $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Wu class of the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel $\begin{cases}\sigma^*(f) = (C(f^!),\varphi = e^{\%}\varphi_{\tilde{M}}[M]\in Q^n(C(f^!))) \\ \sigma_*(f,b) = (C(f^!),\psi = e_{\%}\psi_F[X]\in Q_n(C(f^!))) \end{cases}$

$$\begin{cases} v_r(\varphi) : H^{n-r}(C(f^!)) = K^{n-r}(M) \longrightarrow H^{n-2r}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(X)], (-)^{n-r}) \\ v^r(\psi) : H^{n-r}(C(f^!)) = K^{n-r}(M) \longrightarrow H_{2r-n}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(X)], (-)^{n-r}) \end{cases}$$

sends the Poincaré dual $x \in K^{n-r}(M)$ of the Hurewicz image in $H_{r+1}(\tilde{f}) = K_r(M)$ of $(h,g) \in \pi_{r+1}(f) = \pi_{r+1}(\tilde{f})$ to

$$\begin{cases} v_r(\varphi)(x) = (j(\nu_g),0) + (1+T)\mu(g) \in H^{n-2r}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(X)], (-)^{n-r}) \\ v^r(\psi)(x) = (j(\nu_h,\nu_g),0) + \mu(g) \in H_{2r-n}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(X)], (-)^{n-r}) \ . \end{cases}$$

<u>Proof</u>: The expression for $v_r(\varphi)(x)$ is immediate from Proposition 4.2 i), so only the normal bundle case need be considered. The commutative diagram of maps of spaces

$$\begin{array}{ccc} S^r & \xrightarrow{\ g\ } & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ D^{r+1} & \xrightarrow{\ h\ } & X \end{array}$$

is covered by a commutative diagram of bundle maps

$$\begin{array}{ccc} g^*\nu_M & \longrightarrow & \nu_M \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle b} \\ h^*\nu_X & \longrightarrow & \nu_X \end{array}$$

with $h^*\nu_X$ a trivial bundle and $c: g^*\nu_M \longrightarrow h^*\nu_X$ giving rise to $\nu_h : D^{r+1} \longrightarrow BSO$.
There is induced a commutative diagram of Thom π-spaces and π-maps ($\pi = \pi_1(X)$)

$$\begin{array}{ccc} T\pi(g^*\nu_M) & \longrightarrow & T\pi(\nu_M) \\ {\scriptstyle T\pi(c)}\downarrow & & \downarrow{\scriptstyle T\pi(b)} \\ T\pi(h^*\nu_X) & \longrightarrow & T\pi(\nu_X) \end{array}$$

whose $S\pi$-dual is a π-homotopy commutative diagram of π-maps

$$\begin{array}{ccc} \Sigma^p T\pi(\nu_g) & \xleftarrow{\ G\ } & \Sigma^p \tilde{M}_+ \\ {\scriptstyle I(\nu_h)}\uparrow & & \uparrow{\scriptstyle F} \\ \Sigma^p(\bigvee_\pi S^n) & \xleftarrow{\ H\ } & \Sigma^p \tilde{X}_+ \end{array}$$

for $p \geq 0$ sufficiently large. Applying the sum formula for the quadratic construction of Proposition 2.5 iii) we have

$$\begin{aligned} \varepsilon_{\%}^! \psi_F[X] + \psi_G[M] &= \psi_{GF}[X] \\ &= \psi_{I(\nu_h)H}[X] = \psi_{I(\nu_h)}(1) + I(\nu_h)_* \psi_H[X] \in Q_n(\dot{C}(T\pi(\nu_g))) \ . \end{aligned}$$

The disc theorem for geometric Poincaré complexes (Theorem 2.4 of Wall [4]) provides a homotopy equivalence

$$X \longrightarrow Y \cup_k e^n$$

with Y a homologically $(n-1)$-dimensional complex and $k: S^{n-1} \longrightarrow Y$ some map. Passing to the universal covers, adjoining basepoints and collapsing $\tilde{Y}$ there is obtained a π-map

$$H : \tilde{X}_+ \longrightarrow (\tilde{Y} \cup_{\tilde{k}} (\pi \times e^n))_+ \longrightarrow \bigvee_\pi S^n$$

representing the $S\pi$-dual of $T\pi(h^*\nu_X) \longrightarrow T\pi(\nu_X)$, so that

$$\psi_H = 0 : H_n^\pi(\tilde{X}; {}^w\mathbb{Z}) \longrightarrow Q_n(\dot{C}(\bigvee_\pi S^n)) \qquad (w = w(X)).$$

Applying the rth quadratic Wu class v^r to

$$\varepsilon_{\%}^! \psi_F[X] = \psi_{I(\nu_h)}(1) - \psi_G[M] \in Q_n(\dot{C}(T\pi(\nu_g)))$$

we obtain the desired expression for $v^r(\psi)(x)$.

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At this point it is instructive to compare the approaches taken by Wall [5] and Browder [2] to the problem of performing framed surgery on an element $\alpha \in \pi_{r+1}(f)$ for some n-dimensional normal bundle map $(f,b): M \longrightarrow X$. Theorem 1.1 of Wall [5] establishes that for $r \leq n-2$ every $\alpha \in \pi_{r+1}(f)$ determines a regular homotopy class of framed immersions $g: S^r \looparrowright M$ together with a prescribed null-homotopy $h: D^{r+1} \longrightarrow X$ of $fg: S^r \longrightarrow X$, such that $(\nu_h, \nu_g) = 0 \in \pi_{r+1}(BSO, BSO(n-r))$. Surgery on α is possible if and only if this class contains an embedding, so that on the chain level the surgery obstruction is $v^r(\psi)(x) = \mu(g) \in H_{2r-n}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(X)], (-)^{n-r})$. On the other hand, Theorem IV.1.6 of Browder [2] assumes that $\alpha \in \pi_{r+1}(f)$ is already represented by an embedding $g: S^r \hookrightarrow M$ with a null-homotopy $h: D^{r+1} \longrightarrow X$ of $fg: S^r \longrightarrow X$, so that $\mu(g) = 0$. Surgery on α is possible if and only if $(\nu_h, \nu_g) = 0 \in \pi_{r+1}(BSO, BSO(n-r))$ $(= \pi_r(SO/SO(n-r)) = \pi_r(V_{k,n-r})$, k large), so that the chain level surgery obstruction is $v^r(\psi)(x) = (j(\nu_h,\nu_g),0)$. In Proposition 7.9 i) below we shall interpret the ε-quadratic Wu class $v^r(\psi)(x) \in H_{2r-n}(\mathbb{Z}_2; A, (-)^{n-r}\varepsilon)$ associated to an abstract n-dimensional ε-quadratic Poincaré complex over A $(C, \psi \in Q_n(C,\varepsilon))$ as the obstruction to performing algebraic surgery on $x \in H_r(C)$.

Given an $(i-1)$-connected $\begin{cases} 2i- \\ 2i+1- \end{cases}$ dimensional normal bundle map for $\begin{cases} i \geqslant 3 \\ i \geqslant 2 \end{cases}$ $(f,b):M \longrightarrow X$ let

$$\theta(f,b) = \begin{cases} (K_i(M),\lambda,\mu) \\ (H_{(-)^i}(K_{i+1}(U,\partial U));K_{i+1}(U,\partial U),K_{i+1}(M_0,\partial U)) \end{cases}$$

be the non-singular $(-)^i$quadratic $\begin{cases} \text{form} \\ \text{formation} \end{cases}$ over $\mathbb{Z}[\pi_1(X)]$ with the $w(X)$-twisted involution obtained in $\begin{cases} \S 5 \\ \S 6 \end{cases}$ of Wall [5] as the surgery obstruction kernel, using geometrically defined intersection and self-intersection forms. The odd-dimensional terminology involves the union U of disjoint framed embeddings $S^i \times D^{i+1} \subset M$ such that the images $f(S^i \times D^{i+1}) \subset X$ are contractible, and such that the corresponding elements of $K_i(M)$ are a set of generators, with $M_0 = \overline{M \backslash U} \subset M$. The quadratic kernel $\sigma_*(f,b)$ is the i-fold skew-suspension of a $\begin{cases} 0- \\ 1- \end{cases}$ dimensional $(-)^i$quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$

$$\sigma_*(f,b) = \bar{S}^i \sigma_i(f,b)$$

(as in Proposition 2.12), and $\sigma_i(f,b)$ can be regarded as a non-singular $(-)^i$quadratic $\begin{cases} \text{form} \\ \text{formation} \end{cases}$ by Proposition $\begin{cases} 1.5 \\ 1.8 \end{cases}$.

<u>Proposition 4.4</u> The surgery obstruction kernel of a highly-connected n-dimensional normal bundle map $(f,b):M \longrightarrow X$ agrees with the quadratic kernel defined using a geometric Umkehr map $F \in \{\tilde{X}_+,\tilde{M}_+\}_\pi$ $(\pi = \pi_1(X))$

$$\theta(f,b) = \sigma_i(f,b) \quad (n = 2i \text{ or } 2i+1 \geqslant 5).$$

<u>Proof</u>: Consider first the case $n = 2i$. Now $C(f^!)$ is given up to chain equivalence by

$$C(f^!) : \ldots \longrightarrow 0 \longrightarrow K_i(M) \longrightarrow 0 \longrightarrow \ldots,$$

and the quadratic kernel is given by

$$\sigma_*(f,b) = (C(f^!),\psi = e_\%\psi_F[X] \in Q_{2i}(C(f^!)) = \operatorname{coker}(1-T_{(-)^i}:\operatorname{Hom}_{\mathbb{Z}[\pi]}(K_i(M),K_i(M)^*) \longrightarrow \operatorname{Hom}_{\mathbb{Z}[\pi]}(K_i(M),K_i(M)^*)),$$

identifying $K_i(M) = K^i(M)$ by Poincaré duality. By Theorem 1.1 of Wall [5]

every element $x \in K_i(M)$ is represented by a framed immersion $g:S^i \looparrowright M^{2i}$ together with a null-homotopy $h:D^{i+1} \longrightarrow X$ of $fg:S^i \longrightarrow X$, and Propositions 4.2,4.3 allow the identification

$$\psi(x)(x) = \mu(g) \in H_0(\mathbb{Z}_2;\mathbb{Z}[\pi_1(X)],(-)^i) .$$

Thus $\theta(f,b) = \sigma_i(f,b)$ if $n = 2i$.

In the case $n = 2i+1$ we have that up to chain equivalence

$$C(f^!) : \ldots \longrightarrow 0 \longrightarrow K_{i+1}(M,U) \longrightarrow K_i(U) \longrightarrow 0 \longrightarrow \ldots ,$$

so that $\sigma_i(f,b)$ is a non-singular $(-)^i$quadratic formation over $\mathbb{Z}[\pi_1(X)]$

$$\sigma_i(f,b) = (H_{(-)^i}(K_i(U)^*);K_i(U)^*,K_{i+1}(M,U)) .$$

Identifying $K_{i+1}(U,\partial U) = K^i(U) = K_i(U)^*$ by Poincaré duality and the universal coefficient theorem we can write the inclusion of the lagrangian

$$K_{i+1}(M,U) \longrightarrow K_i(U)^* \oplus K_i(U)$$

as the map

$$K_{i+1}(M,U) = K_{i+1}(M_0,\partial U) \longrightarrow K_i(\partial U) = K_{i+1}(U,\partial U) \oplus K_{i+1}(U,\partial U)^*$$

appearing in the definition of $\theta(f,b)$. Thus $\theta(f,b) = \sigma_i(f,b)$ if $n = 2i+1$.

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§5. Algebraic cobordism

We define an equivalence relation on algebraic Poincaré complexes modelled on the cobordism of manifolds, and also called cobordism. In §6 we show that a $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ bordism of $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ maps of geometric Poincaré complexes determines an algebraic cobordism of the kernel $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complexes. The cobordism classes of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes over A define an abelian group $\begin{cases}L^n(A,\varepsilon)\\ L_n(A,\varepsilon)\end{cases}$ $(n\geqslant 0)$ under the direct sum $\oplus$. In §7 we shall analyze algebraic cobordism using an algebraic surgery technique modelled on the surgery of manifolds. The quadratic L-groups will be identified with the surgery obstruction groups of Wall [5]

$$L_n(\mathbb{Z}[\pi],1) = L_n(\pi)\ .$$

In §8 the cobordism class of the quadratic kernel of a normal bundle map $(f,b):M \longrightarrow X$ will be identified with the surgery obstruction of Wall [5]

$$\sigma_*(f,b) = \theta(f,b) \in L_n(\mathbb{Z}[\pi_1(X)],1) = L_n(\pi_1(X))\ ,$$

and it will be shown that the chain level effect of a geometric surgery is an algebraic surgery.

Let A,ε be as in §1. Given an A-module chain map

$$f : C \longrightarrow D$$

define the <u>relative Q-groups</u>

$$\begin{cases} Q^{n+1}_{[i,j]}(f,\varepsilon) = H_{n+1}(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W[i,j],C(f^t\otimes_A f))) \\ Q_{n+1}^{[i,j]}(f,\varepsilon) = H_{n+1}(W[i,j]\otimes_{\mathbb{Z}[\mathbb{Z}_2]}C(f^t\otimes_A f))) \end{cases} \quad (-\infty\leqslant i\leqslant j\leqslant\infty,\ n\geqslant 0)$$

with $C(f^t\otimes_A f)$ the algebraic mapping cone of the $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f^t\otimes_A f:C^t\otimes_A C \longrightarrow D^t\otimes_A D$, taking $T\in\mathbb{Z}_2$ to act by the ε-transposition T_ε.

An element $\begin{cases}(\delta\varphi,\varphi)\in Q^{n+1}_{[i,j]}(f,\varepsilon)\\ (\delta\psi,\psi)\in Q_{n+1}^{[i,j]}(f,\varepsilon)\end{cases}$ is represented by a collection of chains

$$\begin{cases} \{(\delta\varphi,\varphi)_s = (\delta\varphi_s,\varphi_s)\in(D^t\otimes_A D)_{n+s+1}\oplus(C^t\otimes_A C)_{n+s}\,|\,i\leqslant s\leqslant j\} \\ \{(\delta\psi,\psi)_s = (\delta\psi_s,\psi_s)\in(D^t\otimes_A D)_{n-s+1}\oplus(C^t\otimes_A C)_{n-s}\,|\,i\leqslant s\leqslant j\} \end{cases}$$

such that

$$\begin{cases} d(\delta\varphi,\varphi)_s \equiv (d_{D^t\otimes_A D}(\delta\varphi_s) + (-)^{n+s}(\delta\varphi_{s-1} + (-)^s T_\varepsilon\delta\varphi_{s-1}) + (-)^n(f^t\otimes_A f)(\varphi_s), \\ \qquad d_{C^t\otimes_A C}(\varphi_s) + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon\varphi_{s-1})) \\ \qquad = 0\in(D^t\otimes_A D)_{n+s}\oplus(C^t\otimes_A C)_{n+s-1} \quad (i\leqslant s\leqslant j,\ \delta\varphi_{i-1}=0,\ \varphi_{i-1}=0) \\ d(\delta\psi,\psi)_s \equiv (d_{D^t\otimes_A D}(\delta\psi_s) + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T_\varepsilon\delta\psi_{s+1}) + (-)^n(f^t\otimes_A f)(\psi_s), \\ \qquad d_{C^t\otimes_A C}(\psi_s) + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T_\varepsilon\psi_{s+1})) \\ \qquad = 0\in(D^t\otimes_A D)_{n-s}\oplus(C^t\otimes_A C)_{n-s-1} \quad (i\leqslant s\leqslant j,\ \delta\psi_{j+1}=0,\ \psi_{j+1}=0)\ . \end{cases}$$

For $\varepsilon = 1\in A$ we shall write $\begin{cases}Q^{n+1}_{[i,j]}(f,1) = Q^{n+1}_{[i,j]}(f)\\ Q_{n+1}^{[i,j]}(f,1) = Q_{n+1}^{[i,j]}(f)\end{cases}$.

<u>Proposition 5.1</u> For any A-module chain map $f:C\longrightarrow D$ there is defined a long exact sequence of Q-groups

$$\begin{cases} \dots\longrightarrow Q^{n+1}_{[i,j]}(f,\varepsilon)\longrightarrow Q^n_{[i,j]}(C,\varepsilon)\xrightarrow{f^\%} Q^n_{[i,j]}(D,\varepsilon)\longrightarrow Q^n_{[i,j]}(f,\varepsilon)\longrightarrow\dots \\ \dots\longrightarrow Q_{n+1}^{[i,j]}(f,\varepsilon)\longrightarrow Q_n^{[i,j]}(C,\varepsilon)\xrightarrow{f_\%} Q_n^{[i,j]}(D,\varepsilon)\longrightarrow Q_n^{[i,j]}(f,\varepsilon)\longrightarrow\dots\ , \end{cases}$$

with

$$\begin{cases} Q^{n+1}_{[i,j]}(f,\varepsilon)\longrightarrow Q^n_{[i,j]}(C,\varepsilon);(\delta\varphi,\varphi)\longmapsto\varphi\ ,\ Q^n_{[i,j]}(D,\varepsilon)\longrightarrow Q^n_{[i,j]}(f,\varepsilon);\delta\varphi\longmapsto(\delta\varphi,0) \\ Q_{n+1}^{[i,j]}(f,\varepsilon)\longrightarrow Q_n^{[i,j]}(C,\varepsilon);(\delta\psi,\psi)\longmapsto\psi\ ,\ Q_n^{[i,j]}(D,\varepsilon)\longrightarrow Q_n^{[i,j]}(f,\varepsilon);\delta\psi\longmapsto(\delta\psi,0)\ . \end{cases}$$

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An (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ pair over A $\left\{\begin{array}{l}(f:C\longrightarrow D,(\delta\varphi,\varphi))\\ (f:C\longrightarrow D,(\delta\psi,\psi))\end{array}\right.$ $(n\geqslant 0)$

is a chain map $f:C\longrightarrow D$ from an n-dimensional A-module chain complex C to an (n+1)-dimensional A-module chain complex D, together with a relative $\left\{\begin{array}{l}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{array}\right.$ class $\left\{\begin{array}{l}(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon) = Q^{n+1}_{[0,\infty]}(f,\varepsilon)\\ (\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon) = Q^{[0,\infty]}_{n+1}(f,\varepsilon)\end{array}\right.$, and it is Poincaré if the relative homology class $\left\{\begin{array}{l}(\delta\varphi_0,\varphi_0)\in H_{n+1}(f\overset{t}{\otimes}_A f)\\ ((1+T_\varepsilon)\delta\psi_0,(1+T_\varepsilon)\psi_0)\in H_{n+1}(f\overset{t}{\otimes}_A f)\end{array}\right.$ induces A-module isomorphisms

$$H^r(D,C)\equiv H^r(f)\longrightarrow H_{n+1-r}(D)\qquad (0\leqslant r\leqslant n+1)$$

(Poincaré-Lefschetz duality) via the slant product

$$\backslash : H^r(f)\otimes_{\mathbb{Z}}H_{n+1}(f\overset{t}{\otimes}_A f)\longrightarrow H_{n+1-r}(D)\ ;\ (g,h)\otimes(u\otimes v,x\otimes y)\longmapsto \overline{g(u)}v+\overline{h(x)}f(y)$$
$$((g,h)\in D^r\oplus C^{r-1}\ ,\ u\otimes v\in(D\overset{t}{\otimes}_A D)_{n+1},\ x\otimes y\in(C\overset{t}{\otimes}_A C)_n)\ .$$

The boundary of an (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré pair over A $\left\{\begin{array}{l}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{array}\right.$ is the n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complex over A $\left\{\begin{array}{l}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{array}\right.$. A cobordism of n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complexes $\left\{\begin{array}{l}(C,\varphi),(C',\varphi')\\ (C,\psi),(C',\psi')\end{array}\right.$ is an (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré pair with boundary $\left\{\begin{array}{l}(C,\varphi)\oplus(C',-\varphi')\\ (C,\psi)\oplus(C',-\psi')\end{array}\right.$, say $\left\{\begin{array}{l}((f\ f'):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ f'),\varepsilon))\\ ((f\ f'):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f\ f'),\varepsilon))\end{array}\right.$.

In Proposition 5.2 below we shall prove that cobordism is an equivalence relation on algebraic Poincaré complexes, such that the cobordism classes define abelian groups under the direct sum $\oplus$. The verification of the transitivity of cobordism requires the following algebraic glueing operation. A geometric cobordism gives rise to an algebraic cobordism, and the glueing of geometric cobordisms gives rise to the glueing of algebraic cobordisms.

Our notion of cobordism of symmetric Poincaré cobordism is a chain homotopy invariant version of the "algebraic Poincaré bordism" of Mishchenko [2].

Define the union of adjoining $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ cobordisms

$$\left\{\begin{array}{l}c = ((f_C\ f_{C'}):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f_C\ f_{C'}),\varepsilon))\\ c = ((f_C\ f_{C'}):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f_C\ f_{C'}),\varepsilon))\ ,\end{array}\right.$$
$$\left\{\begin{array}{l}c' = ((f'_{C'}\ f'_{C''}):C'\oplus C''\longrightarrow D',(\delta\varphi',\varphi'\oplus-\varphi'')\in Q^{n+1}((f'_{C'}\ f'_{C''}),\varepsilon))\\ c' = ((f'_{C'}\ f'_{C''}):C'\oplus C''\longrightarrow D',(\delta\psi',\psi'\oplus-\psi'')\in Q_{n+1}((f'_{C'}\ f'_{C''}),\varepsilon))\end{array}\right.$$

to be the $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ cobordism

$$\left\{\begin{array}{l}c\cup c' = ((f''_C\ f''_{C''}):C\oplus C''\longrightarrow D'',(\delta\varphi'',\varphi\oplus-\varphi'')\in Q^{n+1}((f''_C\ f''_{C''}),\varepsilon))\\ c\cup c' = ((f''_C\ f''_{C''}):C\oplus C''\longrightarrow D'',(\delta\psi'',\psi\oplus-\psi'')\in Q_{n+1}((f''_C\ f''_{C''}),\varepsilon))\end{array}\right.$$

given by

$$d_{D''} = \begin{pmatrix} d_D & (-)^{r-1}f_{C'} & 0\\ 0 & d_{C'} & 0\\ 0 & (-)^{r-1}f'_{C'} & d_{D'}\end{pmatrix}$$
$$: D''_r = D_r\oplus C'_{r-1}\oplus D'_r \longrightarrow D''_{r-1} = D_{r-1}\oplus C'_{r-2}\oplus D'_{r-1}\ ,$$

$$f''_C = \begin{pmatrix} f_C\\ 0\\ 0\end{pmatrix} : C_r\longrightarrow D''_r = D_r\oplus C'_{r-1}\oplus D'_r$$

$$f''_{C''} = \begin{pmatrix} 0\\ 0\\ f'_{C''}\end{pmatrix} : C''_r\longrightarrow D''_r = D_r\oplus C'_{r-1}\oplus D'_r$$

$$\left\{\begin{array}{l}\delta\varphi''_s = \begin{pmatrix}\delta\varphi_s & 0 & 0\\ (-)^{n-r}\varphi'_s f^*_{C'} & (-)^{n-r+s+1}T_\varepsilon\varphi'_{s-1} & 0\\ 0 & (-)^s f'_{C'}\varphi'_s & \delta\varphi'_s\end{pmatrix}\\ \quad : D''^{n-r+s+1} = D^{n-r+s+1}\oplus C'^{n-r+s}\oplus D'^{n-r+s+1}\longrightarrow D''_r = D_r\oplus C'_{r-1}\oplus D'_r\quad (s\geqslant 0)\\ \delta\psi''_s = \begin{pmatrix}\delta\psi_s & 0 & 0\\ (-)^{n-r}\psi'_s f^*_{C'} & (-)^{n-r-s}T_\varepsilon\psi'_{s+1} & 0\\ 0 & (-)^s f'_{C'}\psi'_s & \delta\psi'_s\end{pmatrix}\\ \quad : D''^{n-r-s+1} = D^{n-r-s+1}\oplus C'^{n-r-s}\oplus D'^{n-r-s+1}\longrightarrow D''_r = D_r\oplus C'_{r-1}\oplus D'_r\quad (s\geqslant 0).\end{array}\right.$$

We shall normally write

$$D'' = D\cup_{C'}D'\ ,\ \left\{\begin{array}{l}\delta\varphi'' = \delta\varphi\cup_{\varphi'}\delta\varphi'\\ \delta\psi'' = \delta\psi\cup_{\psi'}\delta\psi'\ .\end{array}\right.$$

(Several remarks about the union operation are in order:

i) Given A-module chain maps $(f_C\ f_{C'}):C\oplus C'\longrightarrow D$, $(f'_{C'}\ f'_{C''}):C'\oplus C''\longrightarrow D'$ let $\begin{cases} Q^{n+1} \\ Q_{n+1}\end{cases}$ be the relative groups appearing in the long exact sequence

$$\begin{cases} \dots\longrightarrow Q^{n+1}(C',\varepsilon)\longrightarrow Q^{n+1}\longrightarrow Q^{n+1}((f_C\ f_{C'}),\varepsilon)\oplus Q^{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)\longrightarrow Q^{n}(C',\varepsilon)\longrightarrow\dots \\ \dots\longrightarrow Q_{n+1}(C',\varepsilon)\longrightarrow Q_{n+1}\longrightarrow Q_{n+1}((f_C\ f_{C'}),\varepsilon)\oplus Q_{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)\longrightarrow Q_{n}(C',\varepsilon)\longrightarrow\dots \end{cases}$$

The definition of the union (as above) depends on a choice of lift of

$$\begin{cases} (\delta\varphi,\varphi\oplus-\varphi')\oplus(\delta\varphi',\varphi'\oplus-\varphi'')\in\ker(Q^{n+1}((f_C\ f_{C'}),\varepsilon)\oplus Q^{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)\longrightarrow Q^{n}(C',\varepsilon)) \\ (\delta\psi,\psi\oplus-\psi')\oplus(\delta\psi',\psi'\oplus-\psi'')\in\ker(Q_{n+1}((f_C\ f_{C'}),\varepsilon)\oplus Q_{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)\longrightarrow Q_{n}(C',\varepsilon)) \end{cases}$$

to some element of $\begin{cases} Q^{n+1} \\ Q_{n+1}\end{cases}$, and then evaluating the abelian group morphism

$$\begin{cases} Q^{n+1}\longrightarrow Q^{n+1}((f''_C\ f''_{C''}),\varepsilon) \\ Q_{n+1}\longrightarrow Q_{n+1}((f''_C\ f''_{C''}),\varepsilon) \end{cases}$$

induced by a $\mathbb{Z}_2$-isovariant chain map. There is thus an indeterminacy in the union operation, since adjoining $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré cobordisms determine only an element of $\begin{cases} \operatorname{coker}(Q^{n+1}(C',\varepsilon)\longrightarrow Q^{n+1}) \\ \operatorname{coker}(Q_{n+1}(C',\varepsilon)\longrightarrow Q_{n+1}) \end{cases}$.

ii) There is a neater expression for the union in the special case when all the chain maps involved are direct inclusions.

An $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ pair over A $\begin{cases}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon)) \\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{cases}$ is <u>direct</u>

if each $f\in\operatorname{Hom}_A(C_r,D_r)$ $(r\in\mathbb{Z})$ is a split monomorphism, that is the inclusion of a direct summand. The <u>direct union</u> of adjoining $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ direct cobordisms

$$\begin{cases} c=((f_C\ f_{C'}):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f_C\ f_{C'}),\varepsilon)) \\ c=((f_C\ f_{C'}):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f_C\ f_{C'}),\varepsilon)) \end{cases},$$

$$\begin{cases} c'=((f'_{C'}\ f'_{C''}):C'\oplus C''\longrightarrow D',(\delta\varphi',\varphi'\oplus-\varphi'')\in Q^{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)) \\ c'=((f'_{C'}\ f'_{C''}):C'\oplus C''\longrightarrow D',(\delta\psi',\psi'\oplus-\psi'')\in Q_{n+1}((f'_{C'}\ f'_{C''}),\varepsilon)) \end{cases}$$

is the direct cobordism

$$\begin{cases} c\bar{\cup}c' = ((\bar f''_C\ \bar f''_{C''}):C\oplus C''\longrightarrow\bar D'',(\overline{\delta\varphi''},\varphi\oplus-\varphi'')\in Q^{n+1}((\bar f''_C\ \bar f''_{C''}),\varepsilon)) \\ c\bar{\cup}c' = ((\bar f''_C\ \bar f''_{C''}):C\oplus C''\longrightarrow\bar D'',(\overline{\delta\psi''},\psi\oplus-\psi'')\in Q_{n+1}((\bar f''_C\ \bar f''_{C''}),\varepsilon)) \end{cases}$$

defined by

$$\bar f''_C=\begin{pmatrix}1\\0\end{pmatrix}:C_r\longrightarrow\bar D''_r=\operatorname{coker}(\begin{pmatrix}f_{C'}\\f'_{C'}\end{pmatrix}:C'_r\longrightarrow D_r\oplus D'_r)\ ,\quad d_{\bar D''}=[d_D\oplus d_{D'}]$$

$$\bar f''_{C''}=\begin{pmatrix}0\\1\end{pmatrix}:C''_r\longrightarrow\bar D''_r\ ,\quad \begin{cases}\overline{\delta\varphi''}=[\delta\varphi\oplus\delta\varphi'] \\ \overline{\delta\psi''}=[\delta\psi\oplus\delta\psi']\end{cases}.$$

The direct union for symmetric cobordisms is due to Mishchenko [2], where only the direct symmetric cobordisms were considered. A restriction to direct complexes and direct complexes does not involve a loss of generality, since every $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ cobordism $\begin{cases}c=((f\ f'):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ f'),\varepsilon)) \\ c=((f\ f'):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f\ f'),\varepsilon))\end{cases}$

is homotopy equivalent (in a sense to be made precise) to a direct cobordism $\begin{cases}\bar c=((\bar f\ \bar f'):C\oplus C'\longrightarrow\bar D,(\overline{\delta\varphi},\varphi\oplus-\varphi')\in Q^{n+1}((\bar f\ \bar f'),\varepsilon)) \\ \bar c=((\bar f\ \bar f'):C\oplus C'\longrightarrow\bar D,(\overline{\delta\psi},\psi\oplus-\psi')\in Q_{n+1}((\bar f\ \bar f'),\varepsilon))\end{cases}$ with $\bar D=M(f\ f')$ the algebraic mapping cylinder of $(f\ f'):C\oplus C'\longrightarrow D$. Furthermore, if c,c' are adjoining $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ cobordisms then

$$(\overline{c\cup c'})=(\bar c)\bar{\cup}(\bar c')$$

up to homotopy equivalence, so that the direct union corresponds to the indirect union.

iii) For $n=0,1$ the union of $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ cobordisms can be interpreted as a glueing operation on $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms and formations, cf. the discussion following Proposition 5.7 below).

Proposition 5.2 Cobordism is an equivalence relation on n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes over A, such that homotopy equivalent complexes are cobordant. The cobordism classes define an abelian group, the n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ L-group of A $\begin{cases}L^n(A,\varepsilon)\\ L_n(A,\varepsilon)\end{cases}$ $(n\geqslant 0)$ with addition and inverses by

$$\begin{cases}(C,\varphi)+(C',\varphi') = (C\oplus C',\varphi\oplus\varphi')\ , \quad -(C,\varphi) = (C,-\varphi)\in L^n(A,\varepsilon)\\ (C,\psi)+(C',\psi') = (C\oplus C',\psi\oplus\psi')\ , \quad -(C,\psi) = (C,-\psi)\in L_n(A,\varepsilon)\ .\end{cases}$$

Proof: Given a homotopy equivalence of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes over A

$$\begin{cases}f : (C,\varphi)\longrightarrow(C',\varphi')\\ f : (C,\psi)\longrightarrow(C',\psi')\end{cases}$$

let $\begin{cases}\varphi\in\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C^t\otimes_A C)_n\\ \psi\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C))_n\end{cases}$ be a cycle representing $\begin{cases}\varphi\in Q^n(C,\varepsilon)\\ \psi\in Q_n(C,\varepsilon)\end{cases}$, so that

$$\begin{cases}\varphi' = f^{\%}(\varphi)\in\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C'^t\otimes_A C')_n\\ \psi' = f_{\%}(\psi)\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C'^t\otimes_A C'))_n\end{cases}$$

is a cycle representing $\begin{cases}\varphi'\in Q^n(C',\varepsilon)\\ \psi'\in Q_n(C',\varepsilon)\end{cases}$. There is then defined a cobordism

$$\begin{cases}((f\ 1):C\oplus C'\longrightarrow C',(0,\varphi\oplus-\varphi')\in Q^{n+1}((f\ 1),\varepsilon))\\ ((f\ 1):C\oplus C'\longrightarrow C',(0,\psi\oplus-\psi')\in Q_{n+1}((f\ 1),\varepsilon))\end{cases}\text{ from }\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}\text{ to }\begin{cases}(C',\varphi')\\ (C',\psi')\end{cases}.$$

This verifies that homotopy equivalent algebraic Poincaré complexes are cobordant, and in particular that cobordism is reflexive.

If $\begin{cases}((f\ f'):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ f'),\varepsilon))\\ ((f\ f'):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f\ f'),\varepsilon))\end{cases}$ is a cobordism,

then so is $\begin{cases}((f'\ f):C'\oplus C\longrightarrow D,(-\delta\varphi,\varphi'\oplus-\varphi)\in Q^{n+1}((f'\ f),\varepsilon))\\ ((f'\ f):C'\oplus C\longrightarrow D,(-\delta\psi,\psi'\oplus-\psi)\in Q_{n+1}((f'\ f),\varepsilon))\end{cases}$, thus verifying the symmetry of cobordism.

The union operation ensures that cobordism is transitive.

We shall denote the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ L-groups by $\begin{cases}L^n(A,1) = L^n(A)\\ L_n(A,1) = L_n(A)\end{cases}$.

In §7 below we shall establish that $L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon) = L_{n+4}(A,\varepsilon)$ $(n\geqslant 0)$. The quadratic L-groups of a group ring $A = \mathbb{Z}[\pi]$ will be identified in §8 with the surgery obstruction groups of Wall [5], $L_n(\mathbb{Z}[\pi]) = L_n(\pi)$. The cobordism class of the quadratic kernel of a normal bundle map $(f,b):M\longrightarrow X$ will be identified with the surgery obstruction, $\sigma_*(f,b) = \theta(f,b)\in L_n(\mathbb{Z}[\pi_1(X)]) = L_n(\pi_1(X))$.

The symmetric L-groups $L^n(A)$ coincide with the "algebraic Poincaré bordism" groups $\Omega_n(A)$ of Mishchenko [2], except that $\Omega_n(A)$ is defined using only f.g. free (rather than f.g. projective) A-module chain complexes - the difference this makes will be considered in §12 below.

The correspondences between low-dimensional $(n = 0,1)$ algebraic Poincaré and forms and formations of §1 will be pursued further in §7 below, and $\begin{cases}L^0(A,\varepsilon)\\ L_0(A,\varepsilon)\end{cases}$ (resp. $\begin{cases}L^1(A,\varepsilon)\\ L_1(A,\varepsilon)\end{cases}$) will be identified with a Witt group of stable isomorphism classes of non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms (resp. formations) over A.

The ε-symmetrization of an (n+1)-dimensional ε-quadratic (Poincaré) pair over A $(f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))$ is the (n+1)-dimensional ε-symmetric (Poincaré) pair

$$(1+T_\varepsilon)(f:C\longrightarrow D,(\delta\psi,\psi)) = (f:C\longrightarrow D,(1+T_\varepsilon)(\delta\psi,\psi)\in Q^{n+1}(f,\varepsilon))\ ,$$

where

$$(1+T_\varepsilon)(\delta\psi,\psi)_s = \begin{cases}((1+T_\varepsilon)\delta\psi_0,(1+T_\varepsilon)\psi_0)\in(D^t\otimes_A D)_{n+1}\oplus(C^t\otimes_A C)_n & s=0\\ (0,0)\in(D^t\otimes_A D)_{n+s+1}\oplus(C^t\otimes_A C)_{n+s} & s\geqslant 1\ .\end{cases}$$

Thus the ε-symmetrization of a null-cobordant ε-quadratic Poincaré complex is a null-cobordant ε-symmetric Poincaré complex, and there is defined an ε-symmetrization map in the L-groups

$$(1+T_\varepsilon)\ :\ L_n(A,\varepsilon)\longrightarrow L^n(A,\varepsilon)\ ;\ (C,\psi\in Q_n(C,\varepsilon))\longrightarrow(C,(1+T_\varepsilon)\psi\in Q^n(C,\varepsilon)).$$

In §11 we shall prove that the ε-symmetrization maps $(1+T_\varepsilon):L_n(A,\varepsilon)\longrightarrow L^n(A,\varepsilon)$ are isomorphisms modulo 8-torsion.

Proposition 5.3 If A is such that there exists a central element $a\in A$ with

$$a+\bar{a}=1\in A$$

then the ε-symmetrization maps

$$(1+T_\varepsilon):L_n(A,\varepsilon)\longrightarrow L^n(A,\varepsilon)\qquad(n\geq 0)$$

are isomorphisms. In particular, this is the case if $1/2\in A$.

Proof: For such A already the Q-group ε-symmetrization maps

$$(1+T_\varepsilon):Q_n(C,\varepsilon)\longrightarrow Q^n(C,\varepsilon)$$

are isomorphisms for any A-module chain complex C, with inverse

$$Q^n(C,\varepsilon)\longrightarrow Q_n(C,\varepsilon)\ ;\ \varphi\longmapsto\psi\ ,\ \psi_s=\begin{cases}(1\otimes a)\varphi_0\in(C^t\otimes_A C)_n & s=0\\ 0\in(C^t\otimes_A C)_{n+s} & s\geq 1\end{cases}\quad .$$

Thus there is no difference between ε-quadratic and ε-symmetric complexes over A.

Similarly, the ε-symmetrization maps

$$(1+T_\varepsilon):Q_n(f,\varepsilon)\longrightarrow Q^n(f,\varepsilon)$$

are isomorphisms for any chain map $f:C\longrightarrow D$ of A-module chain complexes, and there is no difference between ε-quadratic and ε-symmetric pairs over A.

[]

The skew-suspension of an $(n+1)$-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ (Poincaré) pair over A $\begin{Bmatrix}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{Bmatrix}$ is the $(n+3)$-dimensional $\begin{Bmatrix}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{Bmatrix}$ (Poincaré) pair over A

$$\begin{cases}\bar{S}(f:C\longrightarrow D,(\delta\varphi,\varphi))=(Sf:SC\longrightarrow SD,\bar{S}(\delta\varphi,\varphi)\in Q^{n+3}(Sf,-\varepsilon))\\ \bar{S}(f:C\longrightarrow D,(\delta\psi,\psi))=(Sf:SC\longrightarrow SD,\bar{S}(\delta\psi,\psi)\in Q_{n+3}(Sf,-\varepsilon))\end{cases},$$

with $\begin{cases}\bar{S}:Q^{n+1}(f,\varepsilon)\longrightarrow Q^{n+3}(Sf,-\varepsilon)\\ \bar{S}:Q_{n+1}(f,\varepsilon)\longrightarrow Q_{n+3}(Sf,-\varepsilon)\end{cases}$ the relative version of the isomorphism defined in the absolute case in §1. Thus the skew-suspension of a null-cobordant n-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ Poincaré complex is a null-cobordant $(n+2)$-dimensional $\begin{Bmatrix}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{Bmatrix}$ Poincaré complex, and there are defined skew-suspension maps in the L-groups

$$\begin{cases}\bar{S}:L^n(A,\varepsilon)\longrightarrow L^{n+2}(A,-\varepsilon)\ ;\ (C,\varphi)\longmapsto(SC,\bar{S}\varphi)\\ \bar{S}:L_n(A,\varepsilon)\longrightarrow L_{n+2}(A,-\varepsilon)\ ;\ (C,\psi)\longmapsto(SC,\bar{S}\psi)\end{cases}\qquad(n\geq 0)\quad .$$

In §7 below we shall prove that $\bar{S}:L_n(A,\varepsilon)\longrightarrow L_{n+2}(A,-\varepsilon)$ is an isomorphism for all $A,\varepsilon,n\geq 0$ (Proposition 7.3). It will also be proved that $\bar{S}:L^n(A,\varepsilon)\longrightarrow L^{n+2}(A,-\varepsilon)$ is an isomorphism if A is noetherian of finite global dimension m and $n\geq 2m$ (Proposition 7.4). In §14 we shall construct examples such that $\bar{S}:L^n(A,\varepsilon)\longrightarrow L^{n+2}(A,-\varepsilon)$ is not an isomorphism (Propositions 14.3, 14.4, 14.9).

We shall now define the notion of homotopy equivalence appropriate to algebraic Poincaré pairs. It turns out that the homotopy equivalence classes of $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pairs are in a natural one-one correspondence with the homotopy equivalence classes of certain $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes (Proposition 5.4). This allows for considerable conceptual simplification, giving the $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ L-groups an expression entirely in terms of $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes. In §7 below we shall use this expression to identify the low-dimensional L-groups with stable isomorphism groups of forms and formations, and to establish the 4-periodicity of the quadratic L-groups, $L_n = L_{n+4}$.

Define a homotopy equivalence of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ pairs over A

$$\begin{cases}(g,h;k) : (f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^n(f,\varepsilon))\longrightarrow(f':C'\longrightarrow D',(\delta\varphi',\varphi')\in Q^n(f',\varepsilon))\\ (g,h;k) : (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_n(f,\varepsilon))\longrightarrow(f':C'\longrightarrow D',(\delta\psi',\psi')\in Q_n(f',\varepsilon))\end{cases}$$

to be a triple $(g,h;k)$ consisting of chain equivalences

$$g : C\longrightarrow C' \ ,\ h : D\longrightarrow D'$$

together with a chain homotopy

$$k : f'g\simeq hf : C\longrightarrow D'$$

such that

$$\begin{cases}(g,h;k)^{\%}(\delta\varphi,\varphi) = (\delta\varphi',\varphi')\in Q^n(f',\varepsilon)\\ (g,h;k)_{\%}(\delta\psi,\psi) = (\delta\psi',\psi')\in Q_n(f',\varepsilon)\ ,\end{cases}$$

where

$$\begin{cases}(g,h;k)^{\%}(\delta\varphi,\varphi)_s = ((h\otimes_A h)(\delta\varphi_s)+(-)^{n-1}(hf\otimes k)(\varphi_s)+(-)^p(k\otimes f'g)(\varphi_s)\\ \qquad +(-)^{n+p-1}(k\otimes k)(\varphi_{s-1}),(g\otimes g)(\varphi_s))\in(D'^t\otimes_A D')_{n+s}\oplus(C'^t\otimes_A C')_{n+s-1}\\ (g,h;k)_{\%}(\delta\psi,\psi)_s = ((h\otimes_A h)(\delta\psi_s)+(-)^{n-1}(hf\otimes k)(\psi_s)+(-)^p(k\otimes f'g)(\psi_s)\\ \qquad +(-)^{n+p}(k\otimes k)(\psi_{s+1})\ ,\ (g\otimes g)(\psi_s))\in(D'^t\otimes_A D')_{n-s}\oplus(C'^t\otimes_A C')_{n-s-1}\end{cases}$$

$$(s\geqslant 0,\quad (D'^t\otimes_A D')_r = \sum_{p+q=r} D'_p{}^t\otimes_A D'_q\ )\ .$$

An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ is connected if

$$\begin{cases}H_0(\varphi_0:C^{n-*}\longrightarrow C) = 0\\ H_0((1+T_\varepsilon)\psi_0:C^{n-*}\longrightarrow C) = 0\ .\end{cases}$$

In particular, Poincaré complexes are connected. For $n = 0$ connected = Poincaré.

Define the boundary of a connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ $(n\geqslant 1)$ to be the $(n-1)$-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over A

$$\begin{cases}\partial(C,\varphi) = (\partial C,\partial\varphi\in Q^{n-1}(\partial C,\varepsilon))\\ \partial(C,\psi) = (\partial C,\partial\psi\in Q_{n-1}(\partial C,\varepsilon))\end{cases}$$

given by

$$d_{\partial C} = \begin{cases}\begin{pmatrix} d_C & (-)^r\varphi_0\\ 0 & (-)^r d_C^*\end{pmatrix}\\ \begin{pmatrix} d_C & (-)^r(1+T_\varepsilon)\psi_0\\ 0 & (-)^r d_C^*\end{pmatrix}\end{cases} : \partial C_r = C_{r+1}\oplus C^{n-r}\longrightarrow \partial C_{r-1} = C_r\oplus C^{n-r+1}$$

$$\begin{cases}\partial\varphi_0 = \begin{pmatrix}(-)^{n-r-1}T_\varepsilon\varphi_1 & (-)^{r(n-r-1)}\varepsilon\\ 1 & 0\end{pmatrix} : \partial C^{n-r-1} = C^{n-r}\oplus C_{r+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r}\\ \partial\psi_0 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} : \partial C^{n-r-1} = C^{n-r}\oplus C_{r+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r}\end{cases}$$

$$\begin{cases}\partial\varphi_s = \begin{pmatrix}(-)^{n-r+s-1}T_\varepsilon\varphi_{s+1} & 0\\ 0 & 0\end{pmatrix} : \partial C^{n-r+s-1} = C^{n-r+s}\oplus C_{r-s+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r}\\ \partial\psi_s = \begin{pmatrix}(-)^{n-r-s}T_\varepsilon\psi_{s-1} & 0\\ 0 & 0\end{pmatrix} : \partial C^{n-r+s-1} = C^{n-r-s}\oplus C_{r+s+1}\longrightarrow \partial C_r = C_{r+1}\oplus C^{n-r}\end{cases}\ (s\geqslant 1)\ .$$

(Motivation: let M be an n-dimensional manifold with boundary ∂M, and let $\sigma^*(M,\partial M) = (f:C(\widetilde{\partial M})\longrightarrow C(\widetilde{M}),\varphi_{\widetilde{M},\widetilde{\partial M}}[M]\in Q^n(f))$ be the associated n-dimensional symmetric Poincaré pair over $\mathbb{Z}[\pi_1(M)]$ defined in §6 below, with boundary $\sigma^*(\partial M) = (C(\widetilde{\partial M}),\varphi_{\widetilde{\partial M}}[\partial M]\in Q^{n-1}(C(\widetilde{\partial M})))$. Then the n-dimensional symmetric complex $(C,\varphi) = (\dot{C}(\widetilde{M}/\widetilde{\partial M}),\dot{\varphi}_{\widetilde{M}/\widetilde{\partial M}}[M]\in Q^n(\dot{C}(\widetilde{M}/\widetilde{\partial M})))$ obtained from $\sigma^*(M,\partial M)$ by collapsing $\sigma^*(\partial M)$ has boundary $\partial(C,\varphi)$ homotopy equivalent to $\sigma^*(\partial M)$).

An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ is <u>contractible</u> if it is homotopy equivalent to 0, that is if $H_*(C) = 0$.

<u>Proposition 5.4</u> i) There is a natural one-one correspondence between the homotopy equivalence classes of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pairs over A and the homotopy equivalence classes of connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes which preserves boundaries. Poincaré pairs with contractible boundaries correspond to Poincaré complexes.

ii) A connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex is Poincaré if and only if its boundary is a contractible (n-1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex.

iii) An n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex is null-cobordant if and only if it is homotopy equivalent to the boundary of a connected (n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex.

<u>Proof</u>: i) Given a connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ with boundary $\begin{cases}\partial(C,\varphi) = (\partial C,\partial\varphi)\\ \partial(C,\psi) = (\partial C,\partial\psi)\end{cases}$ there is defined an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pair $\begin{cases}(i_C:\partial C\longrightarrow C^{n-*},(0,\partial\varphi)\in Q^n(i_C,\varepsilon))\\ (i_C:\partial C\longrightarrow C^{n-*},(0,\partial\psi)\in Q_n(i_C,\varepsilon))\end{cases}$, with

$$i_C = (0\ \ 1) : \partial C_r = C_{r+1}\oplus C^{n-r}\longrightarrow (C^{n-*})_r = C^{n-r} .$$

Given a homotopy equivalence of connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes $\begin{cases}f:(C,\varphi)\longrightarrow(C',\varphi')\\ f:(C,\psi)\longrightarrow(C',\psi')\end{cases}$ choose cycle representatives

$$\begin{cases}\varphi\in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_n,\ \varphi'\in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_n\\ \psi\in(\dot{W}\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C^*,C))_n,\ \psi'\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C'^*,C'))_n\end{cases}$$

, so that

$$\begin{cases}f^{\%}(\varphi)-\varphi' = d(\nu)\in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_n\\ f_{\%}(\psi)-\psi' = d(\chi)\in (W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C'^*,C'))_n\end{cases}$$

for some chain $\begin{cases}\nu\in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_{n+1}\\ \chi\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C^*,C))_{n+1}\end{cases}$, with

$$\begin{cases}f\varphi_s f^* - \varphi'_s = d_{C'}\nu_s + (-)^r\nu_s d^*_{C'} + (-)^{n+s}(\nu_{s-1} + (-)^s T_\varepsilon \nu_{s-1}) : C'^{n-r+s}\longrightarrow C'_r\\ f\psi_s f^* - \psi'_s = d_{C'}\chi_s + (-)^r\chi_s d^*_{C'} + (-)^{n-s}(\chi_{s+1} + (-)^{s+1} T_\varepsilon \chi_{s+1}) : C'^{n-r-s}\longrightarrow C'_r\end{cases}$$

$$(s\geq 0,\ \nu_{-1} = 0) .$$

Let $f':C'\longrightarrow C$ be a chain homotopy inverse for $f:C\longrightarrow C'$, and let $g:f'f\simeq 1:C\longrightarrow C$ be a chain homotopy, with

$$f'f - 1 = d_C g + g d_C : C_r\longrightarrow C_r \qquad (g\in \mathrm{Hom}_A(C_r,C_{r+1})) .$$

Then the A-module morphisms

$$\begin{cases}\partial f = \begin{pmatrix} f & -f\varphi_0 g^* + (-)^r\nu_0 f'^* \\ 0 & f'^*\end{pmatrix} : \partial C_r = C_{r+1}\oplus C^{n-r}\longrightarrow \partial C'_r = C'_{r+1}\oplus C'^{n-r}\\ \partial f = \begin{pmatrix} f & -f(1+T_\varepsilon)\psi_0 g^* + (-)^r(1+T_\varepsilon)\chi_0 f'^* \\ 0 & f'^*\end{pmatrix} : \partial C_r = C_{r+1}\oplus C^{n-r}\longrightarrow \partial C'_r = C'_{r+1}\oplus C'^{n-r}\end{cases}$$

are such that

$$\begin{cases}(\partial f,1;0) : (i_C:\partial C\longrightarrow C^{n-*},(0,\partial\varphi))\longrightarrow(i_{C'}:\partial C'\longrightarrow C'^{n-*},(0,\partial\varphi'))\\ (\partial f,1;0) : (i_C:\partial C\longrightarrow C^{n-*},(0,\partial\psi))\longrightarrow(i_{C'}:\partial C'\longrightarrow C'^{n-*},(0,\partial\psi'))\end{cases}$$

is a homotopy equivalence of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pairs over A.

(The definition of the boundary $\begin{cases}\partial(C,\varphi)\\ \partial(C,\psi)\end{cases}$ depends on a choice of cycle representative for $\begin{cases}\varphi\in Q^n(C,\varepsilon)\\ \psi\in Q_n(C,\varepsilon)\end{cases}$. In particular, we have just shown that a different choice of representative defines a homotopy equivalent complex).

Conversely, given an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pair over A $\begin{cases}(f:\partial C \longrightarrow C,(\varphi,\partial\varphi)\in Q^n(f,\varepsilon))\\ (f:\partial C \longrightarrow C,(\psi,\partial\psi)\in Q_n(f,\varepsilon))\end{cases}$ define a connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C',\varphi'\in Q^n(C',\varepsilon))\\ (C',\psi'\in Q_n(C',\varepsilon))\end{cases}$ by $C' = C(f)$ and

$$\begin{cases}\varphi'_s = \begin{pmatrix}\varphi_s & 0\\ (-)^{n-r-1}\partial\varphi_s f^* & (-)^{n-r+s}T_\varepsilon\partial\varphi_{s-1}\end{pmatrix} : C'^{n-r+s} = C^{n-r+s}\oplus\partial C^{n-r+s-1} \longrightarrow C'_r = C_r\oplus\partial C_{r-1} \quad (s\geq 0)\\[2ex] \psi'_s = \begin{pmatrix}\psi_s & 0\\ (-)^{n-r-1}\partial\psi_s f^* & (-)^{n-r-s-1}T_\varepsilon\partial\psi_{s+1}\end{pmatrix} : C'^{n-r-s} = C^{n-r-s}\oplus\partial C^{n-r-s-1} \longrightarrow C'_r = C_r\oplus\partial C_{r-1} \quad (s\geq 0)\ .\end{cases}$$

(This is an algebraic analogue of the Thom complex construction in topology, being just the collapsing of the boundary $\begin{cases}(\partial C,\partial\varphi)\\ (\partial C,\partial\psi)\end{cases}$). There is defined a homotopy equivalence of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pairs

$$\begin{cases}(\partial g,g;h) : (i_{C'}:\partial C' \longrightarrow C'^{n-*},(0,\partial\varphi')) \longrightarrow (f:\partial C \longrightarrow C,(\varphi,\partial\varphi))\\ (\partial g,g;h) : (i_{C'}:\partial C' \longrightarrow C'^{n-*},(0,\partial\psi')) \longrightarrow (f:\partial C \longrightarrow C,(\psi,\partial\psi))\end{cases} ,$$

with

$$g = \begin{cases}(\varphi_0 \ \ f\partial\varphi_0)\\ ((1+T_\varepsilon)\psi_0 \ \ f(1+T_\varepsilon)\partial\psi_0)\end{cases} : C'^{n-r} = C^{n-r}\oplus\partial C^{n-r-1} \longrightarrow C_r$$

$$\partial g = \begin{cases}(0 \ \ 1 \ \ 0 \ \ \partial\varphi_0)\\ (0 \ \ 1 \ \ 0 \ \ (1+T_\varepsilon)\partial\psi_0)\end{cases} : \partial C'_r = C_{r+1}\oplus\partial C_r\oplus C^{n-r}\oplus\partial C^{n-r-1} \longrightarrow \partial C_r$$

$$h = ((-)^r \ \ 0 \ \ 0 \ \ 0) : \partial C'_r = C_{r+1}\oplus\partial C_r\oplus C^{n-r}\oplus\partial C^{n-r-1} \longrightarrow C_{r+1}\ .$$

ii) Given a connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ we can identify $\begin{cases}S\partial C = C(\varphi_0:C^{n-*}\longrightarrow C)\\ S\partial C = C((1+T_\varepsilon)\psi_0:C^{n-*}\longrightarrow C)\end{cases}$, so that ∂C is chain contractible if and only if $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ is Poincaré.

iii) is immediate from i). []

An n-dimensional ε-quadratic complex $(C,\psi\in Q_n(C,\varepsilon))$ is <u>highly-connected</u> if $H_r(C)=H^r(C)=0,\ r\neq\begin{cases}i\\ i,i+1\end{cases}$ and $\begin{cases}-\\ H_{i-1}((1+T_\varepsilon)\psi_0:C^{2i+1-*}\longrightarrow C) = 0\end{cases}$ for $n = \begin{cases}2i\\ 2i+1\end{cases}$.

Highly-connected complexes are connected and have highly-connected boundaries. The quadratic kernel $\sigma_*(f,b)$ of a highly-connected n-dimensional normal map $(f,b):M\longrightarrow X$ ($K_r(M) = 0$ for $2r<n$) is a highly-connected n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$.

Define the <u>boundary</u> of $\begin{cases}\text{an } \varepsilon\text{-quadratic form } (M,\psi)\\ \text{a split } \varepsilon\text{-quadratic formation } (F,(\binom{\gamma}{\mu},\theta)G)\end{cases}$ to be the non-singular $\begin{cases}\text{split } (-\varepsilon)\text{-quadratic formation}\\ \varepsilon\text{-quadratic form}\end{cases}$

$$\begin{cases}\partial(M,\psi) = (M,(\begin{pmatrix}1\\ \psi+\varepsilon\psi^*\end{pmatrix},\psi)M)\\ \partial(F,G) = (G^\perp/G,\psi^\perp/\psi) = (\ker(\varepsilon\mu^* \ \gamma^*):F\oplus F^*\longrightarrow G^*)/\mathrm{im}(\binom{\gamma}{\mu}:G\longrightarrow F\oplus F^*),\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix})\ .\end{cases}$$

<u>Proposition 5.5</u> The homotopy equivalence classes of highly-connected $\begin{cases}2i\text{-}\\ (2i+1)\text{-}\end{cases}$ dimensional ε-quadratic complexes over A are in a natural one-one correspondence with the $\begin{cases}\text{isomorphism}\\ \text{stable isomorphism}\end{cases}$ classes of $\begin{cases}(-)^i\varepsilon\text{-quadratic forms}\\ \text{split } (-)^i\varepsilon\text{-quadratic formations}\end{cases}$ over A. The correspondence preserves the boundary operations, and sends Poincaré complexes to non-singular $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$.

<u>Proof</u>: Immediate from Proposition 1.5,1.8 i).

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In Ranicki [1] there were defined groups $U_n(A)$ for $n \pmod 4$, as the sets of equivalence classes of non-singular $(-)^i$quadratic $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over A if $n = \begin{cases}2i\\ 2i+1\end{cases}$ under the equivalence relation

$$\begin{cases}(M,\psi) \sim (M',\psi') \\ (M,\psi;F,G) \sim (M',\psi';F',G')\end{cases} \text{ if there exist } \begin{cases}(-)^i \\ (-)^{i+1}\end{cases}\text{-quadratic } \begin{cases}\text{formations}\\ \text{forms}\end{cases}$$

$$\begin{cases}(N,\chi;H,K),(N',\chi';H',K') \\ (N,\chi),(N',\chi')\end{cases} \text{ and } \begin{cases}\text{an isomorphism}\\ \text{a stable isomorphism}\end{cases}$$

$$\begin{cases}f : (M,\psi)\oplus\partial(N,\chi;H,K) \longrightarrow (M',\psi')\oplus\partial(N',\chi';H',K') \\ [f]:(M,\psi;F,G)\oplus\partial(N,\chi) \longrightarrow (M',\psi';F',G')\oplus\partial(N',\chi')\end{cases}.$$

The <u>boundary</u> of $\begin{cases}(N,\chi;H,K)\\ (N,\chi)\end{cases}$ is the non-singular $(-)^i$quadratic $\begin{cases}\text{form}\\ \text{formation}\end{cases}$

$$\begin{cases}\partial(N,\chi;H,K) = (K^\perp/K,\chi^\perp/\chi) \\ \partial(N,\chi) = (H_{(-)^i}(N);N,\Gamma_{(N,\chi)}) \ , \ \Gamma_{(N,\chi)} = \{(x,(\chi+(-)^{i+1}\chi^*)(x))\in N\oplus N^* \mid x\in N\}.\end{cases}$$

Addition and inverses are given by

$$\begin{cases}(M,\psi) + (M',\psi') = (M\oplus M',\psi\oplus\psi') \quad , \quad -(M,\psi) = (M,-\psi) \in U_{2i}(A) \\ (M,\psi;F,G) + (M',\psi';F',G') = (M\oplus M',\psi\oplus\psi';F\oplus F',G\oplus G') \quad , \\ \qquad -(M,\psi;F,G) = (M,-\psi;F,G) \in U_{2i+1}(A) \quad ,\end{cases}$$

<u>Proposition 5.6</u> There is defined an abelian group morphism

$$\begin{cases}U_{2i}(A) \longrightarrow L_0(A,(-)^i) \ ; \ (M,\psi) \longmapsto (C,\psi\in Q_0(C,(-)^i)) \\ U_{2i+1}(A) \longrightarrow L_1(A,(-)^i) \ ; \ (M,\psi;F,G) \longmapsto (C,\psi\in Q_1(C,(-)^i))\end{cases},$$

sending a non-singular $(-)^i$quadratic $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ to the cobordism class of $\begin{cases}\text{the}\\ \text{any}\end{cases}$ associated $\begin{cases}0\text{-}\\ 1\text{-}\end{cases}$dimensional $(-)^i$quadratic Poincaré complex.

<u>Proof</u>: It is immediate from Proposition 5.5 that $U_{2i}(A) \longrightarrow L_0(A,(-)^i)$ is well-defined.

A non-singular $(-)^i$quadratic formation over A $(M,\psi;F,G)$ is such that $(M,\psi;F,G) = 0 \in U_{2i+1}(A)$ if and only if there exists a stable isomorphism

$$[f] : (M,\psi;F,G)\oplus\partial(N,\chi) \longrightarrow \partial(N',\chi')$$

for some $(-)^{i+1}$quadratic forms (N,χ), (N',χ'). (In Proposition 7.8 we shall prove that we can take $(N,\chi) = 0$ in the above). Every formation is isomorphic to one of the type $(H_{(-)^i}(F);F,G)$ by Proposition 1.6, and every stable isomorphism

$$[f] : (H_{(-)^i}(F);F,G)\oplus\partial(N,\chi) \longrightarrow \partial(N',\chi')$$

lifts to a stable isomorphism of split $(-)^i$quadratic formations

$$[\alpha,\beta,\psi] : (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)\oplus(N,(\begin{pmatrix}1\\ (\chi+(-)^i\chi^*)\end{pmatrix},\chi)N) \longrightarrow (N',(\begin{pmatrix}1\\ (\chi'+(-)^i\chi'^*)\end{pmatrix},\tilde{\chi}')N')$$

for any hessian $(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\theta)\in Q_{(-)^{i+1}}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G\longrightarrow F\oplus F^*)$ for G, with a corresponding $(-)^{i+1}$quadratic form $(N',\tilde{\chi}'\in Q_{(-)^{i+1}}(N'))$ such that

$$\tilde{\chi}' + (-)^{i+1}\tilde{\chi}'^* = \chi' + (-)^{i+1}\chi'^* : N' \longrightarrow N'^* ,$$

by the proof of Proposition 1.8 ii). We thus have a stable isomorphism of non-singular split $(-)^{i+1}$quadratic formations

$$[\alpha,\beta,\psi] : (F,G)\oplus\partial(N,\chi) \longrightarrow \partial(N',\tilde{\chi}') ,$$

so that $(F,G) = 0 \in L_1(A,(-)^i)$ by Proposition 5.5. It follows that

$$U_{2i+1}(A) \longrightarrow L_1(A,(-)^i) \ ; \ (H_{(-)^i}(F);F,G) \longmapsto (F,G)$$

is well-defined, using any choice of hessian $(\begin{pmatrix}0&1\\0&0\end{pmatrix},\theta)$ to lift a non-singular $(-)^i$quadratic formation $(H_{(-)^i}(F);F,G)$ to a split formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$, thus obtaining a 1-dimensional $(-)^i$quadratic Poincaré complex.

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The composite

$$\begin{cases}U_{2i}(A) \longrightarrow L_0(A,(-)^i) \xrightarrow{\bar{S}^i} L_{2i}(A) \\ U_{2i+1}(A) \longrightarrow L_1(A,(-)^i) \xrightarrow{\bar{S}^i} L_{2i+1}(A)\end{cases}$$

defines a morphism

$$U_n(A) \longrightarrow L_n(A) \quad (n \geqslant 0)$$

which we shall prove to be an isomorphism in §7 below.

The correspondence of Proposition 5.4 i) shows that up to homotopy equivalence the cobordisms of n-dimensional $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ Poincaré complexes over A may be considered as quadruples

(connected (n+1)-dimensional $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ complex over A $\left\{\begin{matrix}(D,\nu)\\ (D,\chi)\end{matrix}\right.$,

n-dimensional $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ Poincaré complexes over A $\left\{\begin{matrix}(C,\varphi),(C',\varphi')\\ (C,\psi),(C',\psi')\end{matrix}\right.$,

homotopy equivalence $\left\{\begin{matrix}(f\ f') : (C,\varphi)\oplus(C',-\varphi') \longrightarrow \partial(D,\nu)\\ (f\ f') : (C,\psi)\oplus(C',-\psi') \longrightarrow \partial(D,\chi)\end{matrix}\right.$),

which we shall also call cobordisms. The union operation considered above associates a new cobordism to adjoining cobordisms of this type

$$((D,\nu);(C,\varphi),(C',\varphi'),(f\ f'))\cup((D',\nu');(C',\varphi'),(C'',\varphi''),(\tilde{f}'\ f''))$$
$$= ((D\cup_{C'}D',\nu\cup_{\varphi'}\nu');(C,\varphi),(C'',\varphi''),(\tilde{f}\ \tilde{f}'')) .$$

In particular, given connected (n+1)-dimensional ε-symmetric complexes (D,ν), (D',ν') and a homotopy equivalence $g:\partial(D,\nu)\longrightarrow\partial(D',\nu')$ we can glue together so as to obtain the (n+1)-dimensional ε-symmetric Poincaré complex $(D,\nu)\cup_g(D',-\nu')$ appearing in the cobordism

$$((D,\nu);0,-\partial(D,\nu),(0\ 1))\cup((D',-\nu');-\partial(D,\nu),0,(g\ 0)) = ((D,\nu)\cup_g(D',-\nu');0,0,(0\ 0)).$$

Similarly for the ε-quadratic case. (There is then a sum formula

$$\left\{\begin{matrix}((D,\nu)\cup_g(D',-\nu'))\oplus((D',\nu')\cup_{g'}(D'',-\nu'')) = ((D,\nu)\cup_{g'g}(D'',-\nu''))\in L^n(A,\varepsilon)\\ ((D,\chi)\cup_g(D',-\chi'))\oplus((D',\chi')\cup_{g'}(D'',-\chi'')) = ((D,\chi)\cup_{g'g}(D'',-\chi''))\in L_n(A,\varepsilon)\end{matrix}\right. \quad (n\geqslant 0) ,$$

which includes as special cases a sum formula for formations (cf. Proposition 7.7)

$$\left\{\begin{matrix}(M,\varphi;F,G)\oplus(M,\varphi;G,H) = (M,\varphi;F,H)\in L^1(A,\varepsilon)\\ (M,\psi;F,G)\oplus(M,\psi;G,H) = (M,\psi;F,H)\in L_1(A,\varepsilon)\end{matrix}\right.).$$

The formulation of the union operation entirely in terms of $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ complexes (i.e. not involving pairs) has the advantage that in the low-dimensional cases n = 0,1 it translates directly into the language of forms and formations, using the correspondences of Propositions 1.5,1.7,1.8.

We shall now give an explicit description of the union operation for ε-quadratic forms and formations, which was obtained in Ranicki [5] prior to the chain complex theory. We shall need this union operation in §10 below in order to describe the relative ε-quadratic L-groups $L_n(f,\varepsilon)$ of a morphism of rings with involution $f:A\longrightarrow B$ in terms of ε-quadratic forms and formations.

Given ε-quadratic formations over A $(M,\psi;F,G)$, $(M',\psi';F',G')$ and an isomorphism of the boundary ε-quadratic forms

$$f : \partial(M,\psi;F,G) = (G^{\perp}/G,\psi^{\perp}/\psi) \longrightarrow \partial(M',-\psi';F',G') = (G'^{\perp}/G',-\psi'^{\perp}/\psi')$$

define the <u>union</u> non-singular ε-quadratic formation

$$(M,\psi;F,G)\cup_f(M',\psi';F',G') = (M\oplus M',\psi\oplus\psi';F\oplus F',G\oplus \mathrm{im}(\begin{pmatrix} j\\ j'f\end{pmatrix}:G^{\perp}/G\longrightarrow M\oplus M')\oplus G') ,$$

with $j\in \mathrm{Hom}_A(G^{\perp}/G,M)$, $j'\in \mathrm{Hom}_A(G'^{\perp}/G',M')$ the direct inclusions appearing in any of the isomorphisms of ε-quadratic forms given by Proposition 1.6

$$(i\ j) : H_{\varepsilon}(G)\oplus(G^{\perp}/G,\psi^{\perp}/\psi)\longrightarrow(M,\psi) \ , \ (i'\ j') : H_{\varepsilon}(G')\oplus(G'^{\perp}/G',\psi'^{\perp}/\psi')\longrightarrow(M',\psi') .$$

Given ε-quadratic forms over A (M,ψ), (M',ψ') and a stable isomorphism of the boundary split $(-\varepsilon)$-quadratic formations over A

$$[\alpha,\beta,\sigma] : \partial(M,\psi) = (M,(\begin{pmatrix}1\\ \psi+\varepsilon\psi^*\end{pmatrix},\psi)M)\longrightarrow\partial(M',-\psi') = (M',(\begin{pmatrix}1\\ -(\psi'+\varepsilon\psi'^*)\end{pmatrix},-\psi')M')$$

define the <u>union</u> non-singular ε-quadratic form

$$(M,\psi)\cup_{[\alpha,\beta,\sigma]}(M',\psi') = (M'',\psi'')$$

as follows. We have an isomorphism of split $(-\varepsilon)$-quadratic formations

$$(\alpha,\beta,\sigma) = (\begin{pmatrix} a & a_1\\ a_2 & a_3\end{pmatrix},\begin{pmatrix} b & b_1\\ b_2 & b_3\end{pmatrix},\begin{pmatrix} s & s_1\\ s_2 & s_3\end{pmatrix})$$
$$: (M\oplus P,(\begin{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}\psi+\varepsilon\psi^* & 0\\ 0 & 1\end{pmatrix}\end{pmatrix},\begin{pmatrix}\psi & 0\\ 0 & 0\end{pmatrix})M\oplus P^*)$$
$$\longrightarrow (M'\oplus P',(\begin{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}-(\psi'+\varepsilon\psi'^*) & 0\\ 0 & 1\end{pmatrix}\end{pmatrix},\begin{pmatrix}-\psi' & 0\\ 0 & 0\end{pmatrix})M'\oplus P'^*)$$

for some f.g. projective A-modules P,P'. Let

$$\alpha\sigma\alpha^* = \begin{pmatrix} s' & s'_1 \\ s'_2 & s'_3 \end{pmatrix} : M'^*\oplus P' \longrightarrow M'\oplus P'^* ,$$

and set

$$(M'',\psi'') = (M\oplus M'^*, \begin{pmatrix} \psi & 0 \\ \varepsilon a & s' \end{pmatrix}) \in Q_\varepsilon(M\oplus M'^*)) .$$

This is the glueing of forms appearing in Theorem 6.4 of Wall [9], see also Wall [11], [13].

§6. Geometric cobordism

We shall now develop the relative versions of the constructions in §§ 2,3,4 of algebraic Poincaré complexes from geometric Poincaré complexes. Given an $(n+1)$-dimensional geometric Poincaré pair $(X,\partial X)$ we define an $(n+1)$-dimensional symmetric Poincaré pair over $\mathbb{Z}[\pi_1(X)]$ $\sigma^*(X,\partial X)$ with boundary $\sigma^*(\partial X)$, and given a $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ map of $(n+1)$-dimensional geometric Poincaré pairs $\begin{cases}(f,\partial f):(M,\partial M)\longrightarrow(X,\partial X)\\ ((f,\partial f),(b,\partial b)):(M,\partial M)\longrightarrow(X,\partial X)\end{cases}$ we define an $(n+1)$-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré pair $\begin{cases}\sigma^*(f,\partial f)\\ \sigma_*((f,\partial f),(b,\partial b))\end{cases}$ with boundary $\begin{cases}\sigma^*(\partial f)\\ \sigma_*(\partial f,\partial b)\end{cases}$.

The relative symmetric construction $\varphi_{f:X\to Y}$ defined below is a relative version of the absolute symmetric construction φ_X of Proposition 2.2.

Proposition 6.1 Let π be a group, and give $\mathbb{Z}[\pi]$ the w-twisted involution for some group morphism $w:\pi\longrightarrow\mathbb{Z}_2$.

Given a π-map of π-spaces

$$f : X \longrightarrow Y$$

there are defined in a natural way abelian group morphisms

$$\varphi_f : \dot{H}^\pi_{n+1}(f;{}^w\mathbb{Z}) \longrightarrow Q^{n+1}(f:\dot{C}(X)\longrightarrow\dot{C}(Y)) \quad (n\in\mathbb{Z})$$

such that

i) for each $z\in H^\pi_{n+1}(f;{}^w\mathbb{Z})$

$$\varphi_f(z)_0\backslash - = z\cap - : {}^wH^r_\pi(f)\longrightarrow \dot{H}_{n+1-r}(Y)$$

ii) there is defined a map of long exact sequences

$$\begin{array}{ccccccccc}
\dots\longrightarrow & H^\pi_{n+1}(f;{}^w\mathbb{Z}) & \longrightarrow & \dot{H}^\pi_n(X;{}^w\mathbb{Z}) & \xrightarrow{f_*} & \dot{H}^\pi_n(Y;{}^w\mathbb{Z}) & \longrightarrow & H^\pi_n(f;{}^w\mathbb{Z}) & \longrightarrow\dots \\
 & \downarrow{\scriptstyle\varphi_f} & & \downarrow{\scriptstyle\dot{\varphi}_X} & & \downarrow{\scriptstyle\dot{\varphi}_Y} & & \downarrow{\scriptstyle\varphi_f} & \\
\dots\longrightarrow & Q^{n+1}(f) & \longrightarrow & Q^n(\dot{C}(X)) & \xrightarrow{f^\%} & Q^n(\dot{C}(Y)) & \longrightarrow & Q^n(f) & \longrightarrow\dots
\end{array} .$$

Proof: Choosing a functorial diagonal approximation Δ we have a commutative diagram of abelian group chain complexes and chain maps

$$\begin{array}{ccc} \mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} \dot{C}(X) & \xrightarrow{1\otimes f} & \mathbb{Z}^t \otimes_{\mathbb{Z}[\pi]} \dot{C}(Y) \\ \dot{\varphi}_X = 1\otimes\dot{\Delta}_X \downarrow & & \downarrow \dot{\varphi}_Y = 1\otimes\dot{\Delta}_Y \\ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \dot{C}(X)^t \otimes_{\mathbb{Z}[\pi]} \dot{C}(X)) & \xrightarrow{f^{\%}} & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \dot{C}(Y)^t \otimes_{\mathbb{Z}[\pi]} \dot{C}(Y)) \end{array}$$

with a corresponding chain map of algebraic mapping cones

$$\varphi_f : C(1\otimes f) \longrightarrow C(f^{\%}) .$$

The relative symmetric construction is given by the induced maps in the homology groups

$$\varphi_f : H_{n+1}(C(1\otimes f)) = H^{\pi}_{n+1}(f;{}^w\mathbb{Z}) \longrightarrow H_{n+1}(C(f^{\%})) = Q^{n+1}(f) .$$

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Given a pair of π-spaces (X,Y) we shall write the relative symmetric construction for the inclusion $i:Y \hookrightarrow X$ as

$$\varphi_{X,Y} = \varphi_i : H^{\pi}_{n+1}(X,Y;{}^w\mathbb{Z}) = H^{\pi}_{n+1}(i;{}^w\mathbb{Z}) \longrightarrow Q^{n+1}(i:\dot{C}(Y) \longrightarrow \dot{C}(X)) .$$

An <u>(n+1)-dimensional geometric Poincaré pair</u> $(X,\partial X)$ is a CW pair of connected finitely-dominated CW complexes such that ∂X is an n-dimensional geometric Poincaré complex, together with a group morphism $w(X):\pi_1(X) \longrightarrow \mathbb{Z}_2$ such that $w(\partial X)$ factors as

$$w(\partial X) : \pi_1(\partial X) \longrightarrow \pi_1(X) \xrightarrow{w(X)} \mathbb{Z}_2$$

and with a relative homology class $[X] \in H^{\pi_1(X)}_{n+1}(\widetilde{X},\widetilde{\partial X};{}^{w(X)}\mathbb{Z})$ such that the cap products

$$[X]\cap - : {}^{w(X)}H^r_{\pi_1(X)}(\widetilde{X},\widetilde{\partial X}) \longrightarrow H_{n+1-r}(\widetilde{X}) \quad (0 \leq r \leq n+1)$$

are $\mathbb{Z}[\pi_1(X)]$-module isomorphisms (Poincaré-Lefschetz duality) and

$$\partial_*[X] = [\partial X] \in H^{\pi_1(X)}_n(\widetilde{\partial X};{}^{w(X)}\mathbb{Z}) ,$$

with $\widetilde{X}$ the universal cover of X and $\widetilde{\partial X}$ the induced cover of ∂X.

The relative symmetric construction of Proposition 6.1 gives a relative version of the construction of $\sigma^*(X)$ in Proposition 2.7.

<u>Proposition 6.2</u> Given an (n+1)-dimensional geometric Poincaré pair $(X,\partial X)$ and an oriented cover $\widetilde{X}$ of X with data (π,w) and induced cover $\widetilde{\partial X}$ of ∂X there is defined in a natural way an (n+1)-dimensional symmetric Poincaré pair over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$\sigma^*(X,\partial X) = (i_{\widetilde{X}}:C(\widetilde{\partial X}) \longrightarrow C(\widetilde{X}), \varphi_{\widetilde{X},\widetilde{\partial X}}[X] \in Q^{n+1}(i_{\widetilde{X}}))$$

with boundary $\sigma^*(\partial X) = (C(\widetilde{\partial X}), \varphi_{\widetilde{\partial X}}[\partial X] \in Q^n(C(\widetilde{\partial X})))$, where $i_{\widetilde{X}}$ is the inclusion.

[]

Define the <u>symmetric signature</u> of an n-dimensional geometric Poincaré complex X with respect to an oriented cover $\widetilde{X}$ of X with data (π,w) to be the symmetric Poincaré cobordism class

$$\sigma^*(X) \in L^n(\mathbb{Z}[\pi])$$

with $\sigma^*(X) = (C(\widetilde{X}), \varphi_{\widetilde{X}}[X] \in Q^n(C(\widetilde{X})))$ the n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ with the w-twisted involution constructed in Proposition 2.7. The symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}[\pi])$ is induced via the change of rings maps $\mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{Z}[\pi]$ from the universal symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])$ associated to the universal cover of X.

The construction of Proposition 6.2 and the symmetric signature invariant $\sigma^*(X)\in L^n(\mathbb{Z}[\pi_1(X)])$ are due to Mishchenko [2].

Given a connected space K and a group morphism $w:\pi_1(K)\longrightarrow\mathbb{Z}_2$ let $\Omega_n^P(K,w)$ be the group of geometric Poincaré bordism classes of maps $f:X\longrightarrow K$ from n-dimensional geometric Poincaré complexes X such that the orientation map factors as

$$w(X)\ :\ \pi_1(X)\xrightarrow{\ f\ }\pi_1(K)\xrightarrow{\ w\ }\mathbb{Z}_2\ ,$$

i.e. such that the cover $\widetilde{X}$ of X induced from the universal cover $\widetilde{K}$ of K is oriented with data $(\pi_1(K),w)$.

Proposition 6.3 The symmetric signature defines abelian group morphisms

$$\sigma^*\ :\Omega_n^P(K,w)\longrightarrow L^n(\mathbb{Z}[\pi_1(K)])\ ;\ (f:X\longrightarrow K)\longmapsto\sigma^*(X)\quad(n\geqslant 0)\ .$$

Proof: If $(g;f,f'):(Y;X,X')\longrightarrow K$ is an (n+1)-dimensional geometric Poincaré bordism then the construction of Proposition 6.2 defines an (n+1)-dimensional symmetric Poincaré cobordism over $\mathbb{Z}[\pi_1(K)]$ $\sigma^*(Y;X,X')$ from $\sigma^*(X)$ to $\sigma^*(X')$.

[]

As a special case of the geometric Poincaré bordism invariance of the symmetric signature we have homotopy invariance: if $f:X\longrightarrow X'$ is a homotopy equivalence of n-dimensional geometric Poincaré complexes then

$$\sigma^*(X)=\sigma^*(X')\in L^n(\mathbb{Z}[\pi_1(X)])\ .$$

In §13 we shall show that the simply-connected symmetric signature map

$$\sigma^*\ :\Omega_n^P(pt.)\longrightarrow L^n(\mathbb{Z})=\begin{cases}\mathbb{Z}\\ \mathbb{Z}_2\\ 0\\ 0\end{cases}\text{if } n\equiv\begin{cases}0\\1\\2\\3\end{cases}\pmod 4$$

sends an oriented $\begin{cases}4k-\\(4k+1)-\end{cases}$ dimensional geometric Poincaré complex X ($w(X)=1$) to

$$\begin{cases}\sigma^*(X^{4k})=(\text{signature of }X)\equiv(\text{signature of the Poincaré duality intersection}\\ \qquad\text{form }(H^{2k}(X;\mathbb{R}),\varphi_X[X]_0))\in L^{4k}(\mathbb{Z})=\mathbb{Z}\\ \sigma^*(X^{4k+1})=(\text{deRham invariant of }X)\equiv(\text{deRham invariant of the Seifert linking}\\ \qquad\text{form }(H^{2k+1}(X;\mathbb{Q}/\mathbb{Z}),\varphi_X[X]_0))=(\dim_{\mathbb{Z}_2}H^{2k+1}(X;\mathbb{Z}_2))\\ \qquad\in L^{4k+1}(\mathbb{Z})=\mathbb{Z}_2\end{cases}\ .$$

Much work has been done on the evaluation of the symmetric signature on smooth bordism

$$\sigma^*\ :\Omega_n^{SO}(K)\longrightarrow\Omega_n^P(K,1)\xrightarrow{\ \sigma^*\ }L^n(\mathbb{Z}[\pi_1(K)])$$

in terms of characteristic numbers, for which we refer to Mishchenko [3].

The <u>relative quadratic construction</u> $\Psi_{F,G}$ defined below is the relative version of the absolute quadratic construction $\dot{\Psi}_F$ of Proposition 2.5.

Proposition 6.4 Let π be a group, and give $\mathbb{Z}[\pi]$ the w-twisted involution for some group morphism $w:\pi\longrightarrow\mathbb{Z}_2$.

Given π-maps of finite-dimensional π-spaces $f:X\longrightarrow Y$, $f':X'\longrightarrow Y'$, $F:\Sigma^pX\longrightarrow\Sigma^pX'$, $G:\Sigma^pY\longrightarrow\Sigma^pY'$ for some $p\geqslant 0$ such that the diagram

$$\begin{array}{ccc}\Sigma^pX & \xrightarrow{\ \Sigma^pf\ } & \Sigma^pY\\ F\big\downarrow & & \big\downarrow G\\ \Sigma^pX' & \xrightarrow{\ \Sigma^pf'\ } & \Sigma^pY'\end{array}$$

commutes there are defined in a natural way abelian group morphisms

$$\Psi_{F,G}\ :\ H^\pi_{n+1}(f;{}^w\mathbb{Z})\longrightarrow Q_{n+1}(f':\dot{C}(X')\longrightarrow\dot{C}(Y'))\qquad(n\in\mathbb{Z})$$

such that

i) $h^\%\varphi_f-\varphi_{f'}h_*=(1+T)\Psi_{F,G}\ :\ H^\pi_{n+1}(f;{}^w\mathbb{Z})\longrightarrow Q^{n+1}(f')$,

with $h_*:H^\pi_{n+1}(f;{}^w\mathbb{Z})\longrightarrow H^\pi_{n+1}(f';{}^w\mathbb{Z})$, $h^\%:Q^{n+1}(f)\longrightarrow Q^{n+1}(f')$ the induced maps ,

ii) there is defined a map of long exact sequences

$$\begin{array}{ccccccccc}\dots\longrightarrow & H^\pi_{n+1}(f;{}^w\mathbb{Z}) & \longrightarrow & \dot{H}^\pi_n(X;{}^w\mathbb{Z}) & \xrightarrow{\ f_*\ } & \dot{H}^\pi_n(Y;{}^w\mathbb{Z}) & \longrightarrow & H^\pi_n(f;{}^w\mathbb{Z}) & \longrightarrow\dots\\ & \Psi_{F,G}\big\downarrow & & \dot{\Psi}_F\big\downarrow & & \dot{\Psi}_G\big\downarrow & & \Psi_{F,G}\big\downarrow & \\ \dots\longrightarrow & Q_{n+1}(f') & \longrightarrow & Q_n(\dot{C}(X')) & \xrightarrow{\ f'_\%\ } & Q_n(\dot{C}(Y')) & \longrightarrow & Q_n(f') & \longrightarrow\dots\ ,\end{array}$$

iii) $\Psi_{F,G}$ factorizes through $Q^{[0,p-1]}_{n+1}(f)$

$$\Psi_{F,G}\ :\ H^\pi_{n+1}(f;{}^w\mathbb{Z})\longrightarrow Q^{[0,p-1]}_{n+1}(f')\longrightarrow Q^{[0,\infty]}_{n+1}(f')\equiv Q_{n+1}(f')\ .$$

If $p=0$ then $\Psi_{F,G}=0$.

[]

Given a degree 1 map of (n+1)-dimensional geometric Poincaré pairs

$$(f,\partial f) : (M,\partial M) \longrightarrow (X,\partial X)$$

and a covering $\widetilde{X}$ of X with group of covering translations π define $\mathbb{Z}[\pi]$-module <u>Umkehr</u> chain maps

$$\begin{cases} f^! : C(\widetilde{X}) \longrightarrow C(\widetilde{M}) \\ \partial f^! : C(\widetilde{\partial X}) \longrightarrow C(\widetilde{\partial M}) \end{cases}$$

by applying $\mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi_1(X)]} -$ to the $\mathbb{Z}[\pi_1(X)]$-module Umkehr chain maps

$$\begin{cases} f^! : C(\tilde{\tilde{X}}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\tilde{\tilde{X}},\tilde{\tilde{\partial X}})^{n+1-*} \xrightarrow{\tilde{\tilde{f}}^*} {}^{w(X)}C(\tilde{\tilde{M}},\tilde{\tilde{\partial M}})^{n+1-*} \xrightarrow{[M]\cap -} C(\tilde{\tilde{M}}) \\ \partial f^! : C(\tilde{\tilde{\partial X}}) \xrightarrow{([\partial X]\cap -)^{-1}} {}^{w(X)}C(\tilde{\tilde{\partial X}})^{n-*} \xrightarrow{\tilde{\tilde{\partial f}}^*} {}^{w(X)}C(\tilde{\tilde{\partial M}})^{n-*} \xrightarrow{[\partial M]\cap -} C(\tilde{\tilde{\partial M}}) \end{cases}$$

with $\tilde{\tilde{X}}$ the universal cover of X and $\tilde{\tilde{M}},\tilde{\tilde{\partial M}},\tilde{\tilde{\partial X}}$ the induced covers of $M,\partial M,\partial X$.

There is defined a chain homotopy commutative diagram

$$\begin{array}{ccc} C(\widetilde{\partial X}) & \xrightarrow{i_{\widetilde{X}}} & C(\widetilde{X}) \\ \partial f^! \downarrow & & \downarrow f^! \\ C(\widetilde{\partial M}) & \xrightarrow{i_{\widetilde{M}}} & C(\widetilde{M}) \end{array}$$

with $i_{\widetilde{M}},i_{\widetilde{X}}$ the inclusions, so that there is induced a $\mathbb{Z}[\pi]$-module chain map in the algebraic mapping cones

$$i_{\tilde{f}} : C(\partial f^!) \longrightarrow C(f^!) \ .$$

A <u>geometric Umkehr map</u> for $(f,\partial f)$ is a π-map of pairs of π-spaces

$$(F,\partial F) : (\Sigma^p\widetilde{X}_+,\Sigma^p(\widetilde{\partial X})_+) \longrightarrow (\Sigma^p\widetilde{M}_+,\Sigma^p(\widetilde{\partial M})_+)$$

for some $p\geqslant 0$, which induces the Umkehr $(f^!,\partial f^!)$ on the chain level.

The relative $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ construction of Proposition $\begin{Bmatrix}6.1\\ 6.4\end{Bmatrix}$ can be used to obtain a relative analogue of the $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ kernel $\begin{Bmatrix}\sigma^*(f)\\ \sigma_*(f,F)\end{Bmatrix}$ of Proposition $\begin{Bmatrix}2.8\\ 2.9\end{Bmatrix}$, as follows.

<u>Proposition 6.5</u> Given a degree 1 map of (n+1)-dimensional geometric Poincaré pairs

$$(f,\partial f) : (M,\partial M) \longrightarrow (X,\partial X)$$

and an oriented cover $\widetilde{X}$ of X with data (π,w) there is defined in a natural way a <u>symmetric kernel</u> (n+1)-dimensional symmetric Poincaré pair over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$\sigma^*(f,\partial f) = (i_{\tilde{f}}:C(\partial f^!) \longrightarrow C(f^!), e^{\%}\varphi_{\widetilde{M},\widetilde{\partial M}}[M]\in Q^{n+1}(i_{\tilde{f}}))$$

with boundary $\sigma^*(\partial f)$, and such that

$$\sigma^*(M,\partial M) = \sigma^*(f,\partial f)\oplus\sigma^*(X,\partial X)$$

up to homotopy equivalence of pairs. Given also a geometric Umkehr map

$$(F,\partial F) : (\Sigma^p\widetilde{X}_+,\Sigma^p\widetilde{\partial X}_+) \longrightarrow (\Sigma^p\widetilde{M}_+,\Sigma^p\widetilde{\partial M}_+)$$

there is defined in a natural way a <u>quadratic kernel</u> (n+1)-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi]$

$$\sigma_*(f,\partial f;F,\partial F) = (i_{\tilde{f}}:C(\partial f^!) \longrightarrow C(f^!), e_{\%}\psi_{F,\partial F}[X]\in Q_{n+1}(i_{\tilde{f}}))$$

with boundary $\sigma_*(\partial f,\partial F)$, and such that

$$(1+T)\sigma_*(f,\partial f;F,\partial F) = \sigma^*(f,\partial f) \ .$$

[]

Next, we outline the relative version of the equivariant S-duality theory of §3 required to obtain geometric Umkehr maps for normal bundle maps of pairs. A <u>π-pair</u> (X,Y) is a pair of π-spaces, $Y\subset X$, in which case the suspension $\Sigma(X,Y) = (\Sigma X,\Sigma Y)$ is also a π-pair. Given π-pairs (X,Y),(A,B) let $\{X,Y;A,B\}_\pi$ be the abelian group of stable relative π-homotopy classes of π-maps of π-pairs $(f,g):\Sigma^p(X,Y) \longrightarrow \Sigma^p(A,B)$ $(p\geqslant 0)$. The π-pairs $(X,Y),(X^*,Y^*)$ are <u>relatively Sπ-dual</u> if there is given a {1}-map of pairs

$$(\alpha,\beta) : (D^N,S^{N-1}) \longrightarrow (X\wedge_\pi X^*,Y\wedge_\pi Y^*)$$

such that for every π-spectrum of pairs $(\underline{A},\underline{B})$ the slant products

$$\backslash : \{X,Y;\underline{A},\underline{B}\}_\pi \longrightarrow \{D^N,S^{N-1};\underline{A}\wedge_\pi X^*,\underline{B}\wedge_\pi Y^*\} \ ;$$

$$((f,g):(\Sigma^pX,\Sigma^pY)\longrightarrow(A_p,B_p))$$

$$\longmapsto (((f\wedge 1)\Sigma^p\alpha,(g\wedge 1)\Sigma^p\beta) : (D^{N+p},S^{N+p-1}) \longrightarrow (A_p\wedge_\pi X^*,B_p\wedge_\pi Y^*))$$

$$\backslash : \{X^*,Y^*;\underline{A},\underline{B}\}_\pi \longrightarrow \{D^N,S^{N-1};X\wedge_\pi\underline{A},Y\wedge_\pi\underline{B}\} \ ;$$

$$((f^*,g^*):(\Sigma^pX^*,\Sigma^pY^*)\longrightarrow(A_p,B_p))$$

$$\longmapsto (((1\wedge f^*)\Sigma^p\alpha,(1\wedge g^*)\Sigma^p\beta):(D^{N+p},S^{N+p-1}) \longrightarrow (X\wedge_\pi A_p,Y\wedge_\pi B_p))$$

 are isomorphisms and such that the $\{1\}$-map $\beta : S^{N-1} \longrightarrow Y \wedge_\pi Y^*$ is an absolute $S\pi$-duality map. It then follows that there are defined absolute $S\pi$-duality maps

$$\alpha/\beta \; : \; S^N \longrightarrow X/Y \wedge_\pi X^* \quad , \; \alpha/\beta : \; S^N \longrightarrow X \wedge_\pi X^*/Y^* \; .$$

An n-dimensional geometric Poincaré pair $(X,\partial X)$ can be embedded in (D^{n+k},S^{n+k-1}) (k large) with $X \cap S^{n+k-1} = \partial X \subset S^{n+k-1}$, such that there exists a closed regular neighbourhood E of X in D^{n+k} with $E' = E \cap S^{n+k-1} \subset \partial E$ a closed regular neighbourhood of ∂X in S^{n+k-1}. The inclusions $\overline{\partial E \backslash E'} \hookrightarrow E$, $\partial E' \hookrightarrow E'$ define $(k-1)$-spherical fibrations

$$S^{k-1} \longrightarrow \overline{\partial E \backslash E'} \xrightarrow{\nu_X} E = X$$

$$S^{k-1} \longrightarrow \partial E' \xrightarrow{\nu_{\partial X}} E' = \partial X$$

such that $\nu_{\partial X}$ is the restriction of ν_X to ∂X

$$\nu_{\partial X} : \partial X \hookrightarrow X \xrightarrow{\nu_X} BG(k) \; .$$

The collapsing map of $\{1\}$-pairs

$$(\rho_X,\rho_{\partial X}) : (D^{n+k},S^{n+k-1}) \longrightarrow (D^{n+k}/\overline{D^{n+k}-E}, S^{n+k-1}/\overline{S^{n+k-1}-E'})$$
$$= (E/\overline{\partial E \backslash E'}, E'/\partial E') = (T(\nu_X),T(\nu_{\partial X}))$$

can be used to define a relative $S\pi$-duality map

$$(\alpha_X,\alpha_{\partial X}) : (D^{n+k},S^{n+k-1}) \xrightarrow{(\rho_X,\rho_{\partial X})} (T(\nu_X),T(\nu_{\partial X})) \xrightarrow{\Delta} (\widetilde{X}_+ \wedge_\pi T\pi(\nu_X), \partial\widetilde{X}_+ \wedge_\pi T\pi(\nu_{\partial X}))$$

between the π-pairs $(\widetilde{X}_+,\partial\widetilde{X}_+)$ and $(T\pi(\nu_X),T\pi(\nu_{\partial X}))$ for any covering $\widetilde{X}$ of X with group of covering translations π. Given n-dimensional geometric Poincaré pairs $(M,\partial M)$, $(X,\partial X)$ and any coverings $\widetilde{M},\widetilde{X}$ with the same group of covering translations π we thus have relative $S\pi$-duality isomorphisms

$$\{T\pi(\nu_M),T\pi(\nu_{\partial M});T\pi(\nu_X),T\pi(\nu_{\partial X})\}_\pi \longrightarrow \{D^{n+k},S^{n+k-1};\widetilde{M}_+\wedge_\pi T\pi(\nu_X),\partial\widetilde{M}_+\wedge_\pi T\pi(\nu_{\partial X})\}$$
$$\longrightarrow \{\widetilde{X}_+,\partial\widetilde{X}_+;\widetilde{M}_+,\partial\widetilde{M}_+\}_\pi \; .$$

Thus given a normal bundle map of pairs

$$(f,\partial f;b,\partial b) \; : \; (M,\partial M) \longrightarrow (X,\partial X)$$

and an oriented covering $\widetilde{X}$ of X with data (π,w) the $S\pi$-dual of the π-map of π-pairs

$$(T\pi(b),T\pi(\partial b)) \; : \; (T\pi(\nu_M),T\pi(\nu_{\partial M})) \longrightarrow (T\pi(\nu_X),T\pi(\nu_{\partial X}))$$

is the relative $S\pi$-homotopy class of a relative geometric Umkehr map

$$(F,\partial F) \; : \; (\Sigma^p\widetilde{X}_+,\Sigma^p\partial\widetilde{X}_+) \longrightarrow (\Sigma^p\widetilde{M}_+,\Sigma^p\partial\widetilde{M}_+) \; .$$

The construction of Proposition 6.5 now gives a quadratic kernel n-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi]$

$$\sigma_*(f,\partial f;b,\partial b) = \sigma_*(f,\partial f;F,\partial F)$$

with boundary $\sigma_*(f,b)$.

Define the $\begin{Bmatrix}\text{\underline{symmetric}}\\ \text{\underline{quadratic}}\end{Bmatrix}$ signature $\begin{Bmatrix}\sigma^*(f) \in L^n(\mathbb{Z}[\pi_1(X)])\\ \sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])\end{Bmatrix}$ of a $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ map $\begin{Bmatrix}f:M \longrightarrow X\\ (f,b):M \longrightarrow X\end{Bmatrix}$ of n-dimensional geometric Poincaré complexes to be the $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ Poincaré cobordism class of the $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ kernel

$$\begin{cases}\sigma^*(f) = (C(f^!),e^\%\varphi_{\widetilde{M}}[M] \in Q^n(C(f^!)))\\ \sigma^*(f,b) = (C(f^!),e_\%\psi_F[X] \in Q_n(C(f^!)))\end{cases}$$

defined in $\begin{Bmatrix}§2\\ §3\end{Bmatrix}$.

A $\begin{Bmatrix}\text{\underline{degree 1}}\\ \text{\underline{normal}}\end{Bmatrix}$ $\underline{\text{bordism}}$ between n-dimensional $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ maps $\begin{Bmatrix}f:M \longrightarrow X, \; f':M' \longrightarrow X\\ (f,b):M \longrightarrow X, \; (f',b'):M' \longrightarrow X\end{Bmatrix}$ is a $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ map of (n+1)-dimensional geometric Poincaré cobordisms

$$\begin{cases}(g;f,f') \; : \; (N;M,M') \longrightarrow (X \times I;X\times\{0\},X\times\{1\})\\ ((g;f,f'),(c;b,b')) \; : \; (N;M,M') \longrightarrow (X \times I;X\times\{0\},X\times\{1\})\end{cases} \quad (I = [0,1]) \; .$$

<u>Proposition 6.6</u> i) The $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ signature $\begin{Bmatrix}\sigma^*(f) \in L^n(\mathbb{Z}[\pi_1(X)])\\ \sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])\end{Bmatrix}$ of an n-dimensional $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ map $\begin{Bmatrix}f:M \longrightarrow X\\ (f,b):M \longrightarrow X\end{Bmatrix}$ is a $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ bordism invariant such that

$$\begin{cases}\sigma^*(f) = \sigma^*(M) - \sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])\\ (1+T)\sigma_*(f,b) = \sigma^*(f) \in L^n(\mathbb{Z}[\pi_1(X)])\end{cases} \; .$$

ii) The $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ signature of the composite $\begin{Bmatrix}gf:M \longrightarrow Y\\ (gf,cb):M \longrightarrow Y\end{Bmatrix}$ of n-dimensional $\begin{Bmatrix}\text{degree 1}\\ \text{normal}\end{Bmatrix}$ maps $\begin{Bmatrix}f:M \longrightarrow X, g:X \longrightarrow Y\\ (f,b):M \longrightarrow X, (g,c):X \longrightarrow Y\end{Bmatrix}$ is the sum

$$\begin{cases}\sigma^*(gf) = \sigma^*(f) + \sigma^*(g) \in L^n(\mathbb{Z}[\pi_1(Y)])\\ \sigma_*(gf,cb) = \sigma_*(f,b) + \sigma_*(g,c) \in L_n(\mathbb{Z}[\pi_1(Y)])\end{cases} \; .$$

Proof: The $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel $\begin{cases}\sigma^*(g;f,f')\\ \sigma_*((g;f,f'),(c;b,b'))\end{cases}$ of a $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ bordism $\begin{cases}(g;f,f')\\ ((g;f,f'),(c;b,b'))\end{cases}$ is a $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré cobordism between the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernels $\begin{cases}\sigma^*(f),\sigma^*(f')\\ \sigma_*(f,b),\ \sigma_*(f',b')\end{cases}$. Proposition $\begin{cases}2.8\\ 2.9\end{cases}$ gives that $\begin{cases}\sigma^*(f)\oplus\sigma^*(X)=\sigma^*(M)\\ (1+T)\sigma_*(f,b)=\sigma^*(f)\end{cases}$, and Proposition $\begin{cases}2.11\\ 3.14\end{cases}$ that $\begin{cases}\sigma^*(gf)=\sigma^*(f)\oplus\sigma^*(g)\\ \sigma_*(gf,cb)=\sigma_*(f,b)\oplus\sigma_*(g,c)\end{cases}$, up to homotopy equivalence. By Proposition 5.2 homotopy equivalent $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complexes are cobordant.

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In Proposition 8.1 we shall identify the quadratic signature $\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$ of an n-dimensional normal bundle map $(f,b):M\longrightarrow X$ with the surgery obstruction $\theta(f,b)\in L_n(\pi_1(X))$ obtained by Wall [5] using geometric intersection and self-intersection forms. We have already related the two constructions in Proposition 4.4, and the normal bordism invariance of the quadratic signature (Proposition 6.6 i)) ensures that there is defined a morphism of abelian groups

$$L_n(\pi_1(X))\longrightarrow L_n(\mathbb{Z}[\pi_1(X)])\ ;\ \theta(f,b)\longmapsto\sigma_*(f,b)\ .$$

In §7 below we shall give an algebraic definition of this map, and prove that it is an isomorphism.

In view of the above, the quadratic signature sum formula of Proposition 6.6 ii) may be considered as a homotopy-theoretic version of the sum formulae of §17H of Wall [5] and Theorem 7.0 of Jones[1], as reformulated in the correction to Jones [1] (which made precise the notion of a normal map of geometric Poincaré complexes by including the choice of spherical generator for the top homology classes in the Thom complexes of Spivak normal fibrations, just as we have done in §3 above).

The geometric glueing of adjoining geometric bordisms induces the union operation of §5 on the chain level.

__Proposition 6.7__ Let $\begin{cases}(g;f,f'):(N;M,M')\longrightarrow(X\times[0,1];X\times 0,X\times 1)\\ ((g;f,f'),(c;b,b')):(N;M,M')\longrightarrow(X\times[0,1];X\times 0,X\times 1)\end{cases}$ and $\begin{cases}(g';f',f''):(N';M',M'')\longrightarrow(X\times[1,2];X\times 1,X\times 2)\\ ((g';f',f''),(c';b',b'')):(N';M',M'')\longrightarrow(X\times[1,2];X\times 1,X\times 2)\end{cases}$ be adjoining $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ bordisms of n-dimensional geometric Poincaré complexes, with $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernels $\begin{cases}\sigma^*(g;f,f')\\ \sigma_*((g;f,f'),(c;b,b'))\end{cases}$ and $\begin{cases}\sigma^*(g';f',f'')\\ \sigma_*((g';f',f''),(c';b',b''))\end{cases}$. Then the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel of the geometric union is given up to homotopy equivalence by the algebraic union of the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernels

$$\begin{cases}\sigma^*(g\cup_f g';f,f'')=\sigma^*(g;f,f')\cup_{\sigma^*(f')}\sigma^*(g';f',f'')\\ \sigma_*((g\cup_f g';f,f''),(c\cup_b c';b,b''))\\ \qquad\qquad=\sigma_*((g;f,f'),(c;b,b'))\cup_{\sigma_*(f',b')}\sigma_*((g';f',f''),(c';b',b''))\ .\end{cases}$$

[]

Given adjoining $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ bordisms of n-dimensional geometric Poincaré complexes (as above) we have a map of exact sequences

$$\begin{cases}\begin{array}{ccccccc}\dots\to H^\pi_{n+1}(M';{}^w\mathbb{Z}) & \to & H_{n+1} & \to & H^\pi_{n+1}(N,M\cup M';{}^w\mathbb{Z})\oplus H^\pi_{n+1}(N',M'\cup M'';{}^w\mathbb{Z}) & \to & H^\pi_n(M';{}^w\mathbb{Z})\to\dots\\ \downarrow\varphi_{M'} & & \downarrow\varphi & & \downarrow\varphi_{N,M\cup M'}\oplus\varphi_{N',M'\cup M''} & & \downarrow\varphi_{M'}\\ \dots\to Q^{n+1}(C(\tilde{M}')) & \to & Q^{n+1} & \to & Q^{n+1}((f\ f'))\oplus Q^{n+1}((f'\ f'')) & \to & Q^n(C(\tilde{M}'))\to\dots\\ \dots\to H^\pi_{n+1}(X_1;{}^w\mathbb{Z}) & \to & H_{n+1} & \to & H^\pi_{n+1}(X_{01},X_0\cup X_1;{}^w\mathbb{Z})\oplus H_{n+1}(X_{12},X_1\cup X_2;{}^w\mathbb{Z}) & \to & H^\pi_n(X_1;{}^w\mathbb{Z})\to\dots\\ \downarrow\psi_{F'} & & \downarrow\psi & & \downarrow\psi_{G,F\cup F'}\oplus\psi_{G',F'\cup F''} & & \downarrow\psi_{F'}\\ \dots\to Q_{n+1}(C(\tilde{M}')) & \to & Q_{n+1} & \to & Q_{n+1}((f\ f'))\oplus Q_{n+1}((f'\ f'')) & \to & Q_n(C(\tilde{M}'))\to\dots\end{array}\end{cases}$$

with $(\pi,w)=(\pi_1(X),w_1(X))$, $X_{ij}=X\times[i,j]$. Now $\begin{cases}H^\pi_{n+1}(M';{}^w\mathbb{Z})=0\\ H^\pi_{n+1}(X_1;{}^w\mathbb{Z})=0\end{cases}$, so that there is no indeterminacy in the union of $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré cobordisms arising out of geometry.

§7. Algebraic surgery

The original work of Milnor [4], Wallace [1] and Kervaire-Milnor [1] developed $\begin{Bmatrix}\text{oriented}\\ \text{framed}\end{Bmatrix}$ surgery as a method for killing the homotopy groups of an $\begin{Bmatrix}\text{oriented}\\ \text{framed}\end{Bmatrix}$ manifold M. The framed surgery technique was generalized to surgery to homotopy equivalence on a normal bundle map $(f,b):M\longrightarrow X$ from a compact manifold M to a finite geometric Poincaré complex X (previously $X = S^n$) by Browder [2] and Novikov [1] (for $\pi_1(X) = \{1\}$) and Wall [3],[5] (any $\pi_1(X)$). The manifold may be taken to be smooth, PL, topological, or according to Maunder [1] even a homology manifold. There are also versions for paracompact M and infinite X, due to Taylor [1] and Maumary [1]. Various authors - Levitt [1], Jones [1], Lannes-Latour-Morlet [1], Quinn [3] - went on to consider framed surgery on normal maps of geometric Poincaré complexes. In all cases the surgery obstructions lie in the groups $L_n(\pi_1(X),w(X))$ of Wall [5], or one of the closely related variants described in §12. We shall now develop a theory of surgery on $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ Poincaré complexes, as a method for killing homology.

In §8 we shall show that the chain level effect of $\begin{Bmatrix}\text{oriented}\\ \text{framed}\end{Bmatrix}$ surgery in geometry is $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ surgery in the algebra. The $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ Poincaré cobordism class is the obstruction to killing an $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ Poincaré complex by surgery.

In particular, the surgery obstruction $\theta(f,b)\in L_n(\pi_1(X))$ of an n-dimensional normal bundle map $(f,b):M\longrightarrow X$ coincides with the quadratic signature $\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$, which is the obstruction to quadratic surgery on the quadratic kernel $\sigma_*(f,b)$.

An n-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ pair $\begin{Bmatrix}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^n(f,\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_n(f,\varepsilon))\end{Bmatrix}$ is connected if

$$\begin{cases} H_0\left(\begin{pmatrix}\delta\varphi_0\\ \varphi_0 f^*\end{pmatrix}:D^{n-*}\longrightarrow C(f)\right) = 0\\ H_0\left(\begin{pmatrix}(1+T_\varepsilon)\delta\psi_0\\ (1+T_\varepsilon)\psi_0 f^*\end{pmatrix}:D^{n-*}\longrightarrow C(f)\right) = 0\ .\end{cases}$$

In particular, Poincaré pairs are connected.

Define as follows the connected n-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ complex $\begin{Bmatrix}(C',\varphi'\in Q^n(C',\varepsilon))\\ (C',\psi'\in Q_n(C',\varepsilon))\end{Bmatrix}$ obtained from a connected n-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ complex $\begin{Bmatrix}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{Bmatrix}$ by $\begin{Bmatrix}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\end{Bmatrix}$ surgery on a connected (n+1)-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ pair $\begin{Bmatrix}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{Bmatrix}$.

In the ε-symmetric case let

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-)^{n+1}\varphi_0 f^*\\ (-)^r f & d_D & (-)^r\delta\varphi_0\\ 0 & 0 & (-)^r d_D^*\end{pmatrix}$$

$$: C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\longrightarrow C'_{r-1} = C_{r-1}\oplus D_r\oplus D^{n-r+2}$$

$$\varphi'_0 = \begin{pmatrix} \varphi_0 & 0 & 0\\ (-)^{n-r}fT_\varepsilon\varphi_1 & (-)^{n-r}T_\varepsilon\delta\varphi_1 & (-)^{r(n-r)}\varepsilon\\ 0 & 1 & 0\end{pmatrix}$$

$$: C'^{n-r} = C^{n-r}\oplus D^{n-r+1}\oplus D_{r+1}\longrightarrow C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}$$

$$\varphi'_s = \begin{pmatrix} \varphi_s & 0 & 0\\ (-)^{n-r}fT_\varepsilon\varphi_{s+1} & (-)^{n-r+s}T_\varepsilon\delta\varphi_{s+1} & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$: C'^{n-r+s} = C^{n-r+s}\oplus D^{n-r+s+1}\oplus D_{r-s+1}\longrightarrow C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\quad (s\geqslant 1).$$

In the ε-quadratic case let

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-)^{n+1}(1+T_\varepsilon)\psi_0 f^* \\ (-)^r f & d_D & (-)^r(1+T_\varepsilon)\delta\psi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix}$$

$$: C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \longrightarrow C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2}$$

$$\psi'_0 = \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}$$

$$\psi'_s = \begin{pmatrix} \psi_s & (-)^{r+s} T_\varepsilon \psi_{s-1} f^* & 0 \\ 0 & (-)^{n-r-s+1} T_\varepsilon \delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$: C'^{n-r-s} = C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \qquad (s \geq 1) .$$

(We are using matrix notation as if C,D were f.g. projective chain complexes).

It may be verified that performing $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ surgery using a different cycle representative of $\begin{Bmatrix} (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon) \\ (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon) \end{Bmatrix}$ leads to an isomorphic $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ complex $\begin{Bmatrix} (C',\varphi') \\ (C',\psi') \end{Bmatrix}$. Note that the ε-symmetrization of the ε-quadratic surgery on $(f:C \longrightarrow D,(\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))$ is the ε-symmetric surgery on $(f:C \longrightarrow D,(1+T_\varepsilon)(\delta\psi,\psi) \in Q^{n+1}(f,\varepsilon))$.

Let $\begin{Bmatrix} (C,\varphi \in Q^n(C,\varepsilon)) \\ (C,\psi \in Q_n(C,\varepsilon)) \end{Bmatrix}$ be a connected n-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ complex. The (n-1)-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ complex obtained from (0,0) by surgery on the connected n-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ pair $\begin{Bmatrix} (0:0 \longrightarrow C,(0,\varphi) \in Q^n(0,\varepsilon)) \\ (0:0 \longrightarrow C,(0,\psi) \in Q_n(C,\varepsilon)) \end{Bmatrix}$ is just the boundary $\begin{Bmatrix} \partial(C,\varphi) \\ \partial(C,\psi) \end{Bmatrix}$, as defined in §5 above. We can thus interpret Proposition 5.4 iii) as stating that an n-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ Poincaré complex is null-cobordant if and only if it can be obtained from 0 by surgery and homotopy equivalence. This is a special case of the following result, that cobordism is the equivalence relation on algebraic Poincaré complexes generated by surgery and homotopy equivalence. There is an obvious analogy here with Theorem 1 of Milnor [4], which showed that compact oriented manifolds are cobordant if and only if one can be obtained from the other by a sequence of geometric surgeries. In §8 below we shall make this analogy more precise.

<u>Proposition 7.1</u> i) Algebraic surgery preserves the homotopy type of the boundary, sending algebraic Poincaré complexes to algebraic Poincaré complexes.

ii) The n-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ Poincaré complexes $\begin{Bmatrix} (C,\varphi),(C',\varphi') \\ (C,\psi),(C',\psi') \end{Bmatrix}$ are cobordant if and only if $\begin{Bmatrix} (C',\varphi') \\ (C',\psi') \end{Bmatrix}$ can be obtained from $\begin{Bmatrix} (C,\varphi) \\ (C,\psi) \end{Bmatrix}$ by surgery and homotopy equivalence.

<u>Proof</u>: i) Let $\begin{Bmatrix} (f:C \longrightarrow D,(\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon)),(C',\varphi' \in Q^n(C',\varepsilon)) \\ (f:C \longrightarrow D,(\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon)),(C',\psi' \in Q_n(C',\varepsilon)) \end{Bmatrix}$ be as in the definition of $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ surgery. Then the A-module morphisms

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ (-)^{r(n-r)} f & 0 \end{pmatrix}$$

$$: \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow \partial C'_r = C_{r+1} \oplus D_{r+2} \oplus D^{n-r} \oplus C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1}$$

define a homotopy equivalence of the boundary (n-1)-dimensional $\begin{Bmatrix} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{Bmatrix}$ Poincaré complexes

$$\begin{cases} h : \partial(C,\varphi) \longrightarrow \partial(C',\varphi') \\ h : \partial(C,\psi) \longrightarrow \partial(C',\psi') \end{cases} .$$

Proposition 5.4 ii) states that $\begin{Bmatrix} (C,\varphi) \\ (C,\psi) \end{Bmatrix}$ is Poincaré if and only if $\begin{Bmatrix} \partial(C,\varphi) \\ \partial(C,\psi) \end{Bmatrix}$ is contractible. It follows that $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ is Poincaré if and only if $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ is Poincaré.

ii) Continuing with the above notation assume also that $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ is Poincaré, and define an $(n+1)$-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré pair

$$\begin{cases}((g\ g'):C\oplus C'\longrightarrow D',(0,\varphi\oplus-\varphi')\in Q^{n+1}((g\ g'),\varepsilon))\\((g\ g'):C\oplus C'\longrightarrow D',(0,\psi\oplus-\psi')\in Q_{n+1}((g\ g'),\varepsilon))\end{cases}$$

by

$$d_{D'}=\begin{cases}\begin{pmatrix}d_C & (-)^{n+1}\varphi_0 f^*\\ 0 & (-)^r d_D^*\end{pmatrix}\\ \begin{pmatrix}d_C & (-)^{n+1}(1+T_\varepsilon)\psi_0 f^*\\ 0 & (-)^r d_D^*\end{pmatrix}\end{cases} : D'_r=C_r\oplus D^{n-r+1}\longrightarrow D'_{r-1}=C_{r-1}\oplus D^{n-r+2}$$

$$g=\begin{pmatrix}1\\0\end{pmatrix}: C_r\longrightarrow D'_r=C_r\oplus D^{n-r+1}$$

$$g'=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}: C'_r=C_r\oplus D_{r+1}\oplus D^{n-r+1}\longrightarrow D'_r=C_r\oplus D^{n-r+1}\ .$$

We thus have a cobordism from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ to $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$, so that surgery preserves cobordism classes.

Conversely, suppose given a cobordism of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes $\begin{cases}((f\ f'):C\oplus C'\longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ f'),\varepsilon))\\((f\ f'):C\oplus C'\longrightarrow D,(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f\ f'),\varepsilon))\end{cases}$. Let $\begin{cases}(C'',\varphi'')\\(C'',\psi'')\end{cases}$ be the n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré complex obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on the connected $(n+1)$-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ pair $\begin{cases}(g:C\longrightarrow D',(\delta\varphi',\varphi)\in Q^{n+1}(g,\varepsilon))\\(g:C\longrightarrow D',(\delta\psi',\psi)\in Q_{n+1}(g,\varepsilon))\end{cases}$ defined by

$$d_{D'}=\begin{pmatrix}d_D & (-)^{r-1}f'\\ 0 & d_{C'}\end{pmatrix}: D'_r=D_r\oplus C'_{r-1}\longrightarrow D'_{r-1}=D_{r-1}\oplus C'_{r-2}\qquad (D'=C(f'))$$

$$g=\begin{pmatrix}f\\0\end{pmatrix}: C_r\longrightarrow D'_r=D_r\oplus C'_{r-1}$$

$$\begin{cases}\delta\varphi'_s=\begin{pmatrix}\delta\varphi_s & (-)^s f'\varphi'_s\\ 0 & (-)^{n-r+s}T_\varepsilon\varphi'_{s-1}\end{pmatrix}: D'^{n-r+s+1}=D^{n-r+s+1}\oplus C'^{n-r+s}\longrightarrow D'_r=D_r\oplus C'_{r-1}\\ \qquad\qquad (s\geqslant 0,\ \varphi'_{-1}=0)\\ \delta\psi'_s=\begin{pmatrix}\delta\psi_s & (-)^s f'\psi'_s\\ 0 & (-)^{n-r-s+1}T_\varepsilon\psi'_{s+1}\end{pmatrix}: D'^{n-r-s+1}=D^{n-r-s+1}\oplus C'^{n-r-s}\longrightarrow D'_r=D_r\oplus C'_{r-1}\\ \qquad\qquad (s\geqslant 0)\end{cases}$$

The A-module morphisms

$$\begin{cases}h=(0\ \ 0\ \ 1\ \ 0\ \ 0): C''_r=C_r\oplus D_{r+1}\oplus C'_r\oplus D^{n-r+1}\oplus C'^{n-r}\longrightarrow C'_r\\ h=(0\ \ 0\ \ 1\ \ 0\ \ -T_\varepsilon\psi'_0): C''_r=C_r\oplus D_{r+1}\oplus C'_r\oplus D^{n-r+1}\oplus C'^{n-r}\longrightarrow C'_r\end{cases}$$

define a homotopy equivalence of n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes

$$\begin{cases}h:(C'',\varphi'')\longrightarrow(C',\varphi')\\ h:(C'',\psi'')\longrightarrow(C',\psi')\end{cases}$$

Thus $\begin{cases}(C',\psi')\\(C',\varphi')\end{cases}$ may be obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by an $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery followed by a homotopy equivalence.

[]

We shall prove that certain skew-suspension maps in the L-groups are isomorphisms using the following criterion.

<u>Proposition 7.2</u> The skew-suspension map $\begin{cases}\bar{S}:L^n(A,\varepsilon)\longrightarrow L^{n+2}(A,-\varepsilon)\\ \bar{S}:L_n(A,\varepsilon)\longrightarrow L_{n+2}(A,-\varepsilon)\end{cases}$ is onto (resp. one-one) if for every connected $(n+2)$-(resp. $(n+3)$-)dimensional $\begin{cases}(-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ with a boundary $\begin{cases}\partial(C,\varphi)\\\partial(C,\psi)\end{cases}$ which is contractible (resp. a skew-suspension) it is possible to do $\begin{cases}(-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ surgery on $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ to obtain a skew-suspension.

<u>Proof</u>: It is immediate from Proposition 7.1 ii) that the claimed condition for

$$\begin{cases} \overline{S}: L^n(A,\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \\ \overline{S}: L_n(A,\varepsilon) \longrightarrow L_{n+2}(A,-\varepsilon) \end{cases}$$ to be onto is both necessary and sufficient.

Assume the claimed condition for $\begin{cases} \overline{S}: L^n(A,\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \\ \overline{S}: L_n(A,\varepsilon) \longrightarrow L_{n+2}(A,-\varepsilon) \end{cases}$ to be one-one,

and let $\begin{cases} (C,\varphi) \\ (C,\psi) \end{cases}$ be an n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complex over A such that

$$\begin{cases} \overline{S}(C,\varphi) = 0 \in L^{n+2}(A,-\varepsilon) \\ \overline{S}(C,\psi) = 0 \in L_{n+2}(A,-\varepsilon)\ . \end{cases}$$

By Proposition 5.4 iii) we have that $\begin{cases} \overline{S}(C,\varphi) \\ \overline{S}(C,\psi) \end{cases}$ is homotopy equivalent to the

boundary $\begin{cases} \partial(D,\nu) \\ \partial(D,\chi) \end{cases}$ of a connected (n+3)-dimensional $\begin{cases} (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ complex $\begin{cases} (D,\nu) \\ (D,\chi) \end{cases}$.

By hypothesis, it is possible to do surgery on $\begin{cases} (D,\nu) \\ (D,\chi) \end{cases}$ to obtain the

skew-suspension $\begin{cases} \overline{S}(D',\nu') \\ \overline{S}(D',\chi') \end{cases}$ of a connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complex

$\begin{cases} (D',\nu') \\ (D',\chi') \end{cases}$. It now follows from Proposition 7.1 i) that $\begin{cases} (C,\varphi) \\ (C,\psi) \end{cases}$ is homotopy

equivalent to the boundary $\begin{cases} \partial(D',\nu') \\ \partial(D',\chi') \end{cases}$, and so

$$\begin{cases} (C,\varphi) = 0 \in L^n(A,\varepsilon) \\ (C,\psi) = 0 \in L_n(A,\varepsilon). \end{cases}$$

Therefore the stated condition is sufficient to ensure that the skew-suspension map is one-one.

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As a first application of our algebraic surgery we shall establish the 4-periodicity in the ε-quadratic L-groups, $L_n(A,\varepsilon) = L_{n+4}(A,\varepsilon)$, by analogy with the familiar result that is is always possible to perform framed surgery below the middle dimension and the familiar 4-periodicity $L_n(\pi) = L_{n+4}(\pi)$ of Wall [5].

<u>Proposition 7.3</u> The i-fold skew-suspension map

$$\overline{S}^i : L_{n-2i}(A,(-)^i\varepsilon) \longrightarrow L_n(A,\varepsilon) \quad (n \geqslant 2i)$$

is an isomorphism for all A,ε. The inverse isomorphism

$$\Omega^i = (\overline{S}^i)^{-1} : L_n(A,\varepsilon) \longrightarrow L_{n-2i}(A,(-)^i\varepsilon) \quad (n = 2i \text{ or } 2i{+}1)$$

sends the cobordism class of an n-dimensional ε-quadratic Poincaré complex $(C,\psi\in Q_n(C,\varepsilon))$ to the cobordism class of the (n-2i)-dimensional $(-)^i\varepsilon$-quadratic Poincaré complex corresponding to the non-singular $(-)^i\varepsilon$-quadratic $\begin{cases} \text{form} \\ \text{formation} \end{cases}$

$$\Omega^i(C,\psi) = \begin{cases} \left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T_\varepsilon)\psi_0 & d \end{pmatrix} : C^{i-1}\oplus C_{i+2} \to C^i\oplus C_{i+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right) \\ \left(H_{(-)^i\varepsilon}(C_{i+1}); C_{i+1}, \operatorname{im}\left(\begin{pmatrix} (1+T_\varepsilon)\psi_0 & d \\ \varepsilon d^* & 0 \end{pmatrix} : C^i\oplus C_{i+2} \longrightarrow C_{i+1}\oplus C^{i+1}\right)\right) \end{cases} \text{if } n = \begin{cases} 2i \\ 2i \end{cases}$$

<u>Proof</u>: Given a connected n-dimensional ε-quadratic complex over A $(C,\psi\in Q_n(C,\varepsilon))$ let $(C',\psi'\in Q_n(C',\varepsilon))$ be the connected n-dimensional ε-quadratic complex obtained from (C,ψ) by surgery on the connected (n+1)-dimensional ε-quadratic pair $(f:C \longrightarrow D,(0,\psi)\in Q_{n+1}(f,\varepsilon))$ defined by

$$f = \begin{cases} 1 \\ 0 \end{cases} : C_r \longrightarrow D_r = \begin{cases} C_r & r > n-i \\ 0 & r \leqslant n-i \end{cases} \quad (d_D = d_C)\ .$$

Now

$$H_r(C') = H_{r+1}((1+T_\varepsilon)\psi_0 : C^{n-*} \longrightarrow C) \quad (r \leqslant n-i)$$
$$H^r(C') = 0 \quad (r > i)\ ,$$

and also

$$H_{i-1}((1+T_\varepsilon)\psi'_0 : C'^{n-*} \longrightarrow C') = 0\ .$$

Thus if the boundary $\partial(C,\psi)$ is contractible (resp. an i-fold skew-suspension) then (C',ψ') is the i-fold skew-suspension $\bar{S}^i(C'',\psi'')$ of an $(n-2i)$-dimensional $(-)^i\epsilon$-quadratic complex (C'',ψ'') which is Poincaré (resp. connected). Applying Proposition 7.2 we have that $\bar{S}^i: L_{n-2i}(A,(-)^i\epsilon) \longrightarrow L_n(A,\epsilon)$ is onto (resp. $\bar{S}^i: L_{n-2i-1}(A,(-)^i\epsilon) \longrightarrow L_{n-1}(A,\epsilon)$ is one-one). If $n = 2i$ or $2i+1$ and (C,ψ) is Poincaré define as follows an $(n-2i)$-dimensional $(-)^i\epsilon$-quadratic Poincaré complex $(B,\theta\in Q_{n-2i}(B,(-)^i\epsilon))$ and a homotopy equivalence

$$g : (B,\theta) \longrightarrow (C'',\psi'') .$$

In the case $n = 2i$ let

$$B_0 = \mathrm{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T_\epsilon)\psi_0 & d \end{pmatrix} : C^{i-1}\oplus C_{i+2} \longrightarrow C^i\oplus C_{i+1}\right)^* ,$$

$$g : B_0 \xrightarrow{(\text{projection})^*} (C^i\oplus C_{i+1})^* = C_i\oplus C^{i+1} \xrightarrow{\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}} C''_0 = C_i\oplus C_{i+1}\oplus C^{i+1} ,$$

with $\theta_0\in \mathrm{Hom}_A(B^0,B_0)$ any representative of the $(-)^i\epsilon$-quadratic form

$$\left(B^0, \begin{bmatrix}\psi_0 & d\\ 0 & 0\end{bmatrix}\in Q_{(-)^i\epsilon}(B^0)\right).$$

In the case $n = 2i+1$ let

$$d_B = \epsilon[0\ 1]^* : B_1 = C_{i+1} \longrightarrow B_0 = \mathrm{im}\left(\begin{pmatrix}(1+T_\epsilon)\psi_0 & d\\ \epsilon d^* & 0\end{pmatrix} : C_{i+2}\oplus C^i \longrightarrow C_{i+1}\oplus C^{i+1}\right)^*$$

$$\theta_0 = \begin{cases}[1\ 0] : B^0 \longrightarrow B_1 \\ 0 : B^1 \longrightarrow B_0\end{cases} ,\quad \theta_1 = \begin{bmatrix}\psi_1 + d\psi_0 & 0\\ 0 & 0\end{bmatrix} : B^0 \longrightarrow B_0$$

$$g = \begin{cases}\begin{pmatrix}1\\0\end{pmatrix} : B_1 \longrightarrow C''_1 = C_{i+1}\oplus C_{i+2} \\ \begin{bmatrix}(1+T_\epsilon)\psi_0 & d\\ \epsilon d^* & 0\end{bmatrix}^* : B_0 \longrightarrow C''_0 = C_i\oplus C^{i+2}\end{cases} .$$

If $n = \begin{cases}2i\\2i+1\end{cases}$ the correspondence of Proposition $\begin{cases}1.5\\1.8\end{cases}$ sends (B,θ) to the non-singular $(-)^i\epsilon$-quadratic $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ $\Omega^i(C,\psi)$.

[]

The isomorphism $\Omega^i: L_n(A,\epsilon) \longrightarrow L_{n-2i}(A,(-)^i\epsilon)$ inverse to $\bar{S}^i$ is such that there is defined a commutative diagram

$$\begin{array}{ccc} L_n(A,\epsilon) & \xrightarrow{1+T_\epsilon} & L^n(A,\epsilon) \\ \Omega^i \big\downarrow & & \big\uparrow \bar{S}^i \\ L_{n-2i}(A,(-)^i\epsilon) & \xrightarrow{1+T_{(-)^i\epsilon}} & L^{n-2i}(A,(-)^i\epsilon) \end{array} \qquad (n\geq 2i) .$$

The composite $(1+T_{(-)^i\epsilon})\Omega^i : L_n(A,\epsilon) \longrightarrow L^{n-2i}(A,(-)^i\epsilon)$ sends an n-dimensional ϵ-quadratic Poincaré complex (C,ψ) to (the cobordism class of) the $(n-2i)$-dimensional $(-)^i\epsilon$-symmetric complex (C',φ') such that $\bar{S}^i(C',\varphi')$ is the n-dimensional ϵ-symmetric complex obtained from $(C,(1+T_\epsilon)\psi\in Q^n(C,\epsilon))$ by ϵ-symmetric surgery on the connected $(n+1)$-dimensional ϵ-symmetric pair $(f:C\longrightarrow D,(0,(1+T_\epsilon)\psi)\in Q^{n+1}(f,\epsilon))$, with $f:C\longrightarrow D$ as in the proof of Proposition 7.3. Despite appearances the class $(0,(1+T_\epsilon)\psi)\in Q^{n+1}(f,\epsilon)$ depends on $\psi\in Q_n(C,\epsilon)$, and not just on $(1+T_\epsilon)\psi\in Q^n(C,\epsilon)$. If $\psi'\in Q_n(C,\epsilon)$ is such that

$$(1+T_\epsilon)\psi = (1+T_\epsilon)\psi' \in Q^n(C,\epsilon)$$

then by Proposition 1.2 there exists $\theta\in\hat{Q}^{n+1}(C,\epsilon)$ such that

$$\psi' - \psi = H\theta \in Q_n(C,\epsilon)$$

and

$$(0,(1+T_\epsilon)\psi') - (0,(1+T_\epsilon)\psi) = (f\theta f^*,0)\in Q^{n+1}(f,\epsilon) .$$

It will follow from Proposition 9.1 iii) that

$$\bar{S}^{i-1}(1+T_{(-)^i\epsilon})\Omega^i((C,\psi')\oplus(C,-\psi)) = 0\in L^{n-2}(A,-\epsilon) \quad (n\geqslant 2i\geqslant 2) .$$

In Proposition 14.9 i) we shall construct an example such that

$$(1+T_{(-)^i\epsilon})\Omega^i((C,\psi')\oplus(C,-\psi)) \neq 0 \in \ker(\bar{S}^{i-1}: L^{n-2i}(A,(-)^i\epsilon) \longrightarrow L^{n-2}(A,-\epsilon))$$

(with $n = 4$, $i = 2$, $A = \mathbb{Z}[\mathbb{Z}^4]$, $\epsilon = -1$).

The isomorphism $\Omega^{2k}: L_{4k}(\mathbb{Q}) \longrightarrow L_0(\mathbb{Q})$ leads to the semi-local combinatorial formula for the signature of a 4k-manifold obtained in Ranicki and Sullivan [1].

We shall say that a ring A is <u>m-dimensional</u> if every f.g. A-module M has a f.g. projective A-module resolution of length m

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 .$$

Equivalently, A is a noetherian ring of global dimension m. A ring is 0-dimensional if and only if it is semi-simple.

<u>Proposition 7.4</u> For an m-dimensional ring with involution A the skew-suspension map

$$\bar{S} : L^n(A,\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \quad (n \geq 0)$$

is an isomorphism for $n+2 \geq 2m$, and one-one if $n+3 = 2m$. If A is 0-dimensional then

$$L^{2k+1}(A,\varepsilon) = 0 \quad (k \geq 0) .$$

<u>Proof</u>: Let $p = 2$ if $n+2 \geq 2m$ (resp. $p = 3$ if $n+3 \geq 2m$). Given a connected $(n+p)$-dimensional $(-\varepsilon)$-symmetric complex over A $(C,\varphi \in Q^{n+p}(C,-\varepsilon))$ with a boundary $\partial(C,\varphi)$ which is contractible (resp. a skew-suspension) we have that $H_0(C) = \mathrm{coker}(d:C_1 \longrightarrow C_0)$ is a f.g. A-module, with a f.g. projective A-module resolution of length m

$$0 \longrightarrow D_m \xrightarrow{d} D_{m-1} \xrightarrow{d} \dots \xrightarrow{d} D_1 \xrightarrow{d} D_0 \longrightarrow H_0(C) \longrightarrow 0 .$$

Let $f:C \longrightarrow D$ be a chain map inducing

$$f_* = 1 : H_0(C) \longrightarrow H_0(D) = H_0(C) ,$$

and let $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_{n+p}$ be a cycle representing $\varphi \in Q^{n+p}(C,-\varepsilon)$, so that

$$f^{\%}(\varphi) = \begin{cases} f\varphi_0 f^* : D^m \longrightarrow D_m & \text{if } n+p = 2m,\ s = 0 \\ 0 : D^r \longrightarrow D_{n+p-r+s} & \text{otherwise} \end{cases} .$$

In the case $n+p = 2m$ it follows from the commutative diagram

$$\begin{array}{ccccccc} D^m & \xrightarrow{f^*} & C^m & \xrightarrow{\varphi_0} & C_m & \xrightarrow{f} & D_m \\ \big\downarrow & & \big\downarrow{\scriptstyle d^*} & & \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle d} \\ 0 & \longrightarrow & C^{m+1} & \xrightarrow{(-)^m\varphi_0} & C_{m-1} & \xrightarrow{f} & D_{m-1} \end{array}$$

that

$$f\varphi_0 f^* = 0 \in \mathrm{Hom}_A(D^m,D_m) ,$$

since $d \in \mathrm{Hom}_A(D_m,D_{m-1})$ is one-one. Thus if $n+p \geq 2m$

$$f^{\%}(\varphi) = 0 \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(D^*,D))_{n+p} ,$$

and there is defined a connected $(n+p+1)$-dimensional $(-\varepsilon)$-symmetric pair $(f:C \longrightarrow D,(0,\varphi) \in Q^{n+p+1}(f,-\varepsilon))$. Let $(C',\varphi' \in Q^{n+p}(C',-\varepsilon))$ be the $(n+p)$-dimensional $(-\varepsilon)$-symmetric complex obtained from (C,φ) by surgery on this pair. Now

$$H_0(C') = H_1(f) = 0$$

$$H^{n+p}(C') = H_1(f\varphi_0:C^{n+p-*} \longrightarrow D) = H_1(\varphi_0:C^{n+p-*} \longrightarrow C) = 0 ,$$

so that (C',φ') is the skew-suspension of an $(n+p-2)$-dimensional ε-symmetric complex. Applying the criterion of Proposition 4.2 we have that $\bar{S}:L^n(A,\varepsilon) \longrightarrow L^{n+p}(A,-\varepsilon)$ is onto (resp. that $\bar{S}:L^n(A,\varepsilon) \longrightarrow L^{n+p-1}(A,\varepsilon)$ is one-one).

In particular, if A is a 1-dimensional ring with involution we have that the skew-suspensions

$$\bar{S} : L^n(A,\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \quad (n \geq 0)$$

are isomorphisms. If A is 0-dimensional and (C,φ) is a 1-dimensional ε-symmetric Poincaré complex over A, then the above procedure defines a 2-dimensional ε-symmetric Poincaré pair over A $(f:C \longrightarrow D,(\delta\varphi,\varphi) \in Q^2(f,\varepsilon))$ with $D_0 = H_0(C)$, $D_r = 0$ $(r \neq 0)$ and so

$$L^1(A,\varepsilon) = 0 .$$

[]

It has already been proved in Ranicki [6] that for a 0-dimensional ring with involution A

$$L_{2k+1}(A,\varepsilon) = 0 \quad (k \geq 0) .$$

It is also the case that for 0-dimensional A

$$L\langle v_0\rangle^1(A,\varepsilon) = 0 .$$

In Proposition 14.4 we shall give an example to illustrate the sharpness of the stable range determined above, namely an m-dimensional ring with involution A such that $\bar{S}:L^{2m-4}(A,\varepsilon) \longrightarrow L^{2m-2}(A,-\varepsilon)$ is not an isomorphism $(A = \mathbb{Z}[\mathbb{Z}],\ m = 2,\ \varepsilon = -1)$.

It is claimed in §4 of Mishchenko [2] that the double skew-suspension map in the symmetric L-groups

$$\bar{S}^2 : L^n(A) \longrightarrow L^{n+4}(A)$$

is an isomorphism for every ring with involution A. It is indeed the case that $\bar{S}^2$ is an isomorphism if $1/2 \in A$ (when $L_n(A) = L^n(A)$) or if A is 1-dimensional, by Propositions 7.3,7.4. However, the algebraic surgery technique used to establish such a 4-periodicity for general A is bogus: the new algebraic Poincaré pair $(\bar{C}, {}^0\bar{C}, \bar{d}, \bar{D}^k)$ may not satisfy property a') of §3 (of Mishchenko [2]),

$${}^0\bar{D}^{n-2i-1} = \begin{pmatrix} {}^0D^{n-2i-1} & D^{n-2i-1}\beta \\ 0 & 0 \end{pmatrix} : {}^0\bar{C}^{n-i-1} = C^{n-i-1} \oplus A^* \longrightarrow \bar{C}_{n-i} = C_{n-i} \oplus A$$

need not map $\bar{C}^{n-i-1}$ into ${}^0\bar{C}_{n-i} = {}^0C_{n-i} \subseteq \bar{C}_{n-i}$ if $D^{n-2i-1}\beta \neq 0 : A^* \longrightarrow C_{n-i}$.

For a specific example let us consider the 7-dimensional algebraic Poincaré pair over $\mathbb{Z}$ $(C, {}^0C, d, D^k)$ defined by

$$C_r = \begin{cases} \mathbb{Z} & \text{for } r = 4,5 \\ 0 & \text{otherwise} \end{cases} \quad , \quad {}^0C = 0 \quad , \quad d = 1 : C_5 \longrightarrow C_4$$

$$D^1 = 0 : C^4 \longrightarrow C_4 \ , \ D^2 = \begin{cases} 1 : C^4 \longrightarrow C_5 \\ -1 : C^5 \longrightarrow C_4 \end{cases} , \ D^3 = 2 : C^5 \longrightarrow C_5$$

with $\beta = 1 : A^* = \mathbb{Z} \longrightarrow C^5$. Here $n = 6$, $i = 1$. We shall exhibit various failures of 4-periodicity in the symmetric L-groups in §14 below.

An n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complex $\begin{cases} (C, \varphi \in Q^n(C,\varepsilon)) \\ (C, \psi \in Q_n(C,\varepsilon)) \end{cases}$ is <u>well-connected</u> if

$$H_0(C) = 0 \ ,$$

in which case it is connected.

<u>Proposition 7.5</u> The n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complexes $\begin{cases} (C,\varphi),(C',\varphi') \\ (C,\psi),(C',\psi') \end{cases}$ are cobordant if and only if there exists a homotopy equivalence

$$\begin{cases} f : (C,\varphi) \oplus \partial(D,\nu) \longrightarrow (C',\varphi') \oplus \partial(D',\nu') \\ f : (C,\psi) \oplus \partial(D,\chi) \longrightarrow (C',\psi') \oplus \partial(D',\chi') \end{cases}$$

for some well-connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes $\begin{cases} (D,\nu),(D',\nu') \\ (D,\chi),(D',\chi') \end{cases}$.

<u>Proof</u>: In view of Proposition 5.4 iii) it is sufficient to prove that for every connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complex $\begin{cases} (D,\nu) \\ (D,\chi) \end{cases}$ there exists a homotopy equivalence

$$\begin{cases} f : \partial(D,\nu) \oplus \partial(D',\nu') \longrightarrow \partial(D'',\nu'') \\ f : \partial(D,\chi) \oplus \partial(D',\chi') \longrightarrow \partial(D'',\chi'') \end{cases}$$

for some well-connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes $\begin{cases} (D',\nu'),(D'',\nu'') \\ (D',\chi'),(D'',\chi'') \end{cases}$.

Define a chain map $g: D \longrightarrow D'$ by

$$\begin{array}{ccccccccccc} D : & \dots \longrightarrow & D_{n+2} & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \longrightarrow & \dots \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\ D' : & \dots \longrightarrow & D_{n+2} & \xrightarrow{d} & D_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \ , \end{array}$$

and let $\begin{cases} \nu' = -g^{\%}(\nu) \in Q^{n+1}(D',\varepsilon) \\ \chi' = -g_{\%}(\chi) \in Q_{n+1}(D',\varepsilon) \end{cases}$. Let $\begin{cases} (D'',\nu'') \\ (D'',\chi'') \end{cases}$ be the connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complex obtained from $\begin{cases} (D,\nu) \oplus (D',\nu') \\ (D,\chi) \oplus (D',\chi') \end{cases}$ by surgery on the connected (n+2)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ pair $\begin{cases} ((g\ 1) : D \oplus D' \longrightarrow D', (0, \nu \oplus \nu') \in Q^{n+2}((g\ 1),\varepsilon)) \\ ((g\ 1) : D \oplus D' \longrightarrow D', (0, \chi \oplus \chi') \in Q_{n+2}((g\ 1),\varepsilon)) \end{cases}$.

Now $\begin{cases} (D',\nu') \\ (D',\chi') \end{cases}$ and $\begin{cases} (D'',\nu'') \\ (D'',\chi'') \end{cases}$ are well-connected, and Proposition 7.1 i) shows that $\begin{cases} \partial(D,\nu) \oplus \partial(D',\nu') = \partial(D'',\nu'') \\ \partial(D,\chi) \oplus \partial(D',\chi') = \partial(D'',\chi'') \end{cases}$ up to homotopy equivalence. []

We shall now identify the L-groups $\begin{cases} L^0(A,\varepsilon) \\ L_0(A,\varepsilon) \end{cases}$ (resp. $\begin{cases} L^1(A,\varepsilon) \\ L_1(A,\varepsilon) \end{cases}$).

with the Witt groups of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms (resp. formations) over A.

Witt groups of quadratic forms were first studied by Witt [1].

The Witt group of $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A $\begin{cases} L^\varepsilon(A) \\ L_\varepsilon(A) \end{cases}$ is the abelian group of equivalence classes of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A subject to the relation

$$\begin{cases} (M,\varphi\in Q^\varepsilon(M)) \sim (M',\varphi'\in Q^\varepsilon(M')) \\ (M,\psi\in Q_\varepsilon(M)) \sim (M',\psi'\in Q_\varepsilon(M')) \end{cases} \text{ if there exists an isomorphism}$$

$$\begin{cases} (M,\varphi)\oplus H^\varepsilon(N,\nu) \longrightarrow (M',\varphi')\oplus H^\varepsilon(N',\nu') \\ (M,\psi)\oplus H_\varepsilon(N) \longrightarrow (M',\psi')\oplus H_\varepsilon(N') \end{cases}$$

for some $\begin{cases} \text{metabolic} \\ \text{hyperbolic} \end{cases}$ forms $\begin{cases} H^\varepsilon(N,\nu), H^\varepsilon(N',\nu') \\ H_\varepsilon(N), H_\varepsilon(N') \end{cases}$. Addition and inverses are by

$$\begin{cases} (M,\varphi) + (M',\varphi') = (M\oplus M',\varphi\oplus\varphi') \ , \ -(M,\varphi) = (M,-\varphi) \in L^\varepsilon(A) \\ (M,\psi) + (M',\psi') = (M\oplus M',\psi\oplus\psi') \ , \ -(M,\psi) = (M,-\psi) \in L_\varepsilon(A) \ . \end{cases}$$

Proposition 7.6 $\begin{cases} L^0(A,\varepsilon) = L^\varepsilon(A) \\ L_0(A,\varepsilon) = L_\varepsilon(A) \end{cases}$.

Proof: In view of Propositions 1.5, 7.5 it is sufficient to observe that if $\begin{cases} (D,\nu\in Q^1(D,\varepsilon)) \\ (D,\chi\in Q_1(D,\varepsilon)) \end{cases}$ is a well-connected 1-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complex then

$$\begin{cases} \nu\in Q^1(D,\varepsilon) = Q^\varepsilon(H^1(D)) \\ \chi\in Q_1(D,\varepsilon) = 0 \end{cases} ,$$

and the boundary $\begin{cases} \partial(D,\nu) \\ \partial(D,\chi) \end{cases}$ corresponds to the $\begin{cases} \text{metabolic} \\ \text{hyperbolic} \end{cases}$ form $\begin{cases} H^\varepsilon(H^1(D),\nu) \\ H_\varepsilon(H^1(D)) \end{cases}$.

[]

The even-dimensional U-groups of Ranicki [1] are just the Witt groups of $\pm$quadratic forms

$$U_{2i}(A) = L_{(-)^i}(A) \quad (i(\mathrm{mod}\ 2)) \ ,$$

and Proposition 7.6 shows that the morphism defined in Proposition 5.6

$$U_{2i}(A) \longrightarrow L_0(A,(-)^i) = L_{(-)^i}(A)$$

is in fact an isomorphism.

The Witt group of $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ formations over A $\begin{cases} M^\varepsilon(A) \\ M_\varepsilon(A) \end{cases}$ is the abelian group of equivalence classes of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ formations over A subject to the relation

$(M,\varphi;F,G) \sim (M',\varphi';F',G')$ if there exists a stable isomorphism of the type

$$[f] : (M,\varphi;F,G)\oplus(N,\nu;H,K)\oplus(N,\nu;K,L)\oplus(N',\nu';H',L')$$
$$\longrightarrow (M',\varphi';F',G')\oplus(N',\nu';H',K')\oplus(N',\nu';K',L')\oplus(N,\nu;H,L) \ ,$$

with addition and inverses by

$$(M,\varphi;F,G) + (M',\varphi';F',G') = (M\oplus M',\varphi\oplus\varphi';F\oplus F',G\oplus G') \ , \ -(M,\varphi;F,G) = (M,\varphi;G,F) \in M^\varepsilon(A),$$

and similarly for $M_\varepsilon(A)$.

Proposition 7.7 $\begin{cases} L^1(A,\varepsilon) = M^\varepsilon(A) \\ L_1(A,\varepsilon) = M_\varepsilon(A) \end{cases}$.

Proof: Let $(C,\varphi),(C',\varphi'),(C'',\varphi'')$ be the 1-dimensional ε-symmetric Poincaré complexes associated by Proposition 1.7 to non-singular ε-symmetric formations $(N,\nu;F,G),(N,\nu;G,H),(N,\nu;F,H)$. We have to prove that

$$(C,\varphi)\oplus(C',\varphi') = (C'',\varphi'') \in L^1(A,\varepsilon) \ ,$$

corresponding to the generic sum formula in the Witt group

$$(N,\nu;F,G)\oplus(N,\nu;G,H) = (N,\nu;F,H) \in M^\varepsilon(A) \ ,$$

in order to verify that the abelian group morphism

$$M^\varepsilon(A) \longrightarrow L^1(A,\varepsilon) \ ; \ (N,\nu;F,G) \longmapsto (C,\varphi)$$

is well-defined. Choosing a chain homotopy inverse for $\varphi_0 : C^{1-*} \longrightarrow C$ let

$$\tilde{\varphi} = (\varphi_0^{-1})^{\%}(\varphi) \in Q^1(C^{1-*},\varepsilon) \ ,$$

so that there is defined a homotopy equivalence of 1-dimensional ε-symmetric Poincaré complexes over A

$$\varphi_0 : (C^{1-*},\tilde{\varphi}) \longrightarrow (C,\varphi) \ .$$

(In fact $(C^{1-*},\tilde{\varphi})$ corresponds to the formation $(N,-\nu;G,F)$, which is thus stably isomorphic to $(N,\nu;F,G)$, and there is defined an isomorphism of non-singular ε-symmetric formations

$$(N,\nu;F,G)\oplus(H^\varepsilon(G);G,G^*) \longrightarrow (N,-\nu;G,F)\oplus(H^\varepsilon(F);F,F^*) \quad).$$

Define a chain map $(f\ f'):C^{1-*}\oplus C'\longrightarrow D$ by

$$\begin{array}{ccccccccccc} C^{1-*}\oplus C' & : & \ldots\longrightarrow & 0 & \longrightarrow & G\oplus G & \longrightarrow & F^*\oplus H^* & \longrightarrow & 0 & \longrightarrow\ldots \\ \Big\downarrow{\scriptstyle (f\ f')} & & & & & \Big\downarrow{\scriptstyle (1\ 1)} & & \Big\downarrow & & & \\ D & : & \ldots\longrightarrow & 0 & \longrightarrow & G & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\ldots \end{array}$$

Now (C'',φ'') is homotopy equivalent to the 1-dimensional ε-symmetric Poincaré complex obtained from $(C^{1-*},\varphi)\oplus(C',\varphi')$ by surgery on the connected 2-dimensional ε-symmetric pair $((f\ f'):C^{1-*}\oplus C'\longrightarrow D,(0,\varphi\oplus\varphi')\in Q^2((f\ f'),\varepsilon))$, and so

$$(C,\varphi)\oplus(C',\varphi') = (C^{1-*},\varphi)\oplus(C',\varphi') = (C'',\varphi'')\in L^1(A,\varepsilon)$$

by Proposition 7.1 ii).

The correspondence of Proposition 1.7 can also be used to define a morphism

$$L^1(A,\varepsilon)\longrightarrow M^{\varepsilon}(A)\ ;\ (C,\varphi)\longmapsto(N,\nu;F,G)$$

inverse to $M^{\varepsilon}(A)\longrightarrow L^1(A,\varepsilon)$. This is well-defined provided we can show that for any well-connected 2-dimensional ε-symmetric complex $(D,\nu\in Q^2(D,\varepsilon))$ the non-singular ε-symmetric formation $(M,\varphi;F,G)$ associated to the boundary $\partial(D,\nu)$ is such that

$$(M,\varphi;F,G) = 0\in M^{\varepsilon}(A)\ ,$$

applying Proposition 7.5. It may be assumed that D is a f.g. projective A-module chain complex of the type

$$D\ :\ \ldots\longrightarrow 0\longrightarrow D_2\xrightarrow{\ d\ }D_1\longrightarrow 0\rightarrow\ldots\ ,$$

so that a cycle $\nu\in\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(D^*,D))_2$ is represented by A-module morphisms

$$\nu_0:D^1\longrightarrow D_1\ ,\ \nu_1:D^1\longrightarrow D_2\ ,\ \widetilde{\nu}_1:D^2\longrightarrow D_1\ ,\ \nu_2:D^2\longrightarrow D_2$$

such that

$$\begin{aligned} &\nu_0+\varepsilon\nu_0^*+d\nu_1-\widetilde{\nu}_1d^* = 0\ :\ D^1\longrightarrow D_1 \\ &\nu_1+\varepsilon\widetilde{\nu}_1^*-\nu_2d^* = 0:D^1\longrightarrow D_2 \\ &\widetilde{\nu}_1+\varepsilon\nu_1^*-d\nu_2 = 0\ :\ D^2\longrightarrow D_1 \\ &\nu_2-\varepsilon\nu_2^* = 0\ :\ D^2\longrightarrow D_2\ . \end{aligned}$$

The boundary 1-dimensional ε-symmetric Poincaré complex $\partial(D,\nu)$ corresponds to the non-singular ε-symmetric formation

$$(M,\varphi;F,G) = (H^{\varepsilon}(D_1\oplus D^2,\begin{pmatrix}0&0\\0&-\nu_2\end{pmatrix});D^1\oplus D_2,$$

$$\mathrm{im}(\begin{pmatrix}\bar{\varepsilon}&0\\ \widetilde{\nu}_1^*&1\\ -\nu_0^*&-d\\ d^*&0\end{pmatrix}:D^1\oplus D_2\longrightarrow D^1\oplus D_2\oplus D_1\oplus D^2))\ .$$

The A-module automorphism

$$f = \begin{pmatrix}\bar{\varepsilon}&0&0&0\\ \widetilde{\nu}_1^*&1&0&0\\ -\nu_0^*&-d&\bar{\varepsilon}&\nu_1^*\\ d^*&0&0&1\end{pmatrix}:M = D^1\oplus D_2\oplus D_1\oplus D^2\longrightarrow D^1\oplus D_2\oplus D_1\oplus D^2$$

defines an isomorphism of non-singular ε-symmetric formations over A

$$f\ :\ (M,\varphi;H,F)\longrightarrow(M,\varphi;H,G)\ ,$$

where $H = D_2\oplus D_1\subseteq M = D^1\oplus D_2\oplus D_1\oplus D^2$. It follows that

$$(M,\varphi;F,G) = (M,\varphi;F,H)\oplus(M,\varphi;H,G) = (M,\varphi;F,H)\oplus(M,\varphi;H,F) = 0\in M^{\varepsilon}(A)\ .$$

The correspondence of Proposition 1.8 can be similarly used to define inverse isomorphisms of the ε-quadratic Witt groups

$$L_1(A,\varepsilon)\longrightarrow M_{\varepsilon}(A)\quad,\quad M_{\varepsilon}(A)\longrightarrow L_1(A,\varepsilon)$$

provided it can be shown that the cobordism class of the 1-dimensional ε-quadratic Poincaré complex associated to a non-singular split ε-quadratic formation $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ depends only on the underlying ε-quadratic formation $(H_{\varepsilon}(F);F,\mathrm{im}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}:G\longrightarrow F\oplus F^*))$ and not on the choice of hessian $\theta\in Q_{-\varepsilon}(G)$. This has already been carried out in the proof of Proposition 5.6.

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The odd-dimensional U-groups of Ranicki [1] are just the Witt groups of $\pm$quadratic formations

$$U_{2i+1}(A) = M_{(-)^i}(A)\quad (i(\mathrm{mod}\ 2))\ ,$$

and Proposition 7.7 shows that the morphism defined in Proposition 5.6

$$U_{2i+1}(A)\longrightarrow L_1(A,(-)^i) = M_{(-)^i}(A)$$

is in fact an isomorphism.

Propositions 5.6, 7.3, 7.6, 7.7 together allow us to identify

$$U_n(A) = L_{n-2i}(A,(-)^i) = L_n(A) \qquad (n = 2i \text{ or } 2i+1)\ .$$

It was shown in Ranicki [1] that the U-groups of a group ring $A = \mathbb{Z}[\pi]$ are the L-groups $L_n(\pi)$ of Wall [5] (modulo the K-theoretic hypotheses, cf. §12), so that

$$U_n(\mathbb{Z}[\pi]) = L_n(\mathbb{Z}[\pi]) = L_n(\pi)\ .$$

In §8 we shall combine this with the result of Proposition 4.4 to identify the quadratic signature $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of an n-dimensional normal bundle map $(f,b):M \longrightarrow X$ with the surgery obstruction $\theta(f,b) \in L_n(\pi_1(X))$.

The following result has strong implications for the elementary unitary groups of A (cf. Proposition 12.2), generalizing the "normal form" of Sharpe [1].

<u>Proposition 7.8</u> A non-singular ε-quadratic formation over A $(M,\psi;F,G)$ is such that

$$(M,\psi;F,G) = 0 \in L_1(A,\varepsilon)$$

if and only if there exists a stable isomorphism

$$[f] : (M,\psi;F,G) \longrightarrow \partial(N,\chi)$$

for some $(-\varepsilon)$-quadratic form $(N,\chi \in Q_{-\varepsilon}(N))$.

<u>Proof:</u> A 1-dimensional ε-quadratic Poincaré complex over A $(C,\psi \in Q_1(C,\varepsilon))$ is null-cobordant if and only if it is homotopy equivalent to the boundary $\partial(D,\chi)$ of a connected 2-dimensional ε-quadratic complex $(D,\chi \in Q_2(D,\varepsilon))$, by Proposition 5.4 iii). Let (D',χ') be the connected 2-dimensional ε-quadratic complex obtained from (D,χ) by ε-quadratic surgery on $(g:D \longrightarrow E,(0,\chi) \in Q_3(g,\varepsilon))$, with $g:D \longrightarrow E$ the chain map defined by

$$\begin{array}{ccccccccccc}
D: & \dots \longrightarrow & D_3 & \xrightarrow{d} & D_2 & \xrightarrow{d} & D_1 & \xrightarrow{d} & D_0 & \longrightarrow & \dots \\
\downarrow g & & \downarrow 1 & & \downarrow 1 & & \downarrow & & \downarrow & & \\
E: & \dots \longrightarrow & D_3 & \xrightarrow{d} & D_2 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Then (D',χ') is the skew-suspension of a 0-dimensional $(-\varepsilon)$-quadratic complex, and $\partial(D',\chi')$ is homotopy equivalent to (C,ψ) by Proposition 7.1 i). The non-singular ε-quadratic formation $(M,\psi;F,G)$ associated to (C,ψ) is thus stably isomorphic to the boundary $\partial(N,\chi)$ of the $(-\varepsilon)$-quadratic form

$$(N,\chi) = (H^1(D'),\chi' \in Q_{-\varepsilon}(H^1(D')))\ .$$

[]

Intuitively, $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ surgery on $\left\{\begin{array}{l}(f:C \longrightarrow D,(\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon))\\ (f:C \longrightarrow D,(\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\end{array}\right.$ kills $\operatorname{im}(f^*:H^*(D) \longrightarrow H^*(C))$, whereas a geometric surgery only kills individual (co)homology classes. In §8 we shall prove that the chain level effect of a geometric surgery on an r-dimensional spherical homology class in an n-dimensional manifold is an algebraic surgery on a connected algebraic pair such that

$$H^s(D) = \begin{cases} A & s = n-r \\ 0 & s \neq n-r\ . \end{cases}$$

We shall now break down a general algebraic surgery into a sequence of such elementary surgeries (subject to a necessary K-theoretic restriction).

An algebraic surgery on a connected (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ pair over A $\left\{\begin{array}{l}(f:C \longrightarrow D,(\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon))\\ (f:C \longrightarrow D,(\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\end{array}\right.$ is <u>elementary of type (r,n-r-1)</u> if $D = S^{n-r}A$, that is $D_s = \begin{cases} A & s = n-r \\ 0 & s \neq n-r \end{cases}$. Such a surgery will be said to <u>kill</u> the (co)homology class $f^*(1) \in H^{n-r}(C)$ ($= H_r(C)$ if $\left\{\begin{array}{l}(C,\varphi)\\ (C,\psi)\end{array}\right.$ is Poincaré).

<u>Proposition 7.9</u> Let $\left\{\begin{array}{l}(C,\varphi \in Q^n(C,\varepsilon))\\ (C,\psi \in Q_n(C,\varepsilon))\end{array}\right.$ be a connected n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complex over A. Then

i) A cohomology class $x \in H^{n-r}(C)$ can be killed by an elementary $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ surgery of type (r,n-r-1) if and only if its $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Wu class vanishes, that is

$$\left\{\begin{array}{l} v_r(\varphi)(x) = 0 \in H^{n-2r}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon) \\ v^r(\psi)(x) = 0 \in H_{2r-n}(\mathbb{Z}_2;A,(-)^{n-r}\varepsilon)\ , \end{array}\right.$$

and in the case $r = n$ $x \in H^0(C)$ generates a direct summand of $H^0(C)$.

ii) If $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ is obtained from $\begin{cases}(C,\psi)\\(C,\varphi)\end{cases}$ by an elementary $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery of type $(r,n-r-1)$ then $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ is homotopy equivalent to an $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ complex obtained from $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ by an elementary $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery of type $(n-r-1,r)$.

iii) If $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ is obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on $\begin{cases}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\(f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{cases}$ such that D has projective class

$$[D] \equiv \sum_{r=-\infty}^{\infty}(-)^r[D_r] = 0 \in \widetilde{K}_0(A)$$

then $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ may be obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by a finite sequence of elementary $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgeries.

<u>Proof</u>: i) The vanishing of the Wu class is just the condition required to represent $x\in H^{n-r}(C)$ by an $(n+1)$-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ pair $\begin{cases}(f:C\longrightarrow S^{n-r}A,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\(f:C\longrightarrow S^{n-r}A,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{cases}$ such that $f^*(1) = x\in H^{n-r}(C)$. This pair is automatically connected if $r<n$, but for $r = n$ it is connected if and only if $x\in H^0(C)$ generates a direct summand.

ii) + iii) follow from the result below on the composition of algebraic surgeries:

<u>Lemma</u> Let $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ be the connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ complex over A obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on $\begin{cases}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\(f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{cases}$.
If $D = C(g)$ is the algebraic mapping cone of a chain map $g:\Omega D'\longrightarrow D''$, for some $(n+1)$-dimensional A-module chain complexes D',D'', then $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ may be obtained by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on some $\begin{cases}(f'':C''\longrightarrow D'',(\delta\varphi'',\varphi'')\in Q^{n+1}(f'',\varepsilon))\\(f'':C''\longrightarrow D'',(\delta\psi'',\psi'')\in Q_{n+1}(f'',\varepsilon))\end{cases}$ from some $\begin{cases}(C'',\varphi'')\\(C'',\psi'')\end{cases}$ obtained by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on some $\begin{cases}(f':C\longrightarrow D',(\delta\varphi',\varphi))\\(f':C\longrightarrow D',(\delta\psi',\psi))\end{cases}$ from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$

Conversely, given connected n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ complexes $\begin{cases}(C,\varphi),(C',\varphi'),(C'',\varphi'')\\(C,\psi),(C',\psi'),(C'',\psi'')\end{cases}$ such that $\begin{cases}(C'',\varphi'')\\(C'',\psi'')\end{cases}$ (resp. $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$) is obtained from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ (resp. $\begin{cases}(C'',\varphi'')\\(C'',\psi'')\end{cases}$) by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on $\begin{cases}(f':C\longrightarrow D',(\delta\varphi',\varphi))\\(f':C\longrightarrow D',(\delta\psi',\psi))\end{cases}$ (resp. $\begin{cases}(f'':C''\longrightarrow D'',(\delta\varphi'',\varphi''))\\(f'':C''\longrightarrow D'',(\delta\psi'',\psi''))\end{cases}$) it is possible to obtain $\begin{cases}(C',\varphi')\\(C',\psi')\end{cases}$ from $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ by $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ surgery on $\begin{cases}(f:C\longrightarrow D,(\delta\varphi,\varphi))\\(f:C\longrightarrow D,(\delta\psi,\psi))\end{cases}$ with $D' = C(g)$ the algebraic mapping cone of a chain map $g:\Omega D'\longrightarrow D''$.

<u>Proof</u>: Let

$$f = \begin{pmatrix}f'\\ \widetilde{f}''\end{pmatrix} : C_r \longrightarrow D_r = D'_r\oplus D''_r$$

$$\begin{cases}\delta\varphi_s = \begin{pmatrix}\delta\varphi'_s & \widetilde{\nu}_s\\ \nu_s & \delta\varphi''_s\end{pmatrix} : D^{n-r+s+1} = D'^{n-r+s+1}\oplus D''^{n-r+s+1}\longrightarrow D_r = D'_r\oplus D''_r\\ \delta\psi_s = \begin{pmatrix}\delta\psi'_s & \widetilde{\chi}_s\\ \chi_s & \delta\psi''_s\end{pmatrix} : D^{n-r-s+1} = D'^{n-r-s+1}\oplus D''^{n-r-s+1}\longrightarrow D_r = D'_r\oplus D''_r\end{cases}\quad (s\geqslant 0)\ ,$$

and define

$$\begin{cases}f'' = (\widetilde{f}''\quad (-)^{n-r+1}g\quad (-)^n\nu_0) : C''_r = C_r\oplus D'_{r+1}\oplus D'^{n-r+1}\longrightarrow D''_r\\ f'' = (\widetilde{f}''\quad (-)^{n-r+1}g\quad (-)^n(\chi_0+(-)^{r(n-r+1)}\varepsilon\widetilde{\chi}^*_0)) : C''_r = C_r\oplus D'_{r+1}\oplus D'^{n-r+1}\longrightarrow D''_r\ .\end{cases}$$

The converse is proved by reversing the argument.

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§8. Geometric surgery

We now relate framed surgery in topology to quadratic surgery in algebra.

The groups $U_n(A)$ of Ranicki [1] were defined as stable isomorphism groups of non-singular $(-)^i$quadratic $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over A if $n \equiv \begin{cases}2i\\ 2i+1\end{cases} \pmod 4$, following the work of Novikov [2]. Proposition $\begin{cases}7.6\\ 7.7\end{cases}$ identifies

$$U_n(A) = \begin{cases}L_0(A,(-)^i)\\ L_1(A,(-)^i)\end{cases} \text{if } n = \begin{cases}2i\\ 2i+1\end{cases},$$

so that the i-fold skew-suspension isomorphism of Proposition 7.3 can be written as

$$\bar{S}^i : U_n(A) = L_{n-2i}(A,(-)^i) \longrightarrow L_n(A) .$$

It was shown in §3 of Ranicki [1] that the U-groups of a group ring $\mathbb{Z}[\pi]$ with the w-twisted involution are the surgery obstruction groups of Wall [5]

$$U_n(\mathbb{Z}[\pi]) = L_n(\pi,w)$$

for any group morphism $w:\pi \longrightarrow \mathbb{Z}_2$, modulo the K-theoretic conditions discussed in §12 below.

Let $(f,b):M \longrightarrow X$ be an n-dimensional normal bundle map, with quadratic kernel the n-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$ with the $w(X)$-twisted involution

$$\sigma_*(f,b) = (C(f^!),e_\%\psi_F[X]\in Q_n(C(f^!)))$$

constructed in §3. Let $n = 2i$ or $2i+1$, and write $(f',b'):M' \longrightarrow X$ for the normal bordant $(i-1)$-connected n-dimensional normal bundle map obtained from $(f,b):M \longrightarrow X$ by framed surgery below the middle dimension, as in Theorem 1.2 of Wall [5]. Proposition 2.12 gives an $(n-2i)$-dimensional $(-)^i$quadratic Poincaré complex $\sigma_i(f',b')$ such that

$$\bar{S}^i\sigma_i(f',b') = \sigma_*(f',b') .$$

It follows from the normal bordism invariance of quadratic signatures of Proposition 6.6 i) that

$$\sigma_*(f',b') = \sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) .$$

Proposition 4.4 identifies the quadratic Poincaré cobordism class $\sigma_i(f',b')\in L_{n-2i}(\mathbb{Z}[\pi_1(X)],(-)^i)$ with the surgery obstruction $\theta(f,b) = \theta(f',b')\in L_n(\pi_1(X),w(X))$ obtained in §§5,6 of Wall [5].

The i-fold skew-suspension isomorphism

$$\bar{S}^i : L_n(\pi_1(X),w(X)) = L_{n-2i}(\mathbb{Z}[\pi_1(X)],(-)^i) \longrightarrow L_n(\mathbb{Z}[\pi_1(X)])$$

is therefore such that

$$\bar{S}^i\theta(f,b) = \bar{S}^i\theta(f',b') = \bar{S}^i\sigma_i(f',b') = \sigma_*(f',b') = \sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) .$$

We have proved:

Proposition 8.1 The quadratic signature $\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$ is the obstruction $\theta(f,b)\in L_n(\pi_1(X),w(X))$ to making an n-dimensional normal bundle map $(f,b):M\longrightarrow X$ a homotopy equivalence by framed surgeries $(n\geqslant 5)$. []

Given an n-dimensional normal bundle map of pairs

$$((f,\partial f),(b,\partial b)) : (M,\partial M) \longrightarrow (X,\partial X)$$

such that $\partial f:\partial M \longrightarrow \partial X$ is a homotopy equivalence we have that the quadratic kernel $\sigma_*((f,\partial f),(b,\partial b))$ is an n-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi_1(X)]$ with contractible boundary $\sigma_*(\partial f,\partial b)$. Such objects are the same up to homotopy equivalence of quadratic Poincaré pairs as quadratic Poincaré complexes (Proposition 5.4). The quadratic Poincaré cobordism class $\sigma_*((f,\partial f),(b,\partial b))\in L_n(\mathbb{Z}[\pi_1(X)])$ is the obstruction to making $((f,\partial f),(b,\partial b))$ a homotopy equivalence of geometric Poincaré pairs by framed surgeries keeping $(\partial f,\partial b)$ fixed $(n\geqslant 5)$.

The explicit inverse $\Omega^i = \bar{S}^{-i} : L_n(A) \longrightarrow L_{n-2i}(A,(-)^i)$ $(n = 2i$ or $2i+1)$ constructed in Proposition 7.3 has an application. Given an n-dimensional normal bundle map $(f,b):M\longrightarrow X$ we can write down a non-singular $(-)^i$quadratic form or formation over $\mathbb{Z}[\pi_1(X)]$ representing the surgery obstruction $\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$, without preliminary geometric surgeries below the middle dimension, that is an instant surgery obstruction. This solves Problem 5 of Shaneson [3].

<u>Proposition 8.2</u> Let $(f,b):M\longrightarrow X$ be an n-dimensional normal bundle map, with quadratic kernel

$$\sigma_*(f,b) = (C(f^!),e_{\%}\psi_F[X]) = (C,\psi\in Q_n(C)) \quad .$$

Then the surgery obstruction $\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$ is the class of the non-singular $(-)^i$quadratic $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over $\mathbb{Z}[\pi_1(X)]$

$$\Omega^i\sigma_*(f,b) = \begin{cases}\left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0\\ (-)^{i+1}(1+T)\psi_0 & d\end{pmatrix}: C^{i-1}\oplus C_{i+2}\longrightarrow C^i\oplus C_{i+1}\right),\begin{bmatrix}\psi_0 & d\\ 0 & 0\end{bmatrix}\right)\\ \left(H_{(-)^i}(C_{i+1});C_{i+1},\operatorname{im}\left(\begin{pmatrix} d & (1+T)\psi_0\\ 0 & d^*\end{pmatrix}: C_{i+2}\oplus C^i\longrightarrow C_{i+1}\oplus C^{i+1}\right)\right)\end{cases}$$

$$\in L_n(\pi_1(X),w(X)) \text{ if } n = \begin{cases}2i\\ 2i+1\end{cases} .$$

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(It is required that all the chain modules C_r appearing in the above be f.g. projective $\mathbb{Z}[\pi_1(X)]$-modules).

Let us recall the elements of geometric surgery.

An <u>elementary</u> $\begin{cases}\underline{\text{oriented}}\\ \underline{\text{framed}}\end{cases}$ <u>surgery of type (r,n-r-1)</u> $(0\leq r\leq n-1)$ on a $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ map $\begin{cases}f:M\longrightarrow X\\ (f,b):M\longrightarrow X\end{cases}$ from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X is determined by the following data:

i) an embedding $g:S^r\hookrightarrow M$ with an oriented normal bundle $\nu_g:S^r\longrightarrow BSO(n-r)$

ii) a null-homotopy $\bar{\nu}_g:D^{r+1}\longrightarrow BSO(n-r)$ of $\nu_g:S^r\longrightarrow BSO(n-r)$, that is an embedding $\bar{g}:S^r\times D^{n-r}\hookrightarrow M$ extending g

iii) a null-homotopy $h:D^{r+1}\longrightarrow X$ of $fg:S^r\longrightarrow X$

and in the framed case also

iv) a relative null-homotopy $(\bar{\nu}_h,\bar{\nu}_g):(D^{r+1},S^r)\wedge I\longrightarrow(BSO,BSO(n-r))$ extending $\bar{\nu}_g$ of the map of pairs $(\nu_h,\nu_g):(D^{r+1},S^r)\longrightarrow(BSO,BSO(n-r))$, with $\nu_h:D^{r+1}\longrightarrow BSO$ the null-homotopy of the classifying map for the stable normal bundle $\nu_g:S^r\longrightarrow BSO(n-r)\longrightarrow BSO$ determined by $b:\nu_M\longrightarrow\nu_X$ and $h:D^{r+1}\longrightarrow X$.

The surgery replaces $\begin{cases}f\\ (f,b)\end{cases}$ by the n-dimensional $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ map $\begin{cases}f':M'\longrightarrow X\\ (f',b'):M'\longrightarrow X\end{cases}$ appearing in the (n+1)-dimensional $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ map of cobordisms

$$\begin{cases}(e;f,f'):(N;M,M')\longrightarrow(X\times I;X\times 0,X\times 1)\\ ((e;f,f'),(a;b,b')):(N;M,M')\longrightarrow(X\times I;X\times 0,X\times 1)\end{cases}$$

defined by

$$N = M\times I\cup_{\bar{g}\times 1}D^{r+1}\times D^{n-r}\quad ,\quad M' = \overline{M\setminus\bar{g}(S^r\times D^{n-r})}\cup D^{r+1}\times S^{n-r-1} ,$$

using $h:D^{r+1}\longrightarrow X$ to extend $f:M\longrightarrow X\times 0$ to a map of pairs $(e,f'):(N,M')\longrightarrow(X\times I,X\times 1)$, and in the framed case using $\bar{\nu}_h$ to extend $b:\nu_M\longrightarrow\nu_{X\times 0}$ to a bundle map of pairs $(a,b'):(\nu_N,\nu_{M'})\longrightarrow(\nu_{X\times I},\nu_{X\times 1})$. The surgery is said to <u>kill</u> $(h,g)\in\pi_{r+1}(f)$.

An elementary geometric surgery induces an elementary algebraic surgery on the chain level, as defined in §7.

Proposition 8.3 Let $\begin{cases} f:M\longrightarrow X,\ f':M'\longrightarrow X \\ (f,b):M\longrightarrow X,\ (f',b'):M'\longrightarrow X\end{cases}$ be n-dimensional $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ maps such that $\begin{cases} f' \\ (f',b')\end{cases}$ is obtained from $\begin{cases} f \\ (f,b)\end{cases}$ by an elementary $\begin{cases}\text{oriented}\\ \text{framed}\end{cases}$ surgery of type $(r,n-r-1)$ killing $(h,g)\in\pi_{r+1}(f)$. Then the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel $\begin{cases}\sigma^*(f')\\ \sigma_*(f',b')\end{cases}$ is obtained from $\begin{cases}\sigma^*(f)\\ \sigma_*(f,b)\end{cases}$ by an elementary $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ surgery of type $(r,n-r-1)$ killing the image of $(h,g)\in\pi_{r+1}(f)$ under the Hurewicz map $\pi_{r+1}(f) = \pi_{r+1}(\tilde f)\longrightarrow H_{r+1}(\tilde f) = K_r(M)$.

Proof: Let $\begin{cases}(e;f,f'):(N;M,M')\longrightarrow(X\times I;X\times 0,X\times 1)\\ ((e;f,f'),(a;b,b')):(N;M,M')\longrightarrow(X\times I;X\times 0,X\times 1)\end{cases}$ be the associated $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ bordism, with $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel (n+1)-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré pair over $\mathbb{Z}[\pi_1(X)]$

$$\begin{cases}\sigma^*(e;f,f') = ((i\ i'):C(f^!)\oplus C(f'^!)\longrightarrow C(e^!),(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((i\ i')))\\ \sigma_*((e;f,f'),(a;b,b'))\\ \qquad = ((i\ i'):C(f^!)\oplus C(f'^!)\longrightarrow C(e^!),(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((i\ i')))\ .\end{cases}$$

Let $k:C(e^!)\longrightarrow C(i')$ be the inclusion of $C(e^!)$ in the algebraic mapping cone of $i':C(f'^!)\longrightarrow C(e^!)$, which is such that $C(i') = S^{n-r}\mathbb{Z}[\pi_1(X)]$ up to chain equivalence. Use the chain homotopy commutative diagram of $\mathbb{Z}[\pi_1(X)]$-module chain maps (with $j: ki(1\ 0)\simeq k(i\ i')$ any chain homotopy)

$$\begin{array}{ccc} C(f^!)\oplus C(f'^!) & \xrightarrow{(i\ i')} & C(e^!)\\ {\scriptstyle(1\ 0)}\downarrow & \searrow{\scriptstyle j} & \downarrow{\scriptstyle k}\\ C(f^!) & \xrightarrow{\bar i = ki} & C(i') = S^{n-r}\mathbb{Z}[\pi_1(X)]\end{array}$$

to define a relative $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{cases}$ class

$$\begin{cases}(\overline{\delta\varphi},\varphi) = ((1\ 0),k;j)^{\%}(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}(\bar i)\\ (\overline{\delta\psi},\psi) = ((1\ 0),k;j)_{\%}(\delta\psi,\psi\oplus-\psi')\in Q_{n+1}(\bar i)\end{cases}\ .$$

The $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ kernel $\begin{cases}\sigma^*(f') = (C(f'^!),\varphi'\in Q^n(C(f'^!)))\\ \sigma_*(f',b') = (C(f'^!),\psi'\in Q_n(C(f'^!)))\end{cases}$ is obtained from $\begin{cases}\sigma^*(f) = (C(f^!),\varphi)\\ \sigma_*(f,b) = (C(f^!),\psi)\end{cases}$ by an elementary $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ surgery on the (n+1)-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ pair $\begin{cases}(\bar i:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\overline{\delta\varphi},\varphi)\in Q^{n+1}(\bar i))\\ (\bar i:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\overline{\delta\psi},\psi)\in Q_{n+1}(\bar i))\end{cases}$.

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We have the following partial converse to Proposition 8.3.

Proposition 8.4 Let $(f,b):M\longrightarrow X$ be an n-dimensional normal bundle map with quadratic kernel $\sigma_*(f,b) = (C(f^!),\psi = e_{\%}\psi_F[X]\in Q_n(C(f^!)))$. If $f:M\longrightarrow X$ is $(r-1)$-connected $(2r\leqslant n)$ and $n\geqslant 5$ it is possible to kill $x\in\pi_{r+1}(f) = K_r(M)$ by an elementary framed surgery if and only if it can be killed by an elementary quadratic surgery on $\sigma_*(f,b)$, that is if and only if

$$v^r(\psi)(x) = 0\in H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi_1(X)],(-)^{n-r})\ (= 0 \text{ if } 2r<n).$$

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The effect on an elementary geometric surgery of a change of framing of the embedded sphere may be measured as follows.

Let $\begin{cases}f:M\longrightarrow X\\ (f,b):M\longrightarrow X\end{cases}$ be a $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ map from an n-dimensional manifold M to an n-dimensional geometric Poincaré complex X. Let $g:S^r\hookrightarrow M$ be an embedding with a null-homotopy $h:D^{r+1}\longrightarrow X$ of $fg:S^r\longrightarrow X$ on which it is possible to perform $\begin{cases}\text{oriented}\\ \text{framed}\end{cases}$ surgery, and let $\begin{cases}f':M'\longrightarrow X,\ f'':M''\longrightarrow X\\ (f',b'):M'\longrightarrow X,\ (f'',b''):M''\longrightarrow X\end{cases}$ be the $\begin{cases}\text{degree 1}\\ \text{normal bundle}\end{cases}$ maps obtained from $\begin{cases}f\\ (f,b)\end{cases}$ by $\begin{cases}\text{oriented}\\ \text{framed}\end{cases}$ surgery using two $\begin{cases}\text{different extensions } \bar g,\bar{\bar g}:S^r\times D^{n-r}\hookrightarrow M \text{ of } g\\ \text{and two different relative null-homotopies } (\bar\nu_h,\bar\nu_g),(\bar{\bar\nu}_h,\bar{\bar\nu}_g) \text{ of } (\nu_h,\nu_g):(D^{r+1},S^r)\longrightarrow(BSO,BSO(n-r))\end{cases}$.

The differences are measured by elements $\begin{cases}\alpha\in\pi_r(SO(n-r))\\ \beta\in\pi_{r+1}(SO/SO(n-r))\end{cases}$.

The $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ kernels $\left\{\begin{array}{l}\sigma^*(f'),\sigma^*(f'')\\ \sigma_*(f',b'),\sigma_*(f'',b'')\end{array}\right.$ are obtained from the $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ kernel $\left\{\begin{array}{l}\sigma^*(f) = (C(f^!),\varphi = e^{\%}\varphi_{\tilde{M}}[M]\in Q^n(C(f^!)))\\ \sigma_*(f,b) = (C(f^!),\psi = e_{\%}\psi_F[X]\in Q_n(C(f^!)))\end{array}\right.$ by elementary $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ surgeries of type $(r,n-r-1)$ on the $(n+1)$-dimensional $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ pairs

$$\left\{\begin{array}{l}(\bar{i}:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\bar{\delta\varphi},\varphi)\in Q^{n+1}(\bar{i})),(\bar{\bar{i}}:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\bar{\bar{\delta\varphi}},\varphi)\in Q^{n+1}(\bar{\bar{i}}))\\ (\bar{i}:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\bar{\delta\psi},\psi)\in Q_{n+1}(\bar{i})),(\bar{\bar{i}}:C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)],(\bar{\bar{\delta\psi}},\psi)\in Q_{n+1}(\bar{\bar{i}}))\end{array}\right.$$

defined in the proof of Proposition 8.3, with $\bar{i} = \bar{\bar{i}} : C(f^!)\longrightarrow S^{n-r}\mathbb{Z}[\pi_1(X)]$.

Now Proposition 5.1 gives exact sequences

$$\left\{\begin{array}{l}Q^{n+1}(S^{n-r}\mathbb{Z}[\pi])\xrightarrow{\gamma} Q^{n+1}(\bar{i})\xrightarrow{\partial} Q^n(C(f^!))\\ Q_{n+1}(S^{n-r}\mathbb{Z}[\pi])\xrightarrow{\gamma} Q_{n+1}(\bar{i})\xrightarrow{\partial} Q_n(C(f^!))\end{array}\right. \qquad (\pi = \pi_1(X)),$$

so that

$$\left\{\begin{array}{l}(\bar{\bar{\delta\varphi}},\varphi) - (\bar{\delta\varphi},\varphi)\in \ker(\partial) = \mathrm{im}(\gamma:Q^{n+1}(S^{n-r}\mathbb{Z}[\pi])\longrightarrow Q^{n+1}(\bar{i}))\\ (\bar{\bar{\delta\psi}},\psi) - (\bar{\delta\psi},\psi)\in \ker(\partial) = \mathrm{im}(\gamma:Q_{n+1}(S^{n-r}\mathbb{Z}[\pi])\longrightarrow Q_{n+1}(\bar{i}))\end{array}\right. .$$

Next, recall from Proposition 4.1 the morphism $\left\{\begin{array}{l}j:\pi_r(SO(n-r))\longrightarrow Q^{n+1}(S^{n-r}\mathbb{Z})\\ j:\pi_{r+1}(SO/SO(n-r))\longrightarrow Q_{n+1}(S^{n-r}\mathbb{Z})\end{array}\right.$.

<u>Proposition 8.5</u> The algebraic effect on an elementary $\left\{\begin{array}{l}\text{oriented}\\ \text{framed}\end{array}\right.$ surgery of type $(r,n-r-1)$ of a change of framing by $\left\{\begin{array}{l}\alpha\in\pi_r(SO(n-r))\\ \beta\in\pi_{r+1}(SO/SO(n-r))\end{array}\right.$ is given by

$$\left\{\begin{array}{l}(\bar{\bar{\delta\varphi}},\varphi) - (\bar{\delta\varphi},\varphi) = \gamma(j(\alpha),0)\in Q^{n+1}(\bar{i})\\ (\bar{\bar{\delta\psi}},\psi) - (\bar{\delta\psi},\psi) = \gamma(j(\beta),0)\in Q_{n+1}(\bar{i})\end{array}\right. ,$$

with

$$\left\{\begin{array}{l}(j(\alpha),0)\in Q^{n+1}(S^{n-r}\mathbb{Z})\oplus H^{n-2r-1}(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},(-)^{n-r}) = Q^{n+1}(S^{n-r}\mathbb{Z}[\pi])\\ (j(\beta),0)\in Q_{n+1}(S^{n-r}\mathbb{Z})\oplus H_{2r-n+1}(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},(-)^{n-r}) = Q_{n+1}(S^{n-r}\mathbb{Z}[\pi])\end{array}\right. .$$

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§9. <u>Lower L-theory</u>

We shall now construct lower $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups $\left\{\begin{array}{l}L^n(A,\varepsilon)\\ L_n(A,\varepsilon)\end{array}\right.$ for $n = -1,-2,\ldots$ extending the sequence of higher L-groups defined for $n\geqslant 0$ in §5 above. We shall do this entirely in an ad hoc manner, to be justified by the extension to the lower L-groups of various results for the higher L-groups. In §10 below we shall associate to a morphism of rings with involution $f:A\longrightarrow B$ a long exact sequence

$$\left\{\begin{array}{l}\ldots\longrightarrow L^n(A,\varepsilon)\xrightarrow{f} L^n(B,f(\varepsilon))\longrightarrow L^n(f,\varepsilon)\xrightarrow{\partial} L^{n-1}(A,\varepsilon)\longrightarrow\ldots\\ \ldots\longrightarrow L_n(A,\varepsilon)\xrightarrow{f} L_n(B,f(\varepsilon))\longrightarrow L_n(f,\varepsilon)\xrightarrow{\partial} L_{n-1}(A,\varepsilon)\longrightarrow\ldots\end{array}\right. \qquad (n\in\mathbb{Z})$$

involving relative L-groups $\left\{\begin{array}{l}L^n(f,\varepsilon)\\ L_n(f,\varepsilon)\end{array}\right.$ $(n\in\mathbb{Z})$. In §11 we shall define products

$$\left\{\begin{array}{l}\otimes : L^m(A,\varepsilon)\otimes_{\mathbb{Z}}L^n(B,\eta)\longrightarrow L^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\\ \otimes : L_m(A,\varepsilon)\otimes_{\mathbb{Z}}L_n(B,\eta)\longrightarrow L_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\end{array}\right. \qquad (m,n\in\mathbb{Z}) .$$

In §12 we shall establish a long exact sequence in both the higher and the lower L-groups relating the L-theories obtained by restricting the projective class and the Whitehead torsion. In §13 we shall describe the relative L-groups $\left\{\begin{array}{l}L^n(A\longrightarrow S^{-1}A,\varepsilon)\\ L_n(A\longrightarrow S^{-1}A,\varepsilon)\end{array}\right.$ $(n\in\mathbb{Z})$ of a localization map $A\longrightarrow S^{-1}A$ in terms of $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ structures on S-torsion A-modules. In §14 we shall show that

$$L^n(A[z,z^{-1}],\varepsilon) = L^n(A,\varepsilon)\oplus L^{n-1}(A,\varepsilon)\oplus ? \qquad (\bar{z} = z^{-1})$$

for $n\leqslant 1$, with $? = 0$ for $n\leqslant -1$, and that

$$L_n(A[z,z^{-1}],\varepsilon) = L_n(A,\varepsilon)\oplus L_{n-1}(A,\varepsilon) \quad (n\in\mathbb{Z}).$$

In §15 we shall construct simplicial Ω-spectra $\left\{\begin{array}{l}\underline{\mathcal{L}}^*(A,\varepsilon)\\ \underline{\mathcal{L}}_*(A,\varepsilon)\end{array}\right.$ such that

$$\left\{\begin{array}{l}\pi_n(\underline{\mathcal{L}}^*(A,\varepsilon)) = L^n(A,\varepsilon)\\ \pi_n(\underline{\mathcal{L}}_*(A,\varepsilon)) = L_n(A,\varepsilon)\end{array}\right. \qquad (n\in\mathbb{Z}) .$$

An $\left\{\begin{matrix} n- \\ (n+1)- \end{matrix}\right.$ dimensional ε-symmetric $\left\{\begin{matrix} \text{complex} \\ \text{pair} \end{matrix}\right.$ over A

$\left\{\begin{matrix} (C,\varphi\in Q^n(C,\varepsilon)) \\ (f:C \longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon)) \end{matrix}\right.$ is <u>even</u> if

$$\left\{\begin{matrix} \hat{v}_0(\varphi) = 0 : H^n(C) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \ ; \ x \longmapsto \varphi_n(x)(x) \\ \hat{v}_0(\delta\varphi,\varphi) = 0 : H^{n+1}(f) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \ ; \ (y,x) \longmapsto (\delta\varphi_{n+1}(y)(y) + (-)^n\varphi_n(x)(x)) \end{matrix}\right.$$

$$(x\in C^n,\ y\in D^{n+1},\ n \geqslant 0) \ .$$

Call n-dimensional even ε-symmetric Poincaré complexes $(C,\varphi),(C',\varphi')$ <u>cobordant</u> if there exists an even (n+1)-dimensional ε-symmetric Poincaré pair $((f\ f'):C\oplus C' \longrightarrow D,(\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ f'),\varepsilon))$ with boundary $(C,\varphi)\oplus(C',-\varphi')$.

<u>Proposition 9.1</u> i) Cobordism is an equivalence relation on n-dimensional even ε-symmetric Poincaré complexes over A, such that homotopy equivalent complexes are cobordant. The cobordism classes define the <u>**n-dimensional even ε-symmetric L-group of A**</u> $L\langle v_0\rangle^n(A,\varepsilon)$ $(n \geqslant 0)$, with addition and inverses by

$$(C,\varphi) + (C',\varphi') = (C\oplus C',\varphi\oplus\varphi') \quad , \quad -(C,\varphi) = (C,-\varphi) \in L\langle v_0\rangle^n(A,\varepsilon) \ .$$

ii) The ε-symmetrization and skew-suspension maps factorize through the even ε-symmetric L-groups

$$\begin{matrix} 1+T_\varepsilon : L_n(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L\langle v_0\rangle^n(A,\varepsilon) \xrightarrow{\text{forget}} L^n(A,\varepsilon) \\ \overline{S} : L^n(A,\varepsilon) \xrightarrow{\overline{S}} L\langle v_0\rangle^{n+2}(A,-\varepsilon) \xrightarrow{\text{forget}} L^{n+2}(A,-\varepsilon) \end{matrix} \quad (n \geqslant 0) \ .$$

iii) The skew-suspension map

$$\overline{S}:L^n(A,\varepsilon) \longrightarrow L\langle v_0\rangle^{n+2}(A,-\varepsilon) \ ; \ (C,\varphi\in Q^n(C,\varepsilon)) \longmapsto (SC,\overline{S}\varphi\in Q^{n+2}(SC,-\varepsilon))$$

is an isomorphism for all $A,\varepsilon,n \geqslant 0$.

<u>Proof</u>: i) Work exactly as in Proposition 5.2.

ii) The $\left\{\begin{matrix} \varepsilon\text{-symmetrization} \\ \text{skew-suspension} \end{matrix}\right.$ of an n-dimensional $\left\{\begin{matrix} \varepsilon\text{-quadratic} \\ \varepsilon\text{-symmetric} \end{matrix}\right.$ Poincaré complex (resp. pair) is an $\left\{\begin{matrix} n- \\ (n+2)- \end{matrix}\right.$ dimensional even $\left\{\begin{matrix} \varepsilon- \\ (-\varepsilon)- \end{matrix}\right.$ symmetric Poincaré complex (resp. pair).

iii) Define as follows a map

$$\Omega : L\langle v_0\rangle^{n+2}(A,-\varepsilon) \longrightarrow L^n(A,\varepsilon)$$

inverse to $\overline{S}:L^n(A,\varepsilon) \longrightarrow L\langle v_0\rangle^{n+2}(A,-\varepsilon)$.

Let $p = \left\{\begin{matrix} 2 \\ 3 \end{matrix}\right.$. Given an (n+p)-dimensional even $(-\varepsilon)$-symmetric $\left\{\begin{matrix} \text{Poincaré} \\ \text{connected} \end{matrix}\right.$ complex over A $(C,\varphi\in Q^{n+p}(C,\varepsilon))$ replace C (if necessary) by a chain equivalent f.g. projective A-module chain complex of the type

$$C : \ldots \longrightarrow 0 \longrightarrow C_{n+p} \xrightarrow{d} C_{n+p-1} \xrightarrow{d} C_{n+p-2} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots$$

and define a chain map $f:C \longrightarrow D$ by

$$f = \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right. : C_r \longrightarrow D_r = \left\{\begin{matrix} C_{n+p} & r = n+p \\ 0 & r \neq n+p \end{matrix}\right. \ .$$

As (C,φ) is even

$$v_0(\varphi) = 0 : H^{n+p}(C) = \text{coker}(d^*:C^{n+p-1} \longrightarrow C^{n+p}) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,-\varepsilon) \ ; \ x \longmapsto \varphi_{n+p}(x)(x),$$

so that $\varphi_{n+p} \in \text{im}((1+T_{-\varepsilon}):Q_{-\varepsilon}(C^{n+p}) \longrightarrow Q^{-\varepsilon}(C^{n+p}))$ and

$$f^{\%}(\varphi) = \varphi_{n+p} = 0 \in Q^{n+p}(D,-\varepsilon) = \text{coker}((1+T_{-\varepsilon}):Q_{-\varepsilon}(C^{n+p}) \longrightarrow Q^{-\varepsilon}(C^{n+p})) \ .$$

It is thus possible to lift $\varphi\in Q^{n+p}(C,-\varepsilon)$ to an element $(\theta,\varphi)\in Q^{n+p+1}(f,-\varepsilon)$. The result of ε-symmetric surgery on $(f:C \longrightarrow D,(\theta,\varphi)\in Q^{n+p+1}(f,-\varepsilon))$ is the skew-suspension $\overline{S}(C',\varphi')$ of an (n+p-2)-dimensional ε-symmetric $\left\{\begin{matrix} \text{Poincaré} \\ \text{connected} \end{matrix}\right.$ complex over A (C',φ').

Use the case p = 2 to define

$$\Omega : L\langle v_0\rangle^{n+2}(A,-\varepsilon) \longrightarrow L^n(A,\varepsilon) \ ; \ (C,\varphi) \longmapsto (C',\varphi') \ ,$$

so that

$$\Omega\overline{S}(C',\varphi') = (C',\varphi') \in L^n(A,\varepsilon) \ .$$

The proof of Proposition 7.1 ii) gives an even cobordism from (C,φ) to $\overline{S}(C',\varphi')$

$$c = ((g\ g'):C\oplus SC' \longrightarrow D',(S\varphi,\varphi\oplus-\overline{S}\varphi')\in Q^{n+3}((g\ g'),-\varepsilon))$$

with $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C \longrightarrow D' = C(\varphi_0 f^*:D^{n+2-*} \longrightarrow C)$, so that

$$\overline{S}\Omega(C,\varphi) = (C,\varphi) \in L\langle v_0\rangle^{n+2}(A,-\varepsilon) \ .$$

If $(\bar\theta,\varphi)\in Q^{n+3}(f,-\varepsilon)$ is a different lift of $\varphi\in Q^{n+2}(C,-\varepsilon)$, with corresponding n-dimensional ε-symmetric Poincaré complex $(\bar C',\bar\varphi'\in Q^n(\bar C',\varepsilon))$, let

$$\bar c = ((\bar g\ \bar g'):C\oplus S\bar C' \longrightarrow \bar D', (\overline{\delta\varphi},\varphi\oplus -\bar S\bar\varphi')\in Q^{n+3}((\bar g\ \bar g'),-\varepsilon))$$

be the corresponding cobordism from (C,φ) to $\bar S(\bar C',\bar\varphi')$. The union

$$c\cup\bar c = ((h\ h'):SC'\oplus S\bar C' \longrightarrow D'', (\delta\varphi'',\bar S\varphi'\oplus -\bar S\bar\varphi')\in Q^{n+3}((h\ h'),-\varepsilon))$$

is a cobordism from $\bar S(C',\varphi')$ to $\bar S(\bar C',\bar\varphi')$, with $D'' = C(\begin{pmatrix} g\\ g'\end{pmatrix}:C\longrightarrow D'\oplus\bar D')$ such that

$$H_0(D'') = H_{n+3}(D'') = 0 \ .$$

It follows that $c\cup\bar c$ is the skew-suspension of a cobordism from (C',φ') to $(\bar C',\bar\varphi')$ and so

$$(C',\varphi') = (\bar C',\bar\varphi') \in L^n(A,\varepsilon) \ .$$

Thus $\Omega(C,\varphi)\in L^n(A,\varepsilon)$ is independent of the choice of lift $(\theta,\varphi)\in Q^{n+p}(f,-\varepsilon)$.

The correspondence of Proposition 5.4 i) restricts to a one-one correspondence between the homotopy equivalence classes of (n+3)-dimensional even (-ε)-symmetric Poincaré pairs over A and the homotopy equivalence classes of connected even (n+3)-dimensional (-ε)-symmetric complexes over A. As in Proposition 5.4 iii) we thus have that an (n+2)-dimensional even (-ε)-symmetric Poincaré complex over A represents 0 in $L\langle v_0\rangle^{n+2}(A,-\varepsilon)$ if and only if it is homotopy equivalent to the boundary $\partial(C,\varphi)$ of a connected (n+3)-dimensional even (-ε)-symmetric complex over A $(C,\varphi\in Q^{n+3}(C,-\varepsilon))$. Let (C',φ') be the connected (n+1)-dimensional ε-symmetric complex over A obtained from (C,φ), as in the case p = 3 above. Now $\partial(C,\varphi)$ is homotopy equivalent to $\bar S\partial(C',\varphi')$ by Proposition 7.1 ii), and so

$$\Omega\partial(C,\varphi) = \Omega\bar S\partial(C',\varphi') = \partial(C',\varphi') = 0\in L^n(A,\varepsilon) \ .$$

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An ε-symmetric $\begin{cases}\text{form}\\ \text{formation}\end{cases}$ over A $\begin{cases}(M,\varphi)\\ (M,\varphi;F,G)\end{cases}$ is even if $(M,\varphi) = (1+T_\varepsilon)(M,\psi)$ for some ε-quadratic form (M,ψ), that is if $\varphi\in\mathrm{im}((1+T_\varepsilon):Q_\varepsilon(M)\longrightarrow Q^\varepsilon(M))$ or equivalently $\varphi(x)(x)\in\mathrm{im}((1+T_\varepsilon):A\longrightarrow A;a\longmapsto a+\varepsilon\bar a)$ $(x\in M)$. In particular, the metabolic ε-symmetric form $H^\varepsilon(L) = (L\oplus L^*,\begin{pmatrix}0&1\\ \varepsilon&0\end{pmatrix}\in Q^\varepsilon(L\oplus L^*))$ is even, and every even ε-symmetric formation is isomorphic to one of the type $(H^\varepsilon(F);F,G)$ by Proposition 1.6.

The <u>Witt group of even ε-symmetric</u> $\begin{cases}\underline{\text{forms}}\\ \underline{\text{formations}}\end{cases}$ <u>over A</u> $\begin{cases}L\langle v_0\rangle^\varepsilon(A)\\ M\langle v_0\rangle^\varepsilon(A)\end{cases}$ is the abelian group with respect to the direct sum ⊕ of equivalence classes of non-singular even ε-symmetric $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ over A, subject to the equivalence relation $\begin{cases}(M,\varphi)\sim(M',\varphi')\\ (M,\varphi;F,G)\sim(M',\varphi';F',G')\end{cases}$ if there exists $\begin{cases}\text{an isomorphism}\\ \text{a stable isomorphism}\end{cases}$ of the type

$$\begin{cases} f : (M,\varphi)\oplus H^\varepsilon(N) \longrightarrow (M',\varphi')\oplus H^\varepsilon(N') \\ [f] : (M,\varphi;F,G)\oplus(N,\nu;H,K)\oplus(N,\nu;K,L)\oplus(N',\nu';H',L') \\ \qquad \longrightarrow (M',\varphi';F',G')\oplus(N',\nu';H',K')\oplus(N',\nu';K',L')\oplus(N,\nu;K,L) \ . \end{cases}$$

The <u>boundary</u> of an $\begin{cases}\text{ε-symmetric form}\\ \text{even ε-symmetric formation}\end{cases}$ over A $\begin{cases}(M,\varphi)\\ (M,\varphi;F,G)\end{cases}$ is the non-singular even $\begin{cases}\text{(-ε)-symmetric formation}\\ \text{ε-symmetric form}\end{cases}$ over A

$$\begin{cases}\partial(M,\varphi) = (H^{-\varepsilon}(M);M,\Gamma_{(M,\varphi)}) \quad \Gamma_{(M,\varphi)} = \{(x,\varphi(x))\,|\,x\in M\} \\ \partial(M,\varphi;F,G) = (G^\perp/G,\varphi^\perp/\varphi) \end{cases} .$$

<u>Proposition 9.2</u> i) The homotopy equivalence classes of $\begin{cases} 0- \\ 1- \end{cases}$ dimensional even ε-symmetric complexes over A which are $\begin{cases} - \\ \text{connected} \end{cases}$ (Poincaré) are in a natural one-one correspondence with the $\begin{cases} \text{isomorphism} \\ \text{stable isomorphism} \end{cases}$ classes of (non-singular) even ε-symmetric $\begin{cases} \text{forms} \\ \text{formations} \end{cases}$ over A.

ii) The $\begin{cases} 0- \\ 1- \end{cases}$ dimensional even ε-symmetric L-group $\begin{cases} L\langle v_0\rangle^0(A,\varepsilon) \\ L\langle v_0\rangle^1(A,\varepsilon) \end{cases}$ is naturally isomorphic to the Witt group of even ε-symmetric $\begin{cases} \text{forms} \\ \text{formations} \end{cases}$ over A $\begin{cases} L\langle v_0\rangle^\varepsilon(A) \\ M\langle v_0\rangle^\varepsilon(A) \end{cases}$.

iii) A non-singular even ε-symmetric $\begin{cases} \text{form} \\ \text{formation} \end{cases}$ over A $\begin{cases} (M,\varphi) \\ (M,\varphi;F,G) \end{cases}$ represents 0 in $\begin{cases} L\langle v_0\rangle^0(A,\varepsilon) \\ L\langle v_0\rangle^1(A,\varepsilon) \end{cases}$ if and only if it is $\begin{cases} \text{isomorphic} \\ \text{stably isomorphic} \end{cases}$ to the boundary $\begin{cases} \partial(N,\nu;H,K) \\ \partial(N,\nu) \end{cases}$ of $\begin{cases} \text{an even } \varepsilon\text{-symmetric formation} \\ \text{a } (-\varepsilon)\text{-symmetric form} \end{cases}$ over A $\begin{cases} (N,\nu;H,K) \\ (N,\nu) \end{cases}$.

<u>Proof</u>: By analogy with Propositions 1.5,1.7,7.6,7.7,7.8.

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The result of Proposition 9.2 iii) for $L\langle v_0\rangle^1(A,\varepsilon)$ will be interpreted in §12 as a reduction identity for the elementary subgroup of the stable unitary group $\varinjlim_m \text{Aut}\, H^\varepsilon(A^m)$ (Proposition 12.2).

Define the <u>lower ε-quadratic L-groups of A</u>

$$L_n(A,\varepsilon) = L_{n+2i}(A,(-)^i\varepsilon) \quad (n \leqslant -1,\ n+2i \geqslant 0)\ ,$$

extending the periodicity $L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon)$ $(n \geqslant 0)$ of Proposition 7.3.

Define the <u>lower ε-symmetric L-groups of A</u>

$$L^n(A,\varepsilon) = \begin{cases} L\langle v_0\rangle^{n+2}(A,-\varepsilon) & n = -1,-2 \\ L_n(A,\varepsilon) & n \leqslant -3 \end{cases}$$

extending the identification $L^n(A,\varepsilon) = L\langle v_0\rangle^{n+2}(A,-\varepsilon)$ $(n \geqslant 0)$ of Proposition 9.1 iii).

Define the <u>skew-suspension</u> maps

$$\begin{cases} \bar{S} : L^n(A,\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \\ \bar{S} : L_n(A,\varepsilon) \longrightarrow L_{n+2}(A,-\varepsilon) \end{cases} \quad (n \leqslant -1)$$

to be the $\pm\varepsilon$-quadratic skew-suspension isomorphism if $\begin{cases} n \leqslant -5 \\ n \leqslant -1 \end{cases}$, and the forgetful map if $\begin{cases} -4 \leqslant n \leqslant -1 \\ - \end{cases}$.

<u>Proposition 9.3</u> The skew-suspension map

$$\bar{S} : L^{-4}(A,\varepsilon) = L_0(A,\varepsilon) \longrightarrow L^{-2}(A,-\varepsilon) = L\langle v_0\rangle^0(A,\varepsilon)$$

is onto, with kernel generated by the non-singular ε-quadratic forms over A of the type

$$(A\oplus A^*, \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \in Q_\varepsilon(A\oplus A^*))\ ,$$

with $a,b \in \hat{H}^1(\mathbb{Z}_2;A,\varepsilon) = \{x\in A \mid x+\varepsilon\bar{x} = 0\}/\{y-\varepsilon\bar{y} \mid y\in A\}$.

<u>Proof</u>: The map is onto because every even ε-symmetric form is the ε-symmetrization of an ε-quadratic form. An element $(M,\psi)\in\ker(\bar{S})$ is represented by a non-singular ε-quadratic form $(M,\psi\in Q_\varepsilon(M))$ such that

$$\psi+\varepsilon\psi^* = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} : M = L\oplus L^* \longrightarrow M^* = L^*\oplus L$$

for a f.g. free A-module L, and (M,ψ) can be expressed as a direct sum of terms like $(A\oplus A^*, \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix})$.

Proposition 9.4 If A,ε are such that $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\equiv\{a\in A|\varepsilon\bar{a}=a\}/\{b+\varepsilon\bar{b}|b\in A\}=0$ then the skew-suspension maps

$$\bar{S}\ :\ L^n(A,-\varepsilon)\longrightarrow L^{n+2}(A,\varepsilon)\qquad(n\in\mathbb{Z})$$

are isomorphisms.

Proof: The case $n\leqslant -5$ has already been considered in Proposition 7.3. For $-4\leqslant n\leqslant -1$ note that the category of even ε-symmetric (resp. even $(-\varepsilon)$-symmetric) $\left\{\begin{array}{l}\text{forms}\\\text{formations}\end{array}\right.$ is equivalent to that of ε-symmetric (resp. $(-\varepsilon)$-quadratic) $\left\{\begin{array}{l}\text{forms}\\\text{formations}\end{array}\right.$ if $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)=0$. For $n\geqslant 0$ we can apply the criterion of Proposition 7.2, as follows. Given a connected $(n+2)$-dimensional ε-symmetric complex over A $(C,\varphi\in Q^{n+2}(C,\varepsilon))$ define a chain map $f:C\longrightarrow D$ by

$$f=\left\{\begin{array}{l}1\\0\end{array}\right.:C_r\longrightarrow D_r=\left\{\begin{array}{ll}C_{n+2}&r=n+2\\0&r\neq n+2\end{array}\right.,$$

assuming C to be such that $C_r=0$ for $r>n+2$ with C_{n+2} f.g. projective. Now $f^{\%}(\varphi)\in Q^{n+2}(D,\varepsilon)=\ker(1-T_\varepsilon)/\mathrm{im}(1+T_\varepsilon)=0$, with T_ε the ε-duality involution on $\mathrm{Hom}_A(C^{n+2},C_{n+2})$. It is thus possible to define a connected ε-symmetric pair $(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+3}(f,\varepsilon))$ with which to do surgery on (C,φ) to obtain the skew-suspension $\bar{S}(C',\varphi')$ of an n-dimensional $(-\varepsilon)$-symmetric complex over A $(C',\varphi'\in Q^n(C',-\varepsilon))$. This proves that $\bar{S}:L^n(A,-\varepsilon)\longrightarrow L^{n+2}(A,\varepsilon)$ (resp. $\bar{S}:L^{n-1}(A,-\varepsilon)\longrightarrow L^{n+1}(A,\varepsilon)$) is onto (resp. one-one) for $n\geqslant 0$ (resp. $n\geqslant 1$).

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(Note that if $\hat{H}^*(\mathbb{Z}_2;A,\varepsilon)=0$ Proposition 9.4 gives that $L^n(A,\pm\varepsilon)=L^{n+2}(A,\mp\varepsilon)$ for $n\in\mathbb{Z}$. This is the case if there exists a central element $a\in A$ such that $a+\bar{a}=1\in A$, e.g. $a=1/2\in A$, as in Proposition 5.3).

Proposition 9.5 Let π be a group, and give the group ring $\mathbb{Z}[\pi]$ the untwisted involution $\bar{g}=g^{-1}$ $(g\in\pi)$. Then the skew-suspension maps

$$\bar{S}\ :\ L^n(\mathbb{Z}[\pi])\longrightarrow L^{n+2}(\mathbb{Z}[\pi],-1)\qquad(n\in\mathbb{Z})$$

are isomorphisms, and the skew-symmetrization map

$$(1+T_{-1})\ :\ L_0(\mathbb{Z}[\pi],-1)\longrightarrow L^0(\mathbb{Z}[\pi],-1)=L^{-2}(\mathbb{Z}[\pi])$$

is onto. If π has no 2-torsion there is defined a split short exact sequence

$$0\longrightarrow L_0(\mathbb{Z},-1)\longrightarrow L_0(\mathbb{Z}[\pi],-1)\xrightarrow{1+T_{-1}}L^0(\mathbb{Z}[\pi],-1)\longrightarrow 0\ .$$

Proof: Immediate from Propositions 9.3, 9.4 since $\hat{H}^0(\mathbb{Z}_2;\mathbb{Z}[\pi],-1)=0$ for any group π, and $\hat{H}^1(\mathbb{Z}_2;\mathbb{Z}[\pi],-1)=\hat{H}^1(\mathbb{Z}_2;\mathbb{Z},-1)$ if π has no 2-torsion.

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($L_0(\mathbb{Z},-1)=\mathbb{Z}_2$, generated by the Arf form $(\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix}1&1\\0&1\end{pmatrix}\in Q_{-1}(\mathbb{Z}\oplus\mathbb{Z}))$).

§10. Relative L-theory

Given a morphism of rings with involution $f:A \longrightarrow B$ and a central unit $\varepsilon \in A$ such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$ and $f(\varepsilon) \in B$ is central we shall now construct relative $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups $\left\{\begin{array}{l}L^n(f,\varepsilon)\\ L_n(f,\varepsilon)\end{array}\right.$ $(n \in \mathbb{Z})$ to fit into a long exact sequence

$$\left\{\begin{array}{l}\ldots \longrightarrow L^{n+1}(f,\varepsilon) \longrightarrow L^n(A,\varepsilon) \xrightarrow{f} L^n(B,\varepsilon) \longrightarrow L^n(f,\varepsilon) \longrightarrow \ldots\\ \ldots \longrightarrow L_{n+1}(f,\varepsilon) \longrightarrow L_n(A,\varepsilon) \xrightarrow{f} L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \longrightarrow \ldots\end{array}\right. \quad (n \in \mathbb{Z})$$

(denoting $f(\varepsilon) \in B$ by ε). The surgery obstruction of an $(n+1)$-dimensional normal map of pairs $(f,\partial f;b,\partial b) : (M,\partial M) \longrightarrow (X,\partial X)$ is an element

$$\sigma_*(f,\partial f;b,\partial b) \in L_{n+1}(\mathbb{Z}[\pi_1(\partial X)] \longrightarrow \mathbb{Z}[\pi_1(X)]) \ .$$

We shall also construct 'hyperquadratic' L-groups $\hat{L}^n(A,\varepsilon)$ $(n \in \mathbb{Z})$, to fit into a long exact sequence

$$\ldots \longrightarrow \hat{L}^{n+1}(A,\varepsilon) \xrightarrow{H} L_n(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L^n(A,\varepsilon) \xrightarrow{J} \hat{L}^n(A,\varepsilon) \longrightarrow \ldots \ .$$

An n-dimensional normal space $(X,\nu_X:X \longrightarrow BG(k),\rho_X \in \pi_{n+k}(T(\nu_X)))$ has a 'hyperquadratic signature' invariant $\hat{\sigma}^*(X) \in \hat{L}^n(\mathbb{Z}[\pi_1(X)])$, such that $H\hat{\sigma}^*(X) \in L_{n-1}(\mathbb{Z}[\pi_1(X)])$ is the obstruction to making X normal bordant to an n-dimensional geometric Poincaré complex with fundamental group $\pi_1(X)$.

We shall also construct 'mod m' $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups $\left\{\begin{array}{l}L^n(A,\varepsilon;\mathbb{Z}_m)\\ L_n(A,\varepsilon;\mathbb{Z}_m)\end{array}\right.$ $(n \in \mathbb{Z}, m \geqslant 1)$

to fit into a long exact sequence

$$\left\{\begin{array}{l}\ldots \longrightarrow L^{n+1}(A,\varepsilon;\mathbb{Z}_m) \xrightarrow{\partial} L^n(A,\varepsilon) \xrightarrow{m} L^n(A,\varepsilon) \longrightarrow L^n(A,\varepsilon;\mathbb{Z}_m) \longrightarrow \ldots\\ \ldots \longrightarrow L_{n+1}(A,\varepsilon;\mathbb{Z}_m) \xrightarrow{\partial} L_n(A,\varepsilon) \xrightarrow{m} L_n(A,\varepsilon) \longrightarrow L_n(A,\varepsilon;\mathbb{Z}_m) \longrightarrow \ldots\end{array}\right. \ .$$

An n-dimensional geometric $\mathbb{Z}_m$-Poincaré complex $(X,\partial X = \bigcup_m \delta X)$ has a 'mod m symmetric signature' invariant $\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)];\mathbb{Z}_m)$, and the surgery obstruction of an n-dimensional normal bundle map $(f,b):M \longrightarrow X$ from a $\mathbb{Z}_m$-manifold M to a $\mathbb{Z}_m$-Poincaré complex X is an element $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)];\mathbb{Z}_m)$.

Given a commutative square of A-module chain complexes

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f'}\downarrow & \Gamma & \downarrow{\scriptstyle g} \\ D' & \xrightarrow{g'} & C' \end{array}$$

define an A-module chain complex $C(\Gamma)$ by

$$d_{C(\Gamma)} = \begin{pmatrix} d_{C'} & (-)^{r-1}g & (-)^r g' & 0 \\ 0 & d_D & 0 & (-)^r f \\ 0 & 0 & d_{D'} & (-)^r f' \\ 0 & 0 & 0 & d_C \end{pmatrix}$$

$$: C(\Gamma)_r = C'_r \oplus D_{r-1} \oplus D'_{r-1} \oplus C_{r-2} \longrightarrow C(\Gamma)_{r-1} = C'_{r-1} \oplus D_{r-2} \oplus D'_{r-2} \oplus C_{r-3} \ .$$

The homology groups of Γ are defined by

$$H_n(\Gamma) = H_n(C(\Gamma)) \quad (n \in \mathbb{Z}) \ ,$$

and are such that there is defined a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H_{n+2}(\Gamma) & \longrightarrow & H_{n+1}(f') & \longrightarrow & H_{n+1}(g) & \longrightarrow & H_{n+1}(\Gamma) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H_{n+1}(f) & \longrightarrow & H_n(C) & \xrightarrow{f} & H_n(D) & \longrightarrow & H_n(f) & \longrightarrow & \ldots \\
 & & \downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\
\ldots & \longrightarrow & H_{n+1}(g') & \longrightarrow & H_n(D') & \xrightarrow{g'} & H_n(C') & \longrightarrow & H_n(g') & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H_{n+1}(\Gamma) & \longrightarrow & H_n(f') & \longrightarrow & H_n(g) & \longrightarrow & H_n(\Gamma) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Given a commutative square Γ of A-module chain complexes (as above) let $\Gamma^t\otimes_A\Gamma$ be the commutative square of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\begin{array}{ccc} C^t\otimes_A C & \xrightarrow{f^t\otimes_A f} & D^t\otimes_A D \\ {\scriptstyle f'^t\otimes_A f'}\downarrow & \Gamma^t\otimes_A\Gamma & \downarrow{\scriptstyle g^t\otimes_A g} \\ D'^t\otimes_A D' & \xrightarrow{g'^t\otimes_A g'} & C'^t\otimes_A C' \end{array}$$

with $T\in\mathbb{Z}_2$ acting by the ε-transposition T_ε, and define the $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{cases}$ groups

$$\begin{cases} Q^n(\Gamma,\varepsilon) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C(\Gamma^t\otimes_A\Gamma))) \\ Q_n(\Gamma,\varepsilon) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}C(\Gamma^t\otimes_A\Gamma)) \end{cases} \quad (n\in\mathbb{Z}) .$$

An element $\begin{cases}(\varphi',\nu,\nu',\varphi)\in Q^{n+2}(\Gamma,\varepsilon)\\ (\psi',\chi,\chi',\psi)\in Q_{n+2}(\Gamma,\varepsilon)\end{cases}$ is represented by a collection of chains

$$\begin{cases}\{(\varphi'_s,\nu_s,\nu'_s,\varphi_s)\in(C'^t\otimes_A C')_{n+s+2}\oplus(D^t\otimes_A D)_{n+s+1}\oplus(D'^t\otimes_A D')_{n+s+1}\oplus(C^t\otimes_A C)_{n+s}\,|\,s\geq 0\}\\ \{(\psi'_s,\chi_s,\chi'_s,\psi_s)\in(C'^t\otimes_A C')_{n-s+2}\oplus(D^t\otimes_A D)_{n-s+1}\oplus(D'^t\otimes_A D')_{n-s+1}\oplus(C^t\otimes_A C)_{n-s}\,|\,s\geq 0\}\end{cases}$$

such that

$$\begin{cases}\begin{aligned} d(\varphi'_s,\nu_s,\nu'_s,\varphi_s) &= (d_{C'^t\otimes_A C'}(\varphi'_s)+(-)^{n+s+1}(\varphi'_{s-1}+(-)^sT_\varepsilon\varphi'_{s-1})+(-)^{n+1}(g^t\otimes_A g(\nu_s)-g'^t\otimes_A g'(\nu'_s)),\\ &\quad d_{D^t\otimes_A D}(\nu_s)+(-)^{n+s}(\nu_{s-1}+(-)^sT_\varepsilon\nu_{s-1})+(-)^n(f^t\otimes_A f)(\varphi_s),\\ &\quad d_{D'^t\otimes_A D'}(\nu'_s)+(-)^{n+s}(\nu'_{s-1}+(-)^sT_\varepsilon\nu'_{s-1})+(-)^n(f'^t\otimes_A f')(\varphi_s),\\ &\quad d_{C^t\otimes_A C}(\varphi_s)+(-)^{n+s-1}(\varphi_{s-1}+(-)^sT_\varepsilon\varphi_{s-1}))\\ &= 0\in(C'^t\otimes_A C')_{n+s+1}\oplus(D^t\otimes_A D)_{n+s}\oplus(D'^t\otimes_A D')_{n+s}\oplus(C^t\otimes_A C)_{n+s-1}\\ d(\psi'_s,\chi_s,\chi'_s,\psi_s) &= (d_{C'^t\otimes_A C'}(\psi'_s)+(-)^{n-s+1}(\psi'_{s+1}+(-)^{s+1}T_\varepsilon\psi'_{s+1})+(-)^{n+1}(g^t\otimes_A g(\chi_s)-g'^t\otimes_A g'(\chi'_s)),\\ &\quad d_{D^t\otimes_A D}(\chi_s)+(-)^{n-s}(\chi_{s+1}+(-)^{s+1}T_\varepsilon\chi_{s+1})+(-)^n(f^t\otimes_A f)(\psi_s),\\ &\quad d_{D'^t\otimes_A D'}(\chi'_s)+(-)^{n-s}(\chi'_{s+1}+(-)^{s+1}T_\varepsilon\chi'_{s+1})+(-)^n(f'^t\otimes_A f')(\psi_s),\\ &\quad d_{C^t\otimes_A C}(\psi_s)+(-)^{n-s-1}(\psi_{s+1}+(-)^{s+1}T_\varepsilon\psi_{s+1}))\\ &= 0\in(C'^t\otimes_A C')_{n-s+1}\oplus(D^t\otimes_A D)_{n-s}\oplus(D'^t\otimes_A D')_{n-s}\oplus(C^t\otimes_A C)_{n-s-1}\end{aligned}\end{cases}$$

$(s\geq 0)$.

An <u>(n+2)-dimensional</u> $\begin{cases}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\end{cases}$ <u>triad over A</u> $\begin{cases}(\Gamma,\Phi)\\ (\Gamma,\Psi)\end{cases}$ $(n\geq 0)$ is a commutative square $\begin{array}{ccc} C & \xrightarrow{f} & D\\ {\scriptstyle f'}\downarrow & \Gamma & \downarrow{\scriptstyle g}\\ D' & \xrightarrow{g'} & C'\end{array}$ of A-module chain complexes such that C is n-dimensional, D and D' are (n+1)-dimensional, and C' is (n+2)-dimensional together with an element $\begin{cases}\Phi=(\varphi',\nu,\nu',\varphi)\in Q^{n+2}(\Gamma,\varepsilon)\\ \Psi=(\psi',\chi,\chi',\psi)\in Q_{n+2}(\Gamma,\varepsilon)\end{cases}$. Such a triad is <u>Poincaré</u> if the (n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ pairs $\begin{cases}(f:C\longrightarrow D,(\nu,\varphi)\in Q^{n+1}(f,\varepsilon))\\ (f:C\longrightarrow D,(\chi,\psi)\in Q_{n+1}(f,\varepsilon))\end{cases}$, $\begin{cases}(f':C\longrightarrow D',(\nu',\varphi)\in Q^{n+1}(f',\varepsilon))\\ (f':C\longrightarrow D',(\chi',\psi)\in Q_{n+1}(f',\varepsilon))\end{cases}$ are Poincaré (in which case $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$ is an n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex) and such that the chain map

$$\begin{cases}\Phi_0 : C'^{n+2-*}\longrightarrow C(\Gamma)\\ (1+T_\varepsilon)\Psi_0 : C'^{n+2-*}\longrightarrow C(\Gamma)\end{cases}$$

defined by

$$\begin{cases}\Phi_0=\begin{pmatrix}\varphi'_0\\ (-)^{n-r}\nu_0 g^*\\ (-)^{n-r}\nu'_0 g'^*\\ \varphi_0 f^* g^*\end{pmatrix} : C'^{n+2-r}\longrightarrow C(\Gamma)_r = C'_r\oplus D_{r-1}\oplus D'_{r-1}\oplus C_{r-2}\\ (1+T_\varepsilon)\Psi_0=\begin{pmatrix}(1+T_\varepsilon)\psi'_0\\ (-)^{n-r}(1+T_\varepsilon)\chi_0 g^*\\ (-)^{n-r}(1+T_\varepsilon)\chi'_0 g'^*\\ (1+T_\varepsilon)\psi_0 f^* g^*\end{pmatrix} : C'^{n+2-r}\longrightarrow C(\Gamma)_r = C'_r\oplus D_{r-1}\oplus D'_{r-1}\oplus C_{r-2}\end{cases}$$

is a chain equivalence.

Proposition 10.1 Given a morphism of rings with involution $f:A\longrightarrow B$ and a central unit $\varepsilon\in A$ such that $\bar{\varepsilon}=\varepsilon^{-1}\in A$ and $\varepsilon = f(\varepsilon)\in B$ is a central unit there are defined relative $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ L-groups $\begin{cases}L^n(f,\varepsilon)\\ L_n(f,\varepsilon)\end{cases}$ $(n\in\mathbb{Z})$ which fit into a morphism of exact sequences

$$\begin{array}{ccccccccccc}
\dots\longrightarrow & L_n(A,\varepsilon) & \xrightarrow{f} & L_n(B,\varepsilon) & \longrightarrow & L_n(f,\varepsilon) & \longrightarrow & L_{n-1}(A,\varepsilon) & \xrightarrow{f} & L_{n-1}(B,\varepsilon) & \longrightarrow\dots\\
& \downarrow{\scriptstyle 1+T_\varepsilon} & & \downarrow{\scriptstyle 1+T_\varepsilon} & & \downarrow{\scriptstyle 1+T_\varepsilon} & & \downarrow{\scriptstyle 1+T_\varepsilon} & & \downarrow{\scriptstyle 1+T_\varepsilon} & \\
\dots\longrightarrow & L^n(A,\varepsilon) & \xrightarrow{f} & L^n(B,\varepsilon) & \longrightarrow & L^n(f,\varepsilon) & \longrightarrow & L^{n-1}(A,\varepsilon) & \xrightarrow{f} & L^{n-1}(B,\varepsilon) & \longrightarrow\dots
\end{array}$$

involving the change of rings map

$$\begin{cases} f\ :\ L^n(A,\varepsilon)\longrightarrow L^n(B,\varepsilon)\ ;\ (C,\varphi\in Q^n(C,\varepsilon))\longmapsto(B\otimes_A C,1\otimes\varphi\in Q^n(B\otimes_A C,\varepsilon))\\ f\ :\ L_n(A,\varepsilon)\longrightarrow L_n(B,\varepsilon)\ ;\ (C,\psi\in Q_n(C,\varepsilon))\longmapsto(B\otimes_A C,1\otimes\psi\in Q_n(B\otimes_A C,\varepsilon))\ .\end{cases}$$

The relative L-groups are such that

$$L_n(f,\varepsilon) = L_{n+2}(f,-\varepsilon)\quad(n\in\mathbb{Z})\ ,\quad L^n(f,\varepsilon)=L_n(f,\varepsilon)\quad(n\leqslant -3)\ .$$

Proof: For $n\geqslant 0$ define $\begin{cases}L^{n+1}(f,\varepsilon)\\ L_{n+1}(f,\varepsilon)\end{cases}$ to be the abelian group of equivalence classes of pairs

(n-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over A $\begin{cases}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{cases}$,

(n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pair over B

$$\begin{cases}(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi)\in Q^{n+1}(g,\varepsilon))\\ (g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi)\in Q_{n+1}(g,\varepsilon))\end{cases})$$

under the equivalence relation <u>relative cobordism</u>

$$\begin{cases}((C,\varphi),(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi)))\sim((C',\varphi'),(g':B\otimes_A C'\longrightarrow D',(\delta\varphi',1\otimes\varphi')))\\ ((C,\psi),(g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi)))\sim((C',\psi'),(g':B\otimes_A C'\longrightarrow D',(\delta\psi',1\otimes\psi')))\end{cases}$$

if there exists a pair

((n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pair over A

$$\begin{cases}((h\ h'):C\oplus C'\longrightarrow E,(\nu,\varphi\oplus-\varphi')\in Q^{n+1}((h\ h'),\varepsilon))\\ ((h\ h'):C\oplus C'\longrightarrow E,(\chi,\psi\oplus-\psi')\in Q_{n+1}((h\ h'),\varepsilon))\end{cases},$$

(n+2)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré triad over B

$$\begin{cases}(\Gamma,(\delta\nu,\delta\varphi\oplus-\delta\varphi',1\otimes\nu,1\otimes(\varphi\oplus-\varphi'))\in Q^{n+2}(\Gamma,\varepsilon))\\ (\Gamma,(\delta\chi,\delta\psi\oplus-\delta\psi',1\otimes\chi,1\otimes(\psi\oplus-\psi'))\in Q_{n+2}(\Gamma,\varepsilon))\end{cases}$$

with Γ a commutative square of B-module chain complexes

$$\begin{array}{ccc}
B\otimes_A(C\oplus C') & \xrightarrow{g\oplus g'} & D\oplus D'\\
{\scriptstyle 1\otimes(h\ h')}\downarrow & \Gamma & \downarrow\\
B\otimes_A E & \longrightarrow & F
\end{array}\quad).$$

The verification that relative cobordism is an equivalence relation proceeds as in the absolute case (Proposition 5.2), transitivity requiring a relative version of the union operation. Addition in $\begin{cases}L^{n+1}(f,\varepsilon)\\ L_{n+1}(f,\varepsilon)\end{cases}$ is by the direct sum $\oplus$, and inverses are obtained by changing the sign of the $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{cases}$ classes.

Define abelian group morphisms

$$\begin{cases}L^{n+1}(B,\varepsilon)\longrightarrow L^{n+1}(f,\varepsilon);(D,\delta\varphi)\longmapsto(0,(0:0\longrightarrow D,(\delta\varphi,0)))\\ L_{n+1}(B,\varepsilon)\longrightarrow L_{n+1}(f,\varepsilon);(D,\delta\psi)\longmapsto(0,(0:0\longrightarrow D,(\delta\psi,0)))\end{cases}$$

$$\begin{cases}L^{n+1}(f,\varepsilon)\longrightarrow L^n(A,\varepsilon);((C,\varphi),(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi)))\longmapsto(C,\varphi)\\ L_{n+1}(f,\varepsilon)\longrightarrow L_n(A,\varepsilon);((C,\psi),(g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi)))\longmapsto(C,\psi)\ .\end{cases}$$

We have to verify the exactness of the sequence

$$\begin{cases}\dots\longrightarrow L^{n+1}(B,\varepsilon)\longrightarrow L^{n+1}(f,\varepsilon)\longrightarrow L^n(A,\varepsilon)\xrightarrow{f}L^n(B,\varepsilon)\longrightarrow\dots\longrightarrow L^0(B,\varepsilon)\\ \dots\longrightarrow L_{n+1}(B,\varepsilon)\longrightarrow L_{n+1}(f,\varepsilon)\longrightarrow L_n(A,\varepsilon)\xrightarrow{f}L_n(B,\varepsilon)\longrightarrow\dots\longrightarrow L_0(B,\varepsilon)\ .\end{cases}$$

Exactness is obvious at $\begin{cases}L^n(A,\varepsilon),L^n(B,\varepsilon)\\ L_n(A,\varepsilon),L_n(B,\varepsilon)\end{cases}$. As for $\begin{cases}L^{n+1}(f,\varepsilon)\\ L_{n+1}(f,\varepsilon)\end{cases}$, consider

$\begin{cases}((C,\varphi),(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi)))\in\ker(L^{n+1}(f,\varepsilon)\longrightarrow L^n(A,\varepsilon))\\ ((C,\psi),(g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi)))\in\ker(L_{n+1}(f,\varepsilon)\longrightarrow L_n(A,\varepsilon))\end{cases}$, so that there exists a null-cobordism over A $\begin{cases}(h:C\longrightarrow E,(\nu,\varphi))\\(h:C\longrightarrow E,(\chi,\psi))\end{cases}$ of $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$. Replacing D by the algebraic mapping cylinder of $g:B\otimes_A C\longrightarrow D$, it may be assumed that g is the inclusion of a direct summand in each dimension. Similarly, we can take $h:C\longrightarrow E$ to be a direct inclusion, and the direct union (as defined in §5) is an

(n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over B

$$\begin{cases}(F,\delta\nu) = (g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi))\cup_{(B\otimes_A C,1\otimes\varphi)}1\otimes(h:C\longrightarrow E,(-\nu,-\varphi))\\(F,\delta\chi) = (g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi))\cup_{(B\otimes_A C,1\otimes\psi)}1\otimes(h:C\longrightarrow E,(-\chi,-\psi))\end{cases}$$

such that there is defined a commutative pushout diagram of B-module chain complexes

$$\begin{array}{ccc}B\otimes_A C & \xrightarrow{\ g\ } & D\\ {\scriptstyle 1\otimes h}\downarrow & & \downarrow{\scriptstyle k}\\ B\otimes_A E & \xrightarrow{\ j\ } & F\end{array}$$

Define also a commutative diagram of B-module chain complexes

$$\begin{array}{ccc}B\otimes_A C & \xrightarrow{\binom{g}{0}} & D\oplus F\\ {\scriptstyle 1\otimes h}\downarrow & \Gamma & \downarrow{\scriptstyle (k\ 1)}\\ B\otimes_A E & \xrightarrow{\ j\ } & F\end{array},$$

so that there is defined an (n+2)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ Poincaré triad over B

$$\begin{cases}(\Gamma,(0,\delta\varphi\oplus-\delta\nu,1\otimes\nu,\varphi)\in Q^{n+2}(\Gamma,\varepsilon))\\(\Gamma,(0,\delta\psi\oplus-\delta\chi,1\otimes\chi,\psi)\in Q_{n+2}(\Gamma,\varepsilon))\end{cases}$$ and so

$$\begin{cases}((C,\varphi),(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi))) = (0,(0:0\to F,(\delta\nu,0)))\in \mathrm{im}(L^{n+1}(B,\varepsilon)\longrightarrow L^{n+1}(f,\varepsilon))\\((C,\psi),(g:B\otimes_A C\longrightarrow D,(\delta\psi,1\otimes\psi))) = (0,(0:0\to F,(\delta\chi,0)))\in \mathrm{im}(L_{n+1}(B,\varepsilon)\longrightarrow L_{n+1}(f,\varepsilon))\end{cases},$$

whence exactness at $\begin{cases}L^{n+1}(f,\varepsilon)\\L_{n+1}(f,\varepsilon)\end{cases}$.

The periodicity in the absolute $\pm\varepsilon$-quadratic L-groups $L_n(A,\varepsilon)=L_{n+2}(A,-\varepsilon)$ $(n\in\mathbb{Z})$ can now be used to define

$$L_n(f,\varepsilon) = L_{n+2i}(f,(-)^i\varepsilon) \quad (n\leqslant 0, n+2i\geqslant 1)$$
$$L^n(f,\varepsilon) = L_n(f,\varepsilon) \quad (n\leqslant -3)\ .$$

Similarly, relative cobordism classes of pairs

(n-dimensional even ε-symmetric Poincaré complex over A (C,φ),
(n+1)-dimensional even ε-symmetric Poincaré pair over B $(g:B\otimes_A C\longrightarrow D,(\delta\varphi,1\otimes\varphi))$)

define relative even ε-symmetric L-groups $L\langle v_0\rangle^{n+1}(f,\varepsilon)$ $(n\geqslant 0)$ which together with the absolute even ε-symmetric L-groups of §9 fit into a long exact sequence

$$\ldots\longrightarrow L\langle v_0\rangle^{n+1}(B,\varepsilon)\longrightarrow L\langle v_0\rangle^{n+1}(f,\varepsilon)\longrightarrow L\langle v_0\rangle^{n}(A,\varepsilon)\xrightarrow{f} L\langle v_0\rangle^{n}(B,\varepsilon)\to\ldots\to L\langle v_0\rangle^{}(B,\varepsilon).$$

The isomorphism of Proposition 9.1 iii) extends to the relative case

$$L^{n+1}(f,\varepsilon) = L\langle v_0\rangle^{n+3}(f,-\varepsilon) \quad (n\geqslant 0)\ .$$

Define

$$L^{n+1}(f,\varepsilon) = L\langle v_0\rangle^{n+3}(f,-\varepsilon) \quad (n = -1,-2)\ ,$$

Define $L^{-2}(f,\varepsilon)$ to be the abelian group

$$L^{-2}(f,\varepsilon) = \mathrm{coker}(\ker(1+T_{-\varepsilon}:L_0(B,-\varepsilon)\longrightarrow L\langle v_0\rangle^0(B,-\varepsilon))\longrightarrow L_0(f,-\varepsilon))\ .$$

Then the exact sequence above terminating at $L^0(B,\varepsilon)$ can be extended to the right by an exact sequence

$$L^0(A,\varepsilon)\xrightarrow{f} L^0(B,\varepsilon)\longrightarrow L^0(f,\varepsilon)\longrightarrow L^{-1}(A,\varepsilon)\xrightarrow{f} L^{-1}(B,\varepsilon)\longrightarrow L^{-1}(f,\varepsilon)\longrightarrow\ldots\ .$$

[]

Ad hoc definitions of the relative quadratic L-groups $L_n(f) = L_n(f,1)$ have been given by Wall [5] (n odd) and Sharpe [2] (n even). In Proposition 10.2 below we shall express these groups in terms of ±quadratic forms and formations.

In §13 below we shall express the relative L-groups of a localization map $A\longrightarrow S^{-1}A$ as the cobordism groups of algebraic Poincaré complexes over A which become contractible over $S^{-1}A$, identifying the low-dimensional cases with Witt groups of linking forms and formations.

The full force of the equivalence relation of Proposition 5.4 i)

(between the homotopy equivalence classes of $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré pairs and

the homotopy equivalence classes of connected $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complexes over A)

allows us to express the relative $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups $\left\{\begin{array}{l}L^{n+1}(f:A\longrightarrow B,\varepsilon)\\ L_{n+1}(f:A\longrightarrow B,\varepsilon)\end{array}\right.$ $(n\geqslant 0)$

as the group of equivalence classes of triples

(n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complex over A $\left\{\begin{array}{l}(C,\varphi)\\ (C,\psi)\end{array}\right.$,

connected (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complex over B $\left\{\begin{array}{l}(D,\nu)\\ (D,\chi)\end{array}\right.$,

homotopy equivalence $\left\{\begin{array}{l}g:B\otimes_A(C,\varphi)\longrightarrow\partial(D,\nu)\\ g:B\otimes_A(C,\psi)\longrightarrow\partial(D,\chi)\end{array}\right.$).

The equivalence relation is defined by

$$\left\{\begin{array}{l}((C,\varphi),(D,\nu),g:B\otimes_A(C,\varphi)\longrightarrow\partial(D,\nu))\sim((C',\varphi'),(D',\nu'),g':B\otimes_A(C',\varphi')\longrightarrow\partial(D',\nu'))\\ ((C,\psi),(D,\chi),g:B\otimes_A(C,\psi)\longrightarrow\partial(D,\chi))\sim((C',\varphi'),(D',\chi'),g':B\otimes_A(C',\psi')\longrightarrow\partial(D',\chi'))\end{array}\right.$$

if there exists a quadruple

(a connected (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complex over A $\left\{\begin{array}{l}(E,\delta\varphi)\\ (E,\delta\psi)\end{array}\right.$,

a homotopy equivalence $\left\{\begin{array}{l}h:\partial(E,\delta\varphi)\longrightarrow(C,-\varphi)\oplus(C',\varphi')\\ h:\partial(E,\delta\psi)\longrightarrow(C,-\psi)\oplus(C',\psi')\end{array}\right.$

a connected (n+2)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complex over B $\left\{\begin{array}{l}(F,\delta\nu)\\ (F,\delta\chi)\end{array}\right.$,

a homotopy equivalence $\left\{\begin{array}{l}B\otimes_A(E,\delta\varphi)\cup_{(g\oplus g')(1\otimes h)}((D,\nu)\oplus(D',-\nu'))\longrightarrow\partial(F,\delta\nu)\\ B\otimes_A(E,\delta\psi)\cup_{(g\oplus g')(1\otimes h)}((D,\chi)\oplus(D',-\chi'))\longrightarrow\partial(F,\delta\chi)\end{array}\right.$).

In the low-dimensional cases this formulation translates directly into the language of forms and formations, and as such applies to the ε-quadratic L-groups $L_{n+1}(f,\varepsilon)$ $(n\in\mathbb{Z})$ and also $L^{n+1}(f,\varepsilon)$ $(n\leqslant 0)$.

<u>Proposition 10.2</u> The relative ε-quadratic group $\left\{\begin{array}{l}L_{2i}(f,\varepsilon)\\ L_{2i+1}(f,\varepsilon)\end{array}\right.$ of a morphism $f:A\longrightarrow B$ is the abelian group of equivalence classes of triples

(non-singular $\left\{\begin{array}{l}\text{split }(-)^{i-1}\varepsilon\text{-quadratic formation}\\ (-)^{i}\varepsilon\text{-quadratic form}\end{array}\right.$ over A $\left\{\begin{array}{l}(F,G)\\ (M,\psi)\end{array}\right.$,

$(-)^i\varepsilon$-quadratic $\left\{\begin{array}{l}\text{form}\\ \text{formation}\end{array}\right.$ over B $\left\{\begin{array}{l}(N,\chi)\\ (N,\chi;H,K)\end{array}\right.$, $\left\{\begin{array}{l}\text{a stable isomorphism}\\ \text{an isomorphism}\end{array}\right.$ of

non-singular $\left\{\begin{array}{l}\text{split }(-)^{i-1}\varepsilon\text{-quadratic formations}\\ (-)^{i}\varepsilon\text{-quadratic forms}\end{array}\right.$ over B

$\left\{\begin{array}{l}[\alpha,\beta,\sigma]\ :\ B\otimes_A(F,G)\longrightarrow\partial(N,\chi)\\ g\ :\ B\otimes_A(M,\psi)\longrightarrow\partial(N,\chi;H,K)\end{array}\right.$)

under the equivalence relation

$$\left\{\begin{array}{l}((F,G),(N,\chi),[\alpha,\beta,\sigma])\sim((F',G'),(N',\chi'),[\alpha',\beta',\sigma'])\\ ((M,\psi),(N,\chi;H,K),g)\sim((M',\psi'),(N',\chi';H',K'),g')\end{array}\right.$$

if there exists a quadruple

($\left\{\begin{array}{l}(-)^{i}\varepsilon\text{-quadratic form}\\ \text{split }(-)^{i}\varepsilon\text{-quadratic formation}\end{array}\right.$ over A $\left\{\begin{array}{l}(M,\psi)\\ (F,G)\end{array}\right.$,

$\left\{\begin{array}{l}\text{stable isomorphism }[\lambda,\mu,\nu]:\partial(M,\psi)\longrightarrow(F,G)\oplus-(F',G')\\ \text{isomorphism }h:\partial(F,G)\longrightarrow(M,\psi)\oplus(M',-\psi')\end{array}\right.$,

$\left\{\begin{array}{l}\text{split }(-)^{i}\varepsilon\text{-quadratic formation}\\ (-)^{i+1}\varepsilon\text{-quadratic form}\end{array}\right.$ over B $\left\{\begin{array}{l}(H,K)\\ (P,\theta)\end{array}\right.$,

$\left\{\begin{array}{l}\text{isomorphism }h:\partial(H,K)\longrightarrow(B\otimes_A(M,\psi))\cup_{([\alpha,\beta,\sigma]\oplus[\alpha',\beta',\sigma'])(1\otimes[\lambda,\mu,\nu])}((N,\chi)\oplus(N',-\chi'))\\ \text{stable isomorphism}\\ \qquad[\lambda,\mu,\nu]:\partial(P,\theta)\longrightarrow(B\otimes_A(F,G))\cup_{(g\oplus g')(1\otimes h)}((N,\chi;H,K)\oplus(N',-\chi';H',K'))\end{array}\right.$).

[]

(We are here using the union operation for forms and formations defined at the end of §5).

As in §7 of Wall [5] there is in fact a yet simpler expression for $L_{2i+1}(f,\varepsilon)$, as the group of equivalence classes of pairs

(non-singular $(-)^i\varepsilon$-quadratic form over A (M,ψ), lagrangian H of $B\otimes_A(M,\psi)$) ,

that is we can take K = 0 in the corresponding part of Proposition 10.2.

(More generally, every element $\begin{cases} ((C,\varphi),(g:B\otimes_A C \longrightarrow D,(\delta\varphi,1\otimes\varphi)))\in L^{n+1}(f:A\to B,\varepsilon) \\ ((C,\psi),(g:B\otimes_A C \longrightarrow D,(\delta\psi,1\otimes\psi)))\in L_{n+1}(f:A\to B,\varepsilon) \end{cases}$ $(n\geqslant 0)$

has a representative such that

$$H_{n+1}(D) = 0 \quad ,$$

which can also be regarded as a triple

(n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over A $\begin{cases}(C,\varphi) \\ (C,\psi)\end{cases}$,

well-connected (n+1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ complex over B $\begin{cases}(D,\nu) \\ (D,\chi)\end{cases}$,

homotopy equivalence $\begin{cases} g:B\otimes_A(C,\varphi) \longrightarrow \partial(D,\nu) \\ g:B\otimes_A(C,\psi)\longrightarrow \partial(D,\chi)\end{cases}$) .

Similarly for $n\leqslant -1$).

The relative ε-symmetric L-groups $L^n(f,\varepsilon)$ $(n\leqslant 1)$ can be expressed in terms of forms and formations, as in the ε-quadratic case (Proposition 10.2). In particular, $L^{-2}(f,\varepsilon)$ is the abelian group of equivalence classes of triples

(non-singular ε-quadratic formation over A $(M,\psi;F,G)$,

even $(-\varepsilon)$-symmetric form over B (N,χ),

stable isomorphism of non-singular ε-quadratic formations over B

$$[h] : B\otimes_A(M,\psi;F,G) \longrightarrow \partial(N,\chi) \;) .$$

In terms of algebraic Poincaré complexes $L^{-2}(f,\varepsilon)$ is the cobordism group of pairs

(1-dimensional ε-quadratic Poincaré complex over A $(C,\psi\in Q_1(C,\varepsilon))$,

2-dimensional ε-symmetric Poincaré pair over B $(g:B\otimes_A C\longrightarrow D,(\delta\varphi,\varphi)\in Q^2(g,\varepsilon))$)

with

$$(\delta\varphi,\varphi) = (1+T_\varepsilon)(\delta\psi',\psi') \in Q^2(g,\varepsilon)$$

for some $(\delta\psi',\psi')\in Q_2(g,\varepsilon)$ such that

$$\psi' - 1\otimes_A\psi = H(\theta)\in Q_1(B\otimes_A C,\varepsilon)$$

for some $\theta\in \hat{Q}^2(B\otimes_A C,\varepsilon)$ with

$$\hat{v}_1(\theta) = 0 : H^1(B\otimes_A C) \longrightarrow \hat{H}^1(\mathbb{Z}_2;B,\varepsilon)$$

(cf. Proposition 1.8 iii)).

Define as follows the <u>n-dimensional ε-hyperquadratic L-group of A</u> $\hat{L}^n(A,\varepsilon)$ $(n\in\mathbb{Z})$. For $n\geqslant 0$ let $\hat{L}^{n+1}(A,\varepsilon)$ be the group of relative cobordism classes of pairs

(n-dimensional ε-quadratic Poincaré complex over A (C,ψ),

(n+1)-dimensional ε-symmetric Poincaré pair over A $(g:C\longrightarrow D,(\nu,(1+T_\varepsilon)\psi))$) .

For $n = -1,-2$ let $\hat{L}^{n+1}(A,\varepsilon)$ be the group of relative cobordism classes of pairs

((n+2)-dimensional $(-\varepsilon)$-quadratic Poincaré complex over A (C,ψ),

(n+3)-dimensional even $(-\varepsilon)$-symmetric Poincaré pair over A $(g:C\to D,(\nu,(1+T_\varepsilon)\psi))$) .

For $n\leqslant -3$ let $\hat{L}^{n+1}(A,\varepsilon) = 0$.

<u>Proposition 10.3</u> i) The ε-hyperquadratic L-groups of A fit into a long exact sequence

$$\ldots \longrightarrow L^{n+1}(A,\varepsilon)\xrightarrow{J}\hat{L}^{n+1}(A,\varepsilon)\xrightarrow{H} L_n(A,\varepsilon)\xrightarrow{1+T_\varepsilon} L^n(A,\varepsilon)\longrightarrow\ldots$$

with

$$J : L^{n+1}(A,\varepsilon)\longrightarrow \hat{L}^{n+1}(A,\varepsilon) ;\ (D,\nu)\longmapsto (0,(0:0\longrightarrow D,(\nu,0)))$$

$$H : \hat{L}^{n+1}(A,\varepsilon)\longrightarrow L_n(A,\varepsilon);((C,\psi),(g:C\longrightarrow D,(\nu,(1+T_\varepsilon)\psi)))\longmapsto (C,\psi) .$$

ii) If there exists $a\in A$ such that $a+\bar{a} = 1\in A$ (e.g. $a = 1/2$) then $\hat{L}^*(A,\varepsilon) = 0$.

iii) Given a morphism of rings with involution $f:A\longrightarrow B$ there are defined relative ε-hyperquadratic L-groups $\hat{L}^n(f,\varepsilon)$ $(n\in\mathbb{Z})$ which fit into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots\longrightarrow & & L_n(A,\varepsilon) & \xrightarrow{f} & L_n(B,\eta) & \longrightarrow & L_n(f,\varepsilon) & \longrightarrow & L_{n-1}(A,\varepsilon) & \longrightarrow\ \ldots \\
 & & \downarrow 1+T_\varepsilon & & \downarrow 1+T_\eta & & \downarrow 1+T_\varepsilon & & \downarrow 1+T_\varepsilon \\
\ldots\longrightarrow & & L^n(A,\varepsilon) & \xrightarrow{f} & L^n(B,\eta) & \longrightarrow & L^n(f,\varepsilon) & \longrightarrow & L^{n-1}(A,\varepsilon) & \longrightarrow\ \ldots \\
 & & \downarrow J & & \downarrow J & & \downarrow J & & \downarrow J \\
\ldots\longrightarrow & & \hat{L}^n(A,\varepsilon) & \xrightarrow{f} & \hat{L}^n(B,\eta) & \longrightarrow & \hat{L}^n(f,\varepsilon) & \longrightarrow & \hat{L}^{n-1}(A,\varepsilon) & \longrightarrow\ \ldots \\
 & & \downarrow H & & \downarrow H & & \downarrow H & & \downarrow H \\
\ldots\longrightarrow & & L_{n-1}(A,\varepsilon) & \xrightarrow{f} & L_{n-1}(B,\eta) & \longrightarrow & L_{n-1}(f,\varepsilon) & \longrightarrow & L_{n-2}(A,\varepsilon) & \longrightarrow\ \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array} \qquad (n\in\mathbb{Z})$$

[]

For $\varepsilon = 1\in A$ we shall write $\hat{L}^n(A,1) = \hat{L}^n(A)$ $(n\in\mathbb{Z})$.

Given a connected space X and a group morphism $w:\pi_1(X)\longrightarrow\mathbb{Z}_2$ let

$$\left\{\begin{array}{l}\Omega^{N,P}_{n+1}(X)\\ \Omega^{P}_{n}(X)\\ \Omega^{N}_{n}(X)\end{array}\right. \text{ be the bordism group of maps } \left\{\begin{array}{l}f:(L,M)\longrightarrow X\\ f:M\longrightarrow X\\ f:N\longrightarrow X\end{array}\right. \text{ from } \left\{\begin{array}{l}(n+1)\text{-dimensional}\\ n\text{-dimensional}\\ n\text{-dimensional}\end{array}\right.$$

$$\left\{\begin{array}{l}\text{normal pairs } (L,M) \text{ with geometric Poincaré boundary } M\\ \text{geometric Poincaré complexes } M\\ \text{normal spaces } N\end{array}\right. \text{ such that the orientation}$$

maps factor as
$$\left\{\begin{array}{l}w(L):\pi_1(L)\xrightarrow{f}\pi_1(X)\xrightarrow{w}\mathbb{Z}_2\\ w(M):\pi_1(M)\xrightarrow{f}\pi_1(X)\xrightarrow{w}\mathbb{Z}_2\\ w(N):\pi_1(N)\xrightarrow{f}\pi_1(X)\xrightarrow{w}\mathbb{Z}_2\end{array}\right.$$
. The work of Levitt [2] Jones [1] and Quinn [3] (- see also §17C of Wall [5]) identifies -

$$\Omega^{N,P}_{n+1}(X) = L_n(\mathbb{Z}[\pi_1(X)]) \quad , \quad \Omega^{N}_{n}(X) = H_n(X;\underline{MSG}) \ (= \varinjlim_k \pi_{n+k}(X_+\wedge MSG(k)))$$

for $n\geqslant 5$, $w = 1$. Following a suggestion of Frank Quinn we shall now define the notions of higher signature appropriate to normal spaces and normal pairs with Poincaré boundary. The latter is particularly relevant to surgery obstruction theory because the mapping cylinder of a normal map of n-dimensional geometric Poincaré complexes $(f,b):M\longrightarrow X$ defines a bordism class $((M(f) = M\cup_{f\times 0}X\times I, M\cup X)\longrightarrow X)\in\Omega^{N,P}_{n+1}(X)$ (cf. Quinn [3]). We shall find this point of view useful in §15.

<u>Proposition 10.4</u> Given a connected space X and a group morphism $w:\pi_1(X)\longrightarrow\mathbb{Z}_2$ there is defined in a natural way a morphism of long exact sequences

$$\begin{array}{ccccccccc}\ldots\longrightarrow & \Omega^{N,P}_{n+1}(X) & \longrightarrow & \Omega^{P}_{n}(X) & \longrightarrow & \Omega^{N}_{n}(X) & \longrightarrow & \Omega^{N,P}_{n}(X) & \longrightarrow\ldots\\ & \downarrow\sigma_* & & \downarrow\sigma^* & & \downarrow\hat\sigma^* & & \downarrow\sigma_* & \\ \ldots\longrightarrow & L_n(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{1+T} & L^n(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{J} & \hat{L}^n(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{H} & L_{n-1}(\mathbb{Z}[\pi_1(X)]) & \longrightarrow\ldots\end{array}$$

such that $\sigma_*:\Omega^{N,P}_{n+1}(X)\longrightarrow L_n(\mathbb{Z}[\pi_1(X)])$ is an isomorphism for $n\geqslant 5$. If X is a geometric Poincaré complex and $(f,b):M\longrightarrow X$ is a normal map then

$$\sigma_*((M(f),M\cup X)\longrightarrow X) = \sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])\ .$$

<u>Proof</u>: We shall use algebraic surgery to define the <u>quadratic signature</u> of an (n+1)-dimensional normal pair

$$(f:M\hookrightarrow L,(\nu_L,\nu_M):(L,M)\longrightarrow BG(k),(\rho_L,\rho_M):(D^{n+k+1},S^{n+k})\longrightarrow(T(\nu_L),T(\nu_M)))$$

with an n-dimensional geometric Poincaré boundary (M,ν_M,ρ_M) to be the cobordism class in $L_n(\mathbb{Z}[\pi])$ $(\pi = \pi_1(L))$ of an n-dimensional quadratic Poincaré complex $\sigma_*(L,M)$ over $\mathbb{Z}[\pi]$ which comes equipped with a canonical cobordism from $(1+T)\sigma_*(L,M)$ to $\sigma^*(M)$. The <u>hyperquadratic signature</u> $\hat\sigma^*(L)\in\hat{L}^{n+1}(\mathbb{Z}[\pi_1(L)])$ of an (n+1)-dimensional normal space

$$(L,\nu_L:L\longrightarrow BG(k),\rho_L:S^{n+k+1}\longrightarrow T(\nu_L))$$

is the cobordism class in $\hat{L}^{n+1}(\mathbb{Z}[\pi])$ of the pair $\hat\sigma^*(L)$ defined by $\sigma_*(L,\emptyset)$ together with the canonical null-cobordism of $(1+T)\sigma_*(L,\emptyset)$, so that $H\hat\sigma^*(L) = \sigma_*(L,\emptyset)$.

Let then (L,M) be an (n+1)-dimensional normal pair with M Poincaré. The relative symmetric construction applied to the fundamental class $[L] = h(\rho_L,\rho_M)\cap U_{\nu_L}\in H^{\pi}_{n+1}(\tilde L,\tilde M;{}^w\mathbb{Z})$ $(w = w(L) = w_1(\nu_L))$ gives an (n+1)-dimensional symmetric pair over $\mathbb{Z}[\pi]$ $(f:C\longrightarrow D,(\theta,\varphi)\in Q^{n+1}(f))$ with Poincaré boundary $(C,\varphi) = (C(\tilde M),\varphi_{\tilde M}[M]) = \sigma^*(M)$, where $f: C = C(\tilde M)\longrightarrow D = C(\tilde L)$ is the inclusion. Let (C',φ') be the n-dimensional symmetric Poincaré complex obtained from (C,φ) by symmetric surgery on $(f:C\longrightarrow D,(\theta,\varphi))$. Proposition 7.1 provides a canonical cobordism between (C,φ) and (C',φ'). We shall now specify a $\mathbb{Z}_2$-hyperhomology class $\psi'\in Q_n(C')$ such that $(1+T)\psi' = \varphi'\in Q^n(C')$, thus defining $\sigma_*(L,M) = (C',\psi')\in L_n(\mathbb{Z}[\pi])$ such that $(1+T)\sigma_*(L,M) = \sigma^*(M)\in L^n(\mathbb{Z}[\pi])$. Let $(D',\theta'\in Q^{n+1}(D'))$ be the (n+1)-dimensional symmetric complex defined by

$$d_{D'} = \begin{pmatrix} d_C & 0\\ (-)^{r-1}f & d_D\end{pmatrix} : D'_r = C_{r-1}\oplus D_r\longrightarrow D'_{r-1} = C_{r-2}\oplus D_{r-1}$$

$$\theta'_s = \begin{pmatrix} \varphi_{s-1} & 0\\ (-)^{n-r}fT\varphi_s & (-)^{n-r+s-1}T\theta_s\end{pmatrix}$$

$$: D'^{n-r+s+1} = C^{n-r+s}\oplus D^{n-r+s+1}\longrightarrow D'_r = C_{r-1}\oplus D_r \quad (s\geqslant 0,\ \varphi_{-1} = 0)$$

which is the complex obtained by applying the absolute symmetric construction to the image of [L] under $H^{\pi}_{n+1}(\tilde L,\tilde M;{}^w\mathbb{Z})\rightarrow\dot H^{\pi}_{n+1}(\tilde L/\tilde M;{}^w\mathbb{Z})$, with $D' = C(f) = \dot C(\tilde L/\tilde M)$. The chain map $g:D^{n+1-*}\longrightarrow D'$ defined by

$$\varepsilon = \begin{pmatrix} (-)^{n+1}\varphi_0 f^* \\ (-)^r\theta_0 \end{pmatrix} : D^{n-r+1} \longrightarrow D'_r = C_{r-1}\oplus D_r$$

is induced by an $S\pi$-dual $G: T\pi(\nu_L)^* \longrightarrow \Sigma^p(\widetilde{L}/\widetilde{M})$ to the fundamental map of (L,M)

$$\alpha_{L,M} : S^{n+k+1} = D^{n+k+1}/S^{n+k} \xrightarrow{\rho_L/\rho_M} T(\nu_L)/T(\nu_M) \xrightarrow{\Delta} \widetilde{L}/\widetilde{M}\wedge_\pi T\pi(\nu_L) \ .$$

Applying the quadratic construction to the homology class $U^*_{\nu_L} \in \dot{H}^\pi_*(T\pi(\nu_L)^*;{}^w\mathbb{Z})$ dual to the Thom class $U_{\nu_L} \in \dot{H}^k_\pi(T\pi(\nu_L);{}^w\mathbb{Z})$ we have a $\mathbb{Z}_2$-hyperhomology class $\psi = \dot{\psi}_G(U^*_{\nu_L}) \in Q_{n+1}(D')$ such that

$$\theta' - g^\%(\dot{\varphi}_{T\pi(\nu_L)^*}(U^*_{\nu_L})) = (1+T)\psi \in Q^{n+1}(D') \ ,$$

and so

$$e^\%(\theta') = (1+T)e_\%(\psi) \in Q^{n+1}(C(g))$$

with $e: D' \longrightarrow C(g)$ the inclusion. Identifying

$$C' = \Omega C(g) \ , \ S\varphi' = e^\%(\theta') \in Q^n(SC')$$

consider the exact sequence of Q-groups

$$Q_n(C') \xrightarrow{\begin{pmatrix} 1+T \\ S \end{pmatrix}} Q^n(C')\oplus Q_{n+1}(SC') \xrightarrow{(S \quad -(1+T))} Q^{n+1}(SC')$$

(or rather the underlying exact sequence of chain complexes) provided by the commutative exact braid of Proposition 1.3 for p = 1 to obtain a $\mathbb{Z}_2$-hyperhomology class $\psi' \in Q_n(C')$ such that

$$(1+T)\psi' = \varphi' \in Q^n(C') \ , \ S\psi' = e_\%\psi \in Q_{n+1}(SC') \ .$$

Finally, set

$$\sigma_*(L,M) = (C', \psi' \in Q_n(C')) \ .$$

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<u>Proposition 10.5</u> The cokernel of the hyperquadratic signature map

$$\hat{\sigma}^* : \Omega^N_{4k+3}(K(\mathbb{Z}_2 * \mathbb{Z},1)) \longrightarrow \hat{L}^{4k+3}(\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}])$$

is infinitely generated, for each $k \geqslant 1$.

<u>Proof</u>: Cappell [2] has shown that $P/P\cap Q$ is infinitely generated, where

$$P = \ker((1+T): L_{4k+2}(\mathbb{Z}[\pi]) \longrightarrow L^{4k+2}(\mathbb{Z}[\pi])) = \operatorname{im}(H: \hat{L}^{4k+3}(\mathbb{Z}[\pi]) \to L_{4k+2}(\mathbb{Z}[\pi]))$$

$$Q = \operatorname{im}(L_{4k+2}(\mathbb{Z}[\mathbb{Z}_2])\oplus L_{4k+2}(\mathbb{Z}[\mathbb{Z}]) \longrightarrow L_{4k+2}(\mathbb{Z}[\pi])) \quad (\pi = \mathbb{Z}_2 * \mathbb{Z}) \ .$$

It follows that

$$\operatorname{coker}(\hat{\sigma}^* : \Omega^N_{4k+3}(K(\mathbb{Z}_2 * \mathbb{Z},1)) \longrightarrow \hat{L}^{4k+3}(\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}]))$$

is infinitely generated.

[]

Our algebraic methods serve also to define the $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ L-groups of A with coefficients in a f.g. abelian group G $\begin{cases} L^n(A,\varepsilon;G) \\ L_n(A,\varepsilon;G)\end{cases}$ $(n\in\mathbb{Z})$, which are characterized by a long exact sequence

$$\begin{cases} \ldots \longrightarrow L^n(A,\varepsilon;G) \longrightarrow L^n(A,\varepsilon;H) \longrightarrow L^n(A,\varepsilon;K) \longrightarrow L^{n-1}(A,\varepsilon;G) \longrightarrow \ldots \\ \ldots \longrightarrow L_n(A,\varepsilon;G) \longrightarrow L_n(A,\varepsilon;H) \longrightarrow L_n(A,\varepsilon;K) \longrightarrow L_{n-1}(A,\varepsilon;G) \longrightarrow \ldots \end{cases} (n\in\mathbb{Z})$$

associated to any short exact sequence of f.g. abelian groups

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0 \ ,$$

and also

$$\begin{cases} L^n(A,\varepsilon;\mathbb{Z}) = L^n(A,\varepsilon) \\ L_n(A,\varepsilon;\mathbb{Z}) = L_n(A,\varepsilon) \end{cases} (n\in\mathbb{Z}) \ .$$

The quadratic L-groups are 4-periodic

$$L_n(A,\varepsilon;G) = L_{n+2}(A,-\varepsilon;G) = L_{n+4}(A,\varepsilon;G) \quad (n\in\mathbb{Z})$$

and

$$L^n(A,\varepsilon;G) = L_n(A,\varepsilon;G) \quad (n \leqslant -3) \ .$$

By the structure theorem for f.g. abelian groups it is sufficient to consider the case of finite cyclic groups G.

Define the <u>n-dimensional</u> $\begin{cases}\underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}}\end{cases}$ <u>mod m L-group of A</u> $\begin{cases} L^n(A,\varepsilon;\mathbb{Z}_m) \\ L_n(A,\varepsilon;\mathbb{Z}_m)\end{cases}$ $(n\in\mathbb{Z})$ as follows. For $n \geqslant 0$ let $\begin{cases} L^{n+1}(A,\varepsilon;\mathbb{Z}_m) \\ L_{n+1}(A,\varepsilon;\mathbb{Z}_m)\end{cases}$ be the group of relative cobordism classes of pairs

(n-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over A $\begin{cases}(C,\varphi) \\ (C,\psi)\end{cases}$,

(n+1)-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré pair over A $\begin{cases}(f: mC \longrightarrow D,(\nu,m\varphi)) \\ (f: mC \longrightarrow D,(\chi,m\psi))\end{cases}$)

with $\begin{cases}(mC,m\varphi) = (C,\varphi)\oplus(C,\varphi)\oplus\ldots\oplus(C,\varphi) \\ (mC,m\psi) = (C,\psi)\oplus(C,\psi)\oplus\ldots\oplus(C,\psi)\end{cases}$ (m terms). The lower mod m L-groups are defined similarly, working as in Proposition 10.1.

Proposition 10.6 The mod m $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups $\left\{\begin{array}{l}L^n(A,\varepsilon;\mathbb{Z}_m)\\ L_n(A,\varepsilon;\mathbb{Z}_m)\end{array}\right.$ $(n\in\mathbb{Z})$ fit into a long exact sequence

$$\left\{\begin{array}{l}\ldots\longrightarrow L^{n+1}(A,\varepsilon;\mathbb{Z}_m)\longrightarrow L^n(A,\varepsilon)\xrightarrow{m} L^n(A,\varepsilon)\longrightarrow L^n(A,\varepsilon;\mathbb{Z}_m)\longrightarrow\ldots\\ \ldots\longrightarrow L_{n+1}(A,\varepsilon;\mathbb{Z}_m)\longrightarrow L_n(A,\varepsilon)\xrightarrow{m} L_n(A,\varepsilon)\longrightarrow L_n(A,\varepsilon;\mathbb{Z}_m)\longrightarrow\ldots\end{array}\right. .$$

[]

For $\varepsilon = 1\in A$ we shall write $\left\{\begin{array}{l}L^n(A,1;\mathbb{Z}_m) = L^n(A;\mathbb{Z}_m)\\ L_n(A,1;\mathbb{Z}_m) = L_n(A;\mathbb{Z}_m)\end{array}\right.$.

An n-dimensional geometric $\mathbb{Z}_m$-Poincaré complex $(X,\partial X = \underset{m}{\sqcup}\delta X)$ has a <u>symmetric mod m signature</u> invariant

$$\sigma^*(X)\in L^n(\mathbb{Z}[\pi_1(X)];\mathbb{Z}_m)$$

constructed by analogy with the absolute case m = 0 (Proposition 2.7). It will follow from the computation of $L^*(\mathbb{Z})$ in §13 that the simply-connected mod m signature map

$$\sigma^* : \Omega^P_n(pt.;\mathbb{Z}_m)\longrightarrow L^n(\mathbb{Z};\mathbb{Z}_m) = \left\{\begin{array}{l}\mathbb{Z}_m\\ \mathbb{Z}_2\otimes_{\mathbb{Z}}\mathbb{Z}_m\\ \mathbb{Z}_2\otimes_{\mathbb{Z}}\mathbb{Z}_m\\ 0\end{array}\right.\quad n\equiv\left\{\begin{array}{l}0\\1\\2\\3\end{array}\right.\pmod 4$$

sends an oriented $\left\{\begin{array}{l}(4k)\text{-}\\ (4k+1)\text{-dimensional}\\ (4k+2)\text{-}\end{array}\right.$ geometric $\mathbb{Z}_m$-Poincaré complex X to the

$$\left\{\begin{array}{l}\text{(signature of X)}\\ \text{(deRham invariant of X)}\ (m\equiv 0 \pmod 2)\\ \text{(deRham invariant of }\delta X)\ (m\equiv 0 \pmod 2)\end{array}\right. .$$

Given a normal map $(f,b):M\longrightarrow X$ of n-dimensional geometric $\mathbb{Z}_m$-Poincaré complexes there is defined a <u>quadratic mod m signature</u>

$$\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)];\mathbb{Z}_m)$$

by analogy with the absolute case m = 0 (Proposition 2.9). For a normal bundle map from a $\mathbb{Z}_m$-manifold M we can identify the quadratic mod m signature with the obstruction to $\mathbb{Z}_m$-surgery, working as in §8. Surgery on $\mathbb{Z}_m$-manifolds is an important ingredient of the work of Sullivan [1], Morgan and Sullivan [1] and Wall [14] on characteristic classes for surgery obstructions.

§11. <u>Products</u>

The tensor product of rings with involution A,B is a ring with involution $A\otimes_{\mathbb{Z}}B$, where

$$\overline{(a\otimes b)} = \bar{a}\otimes\bar{b}\in A\otimes_{\mathbb{Z}}B\qquad (a\in A,\ b\in B) .$$

(For group rings $A = \mathbb{Z}[\pi]$, $B = \mathbb{Z}[\rho]$ we have $A\otimes_{\mathbb{Z}}B = \mathbb{Z}[\pi\times\rho]$). The tensor product of an A-module chain complex C and a B-module chain complex D is an $A\otimes_{\mathbb{Z}}B$-module chain complex $C\otimes_{\mathbb{Z}}D$, with $A\otimes_{\mathbb{Z}}B$ acting by

$$A\otimes_{\mathbb{Z}}B\times C\otimes_{\mathbb{Z}}D\longrightarrow C\otimes_{\mathbb{Z}}D\ ;\ (a\otimes b,x\otimes y)\longmapsto ax\otimes by .$$

If $\varepsilon\in A,\eta\in B$ are central units such that $\bar{\varepsilon} = \varepsilon^{-1}\in A$, $\bar{\eta} = \eta^{-1}\in B$ then $\varepsilon\otimes\eta\in A\otimes_{\mathbb{Z}}B$ is a central unit such that $\overline{(\varepsilon\otimes\eta)} = (\varepsilon\otimes\eta)^{-1}\in A\otimes_{\mathbb{Z}}B$, and there is a natural identification of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$(C^t\otimes_A C)\otimes_{\mathbb{Z}}(D^t\otimes_B D) = (C\otimes_{\mathbb{Z}}D)^t\otimes_{A\otimes_{\mathbb{Z}}B}(C\otimes_{\mathbb{Z}}D)$$

with $T\in\mathbb{Z}_2$ acting by $T_\varepsilon\otimes T_\eta$ on the left hand side and by $T_{\varepsilon\otimes\eta}$ on the right. As in §4 of Chapter XII of Cartan and Eilenberg [1] it is possible to construct a diagonal chain map

$$\Delta : \hat{W}\longrightarrow\hat{W}\otimes_{\mathbb{Z}}\hat{W}$$

for any complete resolution $\hat{W}$ for $\mathbb{Z}_2$ (allowing infinite chains in $\hat{W}\otimes_{\mathbb{Z}}\hat{W}$), and use the restrictions

$$\Delta : W\longrightarrow W\otimes_{\mathbb{Z}}W$$
$$\Delta : W^{-*}\longrightarrow W\otimes_{\mathbb{Z}}W^{-*}$$

to define chain maps

$$\otimes : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C^t\otimes_A C)\otimes_{\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,D^t\otimes_B D)$$
$$\longrightarrow\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,(C\otimes_{\mathbb{Z}}D)^t\otimes_{A\otimes_{\mathbb{Z}}B}(C\otimes_{\mathbb{Z}}D))\ ;\ \varphi\otimes\nu\longmapsto(\varphi\otimes\nu)\Delta$$

$$\otimes : \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C^t\otimes_A C)\otimes_{\mathbb{Z}}(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(D^t\otimes_B D))$$
$$\longrightarrow W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}((C\otimes_{\mathbb{Z}}D)^t\otimes_{A\otimes_{\mathbb{Z}}B}(C\otimes_{\mathbb{Z}}D))\ ;\ \varphi\otimes\psi\longmapsto(\varphi\otimes\psi)\Delta$$

(identifying $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W^{-*},-) = (W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}-)$) and so obtain products

$$\otimes : Q^m(C,\varepsilon)\otimes_{\mathbb{Z}}Q^n(D,\eta)\longrightarrow Q^{m+n}(C\otimes_{\mathbb{Z}}D,\varepsilon\otimes\eta)$$
$$\otimes : Q^m(C,\varepsilon)\otimes_{\mathbb{Z}}Q_n(D,\eta)\longrightarrow Q_{m+n}(C\otimes_{\mathbb{Z}}D,\varepsilon\otimes\eta) .$$

For the standard complete resolution $\hat{W}$ of $\mathbb{Z}_2$

$$\hat{W}_s = \mathbb{Z}[\mathbb{Z}_2] \quad , \quad d : \hat{W}_s \longrightarrow \hat{W}_{s-1} \; ; \; 1_s \longmapsto 1_{s-1} + (-)^s T_{s-1} \qquad (s \in \mathbb{Z})$$

we can take

$$\Delta : \hat{W}_s \longrightarrow (\hat{W} \otimes_{\mathbb{Z}} \hat{W})_s = \sum_{r=-\infty}^{\infty} \hat{W}_r \otimes_{\mathbb{Z}} \hat{W}_{s-r} \; ; \; 1_s \longmapsto \sum_{r=-\infty}^{\infty} 1_r \otimes T^r_{s-r} \qquad (s \in \mathbb{Z}) \; ,$$

giving explicit formula for the products (subject to sign conventions)

$$\begin{aligned}(\varphi \otimes \nu)_s &= \sum_{r=0}^{s} \varphi_r \otimes T^r_\eta \nu_{s-r} \in ((C \otimes_{\mathbb{Z}} D)^t \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D))_{m+n+s} \\ &= \sum_{r=-\infty}^{\infty} (C^t \otimes_A C)_{m+r} \otimes_{\mathbb{Z}} (D^t \otimes_B D)_{n+s-r} \quad (s \geq 0, \varphi \in Q^m(C,\varepsilon), \nu \in Q^n(D,\eta))\end{aligned}$$

$$\begin{aligned}(\varphi \otimes \psi)_s &= \sum_{r=0}^{\infty} \varphi_r \otimes T^r_\eta \psi_{s+r} \in ((C \otimes_{\mathbb{Z}} D)^t \otimes_{A \otimes_{\mathbb{Z}} B} (C \otimes_{\mathbb{Z}} D))_{m+n-s} \\ &= \sum_{r=-\infty}^{\infty} (C^t \otimes_A C)_{m+r} \otimes_{\mathbb{Z}} (D^t \otimes_B D)_{n-s-r} \quad (s \geq 0, \varphi \in Q^m(C,\varepsilon), \psi \in Q_n(D,\eta)).\end{aligned}$$

<u>Proposition11.1</u> There are defined natural pairings in the symmetric and quadratic L-groups

$$\begin{aligned}\otimes &: L^m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L^{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \\ \otimes &: L^m(A,\varepsilon) \otimes_{\mathbb{Z}} L_n(B,\eta) \longrightarrow L_{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \qquad (m,n \geq 0) \\ \otimes &: L_m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L_{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \\ \otimes &: L_m(A,\varepsilon) \otimes_{\mathbb{Z}} L_n(B,\eta) \longrightarrow L_{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta)\end{aligned}$$

such that there is a commutative diagram

$$\begin{array}{ccc} L_m(A,\varepsilon) \otimes_{\mathbb{Z}} L_n(B,\eta) & \xrightarrow{1 \otimes (1+T_\eta)} & L_m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \\ \downarrow{\scriptstyle (1+T_\varepsilon) \otimes 1} & \searrow{\scriptstyle \otimes} & \downarrow{\scriptstyle \otimes} \\ L^m(A,\varepsilon) \otimes_{\mathbb{Z}} L_n(B,\eta) & \xrightarrow{\otimes} & L_{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \\ \downarrow{\scriptstyle 1 \otimes (1+T_\eta)} & & \downarrow{\scriptstyle (1+T_{\varepsilon \otimes \eta})} \\ L^m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) & \xrightarrow{\otimes} & L^{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \end{array} \; .$$

There are defined compatible pairings in the lower L-groups

$$\otimes : L^m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L^{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta)$$

and also in the hyperquadratic L-groups

$$\otimes : \hat{L}^m(A,\varepsilon) \otimes_{\mathbb{Z}} \hat{L}^n(B,\eta) \longrightarrow \hat{L}^{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \qquad (m,n \in \mathbb{Z}) \; .$$

<u>Proof</u>: The product of an m-dimensional ε-symmetric Poincaré complex over A $(C, \varphi \in Q^m(C,\varepsilon))$ and an n-dimensional η-symmetric Poincaré complex over B $(D, \nu \in Q^n(D,\eta))$ is an (m+n)-dimensional $(\varepsilon \otimes \eta)$-symmetric Poincaré complex over $A \otimes_{\mathbb{Z}} B$

$$(C,\varphi) \otimes_{\mathbb{Z}} (D,\nu) = (C \otimes_{\mathbb{Z}} D, \varphi \otimes \nu \in Q^{m+n}(C \otimes_{\mathbb{Z}} D, \varepsilon \otimes \eta)) \; .$$

If $(g : D \longrightarrow E, (\delta\nu, \nu) \in Q^{n+1}(g,\varepsilon))$ is a null-cobordism of (D,ν) then the product

$$(C,\varphi) \otimes (g : D \longrightarrow E, (\delta\nu, \nu)) = (1 \otimes g : C \otimes_{\mathbb{Z}} D \longrightarrow C \otimes_{\mathbb{Z}} E, (\varphi \otimes \delta\nu, \varphi \otimes \nu) \in Q^{m+n+1}(1 \otimes g, \varepsilon \otimes \eta))$$

is a null-cobordism of $(C,\varphi) \otimes (D,\nu)$. Similarly for null-cobordisms of (C,φ), and also for products of other types of algebraic Poincaré complexes. Thus the Q-group products pass to the L-groups in the non-negative dimensions. The above diagram actually commutes on the Q-group level.

If (C,φ) is an even m-dimensional ε-symmetric Poincaré complex over A and (D,ν) is any n-dimensional η-symmetric Poincaré complex over B then $(C,\varphi) \otimes (D,\nu)$ is an even (m+n)-dimensional $(\varepsilon \otimes \eta)$-symmetric Poincaré complex over $A \otimes_{\mathbb{Z}} B$. This defines products

$$\otimes : L\langle v_0 \rangle^m(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L\langle v_0 \rangle^{m+n}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \qquad (m,n \geq 0) \; .$$

In particular, there are products

$$\begin{aligned}\otimes &: L^{-1}(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L^{n-1}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \qquad (n \geq 0) \\ \otimes &: L^{-2}(A,\varepsilon) \otimes_{\mathbb{Z}} L^n(B,\eta) \longrightarrow L^{n-2}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta)\end{aligned}$$

There are also defined products

$$\otimes : L^{-2}(A,\varepsilon) \otimes_{\mathbb{Z}} L^{-2}(B,\eta) \longrightarrow L^{-4}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) = L_0(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \; ;$$

$$(M, \psi - \varepsilon\psi^* \in Q^{-\varepsilon}(M)) \otimes (N, \chi - \eta\chi^* \in Q^{-\eta}(N))$$

$$\longmapsto (M \otimes_{\mathbb{Z}} N, \psi \otimes (\chi - \eta\chi^*) = (\psi - \varepsilon\psi^*) \otimes \chi \in Q_{\varepsilon \otimes \eta}(M \otimes_{\mathbb{Z}} N)) \quad (\psi \in Q_{-\varepsilon}(M), \chi \in Q_{-\eta}(N)) \; ,$$

$$\otimes : L^{-1}(A,\varepsilon) \otimes_{\mathbb{Z}} L^{-2}(B,\eta) \longrightarrow L^{-3}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) = L_1(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta) \; ;$$

$$(M, \psi - \varepsilon\psi^* ; F, G) \otimes (N, \chi - \eta\chi^*) \longmapsto (M \otimes_{\mathbb{Z}} N, \psi \otimes (\chi - \eta\chi^*) ; F \otimes_{\mathbb{Z}} N, G \otimes_{\mathbb{Z}} N) \; .$$

It now only remains to define the product

$$\otimes : L^{-1}(A,\varepsilon) \otimes_{\mathbb{Z}} L^{-1}(B,\eta) \longrightarrow L^{-2}(A \otimes_{\mathbb{Z}} B, \varepsilon \otimes \eta)$$

and the products in the hyperquadratic L-groups.

Given a non-singular even ε-symmetric formation over A $(H^{\varepsilon}(F);F,G)$ write the inclusion of the lagrangian G as a morphism of even ε-symmetric forms

$$\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : (G,0) \longrightarrow H^{\varepsilon}(F) = (F\oplus F^*, \begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix} \in Q^{\varepsilon}(F\oplus F^*)) \quad .$$

There is defined an even 1-dimensional ε-symmetric Poincaré complex over A $(C,\varphi\in Q^1(C,\varepsilon))$ with

$$d = \mu^* : C_1 = F \longrightarrow C_0 = G^* \ , \ \varphi_0 = \begin{cases}\gamma : C^0 = G \longrightarrow C_1 = F\\ \varepsilon\gamma^* : C^1 = F^* \longrightarrow C_0 = G^*\end{cases}$$

$$C_r = 0 \ (r \neq 0,1) \quad , \quad \varphi_s = 0 : C^r \longrightarrow C_{1-r+s} \ ((r,s) \neq (0,0),(1,0)).$$

Similarly, given a non-singular even η-symmetric formation over B $(H^{\eta}(H);H,K)$ write the inclusion of the lagrangian K as

$$\begin{pmatrix}\beta\\ \lambda\end{pmatrix} : (K,0) \longrightarrow H^{\eta}(H) \ ,$$

and let $(D,\nu\in Q^1(D,\eta))$ be the corresponding even 1-dimensional η-symmetric Poincaré complex over B. The product $(C,\varphi)\otimes(D,\nu)$ is an even 2-dimensional $(\varepsilon\otimes\eta)$-symmetric Poincaré complex over $A\otimes_{\mathbb{Z}}B$ which is cobordant to the skew-suspension of an even 0-dimensional $-(\varepsilon\otimes\eta)$-symmetric Poincaré complex over $A\otimes_{\mathbb{Z}}B$ (obtained by working exactly as in the proof of Proposition 9.1 iii)), corresponding to the non-singular even $-(\varepsilon\otimes\eta)$-symmetric form over $A\otimes_{\mathbb{Z}}B$

$$(H^{\varepsilon}(F);F,G)\otimes(H^{\eta}(H);H,K)$$

$$= (\mathrm{coker}(\begin{pmatrix}\mu\otimes 1\\ 1\otimes\lambda\\ \gamma\otimes\beta\end{pmatrix} : G\otimes K \longrightarrow F^*\otimes K\oplus G\otimes H^*\oplus F\otimes H), \begin{bmatrix}0 & \varepsilon\gamma\otimes\beta^* & \varepsilon\otimes\eta\lambda^*\\ -\gamma^*\otimes\eta\beta & 0 & -\varepsilon\mu^*\otimes\eta\\ -1\otimes\lambda & \mu\otimes 1 & 0\end{bmatrix}) \ .$$

This defines a product

$$\otimes \ : \ L^{-1}(A,-\varepsilon)\otimes_{\mathbb{Z}}L^{-1}(B,-\eta) \longrightarrow L^{-2}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta) \ ,$$

as required.

The product in the hyperquadratic L-groups is defined by

$$\otimes \ : \ \hat{L}^m(A,\varepsilon)\otimes_{\mathbb{Z}}\hat{L}^n(B,\eta) \longrightarrow \hat{L}^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta) \ ;$$

$$((C,\psi),(f:C\longrightarrow D,(\varphi,(1+T_{\varepsilon})\psi)))\otimes((E,\chi),(g:E\longrightarrow F,(\nu,(1+T_{\eta})\chi))) \longmapsto$$

$$((C\otimes F\cup_{C\otimes E}D\otimes E,\psi\otimes\nu\cup_{\psi\otimes\chi}\varphi\otimes\chi),(f\otimes 1\cup_{f\otimes g}1\otimes g : C\otimes F\cup_{C\otimes E}D\otimes E\to D\otimes F,(\varphi\otimes\nu,(1+T_{\varepsilon\otimes\eta})(\psi\otimes\nu\cup_{\psi\otimes\chi}\varphi\otimes\chi))),$$

making use of the union operation of §5. (This may be regarded as an algebraic analogue of the Hopf construction in topology).

[]

Proposition 11.2 i) The symmetric signature of the cartesian product $X\times Y$ of geometric Poincaré complexes is

$$\sigma^*(X\times Y) = \sigma^*(X)\otimes\sigma^*(Y) \in L^{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) \ ,$$

where $m = \dim X$, $n = \dim Y$.

ii) The $\begin{Bmatrix}\text{symmetric}\\ \text{quadratic}\end{Bmatrix}$ signature of the kernel of the cartesian product

$$\begin{Bmatrix}f\times g : M\times N \longrightarrow X\times Y\\ (f\times g,b\times c) : M\times N \longrightarrow X\times Y\end{Bmatrix} \text{ of } \begin{Bmatrix}\text{degree } 1\\ \text{normal}\end{Bmatrix} \text{ maps } \begin{Bmatrix}f:M\longrightarrow X, g:N\longrightarrow Y\\ (f,b):M\longrightarrow X,(g,c):N\longrightarrow Y\end{Bmatrix}$$

of geometric Poincaré complexes is

$$\begin{cases}\sigma^*(f\times g) = \sigma^*(f)\otimes\sigma^*(g) + \sigma^*(X)\otimes\sigma^*(g) + \sigma^*(f)\otimes\sigma^*(Y) \in L^{m+n}(\mathbb{Z}[\pi_1(X\times Y)])\\ \sigma_*(f\times g,b\times c) = \sigma_*(f,b)\otimes\sigma_*(g,c) + \sigma^*(X)\otimes\sigma_*(g,c) + \sigma_*(f,b)\otimes\sigma^*(Y) \in L_{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) \ .\end{cases}$$

Proof: i) Choose a functorial diagonal chain approximation Δ. The standard acyclic model proof of the Eilenberg-Zilber theorem gives a functorial chain equivalence on the category (topological spaces) $\times$ (topological spaces)

$$h_{X,Y} : C(X\times Y) \longrightarrow C(X)\otimes_{\mathbb{Z}}C(Y)$$

and the acyclic model argument underlying the Cartan product formula for the Steenrod squares gives a functorial chain homotopy $k_{X,Y} : \Delta^*(\Delta_X\otimes\Delta_Y)h_{X,Y} \simeq h^{\%}_{X,Y}\Delta_{X\times Y}$ in the diagram

$$\begin{array}{ccc}C(X\times Y) & \xrightarrow{\quad h_{X,Y}\quad} & C(X)\otimes_{\mathbb{Z}}C(Y)\\ \Big\downarrow{\scriptstyle \Delta_{X\times Y}} & \searrow{\scriptstyle k_{X,Y}} & \Big\downarrow{\scriptstyle \Delta_X\otimes\Delta_Y}\\ & & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C(X)\otimes_{\mathbb{Z}}C(X))\otimes_{\mathbb{Z}}\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C(Y)\otimes_{\mathbb{Z}}C(Y))\\ & & \Big\downarrow{\scriptstyle \Delta^*}\\ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C(X\times Y)\otimes_{\mathbb{Z}}C(X\times Y)) & \xrightarrow{\quad h^{\%}_{X,Y}\quad} & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,(C(X)\otimes_{\mathbb{Z}}C(Y))\otimes_{\mathbb{Z}}(C(X)\otimes_{\mathbb{Z}}C(Y)))\end{array}$$

with $\Delta : W \longrightarrow W\otimes_{\mathbb{Z}}W$ an algebraic diagonal approximation for $W = C(S^{\infty})$.

The product of an m-dimensional geometric Poincaré complex X and an n-dimensional geometric Poincaré complex Y is an $(m+n)$-dimensional geometric Poincaré complex $X\times Y$, with orientation character

$$w(X\times Y) = w(X)\times w(Y) : \pi_1(X\times Y) = \pi_1(X)\times\pi_1(Y) \longrightarrow \mathbb{Z}_2$$

and fundamental class

$$[X\times Y] = [X]\otimes[Y] \in H^{\pi_1(X\times Y)}_{m+n}(\widetilde{X}\times\widetilde{Y};{}^{w(X\times Y)}\mathbb{Z}) = H^{\pi_1(X)}_m(\widetilde{X};{}^{w(X)}\mathbb{Z})\otimes_{\mathbb{Z}}H^{\pi_1(Y)}_n(\widetilde{Y};{}^{w(Y)}\mathbb{Z}) \ ,$$

where $\widetilde{X},\widetilde{Y}$ are the universal covers of X,Y. It now follows from the chain homotopy invariance of the Q-groups that there is defined a homotopy equivalence of $(m+n)$-dimensional symmetric Poincaré complexes over $\mathbb{Z}[\pi_1(X\times Y)]$

$$\begin{aligned} h_{\widetilde{X},\widetilde{Y}} : \sigma^*(X\times Y) &= (C(\widetilde{X}\times\widetilde{Y}),\varphi_{\widetilde{X}\times\widetilde{Y}}[X\times Y]\in Q^{m+n}(C(\widetilde{X}\times\widetilde{Y}))) \\ &\longrightarrow \sigma^*(X)\otimes\sigma^*(Y) = (C(\widetilde{X})\otimes_{\mathbb{Z}}C(\widetilde{Y}),\varphi_{\widetilde{X}}[X]\otimes\varphi_{\widetilde{Y}}[Y]\in Q^{m+n}(C(\widetilde{X})\otimes_{\mathbb{Z}}C(\widetilde{Y}))) \quad , \end{aligned}$$

and the homotopy invariance of symmetric Poincaré cobordism gives

$$\sigma^*(X\times Y) = \sigma^*(X)\otimes\sigma^*(Y)\in L^{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) \ .$$

ii) Consider first the special case of the product $\begin{cases} f\times 1:M\times N \longrightarrow X\times N \\ (f\times 1,b\times 1):M\times N \longrightarrow X\times N \end{cases}$ of an m-dimensional $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ map $\begin{cases} f:M\longrightarrow X \\ (f,b):M\longrightarrow X\end{cases}$ with an n-dimensional geometric Poincaré complex N. Given an Umkehr chain map for $f:M\longrightarrow X$

$$f^! : C(\widetilde{X}) \xrightarrow{([X]\cap -)^{-1}} {}^{w(X)}C(\widetilde{X})^{m-*} \xrightarrow{\widetilde{f}^*} {}^{w(X)}C(\widetilde{M})^{m-*} \xrightarrow{[M]\cap -} C(\widetilde{M})$$

there is defined an Umkehr chain map for $f\times 1:M\times N\longrightarrow X\times N$

$$(f\times 1)^! : C(\widetilde{X}\times\widetilde{N}) \xrightarrow{h_{\widetilde{X},\widetilde{N}}} C(\widetilde{X})\otimes_{\mathbb{Z}}C(\widetilde{N}) \xrightarrow{f^!\otimes 1} C(\widetilde{M})\otimes_{\mathbb{Z}}C(\widetilde{N}) \xrightarrow{h^{-1}_{\widetilde{M},\widetilde{N}}} C(\widetilde{M}\times\widetilde{N}) \ .$$

There are also defined a chain equivalence

$$h^!_{\widetilde{M},\widetilde{N}} : C((f\times 1)^!) \longrightarrow C(f^!\otimes 1) = C(f^!)\otimes_{\mathbb{Z}}C(\widetilde{N})$$

and a chain homotopy commutative diagram

$$\begin{array}{ccc} C(\widetilde{M}\times\widetilde{N}) & \xrightarrow{e_{\widetilde{f}\times 1}} & C((f\times 1)^!) \\ \downarrow{\scriptstyle h_{\widetilde{M},\widetilde{N}}} & & \downarrow{\scriptstyle h^!_{\widetilde{M},\widetilde{N}}} \\ C(\widetilde{M})\otimes_{\mathbb{Z}}C(\widetilde{N}) & \xrightarrow{e_{\widetilde{f}}\otimes 1} & C(f^!)\otimes_{\mathbb{Z}}C(\widetilde{N}) \end{array}$$

with $e_{\widetilde{f}}$, $e_{\widetilde{f}\times 1}$ the inclusions. The homotopy equivalence of $(m+n)$-dimensional symmetric Poincaré complexes over $\mathbb{Z}[\pi_1(X\times N)]$

$$\begin{aligned} h^!_{\widetilde{M},\widetilde{N}} : \sigma^*(f\times 1) &= (C((f\times 1)^!),e^{\%}_{\widetilde{f}\times 1}\varphi_{\widetilde{M}\times\widetilde{N}}[M\times N]) \\ &\longrightarrow \sigma^*(f)\otimes\sigma^*(N) = (C(f^!)\otimes_{\mathbb{Z}}C(\widetilde{N}),e^{\%}_{\widetilde{f}}\varphi_{\widetilde{M}}[M]\otimes\varphi_{\widetilde{N}}[N]) \end{aligned}$$

implies that

$$\sigma^*(f\times 1) = \sigma^*(f)\otimes\sigma^*(N)\in L^{m+n}(\mathbb{Z}[\pi_1(X\times N)]) \ .$$

Furthermore, if $F:\Sigma^pX_+\longrightarrow\Sigma^pM_+$ is a geometric Umkehr map for $(f,b):M\longrightarrow X$ then

$$F\wedge 1 : \Sigma^p(\widetilde{X}\times\widetilde{N})_+ = \Sigma^p\widetilde{X}_+\wedge\widetilde{N}_+ \longrightarrow \Sigma^p(\widetilde{M}\times\widetilde{N})_+ = \Sigma^p\widetilde{M}_+\wedge\widetilde{N}_+$$

is a geometric Umkehr map for $(f\times 1,b\times 1):M\times N\longrightarrow X\times N$, with quadratic construction $\psi_{F\wedge 1}:H^{\pi_1(X\times N)}_{m+n}(\widetilde{X}\times\widetilde{N};{}^{w(X\times N)}\mathbb{Z})\longrightarrow Q_{m+n}(C(\widetilde{M}\times\widetilde{N}))$ such that

$$h_{\widetilde{M},\widetilde{N}\%}\psi_{F\wedge 1}[X\times N] = \psi_F[X]\otimes\varphi_{\widetilde{N}}[N]\in Q_{m+n}(C(\widetilde{M})\otimes_{\mathbb{Z}}C(\widetilde{N})) \ .$$

The homotopy equivalence of $(m+n)$-dimensional quadratic Poincaré complexes over $\mathbb{Z}[\pi_1(X\times N)])$

$$\begin{aligned} h^!_{\widetilde{M},\widetilde{N}} : \sigma_*(f\times 1,b\times 1) &= (C((f\times 1)^!),e_{\widetilde{f}\times 1\%}\psi_F[X\times N]) \\ &\longrightarrow \sigma_*(f,b)\otimes\sigma^*(N) = (C(f^!)\otimes_{\mathbb{Z}}C(\widetilde{N}),e_{\widetilde{f}\%}\psi_F[X]\otimes\varphi_{\widetilde{N}}[N]) \end{aligned}$$

implies that

$$\sigma_*(f\times 1,b\times 1) = \sigma_*(f,b)\otimes\sigma^*(N)\in L_{m+n}(\mathbb{Z}[\pi_1(X\times N)]) \ .$$

In the general case express the product $\begin{cases}\text{degree 1}\\ \text{normal}\end{cases}$ map as the composite

$$\begin{cases} f\times g : M\times N \xrightarrow{f\times 1} X\times N \xrightarrow{1\times g} X\times Y \\ (f\times g,b\times c) : M\times N \xrightarrow{(f\times 1,b\times 1)} X\times N \xrightarrow{(1\times g,1\times c)} X\times Y \end{cases} ,$$

and apply the sum formula of Proposition $\begin{cases}2.11\\3.14\end{cases}$ to obtain that

$$\begin{cases} \sigma^*(f\times g) = \sigma^*(f\times 1)\oplus\sigma^*(1\times g) = \sigma^*(f)\otimes\sigma^*(N)\oplus\sigma^*(X)\otimes\sigma^*(g) \\ \sigma_*(f\times g,b\times c) = \sigma_*(f\times 1,b\times 1)\oplus\sigma_*(1\times g,1\times c) = \sigma_*(f,b)\otimes\sigma^*(N)\oplus\sigma^*(X)\otimes\sigma_*(g,c) \end{cases}$$

up to homotopy equivalence. Now $\sigma^*(N) = \sigma^*(g) + \sigma^*(Y)\in L^n(\mathbb{Z}[\pi_1(Y)])$, so that

$$\begin{cases} \sigma^*(f\times g) = \sigma^*(f)\otimes\sigma^*(g) + \sigma^*(X)\otimes\sigma^*(g) + \sigma^*(f)\otimes\sigma^*(Y)\in L^{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) \\ \sigma_*(f\times g,b\times c) = \sigma_*(f,b)\otimes\sigma_*(g,c) + \sigma^*(X)\otimes\sigma_*(g,c) + \sigma_*(f,b)\otimes\sigma^*(Y)\in L_{m+n}(\mathbb{Z}[\pi_1(X\times Y)]) \ . \end{cases}$$

[]

The product formula for symmetric signatures of Proposition 11.2 i) is a generalization of the classical product formula for the signature.

The product formula for surgery obstructions (= quadratic signatures) of Proposition 11.2 ii) is a common generalization of the product formulae of Sullivan (for $\pi_1(X) = \{1\}$, $\pi_1(Y) = \{1\}$, proved in Chapter III of Browder [2]), Williamson [1], Shaneson [2] and Morgan [1] (all for $\pi_1(X) = \{1\}$, $f = 1:M\longrightarrow X = M$).

The product formulae for the obstructions to surgery on $\mathbb{Z}_m$-manifolds of Morgan and Sullivan [1] fit into the scheme of Proposition 11.2 ii) (cf. Proposition 10.6).

The algebraic suspension map defined in §1 for any chain complex C

$$S : Q^n(C) \longrightarrow Q^{n+1}(SC)$$

is the evaluation of the product

$$\otimes : Q^1(S\mathbb{Z})\otimes_{\mathbb{Z}} Q^n(C) \longrightarrow Q^{n+1}(SC) \qquad (S\mathbb{Z}\otimes_{\mathbb{Z}} C = SC)$$

on the generator $1 \in Q^1(S\mathbb{Z}) = \mathbb{Z}$. Thus for the chain complex $C = C(X)$ of a topological space X the algebraic suspension is the evaluation of the product

$$\otimes : Q^1(C(D^1,S^0))\otimes_{\mathbb{Z}} Q^n(C(X)) \longrightarrow Q^{n+1}(C(D^1,S^0)\otimes_{\mathbb{Z}} C(X))$$

on the fundamental $\mathbb{Z}_2$-hypercohomology class of (D^1,S^0), cf. Proposition 2.4.

Proposition 11.3 The periodicity isomorphism in the ε-quadratic L-groups is given by taking products with $\sigma^*(\mathbb{C}P^2) \in L^4(\mathbb{Z})$

$$\overline{S}^2 = \sigma^*(\mathbb{C}P^2)\otimes - \ : L_n(A,\varepsilon) \longrightarrow L_{n+4}(A,\varepsilon) \qquad (\mathbb{Z}\otimes_{\mathbb{Z}} A = A)\ .$$

Proof: Removing the fundamental class of $\mathbb{C}P^2$ by symmetric surgery represent $\sigma^*(\mathbb{C}P^2) \in L^4(\mathbb{Z})$ by $(C,\varphi \in Q^4(C))$, with $C_r = \begin{cases} \mathbb{Z} & \text{if } r=2 \\ 0 & \text{if } r\neq 2\end{cases}$, $\varphi_0 = 1 : C^2 \longrightarrow C_2$.

[]

The algebraic 4-periodicity in the ε-quadratic L-groups is thus seen to correspond to the geometric 4-periodicity

$$L_n(\pi) \longrightarrow L_{n+4}(\pi)\ ;\ \sigma_*((f,b):M\to X) \longmapsto \sigma_*((f\times 1,b\times 1):M\times\mathbb{C}P^2 \longrightarrow X\times\mathbb{C}P^2)$$

of Theorem 9.9 of Wall [5].

Proposition 11.4 Product with the generator $E_8 = \sigma_*(Q^8 \to S^8) \in L_8(\mathbb{Z}) = L_0(\mathbb{Z})$ defines a morphism

$$E_8\otimes - \ : L^n(A,\varepsilon) \longrightarrow L_n(A,\varepsilon)$$

such that both the composites with the ε-symmetrization map

$$1+T_\varepsilon : L_n(A,\varepsilon) \longrightarrow L^n(A,\varepsilon)$$

are multiplication by 8. Thus the ε-symmetric L-groups $L^n(A,\varepsilon)$ differ from the ε-quadratic L-groups $L_n(A,\varepsilon)$ only in the 8-torsion, and the ε-hyperquadratic L-groups $\hat{L}^n(A,\varepsilon)$ are of exponent 8.

Proof: $(1+T)E_8$ has signature $8 \in L^0(\mathbb{Z}) = \mathbb{Z}$.

[]

In §17H of Wall [5] it is proved geometrically that for any n-dimensional normal bundle map $(f,b):M \to X$

$$8\sigma_*(f,b) = \sigma_*(M\times Q^8 \to M\times S^8) - \sigma_*(X\times Q^8 \to X\times S^8) \in L_n(\mathbb{Z}[\pi_1(X)])$$

(modulo the difficulties with composition of normal bundle maps, cf. Proposition 6.6).

A ring with involution A is an R-module for some ring with involution R if there is given a morphism of rings with involution

$$R\otimes_{\mathbb{Z}} A \longrightarrow A\ ;\ r\otimes a \longmapsto ra\ .$$

Proposition 11.5 i) If A is an R-module the symmetric L-theory of R $L^*(R)$ acts on the $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ L-theory of A $\begin{cases} L^*(A,\varepsilon) \\ L_*(A,\varepsilon)\end{cases}$ by natural pairings

$$\begin{cases} L^m(R)\otimes_{\mathbb{Z}} L^n(A,\varepsilon) \longrightarrow L^{m+n}(A,\varepsilon) \\ L^m(R)\otimes_{\mathbb{Z}} L_n(A,\varepsilon) \longrightarrow L_{m+n}(A,\varepsilon)\end{cases} \qquad (m,n\in\mathbb{Z})\ ,$$

such that the element $(R,1:R\to R^*;r\mapsto(s\mapsto s\bar{r})) \in L^0(R)$ acts by the identity.

ii) If $f:A \longrightarrow B$ is a morphism of rings with involution which is also a morphism of R-modules for some ring with involution R then $L^*(R)$ acts on the exact sequence

$$\begin{cases} \ldots \longrightarrow L^n(A,\varepsilon) \xrightarrow{f} L^n(B,\varepsilon) \longrightarrow L^n(f,\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon) \longrightarrow \ldots \\ \ldots \longrightarrow L_n(A,\varepsilon) \xrightarrow{f} L_n(B,\varepsilon) \longrightarrow L_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots \end{cases} \qquad (n\in\mathbb{Z})\ .$$

Proof: i) These are just the pairings of Proposition 11.1, composed with the maps induced by $R\otimes_{\mathbb{Z}} A \longrightarrow A$. For example,

$$L^m(R)\otimes_{\mathbb{Z}} L^n(A,\varepsilon) \longrightarrow L^{m+n}(R\otimes_{\mathbb{Z}} A,\varepsilon) \longrightarrow L^{m+n}(A,\varepsilon) \qquad (m,n\in\mathbb{Z})\ .$$

On the chain level the pairing is given by $(C,\varphi)\otimes(D,\psi) \longmapsto (C\otimes_R D,\varphi\otimes\psi)$, with the tensor product $C\otimes_R D$ defined as follows. Each $r1_A \in A$ $(r\in R)$ is central in A (since $(r1_A)a = (\overline{r1_A})(\bar{a}) = a(r1_A) \in A$ $(a\in A)$). Given an R-module M and an A-module N define an A-module

$$M\otimes_R N = M\otimes_{\mathbb{Z}} N/\{rx\otimes y - x\otimes(r1_A)y \mid x\in M,\ y\in N,\ r\in R\}\ ,$$

with A acting by

$$A\times M\otimes_R N \longrightarrow M\otimes_R N\ ;\ (a,x\otimes y) \longmapsto x\otimes ay\ .$$

If M and N are f.g. projective then so is $M\otimes_R N$.

ii) Immediate from i) and Proposition 10.1.

[]

In particular, a commutative ring with involution R is an R-module and the symmetric L-theory $L^*(R)$ is a graded ring with 1 (such that $xy = (-)^{|y||x|}yx$), such that the quadratic L-theory $L_*(R)$ is a graded $L^*(R)$-algebra. The symmetric Witt ring $L^0(R)$ has the usual tensor product action on the quadratic Witt group $L_0(R)$.

In our applications of Proposition 11.5 we shall need to know the symmetric Witt groups $L^0(\mathbb{Z}_m)$ of the finite cyclic rings $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.

Let $m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ be the factorization of m into prime powers, so that

$$\mathbb{Z}_m = \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i^{k_i}} \quad , \quad L^0(\mathbb{Z}_m) = \bigoplus_{i=1}^{m} L^0(\mathbb{Z}_{p_i^{k_i}}) \quad .$$

Lemma 5 of Wall [8] and Theorem 3.3 of Bak [1] on reduction modulo a complete ideal (alias Hensel's lemma) apply to show that the projections

$$\begin{cases} \mathbb{Z}_{2^k} \longrightarrow \mathbb{Z}_8 \ , \ k \geqslant 3 \\ \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_p \ , \ p \text{ odd} \ , \ k \geqslant 1 \end{cases} \text{induce isomorphisms}$$

$$\begin{cases} L^0(\mathbb{Z}_{2^k}) \longrightarrow L^0(\mathbb{Z}_8) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 \\ L^0(\mathbb{Z}_{p^k}) \longrightarrow L^0(\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 1 (\text{mod } 4) \\ \mathbb{Z}_4 & \text{if } p \equiv 3 (\text{mod } 4) \ . \end{cases} \end{cases}$$

Moreover,

$$L^0(\mathbb{Z}_4) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \quad , \quad L^0(\mathbb{Z}_2) = \mathbb{Z}_2 \ .$$

For each integer $m \geqslant 2$ define the number

$$\psi(m) = \text{exponent of } L^0(\mathbb{Z}_m) = \begin{cases} 2 & \text{if } m = d \text{ or } 2d \\ 4 & \text{if } m = 4d, e, 2e \text{ or } 4e \\ 8 & \text{otherwise} \end{cases} ,$$

with

d = a product of odd primes $p \equiv 1 (\text{mod } 4)$

e = a product of odd primes, including at least one $p \equiv 3 (\text{mod } 4)$.

A ring with involution A is of characteristic m if m is an integer $\geqslant 2$ such that $m1 = 0 \in A$, in which case $ma = 0$ for all $a \in A$ and A is a $\mathbb{Z}_m$-module.

Proposition 11.6 The L-groups $\begin{Bmatrix} L^n(A,\varepsilon) \\ L_n(A,\varepsilon) \end{Bmatrix}$ $(n \in \mathbb{Z})$ of a ring with involution A of characteristic m are $L^0(\mathbb{Z}_m)$-modules, and hence of exponent $\psi(m)$.

Proof: A ring A of characteristic m is a $\mathbb{Z}_m$-module.

[]

The symmetric Witt groups $L^0(\hat{\mathbb{Z}}_m)$ of the rings of m-adic integers $\hat{\mathbb{Z}}_m = \varprojlim_k \mathbb{Z}/m^k\mathbb{Z}$ $(m \geqslant 2)$ are computed as follows. Again, express m as $m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ so that

$$\hat{\mathbb{Z}}_m = \bigoplus_{i=1}^{r} \hat{\mathbb{Z}}_{p_i} \quad , \quad L^0(\hat{\mathbb{Z}}_m) = \bigoplus_{i=1}^{r} L^0(\hat{\mathbb{Z}}_{p_i})$$

and

$$L^0(\hat{\mathbb{Z}}_p) = \begin{cases} L^0(\mathbb{Z}_8) = \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ L^0(\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 1 (\text{mod } 4) \\ \mathbb{Z}_4 & \text{if } p \equiv 3 (\text{mod } 4) \ . \end{cases} \end{cases}$$

For each integer $m \geqslant 2$ define the number

$$\hat{\psi}(m) = \text{exponent of } L^0(\hat{\mathbb{Z}}_m) = \begin{cases} 2 \text{ if } m \text{ is a product of odd primes } p \equiv 1 (\text{mod } 4) \\ 4 \text{ if } m \text{ is a product of odd primes at least one} \\ \quad \text{of which is } p \equiv 3 (\text{mod } 4) \\ 8 \text{ if } m \text{ is even} \end{cases}$$

A ring with involution A is of characteristic m^∞ if it can be expressed as an inverse limit

$$A = \varprojlim_k A_k$$

of an inverse system of rings with involution $\{A_k | k \geqslant 1\}$ such that each A_k is of characteristic m^k. For example, a ring of characteristic m^k (for some $k \geqslant 1$) is also of characteristic m^∞.

Proposition 11.7 The L-groups $\begin{Bmatrix} L^n(A,\varepsilon) \\ L_n(A,\varepsilon) \end{Bmatrix}$ $(n \in \mathbb{Z})$ of a ring with involution of characteristic m^∞ are $L^0(\hat{\mathbb{Z}}_m)$-modules, and hence of exponent $\hat{\psi}(m)$.

Proof: A ring A of characteristic m^∞ is a $\hat{\mathbb{Z}}_m$-module.

[]

Let R be a commutative ring, and let π be a finite group. Given an $R[\pi]$-module M let M_R be the R-module defined by the additive group of M with the restriction of the $R[\pi]$-action to $R\subset R[\pi]$. The dual R-module $M_R^* = \mathrm{Hom}_R(M_R,R)$ is to be regarded as an $R[\pi]$-module by

$$R[\pi]\times M_R^* \longrightarrow M_R^* \ ; \ (\sum_{g\in\pi} r_g g, f)\longmapsto (x\longmapsto \sum_{g\in\pi} r_g f(g^{-1}x)) \quad (r_g\in R, x\in M)\ .$$

An $R[\pi]$-module morphism $\varphi: M\longrightarrow M_R^*$ is the same as an R-module morphism

$$\varphi : M_R \longrightarrow M_R^* \ ; \ x\longmapsto (y\longmapsto \varphi(x)(y))$$

such that

$$\varphi(gx)(gy) = \varphi(x)(y)\in R \quad (x,y\in M,\ g\in\pi)\ .$$

Let $L^0(\pi,R)$ be the Witt group of pairs (M,φ) such that M is a f.g. R-projective $R[\pi]$-module together with an element $\varphi\in Q^+(M_R)$ which defines an $R[\pi]$-module isomorphism $\varphi: M\longrightarrow M_R^*$. The natural pairing

$$(\text{f.g. R-projective } R[\pi]\text{-modules})\times(\text{f.g. projective } R[\pi]\text{-modules})$$
$$\longrightarrow (\text{f.g. projective } R[\pi]\text{-modules})\ ;\ (M,N)\longmapsto M_R\otimes_R N$$

extends to L-theory pairings

$$\begin{cases} L^0(\pi,R)\otimes_{\mathbb{Z}} L^n(R[\pi],\varepsilon) \longrightarrow L^n(R[\pi],\varepsilon) \\ L^0(\pi,R)\otimes_{\mathbb{Z}} L_n(R[\pi],\varepsilon) \longrightarrow L_n(R[\pi],\varepsilon) \end{cases} \quad (n\in\mathbb{Z})\ .$$

The equivariant Witt group $L^0(\pi,R)$ was used by Dress [1] to obtain induction theorems for the quadratic L-groups $L_*(R[\pi])$ of finite groups π with R a Dedekind ring - in principle, these methods also apply to the symmetric L-groups $L^*(R[\pi])$.

§12. Change of K-theory

Let X be a subgroup of the $\begin{Bmatrix}\text{projective class}\\ \text{Whitehead torsion}\end{Bmatrix}$ group $\begin{Bmatrix}\tilde{K}_0(A)\\ \tilde{K}_1(A)\end{Bmatrix}$ which is setwise invariant under the duality involution

$$\begin{cases} * : \tilde{K}_0(A)\longrightarrow \tilde{K}_0(A)\ ;\ [P]\longmapsto [P^*] \\ * : \tilde{K}_1(A)\longrightarrow \tilde{K}_1(A)\ ;\ \tau(f:\underset{\sim}{P}\longrightarrow \underset{\sim}{Q})\longmapsto \tau(f^*:\underset{\sim}{Q}^*\longrightarrow \underset{\sim}{P}^*) \end{cases} ,$$

denoting by $\underset{\sim}{P}$ a f.g. free A-module P together with a choice of base. In dealing with based A-modules it is convenient to assume that A is such that f.g. free A-modules have a well-defined rank, as is the case with group rings $A = \mathbb{Z}[\pi]$. Also, we shall assume that $\tau(\varepsilon:A\longrightarrow A)\in X\subseteq \tilde{K}_1(A)$.

The intermediate $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ L-groups of A $\begin{Bmatrix}L_X^n(A,\varepsilon)\\ L_n^X(A,\varepsilon)\end{Bmatrix}$ $(n\in\mathbb{Z})$ are defined for $n\geqslant 0$ to be the cobordism groups of n-dimensional $\begin{Bmatrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{Bmatrix}$ Poincaré complexes over A $\begin{Bmatrix}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{Bmatrix}$ such that C is a finite chain complex of f.g. projective A-modules with projective class

$$[C] = \sum_{r=-\infty}^{\infty}(-)^r[C_r]\in X$$

if $X\subseteq\tilde{K}_0(A)$, and of based f.g. free A-modules with Whitehead torsion

$$\begin{cases}\tau(\varphi_0:\underset{\sim}{C}^{n-*}\longrightarrow \underset{\sim}{C})\in X\\ \tau((1+T_\varepsilon)\psi_0:\underset{\sim}{C}^{n-*}\longrightarrow \underset{\sim}{C})\in X\end{cases}$$

if $X\subseteq\tilde{K}_1(A)$, with corresponding K-theoretic restrictions on the cobordisms.

Furthermore, $\begin{Bmatrix}L_X^{-1}(A,\varepsilon)\\ L_X^{-2}(A,\varepsilon)\end{Bmatrix}$ is the Witt group of non-singular even $(-\varepsilon)$-symmetric $\begin{Bmatrix}\text{formations } (M,\varphi;F,G)\\ \text{forms } (M,\varphi)\end{Bmatrix}$ such that $\begin{Bmatrix}[G]-[F^*]\in X\\ [M]\in X\end{Bmatrix}$ if $X\subseteq\tilde{K}_0(A)$, and such that $\begin{Bmatrix}F,G\\ M\end{Bmatrix}$ are based f.g. free A-modules with $\begin{Bmatrix}\tau(\underset{\sim}{F}\oplus\underset{\sim}{F}^*\xrightarrow{f} M\xrightarrow{g^{-1}}\underset{\sim}{G}\oplus\underset{\sim}{G}^*)\in X\\ \tau(\varphi:\underset{\sim}{M}\longrightarrow\underset{\sim}{M}^*)\in X\end{Bmatrix}$ if $X\subseteq\tilde{K}_1(A)$, where $f:H^\varepsilon(F)\longrightarrow(M,\varphi)$, $g:H^\varepsilon(G)\longrightarrow(M,\varphi)$ are the isomorphisms of ε-symmetric forms extending the inclusions $F\longrightarrow M$, $G\longrightarrow M$ given by Proposition 1.6.

The lower intermediate L-groups are defined by

$$\begin{cases} L^n_X(A,\varepsilon) = L^X_{-n}(A,(-)^n\varepsilon) & (n\leqslant -3) \\ L_n^X(A,\varepsilon) = L^X_{-n}(A,(-)^n\varepsilon) & (n\leqslant -1) \end{cases} .$$

All the results in §§1 - 11 above have obvious intermediate L-group analogues, as do those of §§13 - 18 below. In particular, the ε-quadratic L-groups are 4-periodic

$$L^X_n(A,\varepsilon) = L^X_{n+2}(A,-\varepsilon) = L^X_{n+4}(A,\varepsilon) \qquad (n\in\mathbb{Z})$$

and $\begin{cases} L^0_X(A,\varepsilon) \\ L^1_X(A,\varepsilon) \end{cases}$ (resp. $\begin{cases} L^X_0(A,\varepsilon) \\ L^X_1(A,\varepsilon) \end{cases}$) is the Witt group of non-singular ε-symmetric (resp. ε-quadratic) $\begin{cases} \text{forms} \\ \text{formations} \end{cases}$ over A, with K-theory in X.

The L-groups considered so far are the case $X = \widetilde{K}_0(A)$

$$\begin{cases} L^n(A,\varepsilon) = L^n_{\widetilde{K}_0(A)}(A,\varepsilon) \\ L_n(A,\varepsilon) = L_n^{\widetilde{K}_0(A)}(A,\varepsilon) \end{cases} \quad (n\in\mathbb{Z}) .$$

The intermediate $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ L-groups $\begin{cases} L^n_X(A,1) \\ L_n^X(A,1) \end{cases}$ will be written $\begin{cases} L^n_X(A) \\ L_n^X(A) \end{cases}$.

The groups $U_n(A), V_n(A), W_n(A)$ of Ranicki [1] can be identified with the appropriate intermediate quadratic L-groups

$$U_n(A) = L_n^{\widetilde{K}_0(A)}(A) = L_n(A)$$

$$V_n(A) = L_n^{\{0\}\subseteq\widetilde{K}_0(A)}(A) = L_n^{\widetilde{K}_1(A)}(A)$$

$$W_n(A) = L_n^{\{0\}\subseteq\widetilde{K}_1(A)}(A) .$$

(For $U_n(A)$ this has already been done in Propositions 7.3,7.6,7.7). More generally, the intermediate quadratic L-groups $L_n^X(A)$ can be identified with the corresponding groups $U_n^X(A)$ (for $X\subseteq\widetilde{K}_0(A)$), $V_n^X(A)$ (for $X\subseteq\widetilde{K}_1(A)$) of Ranicki [3], which were defined using ±quadratic forms and formations with K-theory in X. For a group ring $A = \mathbb{Z}[\pi]$ with the w-twisted involution for some group morphism $w:\pi\longrightarrow\mathbb{Z}_2$ we thus have all the various geometric surgery obstruction groups. If $X\subseteq\begin{cases} \widetilde{K}_0(\mathbb{Z}[\pi]) \\ \widetilde{K}_1(\mathbb{Z}[\pi]) \end{cases}$ is a $*$-invariant subgroup

then $L_n^X(\mathbb{Z}[\pi])$ is the obstruction group for surgery on normal bundle maps of $\begin{cases} - \\ \text{finite} \end{cases}$ geometric Poincaré complexes with fundamental group π up to $\begin{cases} \text{proper} \\ - \end{cases}$ homotopy equivalence, with all the $\begin{cases} \text{finiteness obstructions} \\ \text{Whitehead torsion} \end{cases}$ lying in $\begin{cases} X\subseteq\widetilde{K}_0(\mathbb{Z}[\pi]) \\ X/\pi\subseteq Wh(\pi) = \widetilde{K}_1(\mathbb{Z}[\pi])/\{\pi\} \end{cases}$. (We recall that the finiteness obstruction of a geometric Poincaré complex M is the projective class $[C(\widetilde{M})]\in\widetilde{K}_0(\mathbb{Z}[\pi_1(M)])$ (Wall [2]), and that the Whitehead torsion of a finite geometric Poincaré complex M is $\tau([M]\cap- : C(\widetilde{M})^{n-*}\longrightarrow C(\widetilde{M}))\in Wh(\pi_1(M))$). In particular, we have the surgery obstruction groups

$$\begin{cases} L^s_n(\pi,w) = L_n^{\{\pi\}\subseteq\widetilde{K}_1(\mathbb{Z}[\pi])}(\mathbb{Z}[\pi]) \\ L^h_n(\pi,w) = L_n^{\widetilde{K}_1(\mathbb{Z}[\pi])}(\mathbb{Z}[\pi]) \\ L^p_n(\pi,w) = L_n(\mathbb{Z}[\pi]) \end{cases}$$

considered by $\begin{cases} \text{Wall [5]} \\ \text{Shaneson [1] (see also the discussion in §17D of Wall [5]).} \\ \text{Maumary [1]} \end{cases}$

Intermediate surgery obstruction groups $L_n^{X\subseteq\widetilde{K}_1(\mathbb{Z}[\pi])}(\mathbb{Z}[\pi])$ were first considered by Cappell [1].

We shall write $\begin{cases} L^n_X(A,\varepsilon) \\ L_n^X(A,\varepsilon) \end{cases}$ as $\begin{cases} U^n_X(A,\varepsilon) \\ U_n^X(A,\varepsilon) \end{cases}$ for $X\subseteq\widetilde{K}_0(A)$, and as $\begin{cases} V^n_X(A,\varepsilon) \\ V_n^X(A,\varepsilon) \end{cases}$ for $X\subseteq\widetilde{K}_1(A)$, with

$$\begin{cases} U^n(A,\varepsilon) = U^n_{\widetilde{K}_0(A)}(A,\varepsilon) \\ U_n(A,\varepsilon) = U_n^{\widetilde{K}_0(A)}(A,\varepsilon) \end{cases} , \quad \begin{cases} V^n(A,\varepsilon) = V^n_{\widetilde{K}_1(A)}(A,\varepsilon) \\ V_n(A,\varepsilon) = V_n^{\widetilde{K}_1(A)}(A,\varepsilon) \end{cases} \quad (n\in\mathbb{Z})$$

extending the notation of Ranicki [1],[3]. For ε = 1 the notation is contracted in the usual fashion, e.g. $U^n(A,1) = U^n(A)$.

The change of rings exact sequence of Proposition 10.1 has an intermediate version

$$\begin{cases} \dots\longrightarrow L^n_X(A,\varepsilon)\xrightarrow{f} L^n_Y(B,\varepsilon)\longrightarrow L^n_{X,Y}(f,\varepsilon)\longrightarrow L^{n-1}_X(A,\varepsilon)\longrightarrow\dots \\ \dots\longrightarrow L_n^X(A,\varepsilon)\xrightarrow{f} L_n^Y(B,\varepsilon)\longrightarrow L_n^{X,Y}(f,\varepsilon)\longrightarrow L_{n-1}^X(A,\varepsilon)\longrightarrow\dots \end{cases} \quad (n\in\mathbb{Z})$$

for $*$-invariant subgroups $X\subseteq\widetilde{K}_m(A), Y\subseteq\widetilde{K}_m(B)$ $(m = 0,1)$ such that $B\otimes_A X\subseteq Y$.

Mishchenko [2] only considers finite geometric Poincaré complexes in geometry and f.g. free A-module chain complexes in algebra, so that the symmetric Poincaré cobordism groups $\Omega_n(A)$ defined there are $V^n(A)$ rather than $U^n(A) = L^n(A)$. The difference is at most 2-torsion:

Proposition 12.1 The intermediate $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ L-groups of A associated to $*$-invariant subgroups $X \subseteq Y \subseteq \widetilde{K}_m(A)$ ($m = 0$ or 1) are related by a long exact sequence of abelian groups

$$\begin{cases} \dots \longrightarrow L^n_X(A,\varepsilon) \longrightarrow L^n_Y(A,\varepsilon) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Y/X) \longrightarrow L^{n-1}_X(A,\varepsilon) \longrightarrow L^{n-1}_Y(A,\varepsilon) \longrightarrow \dots \\ \dots \longrightarrow L^X_n(A,\varepsilon) \longrightarrow L^Y_n(A,\varepsilon) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Y/X) \longrightarrow L^X_{n-1}(A,\varepsilon) \longrightarrow L^Y_{n-1}(A,\varepsilon) \longrightarrow \dots \end{cases} (n \in \mathbb{Z})$$

involving the reduced Tate $\mathbb{Z}_2$-cohomology groups

$$\hat{H}^n(\mathbb{Z}_2;Y/X) = \{g \in Y/X \mid g^* = (-)^n g\}/\{h + (-)^n h^* \mid h \in Y/X\},$$

which are of exponent 2.

Proof: The first such exact sequence was obtained by Shaneson [1]

$$\dots \longrightarrow L^s_n(\pi) \longrightarrow L^h_n(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Wh(\pi)) \longrightarrow L^s_{n-1}(\pi) \longrightarrow L^h_{n-1}(\pi) \longrightarrow \dots$$

(the Rothenberg sequence) involving a geometric proof of exactness at $\hat{H}^{2i}(\mathbb{Z}_2;Wh(\pi))$. The exactness of the corresponding sequence

$$\dots \longrightarrow L^h_n(\pi) \longrightarrow L^p_n(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L^h_{n-1}(\pi) \longrightarrow L^p_{n-1}(\pi) \longrightarrow \dots$$

was conjectured in §17D of Wall [5]. Theorems 2.3,3.3 of Ranicki [3] gave a unified algebraic proof of both these sequences, which applies also to the more general ε-quadratic sequence above. (The algebraic proof of exactness at $\hat{H}^{2i}$ was an early instance of the glueing of ε-quadratic forms, cf. the proofs of Proposition 5.2 and of the ε-symmetric case below).

The ε-symmetric case with $m = 0$ proceeds as follows.

Given a f.g. projective A-module P define an $(n+1)$-dimensional ε-symmetric Poincaré pair over A

$$(f(P,n):C(P,n) \longrightarrow \delta C(P,n), (\delta\varphi(P,n),\varphi(P,n)) \in Q^{n+1}(f(P,n),\varepsilon)) \quad (n \geqslant 0)$$

with projective classes

$$\begin{aligned} [C(P,n)] &= \sum_{r=-\infty}^{\infty} (-)^r [C(P,n)_r] = (-)^{n-i}([P] + (-)^n [P^*]) \in \widetilde{K}_0(A) \\ [\delta C(P,n)] &= \sum_{r=-\infty}^{\infty} (-)^r [\delta C(P,n)_r] = (-)^{n-i}[P] \in \widetilde{K}_0(A) \end{aligned} \quad (n = 2i \text{ or } 2i+1)$$

by:

if $n = 2i$

$$f(P,2i) = (1\ \ 0) : C(P,2i)_i = P \oplus P^* \longrightarrow \delta C(P,2i)_i = P,$$

$$\varphi(P,2i)_0 = \begin{pmatrix} 0 & (-)^i\varepsilon \\ 1 & 0 \end{pmatrix} : C(P,2i)^i = P^* \oplus P \longrightarrow C(P,2i)_i = P \oplus P^*,$$

$$C(P,2i)_r = \delta C(P,2i)_r = 0 \ (r \neq i), \ \delta\varphi(P,2i) = 0,$$

if $n = 2i+1$

$$C(P,2i+1)_r = \begin{cases} P^* & r = i \\ P & r = i+1 \\ 0 & r \neq i,i+1 \end{cases}, \quad \delta C(P,2i+1)_r = \begin{cases} P & r = i+1 \\ 0 & r \neq i+1 \end{cases}$$

$$d = 0 : C(P,2i+1)_{i+1} = P \longrightarrow C(P,2i+1)_i = P^*, \quad \delta\varphi(P,2i+1) = 0,$$

$$\varphi(P,2i+1)_0 = \begin{cases} 1 : C(P,2i+1)^i = P \longrightarrow C(P,2i+1)_{i+1} = P \\ \varepsilon : C(P,2i+1)^{i+1} = P^* \longrightarrow C(P,2i+1)_i = P^* \end{cases},$$

$$f(P,2i+1) = 1 : C(P,2i+1)_{i+1} = P \longrightarrow C(P,2i+1)_{i+1} = P.$$

Define abelian group morphisms

$$\begin{aligned} \beta &: L^n_Y(A,\varepsilon) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Y/X) ; \ (C,\varphi) \longmapsto [C] \\ \gamma &: L^n_X(A,\varepsilon) \longrightarrow L^n_Y(A,\varepsilon) ; \ (C,\varphi) \longmapsto (C,\varphi) \\ \partial &: \hat{H}^{n+1}(\mathbb{Z}_2;Y/X) \longrightarrow L^n_X(A,\varepsilon) ; \ [P] \longmapsto (C(P,n),\varphi(P,n)) \end{aligned} \quad (n \geqslant 0).$$

The composite

$$\hat{H}^{n+1}(\mathbb{Z}_2;Y/X) \xrightarrow{\partial} L^n_X(A,\varepsilon) \xrightarrow{\gamma} L^n_Y(A,\varepsilon) \quad \text{(I)}$$

is 0, since $(f(P,n):C(P,n) \longrightarrow \delta C(P,n), (\delta\varphi(P,n),\varphi(P,n)))$ is a null-cobordism of $\gamma\partial[P] = (C(P,n),\varphi(P,n))$ with $[\delta C(P,n)] = (-)^{n-i}[P] \in Y \subseteq \widetilde{K}_0(A)$. Given $(C,\varphi) \in \ker\gamma$ there exists a null-cobordism $(f:C \longrightarrow \delta C,(\delta\varphi,\varphi))$ with $[\delta C] \in Y \subseteq \widetilde{K}_0(A)$, and $(C,\varphi) = (-)^{n-i}\partial[\delta C] \in \operatorname{im}(\partial:\hat{H}^{n+1}(\mathbb{Z}_2;Y/X) \longrightarrow L^n_X(A,\varepsilon))$ so that (I) is exact.

The composite

$$L^n_X(A,\varepsilon) \xrightarrow{\gamma} L^n_Y(A,\varepsilon) \xrightarrow{\beta} \hat{H}^n(\mathbb{Z}_2;Y/X) \quad \text{(II)}$$

is 0. Given $(C,\varphi) \in \ker\beta$ there exists a f.g. projective A-module P such that

$$[C] + (-)^{n-i}([P] + (-)^n[P^*]) = 0 \in \widetilde{K}_0(A), \ [P] \in X \subseteq \widetilde{K}_0(A)$$

and

$$(C,\varphi) = (C,\varphi) \oplus \partial[P] \in \operatorname{im}(\gamma : L^n_X(A,\varepsilon) \longrightarrow L^n_Y(A,\varepsilon)),$$

so that (II) is exact.

The composite

$$L_Y^{n+1}(A,\varepsilon) \xrightarrow{\beta} \hat{H}^{n+1}(\mathbb{Z}_2;Y/X) \xrightarrow{\partial} L_X^n(A,\varepsilon) \qquad \text{(III)}$$

is 0, for if $(C,\varphi) \in L_Y^{n+1}(A,\varepsilon)$ and P is a f.g. projective A-module such that $[P] = (-)^{n-i+1}[C] \in \tilde{K}_0(A)$ then $(f(P,n)\oplus 0 : C(P,n) \longrightarrow \delta C(P,n)\oplus C, (\delta\varphi(P,n)\oplus\varphi, \varphi(P,n)))$ is a null-cobordism of $\partial\beta(C,\varphi) = \partial[P] = (C(P,n),\varphi(P,n))$ such that $[\delta C(P,n)\oplus C] \in X \subseteq \tilde{K}_0(A)$. Given $[P] \in \ker\partial$ let $(g:C(P,n) \longrightarrow D, (\theta,\varphi(P,n)) \in Q^{n+1}(g,\varepsilon))$ be a null-cobordism of $\partial[P] = (C(P,n),\varphi(P,n))$ such that $[D] \in X \subseteq \tilde{K}_0(A)$. The union

$$\begin{aligned}(f(P,n):C(P,n) \longrightarrow \delta C(P,n), (\delta\varphi(P,n),\varphi(P,n))) &\cup (g:C(P,n) \longrightarrow D, (\theta,\varphi(P,n))) \\ &= (D',\theta' \in Q^{n+1}(D',\varepsilon))\end{aligned}$$

is an (n+1)-dimensional ε-symmetric Poincaré complex over A such that

$$[D'] = [D] - [C(P,n)] + [\delta C(P,n)] = [P] \in \hat{H}^{n+1}(\mathbb{Z}_2;Y/X)\ .$$

Thus $[P] = \beta(D',\theta') \in \operatorname{im}(\beta : L_Y^{n+1}(A,\varepsilon) \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;Y/X))$, and (III) is exact.

The cases m = 1 , $n \leqslant -1$ may be treated similarly.

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(It follows from Proposition 12.1 that the intermediate ε-hyperquadratic L-groups $\hat{L}_X^n(A,\varepsilon)$ appearing in the long exact sequence

$$\ldots \longrightarrow L_n^X(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L_X^n(A,\varepsilon) \xrightarrow{J} \hat{L}_X^n(A,\varepsilon) \xrightarrow{H} L_{n-1}^X(A,\varepsilon) \longrightarrow \ldots \qquad (n \in \mathbb{Z})$$

are independent of the group X, with

$$\hat{L}_X^n(A,\varepsilon) = \hat{L}_Y^n(A,\varepsilon) \qquad (n \in \mathbb{Z},\ X \subseteq \tilde{K}_m(A),\ Y \subseteq \tilde{K}_p(A),\ m,p \in \{0,1\})\ ).$$

Define the $\begin{cases}\text{ε-symmetric}\\ \text{ε-quadratic}\end{cases}$ unitary group of A to be

$$\begin{cases}\mathcal{U}^*(A,\varepsilon) = \varinjlim_m \operatorname{Aut} H^\varepsilon(A^m) \\ \mathcal{U}_*(A,\varepsilon) = \varinjlim_m \operatorname{Aut} H_\varepsilon(A^m)\end{cases} ,$$

that is the stable group of automorphisms of the $\begin{cases}\text{metabolic even ε-symmetric}\\ \text{hyperbolic ε-quadratic}\end{cases}$ forms

$$\begin{cases} H^\varepsilon(A^m) = (A^m\oplus(A^m)^*, \begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix} \in Q^\varepsilon(A^m\oplus(A^m)^*)) \\ H_\varepsilon(A^m) = (A^m\oplus(A^m)^*, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in Q_\varepsilon(A^m\oplus(A^m)^*)) \end{cases}$$

(which are a cofinal family of objects in the category of non-singular $\begin{cases}\text{even ε-symmetric}\\ \text{ε-quadratic}\end{cases}$ forms over A).

Define the elementary $\begin{cases}\text{ε-symmetric}\\ \text{ε-quadratic}\end{cases}$ unitary group $\begin{cases}\mathcal{EU}^*(A,\varepsilon)\\ \mathcal{EU}_*(A,\varepsilon)\end{cases}$ to be the subgroup of $\begin{cases}\mathcal{U}^*(A,\varepsilon)\\ \mathcal{U}_*(A,\varepsilon)\end{cases}$ generated by the elements of type

i) $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*-1}\end{pmatrix}$, for any automorphism $\alpha : A^m \longrightarrow A^m$

ii) $\begin{cases}\begin{pmatrix}1 & 0\\ \varphi & 1\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ \nu & 1\end{pmatrix}\end{cases}$, for any $\begin{cases}\text{(−ε)-symmetric}\\ \text{even (−ε)-symmetric}\end{cases}$ form $\begin{cases}(A^m,\varphi \in Q^{-\varepsilon}(A^m))\\ (A^m,\nu \in \operatorname{im}((1+T_{-\varepsilon}):Q_{-\varepsilon}(A^m) \longrightarrow Q^{-\varepsilon}(A^m)))\end{cases}$

iii) $\sigma = \begin{pmatrix}0 & \gamma^{-1}\\ \varepsilon\gamma & 0\end{pmatrix}$, where $\gamma : A \longrightarrow A^* ; a \longmapsto (b \longmapsto b\bar{a})$.

Given a $*$-invariant subgroup $X \subseteq \tilde{K}_1(A)$ define

$$\begin{cases}\mathcal{U}_X^*(A,\varepsilon) = \ker(\tau : \mathcal{U}^*(A,\varepsilon) \longrightarrow \tilde{K}_1(A)/X)\\ \mathcal{U}_*^X(A,\varepsilon) = \ker(\tau : \mathcal{U}_*(A,\varepsilon) \longrightarrow \tilde{K}_1(A)/X)\end{cases} ,$$

and let $\begin{cases}\mathcal{EU}_X^*(A,\varepsilon)\\ \mathcal{EU}_*^X(A,\varepsilon)\end{cases}$ be the subgroup of $\begin{cases}\mathcal{EU}^*(A,\varepsilon)\\ \mathcal{EU}_*(A,\varepsilon)\end{cases}$ obtained by restricting the generators of type i) to be such that $\tau(\alpha) \in X \subseteq \tilde{K}_1(A)$.

Proposition 12.2 Let $X \subseteq \widetilde{K}_1(A)$ be a $*$-invariant subgroup. Then

i) the elementary subgroup $\left\{\begin{matrix} \mathcal{E}\mathcal{U}^*_X(A,\varepsilon) \\ \mathcal{E}\mathcal{U}^X_*(A,\varepsilon) \end{matrix}\right.$ contains the commutator subgroup

$\left\{\begin{matrix} [\mathcal{U}^*_X(A,\varepsilon),\mathcal{U}^*_X(A,\varepsilon)] \\ [\mathcal{U}^X_*(A,\varepsilon),\mathcal{U}^X_*(A,\varepsilon)] \end{matrix}\right.$,

ii) up to natural isomorphism of abelian groups

$$\left\{\begin{matrix} V^{-1}_X(A,-\varepsilon) = \mathcal{U}^*_X(A,\varepsilon)/\mathcal{E}\mathcal{U}^*_X(A,\varepsilon) \\ V^X_1(A,\varepsilon) = \mathcal{U}^X_*(A,\varepsilon)/\mathcal{E}\mathcal{U}^X_*(A,\varepsilon) \end{matrix}\right. ,$$

iii) every element $\left\{\begin{matrix} u \in \mathcal{E}\mathcal{U}^*_X(A,\varepsilon) \\ v \in \mathcal{E}\mathcal{U}^X_*(A,\varepsilon) \end{matrix}\right.$ can be represented by a matrix such that

$$\left\{\begin{matrix} u \oplus \sigma \oplus \ldots \oplus \sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}\begin{pmatrix} 1 & \theta^* \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}\begin{pmatrix} 1 & \theta'^* \\ 0 & 1 \end{pmatrix} \\ v \oplus \sigma \oplus \ldots \oplus \sigma = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix}\begin{pmatrix} 1 & \lambda^* \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda'^* \\ 0 & 1 \end{pmatrix} \end{matrix}\right.$$

for some automorphism $\left\{\begin{matrix} \alpha : A^m \longrightarrow A^m \\ \beta : A^m \longrightarrow A^m \end{matrix}\right.$ with $\left\{\begin{matrix} \tau(\alpha) \in X \subseteq \widetilde{K}_1(A) \\ \tau(\beta) \in X \subseteq \widetilde{K}_1(A) \end{matrix}\right.$, and some

$\left\{\begin{matrix} (-\varepsilon)\text{-symmetric} \\ \text{even } (-\varepsilon)\text{-symmetric} \end{matrix}\right.$ forms $\left\{\begin{matrix} ((A^m)^*,\theta),(A^m,\varphi),((A^m)^*,\theta') \\ ((A^m)^*,\lambda),(A^m,\nu),((A^m)^*,\lambda') \end{matrix}\right.$.

Proof: The elementary unitary group $\left\{\begin{matrix} \mathcal{E}\mathcal{U}^*_X(A,\varepsilon) \\ \mathcal{E}\mathcal{U}^X_*(A,\varepsilon) \end{matrix}\right.$ lies in the kernel of the group morphism

$$\left\{\begin{matrix} \mathcal{U}^*_X(A,\varepsilon) \longrightarrow V^{-1}_X(A,-\varepsilon)\ ;\ (u:H^\varepsilon(A^m) \longrightarrow H^\varepsilon(A^m)) \longmapsto (H^\varepsilon(A^m);\underline{A}^m,u(\underline{A}^m)) \\ \mathcal{U}^X_*(A,\varepsilon) \longrightarrow V^X_1(A,\varepsilon)\ ;\ (v:H_\varepsilon(A^m) \longrightarrow H_\varepsilon(A^m)) \longmapsto (H_\varepsilon(A^m);\underline{A}^m,v(\underline{A}^m)) \end{matrix}\right. ,$$

which is well-defined by Proposition $\left\{\begin{matrix} 7.7 \\ 7.6 \end{matrix}\right.$, and onto by Proposition 1.6.

Every element in the kernel is represented by a matrix $\left\{\begin{matrix} u \\ v \end{matrix}\right.$ with an expression

for some stabilization $\left\{\begin{matrix} u \oplus \sigma \oplus \ldots \oplus \sigma \\ v \oplus \sigma \oplus \ldots \oplus \sigma \end{matrix}\right.$ as in iii), by Proposition $\left\{\begin{matrix} 9.2\ \text{iii)} \\ 7.8 \end{matrix}\right.$, so that

it lies in $\left\{\begin{matrix} \mathcal{E}\mathcal{U}^*_X(A,\varepsilon) \\ \mathcal{E}\mathcal{U}^X_*(A,\varepsilon) \end{matrix}\right.$.

[]

The original definition of the odd-dimensional surgery obstruction groups of Wall [5] was given by

$$L^s_{2i+1}(\pi,w) = \mathcal{U}^{\{\pi\}}_*(\mathbb{Z}[\pi],(-)^i)/\mathcal{E}\mathcal{U}^{\{\pi\}}_*(\mathbb{Z}[\pi],(-)^i) \quad (i(\text{mod } 2)) .$$

The inclusion $\left\{\begin{matrix} [\mathcal{U}^*,\mathcal{U}^*] \subseteq \mathcal{E}\mathcal{U}^* \\ [\mathcal{U}_*,\mathcal{U}_*] \subseteq \mathcal{E}\mathcal{U}_* \end{matrix}\right.$ was obtained by $\left\{\begin{matrix} \text{Wasserstein [1]} \\ \text{Wall [5]} \end{matrix}\right.$ using explicit matrix identities. (The quotient $\left\{\begin{matrix} \mathcal{E}\mathcal{U}^*_X(A,\varepsilon)/[\mathcal{U}^*_X(A,\varepsilon),\mathcal{U}^*_X(A,\varepsilon)] \\ \mathcal{E}\mathcal{U}^X_*(A,\varepsilon)/[\mathcal{U}^X_*(A,\varepsilon),\mathcal{U}^X_*(A,\varepsilon)] \end{matrix}\right.$ is generated by σ, so has order at most 2, at least for $A = \mathbb{Z}[\pi]$, $\varepsilon = \pm 1$, $X = \{\pi\} \subseteq \widetilde{K}_1(\mathbb{Z}[\pi])$).

The "Bruhat type" decomposition of elements in $\mathcal{E}\mathcal{U}_*$ given by Proposition 12.2 iii) is the improvement due to Wall [12] on the "normal form" of Sharpe [1].

Sharpe [1] dealt with the "split" unitary groups $\widetilde{\mathcal{U}}_*(A,\varepsilon)$, $\widetilde{\mathcal{E}\mathcal{U}}_*(A,\varepsilon)$ covering $\mathcal{U}_*(A,\varepsilon)$, $\mathcal{E}\mathcal{U}_*(A,\varepsilon)$, whose definitions we can reformulate as follows.

A split ε-quadratic form over A $(M,\widetilde{\psi})$ is a f.g. projective A-module M together with an A-module morphism $\widetilde{\psi} \in \mathrm{Hom}_A(M,M^*)$. A morphism (resp. isomorphism) of split ε-quadratic forms

$$(f,\chi) : (M,\widetilde{\psi}) \longrightarrow (M',\widetilde{\psi}')$$

is an A-module morphism (resp. isomorphism) $f \in \mathrm{Hom}_A(M,M')$ together with a $(-\varepsilon)$-quadratic form $(M,\chi \in Q_{-\varepsilon}(M))$ such that

$$f^*\widetilde{\psi}'f - \widetilde{\psi} = \chi - \varepsilon\chi^* \in \mathrm{Hom}_A(M,M^*) .$$

(For $\varepsilon = \pm 1 \in A$ this is the category of $\pm$forms defined in Ranicki [1], except that there morphisms involved also a lifting of $(M,\chi \in Q_{-\varepsilon}(M))$ to a split $(-\varepsilon)$-quadratic form $(M,\widetilde{\chi} \in \mathrm{Hom}_A(M,M^*))$). A split ε-quadratic form $(M,\widetilde{\psi} \in \mathrm{Hom}_A(M,M^*))$ determines an ε-quadratic form in the sense of §1 $(M,\psi \in Q_\varepsilon(M))$, with $\widetilde{\psi} \longmapsto \psi$ the natural projection, and this defines a one-one correspondence between the isomorphism classes of split ε-quadratic forms over A and the isomorphism classes of ε-quadratic forms over A. Given a f.g. projective A-module L define the hyperbolic split ε-quadratic form over A

$$\widetilde{H}_\varepsilon(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Hom}_A(L \oplus L^*, L^* \oplus L)) .$$

Define the split ε-quadratic unitary group of A

$$\widetilde{\mathcal{U}}_*(A,\varepsilon) = \varinjlim_m \mathrm{Aut}\,\widetilde{H}_\varepsilon(A^m) ,$$

and let $\widetilde{\mathcal{EU}}_*(A,\varepsilon)$ be the subgroup generated by the elements of type

i) $\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1}\end{pmatrix}, 0\right)$, for any automorphism $\alpha : A^m \longrightarrow A^m$

ii) $\left(\begin{pmatrix} 1 & 0 \\ (\psi-\varepsilon\psi^*) & 1\end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & 0\end{pmatrix}\right)$, for any $(-\varepsilon)$-quadratic form $(A^m, \psi\in Q_{-\varepsilon}(A^m))$

iii) $\sigma = \left(\begin{pmatrix} 0 & \gamma^{-1} \\ \varepsilon\gamma & 0\end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0\end{pmatrix}\right)$.

A split unitary automorphism

$$\left(\begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu}\end{pmatrix}, \begin{pmatrix} \theta & \widetilde{\chi} \\ \chi & \widetilde{\theta}\end{pmatrix}\right) : \widetilde{H}_\varepsilon(A^m) \longrightarrow \widetilde{H}_\varepsilon(A^m)$$

determines a non-singular split ε-quadratic formation $(A^m, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)A^m)$.

Conversely, given a non-singular split ε-quadratic formation over A $(F, (\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta)G)$ the inclusion of the lagrangian defines a morphism of split ε-quadratic forms

$$\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}, \theta\right) : (G,0) \longrightarrow \widetilde{H}_\varepsilon(F)$$

which can be extended to an isomorphism

$$\left(\begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu}\end{pmatrix}, \begin{pmatrix} \theta & \widetilde{\chi} \\ \chi & \widetilde{\theta}\end{pmatrix}\right) : \widetilde{H}_\varepsilon(G) \longrightarrow \widetilde{H}_\varepsilon(F) ,$$

working exactly as in Proposition 1.6.

Proposition 12.3 The natural projections $\widetilde{\mathcal{U}}_*(A,\varepsilon) \longrightarrow \mathcal{U}_*(A,\varepsilon)$, $\widetilde{\mathcal{EU}}_*(A,\varepsilon) \longrightarrow \mathcal{EU}_*(A,\varepsilon)$ induce an isomorphism of abelian quotient groups

$$\widetilde{\mathcal{U}}_*(A,\varepsilon)/\widetilde{\mathcal{EU}}_*(A,\varepsilon) \longrightarrow \mathcal{U}_*(A,\varepsilon)/\mathcal{EU}_*(A,\varepsilon) = V_1(A,\varepsilon) .$$

Proof: Immediate from Proposition 5.6.

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(There is also an intermediate version,

$$\widetilde{\mathcal{U}}^X_*(A,\varepsilon)/\widetilde{\mathcal{EU}}^X_*(A,\varepsilon) = \mathcal{U}^X_*(A,\varepsilon)/\mathcal{EU}^X_*(A,\varepsilon) = V^X_1(A,\varepsilon)$$

for any $*$-invariant subgroup $X \subseteq \widetilde{K}_1(A)$).

Returning to the projective L-groups we have:

Proposition 12.4 The $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ L-groups of a semi-simple ring with involution A are such that

$$\begin{cases} U^{2k}(A,\varepsilon) = U^{2k+2}(A,-\varepsilon) \\ U_{2k}(A,\varepsilon) = U_{2k+2}(A,-\varepsilon)\end{cases} (k\geq 0) , \quad \begin{cases} U^{2k+1}(A,\varepsilon) = 0 \\ U_{2k+1}(A,\varepsilon) = 0\end{cases} (k\in\mathbb{Z}) ,$$

and there are defined exact sequences

$$0 \to \hat{H}^{2k+1}(\mathbb{Z}_2;\widetilde{K}_0(A)) \longrightarrow V^{2k}(A,\varepsilon) \longrightarrow U^{2k}(A,\varepsilon) \longrightarrow \hat{H}^{2k}(\mathbb{Z}_2;\widetilde{K}_0(A)) \to V^{2k-1}(A,\varepsilon) \to 0$$
$$0 \to \hat{H}^{2k+1}(\mathbb{Z}_2;\widetilde{K}_0(A)) \longrightarrow V_{2k}(A,\varepsilon) \longrightarrow U_{2k}(A,\varepsilon) \longrightarrow \hat{H}^{2k}(\mathbb{Z}_2;\widetilde{K}_0(A)) \to V_{2k-1}(A,\varepsilon) \to 0 \quad (k\in\mathbb{Z})$$

Proof: A ring A is semi-simple precisely when it is 0-dimensional in the sense of §7 (i.e. noetherian of global dimension 0) so that $U^n(A,\varepsilon) = U^{n+2}(A,-\varepsilon)$ $(n\geq 0)$, $U^{2k+1}(A,\varepsilon) = 0$ $(k\geq 0)$ by Proposition 7.4. The proof of the latter also applies to show that $U^{-1}(A,\varepsilon) = 0$. The result $U_{2k+1}(A,\varepsilon) = 0$ $(k\in\mathbb{Z})$ was obtained in Ranicki [6]. The above exact sequences are now immediate from those of Proposition 12.1.

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The expression $V_{2k+1}(A) = \mathrm{coker}(U_{2k+2}(A) \longrightarrow \hat{H}^0(\mathbb{Z}_2;\widetilde{K}_0(A)))$ was first obtained by Pardon [2].

§13. Localization

Let S be a multiplicative subset of central non-zero divisors in a ring A, so that the localized ring $S^{-1}A$ obtained by inverting S is defined. The localization exact sequence of algebraic K-theory identifies the relative groups $K_n(A\longrightarrow S^{-1}A)$ appearing in the exact sequence

$$\ldots \longrightarrow K_n(A) \longrightarrow K_n(S^{-1}A) \longrightarrow K_n(A\rightarrow S^{-1}A) \longrightarrow K_{n-1}(A) \longrightarrow \ldots \qquad (n\in\mathbb{Z})$$

(where $K_n(A) = K_n$(exact category of f.g. projective A-modules)) with the K-groups $K_n(A,S)=K_{n-1}$(exact category of S-torsion A-modules of homological dimension 1) , namely

$$K_n(A\longrightarrow S^{-1}A) = K_n(A,S) \quad (n\in\mathbb{Z})$$

(Bass [1] for $n\leqslant 1$, Quillen [1] for $n\geqslant 2$).

Given a ring with involution A and a multiplicative subset S of central non-zero divisors stable under the involution we shall now define L-groups $\begin{cases} L^n(A,S,\varepsilon) \\ L_n(A,S,\varepsilon)\end{cases}$ $(n\in\mathbb{Z})$ of $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complexes over A which become contractible over $S^{-1}A$. We shall exhibit natural identifications

$$\begin{cases} L^n_S(A\longrightarrow S^{-1}A,\varepsilon) = L^n(A,S,\varepsilon) \\ L^S_n(A\longrightarrow S^{-1}A,\varepsilon) = L_n(A,S,\varepsilon)\end{cases} \quad (n\in\mathbb{Z}) \quad ,$$

the groups on the left hand side being the relative intermediate $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ L-groups associated (as in §12) to $S = \mathrm{im}(\widetilde{K}_0(A)\longrightarrow \widetilde{K}_0(S^{-1}A))$. We thus obtain a localization exact sequence in algebraic L-theory

$$\begin{cases} \ldots \longrightarrow L^n(A,\varepsilon) \longrightarrow L^n_S(S^{-1}A,\varepsilon) \longrightarrow L^n(A,S,\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon) \longrightarrow \ldots \\ \ldots \longrightarrow L_n(A,\varepsilon) \longrightarrow L^S_n(S^{-1}A,\varepsilon) \longrightarrow L_n(A,S,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \ldots \end{cases} \quad (n\in\mathbb{Z})$$

(In §17 we shall construct a generalization of this exact sequence, relating the L-groups of a ring A to the Γ-groups of complexes which are defined over A and become Poincaré over another ring B, such as arise in codimension 2 surgery obstruction theory). The relative L-groups $\begin{cases} L^n(A,S,\varepsilon) \\ L_n(A,S,\varepsilon)\end{cases}$ for $n\leqslant 1$ will

be interpreted as Witt groups of $S^{-1}A/A$-valued linking forms and formations involving S-torsion A-modules of homological dimension 1. It will thus be possible to express the top sequence as a localization exact sequence of Witt groups

$$\begin{aligned} \ldots &\longrightarrow L^2(A,S,\varepsilon) \longrightarrow M^\varepsilon(A) \longrightarrow M^\varepsilon_S(S^{-1}A) \longrightarrow M\langle v_0\rangle^\varepsilon(A,S) \longrightarrow L^\varepsilon(A) \longrightarrow L^\varepsilon_S(S^{-1}A) \\ &\longrightarrow L\langle v_0\rangle^\varepsilon(A,S) \longrightarrow M\langle v_0\rangle^{-\varepsilon}(A) \longrightarrow M\langle v_0\rangle^{-\varepsilon}_S(S^{-1}A) \longrightarrow M_{-\varepsilon}(A,S) \longrightarrow L\langle v_0\rangle^{-\varepsilon}(A) \\ &\longrightarrow L\langle v_0\rangle^{-\varepsilon}_S(S^{-1}A) \longrightarrow L_{-\varepsilon}(A,S) \longrightarrow M_\varepsilon(A) \longrightarrow M^S_\varepsilon(S^{-1}A) \longrightarrow \widetilde{M}_\varepsilon(A,S) \longrightarrow L_\varepsilon(A) \\ &\longrightarrow L^S_\varepsilon(S^{-1}A) \longrightarrow \widetilde{L}_\varepsilon(A,S) \longrightarrow M_{-\varepsilon}(A) \longrightarrow M^S_{-\varepsilon}(S^{-1}A) \longrightarrow \widetilde{M}_{-\varepsilon}(A,S) \longrightarrow \ldots \end{aligned}$$

which extends to the left as the localization exact sequence in the higher ε-symmetric L-groups (non-periodic in general), and to the right as the 12-periodic localization exact sequence in the ε-quadratic L-groups.

Here, $\begin{cases} L^\varepsilon(A) = L^0(A,\varepsilon) \\ L\langle v_0\rangle^\varepsilon(A,\varepsilon) = L^{-2}(A,-\varepsilon) \\ L_\varepsilon(A) = L_0(A,\varepsilon)\end{cases}$ (resp. $\begin{cases} M^\varepsilon(A) = L^1(A,\varepsilon) \\ M\langle v_0\rangle^\varepsilon(A) = L^{-1}(A,-\varepsilon) \\ M_\varepsilon(A) = L_1(A,\varepsilon)\end{cases}$) is the Witt group of non-singular $\begin{cases}\varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms (resp. formations) over A, and

$\begin{cases} L^\varepsilon_S(S^{-1}A) = L^0_S(S^{-1}A,\varepsilon) \\ L\langle v_0\rangle^\varepsilon_S(S^{-1}A) = L^{-2}_S(S^{-1}A,-\varepsilon) \\ L^S_\varepsilon(S^{-1}A) = L^S_0(S^{-1}A,\varepsilon)\end{cases}$ (resp. $\begin{cases} M^\varepsilon_S(S^{-1}A) = L^1_S(S^{-1}A,\varepsilon) \\ M\langle v_0\rangle^\varepsilon_S(S^{-1}A) = L^{-1}_S(S^{-1}A,-\varepsilon) \\ M^S_\varepsilon(S^{-1}A) = L^S_1(S^{-1}A,\varepsilon)\end{cases}$) is the Witt group of non-singular $\begin{cases}\varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms (resp. formations) over $S^{-1}A$ involving only the f.g. projective $S^{-1}A$-modules which are induced from f.g. projective A-modules. The relative term $\begin{cases} L\langle v_0\rangle^\varepsilon(A,S) = L^0(A,S,\varepsilon) \\ L_\varepsilon(A,S) = L^{-2}(A,S,-\varepsilon) \\ \widetilde{L}_\varepsilon(A,S) = L_0(A,S,\varepsilon)\end{cases}$

(resp. $\begin{cases} M\langle v_0\rangle^{\varepsilon}(A,S) = L^1(A,S,\varepsilon) \\ M_{\varepsilon}(A,S) = L^{-1}(A,S,-\varepsilon) \\ \widetilde{M}_{\varepsilon}(A,S) = L_1(A,S,\varepsilon) \end{cases}$) is the Witt group of non-singular

$\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking forms (resp. formations) over (A,S), to be defined below.

A localization sequence for Witt groups of the type

$$\ldots \longrightarrow M^{\varepsilon}(A) \longrightarrow M^{\varepsilon}(S^{-1}A) \longrightarrow M^{\varepsilon}(A,S) \longrightarrow L^{\varepsilon}(A) \longrightarrow L^{\varepsilon}(S^{-1}A) \longrightarrow L^{\varepsilon}(A,S) \longrightarrow M^{-\varepsilon}(A) \longrightarrow \ldots$$

was first obtained by Karoubi [2],[3] - unfortunately Theorem 1.2 of Karoubi [2] only holds if $1/2 \in A$ (otherwise the condition $q|_L = 0$ is insufficient to ensure that $Q:F \longrightarrow S^{-1}A;x \longmapsto \frac{1}{s^2}\langle\theta_0 j(x),j(x)\rangle$ actually defines an ε-quadratic function $Q:F \longrightarrow A/\{a-\varepsilon\bar{a}|a\in A\}$)- so that the results of Karoubi [2],[3] only hold if $1/2\in A$, in which case the various categories of linking forms over (A,S) all coincide. A localization exact sequence for surgery obstruction groups

$$\ldots \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow L_n(\mathbb{Q}[\pi]) \longrightarrow L_n(\mathbb{Z}[\pi],\mathbb{Z}-\{0\}) \longrightarrow L_{n-1}(\mathbb{Z}[\pi]) \longrightarrow \ldots$$

was first obtained by Pardon [2], following on from the earlier work of Wall [1],[3], Passman and Petrie [1] and Connolly [1]. The methods of Pardon [2] apply to more general localizations $A \longrightarrow S^{-1}A$, as long as $1/2 \in S^{-1}A$ (e.g. if $2\in S$). The localization exact sequence of Witt groups in the case $1/2 \in S^{-1}A$

$$L^{\varepsilon}(A) \longrightarrow L^{\varepsilon}_S(S^{-1}A) \longrightarrow L\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow M\langle v_0\rangle^{-\varepsilon}(A) \longrightarrow M\langle v_0\rangle^{-\varepsilon}_S(S^{-1}A)$$

has also been obtained by Carlsson and Milgram [1]. Localization techniques play an important role in the computations of the quadratic L-groups $L_n(\mathbb{Z}[\pi])$ of finite groups π due to Wall [10] and Bak [1]. More recently, Carlsson and Milgram [2] used localization to compute $M\langle v_0\rangle^{-1}(\mathbb{Z}[\pi])$ for certain finite π.

Let S be a subset of a ring with involution A such that

i) $st \in S \subseteq A$ $(s,t\in S)$

ii) $\bar{s} \in S \subseteq A$ $(s\in S)$

iii) if $sa = 0 \in A$ $(a\in A, s\in S)$ then $a = 0 \in A$

iv) $as = sa \in A$ $(a\in A, s\in S)$

v) $1 \in S$.

The localization $S^{-1}A$ is the ring of equivalence classes of pairs $(a,s)\in A\times S$ under the relation

$$(a,s) \sim (b,t) \text{ if } at = bs \in A .$$

As usual, the equivalence class of (a,s) is denoted by $\frac{a}{s} \in S^{-1}A$. Addition and multiplication in $S^{-1}A$ are given by

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st} \in S^{-1}A \ , \ (\frac{a}{s})(\frac{b}{t}) = \frac{ab}{st} \in S^{-1}A \ ,$$

and there is defined an involution

$$- : S^{-1}A \longrightarrow S^{-1}A \ ; \ \frac{a}{s} \longmapsto \frac{\bar{a}}{\bar{s}} \ .$$

The localization map

$$A \longrightarrow S^{-1}A \ ; \ a \longmapsto \frac{a}{1}$$

is a morphism of rings with involution. If $\varepsilon \in A$ is a central unit such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$ then $(\frac{\varepsilon}{1}) \in S^{-1}A$ is a central unit such that $\overline{(\frac{\varepsilon}{1})} = (\frac{\varepsilon}{1})^{-1} \in S^{-1}A$.

Given an A-module M there is induced an $S^{-1}A$-module

$$S^{-1}M = S^{-1}A \otimes_A M$$

whose elements $\frac{x}{s} = \frac{1}{s} \otimes x$ $(x\in M, s\in S)$ can be regarded as the equivalence classes of pairs $(x,s) \in M\times S$ under the relation

$$(x,s) \sim (y,t) \text{ if } tx = sy \in M .$$

Similarly for right A-modules M. Given (left) A-modules M,N regard $\mathrm{Hom}_A(M,N)$ as a right A-module by

$$\mathrm{Hom}_A(M,N) \times A \longrightarrow \mathrm{Hom}_A(M,N) \ ; \ (f,a) \longmapsto (x \longmapsto f(x)a) \ .$$

The natural right $S^{-1}A$-module morphism

$$S^{-1}\mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N) \;;\; \frac{f}{s} \longmapsto (\frac{x}{t} \longmapsto \frac{f(x)}{st})$$

is an isomorphism if M is a f.g. projective A-module. Regarding the dual $M^* = \mathrm{Hom}_A(M,A)$ as a left A-module by

$$A \times M^* \longrightarrow M^* \;;\; (a,f) \longmapsto (x \longmapsto f(x)\bar{a})$$

(as before) define a natural left A-module morphism

$$S^{-1}(M^*) = S^{-1}\mathrm{Hom}_A(M,A) \longrightarrow (S^{-1}M)^* = \mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}A) \;;$$
$$\frac{f}{s} \longmapsto (\frac{x}{t} \longmapsto \frac{f(x)}{t\bar{s}}) \quad .$$

If M is a f.g. projective A-module this is an isomorphism, allowing us to identify

$$S^{-1}(M^*) = (S^{-1}M)^*$$

so that the notation $S^{-1}M^*$ is unambiguous.

Given an A-module chain complex C there is induced an $S^{-1}A$-module chain complex

$$S^{-1}C = S^{-1}A\otimes_A C \;.$$

Localization is exact, so that we can identify

$$H_*(S^{-1}C) = S^{-1}H_*(C) \;,\; H^*(S^{-1}C) = S^{-1}H^*(C) \;.$$

An A-module chain complex C is <u>S-acyclic</u> if

$$S^{-1}H_*(C) = 0 \;,\; S^{-1}H^*(C) = 0 \;.$$

A finite-dimensional A-module chain complex C is S-acyclic if and only if $S^{-1}C$ is a contractible $S^{-1}A$-module chain complex.

An $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ complex over A $\left\{\begin{matrix}(C,\varphi\in Q^n(C,\varepsilon))\\ (C,\psi\in Q_n(C,\varepsilon))\end{matrix}\right.$ (resp. an $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ pair over A $\left\{\begin{matrix}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon))\end{matrix}\right.$ or a surgery on such a pair) is <u>S-acyclic</u> if the A-module chain complexes C (resp. C,D) are S-acyclic.

Define the <u>n-dimensional</u> $\left\{\begin{matrix}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\end{matrix}\right.$ <u>L-group of (A,S)</u> $\left\{\begin{matrix}L^n(A,S,\varepsilon)\\ L_n(A,S,\varepsilon)\end{matrix}\right.$ $(n\geqslant 0)$ to be the cobordism group of S-acyclic $(n+1)$-dimensional $\left\{\begin{matrix}\text{even }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{matrix}\right.$ Poincaré complexes over A. Here, a cobordism is an S-acyclic $(n+2)$-dimensional $\left\{\begin{matrix}\text{even }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{matrix}\right.$ Poincaré pair over A. The verification that the direct sum $\oplus$ defines an abelian group law on the set of S-acyclic cobordism classes $\left\{\begin{matrix}L^n(A,S,\varepsilon)\\ L_n(A,S,\varepsilon)\end{matrix}\right.$ proceeds exactly as for the set of cobordism classes $\left\{\begin{matrix}L^n(A,\varepsilon)\\ L_n(A,\varepsilon)\end{matrix}\right.$ (Proposition 5.2). Every algebraic property of the L-groups $\left\{\begin{matrix}L^*(A,\varepsilon)\\ L_*(A,\varepsilon)\end{matrix}\right.$ has an S-acyclic counterpart in the L-groups $\left\{\begin{matrix}L^*(A,S,\varepsilon)\\ L_*(A,S,\varepsilon)\end{matrix}\right.$. In particular, by insisting that all the A-module chain complexes concerned be S-acyclic we obtain from §7 an algebraic S-acyclic surgery theory with which to analyze S-acyclic cobordism.

<u>Proposition 13.1</u> i) There are defined natural isomorphisms of abelian groups

$$\left\{\begin{matrix}L^n(A,S,\varepsilon)\longrightarrow L^n_S(A\longrightarrow S^{-1}A,\varepsilon)\\ L_n(A,S,\varepsilon)\longrightarrow L^S_n(A\longrightarrow S^{-1}A,\varepsilon)\end{matrix}\right. \quad (n\geqslant 0)$$

with $S = \mathrm{im}(\widetilde{K}_0(A)\longrightarrow \widetilde{K}_0(S^{-1}A))$. The L-groups of (A,S) fit into an exact sequence

$$\left\{\begin{matrix}\dots\longrightarrow L^n(A,\varepsilon)\longrightarrow L^n_S(S^{-1}A)\xrightarrow{\partial} L^n(A,S,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon)\longrightarrow\dots\\ \dots\longrightarrow L_n(A,\varepsilon)\longrightarrow L^S_n(S^{-1}A)\xrightarrow{\partial} L_n(A,S,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\longrightarrow\dots\end{matrix}\right. \quad (n\geqslant 0)$$

with

$$\left\{\begin{matrix}\partial : L^n_S(S^{-1}A,\varepsilon)\longrightarrow L^n(A,S,\varepsilon) \;;\; S^{-1}(C,\varphi)\longmapsto \partial\bar{S}(C,\varphi)\\ \partial : L^S_n(S^{-1}A,\varepsilon)\longrightarrow L_n(A,S,\varepsilon) \;;\; S^{-1}(C,\psi)\longmapsto \partial\bar{S}(C,\psi)\end{matrix}\right. \;,$$

$$\left\{\begin{matrix}L^n(A,S,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon) = L\langle v_0\rangle^{n+1}(A,-\varepsilon) \;;\; (C,\varphi)\longmapsto (C,\varphi)\\ L_n(A,S,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon) = L_{n+1}(A,-\varepsilon) \;;\; (C,\psi)\longmapsto (C,\psi)\end{matrix}\right. \;.$$

ii) The skew-suspension maps in the ε-quadratic L-groups of (A,S)

$$\bar{S} : L_n(A,S,\varepsilon) \longrightarrow L_{n+2}(A,S,-\varepsilon) \; ; \; (C,\psi) \longmapsto (SC,\bar{S}\psi) \quad (n \geq 0)$$

are isomorphisms for all A,S,ε.

<u>Proof</u>: i) There are defined forgetful maps from the L-groups of (A,S) to the relative intermediate L-groups of the localization map $A \longrightarrow S^{-1}A$

$$\begin{cases} L^n(A,S,\varepsilon) \dashrightarrow L\langle v_0\rangle_S^{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) ; (C,\varphi) \longmapsto ((C,\varphi),(0:S^{-1}C \longrightarrow 0,(0,S^{-1}\varphi))) \\ L_n(A,S,\varepsilon) \longrightarrow L^S_{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) \; ; \; (C,\psi) \longmapsto ((C,\psi),(0:S^{-1}C \longrightarrow 0,(0,S^{-1}\psi))) \end{cases} \quad (n \geq 0) \; .$$

The relative versions of the skew-suspension isomorphisms of Proposition $\begin{cases} 9.1 \text{ iii)} \\ 7.3 \end{cases}$ define skew-suspension isomorphisms

$$\begin{cases} \bar{S} : L^n_S(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L\langle v_0\rangle_S^{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) \\ \bar{S} : L^S_n(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L^S_{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) \end{cases} \quad (n \geq 0) \; .$$

We shall prove that the composites

$$\begin{cases} L^n(A,S,\varepsilon) \longrightarrow L\langle v_0\rangle_S^{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) \xrightarrow{\bar{S}^{-1}} L^n_S(A \longrightarrow S^{-1}A,\varepsilon) \\ L_n(A,S,\varepsilon) \longrightarrow L^S_{n+2}(A \longrightarrow S^{-1}A,-\varepsilon) \xrightarrow{\bar{S}^{-1}} L^S_n(A \longrightarrow S^{-1}A,\varepsilon) \end{cases} \quad (n \geq 0)$$

are isomorphisms by constructing inverses.

(Explicitly, the map $\begin{cases} L^n(A,S,\varepsilon) \longrightarrow L^n_S(A \longrightarrow S^{-1}A,\varepsilon) \\ L_n(A,S,\varepsilon) \longrightarrow L^S_n(A \longrightarrow S^{-1}A,\varepsilon) \end{cases}$ sends the S-acyclic cobordism class of $\begin{cases} (C,\varphi \in Q^{n+1}(C,-\varepsilon)) \\ (C,\psi \in Q_{n+1}(C,-\varepsilon)) \end{cases}$ to the relative cobordism class of the pair

$$\begin{cases} ((C',\varphi' \in Q^{n-1}(C',\varepsilon)),(S^{-1}f':S^{-1}C' \longrightarrow S^{-1}D',(0,S^{-1}\varphi') \in Q^n(S^{-1}f',\varepsilon))) \\ ((C',\psi' \in Q_{n-1}(C',\varepsilon)),(S^{-1}f':S^{-1}C' \longrightarrow S^{-1}D',(0,S^{-1}\psi') \in Q_n(S^{-1}f',\varepsilon))) \end{cases}$$

obtained as follows. Take C to be a f.g. projective A-module chain complex

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots \; ,$$

and let $f:C \longrightarrow D$ be the A-module chain map defined by

$$f = \begin{cases} 1 \\ 0 \end{cases} : C_r \longrightarrow D_r = \begin{cases} C_{n+1} & r = n+1 \\ 0 & r \neq n+1 \end{cases} \; .$$

The $(n+1)$-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ Poincaré complex over A obtained from $\begin{cases} (C,\varphi) \\ (C,\psi) \end{cases}$ by surgery on the connected $(n+2)$-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ pair over A $\begin{cases} (f:C \longrightarrow D,(0,\varphi) \in Q^{n+2}(f,-\varepsilon)) \\ (f:C \longrightarrow D,(0,\psi) \in Q_{n+2}(f,-\varepsilon)) \end{cases}$ is the skew-suspension $\begin{cases} \bar{S}(C',\varphi') \\ \bar{S}(C',\psi') \end{cases}$ of an $(n-1)$-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complex over A $\begin{cases} (C',\varphi' \in Q^{n-1}(C',\varepsilon)) \\ (C',\psi' \in Q_{n-1}(C',\varepsilon)) \end{cases}$. Define an A-module chain complex D' and an A-module chain map $f':C' \longrightarrow D'$ by

$$f' = (0 \;\; 1) : C'_0 = C_1 \oplus C^{n+1} \longrightarrow D'_0 = C^{n+1} \; , \; D'_r = 0 \quad (r \neq 0) \quad).$$

Consider the case $n \geq 1$. Given an $(n-1)$-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complex over A $\begin{cases} (C,\varphi \in Q^{n-1}(C,\varepsilon)) \\ (C,\psi \in Q_{n-1}(C,\varepsilon)) \end{cases}$ and a null-cobordism over $S^{-1}A$ $\begin{cases} (f:S^{-1}C \longrightarrow D,(\delta\varphi,S^{-1}\varphi) \in Q^n(f,\varepsilon)) \\ (f:S^{-1}C \longrightarrow D,(\delta\psi,S^{-1}\psi) \in Q_n(f,\varepsilon)) \end{cases}$ of $\begin{cases} S^{-1}(C,\varphi) \\ S^{-1}(C,\psi) \end{cases}$ such that $[D] \in S \subseteq \tilde{K}_0(S^{-1}A)$, it is possible to find an element $s \in S$ and an n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$

pair over A $\begin{cases} (f':C \longrightarrow D', (\delta\varphi', \varphi) \in Q^n(f', \varepsilon)) \\ (f':C \longrightarrow D', (\delta\psi', \psi) \in Q_n(f', \varepsilon)) \end{cases}$ such that

$$\begin{cases} (sf:S^{-1}C \longrightarrow D, ((s\otimes s)\delta\varphi, S^{-1}\varphi)) = S^{-1}(f':C \longrightarrow D', (\delta\varphi', \varphi)) \\ (sf:S^{-1}C \longrightarrow D, ((s\otimes s)\delta\psi, S^{-1}\psi)) = S^{-1}(f':C \longrightarrow D', (\delta\psi', \psi)) \end{cases}$$

up to homotopy equivalence of n-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ pairs over $S^{-1}A$.

Let $\begin{cases} (C'', \varphi'' \in Q^{n+1}(C'', -\varepsilon)) \\ (C'', \psi'' \in Q_{n+1}(C'', -\varepsilon)) \end{cases}$ be the S-acyclic (n+1)-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$

Poincaré complex over A obtained from the skew-suspension $\begin{cases} \overline{S}(C, \varphi) \\ \overline{S}(C, \psi) \end{cases}$ by surgery on

the connected (n+2)-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ pair over A

$\begin{cases} \overline{S}(f':C \longrightarrow D', (\delta\varphi', \varphi')) \\ \overline{S}(f':C \longrightarrow D', (\delta\psi', \psi')) \end{cases}$. We can now define the inverse isomorphism

$$\begin{cases} L^n_S(A \longrightarrow S^{-1}A, \varepsilon) \longrightarrow L^n(A, S, \varepsilon); ((C, \varphi), (f:S^{-1}C \longrightarrow D, (\delta\varphi, S^{-1}\varphi))) \longmapsto (C'', \varphi'') \\ L_n^S(A \longrightarrow S^{-1}A, \varepsilon) \longrightarrow L_n(A, S, \varepsilon); ((C, \psi), (f:S^{-1}C \longrightarrow D, (\delta\psi, S^{-1}\psi))) \longmapsto (C'', \psi'') \end{cases} \quad (n \geqslant 1) .$$

The construction also works for $n = 0$, taking (-1)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$

complexes to mean 1-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ complexes, and

0-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ pairs to mean 2-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$

pairs $\begin{cases} (f:C \longrightarrow D, (\delta\varphi, \varphi)) \\ (f:C \longrightarrow D, (\delta\psi, \psi)) \end{cases}$ such that $H_0(D) = 0$, $H_2(D) = 0$.

ii) It is immediate from i) and Proposition 7.3 that $\overline{S}:L_n(A, S, \varepsilon) \longrightarrow L_{n+2}(A, S, -\varepsilon)$ is an isomorphism for all A, S, ε, n 0.

[]

We shall say that the pair (A,S) is <u>m-dimensional</u> if every f.g. S-torsion A-module M has a f.g. projective A-module resolution of length m+1

$$0 \longrightarrow P_{m+1} \longrightarrow P_m \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 .$$

For example, if A is (m+1)-dimensional (in the sense of §7) then (A,S) is m-dimensional. If π is a finite group and p is a prime such that $p \nmid |\pi|$ then $(\mathbb{Z}[\pi], \{p^k | k \geqslant 0\})$ is 0-dimensional.

<u>Proposition 13.2</u> If (A,S) is m-dimensional the skew-suspension maps

$$\overline{S} : L^n(A, S, \varepsilon) \longrightarrow L^{n+2}(A, S, -\varepsilon) \quad (n \geqslant 2m+1)$$

are isomorphisms, and there are natural identifications

$$\begin{cases} L^{2i}(A, S, \varepsilon) \\ L^{2i-1}(A, S, \varepsilon) \end{cases} = (\text{S-acyclic cobordism group of S-acyclic} \begin{cases} (2m+1)\text{-} \\ 2m\text{-} \end{cases} \text{dimensional}$$

$$(-)^{i-m-1}\varepsilon\text{-symmetric Poincaré complexes over A}) \quad (i \geqslant m+1)$$

under which $L^n(A, S, \varepsilon) \longrightarrow L^{n-1}(A, \varepsilon)$ $(n \geqslant 2m+1)$ becomes the forgetful map

$$\begin{cases} L^{2i}(A, S, \varepsilon) \longrightarrow L^{2i-1}(A, \varepsilon) ; (C, \varphi) \longmapsto \overline{S}^{i-m-1}(C, \varphi) \\ L^{2i-1}(A, S, \varepsilon) \longrightarrow L^{2i-2}(A, S, \varepsilon) ; (C, \varphi) \longmapsto \overline{S}^{i-m-1}(C, \varphi) \end{cases} \quad (i \geqslant m+1) .$$

In particular, for m = 0

$$L^{2k+1}(A, S, \varepsilon) = 0 \quad (k \geqslant 0) .$$

<u>Proof</u>: In order to identify $\begin{cases} L^{2i}(A, S, \varepsilon) \\ L^{2i-1}(A, S, \varepsilon) \end{cases}$ $(i \geqslant m+1)$ with the S-acyclic cobordism

group of S-acyclic $\begin{cases} (2m+1)\text{-} \\ 2m\text{-} \end{cases}$ dimensional $(-)^{i-m-1}\varepsilon$-symmetric Poincaré complexes over A it suffices (by the S-acyclic counterpart of Proposition 7.2) to show that it is possible to perform S-acyclic surgery on a connected S-acyclic (n+1)-dimensional even $(-\varepsilon)$-symmetric complex over A $(C, \varphi \in Q^{n+1}(C, -\varepsilon))$ $(n \geqslant 2m+1)$ so as to obtain a skew-suspension, killing $H^{n+1}(C)$. Working exactly as in the proof of Proposition 7.4 use a f.g. projective A-module resolution of the f.g. S-torsion A-module $H_0(C)$

$$0 \longrightarrow D_{m+1} \longrightarrow D_m \longrightarrow \dots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow H_0(C) \longrightarrow 0$$

to define a connected S-acyclic $(n+2)$-dimensional $(-\varepsilon)$-symmetric pair over A $(f:C \longrightarrow D,(0,\varphi)\in Q^{n+2}(f,-\varepsilon))$ with which to perform just such a surgery.

In particular, if (A,S) is 0-dimensional we have that $L^{2k+1}(A,S,\varepsilon)$ $(k\geqslant 0)$ is the S-acyclic cobordism group of S-acyclic 0-dimensional $(-)^k\varepsilon$-symmetric Poincaré complexes over A $(C,\varphi\in Q^0(C,(-)^k\varepsilon))$. Now $H_0(C)$ is an S-torsion f.g. projective A-module, and S consists of non-zero divisors, so that $H_0(C) = 0$ and $L^{2k+1}(A,S,\varepsilon) = 0$.

[]

In §1 we identified the homotopy equivalence classes of n-dimensional $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ Poincaré complexes over A for $n = 0$ (resp. 1) with the (stable) isomorphism classes of non-singular $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ forms (resp. formations) over A, going on in §7 to identify the cobordism group $\left\{\begin{matrix}L^0(A,\varepsilon)\\ L_0(A,\varepsilon)\end{matrix}\right.$ (resp. $\left\{\begin{matrix}L^1(A,\varepsilon)\\ L_1(A,\varepsilon)\end{matrix}\right.$) with the Witt group of such objects. We shall now proceed to identify the homotopy equivalence classes of S-acyclic $(n+1)$-dimensional $\left\{\begin{matrix}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{matrix}\right.$ Poincaré complexes over A for $n = 0$ (resp. 1) with the (stable) isomorphism classes of non-singular $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ $S^{-1}A/A$-valued linking forms (resp. formations) on S-torsion A-modules of homological dimension 1, going on to identify the S-acyclic cobordism group $\left\{\begin{matrix}L^0(A,S,\varepsilon)\\ L_0(A,S,\varepsilon)\end{matrix}\right.$ (resp. $\left\{\begin{matrix}L^1(A,S,\varepsilon)\\ L_1(A,S,\varepsilon)\end{matrix}\right.$) with the Witt group of such objects. As a necessary preliminary we shall develop an algebraic theory of linking Wu classes of S-acyclic $\left\{\begin{matrix}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{matrix}\right.$ complexes, an S-acyclic counterpart to the algebraic Wu classes of §1.

An A-module M is <u>S-torsion</u> if

$$S^{-1}M = 0 \ .$$

Given an A-module M let T_SM be the maximal S-torsion submodule of M

$$T_SM = \{x\in M \mid sx = 0 \text{ for some } s\in S\} = \ker(M \longrightarrow S^{-1}M; x \longmapsto \tfrac{x}{1}) \ .$$

An A-module M is S-torsion if and only if

$$T_SM = M \ .$$

A f.g. A-module M is S-torsion if and only if there exists $s\in S$ such that

$$sM = 0 \ .$$

Define the <u>linking pairing</u> of an n-dimensional ε-symmetric complex over A $(C,\varphi\in Q^n(C,\varepsilon))$

$$\varphi_0^S : T_S H^r(C)\times T_S H^{n-r+1}(C) \longrightarrow S^{-1}A/A \ ;\ (x,y)\longmapsto \frac{1}{s}\varphi_0(x)(z)$$
$$(x\in C^r,\ y\in C^{n-r+1},\ z\in C^{n-r},\ s\in S,\ d^*z = sy\in C^{n-r+1})\ ,$$

with the properties

$$\varphi_0^S(x,y+y') = \varphi_0^S(x,y) + \varphi_0^S(x,y')$$
$$\varphi_0^S(x,ay) = a\varphi_0^S(x,y)$$
$$\varphi_0^S(x,y) = (-)^{r(n-r+1)}\varepsilon\overline{\varphi_0^S(y,x)}$$
$$(x\in T_S H^r(C),\ y,y'\in T_S H^{n-r+1}(C),\ a\in A)\ .$$

In particular, the linking pairing of the n-dimensional symmetric Poincaré complex over $\mathbb{Z}$ $\sigma^*(M) = (C(M),\varphi=\varphi_M[M]\in Q^n(C(M)))$ of an oriented n-dimensional manifold M

$$\varphi_0^S : T_S H^r(M)\times T_S H^{n-r+1}(M)\longrightarrow \mathbb{Q}/\mathbb{Z} \qquad (S = \mathbb{Z}-\{0\})$$

agrees (via Poincaré duality) with the pairing

$$T_S H_{n-r}(M)\times T_S H_{r-1}(M)\longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by the geometric linking numbers of torsion homology classes, as originally studied by Brouwer, deRham [1] and Seifert [1].

In §17 Proposition 13.26 we show that the cobordism class $(C,\varphi)\in L^n(A,\varepsilon)$ of an n-dimensional ε-symmetric Poincaré complex over a Dedekind ring A is determined for $n = \begin{cases}2i\\2i-1\end{cases}$ by the $(-)^i\varepsilon$-symmetric $\begin{cases}\text{form over A}\\ \text{linking pairing}\end{cases}$

$$\begin{cases}\varphi_0 : H^i(C)/T_S H^i(C)\times H^i(C)/T_S H^i(C)\longrightarrow A\\ \varphi_0^S : T_S H^i(C)\times T_S H^i(C)\longrightarrow S^{-1}A/A\end{cases} \qquad (S = A-\{0\})\ .$$

An A-module morphism $f\in \mathrm{Hom}_A(P,Q)$ is an <u>S-isomorphism</u> if

$$S^{-1}f : S^{-1}P\longrightarrow S^{-1}Q\ ;\ \frac{x}{s}\longmapsto\frac{f(x)}{s}$$

is an $S^{-1}A$-module isomorphism. If P,Q are f.g. projective A-modules then $f\in\mathrm{Hom}_A(P,Q)$ is an S-isomorphism if and only if $f^*\in\mathrm{Hom}_A(Q^*,P^*)$ is an S-isomorphism.

An <u>h.d. 1 S-torsion A-module</u> M is an S-torsion A-module of homological dimension 1, that is an A-module which admits a f.g. projective A-module resolution

$$0\longrightarrow P_1\xrightarrow{\ d\ }P_0\longrightarrow M\longrightarrow 0$$

such that $d\in\mathrm{Hom}_A(P_1,P_0)$ is an S-isomorphism.

The <u>S-dual</u> of an A-module M is the A-module

$$M\hat{} = \mathrm{Hom}_A(M,S^{-1}A/A)$$

with A acting by

$$A\times M\hat{}\longrightarrow M\hat{}\ ;\ (a,f)\longmapsto (x\longmapsto f(x)\bar{a})\ .$$

The S-dual of an h.d.1 S-torsion A-module M is an h.d.1 S-torsion A-module $M\hat{}$, since the dual of a f.g. projective resolution of M

$$0\longrightarrow P_1\xrightarrow{\ d\ }P_0\longrightarrow M\longrightarrow 0$$

is a f.g. projective resolution of $M\hat{}$

$$0\longrightarrow P_0^*\xrightarrow{\ d^*\ }P_1^*\longrightarrow M\hat{}\longrightarrow 0$$

with

$$P_1^*\longrightarrow M\hat{}\ ;\ f\longmapsto([x]\longrightarrow\frac{1}{s}f(y))\qquad(x\in P_0,\ [x]\in M,\ s\in S,\ y\in P_1,\ sx = dy\in P_0)\ .$$

The natural A-module morphism

$$M\longrightarrow M\hat{}\hat{}\ ;\ x\longmapsto(f\longmapsto\overline{f(x)})$$

is an isomorphism if M is an h.d.1 S-torsion A-module, and if N is another h.d.1 S-torsion A-module the abelian group morphism

$$\mathrm{Hom}_A(M,N)\longrightarrow\mathrm{Hom}_A(N\hat{},M\hat{})\ ;\ f\longmapsto(f\hat{}:g\longmapsto(x\longmapsto gf(x)))$$

is an isomorphism. In particular, the ε-transposition map

$$T_\varepsilon : \mathrm{Hom}_A(M,M\hat{})\longrightarrow\mathrm{Hom}_A(M,M\hat{})\ ;\ \varphi\longmapsto(T_\varepsilon\varphi=\varepsilon\varphi\hat{}:x\longmapsto(y\longmapsto\varepsilon\overline{\varphi(y)(x)}))$$

is an involution, $T_\varepsilon^2 = 1$.

An ε-symmetric linking form over (A,S) (M,λ) is an h.d.1 S-torsion A-module M together with an A-module morphism $\lambda \in \mathrm{Hom}_A(M,M^\wedge)$ such that

$$T_\varepsilon \lambda = \lambda \in \mathrm{Hom}_A(M,M^\wedge) \ .$$

Equivalently, it is possible to regard $\lambda \in \mathrm{Hom}_A(M,M^\wedge)$ as a pairing

$$\lambda \ : \ M\times M \longrightarrow S^{-1}A/A \ ; \ (x,y) \longmapsto \lambda(x,y) \equiv \lambda(x)(y)$$

such that

$$\begin{aligned} &\lambda(x,y+y') = \lambda(x,y) + \lambda(x,y') \\ &\lambda(x,ay) = a\lambda(x,y) \qquad\qquad (x,y,y'\in M,\ a\in A) \\ &\lambda(x,y) = \varepsilon\overline{\lambda(y,x)} \end{aligned}$$

Write $Q^\varepsilon(S^{-1}A/A)$ for the $\mathbb{Z}_2$-cohomology group

$$Q^\varepsilon(S^{-1}A/A) = H^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) = \{b\in S^{-1}A \,|\, b-\varepsilon\bar{b}\in A\}/A \ ,$$

and let $Q^\varepsilon(A,S)$ be the subgroup of $Q^\varepsilon(S^{-1}A/A)$ defined by

$$Q^\varepsilon(A,S) = \{b\in S^{-1}A \,|\, b-\varepsilon\bar{b} = a-\varepsilon\bar{a},\ a\in A\}/A \ .$$

An ε-symmetric linking form over (A,S) (M,λ) is even if for each $x\in M$

$$\lambda(x)(x) \in Q^\varepsilon(A,S) \subseteq Q^\varepsilon(S^{-1}A/A) \ .$$

Write $Q_\varepsilon(S^{-1}A/A)$ for the $\mathbb{Z}_2$-homology group

$$Q_\varepsilon(S^{-1}A/A) = H_0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) = S^{-1}A/\{a+b-\varepsilon\bar{b} \,|\, a\in A, b\in S^{-1}A\},$$

and define also the abelian group

$$Q_\varepsilon(A,S) = \{b\in S^{-1}A \,|\, b = \varepsilon\bar{b}\}/\{a+\varepsilon\bar{a} \,|\, a\in A\} \ .$$

The ε-symmetrization map

$$1+T_\varepsilon \ : \ Q_\varepsilon(S^{-1}A/A) \longrightarrow Q^\varepsilon(S^{-1}A/A) \ ; \ x \longmapsto x+\varepsilon\bar{x}$$

factors as

$$1+T_\varepsilon \ : \ Q_\varepsilon(S^{-1}A/A) \xrightarrow{\ p\ } Q_\varepsilon(A,S) \xrightarrow{\ q\ } Q^\varepsilon(A,S) \xrightarrow{\ r\ } Q^\varepsilon(S^{-1}A/A) \ ,$$

with

$$\begin{aligned} &p \ : \ Q_\varepsilon(S^{-1}A/A) \longrightarrow Q_\varepsilon(A,S) \ ; \ x \longmapsto x+\varepsilon\bar{x} \\ &q \ : \ Q_\varepsilon(A,S) \longrightarrow Q^\varepsilon(A,S) \ ; \ x \longmapsto x \\ &r \ : \ Q^\varepsilon(A,S) \longrightarrow Q^\varepsilon(S^{-1}A/A) \ ; \ x \longmapsto x \end{aligned}$$

An ε-quadratic linking form over (A,S) (M,λ,μ) is an ε-symmetric linking form over (A,S) (M,λ) together with a function

$$\mu \ : \ M \longrightarrow Q_\varepsilon(A,S)$$

such that

$$\begin{aligned} &\text{i)}\ \mu(ax) = a\mu(x)\bar{a} \in Q_\varepsilon(A,S) \\ &\text{ii)}\ \mu(x+y) - \mu(x) - \mu(y) = \lambda(x)(y) + \varepsilon\overline{\lambda(x)(y)} \in Q_\varepsilon(A,S) \\ &\text{iii)}\ q\mu(x) = \lambda(x)(x) \in Q^\varepsilon(A,S) \qquad (x,y\in M,\ a\in A) \ . \end{aligned}$$

Then (M,λ) is an even ε-symmetric linking form over (A,S).

A split ε-quadratic linking form over (A,S) (M,λ,ν) is an ε-symmetric linking form over (A,S) (M,λ) together with a function

$$\nu \ : \ M \longrightarrow Q_\varepsilon(S^{-1}A/A)$$

such that

$$\begin{aligned} &\text{i)}\ \nu(ax) = a\nu(x)\bar{a} \in Q_\varepsilon(S^{-1}A/A) \\ &\text{ii)}\ \nu(x+y) - \nu(x) - \nu(y) = \lambda(x)(y) \in Q_\varepsilon(S^{-1}A/A) \\ &\text{iii)}\ qp\nu(x) = \lambda(x)(x) \in Q^\varepsilon(A,S) \qquad (x,y\in M,\ a\in A) \ . \end{aligned}$$

Then $(M,\lambda,\mu = p\nu : M \longrightarrow Q_\varepsilon(A,S))$ is an ε-quadratic linking form over (A,S).

In Proposition 13.5 below we shall show that every ε-quadratic linking form (M,λ,μ) can be refined (non-uniquely) to a split ε-quadratic linking form (M,λ,ν) such that $\mu = p\nu$, and that if $1/2\in S^{-1}A$ (e.g. if $2\in S$) there is no difference between ε-quadratic and split ε-quadratic linking forms over (A,S).

A morphism of $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking forms over (A,S)

$$\left\{\begin{array}{l} f \ : \ (M,\lambda) \longrightarrow (M',\lambda') \\ f \ : \ (M,\lambda,\mu) \longrightarrow (M',\lambda',\mu') \\ f \ : \ (M,\lambda,\nu) \longrightarrow (M',\lambda',\nu') \end{array}\right.$$

is an A-module morphism $f\in \mathrm{Hom}_A(M,M')$ such that

$$\lambda'(f(x),f(y)) = \lambda(x,y) \in S^{-1}A/A \qquad (x,y \in M)$$

and also

$$\begin{cases} \mu'(f(x)) = \mu(x) \in Q_\varepsilon(A,S) \\ \nu'(f(x)) = \nu(x) \in Q_\varepsilon(S^{-1}A/A) \end{cases} \quad (x \in M) \; .$$

The $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ is <u>non-singular</u> if $\lambda \in \mathrm{Hom}_A(M,M^\wedge)$ is an isomorphism.

In Proposition 13.10 below we shall identify $\begin{cases} L^2(A,S,-\varepsilon) \;\; ((A,S)\ 0\text{-dimensional}) \\ L^0(A,S,\varepsilon) \\ L^{-2}(A,S,-\varepsilon) \\ L_0(A,S,\varepsilon) \end{cases}$

with the Witt group of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking forms over (A,S).

In the first instance, we shall identify the isomorphism classes of linking forms over (A,S) with appropriate equivalence classes of S-acyclic 1-dimensional complexes over A (Proposition 13.4).

Kervaire and Milnor [1] used geometric linking numbers to prove that odd-dimensional surgery obstructions vanish in the simply-connected case. Wall [3] used geometric linking and self-linking numbers to express the odd-dimensional surgery obstructions for a finite fundamental group π in terms of non-singular $\pm$quadratic linking forms over $(\mathbb{Z}[\pi],\mathbb{Z}-\{0\})$. Linking forms were used by Passman and Petrie [1] and Connolly [1] in certain computations of the odd L-groups $L_{2k+1}(\pi)$ of finite groups π. More recently, Pardon [2],[4] has developed a surgery obstruction theory (along the lines of Wall [5]) for normal bundle maps which are rational homotopy equivalences. On the algebraic side, there was obtained in Pardon [2] a localization exact sequence for the surgery

obstruction groups of a finite group π

$$\ldots \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow L_n(\mathbb{Q}[\pi]) \longrightarrow L_n(\mathbb{Z}[\pi],\mathbb{Z}-\{0\}) \longrightarrow L_{n-1}(\mathbb{Z}[\pi]) \longrightarrow \ldots$$

with $\begin{cases} L_{2i}(\mathbb{Z}[\pi],\mathbb{Z}-\{0\}) \\ L_{2i+1}(\mathbb{Z}[\pi],\mathbb{Z}-\{0\}) \end{cases}$ the Witt group of non-singular $(-)^i$quadratic linking $\begin{cases} \text{forms} \\ \text{formations} \end{cases}$ over $(\mathbb{Z}[\pi],\mathbb{Z}-\{0\})$. On the geometric side, Pardon [4] used certain combinations of elementary framed surgeries preserving rational homotopy type ("local surgery on conglomerate Moore spaces") to identify the obstruction to making an $(n-1)$-dimensional normal bundle $(f,b):M \longrightarrow X$ such that $\pi_*(f)\otimes\mathbb{Q} = 0$ a homotopy equivalence $(f',b'):M' \longrightarrow X$ by a normal bordism

$$((g;f,f'),(c;b,b')) : (N;M,M') \longrightarrow (X\times I;X\times 0,X\times 1) \; -$$

such that $\pi_*(g)\otimes\mathbb{Q} = 0$ with an element $\sigma_*^S(f,b) \in L_n(\mathbb{Z}[\pi_1(X)],\mathbb{Z}-\{0\})$ obtained by local surgery below the middle dimension. The S-acyclic quadratic Poincaré cobordism theory developed here provides a unified approach to both the algebra and the geometry. In particular, the local surgery obstruction can be expressed as the S-acyclic cobordism class

$$\sigma_*^S(f,b) = \bar{S}\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)],S) \qquad (S = \mathbb{Z}-\{0\})$$

of the skew-suspension of the quadratic kernel $\sigma_*(f,b) = (C(f^!),\psi_b \in Q_{n-1}(C(f^!)))$, an S-acyclic counterpart to Proposition 8.1. We shall not spell out the connection between our algebra and the geometry of Pardon [4] except to mention that the chain level effect of a local surgery is that of an S-acyclic surgery on an S-acyclic quadratic pair over $\mathbb{Z}[\pi_1(X)]$ $(h:C \longrightarrow D,(\delta\psi,\psi) \in Q_n(h))$ with D an S-acyclic f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex

$$D : \ldots \longrightarrow 0 \longrightarrow D_k \xrightarrow{\;d\;} D_{k-1} \longrightarrow 0 \longrightarrow \ldots \; .$$

Let $T\in\mathbb{Z}_2$ act on the additive groups A, $S^{-1}A$, $S^{-1}A/A$ by $x\longmapsto\varepsilon\bar{x}$ in each case. As in §1 define the $\left\{\begin{array}{l}\mathbb{Z}_2\text{-cohomology}\\ \mathbb{Z}_2\text{-homology}\\ \text{Tate } \mathbb{Z}_2\text{-cohomology}\end{array}\right.$ groups $\left\{\begin{array}{l}H^r(\mathbb{Z}_2;A,\varepsilon)\\ H_r(\mathbb{Z}_2;A,\varepsilon)\\ \hat{H}^r(\mathbb{Z}_2;A,\varepsilon)\end{array}\right.$ $(r\in\mathbb{Z})$.

Similarly define $\left\{\begin{array}{l}H^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\ ,\ H^r(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\\ H_r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\ ,\ H_r(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\\ \hat{H}^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\ ,\ \hat{H}^r(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\end{array}\right.$ $(r\in\mathbb{Z})$. The short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0\longrightarrow A\longrightarrow S^{-1}A\longrightarrow S^{-1}A/A\longrightarrow 0$$

induces a long exact sequence of $\left\{\begin{array}{l}\mathbb{Z}_2\text{-cohomology}\\ \mathbb{Z}_2\text{-homology}\\ \text{Tate } \mathbb{Z}_2\text{-cohomology}\end{array}\right.$ groups

$$\left\{\begin{array}{l}\ldots\longrightarrow H^r(\mathbb{Z}_2;A,\varepsilon)\longrightarrow H^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow H^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\xrightarrow{\delta} H^{r+1}(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\ldots\\ \ldots\longrightarrow H_r(\mathbb{Z}_2;A,\varepsilon)\longrightarrow H_r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow H_r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\xrightarrow{\partial} H_{r-1}(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\ldots\\ \ldots\longrightarrow \hat{H}^r(\mathbb{Z}_2;A,\varepsilon)\longrightarrow \hat{H}^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow \hat{H}^r(\mathbb{Z}_2;S^{-1}A,\varepsilon)\xrightarrow{\hat{\delta}} \hat{H}^{r+1}(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\ldots\end{array}\right.\quad (r\in\mathbb{Z})$$

Define the <u>rth</u> $\left\{\begin{array}{l}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\\ \underline{\varepsilon\text{-hyperquadratic}}\end{array}\right.$ <u>linking Wu class</u> of an S-acyclic (n+1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \varepsilon\text{-hyperquadratic}\end{array}\right.$ complex over A $\left\{\begin{array}{l}(C,\varphi\in Q^{n+1}(C,\varepsilon))\\ (C,\psi\in Q_{n+1}(C,\varepsilon))\\ (C,\theta\in\hat{Q}^{n+1}(C,\varepsilon))\end{array}\right.$ to be the function

$$\left\{\begin{array}{l}v_r^S(\varphi):H^{n-r+1}(C)\longrightarrow H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon);x\longmapsto\frac{1}{s\bar{s}}(\varphi_{n-2r-1}+(-)^{n-r}\varphi_{n-2r}d^*)(y)(\bar{y})\\ v_S^r(\psi):H^{n-r+1}(C)\longrightarrow H_{2r-n}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon);x\longmapsto\frac{1}{s\bar{s}}(\psi_{2r-n+1}+(-)^{n-r}\psi_{2r-n}d^*)(y)(\bar{y})\\ \hat{v}_r^S(\theta):H^{n-r+1}(C)\longrightarrow \hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon);x\longmapsto\frac{1}{s\bar{s}}(\theta_{n-2r-1}+(-)^{n-r}\theta_{n-2r}d^*)(y)(\bar{y})\end{array}\right.$$

$$(x\in C^{n-r+1},\ y\in C^{n-r},\ s\in S,\ sx=d^*y)\ .$$

Motivation: The cohomology classes $x\in H^m(C)$ of an S-acyclic A-module chain complex C are in a natural one-one correspondence with the chain homotopy classes of A-module chain maps $x:C\longrightarrow C_m(A,S)$, where $C_m(A,S)$ is the S-acyclic A-module chain complex defined by

$$C_m(A,S)_i=\left\{\begin{array}{ll}A & i=m\\ S^{-1}A & i=m-1\\ 0 & i\neq m,m-1\end{array}\right.\ ,\quad d:C_m(A,S)_m=A\longrightarrow C_m(A,S)_{m-1}=S^{-1}A\ ;\ a\longmapsto\frac{a}{1}\ .$$

The rth $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \varepsilon\text{-hyperquadratic}\end{array}\right.$ linking Wu class $\left\{\begin{array}{l}v_r^S\\ v_S^r\\ \hat{v}_r^S\end{array}\right.$ of $\left\{\begin{array}{l}(C,\varphi\in Q^{n+1}(C,\varepsilon))\\ (C,\psi\in Q_{n+1}(C,\varepsilon))\\ (C,\theta\in\hat{Q}^{n+1}(C,\varepsilon))\end{array}\right.$ is such that

$$\left\{\begin{array}{l}v_r^S(\varphi)(x)=x^{\%}(\varphi)\in Q^{n+1}(C_{n-r+1}(A,S),\varepsilon)=H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)\\ v_S^r(\psi)(x)=x_{\%}(\psi)\in Q_{n+1}(C_{n-r+1}(A,S),\varepsilon)=H_{2r-n}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)\\ \hat{v}_r^S(\theta)(x)=\hat{x}^{\%}(\theta)\in\hat{Q}^{n+1}(C_{n-r+1}(A,S),\varepsilon)=\hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)\end{array}\right.$$

$$(x:C\longrightarrow C_{n-r+1}(A,S))$$

The $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ linking Wu class $\left\{\begin{array}{l}v_r^S(\varphi)(x)\\ v_S^r(\psi)(x)\end{array}\right.$ is the obstruction to killing $x\in H^{n-r+1}(C)$ $(=H_r(C)$, if $\left\{\begin{array}{l}(C,\varphi)\\ (C,\psi)\end{array}\right.$ is Poincaré) by an S-acyclic $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ surgery on $\left\{\begin{array}{l}(x:C\longrightarrow C_{n-r+1}(A,s),(\delta\varphi,\varphi)\in Q^{n+2}(x,\varepsilon))\\ (x:C\longrightarrow C_{n-r+1}(A,s),(\delta\psi,\psi)\in Q_{n+2}(x,\varepsilon))\end{array}\right.$ for some $s\in S$. Here, $C_m(A,s)$ is the S-acyclic A-module chain complex defined by

$$C_m(A,s)_i=\left\{\begin{array}{ll}A & i=m,m-1\\ 0 & i\neq m,m-1\end{array}\right.\ ,\quad d:C_m(A,s)_m=A\longrightarrow C_m(A,s)_{m-1}=A\ ;\ a\longmapsto as\ .$$

The A-module chain maps $C_m(A,s)\longrightarrow C_m(A,st)$ $(s,t\in S)$ given by

$$C_m(A,s)_i=A\longrightarrow C_m(A,st)_i=A\ ;\ a\longmapsto\left\{\begin{array}{ll}a & i=m\\ at & i=m-1\end{array}\right.$$

define a directed system of S-acyclic A-module chain complexes $\{C_m(A,s)\,|\,s\in S\}$ $(s\leq s'$ if there exists $t\in S$ such that $s'=st\in S)$ with direct limit

$$\varinjlim_{s\in S} C_m(A,s)=C_m(A,S)\ .$$

The linking Wu classes are related to each other by

$$\hat{v}_r^S(J\varphi) : H^{n-r+1}(C) \xrightarrow{v_r^S(\varphi)} H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{J} \hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)$$

$$v_r^S((1+T_\varepsilon)\psi) : H^{n-r+1}(C) \xrightarrow{v_S^r(\psi)} H_{2r-n}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{1+T_\varepsilon} H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)$$

$$v_S^{r-1}(H\theta) : H^{n-r+1}(C) \xrightarrow{\hat{v}_r^S(\theta)} \hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{H} H_{2r-n-1}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)$$

$$(\varphi\in Q^{n+1}(C,\varepsilon),\ \psi\in Q_{n+1}(C,\varepsilon),\ \theta\in\hat{Q}^{n+1}(C,\varepsilon))$$

The linking Wu classes are related to the algebraic Wu classes of §1 by

$$v_r(\varphi) : H^{n-r+1}(C) \xrightarrow{v_r^S(\varphi)} H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{\delta} H^{n-2r+1}(\mathbb{Z}_2;A,(-)^{n-r+1}\varepsilon)$$

$$v^r(\psi) : H^{n-r+1}(C) \xrightarrow{v_S^r(\psi)} H_{2r-n}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{\partial} H_{2r-n-1}(\mathbb{Z}_2;A,(-)^{n-r+1}\varepsilon)$$

$$\hat{v}_r(\theta) : H^{n-r+1}(C) \xrightarrow{\hat{v}_r^S(\theta)} \hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) \xrightarrow{\hat{\delta}} \hat{H}^{n-2r+1}(\mathbb{Z}_2;A,(-)^{n-r+1}\varepsilon)\ .$$

The linking Wu classes satisfy the sum formulae

$$v_r^S(\varphi)(x+y) - v_r^S(\varphi)(x) - v_r^S(\varphi)(y) = \begin{cases} \varphi_0^S(x,y)+\varphi_0^S(y,x) \in H^0(\mathbb{Z}_2;S^{-1}A/A,(-)^{r+1}\varepsilon) & (n=2r) \\ 0 \in H^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) & (n\neq 2r) \end{cases}$$

$$v_S^r(\psi)(x+y) - v_S^r(\psi)(x) - v_S^r(\psi)(y) = \begin{cases} (1+T_\varepsilon)\psi_0^S(x,y) \in H_0(\mathbb{Z}_2;S^{-1}A/A,(-)^{r+1}\varepsilon) & (n=2r) \\ 0 \in H_{2r-n}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon) & (n\neq 2r) \end{cases}$$

$$\hat{v}_r^S(\theta)(x+y) - \hat{v}_r^S(\theta)(x) - \hat{v}_r^S(\theta)(y) = 0 \in \hat{H}^{n-2r}(\mathbb{Z}_2;S^{-1}A/A,(-)^{n-r+1}\varepsilon)$$

$$(x,y\in H^{n-r+1}(C))$$

with $\begin{cases}\varphi_0^S \\ (1+T_\varepsilon)\psi_0^S\end{cases} : H^{r+1}(C)\times H^{r+1}(C) \longrightarrow S^{-1}A/A$ the linking pairing of

$\begin{cases}(C,\varphi\in Q^{2r+1}(C,\varepsilon)) \\ (C,(1+T_\varepsilon)\psi\in Q_{2r+1}(C,\varepsilon))\end{cases}$ $(n=2r)$. Furthermore,

$$v_r^S(\varphi)(x) = \varphi_0^S(x,x) \in H^0(\mathbb{Z}_2;S^{-1}A/A,(-)^{r+1}\varepsilon) \quad (n=2r)\ .$$

<u>Proposition 13.3</u> i) If $\ker(\hat{\delta}:\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0$ there is a natural identification

$L^n(A,S,\varepsilon)$ = (S-acyclic cobordism group of S-acyclic $(n-1)$-dimensional ε-symmetric Poincaré complexes over A) $(n\geq 2)$,

under which $L^n(A,S,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon)$ becomes the forgetful map. In particular, this is the case if $\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon) = 0$, e.g. if $1/2\in S^{-1}A$.

ii) If $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is an isomorphism then the skew-suspension map

$$\bar{S} : L^n(A,S,-\varepsilon)\longrightarrow L^{n+2}(A,S,\varepsilon) \quad (n\geq 0)$$

is an isomorphism. In particular, this is the case if $1/2\in A$.

iii) If $\hat{H}^*(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) = 0$ then the ε-symmetrization map

$$1+T_\varepsilon : L_n(A,S,\varepsilon)\longrightarrow L^n(A,S,\varepsilon) \quad (n\geq 0)$$

is an isomorphism. In particular, this is the case if $1/2\in A$.

<u>Proof</u>: i) By the S-acyclic counterpart of Proposition 7.2 it suffices to show that it is possible to perform S-acyclic surgery on a connected S-acyclic $(n+1)$-dimensional even $(-\varepsilon)$-symmetric complex over A $(C,\varphi\in Q^{n+1}(C,-\varepsilon))$ $(n\geq 2)$ so as to obtain a skew-suspension, killing $H^{n+1}(C)$. As (C,φ) is even we have that for any element $x\in H^{n+1}(C)$

$$\hat{\delta}v_0^S(\varphi)(x) = v_0(\varphi)(x) = 0\in\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)\ ,$$

and so

$$v_0^S(\varphi)(x)\in\ker(\hat{\delta}:\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0\ .$$

Thus $x\in H^{n+1}(C)$ may be represented by an A-module chain map $x:C\longrightarrow C_{n+1}(A,s)$ for some $s\in S$ such that there is defined a connected S-acyclic $(n+2)$-dimensional even $(-\varepsilon)$-symmetric pair over A $(x:C\longrightarrow C_{n+1}(A,s),(\delta\varphi,\varphi)\in Q^{n+2}(x,-\varepsilon))$. S-acyclic surgery on such a pair results in a connected S-acyclic $(n+1)$-dimensional even $(-\varepsilon)$-symmetric complex over A $(C',\varphi'\in Q^{n+1}(C',-\varepsilon))$ with

$$H^{n+1}(C') = H^{n+1}(C)/(x)\ .$$

Now $H^{n+1}(C)$ is a f.g. S-torsion A-module, so that it is possible to kill $H^{n+1}(C)$ by successively killing off a finite set of generators.

ii) Consider the exact sequence of abelian groups

$$\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{\delta}}\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{\delta}}\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)\ .$$

If $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is onto then $\ker(\hat{\delta}:\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon))=0$, and by i) we can identify

$L^{n+2}(A,S,\varepsilon)$ = (S-acyclic cobordism group of S-acyclic $(n+1)$-dimensional ε-symmetric Poincaré complexes over A) $(n\geqslant 0)$.

If $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is one-one then $\mathrm{im}(\hat{\delta}:\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;A,\varepsilon))=0$ and every S-acyclic $(n+1)$-dimensional ε-symmetric complex or pair over A is even. Thus if $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is an isomorphism we can identify

$L^{n+2}(A,S,\varepsilon)$ = (S-acyclic cobordism group of S-acyclic $(n+1)$-dimensional even ε-symmetric Poincaré complexes over A) = $L^n(A,S,-\varepsilon)$ $(n\geqslant 0)$.

iii) If $\hat{H}^*(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)=0$ then $\hat{Q}^*(C,\varepsilon)=0$ and $1+T_\varepsilon:Q_n(C,\varepsilon)\longrightarrow Q^n(C,\varepsilon)$ is an isomorphism for every finite-dimensional S-acyclic A-module chain complex C. []

An $\begin{cases}\text{\underline{ε-symmetric}}\\ \text{\underline{ε-quadratic}}\\ \text{\underline{split ε-quadratic}}\end{cases}$ <u>map</u> (resp. <u>homotopy equivalence</u>) of S-acyclic 1-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes over A

$$\begin{cases} f:(C,\varphi)\longrightarrow(C',\varphi')\\ f:(C,\psi)\longrightarrow(C',\psi')\\ f:(C,\psi)\longrightarrow(C',\psi')\end{cases}$$

is an A-module chain map (resp. chain equivalence)

$$f:C\longrightarrow C'$$

such that

$$\begin{cases} f^{\%}(\varphi)-\varphi'=0\in Q^1(C',\varepsilon)\\ f_{\%}(\psi)-\psi'=H(\theta)\in Q_1(C',\varepsilon)\\ f_{\%}(\psi)-\psi'=H(\theta)\in Q_1(C',\varepsilon)\end{cases}$$

for some Tate $\mathbb{Z}_2$-hypercohomology class $\theta\in\hat{Q}^2(C',\varepsilon)$ such that

$$\begin{cases}\hat{v}_1(\theta)=0:H^1(C')\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)\\ \hat{v}_1^S(\theta)=0:H^1(C')\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\ .\end{cases}$$

An $\begin{cases}\varepsilon\text{-quadratic}\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ map $f:(C,\psi)\longrightarrow(C',\psi')$ determines an $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ map

$$\begin{cases} f:(C,(1+T_\varepsilon)\psi)\longrightarrow(C',(1+T_\varepsilon)\psi')\\ f:(C,\psi)\longrightarrow(C',\psi')\end{cases}$$

since

$$\begin{cases} f^{\%}((1+T_\varepsilon)\psi)-(1+T_\varepsilon)\psi'=(1+T_\varepsilon)H(\theta)=0\in Q^1(C',\varepsilon)\\ \hat{v}_1(\theta):H^1(C')\xrightarrow{\hat{v}_1^S(\theta)=0}\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{\delta}}\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)\ .\end{cases}$$

<u>Proposition 13.4</u> The category of $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S) is naturally equivalent to the opposite of the category of S-acyclic 1-dimensional $\begin{cases}(-\varepsilon)\text{-symmetric}\\ \text{even }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complexes over A and homotopy classes of $\begin{cases}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\\ \text{split }(-\varepsilon)\text{-quadratic}\end{cases}$ maps. Isomorphisms of linking forms are equivalent to homotopy equivalences of complexes. Non-singular linking forms are equivalent to Poincaré complexes.

<u>Proof</u>: The linking pairing of an S-acyclic 1-dimensional $(-\varepsilon)$-symmetric complex over A $(C,\varphi\in Q^1(C,-\varepsilon))$

$$\varphi_0^S : H^1(C)\times H^1(C) \longrightarrow S^{-1}A/A \ ; \ (x,y)\longmapsto \frac{1}{s}\varphi_0(x)(z)$$
$$(x,y\in C^1, z\in C^0, s\in S, d^*z = sy\in C^1)$$

defines an ε-symmetric linking form over (A,S)

$$(M,\lambda) = (H^1(C),\varphi_0^S) \ .$$

The 0th $(-\varepsilon)$-symmetric Wu class of (C,φ) factors as

$$v_0(\varphi) \ : \ H^1(C) \xrightarrow{v_0^S(\varphi)} H^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \xrightarrow{\delta} H^1(\mathbb{Z}_2;A,\varepsilon) \ -$$

and $\ker\delta = Q^\varepsilon(A,S)\subseteq H^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) = Q^\varepsilon(S^{-1}A/A)$, so that the complex (C,φ) is even $(v_0(\varphi) = 0)$ if and only if the linking form (M,λ) is even $(\lambda(x,x) \equiv v_0^S(\varphi)(x)\in Q^\varepsilon(A,S)$ for all $x\in M = H^1(C))$.

The 0th $(-\varepsilon)$-quadratic linking Wu class of an S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complex over A $(C,\psi\in Q_1(C,-\varepsilon))$

$$v_S^0(\psi) \ : \ H^1(C)\longrightarrow H_0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) = Q_\varepsilon(S^{-1}A/A) \ ; \ y\longmapsto \frac{1}{ss}(\psi_1+\psi_0 d^*)(z)(z)$$
$$(y\in C^1, z\in C^0, s\in S, d^*z = sy\in C^1)$$

defines a split ε-quadratic linking form over (A,S)

$$(M,\lambda,\nu) = (H^1(C),(1+T_{-\varepsilon})\psi_0^S, v_S^0(\psi))$$

with associated ε-quadratic linking form over (A,S)

$$(M,\lambda,\mu) = (H^1(C),(1+T_{-\varepsilon})\psi_0^S, pv_S^0(\psi):H^1(C)\longrightarrow Q_\varepsilon(A,S)) \ .$$

A $(-\varepsilon)$-symmetric map of S-acyclic 1-dimensional $(-\varepsilon)$-symmetric complexes over A

$$f \ : \ (C,\varphi)\longrightarrow (C',\varphi')$$

induces contravariantly a morphism of the associated ε-symmetric linking forms over (A,S)

$$f^* \ : \ (H^1(C'),\varphi_0'^S)\longrightarrow (H^1(C),\varphi_0^S) \ .$$

Conversely, every morphism of the associated ε-symmetric linking forms is induced by a $(-\varepsilon)$-symmetric map of complexes.

Let $(C,\psi\in Q_1(C,-\varepsilon)),(C',\psi'\in Q_1(C',-\varepsilon))$ be S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complexes over A. A $(-\varepsilon)$-symmetric map of the $(-\varepsilon)$-symmetrizations

$$f \ : \ (C,(1+T_{-\varepsilon})\psi)\longrightarrow (C',(1+T_{-\varepsilon})\psi')$$

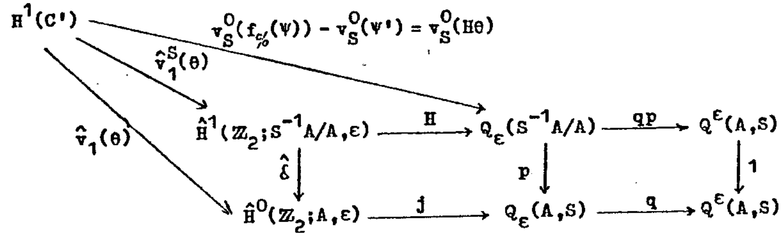
induces contravariantly a morphism of the associated $\begin{cases}\varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S)

$$\begin{cases} f^* \ : \ (H^1(C'),(1+T_{-\varepsilon})\psi_0'^S,pv_S^0(\psi'))\longrightarrow (H^1(C),(1+T_{-\varepsilon})\psi_0^S,pv_S^0(\psi)) \\ f^* \ : \ (H^1(C'),(1+T_{-\varepsilon})\psi_0'^S,v_S^0(\psi'))\longrightarrow (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) \end{cases}$$

if and only if $f:(C,\psi)\longrightarrow (C',\psi')$ is a $\begin{cases}(-\varepsilon)\text{-quadratic}\\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ map, since

$$f_\%(\psi) - \psi' = H(\theta)\in Q_1(C',-\varepsilon)$$

for some $\theta\in \hat{Q}^2(C',-\varepsilon)$ (by the exact sequence of Proposition 1.2) and there is defined a commutative diagram



involving the exact sequences

$$0\longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{H} Q_\varepsilon(S^{-1}A/A)\xrightarrow{qp} Q^\varepsilon(A,S)$$
$$0\longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\xrightarrow{j} Q_\varepsilon(A,S)\xrightarrow{q} Q^\varepsilon(A,S)$$

with

$$H \ : \ \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow Q_\varepsilon(S^{-1}A/A) \ ; \ x\longmapsto x$$
$$j \ : \ \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow Q_\varepsilon(A,S) \ ; \ a\longmapsto \frac{a}{1} \ .$$

Conversely, given an $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking form over (A,S) $\left\{\begin{array}{l}(M,\lambda)\\ (M,\lambda,\mu)\\ (M,\lambda,\nu)\end{array}\right.$ we shall construct an S-acyclic 1-dimensional $\left\{\begin{array}{l}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ complex over A $\left\{\begin{array}{l}(C,\varphi\in Q^1(C,-\varepsilon))\\ (C,\psi\in Q_1(C,-\varepsilon))\\ (C,\psi\in Q_1(C,-\varepsilon))\end{array}\right.$ such that

$$\left\{\begin{array}{l}(H^1(C),\varphi_0^S) = (M,\lambda)\\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,pv_0^S(\psi)) = (M,\lambda,\mu)\\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_0^S(\psi)) = (M,\lambda,\nu)\end{array}\right. ,$$

as follows.

Given an ε-symmetric linking form over (A,S) (M,λ) let

$$0 \longrightarrow C_1 \xrightarrow{\ d\ } C_0 \longrightarrow M^\wedge \longrightarrow 0$$

be a f.g. projective A-module resolution of the S-dual $M^\wedge$. The A-module morphism $\lambda\in \mathrm{Hom}_A(M,M^\wedge)$ can be resolved by a chain map

$$\varphi_0 : C^{1-*} \longrightarrow C$$

such that there is defined a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0 & \xrightarrow{\ d^*\ } & C^1 & \longrightarrow & M & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle \varphi_0} & & \big\downarrow{\scriptstyle -\tilde{\varphi}_0} & & \big\downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{\ d\ } & C_0 & \longrightarrow & M^\wedge & \longrightarrow & 0 \end{array} .$$

We thus have A-module morphisms

$$\varphi_0 : C^0 \longrightarrow C_1 \ ,\ \tilde{\varphi}_0 : C^1 \longrightarrow C_0$$

such that

$$d\varphi_0 + \tilde{\varphi}_0 d^* = 0 : C^0 \longrightarrow C_1$$

and

$$\lambda : M = \mathrm{coker}(d^*:C^0\longrightarrow C^1) \longrightarrow M^\wedge \ ;\ x \longmapsto (y \longmapsto \tfrac{1}{s}\varphi_0(x)(z))$$
$$(x,y\in C^1, z\in C^0, s\in S, d^*z = sy\in C^1) .$$

The relation $T_\varepsilon\lambda = \lambda\in \mathrm{Hom}_A(M,M^\wedge)$ is resolved by a chain homotopy

$$\varphi_1 : T_{-\varepsilon}\varphi_0 \simeq \varphi_0 : C^{1-*} \longrightarrow C \ ,$$

as defined by an A-module morphism

$$\varphi_1 : C^1 \longrightarrow C_1$$

such that

$$\varphi_0 + \varepsilon\tilde{\varphi}_0^* = -\varphi_1 d^* : C^0 \longrightarrow C_1 \ ,\ \tilde{\varphi}_0 + \varepsilon\varphi_0^* = d\varphi_1 : C^1 \longrightarrow C_0 \ .$$

Now

$$d(\varphi_1 + \varepsilon\varphi_1^*) = (\tilde{\varphi}_0 + \varepsilon\varphi_0^*) - \varepsilon(\varphi_0 + \varepsilon\tilde{\varphi}_0^*)^* = 0 : C^1 \longrightarrow C_0$$

and $d\in \mathrm{Hom}_A(C_1,C_0)$ is a monomorphism, so that

$$\varphi_1 + \varepsilon\varphi_1^* = 0 : C^1 \longrightarrow C_1 \ .$$

We have obtained an S-acyclic 1-dimensional $(-\varepsilon)$-symmetric complex over A $(C,\varphi\in Q^1(C,-\varepsilon))$ such that

$$(H^1(C),\varphi_0^S) = (M,\lambda) \ .$$

The chain map $\varphi_0 : C^{1-*} \longrightarrow C$ is a chain equivalence if and only if it induces an A-module isomorphism

$$(\varphi_0)_* = \lambda : H^1(C) = M \longrightarrow H_0(C) = M^\wedge \ ,$$

so that the complex (C,φ) is Poincaré if and only if the linking form (M,λ) is non-singular.

Given an ε-quadratic linking form over (A,S) (M,λ,μ) let

$$0 \longrightarrow C_1 \xrightarrow{\ d\ } C_0 \longrightarrow M^\wedge \longrightarrow 0$$

be a f.g. projective A-module resolution of the S-dual $M^\wedge$ with C_1 a f.g. free A-module. Write the dual resolution for the double S-dual $(M^\wedge)^\wedge = M$ as

$$0 \longrightarrow C^0 \xrightarrow{\ d^*\ } C^1 \xrightarrow{\ e\ } M \longrightarrow 0 \ .$$

Choose a base $\{x_i \mid 1\leq i\leq n\}$ for $C^1 = C_1^*$, and let $\{y_{ij}\in S^{-1}A \mid 1\leq i,j\leq n\}$ be such that

i) $y_{ij} = \varepsilon\bar{y}_{ji} \in S^{-1}A \quad (1\leq i,j\leq n)$

ii) $\lambda(ex_i)(ex_j) = y_{ij} \in S^{-1}A/A \quad (1\leq i<j\leq n)$

iii) $\mu(ex_i) = y_{ii} \in Q_\varepsilon(A,S) \quad (1\leq i\leq n)$.

Give $\mathrm{Hom}_A(C^1,S^{-1}A)$ an A-module structure by

$$A\times \mathrm{Hom}_A(C^1,S^{-1}A)\longrightarrow \mathrm{Hom}_A(C^1,S^{-1}A)\ ;\ (a,f)\longmapsto (x\longmapsto f(x)\bar{a})\ .$$

The A-module morphism

$$\alpha\ :\ C^1\longrightarrow \mathrm{Hom}_A(C^1,S^{-1}A)\ ;\ \sum_{i=1}^{n} a_ix_i\longmapsto (\sum_{j=1}^{n} b_jx_j\longmapsto \sum_{1\leqslant i,j\leqslant n} b_jy_{ij}\bar{a}_i)$$
$$(a_i,b_j\in A)$$

is such that

$$\text{i)}\ \alpha(x)(y)\ =\ \varepsilon\overline{\alpha(y)(x)}\in S^{-1}A/A$$
$$\text{ii)}\ \lambda(ex)(ey)\ =\alpha(x)(y)\in S^{-1}A/A \qquad (x,y\in C^1)$$
$$\text{iii)}\ \mu(ex)\ =\alpha(x)(x)\in Q_\varepsilon(A,S)\qquad .$$

Now

$$\alpha(d^*z)(y)\in A\subseteq S^{-1}A\quad (y\in C^1,z\in C^0)$$
$$\alpha(d^*z)(d^*z)\in \mathrm{im}(1+T_\varepsilon:A\longrightarrow A;a\longmapsto a+\varepsilon\bar{a})\subseteq S^{-1}A\quad (z\in C^0)\quad ,$$

so that there is a well-defined A-module morphism

$$\psi_0\ :\ C^0\longrightarrow (C^1)^*\ =\ C_1\ ;\ z\longmapsto (y\longmapsto \alpha d^*(z)(y))$$

such that

$$d\psi_0\ +\ \psi_1\ +\ \varepsilon\psi_1^*\ =\ 0\ :\ C^0\longrightarrow C_0$$

for some $\psi_1\in \mathrm{Hom}_A(C^0,C_0)$. We have obtained an S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complex over A $(C,\psi\in Q_1(C,-\varepsilon))$ such that

$$(H^1(C),(1+T_{-\varepsilon})\psi_0^S,pv_S^0(\psi))\ =\ (M,\lambda,\mu)\ .$$

Given a split ε-quadratic linking form over (A,S) (M,λ,ν) let $(C,\psi\in Q_1(C,-\varepsilon))$ be the S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complex over A constructed as above, but with $\psi_1\in \mathrm{Hom}_A(C^0,C_0)$ determined by $\nu:M\longrightarrow Q_\varepsilon(S^{-1}A/A)$ as follows. Let $\{z_i\in S^{-1}A\,|\,1\leqslant i\leqslant n\}$ be such that

$$\nu(ex_i)\ =\ z_i\in Q_\varepsilon(S^{-1}A/A)\qquad (1\leqslant i\leqslant n)\ ,$$

and define an A-module morphism

$$\beta:\ C^1\longrightarrow \mathrm{Hom}_A(C^1,S^{-1}A)\ ;\ \sum_{i=1}^{n} a_ix_i\longmapsto (\sum_{j=1}^{n} b_jx_j\longmapsto \sum_{1\leqslant i<j\leqslant n} b_jy_{ij}\bar{a}_i+\sum_{i=1}^{n} b_iz_i\bar{a}_i)$$
$$(a_i,b_j\in A)\qquad .$$

Now

$$\alpha(x)(y)\ =\ \beta(x)(y)+\varepsilon\overline{\beta(y)(x)}\in S^{-1}A \qquad (x,y\in C^1)$$
$$\nu(ex)\ =\ \beta(x)(x)\in Q_\varepsilon(S^{-1}A/A)\qquad ,$$

and there exists an A-module morphism ψ_1 $\mathrm{Hom}_A(C^0,C_0)$ such that

$$\psi_1(z)(z)\ =\ -\beta(d^*z)(d^*z)\in Q_\varepsilon(S^{-1}A)\quad (z\in C^0)$$
$$d\psi_0+\psi_1+\varepsilon\psi_1^*=0\ :\ C^0\longrightarrow C_0\ .$$

The S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complex over A $(C,\psi\in Q_1(C,-\varepsilon))$ is such that

$$(H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi))\ =\ (M,\lambda,\nu)\ .$$

[]

<u>Proposition 13.5</u> i) Every ε-quadratic linking form over (A,S) (M,λ,μ) admits a split ε-quadratic refinement (M,λ,ν) such that

$$\mu\ :\ M\xrightarrow{\ \nu\ } Q_\varepsilon(S^{-1}A/A)\xrightarrow{\ p\ } Q_\varepsilon(A,S)\quad .$$

ii) If A,S,ε are such that

$$\begin{cases}\mathrm{im}(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{\delta}}\hat{H}^1(\mathbb{Z}_2;A,\varepsilon))\ =\ 0\\ \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)\ \text{is an isomorphism}\\ \mathrm{im}(\hat{H}^1(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon))\ =\ 0\end{cases}$$

then the forgetful functor

$$\left(\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}\ \text{linking forms over (A,S)}\right)$$
$$\longrightarrow\left(\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}\ \text{linking forms over (A,S)}\right)$$

is an isomorphism of categories. In particular, this is the case if $1/2\in A$. If $1/2\in S^{-1}A$ (e.g. if $2\in S$) then the forgetful functor

(split ε-quadratic linking forms over (A,S))

$\longrightarrow$ (ε-quadratic linking forms over (A,S))

is an isomorphism of categories.

Proof: i) Immediate from Proposition 13.4.

ii) Let $\widetilde{Q}_\varepsilon(A,S)$ be the subgroup of $Q_\varepsilon(A,S)$ defined by

$$\widetilde{Q}_\varepsilon(A,S) = \{b+\varepsilon\bar{b}\,|\,b\in S^{-1}A\}/\{a+\varepsilon\bar{a}\,|\,a\in A\}\ ,$$

and define abelian group morphisms

$$\widetilde{p} = p| : Q_\varepsilon(S^{-1}A/A) \longrightarrow \widetilde{Q}_\varepsilon(A,S)\ ;\ b \longmapsto b+\varepsilon\bar{b}$$
$$\widetilde{q} = q| : \widetilde{Q}_\varepsilon(A,S) \longrightarrow Q^\varepsilon(A,S)\ ;\ x \longmapsto x\ .$$

By i) we have that for every ε-quadratic linking form over (A,S) (M,λ,μ)

$$\mu(x)\in\widetilde{Q}_\varepsilon(A,S)\subseteq Q_\varepsilon(A,S)\quad(x\in M)\ .$$

The isomorphisms of categories of linking forms may now be deduced from the correspondences of Proposition 13.4 and the exact sequences

$$\begin{cases} 0\longrightarrow Q^\varepsilon(A,S)\xrightarrow{\ r\ } Q^\varepsilon(S^{-1}A/A)\longrightarrow \mathrm{im}(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\ \hat{S}\ }\hat{H}^1(\mathbb{Z}_2;A,\varepsilon))\longrightarrow 0 \\ 0\longrightarrow \ker(\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon))\longrightarrow\widetilde{Q}_\varepsilon(A,S)\xrightarrow{\ \widetilde{q}\ } Q^\varepsilon(A,S) \\ \qquad\longrightarrow \mathrm{coker}(\hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\longrightarrow\hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon))\longrightarrow 0 \\ 0\longrightarrow \mathrm{im}(\hat{H}^1(\mathbb{Z}_2;S^{-1}A,\varepsilon)\longrightarrow\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon))\longrightarrow Q_\varepsilon(S^{-1}A/A)\xrightarrow{\ \widetilde{p}\ }\widetilde{Q}_\varepsilon(A,S)\longrightarrow 0 \end{cases}$$

(which are valid for any A,S,ε).

[]

We have related linking forms over (A,S) to S-acyclic 1-dimensional complexes over A (Proposition 13.4). In §1 we related 1-dimensional complexes over A to formations over A. It is therefore possible to relate linking forms to formations - this was first observed by Wall [1] (in the case $A=\mathbb{Z}$, $S=\mathbb{Z}-\{0\}$).

An <u>S-lagrangian</u> of an $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(K,\alpha\in Q^\varepsilon(K))\\ (K,\beta\in Q_\varepsilon(K))\end{cases}$ is a f.g. projective A-submodule L of K such that $S^{-1}L$ is a lagrangian of the induced $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over $S^{-1}A$ $\begin{cases}S^{-1}(K,\alpha)\\ S^{-1}(K,\beta)\end{cases}$ and is also such that

$$\begin{cases} j^*\alpha j = 0\in Q^\varepsilon(L)\\ j^*\beta j = 0\in Q_\varepsilon(L)\end{cases}\ ,$$

with $j\in\mathrm{Hom}_A(L,K)$ the inclusion.

An $\begin{cases}\underline{\text{(even) }\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\end{cases}$ <u>S-formation over A</u> $\begin{cases}(Q,\varphi;F,G)\\ (Q,\psi;F,G)\end{cases}$ is a non-singular $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(Q,\varphi)\\ (Q,\psi)\end{cases}$ together with a lagrangian F and an S-lagrangian G, such that $S^{-1}F$ and $S^{-1}G$ are complementary lagrangians in $\begin{cases}S^{-1}(Q,\varphi)\\ S^{-1}(Q,\psi)\end{cases}$

$$S^{-1}Q = S^{-1}F\oplus S^{-1}G\ .$$

Then $F\cap G=\{0\}$, and $Q/F+G = \mathrm{coker}(G\longrightarrow Q/F)$ is an h.d.1 S-torsion A-module supporting an $\begin{cases}\text{(even) }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ linking form over (A,S) (as made precise in Proposition 13.6 below). The S-formation $\begin{cases}(Q,\varphi;F,G)\\ (Q,\psi;F,G)\end{cases}$ is <u>non-singular</u> if G is a lagrangian of $\begin{cases}(Q,\varphi)\\ (Q,\psi)\end{cases}$.

An <u>isomorphism</u> of $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ S-formations over A

$$\begin{cases} f : (Q,\varphi;F,G)\longrightarrow(Q',\varphi';F',G')\\ f : (Q,\psi;F,G)\longrightarrow(Q',\psi';F',G')\end{cases}$$

is an isomorphism of $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms over A

$$\begin{cases} f : (Q,\varphi)\longrightarrow(Q',\varphi')\\ f : (Q,\psi)\longrightarrow(Q',\psi')\end{cases}$$

such that

$$f(F) = F'\ ,\ f(G) = G'\ .$$

A <u>stable isomorphism</u> of $\begin{cases}\text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ S-formations over A

$$\begin{cases}[f] : (Q,\varphi;F,G) \longrightarrow (Q',\varphi';F',G') \\ [f] : (Q,\psi;F,G) \longrightarrow (Q',\psi';F',G')\end{cases}$$

is an isomorphism of the type

$$\begin{cases} f : (Q,\varphi;F,G)\oplus(H^{\varepsilon}(P);P,P^*) \longrightarrow (Q',\varphi';F',G')\oplus(H^{\varepsilon}(P');P',P'^*) \\ f : (Q,\psi;F,G)\oplus(H_{\varepsilon}(P);P,P^*) \longrightarrow (Q',\psi';F',G')\oplus(H_{\varepsilon}(P');P',P'^*)\end{cases}$$

for some f.g. projective A-modules P,P'.

A <u>split ε-quadratic S-formation over A</u> $(F,G) \equiv (F,(\binom{\gamma}{\mu},\theta)G)$ is an ε-quadratic S-formation over A $(H_{\varepsilon}(F);F,G)$, with $\binom{\gamma}{\mu} : G \longrightarrow F\oplus F^*$ the inclusion, together with a $(-\varepsilon)$-quadratic form over A $(G,\theta\in Q_{-\varepsilon}(G))$ such that

$$\gamma^*\mu = \theta - \varepsilon\theta^* \in \mathrm{Hom}_A(G,G^*) \ .$$

Then $\mu\in\mathrm{Hom}_A(G,F^*)$ is an S-isomorphism, and $F\oplus F^*/F+G = \mathrm{coker}(\mu:G\longrightarrow F^*)$ is an h.d.1 S-torsion A-module supporting a split $(-\varepsilon)$-quadratic linking form over (A,S) (as made precise in Proposition 13.6 below). Call (F,G) <u>non-singular</u> if $(H_{\varepsilon}(F);F,G)$ is non-singular, that is if the sequence

$$0 \longrightarrow G \xrightarrow{\binom{\gamma}{\mu}} F\oplus F^* \xrightarrow{(\varepsilon\mu^*\ \gamma^*)} G^* \longrightarrow 0$$

is exact.

An <u>isomorphism</u> of split ε-quadratic S-formations over A

$$(\alpha,\beta,\psi) : (F,(\binom{\gamma}{\mu},\theta)G) \longrightarrow (F',(\binom{\gamma'}{\mu'},\theta')G')$$

is a triple

$$(\alpha\in\mathrm{Hom}_A(F,F'),\ \beta\in\mathrm{Hom}_A(G,G'),\ \psi\in Q_{-\varepsilon}(F^*))$$

such that α and β are isomorphisms, satisfying

i) $\mu'\beta = \alpha^{*-1}\mu \in \mathrm{Hom}_A(G,F'^*)$

ii) $\gamma'\beta = \alpha\gamma + \alpha(\psi-\varepsilon\psi^*)^*\mu\in\mathrm{Hom}_A(G,F')$

iii) $\beta^*\theta'\beta - \theta - \mu^*\psi\mu\in\ker(S^{-1}:Q_{-\varepsilon}(G)\longrightarrow Q_{-\varepsilon}(S^{-1}G))$.

A <u>stable isomorphism</u> of split ε-quadratic S-formations over A

$$[\alpha,\beta,\psi] : (F,G) \longrightarrow (F',G')$$

is an isomorphism of the type

$$(\alpha,\beta,\psi) : (F,G)\oplus(P,P^*) \longrightarrow (F',G')\oplus(P',P'^*)$$

for some f.g. projective A-modules P,P' with $(P,P^*) = (P,(\binom{0}{1},0)P^*)$.

<u>Proposition 13.6</u> The isomorphism classes of $\begin{cases}\text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S) are in a natural one-one correspondence with the stable isomorphism classes of $\begin{cases}\text{(even) } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ S-formations over A. Non-singular linking forms correspond to non-singular S-formations.

<u>Proof</u>: The isomorphism classes of (non-singular) $\begin{cases}\text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S) are in a natural one-one correspondence with the $\begin{cases}(-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ homotopy equivalence classes of S-acyclic 1-dimensional $\begin{cases}\text{(even) } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ (-\varepsilon)\text{-quadratic}\end{cases}$ (Poincaré) complexes over A (by Proposition 13.5), which in turn are in a natural one-one correspondence with the stable isomorphism classes of (non-singular) $\begin{cases}\text{(even) } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ S-formations over A (by a straightforward modification of the proofs of Propositions 1.7,1.8)

Explicitly, a (non-singular) $\begin{cases} \text{(even) } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic} \end{cases}$ S-formation over A

$\begin{cases} (Q,\varphi;F,G) \\ (Q,\psi;F,G) \\ (F,(\binom{\gamma}{\mu}),\theta)G) \end{cases}$ determines a (non-singular) $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form

over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ by

$$\begin{cases} \begin{cases} \lambda : M = Q/F+G \longrightarrow M^\wedge \ ; \ x \longmapsto (y \longmapsto \frac{1}{s}\varphi(x)(g)) \\ \qquad (x,y\in Q,\ g\in G,\ s\in S,\ sy-g\in F) \end{cases} \\ \begin{cases} \lambda : M = Q/F+G \longrightarrow M^\wedge \ ; \ x \longmapsto (y \longmapsto \frac{1}{s}(\psi-\varepsilon\psi^*)(x)(g)) \\ \mu : M = Q/F+G \longrightarrow Q_\varepsilon(A,S) \ ; \ y \longmapsto \frac{1}{s}(\psi-\varepsilon\psi^*)(y)(g) - \psi(y)(y) \\ \qquad (x,y\in Q,\ g\in G,\ s\in S,\ sy-g\in F) \end{cases} \\ \begin{cases} \lambda : M = \mathrm{coker}(\mu:G \longrightarrow F^*) \longrightarrow M^\wedge \ ; \ x \longmapsto (y \longmapsto \frac{1}{s}\gamma^*(x)(g)) \\ \nu : M = \mathrm{coker}(\mu:G \longrightarrow F^*) \longrightarrow Q_\varepsilon(S^{-1}A/A) \ ; \ y \longmapsto \frac{1}{s\bar{s}}\theta(g)(g) \\ \qquad (x,y\in F^*,\ g\in G,\ s\in S,\ sy=\mu g\in F^*)\ . \end{cases} \end{cases}$$

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An $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over A $\begin{cases} (K,\alpha\in Q^\varepsilon(K)) \\ (K,\beta\in Q_\varepsilon(K)) \end{cases}$ is <u>non-degenerate</u> if

$\begin{cases} S^{-1}(K,\alpha) \\ S^{-1}(K,\beta) \end{cases}$ is a non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over $S^{-1}A$, that is if

$\begin{cases} \alpha\in \mathrm{Hom}_A(K,K^*) \\ (\beta+\varepsilon\beta^*)\in \mathrm{Hom}_A(K,K^*) \end{cases}$ is an S-isomorphism.

We shall now describe the non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking

forms over (A,S) representing 0 in $\begin{cases} L^0(A,S,\varepsilon) \\ L_0(A,S,\varepsilon) \end{cases}$ (under the correspondence of

Proposition 13.4) in terms of non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A.

The <u>boundary</u> of a non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric form over A} \\ \varepsilon\text{-quadratic} \end{cases}$

$\begin{cases} (K,\alpha\in Q^\varepsilon(K)) \\ (K,\alpha\in \mathrm{im}(1+T_\varepsilon:Q_\varepsilon(K)\longrightarrow Q^\varepsilon(K))) \\ (K,\beta\in Q_\varepsilon(K)) \end{cases}$ is the non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$

linking form over (A,S)

$$\begin{cases} \partial(K,\alpha) = (\partial K,\lambda) \\ \partial(K,\alpha) = (\partial K,\lambda,\mu) \\ \partial(K,\beta) = (\partial K,\lambda,\nu) \end{cases}$$

defined by

$$\lambda : \partial K = \mathrm{coker}(\alpha:K \longrightarrow K^*) \longrightarrow \partial K^\wedge \ ; \ [x] \longmapsto ([y] \longmapsto \frac{1}{s}x(z))$$

$$\mu : \partial K = \mathrm{coker}(\alpha:K \longrightarrow K^*) \longrightarrow Q_\varepsilon(A,S) \ ; \ [y] \longmapsto \frac{1}{s}y(z)$$

$$\nu : \partial K = \mathrm{coker}(\alpha:K \longrightarrow K^*) \longrightarrow Q_\varepsilon(S^{-1}A/A) \ ; \ [y] \longmapsto \frac{1}{s\bar{s}}\beta(z)(z)$$

$(x,y\in K^*,\ z\in K,\ s\in S,\ \alpha(z) = sy\in K^*, \alpha = \beta+\varepsilon\beta^*$ in the ε-quadratic case).

The boundary linking form corresponds to the boundary $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic} \end{cases}$

S-formation over A

$$\begin{cases} \partial(K,\alpha) = (H^{-\varepsilon}(K);K,\Gamma_{(K,\alpha)})\ , \quad \Gamma_{(K,\alpha)} = \{(x,\alpha(x))\in K\oplus K^* \mid x\in K\} \\ \partial(K,\alpha) = (H_{-\varepsilon}(K);K,\Gamma_{(K,\alpha)}) \\ \partial(K,\beta) = (K,(\binom{1}{\beta+\varepsilon\beta^*}),\beta)K) \end{cases}$$

(under the correspondence of Proposition 13.6). The boundary operation on non-degenerate forms over A is thus seen to be a special case of the boundary operation $\partial : (\text{forms}) \longrightarrow (\text{formations})$ of §5. (The boundary operation $\partial : (\text{non-degenerate forms}) \longrightarrow (\text{linking forms})$ can also be expressed in terms of the "dual lattice" construction familiar in the classical theory of quadratic forms, particularly the case $A = \mathbb{Z}$, $S = \mathbb{Z}-\{0\}$, $S^{-1}A = \mathbb{Q}$.

A <u>lattice</u> of a non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over $S^{-1}A$ $\begin{cases} (Q,\varphi) \\ (Q,\varphi) \\ (Q,\psi) \end{cases}$ is a

non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over A $\begin{cases} (K,\alpha) \\ (K,\alpha) \\ (K,\beta) \end{cases}$ defined on a f.g. projective A-submodule K of Q, such that the inclusion $K \longrightarrow Q$ extends to an isomorphism of non-singular forms over $S^{-1}A$

$$\begin{cases} S^{-1}(K,\alpha) \longrightarrow (Q,\varphi) \\ S^{-1}(K,\alpha) \longrightarrow (Q,\varphi) \\ S^{-1}(K,\beta) \longrightarrow (Q,\psi)\ . \end{cases}$$

A non-singular form over $S^{-1}A$ (Q,φ) admits a lattice if and only if Q is isomorphic to $S^{-1}K$ for some f.g. projective A-module K. The <u>dual lattice</u> of a lattice (K,α) in a non-singular form over $S^{-1}A$ (Q,φ) is the A-submodule

$$K^{\#} = \{x \in Q \mid \varphi(x)(K) \subseteq A \subseteq S^{-1}A\} \subseteq Q$$

(with $\varphi = \psi+\varepsilon\psi^*$, $\alpha = \beta+\varepsilon\beta^*$ in the ε-quadratic case). Define a non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (K^{\#}/K,\lambda) \\ (K^{\#}/K,\lambda,\mu) \\ (K^{\#}/K,\lambda,\nu) \end{cases}$ by

$$\begin{aligned} \lambda &: K^{\#}/K \longrightarrow (K^{\#}/K)^{\wedge}\ ;\ [x] \longmapsto ([y] \longmapsto \varphi(x)(y)) \\ \mu &: K^{\#}/K \longrightarrow Q_{\varepsilon}(A,S)\ ;\ [x] \longmapsto \varphi(x)(x) \qquad (x,y \in K^{\#}) \\ \nu &: K^{\#}/K \longrightarrow Q_{\varepsilon}(S^{-1}A/A)\ ;\ [x] \longmapsto \psi(x)(x) \end{aligned}$$

The A-module isomorphism

$$K^{\#} \longrightarrow K^*\ ;\ x \longmapsto (y \longmapsto \varphi(x)(y)) \qquad (x \in K^{\#}, y \in K)$$

sends $K \subseteq K^{\#}$ to $\alpha(K) \subseteq K^*$, so there is induced an A-module isomorphism

$$K^{\#}/K \longrightarrow \partial K = \text{coker}(\alpha : K \longrightarrow K^*)\ .$$

The A-module isomorphism $K^{\#}/K \longrightarrow \partial K$ defines an isomorphism of non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking forms over (A,S)

$$\left(\begin{cases} (K^{\#}/K,\lambda) \longrightarrow \partial(K,\alpha) \\ (K^{\#}/K,\lambda,\mu) \longrightarrow \partial(K,\alpha) \\ (K^{\#}/K,\lambda,\nu) \longrightarrow \partial(K,\beta) \end{cases}\right).$$

A non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over A $\begin{cases} (K,\alpha) \\ (K,\beta) \end{cases}$ is $\begin{cases} \underline{S\text{-metabolic}} \\ \underline{S\text{-hyperbolic}} \end{cases}$ if it admits an S-lagrangian, or equivalently if $\begin{cases} S^{-1}(K,\alpha) \\ S^{-1}(K,\beta) \end{cases}$ is a $\begin{cases} \text{metabolic} \\ \text{hyperbolic} \end{cases}$ $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over $S^{-1}A$ with a lagrangian isomorphic to $S^{-1}L$ for some f.g. projective A-module L.

<u>Proposition 13.7</u> Let $\begin{cases} (C,\varphi \in Q^1(C,-\varepsilon)) \\ (C,\psi \in Q_1(C,-\varepsilon)) \end{cases}$ be an S-acyclic 1-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ Poincaré complex over A, with associated non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) = (H^1(C),\varphi_0^S) \\ (M,\lambda,\nu) = (H^1(C),(1+T_{-\varepsilon})\psi_0^S, v_S^0(\psi))\ . \end{cases}$

i) The S-acyclic cobordism class $\begin{cases} (C,\varphi) \in L^0(A,S,\varepsilon) \\ (C,\psi) \in L_0(A,S,\varepsilon) \end{cases}$ depends only on the isomorphism class of $\begin{cases} (M,\lambda) \\ (M,\lambda,\nu) \end{cases}$.

ii) $\begin{cases} (C,\varphi) = 0 \in L^0(A,S,\varepsilon) \\ (C,\psi) = 0 \in L_0(A,S,\varepsilon) \end{cases}$ if and only if $\begin{cases} (M,\lambda) \\ (M,\lambda,\nu) \end{cases}$ is isomorphic to the boundary $\begin{cases} \partial(K,\alpha) \\ \partial(K,\beta) \end{cases}$ of an $\begin{cases} S\text{-metabolic} \\ S\text{-hyperbolic} \end{cases}$ non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over A $\begin{cases} (K,\alpha) \\ (K,\beta) \end{cases}$.

<u>Proof</u>: An S-acyclic 1-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré complex over A $\begin{cases}(C,\varphi\in Q^1(C,-\varepsilon))\\ (C,\psi\in Q_1(C,-\varepsilon))\end{cases}$ represents 0 in $\begin{cases}L^0(A,S,\varepsilon)\\ L_0(A,S,\varepsilon)\end{cases}$ if and only if it is homotopy equivalent to the boundary $\begin{cases}\partial(D,\eta)\\ \partial(D,\xi)\end{cases}$ of a connected S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D,\eta\in Q^2(D,-\varepsilon))\\ (D,\xi\in Q_2(D,-\varepsilon))\end{cases}$ with D a f.g. projective A-module chain complex of the type

$$D : \ldots \longrightarrow 0 \longrightarrow D_2 \xrightarrow{d} D_1 \xrightarrow{d} D_0 \longrightarrow 0 \longrightarrow \ldots ,$$

by the S-acyclic counterpart of Proposition 5.4 iii).

Let $\begin{cases}(C,\varphi) = \partial(D,\eta)\\ (C,\psi) = \partial(D,\xi)\end{cases}$ be an S-acyclic boundary, as above. The associated $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form is the boundary

$$\begin{cases}(H^1(C),\varphi_0^S) = \partial(K,\alpha)\\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) = \partial(K,\beta)\end{cases}$$

of the non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A

$$\begin{cases}(K,\alpha) = (\mathrm{coker}(\begin{pmatrix}d^*\\ \eta_0\end{pmatrix} : D^0 \longrightarrow D^1\oplus D_2), \begin{bmatrix}\eta_0+d\eta_1 & d\\ \varepsilon d^* & 0\end{bmatrix}\in Q^\varepsilon(K))\\ (K,\beta) = (\mathrm{coker}(\begin{pmatrix}d^*\\ (1+T_{-\varepsilon})\xi_0\end{pmatrix}) : D^0 \longrightarrow D^1\oplus D_2), \begin{bmatrix}\xi_0 & d\\ 0 & 0\end{bmatrix}\in Q_\varepsilon(K))\end{cases}$$

(which is obtained from $\begin{cases}(D,\eta)\\ (D,\xi)\end{cases}$ by surgery on $H^2(D)$). Moreover, the morphism of $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms over A

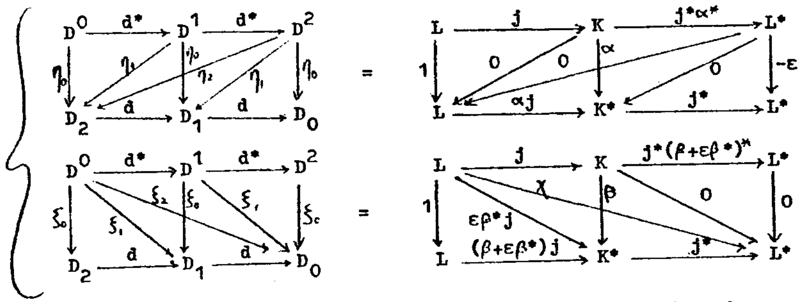
$$\begin{cases}\begin{bmatrix}0\\ 1\end{bmatrix} : (D_2,0) \longrightarrow (K,\alpha)\\ \begin{bmatrix}0\\ 1\end{bmatrix} : (D_2,0) \longrightarrow (K,\beta)\end{cases}$$

is the inclusion of an S-lagrangian in $\begin{cases}(K,\alpha)\\ (K,\beta)\end{cases}$.

Conversely, let $\begin{cases}(K,\alpha)\\ (K,\beta)\end{cases}$ be an $\begin{cases}\text{S-metabolic}\\ \text{S-hyperbolic}\end{cases}$ non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A, and let

$$\begin{cases}j : (L,0) \longrightarrow (K,\alpha)\\ j : (L,0) \longrightarrow (K,\beta)\end{cases}$$

be the inclusion of an S-lagrangian. Define a connected S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D,\eta\in Q^2(D,-\varepsilon))\\ (D,\xi\in Q_2(D,-\varepsilon))\end{cases}$ by



for any $\beta\in \mathrm{Hom}_A(K,K^*)$ representing $\beta\in Q_\varepsilon(K)$, and any $\chi\in \mathrm{Hom}_A(L,L^*)$ such that

$$j^*\beta j = \chi - \varepsilon\chi^* \in \mathrm{Hom}_A(L,L^*) .$$

The boundary $\begin{cases}\partial(K,\alpha)\\ \partial(K,\beta)\end{cases}$ is the non-singular $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form associated to the boundary S-acyclic 1-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$

Poincaré complex over A $\begin{cases}(C,\varphi) = \partial(D,\eta) \\ (C,\psi) = \partial(D,\xi)\end{cases}$,

$$\begin{cases}\partial(K,\alpha) = (H^1(C),\eta_0^S) \\ \partial(K,\beta) = (H^1(C),(1+T_{-\varepsilon})\xi_0^S,v_S^0(\xi))\ .\end{cases}$$

It remains to show that if $\begin{cases}(C,\varphi),(C',\varphi') \\ (C,\psi),(C',\psi')\end{cases}$ are S-acyclic 1-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré complexes over A such that there exists an isomorphism of the associated non-singular $\begin{cases}\text{even } \varepsilon\text{-symmetric} \\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S)

$$\begin{cases}(H^1(C),\varphi_0^S) \longrightarrow (H^1(C'),\varphi_0'^S) \\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) \longrightarrow (H^1(C'),(1+T_{-\varepsilon})\psi_0'^S,v_S^0(\psi'))\end{cases}$$

then

$$\begin{cases}(C,\varphi) = (C',\varphi') \in L^0(A,S,\varepsilon) \\ (C,\psi) = (C',\psi') \in L_0(A,S,\varepsilon)\ .\end{cases}$$

Given such an isomorphism we have (by Proposition 13.3) a $\begin{cases}(-\varepsilon)\text{-symmetric} \\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ homotopy equivalence

$$\begin{cases}f : (C,\varphi) \longrightarrow (C',\varphi') \\ f : (C,\psi) \longrightarrow (C',\psi')\ ,\end{cases}$$

with

$$\begin{cases}f^{\%}(\varphi) = \varphi' \in Q^1(C',-\varepsilon) \\ f_{\%}(\psi) = \psi' + H(\theta) \in Q_1(C',-\varepsilon)\end{cases}$$

for some $\theta \in \hat{Q}^2(C',-\varepsilon)$ such that

$$\hat{v}_1^S(\theta) = 0 : H^1(C') \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\ .$$

There is defined an S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré pair over A

$$\begin{cases}((f\ 1) : C\oplus C' \longrightarrow C', (\delta\varphi, \varphi\oplus -\varphi') \in Q^2((f\ 1),-\varepsilon)) \\ ((f\ 1) : C\oplus C' \longrightarrow C', (\delta\psi, \psi\oplus -(\psi'+H(\theta)) \in Q_2((f\ 1),-\varepsilon))\end{cases}$$

so that

$$\begin{cases}(C,\varphi) = (C',\varphi') \in L^0(A,S,\varepsilon) \\ (C,\psi) = (C',\psi'+H(\theta)) \in L_0(A,S,\varepsilon)\ .\end{cases}$$

We shall prove that $(C',\psi'+H(\theta)) = (C',\psi') \in L_0(A,S,\varepsilon)$ using the language of S-formations (Proposition 13.6), as follows.

Given non-singular split $(-\varepsilon)$-quadratic S-formations over A $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G),(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta')G)$ such that

$$\theta' - \theta \in \ker(S^{-1} : Q_\varepsilon(G) \longrightarrow Q_\varepsilon(S^{-1}G))$$

we have to show that the non-singular split $(-\varepsilon)$-quadratic formation over A $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)\oplus(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},-\theta')G)$ is stably isomorphic to the boundary $\partial(K,\beta) = (K,(\begin{pmatrix}1\\ \beta+\varepsilon\beta^*\end{pmatrix},\beta)K)$ of an S-hyperbolic non-degenerate ε-quadratic form over A $(K,\beta\in Q_\varepsilon(K))$. Extend the inclusion of the lagrangian

$$\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : (G,0) \longrightarrow H_{-\varepsilon}(F)$$

to an isomorphism of hyperbolic $(-\varepsilon)$-quadratic forms over A

$$\begin{pmatrix}\gamma & \tilde{\gamma}\\ \mu & \tilde{\mu}\end{pmatrix} : H_{-\varepsilon}(G) = (G\oplus G^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}) \longrightarrow H_{-\varepsilon}(F) = (F\oplus F^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$$

(using Proposition 1.5). Define a non-degenerate ε-quadratic form over A

$$(K,\beta) = (G\oplus F,\begin{pmatrix}-\theta & 0\\ \mu & 0\end{pmatrix}+\begin{pmatrix}-\gamma\tilde{\mu}^* & -\mu\tilde{\gamma}^*\\ \tilde{\mu} & \tilde{\mu}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \theta'-\theta\end{pmatrix}\begin{pmatrix}-\tilde{\mu}^*\gamma & \tilde{\mu}^*\\ -\varepsilon\tilde{\gamma}^*\mu & \tilde{\mu}^*\end{pmatrix}\in Q_\varepsilon(G\oplus F))\ .$$

For some $s\in S$ there is defined a morphism of ε-quadratic forms over A

$$\begin{pmatrix}0\\ s\end{pmatrix} : (F,0) \longrightarrow (K,\beta) = (G\oplus F,\beta)\ ,$$

which is the inclusion of an S-lagrangian so that (K,β) is S-hyperbolic.

The isomorphism of non-singular split $(-\varepsilon)$-quadratic formations over A

$$(a,b,c) = \left(\begin{pmatrix} \gamma & -1 & -\varepsilon\tilde{\gamma} & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & \tilde{\mu} & 1 \\ \mu & 0 & \tilde{\mu} & 0 \end{pmatrix}, \begin{pmatrix} -\tilde{\mu}^*\gamma & \tilde{\mu}^* & 1 & 0 \\ -\varepsilon\tilde{\gamma}^*\mu & \tilde{\mu}^* & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma\tilde{\gamma}^*\mu & \varepsilon\tilde{\gamma}\mu^* & \gamma & -1 \end{pmatrix}, \begin{pmatrix} 0 & \tilde{\gamma}^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma^* & \theta'-\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

$$: \partial(G\oplus F,\beta)\oplus(G^*\oplus F^*,G\oplus F) \longrightarrow (F,(\begin{pmatrix} -\gamma \\ \mu \end{pmatrix},-\theta)G)\oplus(F,(\begin{pmatrix} \gamma \\ \mu \end{pmatrix},\theta')G)\oplus(F^*\oplus F^*,F\oplus F)$$

defines a stable isomorphism

$$[a,b,c] : \partial(K,\beta) \longrightarrow (F,(\begin{pmatrix} -\gamma \\ \mu \end{pmatrix},-\theta)G)\oplus(F,(\begin{pmatrix} \gamma \\ \mu \end{pmatrix},\theta')G) .$$

It follows that the S-acyclic 1-dimensional $(-\varepsilon)$-quadratic Poincaré complexes over A $(C,\psi),(C,\psi')$ associated to $(F,(\begin{pmatrix} \gamma \\ \mu \end{pmatrix},\theta)G),(F,(\begin{pmatrix} \gamma \\ \mu \end{pmatrix},\theta')G)$ are S-acyclic cobordant

$$(C,\psi) = (C,\psi') \in L_0(A,S,\varepsilon) .$$

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A <u>sublagrangian</u> of a non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$ is a submodule L of M such that

i) L and M/L are h.d. 1 S-torsion A-modules

ii) the inclusion $j \in \mathrm{Hom}_A(L,M)$ defines a morphism of $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadrati} \end{cases}$ linking forms over (A,S)

$$\begin{cases} j : (L,0) \longrightarrow (M,\lambda) \\ j : (L,0,0) \longrightarrow (M,\lambda,\mu) \\ j : (L,0,0) \longrightarrow (M,\lambda,\nu) \end{cases}$$

iii) the A-module morphism

$$[\lambda] : M/L \longrightarrow L^\wedge \ ; \ [x] \longmapsto (y \longmapsto \lambda(x)(y))$$

is onto.

The <u>annihilator</u> of a sublagrangian L is the submodule

$$L^\perp = \ker(j^\wedge\lambda : M \longrightarrow L^\wedge \ ; \ x \longmapsto (y \longmapsto \lambda(x)(y)))$$

of M, such that $L \subseteq L^\perp$ and $L^\perp, L^\perp/L$ are h.d.1 S-torsion A-modules with

$$L^\perp/L = \ker([\lambda] : M/L \longrightarrow L^\wedge) .$$

A <u>lagrangian</u> is a sublagrangian L such that $[\lambda] \in \mathrm{Hom}_A(M/L,L^\wedge)$ is an isomorphism, that is

$$L^\perp = L .$$

A non-singular $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \text{(split) } \varepsilon\text{-quadratic} \end{cases}$ linking form is $\begin{cases} \underline{\text{(even) metabolic}} \\ \underline{\text{(split) hyperbolic}} \end{cases}$ if it admits a lagrangian.

<u>Proposition 13.8</u> i) Given a sublagrangian L of a non-singular

$$\left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right. \text{ linking form over } (A,S) \left\{\begin{array}{l}(M,\lambda)\\ (M,\lambda,\mu)\\ (M,\lambda,\nu)\end{array}\right. \text{ there is defined a}$$

$$\text{non-singular} \left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right. \text{ linking form over } (A,S) \left\{\begin{array}{l}(L^{\perp}/L,\lambda^{\perp}/\lambda)\\ (L^{\perp}/L,\lambda^{\perp}/\lambda,\mu|)\\ (L^{\perp}/L,\lambda^{\perp}/\lambda,\nu|)\end{array}\right.$$

$$\text{such that} \left\{\begin{array}{l}(M,\lambda)\oplus(L^{\perp}/L,-\lambda^{\perp}/\lambda)\\ (M,\lambda,\mu)\oplus(L^{\perp}/L,-\lambda^{\perp}/\lambda,-\mu|)\\ (M,\lambda,\nu)\oplus(L^{\perp}/L,-\lambda^{\perp}/\lambda,-\nu|)\end{array}\right. \text{ is } \left\{\begin{array}{l}\text{(even) metabolic}\\ \text{hyperbolic}\\ \text{split hyperbolic}\end{array}\right. \text{, with lagrangian}$$

$$\Delta = \{(x,[x]) \in M\oplus L^{\perp}/L \,|\, x\in L^{\perp}\}\ .$$

ii) A non-singular $\left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \text{(split) } \varepsilon\text{-quadratic}\end{array}\right.$ linking form over (A,S) is $\left\{\begin{array}{l}\text{(even) metabolic}\\ \text{(split) hyperbolic}\end{array}\right.$ if and only if the associated $\left\{\begin{array}{l}(-\varepsilon)\text{-symmetric}\\ \text{(split) } (-\varepsilon)\text{-quadratic}\end{array}\right.$ homotopy equivalence class of S-acyclic 1-dimensional $\left\{\begin{array}{l}\text{(even) } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ Poincaré complexes over A contains the boundary $\left\{\begin{array}{l}(C,\varphi\in Q^1(C,-\varepsilon))\\ (C,\psi\in Q_1(C,-\varepsilon))\end{array}\right.$ of an S-acyclic 2-dimensional $\left\{\begin{array}{l}\text{(even) } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ Poincaré pair over A $\left\{\begin{array}{l}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^2(f,-\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_2(f,-\varepsilon))\end{array}\right.$ with $H_2(D) = 0$.

<u>Proof</u>: ii) Let $\left\{\begin{array}{l}(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^2(f,-\varepsilon))\\ (f:C\longrightarrow D,(\delta\psi,\psi)\in Q_2(f,-\varepsilon))\end{array}\right.$ be an S-acyclic 2-dimensional $\left\{\begin{array}{l}\text{(even) } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ Poincaré pair over A such that $H_2(D) = 0$.

The non-singular $\left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking form over (A,S) $\left\{\begin{array}{l}(H^1(C),\varphi_0^S)\\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi))\end{array}\right.$ associated to the boundary $\left\{\begin{array}{l}(C,\varphi\in Q^1(C,-\varepsilon))\\ (C,\psi\in Q_1(C,-\varepsilon))\end{array}\right.$ is $\left\{\begin{array}{l}\text{(even) metabolic}\\ \text{split hyperbolic}\end{array}\right.$, with lagrangian

$$L = \mathrm{im}(f^*:H^1(D)\longrightarrow H^1(C))\subseteq H^1(C)\ .$$

The correspondence of Proposition 13.4 associates to a metabolic (even) ε-symmetric linking form over (A,S) (M,λ) with lagrangian L a map of S-acyclic 1-dimensional (even) $(-\varepsilon)$-symmetric complexes over A

$$f : (C,\varphi) \longrightarrow (D,0)\ ,$$

with $f:C\longrightarrow D$ a chain map of f.g. projective A-module chain complexes

$$\begin{array}{ccccccccccc} C: & \ldots & \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{d} & C_0 & \longrightarrow & 0 & \longrightarrow \ldots\\ \downarrow f & & & & & \downarrow f & & \downarrow \tilde{f} & & & \\ D: & \ldots & \longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{\tilde{d}} & D_0 & \longrightarrow & 0 & \longrightarrow \ldots \end{array}$$

resolving

$$f^* = \text{inclusion} : H^1(D) = L \longrightarrow H^1(C) = M\ .$$

From the exact sequence of Proposition 5.1 we have

$$\varphi\in \ker(f^{\%}:Q^1(C,-\varepsilon)\longrightarrow Q^1(D,-\varepsilon)) = \mathrm{im}(\partial:Q^2(f,-\varepsilon)\longrightarrow Q^1(C,-\varepsilon))\ ,$$

so that there exists an S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric Poincaré pair over A $(f:C\longrightarrow D,(\delta\varphi,\varphi)\in Q^2(f,-\varepsilon))$ such that $H_2(D) = 0$, with boundary (C,φ). Thus a non-singular (even) ε-symmetric linking form over (A,S) (M,λ) is metabolic if and only if any associated S-acyclic 1-dimensional $(-\varepsilon)$-symmetric Poincaré complex over A (C,φ) is such a boundary.

The correspondence of Proposition 13.4 associates to a $\begin{cases}\text{hyperbolic}\\ \text{split hyperbolic}\end{cases}$ $\begin{cases}\varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S) $\begin{cases}(M,\lambda,\mu)\\ (M,\lambda,\nu)\end{cases}$ with lagrangian L a $\begin{cases}(-\varepsilon)\text{-quadratic}\\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ map of S-acyclic 1-dimensional $(-\varepsilon)$-quadratic complexes over A

$$f : (C,\psi) \longrightarrow (D,0)$$

with $f:C \longrightarrow D$ exactly as in the ε-symmetric case above. It is possible to choose resolutions such that $\tilde{f} \in \mathrm{Hom}_A(C_0,D_0)$ is an isomorphism. (Explicitly, given a f.g. projective A-module resolution for $M^\wedge$

$$0 \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow M^\wedge \longrightarrow 0$$

write the dual resolution of $(M^\wedge)^\wedge = M$ as

$$0 \longrightarrow C^0 \xrightarrow{d^*} C^1 \xrightarrow{e} M \longrightarrow 0 \ .$$

Define a f.g. projective A-module

$$P = e^{-1}(L) \subseteq C^1 \ ,$$

write $g \in \mathrm{Hom}_A(P,C^1)$ for the inclusion, and let $h \in \mathrm{Hom}_A(C^0,P)$ be the restriction of $d^* = gh \in \mathrm{Hom}_A(C^0,C^1)$. The S-acyclic 1-dimensional f.g. projective A-module chain complex

$$D : \ldots \longrightarrow 0 \longrightarrow D_1 \xrightarrow{\tilde{d}} D_0 \longrightarrow 0 \longrightarrow \ldots$$

defined by

$$\tilde{d} = h^* : D_1 = P^* \longrightarrow D_0 = C_0$$

is a resolution of $L^\wedge$

$$0 \longrightarrow D_1 \xrightarrow{d} D_0 \longrightarrow L^\wedge \longrightarrow 0 \ .$$

The A-module chain map $f:C \longrightarrow D$ defined by

$$\begin{array}{ccccccccc}
C : & \ldots \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{d} & C_0 & \longrightarrow 0 \longrightarrow & \ldots \\
f\big\downarrow & & & & \big\downarrow f = g^* & & \big\downarrow \tilde{f} = 1 & & \\
D : & \ldots \longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{\tilde{d}} & D_0 & \longrightarrow 0 \longrightarrow & \ldots
\end{array}$$

is a resolution of

$$f^* = \text{inclusion} : H^1(D) = L \longrightarrow H^1(C) = M$$

with $\tilde{f} \in \mathrm{Hom}_A(C_0,D_0)$ an isomorphism). By the definition of $\begin{cases}(-\varepsilon)\text{-quadratic}\\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ map we have that

$$f_\%(\psi) = H(\theta) \in Q_1(D,-\varepsilon)$$

for some Tate $\mathbb{Z}_2$-hypercohomology class $\theta \in \hat{Q}^2(D,-\varepsilon)$ such that

$$\begin{cases}\hat{v}_1(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\\ \hat{v}_1^S(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \ .\end{cases}$$

The classes $\psi \in Q_1(C,-\varepsilon)$, $\theta \in \hat{Q}^2(D,-\varepsilon)$ are represented on the chain level by A-module morphisms

$$\psi_0 : C^0 \longrightarrow C_1 \ , \ \tilde{\psi}_0 : C^1 \longrightarrow C_0 \ , \ \psi_1 : C^0 \longrightarrow C_0$$

$$\theta_0 : D^1 \longrightarrow D_1 \ , \ \theta_{-1} : D^0 \longrightarrow D_1 \ , \ \tilde{\theta}_{-1} : D^1 \longrightarrow D_0 \ , \ \theta_{-2} : D^0 \longrightarrow D_0$$

such that

$$d\psi_0 + \tilde{\psi}_0 d^* + \psi_1 + \varepsilon\psi_1^* = 0 \ , \ \tilde{d}\theta_{-1} + \tilde{\theta}_{-1}\tilde{d}^* + \theta_{-2} + \varepsilon\theta_{-2}^* = 0 \ ,$$
$$\theta_0 - \varepsilon\theta_0^* = 0 \ , \ \tilde{d}\theta_0 - \theta_{-1} + \varepsilon\tilde{\theta}_{-1}^* = 0 \ , \ \theta_0\tilde{d}^* + \tilde{\theta}_{-1} - \varepsilon\theta_{-1}^* = 0 \ ,$$
$$f\psi_0\tilde{f}^* = \theta_{-1} \ , \ \tilde{f}\tilde{\psi}_0 f^* = \tilde{\theta}_{-1} \ , \ \tilde{f}\psi_1\tilde{f}^* = \theta_{-2} \ .$$

The vanishing of the $(-\varepsilon)$-hyperquadratic $\begin{cases}-\\ \text{linking}\end{cases}$ Wu class

$$\begin{cases}\hat{v}_1(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \ ; \ x \longmapsto \theta_0(x)(x)\\ \hat{v}_1^S(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \ ; \ x \longmapsto \dfrac{1}{s\bar{s}}(\theta_{-2} + \tilde{\theta}_{-1}\tilde{d}^*)(y)(y)\end{cases}$$
$$(x \in D^1,\ y \in D^0,\ s \in S,\ sx = \tilde{d}^*y \in D^1)$$

implies that there exists $\chi \in \mathrm{Hom}_A(D^1,D_1)$ such that

$$\begin{cases}\theta_0 = \chi + \varepsilon\chi^* \in \mathrm{im}(1+T_\varepsilon : Q_\varepsilon(D^1) \longrightarrow Q^\varepsilon(D^1))\\ \tilde{f}(\psi_1 + \tilde{\psi}_0 d^*)\tilde{f}^* - \tilde{d}\chi\tilde{d}^* \in \ker(S^{-1} : Q_\varepsilon(D^0) \longrightarrow Q^\varepsilon(S^{-1}D^0)) \ .\end{cases}$$

Define $\theta' \in \hat{Q}^2(C,-\varepsilon)$ by

$$\theta'_{-2} = \tilde{f}^{-1}\tilde{d}\chi\tilde{d}^*\tilde{f}^{*-1} - (\psi_1 + \tilde{\psi}_0 d^*) : C^0 \longrightarrow C_0$$
$$\theta'_s = 0 : C^{2-r+s} \longrightarrow C_r \quad (s \geqslant -1) \ ,$$

and let

$$\psi' = \psi + H(\theta') \in Q_1(C,-\varepsilon) \ .$$

Then

$$f_{\%}(\psi') = 0 \in Q_1(D,-\varepsilon)$$

and

$$\begin{cases} \hat{v}_1(\theta') = 0 : H^1(C) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \\ \hat{v}_1^S(\theta') = 0 : H^1(C) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \end{cases} .$$

We thus have a $\begin{cases} (-\varepsilon)\text{-quadratic} \\ \text{split } (-\varepsilon)\text{-quadratic} \end{cases}$ homotopy equivalence of S-acyclic 1-dimensional $(-\varepsilon)$-quadratic Poincaré complexes over A

$$1 : (C,\psi) \longrightarrow (C,\psi')$$

with (C,ψ') the boundary of an S-acyclic 2-dimensional $(-\varepsilon)$-quadratic Poincaré pair over A $(f:C \longrightarrow D, (\delta\psi',\psi') \in Q_2(f,-\varepsilon))$.

[]

Next, we relate the (sub)lagrangians of the boundary of a non-degenerate form over A to morphisms of non-degenerate forms over A which become isomorphisms over $S^{-1}A$.

An <u>S-isomorphism</u> of non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A

$$\begin{cases} f : (K,\alpha) \longrightarrow (K',\alpha') \\ f : (K,\beta) \longrightarrow (K',\beta') \end{cases}$$

is an A-module S-isomorphism $f \in \mathrm{Hom}_A(K,K')$ such that

$$\begin{cases} f^*\alpha' f - \alpha \in \ker(S^{-1}:Q^{\varepsilon}(K) \longrightarrow Q^{\varepsilon}(S^{-1}K)) = 0 \\ f^*\beta' f - \beta \in \ker(S^{-1}:Q_{\varepsilon}(K) \longrightarrow Q_{\varepsilon}(S^{-1}K)) \ (\neq 0, \text{ in general}) \end{cases} .$$

Then $\begin{cases} S^{-1}f : S^{-1}(K,\alpha) \longrightarrow S^{-1}(K',\alpha') \\ S^{-1}f : S^{-1}(K,\beta) \longrightarrow S^{-1}(K',\beta') \end{cases}$ is an isomorphism of non-singular $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over $S^{-1}A$. Note that if $1/2 \in S^{-1}A$ the S-isomorphism classes of non-degenerate ε-quadratic forms over A coincide with the

S-isomorphism classes of non-degenerate even ε-symmetric forms over A. (An S-isomorphism of the ε-symmetrizations

$$f : (K,\beta+\varepsilon\beta^*) \longrightarrow (K',\beta'+\varepsilon\beta'^*)$$

defines an S-isomorphism of non-degenerate ε-quadratic forms over A

$$f : (K,\beta) \longrightarrow (K',\beta')$$

since

$$S^{-1}(f^*\beta' f - \beta) = \tfrac{1}{2}(f^*\beta' f - \beta) - \tfrac{1}{2}\varepsilon(f^*\beta' f - \beta)^* = 0 \in Q_{\varepsilon}(S^{-1}K)) \ .$$

An <u>S-equivalence</u> of S-isomorphisms of non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A

$$\begin{cases} (g,g') : (f:(K,\alpha) \longrightarrow (K',\alpha')) \longrightarrow (\tilde{f}:(\tilde{K},\tilde{\alpha}) \longrightarrow (\tilde{K}',\tilde{\alpha}')) \\ (g,g') : (f:(K,\beta) \longrightarrow (K',\beta')) \longrightarrow (\tilde{f}:(\tilde{K},\tilde{\beta}) \longrightarrow (\tilde{K}',\tilde{\beta}')) \end{cases}$$

is defined by S-isomorphisms

$$\begin{cases} g : (K,\alpha) \longrightarrow (\tilde{K},\tilde{\alpha}) \ , \ g' : (K',\alpha') \longrightarrow (\tilde{K}',\tilde{\alpha}') \\ g : (K,\beta) \longrightarrow (\tilde{K},\tilde{\beta}) \ , \ g' : (K',\beta') \longrightarrow (\tilde{K}',\tilde{\beta}') \end{cases}$$

such that $g \in \mathrm{Hom}_A(K,\tilde{K})$, $g' \in \mathrm{Hom}_A(K',\tilde{K}')$ are isomorphisms and such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\ f\ } & K' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \tilde{K} & \xrightarrow{\ \tilde{f}\ } & \tilde{K}' \end{array}$$

commutes.

A non-singular $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \text{(split) } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \end{cases}$ is <u>stably</u> $\begin{cases} \underline{\text{(even) metabolic}} \\ \underline{\text{(split) hyperbolic}} \end{cases}$ if there exists an isomorphism

$$\begin{cases} f : (M,\lambda)\oplus(N,\varphi) \longrightarrow (N',\varphi') \\ f : (M,\lambda,\mu)\oplus(N,\varphi,\psi) \longrightarrow (N',\varphi',\psi') \end{cases}$$

for some $\begin{cases} \text{(even) metabolic} \\ \text{(split) hyperbolic} \end{cases}$ linking forms $\begin{cases} (N,\varphi),(N',\varphi') \\ (N,\varphi,\psi),(N',\varphi',\psi') \end{cases}$.

Proposition 13.9 i) The sublagrangians L of the boundary $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$

linking form over (A,S) $\begin{cases} \partial(K,\alpha) = (M,\lambda) \\ \partial(K,\alpha) = (M,\lambda,\mu) \\ \partial(K,\beta) = (M,\lambda,\nu) \end{cases}$ of a non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$

form over A $\begin{cases} (K,\alpha\in Q^{\varepsilon}(K)) \\ (K,\alpha\in \mathrm{im}(1+T_{\varepsilon}:Q_{\varepsilon}(K)\longrightarrow Q^{\varepsilon}(K))) \\ (K,\beta\in Q_{\varepsilon}(K)) \end{cases}$ are in a natural one-one

correspondence with the S-equivalence classes of S-isomorphisms of -

non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric forms over A} \\ \varepsilon\text{-quadratic} \end{cases}$

$$\begin{cases} f : (K,\alpha) \longrightarrow (K',\alpha') \\ f : (K,\alpha) \longrightarrow (K',\alpha') \\ f : (K,\beta) \longrightarrow (K',\beta') \end{cases} ,$$

with

$$L = \mathrm{coker}(f:K \longrightarrow K') , \quad \begin{cases} (L^{\perp}/L,\lambda^{\perp}/\lambda) = \partial(K',\alpha') \\ (L^{\perp}/L,\lambda^{\perp}/\lambda,\mu|) = \partial(K',\alpha') \\ (L^{\perp}/L,\lambda^{\perp}/\lambda,\nu|) = \partial(K',\beta') . \end{cases}$$

Lagrangians L correspond to S-isomorphisms with $\begin{cases} (K',\alpha') \\ (K',\alpha') \\ (K',\beta') \end{cases}$ non-singular.

ii) A non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \\ (M,\lambda,\nu) \end{cases}$

is stably $\begin{cases} \text{even metabolic} \\ \text{hyperbolic} \\ \text{split hyperbolic} \end{cases}$ if and only if it is isomorphic to the

boundary $\begin{cases} \partial(K,\alpha) \\ \partial(K,\alpha) \\ \partial(K,\beta) \end{cases}$ of an $\begin{cases} \text{S-metabolic} \\ \text{S-metabolic} \\ \text{S-hyperbolic} \end{cases}$ non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$

form over A $\begin{cases} (K,\alpha) \\ (K,\alpha) . \\ (K,\beta) \end{cases}$

Proof: i) Given an S-isomorphism of non-degenerate $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$

forms over A

$$\begin{cases} f : (K,\alpha) \longrightarrow (K',\alpha') \\ f : (K,\beta) \longrightarrow (K',\beta') \end{cases}$$

define a sublagrangian L for the boundary $\begin{cases} \text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic}) \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$

linking form over (A,S) $\begin{cases} \partial(K,\alpha) \\ \partial(K,\beta) \end{cases}$ by the resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & K' & \longrightarrow & L & \longrightarrow & 0 \\ & & \big\downarrow 1 & & \big\downarrow f^{*}\alpha' & & \big\downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{\alpha} & K^{*} & \longrightarrow & M & \longrightarrow & 0 , \end{array}$$

with $\alpha = \beta + \varepsilon\beta^{*} \in \mathrm{Hom}_A(K,K^{*})$, $\alpha' = \beta' + \varepsilon\beta'^{*} \in \mathrm{Hom}_A(K',K'^{*})$ in the ε-quadratic case. An S-equivalence of S-isomorphisms of non-degenerate

$\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A

$$\begin{cases} (g,g') : (f:(K,\alpha) \longrightarrow (K',\alpha')) \longrightarrow (\tilde{f}:(\tilde{K},\tilde{\alpha}) \longrightarrow (\tilde{K}',\tilde{\alpha}')) \\ (g,g') : (f:(K,\beta) \longrightarrow (K',\beta')) \longrightarrow (\tilde{f}:(\tilde{K},\tilde{\beta}) \longrightarrow (\tilde{K}',\tilde{\beta}')) \end{cases}$$

induces an isomorphism of $\begin{cases} \text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic}) \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking forms over (A,S)

$$\begin{cases} h : \partial(K,\alpha) \longrightarrow \partial(\tilde{K},\tilde{\alpha}) \\ h : \partial(K,\beta) \longrightarrow \partial(\tilde{K},\tilde{\beta}) \end{cases}$$

such that

$$h(L) = \tilde{L} \subseteq \tilde{M} \ ,$$

where $h \in \mathrm{Hom}_A(M,\tilde{M})$ is the isomorphism with resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\begin{cases}\alpha \\ \beta+\varepsilon\beta^*\end{cases}} & K^* & \longrightarrow & M & \longrightarrow & 0 \\ & & \Big\downarrow g & & \Big\downarrow g^{*-1} & & \Big\downarrow h & & \\ 0 & \longrightarrow & \tilde{K} & \xrightarrow{\begin{cases}\tilde{\alpha} \\ \tilde{\beta}+\varepsilon\tilde{\beta}^*\end{cases}} & \tilde{K}^* & \longrightarrow & \tilde{M} & \longrightarrow & 0 \ . \end{array}$$

Conversely, suppose given a non-degenerate $\begin{cases} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric form} \\ \varepsilon\text{-quadratic} \end{cases}$ over A $\begin{cases} (K,\alpha) \\ (K,\alpha) \\ (K,\beta) \end{cases}$ and a sublagrangian L of the boundary $\begin{cases} \partial(K,\alpha) \\ \partial(K,\alpha) \\ \partial(K,\beta) \end{cases}$. Define an S-acyclic 1-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ Poincaré complex over A $\begin{cases} (C,\varphi\in Q^1(C,-\varepsilon)) \\ (C,\psi\in Q_1(C,-\varepsilon)) \\ (C,\psi\in Q_1(C,-\varepsilon)) \end{cases}$ with associated non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking

form over (A,S) $\begin{cases} \partial(K,\alpha) \\ \partial(K,\alpha) \\ \partial(K,\beta) \end{cases}$ by

$$\begin{cases} d = \begin{cases} \alpha \\ \alpha \\ \beta+\varepsilon\beta^* \end{cases} : C_1 = K \longrightarrow C_0 = K^* \ , \ C_r = 0 \ (r \neq 0,1) \\ \varphi_0 = \begin{cases} 1 : C^0 = K \longrightarrow C_1 = K \\ -\varepsilon : C^1 = K^* \longrightarrow C_0 = K^* \end{cases} , \ \varphi_1 = 0 : C^1 = K^* \longrightarrow C_1 = K \\ \psi_0 = \begin{cases} 1 : C^0 = K \longrightarrow C_1 = K \\ 0 : C^1 = K^* \longrightarrow C_0 = K^* \end{cases} , \ \psi_1 = -\tilde{\beta} : C^0 = K \longrightarrow C_0 = K^* \\ \qquad \text{for any } \tilde{\beta}\in \mathrm{Hom}_A(K,K^*) \text{ such that } \begin{cases} \alpha = \tilde{\beta}+\varepsilon\tilde{\beta}^* \in \mathrm{Hom}_A(K,K^*) \\ \beta = \tilde{\beta} \in Q_\varepsilon(K) \end{cases} \end{cases}$$

Let $e \in \mathrm{Hom}_A(K^*,M)$ be the natural projection

$$e : K^* \longrightarrow M = \begin{cases} \mathrm{coker}(\alpha:K \longrightarrow K^*) \\ \mathrm{coker}(\alpha:K \longrightarrow K^*) \\ \mathrm{coker}(\beta+\varepsilon\beta^*:K \longrightarrow K^*) \ , \end{cases}$$

define a f.g. projective A-module

$$K' = e^{-1}(L) \subseteq K^*$$

and define also A-module morphisms

$$f = \begin{cases} \alpha| \\ \alpha| \\ \beta+\varepsilon\beta^*| \end{cases} : K \longrightarrow K'$$

$$g = \text{inclusion} : K' \longrightarrow K^* \ .$$

The A-module chain map $h:C \longrightarrow D$ defined by

$$\begin{array}{cccccccccc} C : & \dots \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{\alpha} & K^* & \longrightarrow & 0 & \longrightarrow \dots \\ h \Big\downarrow & & & & \Big\downarrow g^* & & \Big\downarrow 1 & & & \\ D : & \dots \longrightarrow & 0 & \longrightarrow & K'^* & \xrightarrow{f^*} & K^* & \longrightarrow & 0 & \longrightarrow \dots \end{array}$$

(with $\alpha = \beta+\varepsilon\beta^* \in \mathrm{Hom}_A(K,K^*)$ in the ε-quadratic case) is a resolution of

$$h_* = (\text{inclusion})\hat{} : H_0(C) = M\hat{} \longrightarrow H_0(D) = L\hat{}$$

The morphism of $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S) defined by the inclusion

$$\begin{cases}(L,0) \longrightarrow \partial(K,\alpha)\\ (L,0,0) \longrightarrow \partial(K,\alpha)\\ (L,0,0) \longrightarrow \partial(K,\beta)\end{cases}$$

is associated (by Proposition 13.4) to an $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ map

$$\begin{cases}h : (C,\varphi) \longrightarrow (D,0)\\ h : (C,\psi) \longrightarrow (D,0)\\ h : (C,\psi) \longrightarrow (D,0)\ ,\end{cases}$$

with

$$h_{\%}(\psi) = H(\theta) \in Q_1(D,-\varepsilon)$$

for some $\theta \in \hat{Q}^2(D,-\varepsilon)$ such that

$$\begin{cases}\hat{v}_1(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\\ \hat{v}_1^S(\theta) = 0 : H^1(D) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\ .\end{cases}$$

Working exactly as in the proof of Proposition 13.8 it is possible to replace $\psi \in Q_1(C,-\varepsilon)$ by $\psi + H(\theta') \in Q_1(C,-\varepsilon)$ for some $\theta' \in \hat{Q}^2(C,-\varepsilon)$ such that $\begin{cases}\hat{v}_1(\theta') = 0\\ \hat{v}_1^S(\theta') = 0\end{cases}$, to ensure that

$$h_{\%}(\psi) = 0 \in Q_1(D,-\varepsilon)\ .$$

It follows from $\begin{cases}h^{\%}(\varphi) = 0 \in Q^1(D,-\varepsilon)\\ h_{\%}(\psi) = 0 \in Q_1(D,-\varepsilon)\\ h_{\%}(\psi) = 0 \in Q_1(D,-\varepsilon)\end{cases}$ that there exists a connected S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ pair over A $\begin{cases}(h:C \longrightarrow D,(\delta\varphi,\varphi) \in Q^2(h,-\varepsilon))\\ (h:C \longrightarrow D,(\delta\psi,\psi) \in Q_2(h,-\varepsilon))\ .\\ (h:C \longrightarrow D,(\delta\psi,\psi) \in Q_2(h,-\varepsilon))\end{cases}$

Define a non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric form over } A\\ \varepsilon\text{-quadratic}\end{cases}$ $\begin{cases}(K',\alpha')\\ (K',\alpha')\\ (K',\beta')\end{cases}$ by

$$\begin{cases}\alpha' = -\delta\varphi_0^*\\ \alpha' = -(\delta\psi_0+\varepsilon\delta\psi_0^*)^* : D^1 = K' \longrightarrow D_1 = K'^*\ .\\ \beta' = -\delta\psi_0^*\end{cases}$$

The S-isomorphism of non-degenerate forms

$$\begin{cases}f : (K,\alpha) \longrightarrow (K',\alpha')\\ f : (K,\alpha) \longrightarrow (K',\alpha')\\ f : (K,\beta) \longrightarrow (K',\beta')\end{cases}$$

determines the sublagrangian L of $\begin{cases}\partial(K,\alpha)\\ \partial(K,\alpha)\ .\\ \partial(K,\beta)\end{cases}$

ii) We need a preliminary result.

<u>Lemma</u> Given an S-isomorphism of non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric forms over } A\\ \varepsilon\text{-quadratic}\end{cases}$

$$\begin{cases}f : (K,\alpha) \longrightarrow (K',\alpha')\\ f : (K,\alpha) \longrightarrow (K',\alpha')\\ f : (K,\beta) \longrightarrow (K',\beta')\end{cases}$$

there is defined a lagrangian L for the boundary $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S) $\begin{cases}\partial(K,\alpha)\oplus\partial(K',-\alpha')\\ \partial(K,\alpha)\oplus\partial(K',-\alpha')\ .\\ \partial(K,\beta)\oplus\partial(K',-\beta')\end{cases}$

<u>Proof</u>: The S-isomorphism of non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ forms over A

$$\begin{cases} \begin{pmatrix} \alpha'^* f^* & 0 \\ f^* & 1 \end{pmatrix} : (K,\alpha)\oplus(K',-\alpha') \longrightarrow \left(K'^*\oplus K', \begin{pmatrix} 0 & 1 \\ \varepsilon & -\alpha' \end{pmatrix}\right) \\ \begin{pmatrix} (\beta'+\varepsilon\beta'^*)^* & 0 \\ f^* & 1 \end{pmatrix} : (K,\beta)\oplus(K',-\beta') \longrightarrow \left(K'^*\oplus K', \begin{pmatrix} 0 & 1 \\ 0 & -\beta' \end{pmatrix}\right) \end{cases}$$

has non-singular range, so that $\begin{cases}\partial(K,\alpha)\oplus\partial(K',-\alpha') \\ \partial(K,\beta)\oplus\partial(K',-\beta')\end{cases}$ has a lagrangian by i).

[]

Let $\begin{cases}(K,\alpha) \\ (K,\beta)\end{cases}$ be an $\begin{cases}S\text{-metabolic} \\ S\text{-hyperbolic}\end{cases}$ non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ form over A, and let $j\in \mathrm{Hom}_A(L,K)$ be the inclusion of an S-lagrangian L.

As $\begin{cases} j^*\alpha \in \mathrm{Hom}_A(K,L^*) \\ j^*(\beta+\varepsilon\beta^*)\in \mathrm{Hom}_A(K,L^*)\end{cases}$ becomes onto over $S^{-1}A$ there exist $k\in \mathrm{Hom}_A(L^*,K)$, $s\in S$ such that

$$\begin{cases} j^*\alpha k = s\in \mathrm{Hom}_A(L^*,L^*) \\ j^*(\beta+\varepsilon\beta^*)k = s\in \mathrm{Hom}_A(L^*,L^*)\ . \end{cases}$$

Applying the Lemma to the S-isomorphism of non-degenerate forms

$$\begin{cases} (j\ k) : (K',\alpha') = \left(L\oplus L^*, \begin{pmatrix} 0 & s \\ \varepsilon\bar{s} & k^*\alpha k \end{pmatrix}\right) \longrightarrow (K,\alpha) \\ (j\ k) : (K',\beta') = \left(L\oplus L^*, \begin{pmatrix} 0 & s \\ 0 & k^*\beta k \end{pmatrix}\right) \longrightarrow (K,\beta) \end{cases}$$

we have that $\begin{cases}\partial(K,\alpha)\oplus\partial(K',-\alpha') \\ \partial(K,\beta)\oplus\partial(K',-\beta')\end{cases}$ is a $\begin{cases}\text{(hyperbolic) even metabolic} \\ \text{split hyperbolic}\end{cases}$ linking form over (A,S). Furthermore, there is defined an S-isomorphism of non-degenerate forms

$$\begin{cases} \begin{pmatrix} \bar{s} & 0 \\ 0 & 1 \end{pmatrix} : (K',\alpha') \longrightarrow \left(L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & k^*\alpha k \end{pmatrix}\right) \\ \begin{pmatrix} \bar{s} & 0 \\ 0 & 1 \end{pmatrix} : (K',\beta') \longrightarrow \left(L\oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & k^*\beta k \end{pmatrix}\right) \end{cases}$$

with non-singular range, so that $\begin{cases}(K',\alpha') \\ (K',\beta')\end{cases}$ is $\begin{cases}\text{(hyperbolic) even metabolic} \\ \text{split hyperbolic}\end{cases}$ by i).

We have shown that $\begin{cases}(K,\alpha) \\ (K,\beta)\end{cases}$ is stably $\begin{cases}\text{(hyperbolic) even metabolic} \\ \text{split hyperbolic}\end{cases}$.

It remains to prove the converse, that a stably $\begin{cases}\text{even metabolic} \\ \text{hyperbolic} \\ \text{split hyperbolic}\end{cases}$ linking form over (A,S) is isomorphic to the boundary of an $\begin{cases}S\text{-metabolic} \\ \text{even } S\text{-metabolic} \\ S\text{-hyperbolic}\end{cases}$ non-degenerate form over A.

Let $\begin{cases}(M,\lambda) \\ (M,\lambda,\nu)\end{cases}$ be a non-singular $\begin{cases}\text{even } \varepsilon\text{-symmetric} \\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S).

By Proposition 13.4 there exists an S-acyclic 1-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic}\end{cases}$ Poincare complex over A $\begin{cases}(C,\varphi\in Q^1(C,-\varepsilon)) \\ (C,\psi\in Q_1(C,-\varepsilon))\end{cases}$ such that

$$\begin{cases} (H^1(C),\varphi_0^S) = (M,\lambda) \\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) = (M,\lambda,\nu)\ . \end{cases}$$

The S-acyclic cobordism class $\begin{cases}(C,\varphi)\in L^0(A,S,\varepsilon) \\ (C,\psi)\in L_0(A,S,\varepsilon)\end{cases}$ depends only on the isomorphism class of $\begin{cases}(M,\lambda) \\ (M,\lambda,\nu)\end{cases}$ (Proposition 13.7 i)), vanishing if $\begin{cases}(M,\lambda) \\ (M,\lambda,\nu)\end{cases}$ is $\begin{cases}\text{even metabolic} \\ \text{split hyperbolic}\end{cases}$ (Proposition 13.8 ii)). It follows that $\begin{cases}(C,\varphi) = 0\in L^0(A,S,\varepsilon) \\ (C,\psi) = 0 \in L_0(A,S,\varepsilon)\end{cases}$ if $\begin{cases}(M,\lambda) \\ (M,\lambda,\nu)\end{cases}$ is stably $\begin{cases}\text{even metabolic} \\ \text{split hyperbolic}\end{cases}$, and hence that $\begin{cases}(M,\lambda) \\ (M,\lambda,\nu)\end{cases}$ is isomorphic to the boundary $\begin{cases}\partial(K,\alpha)\\ \partial(K,\beta)\end{cases}$ of an $\begin{cases}S\text{-metabolic}\\ S\text{-hyperbolic}\end{cases}$ non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(K,\alpha)\\ (K,\beta)\end{cases}$ (Proposition 13.7 ii)). It may be similarly verified that if (M,λ) is the ε-symmetrization of a stably hyperbolic ε-quadratic linking form over (A,S) (M,λ,μ) then it is possible to find an S-metabolic non-degenerate ε-symmetric form over (K,α) which is even and such that (M,λ,μ) is isomorphic to the boundary $\partial(K,\alpha)$.

[]

Define the Witt group of $\begin{cases}\underline{\varepsilon\text{-symmetric}}\\ \underline{\text{even } \varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\\ \underline{\text{split } \varepsilon\text{-quadratic}}\end{cases}$ linking forms over (A,S)

$\begin{cases}L^{\varepsilon}(A,S)\\ L\langle v_0\rangle^{\varepsilon}(A,S)\\ L_{\varepsilon}(A,S)\\ \widetilde{L}_{\varepsilon}(A,S)\end{cases}$ to be the abelian group of stable isomorphism classes of

non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking forms over (A,S), the stability being

with respect to the $\begin{cases}\text{metabolic}\\ \text{even metabolic}\\ \text{hyperbolic}\\ \text{split hyperbolic}\end{cases}$ linking forms. Addition is by the direct

sum $\oplus$, and inverses are given by

$$\begin{cases}-(M,\lambda) = (M,-\lambda) \in L^{\varepsilon}(A,S)\\ -(M,\lambda) = (M,-\lambda) \in L\langle v_0\rangle^{\varepsilon}(A,S)\\ -(M,\lambda,\mu) = (M,-\lambda,-\mu) \in L_{\varepsilon}(A,S)\\ -(M,\lambda,\nu) = (M,-\lambda,-\nu) \in \widetilde{L}_{\varepsilon}(A,S)\ ,\end{cases}$$

since the diagonal $\Delta = \{(x,x)\in M\oplus M\,|\,x\in M\}$ is a lagrangian of $\begin{cases}(M,\lambda)\oplus(M,-\lambda)\\ (M,\lambda)\oplus(M,-\lambda)\\ (M,\lambda,\mu)\oplus(M,-\lambda,-\mu)\\ (M,\lambda,\nu)\oplus(M,-\lambda,-\nu)\end{cases}$

(Proposition 13.8 i)). There are defined forgetful maps

$$L\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow L^{\varepsilon}(A,S)\ ;\ (M,\lambda) \longmapsto (M,\lambda)$$
$$L_{\varepsilon}(A,S) \longrightarrow L\langle v_0\rangle^{\varepsilon}(A,S)\ ;\ (M,\lambda,\mu) \longrightarrow (M,\lambda)$$
$$\widetilde{L}_{\varepsilon}(A,S) \longrightarrow L_{\varepsilon}(A,S)\ ;\ (M,\lambda,\nu) \longmapsto (M,\lambda,\mu = p\nu)\ .$$

A non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over $S^{-1}A$ $\begin{cases}(Q,\varphi)\\ (Q,\psi)\end{cases}$ such that $[Q]\in S = \mathrm{im}(\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(S^{-1}A))$ is isomorphic to $\begin{cases}S^{-1}(K,\alpha)\\ S^{-1}(K,\beta)\end{cases}$ for some non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(K,\alpha)\\ (K,\beta)\end{cases}$. It follows from Proposition 13.9 ii) that the boundary operation

$$\partial\ :\ (\text{non-degenerate forms over } A) \longrightarrow (\text{linking forms over } (A,S))$$

can be used to define abelian group morphisms

$$\partial\ :\ L^{\varepsilon}_S(S^{-1}A) \longrightarrow L\langle v_0\rangle^{\varepsilon}(A,S)\ ;\ S^{-1}(K,\alpha) \longrightarrow \partial(K,\alpha)$$
$$\partial\ :\ L\langle v_0\rangle^{\varepsilon}_S(S^{-1}A) \longrightarrow L_{\varepsilon}(A,S)\ ;\ S^{-1}(K,\alpha) \longrightarrow \partial(K,\alpha)$$
$$\partial\ :\ L^S_{\varepsilon}(S^{-1}A) \longrightarrow \widetilde{L}_{\varepsilon}(A,S)\ ;\ S^{-1}(K,\beta) \longrightarrow \partial(K,\beta)\ .$$

There is also defined a morphism

$$\partial\ :\ L^{\varepsilon}_S(S^{-1}A) \longrightarrow L^{\varepsilon}(A,S)\ ;\ S^{-1}(K,\alpha) \longmapsto \partial(K,\alpha)\ ,$$

namely the composite

$$L^{\varepsilon}_S(S^{-1}A) \xrightarrow{\ \partial\ } L\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow L^{\varepsilon}(A,S)\ .$$

The correspondence of Proposition 13.6 associates to a non-singular $\left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ linking form over (A,S) $\left\{\begin{array}{l}(M,\lambda)\\ (M,\lambda,\mu)\end{array}\right.$ a stable isomorphism class of non-singular $\left\{\begin{array}{l}\text{(even) } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ formations over A $\left\{\begin{array}{l}(Q,\varphi;F,G)\\ (Q,\psi;F,G)\end{array}\right.$, and it follows from Proposition 13.9 ii) that there are defined abelian group morphisms

$$L^{\varepsilon}(A,S) \longrightarrow M^{-\varepsilon}(A) \ ; \ (M,\lambda) \longmapsto (Q,\varphi;F,G)$$
$$L\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow M\langle v_0\rangle^{-\varepsilon}(A) \ ; \ (M,\lambda) \longmapsto (Q,\varphi;F,G)$$
$$L_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \ ; \ (M,\lambda,\mu) \longmapsto (Q,\psi;F,G) \ .$$

There is also defined an abelian group morphism

$$\widetilde{L}_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \ ; \ (M,\lambda,\nu) \longmapsto (Q,\psi;F,G) \ ,$$

namely the composite

$$\widetilde{L}_{\varepsilon}(A,S) \longrightarrow L_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \ .$$

Define abelian groups $\left\{\begin{array}{l}L^{2k}(A,S,\varepsilon)\\ L_{2k}(A,S,\varepsilon)\end{array}\right.$ $(k \leqslant -1)$ by

$$L_{2k}(A,S,\varepsilon) = L_{2k+2i}(A,S,(-)^{i}\varepsilon) \qquad (k+i \geqslant 0)$$

(extending the periodicity of Proposition 13.1 ii))

$$L^{-2}(A,S,\varepsilon) = L_{-\varepsilon}(A,S)$$
$$L^{2k}(A,S,\varepsilon) = L_{2k}(A,S,\varepsilon) \quad (k \leqslant -2) \ .$$

<u>Proposition 13.10</u> i) The localization exact sequence of algebraic Poincaré cobordism groups

$$L^{2k}(A,(-)^{k}\varepsilon) \longrightarrow L^{2k}_{S}(S^{-1}A,(-)^{k}\varepsilon) \longrightarrow L^{2k}(A,S,(-)^{k}\varepsilon) \longrightarrow L^{2k-1}(A,(-)^{k}\varepsilon) \longrightarrow L^{2k-1}_{S}(S^{-1}A,(-)^{k}\varepsilon) \qquad (*)_{2k}$$

is naturally isomorphic for $\left\{\begin{array}{l}k=0\\ k=-1\\ k \leqslant -2\end{array}\right.$ to a localization exact sequence of Witt groups

$$\left\{\begin{array}{l}L^{\varepsilon}(A) \to L^{\varepsilon}_{S}(S^{-1}A) \to L\langle v_0\rangle^{\varepsilon}(A,S) \to M\langle v_0\rangle^{-\varepsilon}(A) \to M\langle v_0\rangle^{-\varepsilon}_{S}(S^{-1}A)\\ L\langle v_0\rangle^{\varepsilon}(A) \to L\langle v_0\rangle^{\varepsilon}_{S}(S^{-1}A) \to L_{\varepsilon}(A,S) \to M_{-\varepsilon}(A) \to M^{S}_{-\varepsilon}(S^{-1}A)\\ L_{\varepsilon}(A) \to L^{S}_{\varepsilon}(S^{-1}A) \to \widetilde{L}_{\varepsilon}(A,S) \to M_{-\varepsilon}(A) \to M^{S}_{-\varepsilon}(S^{-1}A) \ .\end{array}\right.$$

ii) There are defined natural abelian group morphisms

$$L^{\varepsilon}(A,S) \longrightarrow L^{2k}(A,S,(-)^{k}\varepsilon) \quad (k \geqslant 1)$$

for all A,S,ε. If $\left\{\begin{array}{l}(A,S) \text{ is 0-dimensional}\\ \ker(\hat{\delta}:\hat{H}^{1}(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \to \hat{H}^{0}(\mathbb{Z}_2;A,\varepsilon)) = 0\end{array}\right.$ then for $\left\{\begin{array}{l}k \geqslant 1\\ k=1\end{array}\right.$ these are isomorphisms, and $(*)_{2k}$ is naturally isomorphic to a localization exact sequence of Witt groups

$$\left\{\begin{array}{l}L^{\varepsilon}(A) \to L^{\varepsilon}_{S}(S^{-1}A) \to L^{\varepsilon}(A,S) \to M^{-\varepsilon}(A) \to M^{-\varepsilon}_{S}(S^{-1}A) \to 0\\ L^{2}(A,-\varepsilon) \to L^{2}_{S}(S^{-1}A,-\varepsilon) \to L^{\varepsilon}(A,S) \to M^{-\varepsilon}(A) \to M^{-\varepsilon}_{S}(S^{-1}A) \ .\end{array}\right.$$

iii) For all A,S,ε the forgetful map of Witt groups

$$\widetilde{L}_{\varepsilon}(A,S) \longrightarrow L_{\varepsilon}(A,S) \ ; \ (M,\lambda,\nu) \longmapsto (M,\lambda,\mu = p\nu)$$

is onto, and there are natural identifications

$$\operatorname{coker}(\partial: L^{S}_{\varepsilon}(S^{-1}A) \to \widetilde{L}_{\varepsilon}(A,S)) = \operatorname{coker}(\partial: L\langle v_0\rangle^{\varepsilon}_{S}(S^{-1}A) \to L_{\varepsilon}(A,S))$$
$$= \ker(M_{-\varepsilon}(A) \to M^{S}_{-\varepsilon}(S^{-1}A)) \ .$$

If (A,S) is 0-dimensional

$$\ker(L^{\varepsilon}(A,S) \to M^{-\varepsilon}(A)) = \ker(L\langle v_0\rangle^{\varepsilon}(A,S) \to M\langle v_0\rangle^{-\varepsilon}(A))$$
$$= \operatorname{coker}(L^{\varepsilon}(A) \to L^{\varepsilon}_{S}(S^{-1}A)) \ .$$

If $\begin{cases} \mathrm{im}(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \xrightarrow{\hat{\delta}} \hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0 \\ \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \to \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon) \text{ is an isomorphism the forgetful map identifies} \\ \mathrm{im}(\hat{H}^1(\mathbb{Z}_2;S^{-1}A,\varepsilon) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)) = 0 \end{cases}$

$$\begin{cases} L\langle v_0\rangle^{\varepsilon}(A,S) = L^{\varepsilon}(A,S) \\ L_{\varepsilon}(A,S) = L\langle v_0\rangle^{\varepsilon}(A,S) \\ \tilde{L}_{\varepsilon}(A,S) = L_{\varepsilon}(A,S) \end{cases} .$$

In particular, if $1/2 \in S^{-1}A$

$$\tilde{L}_{\varepsilon}(A,S) = L_{\varepsilon}(A,S) \ ,$$

and if $1/2 \in A$ then

$$\tilde{L}_{\varepsilon}(A,S) = L_{\varepsilon}(A,S) = L\langle v_0\rangle^{\varepsilon}(A,S) = L^{\varepsilon}(A,S) \ .$$

<u>Proof</u>: i) An S-acyclic 1-dimensional $\begin{cases} \text{even } (-\varepsilon)\text{-symmetric} \\ (-\varepsilon)\text{-quadratic} \end{cases}$ Poincaré complex over A $\begin{cases} (C,\varphi\in Q^1(C,-\varepsilon)) \\ (C,\psi\in Q_1(C,-\varepsilon)) \end{cases}$ represents 0 in $\begin{cases} L^0(A,S,\varepsilon) \\ L_0(A,S,\varepsilon) \end{cases}$ if and only if the associated non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (H^1(C),\varphi_0^S) \\ (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) \end{cases}$ represents 0 in $\begin{cases} L\langle v_0\rangle^{\varepsilon}(A,S) \\ L_{\varepsilon}(A,S) \end{cases}$ (by Propositions 13.7 ii), 13.9 ii)). It follows that the correspondence of Proposition 13.4 defines abelian group isomorphisms

$$\begin{cases} L^0(A,S,\varepsilon) \longrightarrow L\langle v_0\rangle^{\varepsilon}(A,S) \ ; \ (C,\varphi) \longmapsto (H^1(C),\varphi_0^S) \\ L_0(A,S,\varepsilon) \longrightarrow \tilde{L}_{\varepsilon}(A,S) \ ; \ (C,\psi) \longmapsto (H^1(C),(1+T_{-\varepsilon})\psi_0^S,v_S^0(\psi)) \ . \end{cases}$$

The exactness of the Witt group sequences

$$\begin{cases} L^{\varepsilon}(A) \longrightarrow L_S^{\varepsilon}(S^{-1}A) \xrightarrow{\partial} L\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow M\langle v_0\rangle^{-\varepsilon}(A) \longrightarrow M\langle v_0\rangle_S^{-\varepsilon}(S^{-1}A) \\ L_{\varepsilon}(A) \longrightarrow L^S_{\varepsilon}(S^{-1}A) \xrightarrow{\partial} \tilde{L}_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \longrightarrow M^S_{-\varepsilon}(S^{-1}A) \end{cases}$$

may now be deduced from the exactness of $(*)_0$ and $(*)_{-4}$ (Proposition 13.1 i)), or else may be established directly using Proposition 13.9 ii). The direct method applies also to the exactness of $(*)_{-2}$

$$L\langle v_0\rangle^{\varepsilon}(A) \longrightarrow L\langle v_0\rangle_S^{\varepsilon}(S^{-1}A) \xrightarrow{\partial} L_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \longrightarrow M^S_{-\varepsilon}(S^{-1}A) \ .$$

ii) Define abelian group morphisms

$$L^{\varepsilon}(A,S) \longrightarrow L^{2k}(A,S,(-)^k\varepsilon) \ ; \ (M,\lambda) \longmapsto \bar{S}^k(C,\varphi) \quad (k \geqslant 1)$$

by sending a non-singular ε-symmetric linking form over (A,S) (M,λ) to the k-fold skew-suspension of an S-acyclic 1-dimensional $(-\varepsilon)$-symmetric Poincaré complex over A $(C,\varphi\in Q^1(C,-\varepsilon))$ such that

$$(H^1(C),\varphi_0^S) = (M,\lambda) \ ,$$

as given by Proposition 13.4. The S-acyclic cobordism class $\bar{S}^k(C,\varphi) \in L^{2k}(A,S,(-)^k\varepsilon)$ depends only on the isomorphism class of (M,λ) (which may be proved exactly as in Proposition 13.7 i)), and vanishes if (M,λ) is stably metabolic (Proposition 13.8 ii)), so that the morphisms are well-defined. If $\begin{cases} (A,S) \text{ is 0-dimensional} \\ \ker(\hat{\delta}:\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)) = 0 \end{cases}$ then by Proposition $\begin{cases} 13.2 \\ 13.3 \text{ i)} \end{cases}$ there are natural identifications

$$L^{2k}(A,S,(-)^k\varepsilon) = (\text{S-acyclic cobordism group of S-acyclic 1-dimensional } (-\varepsilon)\text{-symmetric Poincaré complexes over } A) \quad \left(\begin{cases} k \geqslant 1 \\ k = 1 \end{cases}\right),$$

so that the morphisms are onto. Moreover, if $(M,\lambda) \in \ker(L^{\varepsilon}(A,S) \longrightarrow L^2(A,S,-\varepsilon))$ then (C,φ) is homotopy equivalent to the boundary $\partial(D,\eta)$ of a connected S-acyclic 2-dimensional $(-\varepsilon)$-symmetric complex over A $(D,\eta\in Q^2(D,-\varepsilon))$, and the proof of Proposition 13.9 ii) generalizes to show that (M,λ) is stably metabolic, so that the morphisms are also one-one, and hence isomorphisms.

iii) Immediate from i), ii) and Proposition 13.5 ii).

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An $\begin{cases} \underline{\text{(even) } \varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}} \end{cases}$ <u>linking formation over (A,S)</u> $\begin{cases} (M,\lambda;F,G) \\ (M,\lambda,\mu;F,G) \end{cases}$

is a non-singular $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S) $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \end{cases}$ together with a lagrangian F and a sublagrangian G. The linking formation is <u>non-singular</u> if G is a lagrangian.

An <u>isomorphism</u> of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking formations over (A,S)

$$\begin{cases} f : (M,\lambda;F,G) \longrightarrow (M',\lambda';F',G') \\ f : (M,\lambda,\mu;F,G) \longrightarrow (M',\lambda',\mu';F',G') \end{cases}$$

is an isomorphism of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking forms over (A,S)

$$\begin{cases} f : (M,\lambda) \longrightarrow (M',\lambda') \\ f : (M,\lambda,\mu) \longrightarrow (M',\lambda',\mu') \end{cases}$$

such that

$$f(F) = F' \ , \ f(G) = G' \ .$$

A <u>sublagrangian</u> of an $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking formation over (A,S) $\begin{cases} (M,\lambda;F,G) \\ (M,\lambda,\mu;F,G) \end{cases}$ is a sublagrangian H of $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \end{cases}$ such that

i) $H \subseteq G$, with G/H an h.d.1 S-torsion A-module

ii) $F \cap H = \{0\}$, $M = F + H^{\perp}$.

An <u>elementary equivalence</u> of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking formations over (A,S) is the transformation

$$\begin{cases} (M,\lambda;F,G) \longmapsto (M',\lambda';F',G') \\ (M,\lambda,\mu;F,G) \longmapsto (M',\lambda',\mu';F',G') \end{cases}$$

determined by a sublagrangian H of $\begin{cases} (M,\lambda;F,G) \\ (M,\lambda,\mu;F,G) \end{cases}$, with

$$\begin{cases} (M',\lambda';F',G') = (H^{\perp}/H,\lambda^{\perp}/\lambda;F\cap H^{\perp},G/H) \\ (M',\lambda',\mu';F',G') = (H^{\perp}/H,\lambda^{\perp}/\lambda,\mu|;F\cap H^{\perp},G/H) \end{cases}$$

(where $F \cap H^{\perp}$ stands for the image of the injection $F \cap H^{\perp} \longrightarrow H^{\perp}/H \ ; \ x \longmapsto [x]$). Note that there are natural identifications of S-torsion A-modules

$$F' \cap G' = F \cap G \ , \ M'/F'+G' = M/F+G \ , \ G'^{\perp}/G' = G^{\perp}/G$$

- in general, only $G^{\perp}/G$ is h.d.1. Elementary equivalences and isomorphisms generate an equivalence relation on the set of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ linking formations over (A,S), which we shall call <u>stable equivalence</u>.

In Proposition 13.12 below we shall identify the stable equivalence classes of (non-singular, resp. even) ε-symmetric linking formations over (A,S) with the homotopy equivalence classes of connected S-acyclic 2-dimensional (Poincaré, resp. even) $(-\varepsilon)$-symmetric complexes over A.

Given an h.d.1 S-torsion A-module L define the $\begin{cases} \underline{\text{metabolic } \varepsilon\text{-symmetric}} \\ \underline{\text{hyperbolic } \varepsilon\text{-quadratic}} \\ \underline{\text{hyperbolic split } \varepsilon\text{-quadratic}} \end{cases}$ <u>linking form over (A,S) on L</u> to be the non-singular $\begin{cases} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$ linking form over (A,S)

$$\begin{cases} H^{\varepsilon}(L) = (L \oplus L^{\wedge},\lambda) \\ H_{\varepsilon}(L) = (L \oplus L^{\wedge},\lambda,\mu) \\ \widetilde{H}_{\varepsilon}(L) = (L \oplus L^{\wedge},\lambda,\nu) \end{cases}$$

with

$$\lambda : L\oplus L^{\wedge} \times L\oplus L^{\wedge} \longrightarrow S^{-1}A/A \ ; \ ((x,f),(x',f')) \longmapsto f(x') + \varepsilon\overline{f'(x)}$$
$$\mu : L\oplus L^{\wedge} \longrightarrow Q_{\varepsilon}(A,S) \ ; \ (x,f) \longmapsto f(x) + \varepsilon\overline{f(x)}$$
$$\nu : L\oplus L^{\wedge} \longrightarrow Q_{\varepsilon}(S^{-1}A/A) \ ; \ (x,f) \longmapsto f(x) \quad ,$$

for which both L and $L^{\wedge}$ are lagrangians.

A <u>split ε-quadratic linking formation over (A,S)</u> $(F,G) \equiv (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is an ε-quadratic linking formation over (A,S) $(H_{\varepsilon}(F);F,G)$, where $\begin{pmatrix}\gamma\\ \mu\end{pmatrix} : G \longrightarrow F\oplus F^{\wedge}$ is the inclusion, together with a function $\theta : G \longrightarrow Q_{-\varepsilon}(A,S)$ such that $(G,\gamma^{\wedge}\mu \in \mathrm{Hom}_A(G,G^{\wedge}),\theta)$ is a $(-\varepsilon)$-quadratic linking form over (A,S).
(It can be shown that every ε-quadratic linking formation over (A,S) is stably equivalent to one of the type $(H_{\varepsilon}(F);F,G)$, and that $(H_{\varepsilon}(F);F,G)$ supports a split ε-quadratic linking formation over (A,S) (F,G) if and only if G is a sublagrangian of the hyperbolic split ε-quadratic linking form $\widetilde{H}_{\varepsilon}(F)$.)
In Proposition 13.12 we shall identify the homotopy equivalence classes of S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complexes over A with appropriate equivalence classes of split ε-quadratic linking formations over (A,S).

A split ε-quadratic linking formation over (A,S) (F,G) is <u>non-singular</u> if G is a lagrangian of $\widetilde{H}_{\varepsilon}(F)$, that is if the sequence of h.d.1 S-torsion A-modules

$$0 \longrightarrow G \xrightarrow{\begin{pmatrix}\gamma\\ \mu\end{pmatrix}} F\oplus F^{\wedge} \xrightarrow{(\varepsilon\mu^{\wedge}\ \gamma^{\wedge})} G^{\wedge} \longrightarrow 0$$

is exact.

An <u>isomorphism</u> of split ε-quadratic linking formations over (A,S)

$$(\alpha,\beta,\varphi,\psi) : (F,G) \longrightarrow (F',G')$$

is a quadruple

$$(\alpha \in \mathrm{Hom}_A(F,F'),\beta \in \mathrm{Hom}_A(G,G'),\varphi \in \mathrm{Hom}_A(F^{\wedge},F),\psi : F^{\wedge} \longrightarrow Q_{-\varepsilon}(A,S))$$

such that α and β are isomorphisms, $(F^{\wedge},\varphi,\psi)$ is a $(-\varepsilon)$-quadratic linking form over (A,S), and

i) $\alpha^{\wedge -1}\mu = \mu'\beta \in \mathrm{Hom}_A(G,F')$

ii) $\alpha\gamma + \alpha\varphi^{\wedge}\mu = \gamma'\beta \in \mathrm{Hom}_A(G,F')$

iii) $\theta + \psi\mu = \theta'\beta : G \longrightarrow Q_{-\varepsilon}(A,S)$

(The A-module isomorphism

$$f = \begin{pmatrix}\alpha & \alpha\varphi\\ 0 & \alpha^{\wedge -1}\end{pmatrix} : F\oplus F^{\wedge} \longrightarrow F'\oplus F'^{\wedge}$$

defines an isomorphism of the underlying ε-quadratic linking formations over (A,S)

$$f : (H_{\varepsilon}(F);F,G) \longrightarrow (H_{\varepsilon}(F');F',G') \ ,$$

as well as an isomorphism of hyperbolic split ε-quadratic linking forms over (A,S)

$$f : \widetilde{H}_{\varepsilon}(F) \longrightarrow \widetilde{H}_{\varepsilon}(F') \ .$$

Conversely, given split ε-quadratic linking formations over (A,S) (F,G),(F',G') and an isomorphism of hyperbolic split ε-quadratic linking forms over (A,S) $f : \widetilde{H}_{\varepsilon}(F) \longrightarrow \widetilde{H}_{\varepsilon}(F')$ such that $f(F) = F'$, $f(G) = G'$ it is possible to define an isomorphism $(\alpha,\beta,\varphi,\psi) : (F,G) \longrightarrow (F',G')$ such that $f = \begin{pmatrix}\alpha & \alpha\varphi\\ 0 & \alpha^{\wedge -1}\end{pmatrix} : F\oplus F^{\wedge} \longrightarrow F'\oplus F'^{\wedge}$)

(There is an evident analogy between the theory of forms and formations over A (as developed in §1) and the theory of linking forms and linking formations over (A,S), except for the following minor discrepancy. Given a sublagrangian L in a split ε-quadratic linking form over (A,S) (M,λ,ν) define a <u>hessian</u> to be a $(-\varepsilon)$-quadratic linking form over (A,S) (L,φ,θ) together with an A-module morphism $\psi \in \mathrm{Hom}_A(M,M^{\wedge})$ such that

$$\varphi(u)(v) = \psi(u)(v) \in S^{-1}A/A \quad (u,v \in L)$$
$$\lambda(x)(y) = \psi(x)(y) + \varepsilon\overline{\psi(y)(x)} \in S^{-1}A/A \ , \ \nu(x) = \psi(x)(x) \in Q_{\varepsilon}(S^{-1}A/A) \quad (x,y \in M)$$

In particular, a split ε-quadratic linking formation over (A,S) (F,G) is an ε-quadratic linking formation over (A,S) $(H_{\varepsilon}(F);F,G)$ together with a choice of hessian $(G,\gamma^{\wedge}\mu,\theta)$ for G in $\widetilde{H}_{\varepsilon}(F)$, where

$$\psi = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} : F\oplus F^{\wedge} \longrightarrow (F\oplus F^{\wedge})^{\wedge} = F^{\wedge}\oplus F \ ; \ (x,f) \longmapsto ((x',f') \longmapsto f(x')) \ .$$

The non-singular split quadratic linking form over $(\mathbb{Z},\mathbb{Z}-\{0\})$ (M,λ,ν) defined by

$$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\lambda : \mathbb{Z}_2\oplus\mathbb{Z}_2 \times \mathbb{Z}_2\oplus\mathbb{Z}_2 \longrightarrow \mathbb{Q}/\mathbb{Z} \; ; \; ((a,b),(a',b')) \longmapsto \tfrac{1}{2}(ab' + ba' + bb')$$

$$\nu : \mathbb{Z}_2\oplus\mathbb{Z}_2 \longrightarrow Q_{+1}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \; ; \; (a,b) \longmapsto \tfrac{1}{4}(2ab + b^2)$$

$$(a,a',b,b' \in \mathbb{Z}_2 \equiv \mathbb{Z}/2\mathbb{Z})$$

has a lagrangian $L = \mathbb{Z}_2\oplus 0$ which does not admit a hessian, since $\lambda \neq \psi + \psi^*$ for any $\psi \in \mathrm{Hom}_{\mathbb{Z}}(M,M^\wedge)$. Thus hessians need not exist, unlike the situation obtaining in §1. However, a sublagrangian G in a hyperbolic ε-quadratic linking form over (A,S) $H_\varepsilon(F)$ is a sublagrangian in the hyperbolic split ε-quadratic linking form over (A,S) $\widetilde{H}_\varepsilon(F)$ if and only if it admits a hessian. It is conceivable that there exists a different notion of hessian for sublagrangians in arbitrary split ε-quadratic linking forms, which agrees with the one above for hyperbolic split ε-quadratic linking forms. The extra structure carried by a split ε-quadratic linking formation (F,G) corresponding to the choice of hessian for G in $\widetilde{H}_\varepsilon(F)$ (i.e. the $(-\varepsilon)$-quadratic linking form $(G,\gamma^\wedge\mu,\theta)$) was first obtained by Pardon [2].

A sublagrangian H of a split ε-quadratic linking formation over (A,S) (F,G) is a sublagrangian H of the underlying ε-quadratic linking formation $(H_\varepsilon(F);F,G)$ such that

$$\theta|_H = 0 : H \longrightarrow Q_{-\varepsilon}(A,S) \; .$$

An elementary equivalence of split ε-quadratic linking formations over (A,S) is the transformation

$$(F,G) \longmapsto (F',G')$$

determined by a sublagrangian H of (F,G), with

$$F' = F \cap H^\perp = \ker(j^\wedge\mu^\wedge : F \longrightarrow H^\wedge) \qquad (j = \text{inclusion} : H \longrightarrow G)$$

$$G' = G/H = \mathrm{coker}(j : H \longrightarrow G)$$

$$\gamma' : G' \longrightarrow F' \; ; \; [x] \longmapsto \gamma(x)$$

$$\mu' : G' \longrightarrow F'^\wedge \; ; \; [x] \longmapsto (y \longmapsto \mu(x)(y))$$

$$\theta' : G' \longrightarrow Q_{-\varepsilon}(A,S) \; ; \; [x] \longmapsto \theta(x) \qquad (x\in G, y \in F') \; .$$

(Then $(H_\varepsilon(F');F',\mathrm{im}(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}:G' \longrightarrow F'\oplus F'^\wedge))$ is isomorphic to the ε-quadratic linking formation obtained from $(H_\varepsilon(F);F,\mathrm{im}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}):G \longrightarrow F\oplus F^\wedge))$ by the elementary equivalence determined by H). Elementary equivalences and isomorphisms generate an equivalence relation on the set of split ε-quadratic linking formations over (A,S), which we shall call stable equivalence.

Prior to the identification of equivalence classes of linking formations over (A,S) with equivalence classes of S-acyclic 2-dimensional complexes over A we need some preliminary results on the homotopy classification of 2-dimensional complexes.

A 2-dimensional A-module chain complex C is in normal form if it is a f.g. projective A-module chain complex with $C_r = 0$ $(r \neq 0,1,2)$

$$C : \ldots \longrightarrow 0 \longrightarrow C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots \; .$$

A connected 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ is in normal form if the chain complex C is in normal form and $\begin{cases}\varphi \in Q^2(C,\varepsilon)\\ \psi \in Q_2(C,\varepsilon)\end{cases}$ has a chain representative $\begin{cases}\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_2\\ \psi \in (W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C^*,C))_2\end{cases}$ such that $\begin{cases}\varphi_0 \in \mathrm{Hom}_A(C^0,C_2)\\ \psi_0 \in \mathrm{Hom}_A(C^0,C_2)\end{cases}$ is an isomorphism and $\begin{cases}\varphi_1 = 0 \in \mathrm{Hom}_A(C^2,C_1)\\ \psi_1 = 0 \in \mathrm{Hom}_A(C^1,C_0),\ \psi_0 = 0 \in \mathrm{Hom}_A(C^2,C_0).\end{cases}$

An $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ in normal form is Poincaré if and only if $\begin{cases}\varphi_0 \in \mathrm{Hom}_A(C^1,C_1)\\ (1+T_\varepsilon)\psi_0 \in \mathrm{Hom}_A(C^1,C_1)\end{cases}$ is an isomorphism.

A <u>stable isomorphism</u> of connected 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes over A in normal form

$$\begin{cases}[f] : (C,\varphi) \longrightarrow (C',\varphi')\\ [f] : (C,\psi) \longrightarrow (C',\psi')\end{cases}$$

is an isomorphism of $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes

$$\begin{cases}f : (C,\varphi)\oplus C^{\varepsilon}(P) \longrightarrow (C',\varphi')\oplus C^{\varepsilon}(P')\\ f : (C,\psi)\oplus C_{\varepsilon}(P) \longrightarrow (C',\psi')\oplus C_{\varepsilon}(P')\end{cases}$$

for some f.g. projective A-modules P,P', with $\begin{cases}C^{\varepsilon}(P) = (D,\eta\in Q^2(D,\varepsilon))\\ C_{\varepsilon}(P) = (D,\xi\in Q_2(D,\varepsilon))\end{cases}$ the contractible 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A in normal form defined by

$$D : \ldots \longrightarrow 0 \longrightarrow P \xrightarrow{\begin{pmatrix}0\\-\varepsilon\end{pmatrix}} P^*\oplus P \xrightarrow{(1\ \ 0)} P^* \longrightarrow 0 \longrightarrow \ldots$$

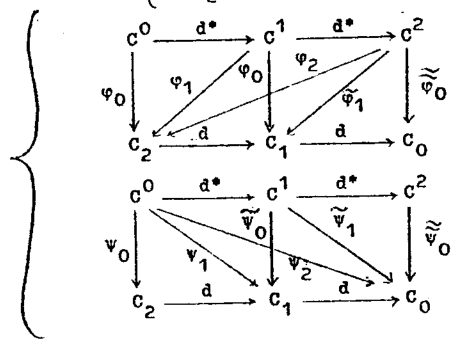
$$\begin{cases}\eta_0 = \begin{cases}1 : D^0 = P \longrightarrow D_2 = P\\ \begin{pmatrix}0 & 1\\ -\varepsilon & 0\end{pmatrix} : D^1 = P\oplus P^* \longrightarrow D_1 = P^*\oplus P\\ \varepsilon : D^2 = P^* \longrightarrow D_0 = P^*\end{cases} ,\ \eta_s = 0 : D^{2-r+s} \longrightarrow D_r\ (s \geqslant 0)\\ \xi_0 = \begin{cases}1 : D^0 = P \longrightarrow D_2 = P\\ \begin{pmatrix}0 & 0\\ -\varepsilon & 0\end{pmatrix} : D^1 = P\oplus P^* \longrightarrow D_1 = P^*\oplus P\\ 0 : D^2 = P^* \longrightarrow D_0 = P^*\end{cases} ,\ \xi_s = 0 : D^{2-r-s} \longrightarrow D_r\ (s \geqslant 0)\end{cases}$$

and similarly for $\begin{cases}C^{\varepsilon}(P')\\ C_{\varepsilon}(P')\end{cases}$.

<u>Proposition 13.11</u> The homotopy equivalence classes of connected 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes over A are in a natural one-one correspondence with the stable isomorphism classes of connected 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complexes over A in normal form.

<u>Proof</u>: A stable isomorphism is a homotopy equivalence. Therefore it is sufficient to prove that every connected 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex is homotopy equivalent to one in normal form, and that homotopy equivalent complexes determine stably isomorphic complexes in normal form.

Every 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(C,\varphi\in Q^2(C,\varepsilon))\\ (C,\psi\in Q_2(C,\varepsilon))\end{cases}$ is homotopy equivalent to one in which the chain complex C is in normal form, and for such C the class $\begin{cases}\varphi\in Q^2(C,\varepsilon)\\ \psi\in Q_2(C,\varepsilon)\end{cases}$ is represented by A-module morphisms



such that

$$\left\{\begin{array}{l}\left\{\begin{array}{l} d\varphi_0 - \tilde{\varphi}_0 d^* = 0 : C^0 \longrightarrow C_1 \ , \ d\tilde{\varphi}_0 + \tilde{\tilde{\varphi}}_0 d^* = 0 : C^1 \longrightarrow C_0 \\ \varphi_1 d^* + \varphi_0 - \varepsilon\tilde{\tilde{\varphi}}_0^* = 0 : C^0 \longrightarrow C_2 \ , \ d\varphi_1 - \tilde{\varphi}_1 d^* + \tilde{\varphi}_0 + \varepsilon\tilde{\varphi}_0^* = 0 : C^1 \longrightarrow C_1 \ , \\ d\tilde{\varphi}_1 + \tilde{\tilde{\varphi}}_0 - \varepsilon\varphi_0^* = 0 : C^2 \longrightarrow C_0 \ , \ \varphi_2 d^* - \varphi_1 - \varepsilon\tilde{\varphi}_1^* = 0 : C^1 \longrightarrow C_2 \ , \\ d\varphi_2 - \tilde{\varphi}_1 - \varepsilon\varphi_1^* = 0 : C^2 \longrightarrow C_1 \ , \ \varphi_2 - \varepsilon\varphi_2^* = 0 : C^2 \longrightarrow C_2 \end{array}\right. \\ \left\{\begin{array}{l} d\psi_0 - \tilde{\psi}_0 d^* - \psi_1 + \varepsilon\tilde{\psi}_1^* = 0 : C^0 \longrightarrow C_1 \ , \\ d\tilde{\psi}_0 + \tilde{\tilde{\psi}}_0 d^* - \tilde{\psi}_1 + \varepsilon\psi_1^* = 0 : C^1 \longrightarrow C_0 \ , \\ d\psi_1 + \tilde{\psi}_1 d^* + \psi_2 + \varepsilon\psi_2^* = 0 : C^0 \longrightarrow C_0 \ . \end{array}\right. \end{array}\right.$$

Such a complex $\left\{\begin{array}{l}(C,\varphi) \\ (C,\psi)\end{array}\right.$ is connected if and only if the A-module morphism

$\left\{\begin{array}{l}(d \ \ \tilde{\tilde{\varphi}}_0) : C_1 \oplus C^2 \longrightarrow C_0 \\ (d \ \ \tilde{\tilde{\psi}}_0 + \varepsilon\psi_0^*) : C_1 \oplus C^2 \longrightarrow C_0\end{array}\right.$ is onto, in which case we shall construct a homoto

equivalent complex $\left\{\begin{array}{l}(C',\varphi' \in Q^2(C',\varepsilon)) \\ (C',\psi' \in Q_2(C',\varepsilon))\end{array}\right.$ in normal form, as follows.

Define a connected 2-dimensional ε-symmetric complex over A

$(C',\varphi' \in Q^2(C',\varepsilon))$ in normal form by

$$d' = \begin{pmatrix} d \\ 0 \end{pmatrix} : C'_2 = C_2 \longrightarrow C'_1 = \ker((d \ \ -\varphi_0^*) : C_1 \oplus C^2 \longrightarrow C_0) \ ,$$

$$d' = (0 \ \ 1) : C'_1 \longrightarrow C'_0 = C^2 \ , \ \varphi'_0 = 1 : C'^2 = C^2 \longrightarrow C'_0 = C^2 \ ,$$

$$\tilde{\varphi}'_0 = \begin{bmatrix} \tilde{\varphi}_0 - \tilde{\varphi}_1 d^* & d \\ -\varepsilon d^* & 0 \end{bmatrix} : C'^1 = \operatorname{coker}\left(\begin{pmatrix} d^* \\ -\varphi_0 \end{pmatrix} : C^0 \longrightarrow C^1 \oplus C_2\right) \longrightarrow C'_1 \ ,$$

$$\tilde{\tilde{\varphi}}'_0 = \varepsilon : C'^0 = C_2 \longrightarrow C'_2 = C_2 \ , \ \varphi'_1 = [\varphi_2 d^* \ \ 0] : C'^1 \longrightarrow C'_2 = C_2$$

$$\tilde{\varphi}'_1 = 0 : C'^2 = C_2 \longrightarrow C'_1 \ , \ \varphi'_2 = \varphi_2 : C'^2 = C^2 \longrightarrow C'_2 = C_2 \ .$$

The chain equivalence $f : C' \longrightarrow C$ given by

$$\begin{array}{ccccccccccc} C' : & \dots \longrightarrow & 0 & \longrightarrow & C'_2 & \xrightarrow{d'} & C'_1 & \xrightarrow{d'} & C'_0 & \longrightarrow 0 \longrightarrow & \dots \\ f \big\downarrow & & & & 1 \big\downarrow & & (1 \ 0) \big\downarrow & & \varphi_0^* \big\downarrow & & \\ C : & \dots \longrightarrow & 0 & \longrightarrow & C_2 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \longrightarrow 0 \longrightarrow & \dots \end{array}$$

defines a homotopy equivalence of 2-dimensional ε-symmetric complexes

$$f : (C',\varphi') \longrightarrow (C,\varphi) \ .$$

Given $(C,\psi \in Q_2(C,\varepsilon))$ as above we shall define first an auxiliary 2-dimensional ε-quadratic complex over A $(C'',\psi'' \in Q_2(C'',\varepsilon))$ by

$$d'' = \begin{pmatrix} d \\ 0 \end{pmatrix} : C''_2 = C_2 \longrightarrow C''_1 = C_1 \oplus C^2 \ ,$$

$$d'' = \begin{pmatrix} d & -(\psi_0 + \varepsilon\tilde{\tilde{\psi}}_0^*)^* \\ 0 & 1 \end{pmatrix} : C''_1 = C_1 \oplus C^2 \longrightarrow C''_0 = C_0 \oplus C^2$$

$$\psi''_0 = (0 \ \ 1) : C''^0 = C^0 \oplus C_2 \longrightarrow C''_2 = C_2$$

$$\tilde{\psi}''_0 = \begin{pmatrix} \tilde{\psi}_0 & 0 \\ -\varepsilon d^* & 0 \end{pmatrix} : C''^1 = C^1 \oplus C_2 \longrightarrow C''_1 = C_1 \oplus C^2$$

$$\tilde{\tilde{\psi}}''_0 = 0 : C''^2 = C^2 \longrightarrow C''_0 = C_0 \oplus C^2$$

$$\psi''_1 = \begin{pmatrix} -\tilde{\psi}_0 d^* & d \\ 0 & 0 \end{pmatrix} : C''^0 = C^0 \oplus C_2 \longrightarrow C''_1 = C_1 \oplus C^2$$

$$\tilde{\psi}''_1 = 0 : C''^1 = C^1 \oplus C_2 \longrightarrow C''_0 = C_0 \oplus C^2$$

$$\psi''_2 = \begin{pmatrix} \psi_2 + \tilde{\psi}_1 d^* & 0 \\ 0 & 0 \end{pmatrix} : C''^0 = C^0 \oplus C_2 \longrightarrow C''_0 = C_0 \oplus C^2$$

The chain equivalence $f'' : C \longrightarrow C''$ given by

$$\begin{array}{ccccccccccc} C : & \dots \longrightarrow & 0 & \longrightarrow & C_2 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \longrightarrow 0 \longrightarrow & \dots \\ f'' \big\downarrow & & & & 1 \big\downarrow & & \big\downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \big\downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \\ C'' : & \dots \longrightarrow & 0 & \longrightarrow & C''_2 & \xrightarrow{d''} & C''_1 & \xrightarrow{d''} & C''_0 & \longrightarrow 0 \longrightarrow & \dots \end{array}$$

defines a homotopy equivalence of 2-dimensional ε-quadratic complexes over A

$$f'' : (C,\psi) \longrightarrow (C'',\psi'') \ .$$

Define a 2-dimensional A-module chain complex C' in normal form

$$C' : \dots \longrightarrow 0 \longrightarrow C'_2 \xrightarrow{d'} C'_1 \xrightarrow{d'} C'_0 \longrightarrow 0 \longrightarrow \dots$$

by

$$d' = \begin{pmatrix} d \\ 0 \end{pmatrix} : C'_2 = C_2 \longrightarrow C'_1 = \ker((d \ \ \tilde{\tilde{\psi}}_0 + \varepsilon\psi_0^*) : C_1 \oplus C^2 \longrightarrow C_0)$$

$$d' = (0 \ \ 1) : C'_1 \longrightarrow C'_0 = C^2 \ .$$

Choose a splitting map $\begin{pmatrix} j \\ k \end{pmatrix} : C_0 \longrightarrow C_1 \oplus C^2$ for $(d \ \widetilde{\widetilde{\psi}}_0 + \varepsilon\psi_0^*) : C_1 \oplus C^2 \longrightarrow C_0$, such that

$$dj + (\widetilde{\widetilde{\psi}}_0 + \varepsilon\psi_0^*)k = 1 : C_0 \longrightarrow C_0 \ ,$$

and define a chain equivalence $f' : C'' \longrightarrow C'$

$$\begin{array}{ccccccccccc} C'' : & \dots \longrightarrow 0 \longrightarrow & C''_2 & \xrightarrow{d''} & C''_1 & \xrightarrow{d''} & C''_0 & \longrightarrow 0 \longrightarrow \dots \\ f' \big\downarrow & & f'_2 \big\downarrow & & f'_1 \big\downarrow & & f'_0 \big\downarrow & \\ C' : & \dots \longrightarrow 0 \longrightarrow & C'_2 & \xrightarrow{d'} & C'_1 & \xrightarrow{d'} & C'_0 & \longrightarrow 0 \longrightarrow \dots \end{array}$$

by

$$f'_0 = (-k \ -\overline{\varepsilon}) : C''_0 = C_0 \oplus C^2 \longrightarrow C'_0 = C^2$$

$$f'_1 = \begin{pmatrix} 1 - jd & \overline{\varepsilon} j (\widetilde{\widetilde{\psi}}_0 + \varepsilon\psi_0^*) \\ -kd & \overline{\varepsilon} k (\widetilde{\widetilde{\psi}}_0 + \varepsilon\psi_0^*) - \overline{\varepsilon} \end{pmatrix}$$
$$: C''_1 = C_1 \oplus C^2 \longrightarrow C'_1 = \ker((d \ \widetilde{\widetilde{\psi}}_0 + \varepsilon\psi_0^*) : C_1 \oplus C^2 \longrightarrow C_0)$$

$$f'_2 = 1 : C''_2 = C_2 \longrightarrow C'_2 = C_2 \ .$$

The connected 2-dimensional ε-quadratic complex over A $(C', \psi' \in Q_2(C', \varepsilon))$ defined by

$$\psi' = f'_\%(\psi'') \in Q_2(C', \varepsilon)$$

is in normal form, and there is defined a homotopy equivalence

$$f = f'f'' : (C, \psi) \longrightarrow (C', \psi') \ .$$

The above procedure associates to an isomorphism class of connected 2-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes over A $\begin{cases} (C,\varphi) \\ (C,\psi) \end{cases}$ with C in normal form an isomorphism class of connected 2-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes over A $\begin{cases} (C',\varphi') \\ (C',\psi') \end{cases}$ in normal form, preserving the homotopy type and also the direct sum $\oplus$. In particular, if C is a chain contractible 2-dimensional A-module chain complex in normal form then it is isomorphic to

$$\dots \longrightarrow 0 \longrightarrow P \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} P \oplus Q \xrightarrow{(0 \ 1)} Q \longrightarrow 0 \longrightarrow \dots$$

for some f.g. projective A-modules P, Q and $\begin{cases} (C',\varphi') \\ (C',\psi') \end{cases}$ is isomorphic to $\begin{cases} C^\varepsilon(P) \\ C_\varepsilon(P) \end{cases}$, and hence stably isomorphic to 0. It will follow from the Lemma below that homotopy equivalent complexes $\begin{cases} (C,\varphi), (\widetilde{C},\widetilde{\varphi}) \\ (C,\psi), (\widetilde{C},\widetilde{\psi}) \end{cases}$ with C and $\widetilde{C}$ in normal form determine stably isomorphic complexes in normal form $\begin{cases} (C',\varphi'), (\widetilde{C}',\widetilde{\varphi}') \\ (C',\psi'), (\widetilde{C}',\widetilde{\psi}') \end{cases}$.

<u>Lemma</u> Let $\begin{cases} (C,\varphi), (\widetilde{C},\widetilde{\varphi}) \\ (C,\psi), (\widetilde{C},\widetilde{\psi}) \end{cases}$ be 2-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes over A with C and $\widetilde{C}$ in normal form. There exists a homotopy equivalence

$$\begin{cases} f : (C,\varphi) \longrightarrow (\widetilde{C},\widetilde{\varphi}) \\ f : (C,\psi) \longrightarrow (\widetilde{C},\widetilde{\psi}) \end{cases}$$

if and only if there exists an isomorphism

$$\begin{cases} i : (C,\varphi) \oplus (D,0) \longrightarrow (\widetilde{C},\widetilde{\varphi}) \oplus (\widetilde{D},0) \\ i : (C,\psi) \oplus (D,0) \longrightarrow (\widetilde{C},\widetilde{\psi}) \oplus (\widetilde{D},0) \end{cases}$$

for some contractible 2-dimensional A-module chain complexes $D, \widetilde{D}$ in normal form.

<u>Proof</u>: Given such an isomorphism i define a homotopy equivalence f by the composition

$$f : C \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} C \oplus D \xrightarrow{i} \widetilde{C} \oplus \widetilde{D} \xrightarrow{(1 \ 0)} \widetilde{C} \ .$$

Conversely, let f be a homotopy equivalence of the type stated. The algebraic mapping cone $C(f)$, as defined by

$$d_{C(f)} = \begin{pmatrix} \widetilde{d} & (-)^{r-1} f \\ 0 & d \end{pmatrix} : C(f)_r = \widetilde{C}_r \oplus C_{r-1} \longrightarrow C(f)_{r-1} = \widetilde{C}_{r-1} \oplus C_{r-2} \ ,$$

is chain contractible. Choose a chain contraction

$$\Gamma' = \begin{pmatrix} \tilde{g} & h \\ \tilde{f} & g \end{pmatrix} : C(f)_r = \tilde{C}_r \oplus C_{r-1} \longrightarrow C(f)_{r+1} = \tilde{C}_{r+1} \oplus C_r \ ,$$

so that

$$d_{C(f)}\Gamma' + \Gamma' d_{C(f)} = 1 : C(f)_r \longrightarrow C(f)_r \ ,$$

and define chain contractible 2-dimensional A-module chain complexes $D,\tilde{D}$ in normal form by

$$d_D = \begin{cases} \begin{pmatrix} \tilde{d} & -f \\ 0 & d \end{pmatrix} : D_2 = \mathrm{coker}\left(\begin{pmatrix} f \\ d \end{pmatrix} : C_2 \longrightarrow \tilde{C}_2 \oplus C_1\right) \longrightarrow D_1 = \tilde{C}_1 \oplus C_0 \\ \begin{pmatrix} \tilde{d} & f \end{pmatrix} : D_1 = \tilde{C}_1 \oplus C_0 \longrightarrow D_0 = \tilde{C}_0 \ , \quad D_r = 0 \ (r \neq 0,1,2) \end{cases}$$

$$d_{\tilde{D}} = \begin{cases} \begin{pmatrix} 1 \\ d \end{pmatrix} : \tilde{D}_2 = C_1 \longrightarrow \tilde{D}_1 = C_1 \oplus C_0 \ , \quad \tilde{D}_r = 0 \ (r \neq 0,1,2) \\ \begin{pmatrix} d & -1 \end{pmatrix} : \tilde{D}_1 = C_1 \oplus C_0 \longrightarrow \tilde{D}_0 = C_0 \end{cases}$$

The isomorphism of A-module chain complexes

$$i : C \oplus D \longrightarrow \tilde{C} \oplus \tilde{D}$$

given by

$$i = \begin{cases} \begin{pmatrix} f & \begin{bmatrix} 1 - f\tilde{f} & -fg \\ -d\tilde{f} & 1 - dg \end{bmatrix} \\ 0 & \end{pmatrix} \\ \quad : C_2 \oplus D_2 = C_2 \oplus \mathrm{coker}\left(\begin{pmatrix} f \\ d \end{pmatrix} : C_2 \longrightarrow \tilde{C}_2 \oplus C_1\right) \longrightarrow \tilde{C}_2 \oplus \tilde{D}_2 = \tilde{C}_2 \oplus C_1 \ , \\ \begin{pmatrix} f & 1 + (-)^{r-1} f\tilde{f} & (-)^{r-1} fg \\ 1 & (-)^{r-1}\tilde{f} & (-)^{r-1} g \\ 0 & 0 & 1 \end{pmatrix} \\ \quad : C_r \oplus D_r = C_r \oplus \tilde{C}_r \oplus C_{r-1} \longrightarrow \tilde{C}_r \oplus \tilde{D}_r = \tilde{C}_r \oplus C_r \oplus C_{r-1} \quad (r = 0,1) \end{cases}$$

defines an isomorphism of 2-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ complexes

$$\begin{cases} i : (C,\varphi) \oplus (D,0) \longrightarrow (\tilde{C},\tilde{\varphi}) \oplus (\tilde{D},0) \\ i : (C,\psi) \oplus (D,0) \longrightarrow (\tilde{C},\tilde{\psi}) \oplus (\tilde{D},0) \ . \end{cases}$$

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An $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ S-form over A $\begin{cases} (K,\alpha;L) \\ (K,\beta;L) \end{cases}$ is a non-degenerate $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ form over A $\begin{cases} (K,\alpha \in Q^{\varepsilon}(K)) \\ (K,\beta \in Q_{\varepsilon}(K)) \end{cases}$ together with an S-lagrangian L.

The S-form $\begin{cases} (K,\alpha;L) \\ (K,\beta;L) \end{cases}$ is non-singular if the form $\begin{cases} (K,\alpha) \\ (K,\beta) \end{cases}$ is non-singular.

An isomorphism of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ S-forms over A

$$\begin{cases} f : (K,\alpha;L) \longrightarrow (K',\alpha';L') \\ f : (K,\beta;L) \longrightarrow (K',\beta';L') \end{cases}$$

is an isomorphism of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ forms over A

$$\begin{cases} f : (K,\alpha) \longrightarrow (K',\alpha') \\ f : (K,\beta) \longrightarrow (K',\beta') \end{cases}$$

such that

$$f(L) = L' \ .$$

A stable isomorphism of $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ S-forms over A

$$\begin{cases} [f] : (K,\alpha;L) \longrightarrow (K',\alpha';L') \\ [f] : (K,\beta;L) \longrightarrow (K',\beta';L') \end{cases}$$

is an isomorphism of S-forms

$$\begin{cases} f : (K,\alpha;L) \oplus (M,\varphi;N) \longrightarrow (K',\alpha';L') \oplus (M',\varphi';N') \\ f : (K,\beta;L) \oplus (M,\psi;N) \longrightarrow (K',\beta';L') \oplus (M',\psi';N') \end{cases}$$

for some non-singular $\begin{cases} \text{(even) } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ S-forms over A $\begin{cases} (M,\varphi;N),(M',\varphi';N') \\ (M,\psi;N),(M',\psi';N') \end{cases}$

such that N is a lagrangian of $\begin{cases} (M,\varphi) \\ (M,\psi) \end{cases}$ and N' is a lagrangian of $\begin{cases} (M',\varphi') \\ (M',\psi') \end{cases}$.

<u>Proposition 13.12</u> i) The stable equivalence classes of $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$

linking formations over (A,S) are in a natural one-one correspondence with the stable isomorphism classes of $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ S-forms over A.

Non-singular linking formations correspond to non-singular S-forms.

ii) The stable equivalence classes of (even) ε-symmetric linking formations over (A,S) $(M,\lambda;F,G)$ are in a natural one-one correspondence with the homotopy equivalence classes of connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complexes over A $(C,\varphi\in Q^2(C,-\varepsilon))$. Under this correspondence the exact sequence

$$0 \longrightarrow H^1(C) \xrightarrow{\varphi_0} H_1(C) \longrightarrow H_1(\varphi_0) \longrightarrow H^2(C) \xrightarrow{\varphi_0} H_0(C) \longrightarrow 0$$

can be identified with

$$0 \longrightarrow F\cap G \longrightarrow F\cap G^{\perp} \longrightarrow G^{\perp}/G \longrightarrow M/F+G \longrightarrow M/F+G^{\perp} \longrightarrow 0\ ,$$

and

$$v_S^0(\varphi)\ :\ H^2(C) = M/F+G \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\ ;\ x \longmapsto \lambda(x)(x)\ .$$

Non-singular linking formations correspond to Poincaré complexes.

iii) The homotopy equivalence classes of connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complexes over A $(C,\psi\in Q_2(C,-\varepsilon))$ naturally project onto the stable equivalence classes of split ε-quadratic linking formations over (A,S) $(F,(\binom{\gamma}{\mu}),\theta)G)$. If the complexes (C,ψ), (C',ψ') project to the same stable equivalence class then (C',ψ') is homotopy equivalent to a complex obtained from (C,ψ) by an S-acyclic $(-\varepsilon)$-quadratic surgery preserving the $(-\varepsilon)$-symmetric homotopy type, and

$$pv_S^1(\psi) = pv_S^1(\psi')\ :\ H^1(C) = \ker(\mu:G\longrightarrow F^{\wedge}) \longrightarrow Q_{-\varepsilon}(A,S)\ ;\ x \longmapsto \theta(x)\ .$$

Poincaré complexes project to non-singular linking formations.

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(Before embarking on the proof of Proposition 13.12 we remark on the similarity between these correspondences and those of

(linking forms over (A,S))

$\longleftrightarrow$ (S-acyclic 1-dimensional complexes over A) (Proposition 13.4) ,

(linking forms over (A,S)) $\longleftrightarrow$ (S-formations over A) (Proposition 13.6) ,

(formations over A) $\longleftrightarrow$ (1-dimensional complexes over A)

(Propositions 1.7, 1.8) .

In particular, given a connected 1-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ complex over A $\left\{\begin{array}{l}(C,\varphi\in Q^1(C,\varepsilon))\\ (C,\psi\in Q_1(C,\varepsilon))\end{array}\right.$ and an associated $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ formation over A $\left\{\begin{array}{l}(M,\alpha;F,G)\\ (F,(\binom{\gamma}{\mu}),\theta)G)\end{array}\right.$ we can identify the exact sequence

$$0 \longrightarrow H^0(C) \xrightarrow{\varphi_0} H_1(C) \longrightarrow H_1(\varphi_0) \longrightarrow H^1(C) \xrightarrow{\varphi_0} H_0(C) \longrightarrow 0$$

with

$$0 \longrightarrow F\cap G \longrightarrow F\cap G^{\perp} \longrightarrow G^{\perp}/G \longrightarrow M/F+G \longrightarrow M/F+G^{\perp} \longrightarrow 0$$

(take $\varphi = (1+T_\varepsilon)\psi$ in the ε-quadratic case), and the $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Wu classes are given by

$$\left\{\begin{array}{l} v_0(\varphi)\ :\ H^1(C) = M/F+G \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\ ;\ [x] \longmapsto \alpha(x)(x) \quad (x\in M)\\ v^1(\psi)\ :\ H^0(C) = F\cap G = \ker(\mu:G\longrightarrow F^*) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)\ ;\ y \longmapsto \theta(y)(y)).\end{array}\right.$$

<u>Proof</u>: i) Given an $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\end{array}\right.$

we have from Proposition 13.9 that the $\left|\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ linking form over (A,S)

$\left\{\begin{array}{l}(M,\lambda)\\ (M,\lambda,\mu)\end{array}\right.$ is isomorphic to the boundary $\partial(K,\alpha)$ of an S-metabolic non-degenerate

$\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ form over A $\left\{\begin{array}{l}(K,\alpha\in Q^{\varepsilon}(K))\\ (K,\alpha\in \mathrm{im}(1+T_{\varepsilon}:Q_{\varepsilon}(K)\longrightarrow Q^{\varepsilon}(K))\end{array}\right.$, and that

$$F = \mathrm{coker}(f:K\longrightarrow K_F)\subseteq M = \mathrm{coker}(\alpha:K\longrightarrow K^*)$$

$$G = \mathrm{coker}(g:K\longrightarrow K_G)\subseteq M = \mathrm{coker}(\alpha:K\longrightarrow K^*)$$

for some isomorphism of non-degenerate $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ forms over A

$$f : (K,\alpha)\longrightarrow (K_F,\alpha_F)\ ,\ g : (K,\alpha)\longrightarrow(K_G,\alpha_G)$$

with (K_F,α_F) non-singular. The $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ S-form over A associated to

$\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\end{array}\right.$ is to be

$$(K_F\oplus K_G,\begin{pmatrix}\alpha_F & 0\\ 0 & -\alpha_G\end{pmatrix}\in Q^{\varepsilon}(K_F\oplus K_G);\mathrm{im}(\begin{pmatrix}f\\ g\end{pmatrix}: K\longrightarrow K_F\oplus K_G))\ .$$

We defer to ii) the proof that the stable isomorphism class of this S-form is is independent of the choice of non-degenerate form (K,α) such that $\left\{\begin{array}{l}(M,\lambda) = \partial(K,\alpha)\\ (M,\lambda,\mu)=\partial(K,\alpha)\end{array}\right.$

Given an $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\end{array}\right.$

and a sublagrangian H write the linking formation obtained by elementary equivalence as

$$\left\{\begin{array}{l}(M',\lambda';F',G') = (H^{\perp}/H,\lambda^{\perp}/\lambda;F\cap H^{\perp},G/H)\\ (M',\lambda',\mu';F',G') = (H^{\perp}/H,\lambda^{\perp}/\lambda,\mu|;F\cap H^{\perp},G/H)\ .\end{array}\right.$$

Continuing the previous terminology, let

$$h : (K,\alpha)\longrightarrow(K',\alpha')$$

be the S-isomorphism of non-degenerate $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ forms over A

associated to H by Proposition 13.9 i), with

$$H = \mathrm{coker}(h:K\longrightarrow K')\subseteq M = \mathrm{coker}(\alpha:K\longrightarrow K^*)\ ,\ \left\{\begin{array}{l}(M',\lambda') = \partial(K',\alpha')\\ (M',\lambda',\mu') = \partial(K',\alpha')\ .\end{array}\right.$$

As $H\subseteq G$ there is also defined an S-isomorphism of non-degenerate forms over A

$$g' : (K',\alpha')\longrightarrow(K_G,\alpha_G)$$

such that

$$g = g'h : (K,\alpha)\xrightarrow{h}(K',\alpha')\xrightarrow{g'}(K_G,\alpha_G)\ .$$

The composite

$$F\xrightarrow{[\text{inclusion}]}M/H^{\perp}\xrightarrow{[\lambda]}H^{\wedge}$$

is onto, with resolution

$$\begin{array}{ccccccccc}0 & \longrightarrow & K & \xrightarrow{f} & K_F & \longrightarrow & F & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \alpha'h} & & \downarrow{\scriptstyle f^*\alpha_F} & & \downarrow & & \\ 0 & \longrightarrow & K'^* & \xrightarrow{h^*} & K^* & \longrightarrow & H^{\wedge} & \longrightarrow & 0\end{array}$$

Thus $F' = \ker(F\longrightarrow H^{\wedge})$ has resolution

$$0\longrightarrow K\xrightarrow{e}J\longrightarrow F'\longrightarrow 0$$

with

$$e = \begin{pmatrix}f\\ \alpha'h\end{pmatrix} : K\longrightarrow J = \ker((f^*\alpha_F\ -h^*) : K_F\oplus K'^*\longrightarrow K^*)\ .$$

Define a non-singular $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ form over A

$$(R,\rho) = (K_F\oplus K'^*\oplus K',\begin{pmatrix}\alpha_F & 0 & 0\\ 0 & 0 & 1\\ 0 & \varepsilon & \alpha'\end{pmatrix}\in Q^{\varepsilon}(K_F\oplus K'^*\oplus K'))\ ,$$

and let L be the sublagrangian of (R,ρ) given by

$$L = \mathrm{im}(\begin{pmatrix} f \\ -\alpha'^*h \\ h \end{pmatrix} : K \longrightarrow K_F\oplus K'^*\oplus K') ,$$

so that

$$(L^{\perp}/L,\rho^{\perp}/\rho) = (K_{F'},\alpha_{F'})$$

is also a non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ form over A. The S-isomorphism of f.g. projective A-modules

$$f' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : K' \longrightarrow K_{F'} = \frac{\ker((f^*\alpha_F\ \ \varepsilon h^*\ \ 0) : K_F\oplus K'^*\oplus K' \longrightarrow K^*)}{\mathrm{im}(\begin{pmatrix} f \\ -\alpha'^*h \\ h \end{pmatrix} : K \longrightarrow K_F\oplus K'^*\oplus K')}$$

defines an S-isomorphism of non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ forms over A

$$f' : (K',\alpha') \longrightarrow (K_{F'},\alpha_{F'})$$

such that the associated lagrangian of $\partial(K',\alpha') = \begin{cases}(M',\lambda')\\ (M',\lambda',\mu')\end{cases}$ is precisely

$$\mathrm{coker}(f':K' \longrightarrow K_{F'}) = \mathrm{coker}(e:K \longrightarrow J) = F' .$$

Thus

$$(K_{F'}\oplus K_G,\begin{pmatrix}\alpha_{F'} & 0 \\ 0 & -\alpha_G\end{pmatrix}\in Q^{\varepsilon}(K_{F'}\oplus K_G);\mathrm{im}(\begin{pmatrix} f' \\ g' \end{pmatrix} : K' \longrightarrow K_{F'}\oplus K_G))$$

is the $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ S-form over A associated to $\begin{cases}(M',\lambda';F',G')\\ (M',\lambda',\mu';F',G')\end{cases}$.

Define an $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ S-form over A

$$(Q,\varphi;P) = (K_F\oplus K'^*\oplus K'\oplus K_G,\begin{pmatrix}\alpha_F & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 1 & -\alpha' & 0 \\ 0 & 0 & 0 & -\alpha_G\end{pmatrix};$$

$$\mathrm{im}(\begin{pmatrix} f & 0 \\ -\alpha'h & -\alpha' \\ 0 & 1 \\ g & g' \end{pmatrix} : K\oplus K' \longrightarrow K_F\oplus K'^*\oplus K'\oplus K_G)) .$$

By Proposition 1.6 the inclusion of the sublagrangian

$$\begin{pmatrix} f \\ -\alpha'h \\ h \\ 0 \end{pmatrix} : (K,0) \longrightarrow (Q,\varphi)$$

extends to an isomorphism of $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ forms over A

$$H^{\varepsilon}(K,\theta)\oplus(K^{\perp}/K,\varphi^{\perp}/\varphi) = (K\oplus K^*,\begin{pmatrix}0 & 1 \\ \varepsilon & \theta\end{pmatrix})\oplus(K_{F'}\oplus K_G,\begin{pmatrix}\alpha_{F'} & 0 \\ 0 & -\alpha_G\end{pmatrix}) \longrightarrow (Q,\varphi)$$

sending $K\oplus\mathrm{im}(\begin{pmatrix} f' \\ g' \end{pmatrix} : K' \longrightarrow K_{F'}\oplus K_G)$ to P, for some $\begin{cases}\theta\in Q^{\varepsilon}(K^*)\\ \theta\in\mathrm{im}(1+T_{\varepsilon}:Q_{\varepsilon}(K^*)\longrightarrow Q^{\varepsilon}(K^*))\end{cases}$.

Thus there is defined an isomorphism of $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ forms over A

$$(H^{\varepsilon}(K,\theta);K)\oplus(K_{F'}\oplus K_G,\begin{pmatrix}\alpha_{F'} & 0 \\ 0 & -\alpha_G\end{pmatrix});\mathrm{im}(\begin{pmatrix} f' \\ g' \end{pmatrix}:K' \longrightarrow K_{F'}\oplus K_G)) \longrightarrow (Q,\varphi;P) .$$

There is also defined an isomorphism of $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\end{cases}$ forms over A

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & g' & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & g'^*\alpha_G \end{pmatrix}$$

$$: (Q,\varphi;P) \longrightarrow (K_F\oplus K_G, \begin{pmatrix} \alpha_F & 0 \\ 0 & -\alpha_G \end{pmatrix}; \mathrm{im}(\begin{pmatrix} f \\ g \end{pmatrix}: K \longrightarrow K_F\oplus K_G))\oplus(H^{\varepsilon}(K');K')$$

Thus the $\left\{\begin{matrix} \varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric} \end{matrix}\right.$ S-forms over A associated to stably equivalent $\left\{\begin{matrix} \text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{matrix}\right.$ linking formations over (A,S) $\left\{\begin{matrix} (M,\lambda;F,G),(M',\lambda';F',G') \\ (M,\lambda,\mu;F,G),(M',\lambda',\mu';F',G') \end{matrix}\right.$ are related by a stable isomorphism

$$(K_F\oplus K_G, \begin{pmatrix} \alpha_F & 0 \\ 0 & -\alpha_G \end{pmatrix}; \mathrm{im}(\begin{pmatrix} f \\ g \end{pmatrix}: K \longrightarrow K_F\oplus K_G))$$

$$\longrightarrow (K_{F'}\oplus K_{G'}, \begin{pmatrix} \alpha_{F'} & 0 \\ 0 & -\alpha_{G'} \end{pmatrix}; \mathrm{im}(\begin{pmatrix} f' \\ g' \end{pmatrix}: K' \longrightarrow K_{F'}\oplus K_{G'})) .$$

Given a split ε-quadratic linking formation over (A,S) $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ we shall obtain an ε-quadratic S-form over A $(K,\beta;L)$, as follows. Let $u\in\mathrm{Hom}_A(L',L^*)$ be an S-isomorphism of f.g. projective A-modules such that there is defined a resolution for F

$$0 \longrightarrow L \xrightarrow{u^*} L'^* \longrightarrow F \longrightarrow 0 .$$

Let $e\in\mathrm{Hom}_A(L^*\oplus L'^*, F\oplus F^\wedge)$ be the projection appearing in the corresponding resolution for $F\oplus F^\wedge$

$$0 \longrightarrow L\oplus L' \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & 0 \end{pmatrix}} L^*\oplus L'^* \xrightarrow{\ e\ } F\oplus F^\wedge \longrightarrow 0 ,$$

and define a f.g. projective A-module

$$K = e^{-1}(G) \subseteq L^*\oplus L'^* .$$

The inclusion $(j\ k) : L\oplus L' \longrightarrow K$ defines a resolution for $G = \mathrm{coker}(j\ k)$, with $\begin{pmatrix}\gamma\\ \mu\end{pmatrix}: G \longrightarrow F\oplus F^\wedge$ resolved by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L' & \xrightarrow{(j\ k)} & K & \longrightarrow & G & \longrightarrow & 0 \\ & & \big\downarrow 1 & & \big\downarrow \begin{pmatrix} j^*\alpha \\ k^*\alpha \end{pmatrix} & & \big\downarrow \begin{pmatrix}\gamma\\ \mu\end{pmatrix} & & \\ 0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & 0 \end{pmatrix}} & L^*\oplus L'^* & \xrightarrow{\ e\ } & F\oplus F^\wedge & \longrightarrow & 0 \end{array}$$

for some non-degenerate even ε-symmetric form over A $(K,\alpha\in Q^{\varepsilon}(K))$. As $\mathrm{im}(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}: G \longrightarrow F\oplus F^\wedge)$ is a sublagrangian of the hyperbolic split ε-quadratic linking form over (A,S)

$$\widetilde{H}_{\varepsilon}(F) = \partial(L\oplus L', \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}\in Q_{\varepsilon}(L\oplus L'))$$

we have that

$$\alpha = \beta + \varepsilon\beta^* \in \mathrm{Hom}_A(K,K^*)$$

for some non-degenerate ε-quadratic form over A $(K,\beta\in Q_{\varepsilon}(K))$ - as in Proposition 13.9 i) this is only determined up to S-isomorphism, i.e only $[\beta]\in Q_{\varepsilon}(K)/\ker(S^{-1}:Q_{\varepsilon}(K)\longrightarrow Q_{\varepsilon}(S^{-1}K))$ is determined. Proposition 13.6 associates to the $(-\varepsilon)$-quadratic linking form over (A,S)

$$(G,\gamma^\wedge\mu\in\mathrm{Hom}_A(G,G^\wedge),\theta:G\longrightarrow Q_{-\varepsilon}(A,S))$$

an ε-quadratic S-formation over A

$$(K^*\oplus K, \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix}\in Q_{\varepsilon}(K^*\oplus K); K^*, \mathrm{im}(\begin{pmatrix} -(\beta+\varepsilon\beta^*)^*j & 0 \\ j & k \end{pmatrix}: L\oplus L' \longrightarrow K^*\oplus K))$$

with

$$\theta : G = \mathrm{coker}((j\ k) : L\oplus L' \longrightarrow K) \longrightarrow Q_{-\varepsilon}(A,S) ;$$

$$x \longmapsto \frac{\varepsilon u^*(y)(y')}{s\bar{s}} - \beta(x)(x)$$

$$(x\in K,\ s\in S,\ y\in L,\ y'\in L',\ sx = jy + ky'\in K) ,$$

for a unique ε-quadratic form over A $(K,\beta\in Q_{\varepsilon}(K))$ in the prescribed S-isomorphism class. The ε-quadratic S-form over A associated to $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$ is defined to be

$$(K,\beta\in Q_\varepsilon(K);\mathrm{im}(j:L\longrightarrow K))\ .$$

Conversely, given an $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ S-form over A $\begin{cases}(K,\alpha;L)\\ (K,\beta;L)\end{cases}$ we shall define an $\begin{cases}\text{even }\varepsilon\text{-symmetric }(\varepsilon\text{-quadratic})\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) $\begin{cases}(M,\lambda;F,G)\ ((M,\lambda,\mu;F,G))\\ (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)\end{cases}$, as follows.

Let $j\in \mathrm{Hom}_A(L,K)$. The inclusion of the lagrangian

$$\begin{cases} j\ :\ (S^{-1}L,0)\longrightarrow S^{-1}(K,\alpha)\\ j\ :\ (S^{-1}L,0)\longrightarrow S^{-1}(K,\beta)\end{cases}$$

can be extended (by Proposition 1.6) to an isomorphism of non-singular $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms over $S^{-1}A$

$$\begin{cases}(j\ j')\ :\ (S^{-1}L\oplus S^{-1}L^*,\begin{pmatrix}0 & 1\\ \varepsilon & j'^*\alpha j'\end{pmatrix})\longrightarrow S^{-1}(K,\alpha)\\ (j\ j')\ :\ (S^{-1}L\oplus S^{-1}L^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})\longrightarrow S^{-1}(K,\beta)\end{cases}$$

for some $j'\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*,S^{-1}K)$ such that

$$\begin{cases} j^*\alpha j' = 1\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*,S^{-1}L^*)\\ j^*(\beta+\varepsilon\beta^*)j' = 1\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*,S^{-1}L^*)\ .\end{cases}$$

Given a f.g. projective A-module L' such that the induced $S^{-1}A$-module $S^{-1}L'$ is isomorphic to $S^{-1}L^*$ there exists $k\in\mathrm{Hom}_A(L',K)$ such that $u = j^*\alpha k\in \mathrm{Hom}_A(L',L^*)$ is an S-isomorphism, with

$$j'u = k\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L',S^{-1}K).$$

(For example, take $L' = L^*$ so that $j' = \frac{k}{s}\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*,S^{-1}K)$ for some $s\in S$, $k\in \mathrm{Hom}_A(L^*,K)$, and $u = j^*\alpha k = s\in \mathrm{Hom}_A(L^*,L^*)$ is indeed an S-isomorphism).

In the ε-quadratic case we have

$$j'^*\beta j' = 0\in Q_\varepsilon(S^{-1}L^*)\quad ,$$

so that $k^*\beta k\in \ker(S^{-1}:Q_\varepsilon(L')\longrightarrow Q_\varepsilon(S^{-1}L'))$ and

$$k^*\beta k = \frac{\chi}{t} - \frac{\varepsilon\chi^*}{t}\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L',S^{-1}L'^*)$$

for some $t\in S$, $\chi\in \mathrm{Hom}_A(L',L'^*)$. Replacing k by $kt\in \mathrm{Hom}_A(L',K)$ we can ensure that

$$k^*\beta k = 0\in Q_\varepsilon(L')\ .$$

The S-isomorphism of non-degenerate $\begin{cases}\text{(even) }\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms over A

$$\begin{cases}\begin{pmatrix}u^* & 0\\ 0 & 1\end{pmatrix}\ :\ (L\oplus L',\begin{pmatrix}0 & u\\ \varepsilon u^* & k^*k\end{pmatrix})\longrightarrow (L'^*\oplus L',\begin{pmatrix}0 & 1\\ \varepsilon & k^*k\end{pmatrix})\\ \begin{pmatrix}u^* & 0\\ 0 & 1\end{pmatrix}\ :\ (L\oplus L',\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix})\longrightarrow (L'^*\oplus L',\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})\end{cases}$$

has non-singular range, corresponding by Proposition 13.9 i) to a lagrangian

$$F = \mathrm{coker}(u^*:L\longrightarrow L'^*)\subseteq M$$

of the boundary $\begin{cases}\text{even }\varepsilon\text{-symmetric }(\varepsilon\text{-quadratic})\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S)

$$\begin{cases}\partial(L\oplus L',\begin{pmatrix}0 & u\\ \varepsilon u^* & k^*\alpha k\end{pmatrix}) = (M,\lambda)\ \ (= (M,\lambda,\mu))\\ \partial(L\oplus L',\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}) = \widetilde{H}_\varepsilon(F) = (M,\lambda,\nu)\end{cases}\ .$$

The inclusion of the lagrangian $F\longrightarrow M$ has resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix}u^* & 0\\ 0 & 1\end{pmatrix}} & L'^*\oplus L' & \longrightarrow & F & \longrightarrow & 0\\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \begin{pmatrix}0 & u\\ \varepsilon & k^*\alpha k\end{pmatrix}} & & \downarrow & & \\ 0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix}0 & u\\ \varepsilon u^* & k^*\alpha k\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow & M & \longrightarrow & 0\end{array}\ ,$$

with $k^*\alpha k = k^*(\beta+\varepsilon\beta^*)k = 0\in \mathrm{Hom}_A(L',L'^*)$ in the ε-quadratic case.

The S-isomorphism of non-degenerate $\left\{\begin{array}{l}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ forms over A

$$\left\{\begin{array}{l}(j\ k) : (L\oplus L', \begin{pmatrix} 0 & u \\ \varepsilon u^* & k^*\alpha k\end{pmatrix}) \longrightarrow (K,\alpha)\\ (j\ k) : (L\oplus L', \begin{pmatrix} 0 & u \\ 0 & 0\end{pmatrix}) \longrightarrow (K,\beta)\end{array}\right.$$

corresponds by Proposition 13.9 i) to a sublagrangian

$$G = \mathrm{coker}((j\ k):L\oplus L' \longrightarrow K) \subseteq M$$

of the $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic})\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking form $\left\{\begin{array}{l}(M,\lambda)\ ((M,\lambda,\mu))\\ (M,\lambda,\nu) = \widetilde{H}_\varepsilon(F)\end{array}\right.$, such that the inclusion $G \longrightarrow M$ has resolution

$$\left\{\begin{array}{l}
\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L' & \xrightarrow{(j\ k)} & K & \longrightarrow & G & \longrightarrow & 0\\
 & & \Big\downarrow 1 & & \Big\downarrow \begin{pmatrix} j^*\alpha \\ k^*\alpha\end{pmatrix} & & \Big\downarrow & & \\
0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & k^*\alpha k\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow & M & \longrightarrow & 0
\end{array}\\[2ex]
\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L' & \xrightarrow{(j\ k)} & K & \longrightarrow & G & \longrightarrow & 0\\
 & & \Big\downarrow 1 & & \Big\downarrow \begin{pmatrix} j^*(\beta+\varepsilon\beta^*) \\ k^*(\beta+\varepsilon\beta^*)\end{pmatrix} & & \Big\downarrow \begin{pmatrix}\gamma\\ \mu\end{pmatrix} & & \\
0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & 0\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow & F\oplus F^\wedge & \longrightarrow & 0
\end{array}
\end{array}\right. .$$

In the (even) ε-symmetric case we thus have an even ε-symmetric (ε-quadratic) linking formation over (A,S) $(M,\lambda;F,G)$ $((M,\lambda,\mu;F,G))$. In the ε-quadratic case we have a split ε-quadratic linking formation over (A,S) $(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)$, with $(G,\gamma\mu\in \mathrm{Hom}_A(G,G^\wedge),\theta:G\longrightarrow Q_{-\varepsilon}(A,S))$ the $(-\varepsilon)$-quadratic linking form over (A,S) associated by Proposition 13.6 to the ε-quadratic S-formation over A

$$(H_\varepsilon(K^*);K^*,\mathrm{im}(\begin{pmatrix} -\bar{\varepsilon}\beta j & \beta^* k \\ j & k\end{pmatrix}:L\oplus L' \longrightarrow K^*\oplus K)) .$$

(In the even ε-symmetric case Proposition 13.7 actually gives an extension of $j\in \mathrm{Hom}_{S^{-1}A}(S^{-1}L,S^{-1}K)$ to an isomorphism of non-singular even ε-symmetric forms over $S^{-1}A$

$$(j\ j') : H^\varepsilon(S^{-1}L) = (S^{-1}L\oplus S^{-1}L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0\end{pmatrix}) \longrightarrow S^{-1}(K,\alpha) ,$$

leading to an S-isomorphism of non-degenerate even ε-symmetric forms over A

$$(j\ k) : (L\oplus L^*, \begin{pmatrix} 0 & u \\ \varepsilon u^* & 0\end{pmatrix}) \longrightarrow (K,\alpha) .$$

In this way it can be proved that every ε-quadratic linking formation over (A,S) $(M,\lambda,\mu;F,G)$ is stably equivalent to one of the type $(H_\varepsilon(F);F,G)$).

It remains to investigate the effect on the $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic})\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\ ((M,\lambda,\mu;F,G))\\ (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G)\end{array}\right.$ of the choice of extension $(j\ k)$. Given two such extensions

$$\left\{\begin{array}{l}(j\ k_r) : (L\oplus L'_r, \begin{pmatrix} 0 & u_r \\ \varepsilon u_r^* & k_r^*\alpha k_r\end{pmatrix}) \longrightarrow (K,\alpha)\\ (j\ k_r) : (L\oplus L'_r, \begin{pmatrix} 0 & u_r \\ 0 & 0\end{pmatrix}) \longrightarrow (K,\beta)\end{array}\right. \qquad (r = 1,2)$$

write $\left\{\begin{array}{l}(M_r,\lambda_r;F_r,G_r)\ ((M_r,\lambda_r,\mu_r;F_r,G_r))\\ (F_r,(\begin{pmatrix}\gamma_r\\ \mu_r\end{pmatrix},\theta_r)G_r)\end{array}\right.$ for the corresponding linking formations.

There exist a f.g. projective A-module L'_3, S-isomorphisms $v_r\in \mathrm{Hom}_A(L'_3,L'_r)$ (r = 1,2) and an A-module morphism $h\in \mathrm{Hom}_A(L'_3,L)$ such that

$$u_1v_1 = u_2v_2 \in \mathrm{Hom}_A(L'_3,L^*) \quad , \quad k_2v_2 - k_1v_1 = jh \in \mathrm{Hom}_A(L'_3,K)$$

$$\left\{\begin{array}{l}v_1^*u_1^*h \in Q^{-\varepsilon}(L'_3) = \ker(1-T_{-\varepsilon}:\mathrm{Hom}_A(L'_3,L'^*_3) \longrightarrow \mathrm{Hom}_A(L'_3,L'^*_3))\\ v_1^*u_1^*h \in \mathrm{im}(1+T_{-\varepsilon}:Q_{-\varepsilon}(L'_3) \longrightarrow Q^{-\varepsilon}(L'_3))\end{array}\right. .$$

(For example, let $L_3^! = L^*$ and apply the uniqueness clause of Proposition 1.6 to obtain an $S^{-1}A$-module morphism $g \in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*, S^{-1}L)$ such that

$$k_2u_2^{-1} - k_1u_1^{-1} = jg \in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*, S^{-1}K) \ ,$$

$$\begin{cases} g \in Q^{-\varepsilon}(S^{-1}L^*) \\ g \in \mathrm{im}(1+T_{-\varepsilon} : Q_{-\varepsilon}(S^{-1}L^*) \longrightarrow Q^{-\varepsilon}(S^{-1}L^*)) \ . \end{cases}$$

There exist $s \in S$, S-isomorphisms $v_r \in \mathrm{Hom}_A(L^*, L_r)$ $(r = 1,2)$, and an A-module morphism $h \in \mathrm{Hom}_A(L^*, L)$ such that

$$u_r^{-1} = \frac{v_r}{s} \in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*, S^{-1}L_r^!) \quad (r = 1,2)$$

$$g = \frac{h}{s} \in \mathrm{Hom}_{S^{-1}A}(S^{-1}L^*, S^{-1}L)$$

$$\begin{cases} \bar{s}h \in Q^{-\varepsilon}(L^*) \\ \bar{s}h \in \mathrm{im}(1+T_{-\varepsilon} : Q_{-\varepsilon}(L^*) \longrightarrow Q^{-\varepsilon}(L^*)) \end{cases} \quad).$$

We shall consider separately the effect on linking formations of the transformations

$$(L_1^!, u_1, k_1) \longmapsto (L_3^!, u_1v_1, k_1v_1) \longmapsto (L_3^!, u_1v_1, k_1v_1 + jh) = (L_3^!, u_1v_1, k_2v_2)$$
$$\longmapsto (L_2^!, u_2, k_2) \ .$$

If the choices $(L_1^!, u_1, k_1)$, $(L_2^!, u_2, k_2)$ are related by an S-isomorphism $v \in \mathrm{Hom}_A(L_1^!, L_2^!)$ such that

$$u_1 = u_2v \in \mathrm{Hom}_A(L_1^!, L^*) \ , \ k_1 = k_2v \in \mathrm{Hom}_A(L_1^!, K)$$

then the sublagrangian H of $\begin{cases} (M_1,\lambda_1;F_1,G_1) \quad ((M_1,\lambda_1,\mu_1;F_1,G_1)) \\ (F_1,\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1) \end{cases}$ defined by the resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L_1^! & \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}} & L\oplus L_2^! & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle (j\ k_2)} & & \downarrow & & \\ 0 & \longrightarrow & L\oplus L_1^! & \xrightarrow{(j\ k_1)} & K & \longrightarrow & G_1 & \longrightarrow & 0 \end{array}$$

is such that

$$\begin{cases} (H^\perp/H, \lambda_1^\perp/\lambda_1; F_1\cap H^\perp, G_1/H) = (M_2,\lambda_2;F_2,G_2) \\ \qquad ((H^\perp/H, \lambda_1^\perp/\lambda_1, \mu_1|; F_1\cap H^\perp, G_1/H) = (M_2,\lambda_2,\mu_2;F_2,G_2)) \\ (F_1\cap H^\perp, (\begin{pmatrix}[\gamma_1]\\ [\mu_1]\end{pmatrix}, [\theta_1])G_1/H) = (F_2, (\begin{pmatrix}\gamma_2\\ \mu_2\end{pmatrix}, \theta_2)G_2) \ . \end{cases}$$

Thus the linking formations associated to the choices $(L_1^!, u_1, k_1)$, $(L_2^!, u_2, k_2)$ are stably equivalent.

If the choices $(L_1^!, u_1, k_1)$, $(L_2^!, u_2, k_2)$ are related by

$$L_1^! = L_2^! \ , \ u_1 = u_2 \ , \ k_2 = k_1 + jh$$

for some $h \in \mathrm{Hom}_A(L_1^!, L)$ such that

$$\begin{cases} u_1^*h \in Q^{-\varepsilon}(L_1^!) \\ u_1^*h \in \mathrm{im}(1+T_{-\varepsilon} : Q_{-\varepsilon}(L_1^!) \longrightarrow Q^{-\varepsilon}(L_1^!)) \end{cases}$$

then there is defined an isomorphism of $\begin{cases} \text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic}) \\ \text{split } \varepsilon\text{-quadratic} \end{cases}$

linking formations over (A,S)

$$\begin{cases} f : (M_1,\lambda_1;F_1,G_1) \longrightarrow (M_2,\lambda_2;F_2,G_2) \quad (f : (M_1,\lambda_1,\mu_1;F_1,G_1) \longrightarrow (M_2,\lambda_2,\mu_2;F_2,G_2)) \\ (1,g,\varphi,\psi) : (F_1,(\begin{pmatrix}\gamma_1\\ \mu_1\end{pmatrix},\theta_1)G_1) \longrightarrow (F_2,(\begin{pmatrix}\gamma_2\\ \mu_2\end{pmatrix},\theta_2)G_2) \end{cases}$$

with

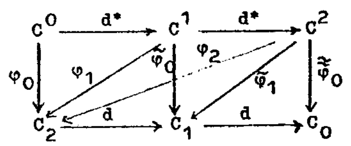
$$\begin{cases} \begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L_1^! & \xrightarrow{\begin{pmatrix}0 & u_1\\ \varepsilon u_1^* & k_1^*\alpha k_1\end{pmatrix}} & L^*\oplus L_1^{!*} & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \begin{pmatrix}1 & -h\\ 0 & 1\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix}1 & 0\\ h^* & 1\end{pmatrix}} & & \downarrow & & \\ 0 & \longrightarrow & L\oplus L_2^! & \xrightarrow{\begin{pmatrix}0 & u_2\\ \varepsilon u_2^* & k_2^*\alpha k_2\end{pmatrix}} & L^*\oplus L_2^{!*} & \longrightarrow & M_2 & \longrightarrow & 0 \end{array} \\ \begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L_1^! & \xrightarrow{(j\ k_1)} & K & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \begin{pmatrix}1 & -h\\ 0 & 1\end{pmatrix}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L\oplus L_2^! & \xrightarrow{(j\ k_2)} & K & \longrightarrow & G_2 & \longrightarrow & 0 \end{array} \end{cases} ,$$

and $(F_1^{\wedge} = \mathrm{coker}(u_1 : L_1^! \longrightarrow L^*), \varphi \in \mathrm{Hom}_A(F_1^{\wedge}, F_1), \psi : F_1^{\wedge} \longrightarrow Q_{-\varepsilon}(A,S))$ the $(-\varepsilon)$-quadratic linking form over (A,S) associated by Proposition 13.7 to the ε-quadratic S-formation over A

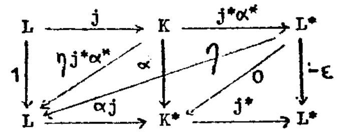
$$(H_{\varepsilon}(L); L, \mathrm{im}(\begin{pmatrix} -\bar{\varepsilon}h \\ u_1 \end{pmatrix} : L_1^! \longrightarrow L \oplus L^*)) .$$

This completes the verification that the stable equivalence class of the linking formation $\begin{cases} (M,\lambda;F,G) \ ((M,\lambda,\mu;F,G)) \\ (F,G) \end{cases}$ is independent of the choice of extension $(j\ k)$.

ii) A connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complex over A $(C, \varphi \in Q^2(C,-\varepsilon))$ is homotopy equivalent to one in normal form (Proposition 13.11). Given such a complex in normal form we shall construct an (even) ε-symmetric linking formation over (A,S) $(M,\lambda;F,G)$, as follows. Choose a cycle representative $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*,C))_2$ with $\varphi_0 \in \mathrm{Hom}_A(C^0, C_2)$ an isomorphism and $\varphi_1 = 0 \in \mathrm{Hom}_A(C^2, C_1)$. Using $\varphi_0 \in \mathrm{Hom}_A(C^0, C_2)$ as an identification express the diagram



as



with $j \in \mathrm{Hom}_A(L,K), \alpha \in \mathrm{Hom}_A(K,K^*), \eta \in \mathrm{Hom}_A(L^*,L)$ satisfying

$$j^*\alpha j = 0 \in \mathrm{Hom}_A(L,L^*)$$
$$\alpha - \varepsilon\alpha^* + \alpha j \eta j^* \alpha^* = 0 \in \mathrm{Hom}_A(K,K^*)$$
$$\eta + \varepsilon\eta^* = 0 \in \mathrm{Hom}_A(L^*,L) .$$

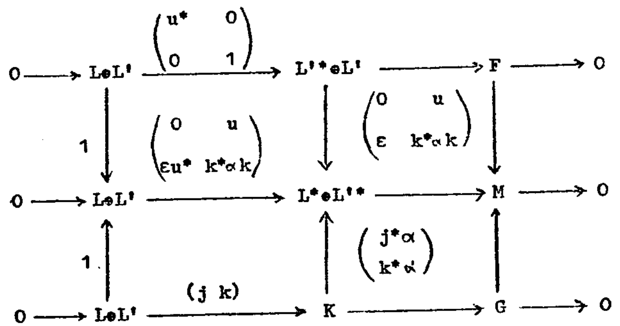
The sequence of f.g. projective A-modules and A-module morphisms

$$0 \longrightarrow L \xrightarrow{\ j\ } K \xrightarrow{\ j^*\alpha\ } L^* \longrightarrow 0$$

becomes exact over $S^{-1}A$, so that there exist a f.g. projective A-module L' and an A-module morphism $k \in \mathrm{Hom}_A(L',K)$ such that $u = j^*\alpha k \in \mathrm{Hom}_A(L',L^*)$ is an S-isomorphism. Let (M,λ) be the non-singular (even) ε-symmetric linking form over (A,S) associated by Proposition 13.6 to the non-singular (even) $(-\varepsilon)$-symmetric S-formation over A

$$(L \oplus L' \oplus L^* \oplus L'^*, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\varepsilon & 0 & \eta & 0 \\ 0 & -\varepsilon & 0 & 0 \end{pmatrix}; L \oplus L', \mathrm{im}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & u \\ \varepsilon u^* & k^*\alpha k \end{pmatrix} : L \oplus L' \longrightarrow L \oplus L' \oplus L^* \oplus L'^*))$$

Define a lagrangian F and a sublagrangian G of (M,λ) by the resolutions



Then $(M,\lambda;F,G)$ is the (even) ε-symmetric linking formation over (A,S) associated to the complex (C,φ).

Replacing $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*,C))_2$ by a different cycle representative $\varphi' \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*,C))_2$ of $\varphi \in Q^2(C,-\varepsilon)$ replaces α, η by α', η' such that

$$\alpha' - \alpha = \alpha j \chi j^* \alpha^* \in \mathrm{Hom}_A(K,K^*)$$
$$\eta - \eta' = \chi - \varepsilon\chi^* \in \mathrm{Hom}_A(L^*,L)$$

for some $\chi \in \mathrm{Hom}_A(L^*,L)$. The A-module isomorphism $f \in \mathrm{Hom}_A(M,M')$ given by the resolution

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & k^*\alpha k\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow 1 & & \Big\downarrow \begin{pmatrix} 1 & 0 \\ u^*\chi & 1\end{pmatrix} & & \Big\downarrow f & & \\
0 & \longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix} 0 & u \\ \varepsilon u^* & k^*\alpha' k\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow & M' & \longrightarrow & 0
\end{array}$$

defines an isomorphism of the associated (even) ε-symmetric linking formations over (A,S)

$$f : (M,\lambda;F,G) \longrightarrow (M',\lambda';F',G') .$$

The verification that the stable equivalence class of $(M,\lambda;F,G)$ is independent of the choice of $(L',u\in \mathrm{Hom}_A(L',L^*),k\in\mathrm{Hom}_A(L',K))$ proceeds exactly as in the proof of i) - indeed, if (C,φ) is even then $\varphi\in Q^2(C,-\varepsilon)$ has a cycle representative with $\eta = 0\in\mathrm{Hom}_A(L^*,L)$, in which case $(K,\alpha;\mathrm{im}(j:L\longrightarrow K))$ is an ε-symmetric S-form over A and $(M,\lambda;F,G)$ is the associated even ε-symmetric linking formation over (A,S). Moreover, if $(C,\varphi)=C^{-\varepsilon}(P)$ for some f.g. projective A-module P we can take

$$(L',u,k) = (P^*,1\in\mathrm{Hom}_A(P^*,P^*),\begin{pmatrix}1\\0\end{pmatrix}:P^*\longrightarrow P^*\oplus P)$$

so that $(M,\lambda;F,G) = 0$ up to stable equivalence.

Thus the stable equivalence class of $(M,\lambda;F,G)$ depends only on the stable isomorphism class of the complex (C,φ) in normal form, which by Proposition 13.11 is the same as the homotopy equivalence class of (C,φ).

Conversely, given an (even) ε-symmetric linking formation over (A,S) $(M,\lambda;F,G)$ we shall construct a connected S-acyclic 2-dimensional $(-\varepsilon)$-symmetric complex over A $(C,\varphi\in Q^2(C,-\varepsilon))$ in normal form, such that $(M,\lambda;F,G)$ is in the stable equivalence class determined by (C,φ), as follows.

Let $(D,\eta\in Q^1(D,-\varepsilon))$ be an S-acyclic 1-dimensional (even) $(-\varepsilon)$-symmetric Poincaré complex over A associated to the non-singular (even) ε-symmetric linking form over (A,S) (M,λ) by Proposition 13.4, with D an S-acyclic 1-dimensional f.g. projective A-module chain complex

$$D : \ldots\longrightarrow 0 \longrightarrow D_1 \xrightarrow{d} D_0 \longrightarrow 0 \longrightarrow \ldots ,$$

such that

$$(H^1(D),\eta_0^S) = (M,\lambda) .$$

Let $e\in\mathrm{Hom}_A(D^1,M)$ be the projection appearing in the resolution

$$0 \longrightarrow D^0 \xrightarrow{d^*} D^1 \xrightarrow{e} M \longrightarrow 0 ,$$

and define f.g. projective A-modules

$$D_1' = (e^{-1}(F))^* ,\quad D_1'' = (e^{-1}(G))^* .$$

Let $f'\in\mathrm{Hom}_A(D_1,D_1')$, $f''\in\mathrm{Hom}_A(D_1,D_1'')$ be the duals of the inclusions

$$f'^* : D'^1 = e^{-1}(F) \longrightarrow D^1 ,\quad f''^* : D''^1 = e^{-1}(G) \longrightarrow D^1 ,$$

and let $d'\in\mathrm{Hom}_A(D_1',D_0)$, $d''\in\mathrm{Hom}_A(D_1'',D_0)$ be the duals of the restrictions

$$d'^* = d^*| : D^0 \longrightarrow D'^1 ,\quad d''^* = d^*| : D^0 \longrightarrow D''^1$$

(which are well-defined since $\mathrm{im}(d^*:D^0\longrightarrow D^1) = e^{-1}(0)\subseteq e^{-1}(F)\cap e^{-1}(G)$).

The A-module chain maps $f':D\longrightarrow D'$, $f'':D\longrightarrow D''$ defined by

$$\begin{array}{lccccccccc}
D: & \ldots\longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{d} & D_0 & \longrightarrow & 0 & \longrightarrow\ldots \\
\Big\downarrow f' & & & & \Big\downarrow f' & & \Big\downarrow 1 & & & \\
D': & \ldots\longrightarrow & 0 & \longrightarrow & D_1' & \xrightarrow{d'} & D_0 & \longrightarrow & 0 & \longrightarrow\ldots
\end{array}$$

$$\begin{array}{lccccccccc}
D: & \ldots\longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{d} & D_0 & \longrightarrow & 0 & \longrightarrow\ldots \\
\Big\downarrow f'' & & & & \Big\downarrow f'' & & \Big\downarrow 1 & & & \\
D'': & \ldots\longrightarrow & 0 & \longrightarrow & D_1'' & \xrightarrow{d''} & D_0 & \longrightarrow & 0 & \longrightarrow\ldots
\end{array}$$

are resolutions of the inclusions

$$f'^* : H^1(D') = F \longrightarrow H^1(D) = M ,$$
$$f''^* : H^1(D'') = G \longrightarrow H^1(D) = M .$$

The inclusions define morphisms of (even) ε-symmetric linking forms over (A,S)

$$(F,0)\longrightarrow(M,\lambda)\ ,\ (G,0)\longrightarrow(M,\lambda)\ ,$$

corresponding by Proposition 13.4 to maps of S-acyclic 1-dimensional (even) $(-\varepsilon)$-symmetric complexes over A

$$f' : (D,\eta)\longrightarrow(D',0)\ ,\ f'' : (D,\eta)\longrightarrow(D'',0)\ .$$

Thus there are defined an S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric Poincaré pair over A $(f':D\longrightarrow D',(\delta\eta',\eta)\in Q^2(f',-\varepsilon))$ and a connected S-acyclic (even) $(-\varepsilon)$-symmetric pair over A $(f'':D\longrightarrow D'',(\delta\eta'',\eta)\in Q^2(f'',-\varepsilon))$. The union

$$(C,\varphi) = (D'\cup_D D'',-\delta\eta'\cup_\eta\delta\eta''\in Q^2(D'\cup_D D'',-\varepsilon))$$

(as defined in §5) is a connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complex over A. Next, we show how to recover the stable equivalence class of $(M,\lambda;F,G)$ from (C,φ).

The relative $\mathbb{Z}_2$-hypercohomology classes $(\delta\eta',\eta)\in Q^2(f',-\varepsilon)$, $(\delta\eta'',\eta)\in Q^2(f'',-\varepsilon)$ are represented by A-module morphisms

$$\eta_0 : D^0\longrightarrow D_1\ ,\ \tilde{\eta}_0 : D^1\longrightarrow D_0\ ,\ \eta_1 : D^1\longrightarrow D_1\ ,\ \delta\eta_0' : D'^1\longrightarrow D_1'$$
$$\delta\eta_0'' : D''^1\longrightarrow D_1''$$

such that

$$d\eta_0+\tilde{\eta}_0d^* = 0 : D^0\longrightarrow D_0\ ,\ d\eta_1-\tilde{\eta}_0-\varepsilon\eta_0^* = 0 : D^1\longrightarrow D_1\ ,$$
$$\eta_1+\varepsilon\eta_1^* = 0 : D^1\longrightarrow D_1\ ,\ \eta_1d^*+\eta_0+\varepsilon\tilde{\eta}_0^* = 0 : D^0\longrightarrow D_1$$
$$f'\eta_0 = -\delta\eta_0'd'^* : D^0\longrightarrow D_1'\ ,\ \tilde{\eta}_0f'^* = d'\delta\eta_0' : D'^1\longrightarrow D_0\ ,$$
$$f'\eta_1f'^* = \delta\eta_0'-\varepsilon\delta\eta_0'^* : D'^1\longrightarrow D_1'\ ,\ f''\eta_0 = -\delta\eta_0''d''^* : D^0\longrightarrow D_1''\ ,$$
$$\tilde{\eta}_0f''^* = d''\delta\eta_0'' : D''^1\longrightarrow D_0\ ,\ f''\eta_1f''^* = \delta\eta_0''-\varepsilon\delta\eta_0''^* : D''^1\longrightarrow D_1''\ .$$

Define a connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complex over A $(C',\varphi'\in Q^2(C',-\varepsilon))$ by

$$d_{C'} = \begin{cases}\begin{pmatrix}f'\\ f''\end{pmatrix} : C_2' = D_1\longrightarrow C_1' = D_1'\oplus D_1''\\ (d'\ \ -d'') : C_1' = D_1'\oplus D_1''\longrightarrow C_0' = D_0\end{cases}\qquad C_r' = 0\ (r\neq 0,1,2)$$

$$\varphi_0' = \begin{cases}\eta_0 : C'^0 = D^0\longrightarrow C_2' = D_1\\ \begin{pmatrix}-\delta\eta_0' & 0\\ 0 & \delta\eta_0''\end{pmatrix} : C'^1 = D'^1\oplus D''^1\longrightarrow C_1' = D_1'\oplus D_1''\\ \tilde{\eta}_0 : C'^2 = D^1\longrightarrow C_0' = D_0\end{cases}$$

$$\varphi_1' = \begin{cases}(\eta_1f'^*\ \ 0) : C'^1 = D'^1\oplus D''^1\longrightarrow C_2' = D_1\\ \begin{pmatrix}0\\ f''\eta_1\end{pmatrix} : C'^2 = D^1\longrightarrow C_1' = D_1'\oplus D_1''\end{cases}$$

$$\varphi_2' = \eta_1 : C'^2 = D^1\longrightarrow C_2' = D_1\ .$$

The A-module chain equivalence $h:C\longrightarrow C'$ given by

$$\begin{array}{ccccccccccc}
C : & \dots\longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{\begin{pmatrix}-f'\\ d\\ -f''\end{pmatrix}} & D_1'\oplus D_0\oplus D_1'' & \xrightarrow{\begin{pmatrix}d' & 1 & 0\\ 0 & 1 & d''\end{pmatrix}} & D_0\oplus D_0 & \longrightarrow 0\longrightarrow & \dots\\
h\ \downarrow & & & & \downarrow -1 & & \downarrow \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} & & \downarrow (1\ \ -1) & & \\
C' : & \dots\longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{\begin{pmatrix}f'\\ f''\end{pmatrix}} & D_1'\oplus D_1'' & \xrightarrow{(d'\ \ -d'')} & D_0 & \longrightarrow 0\longrightarrow & \dots
\end{array}$$

defines a homotopy equivalence of connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complexes over A

$$h : (C,\varphi)\longrightarrow(C',\varphi')\ .$$

Now apply the method of the proof of Proposition 13.1 to obtain a complex $(C'',\varphi''\in Q^2(C'',-\varepsilon))$ in normal form which is homotopy equivalent to (C',φ'), with

$$d_{C''} = \begin{cases} \begin{pmatrix} f' \\ f'' \\ 0 \end{pmatrix} : C''_2 = D_1 \longrightarrow C''_1 = \ker((d' \;\; -d'' \;\; -\eta_0^*) : D'_1 \oplus D''_1 \oplus D^1 \longrightarrow D_0) \\ (0 \;\; 0 \;\; 1) : C''_1 \longrightarrow C''_0 = D^1 \end{cases}$$

$$\varphi''_0 = \begin{cases} 1 : C''^0 = D_1 \qquad C''_2 = D_1 . \\ \begin{pmatrix} -\delta\eta'_0 & 0 & f' \\ -f''\eta_1 f'^* & \varepsilon\delta\eta''^*_0 & f'' \\ \varepsilon f'^* & \varepsilon f''^* & 0 \end{pmatrix} : C''^1 = \operatorname{coker}\left(\begin{pmatrix} d'^* \\ -d''^* \\ -\eta_0 \end{pmatrix} : D^0 \longrightarrow D'^1 \oplus D''^1 \oplus D_1\right) \longrightarrow C''_1 \\ -\varepsilon : C''^2 = D^1 \longrightarrow C''_0 = D^1 \end{cases}$$

$$\varphi''_1 = \begin{cases} (\eta_1 f'^* \;\; \eta_1 f''^* \;\; 0) : C''^1 \longrightarrow C''_2 = D^1 \\ 0 : C''^2 \longrightarrow C''_1 \end{cases}$$

$$\varphi''_2 = \eta_1 : C''^2 = D^1 \longrightarrow C''_2 = D_1 .$$

As before, write

$$d^*_{C''} = j : C''^0 = L \longrightarrow C''^1 = K$$

$$\varphi''_0 = \alpha : C''^1 = K \longrightarrow C''_1 = K^* ,$$

and let $i \in \operatorname{Hom}_A(D'_1, D_1)$, $s \in S$ be such that

$$f'^{-1} = \frac{i}{s} : S^{-1}D'_1 \longrightarrow S^{-1}D_1$$

so that the A-module morphism

$$k = \begin{pmatrix} i^* \\ 0 \\ 0 \end{pmatrix} : L^* = D^1 \longrightarrow K = \operatorname{coker}\left(\begin{pmatrix} d'^* \\ -d''^* \\ -\eta_0 \end{pmatrix} : D^0 \longrightarrow D'^1 \oplus D''^1 \oplus D_1\right)$$

is such that

$$j^*\alpha k = \varepsilon\bar{s} : L^* \longrightarrow L^* .$$

The (even) ε-symmetric linking formation over (A,S) $(M',\lambda';F',G')$ associated to the complex $(C,\varphi \in Q^2(C,-\varepsilon))$ is described by the resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L\oplus L^* & \xrightarrow{\begin{pmatrix} \bar{s} & 0 \\ 0 & 1 \end{pmatrix}} & L\oplus L^* & \longrightarrow & F' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \begin{pmatrix} 0 & \varepsilon\bar{s} \\ 1 & k^*\alpha k \end{pmatrix}} & & \downarrow & & \\ 0 & \longrightarrow & L\oplus L^* & \xrightarrow{\begin{pmatrix} 0 & \varepsilon\bar{s} \\ s & k^*\alpha k \end{pmatrix}} & L^*\oplus L & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle \begin{pmatrix} j^*\alpha \\ k^*\alpha \end{pmatrix}} & & \uparrow & & \\ 0 & \longrightarrow & L\oplus L^* & \xrightarrow{(j \;\; k)} & K & \longrightarrow & G' & \longrightarrow & 0 \end{array} .$$

Let H' be the sublagrangian of $(M',\lambda';F',G')$ with resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1\oplus D^1 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & i^* \end{pmatrix}} & D_1\oplus D'^1 & \longrightarrow & H' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}} & & \downarrow & & \\ 0 & \longrightarrow & L\oplus L^* & \xrightarrow{(j \;\; k)} & K & \longrightarrow & G' & \longrightarrow & 0 \end{array} .$$

There is defined an isomorphism of (even) ε-symmetric linking formations over (A,S)

$$(M,\lambda;F,G) \longrightarrow (H'^{\perp}/H', \lambda'^{\perp}/\lambda'; F'\cap H'^{\perp}, H'/G') ,$$

so that $(M',\lambda';F',G')$ is stably equivalent to $(M,\lambda;F,G)$.

Next, we consider the effect on the complex $(C,\varphi) = (D'\cup_D D'', -\delta\eta' \cup_\eta \delta\eta'')$ of the elementary equivalence

$$(M,\lambda;F,G) \longmapsto (\bar{M},\bar{\lambda};\bar{F},\bar{G}) = (H^{\perp}/H, \lambda^{\perp}/\lambda; F\cap H^{\perp}, G/H)$$

for some sublagrangian H of $(M,\lambda;F,G)$. Let the inclusion $H \longrightarrow G$ have resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^0 & \xrightarrow{d'''^*} & D'''^1 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow g^* & & \downarrow & & \\ 0 & \longrightarrow & D^0 & \xrightarrow{d''^*} & D''^1 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

with $g^* \in \mathrm{Hom}_A(D'''^1, D''^1)$ the inclusion of $D'''^1 = e^{-1}(H)$ in $D''^1 = e^{-1}(G) \subseteq D^1$, and $e \in \mathrm{Hom}_A(D^1, M)$ is the projection (as above). The A-module chain map $f'' : D''$ D'' defined by

$$\begin{array}{ccccccccccc} D: & \dots \longrightarrow & 0 & \longrightarrow & D_1 & \xrightarrow{d} & D_0 & \longrightarrow & 0 & \longrightarrow & \dots \\ f''' \downarrow & & & & \downarrow gf'' & & \downarrow 1 & & & & \\ D''': & \dots \longrightarrow & 0 & \longrightarrow & D'''_1 & \xrightarrow{d'''} & D_0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

is such that there exists a connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric pair over A $(f''' : D \longrightarrow D''', (\delta\eta''', \eta) \in Q^2(f'', -\varepsilon))$. The S-acyclic 1-dimensional (even) $(-\varepsilon)$-symmetric Poincaré complex over A $(\bar{D}, \bar{\eta} \in Q^1(\bar{D}, -\varepsilon))$ obtained from (D, η) by $(-\varepsilon)$-symmetric surgery on $(f''' : D \longrightarrow D''', (\delta\eta''', \eta))$ has associated non-singular (even) ε-symmetric linking form over (A,S)

$$(H^1(\bar{D}), \bar{\eta}_0^S) = (\bar{M}, \bar{\lambda})$$

Define A-module chain complexes

$$\bar{D}' = C(f'\eta_0 f'''^* : D'''^{1-*} \longrightarrow D'), \quad \bar{D}'' = \Omega C(g : D'' \longrightarrow D''')$$

and A-module chain maps

$$\bar{f}' : \bar{D} \longrightarrow \bar{D}', \quad \bar{f}'' : \bar{D} \longrightarrow \bar{D}'',$$

with

$$\bar{f}' = \begin{pmatrix} f' & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \bar{D}_r = D_r \oplus D'''_{r+1} \oplus D'''^{2-r} \longrightarrow \bar{D}'_r = D'_r \oplus D'''^{2-r}$$

$$\bar{f}'' = \begin{pmatrix} f'' & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \bar{D}_r = D_r \oplus D'''_{r+1} \oplus D'''^{2-r} \longrightarrow \bar{D}''_r = D''_r \oplus D'''_{r+1},$$

so that

$$\bar{f}'^* = \text{inclusion} : H^1(\bar{D}') = \bar{F} \longrightarrow H^1(\bar{D}) = \bar{M}$$

$$\bar{f}''^* = \text{inclusion} : H^1(\bar{D}'') = \bar{G} \longrightarrow H^1(\bar{D}) = \bar{M}.$$

There exist connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric pairs over A

$$(\bar{f}' : \bar{D} \longrightarrow \bar{D}', (\overline{\delta\eta}', \bar{\eta}) \in Q^2(\bar{f}', -\varepsilon)),$$

$$(\bar{f}'' : \bar{D} \longrightarrow \bar{D}'', (\overline{\delta\eta}'', \bar{\eta}) \in Q^2(\bar{f}'', -\varepsilon))$$

and the union

$$(\bar{C}, \bar{\varphi}) = (\bar{D}' \cup_{\bar{D}} \bar{D}'', -\overline{\delta\eta}' \cup_{\bar{\eta}} \overline{\delta\eta}'' \in Q^2(\bar{D}' \cup_{\bar{D}} \bar{D}'', -\varepsilon))$$

is a connected S-acyclic 2-dimensional (even) $(-\varepsilon)$-symmetric complex over A associated to $(\bar{M}, \bar{\lambda}; \bar{F}, \bar{G})$. It may be verified that $(\bar{C}, \bar{\varphi})$ is homotopy equivalent to (C, φ), the complex associated to $(M, \lambda; F, G)$.

This completes the proof of ii). It remains to complete the proof of i). Given an $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M, \lambda; F, G) \\ (M, \lambda, \mu; F, G)\end{array}\right.$ let $(K, \alpha), (K', \alpha')$ be non-degenerate $\left\{\begin{array}{l}\varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ forms over A such that

$$(M, \lambda) = \partial(K, \alpha) = \partial(K', \alpha'),$$

so that

$$F = \mathrm{coker}(f : K \longrightarrow K_F) = \mathrm{coker}(f' : K' \longrightarrow K'_F)$$

$$G = \mathrm{coker}(g : K \longrightarrow K_G) = \mathrm{coker}(g' : K' \longrightarrow K'_G)$$

for some S-isomorphisms of non-degenerate $\left\{\begin{array}{l}\varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ forms over A

$$f : (K, \alpha) \longrightarrow (K_F, \alpha_F), \quad f' : (K', \alpha') \longrightarrow (K'_F, \alpha'_F)$$

$$g : (K, \alpha) \longrightarrow (K_G, \alpha_G), \quad g' : (K', \alpha') \longrightarrow (K'_G, \alpha'_G).$$

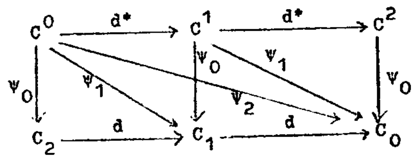
We have to show that the associated $\left\{\begin{array}{l}\varepsilon\text{-symmetric} \\ \text{even } \varepsilon\text{-symmetric}\end{array}\right.$ S-forms over A

$$(K_F \oplus K_G, \begin{pmatrix} \alpha_F & 0 \\ 0 & \alpha_G \end{pmatrix}; \mathrm{im}(\begin{pmatrix} f \\ g \end{pmatrix} : K \longrightarrow K_F \oplus K_G))$$

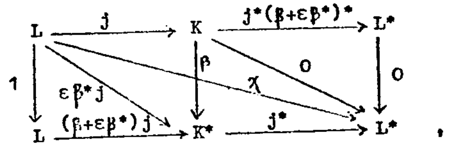
$$(K'_F \oplus K'_G, \begin{pmatrix} \alpha'_F & 0 \\ 0 & \alpha'_G \end{pmatrix}; \mathrm{im}(\begin{pmatrix} f' \\ g' \end{pmatrix} : K' \longrightarrow K'_F \oplus K'_G))$$

are stably isomorphic. The S-acyclic connected 2-dimensional $\left\{\begin{array}{l}- \\ \text{even}\end{array}\right.$ $(-\varepsilon)$-symmetric complexes over A in normal form obtained from the S-forms (as in i)) are homotopy equivalent, since they correspond to the same linking formation, and are therefore stably isomorphic (by Proposition 13.11). It follows that the S-forms are stably isomorphic.

iii) A connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complex over A $(C,\psi\in Q_2(C,-\varepsilon))$ is homotopy equivalent to one in normal form (by Proposition 13.10). Given such a complex in normal form we shall construct a split ε-quadratic linking formation over (A,S) $(F,G) = (F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}\theta)G)$, as follows. Choose a cycle representative $\psi\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C^*,C))_2$ with $\psi_0\in\mathrm{Hom}_A(C^0,C_2)$ an isomorphism, $\psi_0 = 0\in\mathrm{Hom}_A(C^2,C_0)$, $\psi_1 = 0\in\mathrm{Hom}_A(C^1,C_0)$. Using ψ_0 as an identification write the diagram



as



with $j\in\mathrm{Hom}_A(L,K)$, $\beta\in\mathrm{Hom}_A(K,K^*)$, $\chi\in\mathrm{Hom}_A(L,L^*)$ such that

$$j^*\beta j = \chi - \varepsilon\chi^*\in\mathrm{Hom}_A(L,L^*)\ .$$

Let (F,G) be the split ε-quadratic linking formation over (A,S) associated by i) to the ε-quadratic S-form over A $(K,\beta\in Q_\varepsilon(K);\mathrm{im}(j:L\longrightarrow K))$.

Replacing $\psi\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(C^*,C))_2$ by a different cycle representative in normal form $\psi'\in(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}\mathrm{Hom}_A(C^*,C))_2$ replaces β,χ by β',χ' such that

$$\beta'-\beta = \omega-\varepsilon\omega^*\in\mathrm{Hom}_A(K,K^*)\ ,\ \chi'-\chi = j^*\omega j+\eta+\varepsilon\eta^*\in\mathrm{Hom}_A(L,L^*)$$

for some $\omega\in\mathrm{Hom}_A(K,K^*)$, $\eta\in\mathrm{Hom}_A(L,L^*)$. Neither the ε-quadratic S-form $(K,\beta;L)$ nor the split ε-quadratic linking formation (F,G) are affected by such a change.

In particular, for the contractible S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complex over A $(C,\psi) = C_{-\varepsilon}(P)$ (P = a f.g. projective A-module) we have the ε-quadratic S-form over A

$$(K,\beta;L) = (P\oplus P^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};P)\ ,$$

corresponding by i) to the stable equivalence class of the 0 split ε-quadratic linking formation over (A,S) (take $k = \begin{pmatrix}1\\ 0\end{pmatrix}: L = P\longrightarrow K = P\oplus P^*$) .

Thus the stable equivalence class of the linking formation (F,G) associated to (C,ψ) depends only on the stable isomorphism class of (C,ψ), which by Proposition 13.11 is the same as the homotopy equivalence class of (C,ψ).

Conversely, given a split ε-quadratic linking formation over (A,S) (F,G) we shall construct a connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complex over A (C,ψ) in normal form, such that (F,G) is in the stable equivalence class determined by (C,ψ).

Let $(K,\beta;L)$ be an ε-quadratic S-form over A associated by i) to (F,G). Let $j\in\mathrm{Hom}_A(L,K)$ be the inclusion. For any lift $\beta\in\mathrm{Hom}_A(K,K^*)$ of $\beta\in Q_\varepsilon(K)$ there exist $\chi\in\mathrm{Hom}_A(L,L^*)$ such that

$$j^*\beta j = \chi-\varepsilon\chi^*\in\mathrm{Hom}_A(L,L^*)\ .$$

Given such a choice $(\beta,\chi)\in\mathrm{Hom}_A(K,K^*)\oplus\mathrm{Hom}_A(L,L^*)$ define $(C,\psi\in Q_2(C,-\varepsilon))$ by

$$d = \begin{cases}(\beta+\varepsilon\beta^*)j : C_2 = L\longrightarrow C_1 = K^*\\ j^* : C_1 = K^*\longrightarrow C_0 = L^*\end{cases}\ ,\quad C_r = 0\ (r\neq 0,1,2)$$

$$\psi_0 = \begin{cases}1 : C^0 = L\longrightarrow C_2 = L\\ \beta : C^1 = K\longrightarrow C_1 = K^*\\ 0 : C^2 = L^*\longrightarrow C_0 = L^*\end{cases}\ ,\quad \psi_1 = \begin{cases}\varepsilon\beta^*j : C^0 = L\longrightarrow C_1 = K^*\\ 0 : C^1 = K\longrightarrow C_0 = L^*\end{cases}$$

$$\psi_2 = \chi : C^0 = L\longrightarrow C_0 = L^*\ .$$

The method of proof of i) shows that the homotopy equivalence class of (C,ψ) depends only on the stable equivalence class of (F,G) together with a choice of hessian $(\beta,\chi)\in Q_\varepsilon(K,L)$, where

$$Q_\varepsilon(K,L) = \{(\beta,\chi)\in\mathrm{Hom}_A(K,K^*)\oplus\mathrm{Hom}_A(L,L^*)\,|\,j^*\beta j = \chi-\varepsilon\chi^*\}/\{(\omega-\varepsilon\omega^*,j^*\omega j+\eta+\varepsilon\eta^*)\,|\,(\omega,\eta)\in\mathrm{Hom}_A(K,K^*)\oplus\mathrm{Hom}_A(L,L^*)\}\ .$$

(Define a <u>split ε-quadratic S-form over A</u> $(K,\beta;L,\chi)$ to be a non-degenerate ε-quadratic form over A $(K,\beta\in Q_\varepsilon(K))$ together with an S-lagrangian L and a choice of hessian $(\beta,\chi)\in Q_\varepsilon(K,L)$. The homotopy equivalence classes of connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complexes are in a natural one-one correspondence with the stable isomorphism classes of split ε-quadratic S-forms over A).

It remains to show that if $(C,\psi),(C,\tilde\psi)$ are the complexes associated to different choices $\chi,\tilde\chi\in \mathrm{Hom}_A(L,L^*)$ such that

$$j^*\beta j = \chi-\varepsilon\chi^* = \tilde\chi-\varepsilon\tilde\chi^*\in \mathrm{Hom}_A(L,L^*)$$

then $(C,\tilde\psi)$ is homotopy equivalent to a complex obtained from (C,ψ) by an S-acyclic $(-\varepsilon)$-quadratic surgery. As before, let L' be a f.g. projective A-module and let $k\in \mathrm{Hom}_A(L',L^*)$ be such that $u=j^*(\beta+\varepsilon\beta^*)k\in \mathrm{Hom}_A(L',L^*)$ is an S-isomorphism and $k^*\beta k=0\in Q_\varepsilon(L')$. Also, let $\chi'\in \mathrm{Hom}_A(L',L'^*)$ be such that

$$k^*\beta k = \chi'-\varepsilon\chi'^*\in \mathrm{Hom}_A(L',L'^*)\ .$$

Let $(C',\psi'\in Q_2(C',-\varepsilon))$ be the connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complex over A associated to the ε-quadratic S-form over A

$$(K,\beta;\mathrm{im}(k:L'\longrightarrow K))$$

with choice of hessian $(\beta,\chi')\in Q_\varepsilon(K,L')$, corresponding by i) to the split ε-quadratic linking formation over (A,S) $(F^\wedge,(\begin{pmatrix}\mu\\-\varepsilon\chi\end{pmatrix},\theta)G)$. Let $(C'',\psi''\in Q_2(C'',-\varepsilon))$ be the connected S-acyclic 2-dimensional $(-\varepsilon)$-quadratic complex over A obtained from (C,ψ) by surgery on the connected S-acyclic 3-dimensional $(-\varepsilon)$-quadratic pair over A $(f:C\longrightarrow D,(\delta\psi,\psi)\in Q_3(f,-\varepsilon))$ defined by

$$d_D=\varepsilon u^*:D_2=L\longrightarrow D_1=L'^*\quad,\quad D_r=0\ (r\neq 1,2)$$

$$f=\begin{cases}1:C_2=L\longrightarrow D_2=L\\ k^*:C_1=K^*\longrightarrow D_1=L'^*\end{cases}\ ,\quad \delta\psi_1=-\chi':D^1=L'\longrightarrow D_1=L'^*$$

$$\delta\psi_0=0:D^r\longrightarrow D_{3-r}\quad (r=1,2)\ .$$

The A-module chain equivalence

$$h:C''\longrightarrow C'$$

given by

$$\begin{array}{ccccccccc}
C'': & \dots\longrightarrow 0\longrightarrow & L\oplus L' & \xrightarrow{\begin{pmatrix}(\beta+\varepsilon\beta^*)j & -(\beta+\varepsilon\beta^*)k\\ 1 & 0\\ 0 & \bar\varepsilon u\end{pmatrix}} & K^*\oplus L\oplus L^* & \xrightarrow{\begin{pmatrix}j^* & 0 & \varepsilon\\ -k^* & \varepsilon u^* & 0\end{pmatrix}} & L^*\oplus L'^* & \longrightarrow 0\longrightarrow\dots\\
h\downarrow & & \downarrow{\scriptstyle(0\ \ 1)} & & \downarrow{\scriptstyle(-1\ \ (\beta+\varepsilon\beta^*)j\ \ 0)} & & \downarrow{\scriptstyle(0\ \ 1)} & \\
C': & \dots\longrightarrow 0\longrightarrow & L' & \xrightarrow{(\beta+\varepsilon\beta^*)k} & K^* & \xrightarrow{k^*} & L'^* & \longrightarrow 0\longrightarrow\dots
\end{array}$$

defines a homotopy equivalence

$$h:(C'',\psi'')\longrightarrow(C',\psi')\ .$$

Thus (C,ψ) is homotopy equivalent to a complex obtained from (C',ψ') by S-acyclic $(-\varepsilon)$-quadratic surgery. Now (C',ψ') is independent of the choices of $\chi,\tilde\chi\in \mathrm{Hom}_A(L,L^*)$. It now follows from Proposition 7.9 (or rather the Lemma used in the proof, concerning the composition of algebraic surgeries) that $(C,\tilde\psi)$ is homotopy equivalent to a complex obtained from (C,ψ) by S-acyclic $(-\varepsilon)$-quadratic surgery.

[]

A <u>non-degenerate</u> $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ <u>formation over A</u> $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ is a non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(K,\alpha\in Q^{\varepsilon}(K))\\(K,\beta\in Q_{\varepsilon}(K))\end{cases}$ together with a lagrangian I and an S-lagrangian J. The $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over $S^{-1}A$ $\begin{cases}S^{-1}(K,\alpha;I,J)\\S^{-1}(K,\beta;I,J)\end{cases}$ is non-singular, and it is stably isomorphic to 0 precisely when $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ is an S-formation. A non-degenerate formation $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ is <u>non-singular</u> if J is a lagrangian of $\begin{cases}(K,\alpha)\\(K,\beta)\end{cases}$.

The <u>boundary</u> $\begin{cases}\partial(K,\alpha;I,J)\\ \partial(K,\beta;I,J)\end{cases}$ of a non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over A $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ is the stable equivalence class of the non-singular $\begin{cases}\text{even } \varepsilon\text{-symmetric } (\varepsilon\text{-quadratic})\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formations over (A,S) $\begin{cases}(M,\lambda;F,G)\ ((M,\lambda,\mu;F,G))\\(F,G)\end{cases}$ associated by Proposition 13.11 i) to the non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ S-form over A $\begin{cases}(K,\alpha;J)\\(K,\beta;J)\end{cases}$. As usual, $\begin{cases}\partial(K,\alpha;I,J)=0\\ \partial(K,\beta;I,J)=0\end{cases}$ if and only if $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ is non-singular.

(The boundary operation

$$\partial : (\text{non-degenerate formations}) \longrightarrow (\text{linking formations})$$

can also be expressed in terms of the "dual lattice" construction, as follows.

A lattice $\begin{cases}(K,\alpha)\\(K,\beta)\end{cases}$ of a non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over $S^{-1}A$ $\begin{cases}(Q,\varphi)\\(Q,\psi)\end{cases}$ is <u>non-singular</u> if it is a non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A, or equivalently if it is self-dual

$$\begin{cases}K^{\#}\equiv\{x\in Q\,|\,\varphi(x)(K)\subseteq A\subseteq S^{-1}A\}=K\\ K^{\#}\equiv\{x\in Q\,|\,(\psi+\varepsilon\psi^*)(x)(K)\subseteq A\subseteq S^{-1}A\}=K\ .\end{cases}$$

Given a non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over A $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ there exist non-singular lattices $\begin{cases}(K_I,\alpha_I),(K_J,\alpha_J)\\(K_I,\beta_I),(K_J,\beta_J)\end{cases}$ of $\begin{cases}S^{-1}(K,\alpha)\\S^{-1}(K,\beta)\end{cases}$ such that $K_I\cap S^{-1}I=I$ is a lagrangian of $\begin{cases}(K_I,\alpha_I)\\(K_I,\beta_I)\end{cases}$ and $K_J\cap S^{-1}J=J$ is a lagrangian of $\begin{cases}(K_J,\alpha_J)\\(K_J,\beta_J)\end{cases}$. Also, there exists a non-degenerate lattice $\begin{cases}(K',\alpha')\\(K',\beta')\end{cases}$ of $\begin{cases}S^{-1}(K,\alpha)\\S^{-1}(K,\beta)\end{cases}$ such that

$$K'\subseteq K_I\cap K_J\subseteq S^{-1}K\ ,$$

and such that the inclusions $K'\longrightarrow K_I$, $K'\longrightarrow K_J$ define morphisms of non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ forms over A

$$\begin{cases}(K',\alpha')\longrightarrow(K_I,\alpha_I)\ ,\ (K',\alpha')\longrightarrow(K_J,\alpha_J)\\(K',\beta')\longrightarrow(K_I,\beta_I)\ ,\ (K',\beta')\longrightarrow(K_J,\beta_J)\end{cases}\ ,$$

with

$$\begin{cases}\partial(K,\alpha;I,J)=(K'^{\#}/K',\alpha'^{\#}/\alpha';K_I/K',K_J/K')\\ \partial(K,\beta;I,J)=(K_I/K',K_J/K')\ ,\ (K'^{\#}/K',\beta'^{\#}/\beta')=\widetilde{H}_{\varepsilon}(K_I/K')\ .\end{cases}\quad)$$

The <u>boundary</u> of an $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S) $\begin{cases}(M,\lambda)\\ (M,\lambda)\\ (M,\lambda,\mu)\end{cases}$ is the non-singular $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\\ \text{split } (-\varepsilon)\text{-quadratic}\end{cases}$ linking formation over (A,S)

$$\begin{cases}\partial(M,\lambda) = (H^{-\varepsilon}(M);M,\Gamma_{(M,\lambda)})\\ \partial(M,\lambda) = (H_{-\varepsilon}(M);M,\Gamma_{(M,\lambda)})\\ \partial(M,\lambda,\mu) = (M,(\binom{1}{\lambda},\mu)M)\end{cases} \qquad \Gamma_{(M,\lambda)} = \{(x,\lambda(x))\in M\oplus M^\wedge \mid x\in M\} .$$

If $\begin{cases}(K,\alpha;I,J)\\ (K,\alpha;I,J)\\ (K,\beta;I,J)\end{cases}$ is the $\begin{cases}(-\varepsilon)\text{-symmetric}\\ \text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ S-formation over A associated to $\begin{cases}(M,\lambda)\\ (M,\lambda)\\ (M,\lambda,\mu)\end{cases}$ by Proposition 13.6 then $\begin{cases}\partial(M,\lambda)\\ \partial(M,\lambda)\\ \partial(M,\lambda,\mu)\end{cases}$ agrees with the boundary $\begin{cases}\partial(K,\alpha;I,J)\\ \partial(K,\alpha;I,J)\\ \partial(K,\beta;I,J)\end{cases}$ obtained by regarding $\begin{cases}(K,\alpha;I,J)\\ (K,\alpha;I,J)\\ (K,\beta;I,J)\end{cases}$ as a non-degenerate formation over A. (There is an evident analogy between the boundary operation

$$\partial : (\text{linking forms}) \longrightarrow (\text{linking formations})$$

and the boundary operation

$$\partial : (\text{forms}) \longrightarrow (\text{formations})$$

of §5. To complete the analogy we can also define a boundary operation

$$\partial : (\text{linking formations}) \longrightarrow (\text{linking forms})$$

corresponding to

$$\partial : (\text{formations}) \longrightarrow (\text{forms}) .$$

The <u>boundary</u> of an $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) $\begin{cases}(M,\lambda;F,G)\\ (M,\lambda,\nu;F,G)\\ (F,G)\end{cases}$ is the non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S)

$$\begin{cases}\partial(M,\lambda;F,G) = (G^\perp/G,\lambda^\perp/\lambda)\\ \partial(M,\lambda,\mu;F,G) = (G^\perp/G,\lambda^\perp/\lambda,\mu|)\\ \partial(F,G) = (G^\perp/G,\lambda^\perp/\lambda,\nu|) \quad (\tilde{H}_\varepsilon(F) = (F\oplus F^\wedge,\lambda,\nu))\end{cases} .$$

An $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking form over (A,S) is non-singular (resp. represents 0 in the Witt group $\begin{cases}L^\varepsilon(A,S)\\ L\langle v_0\rangle^\varepsilon(A,S)\\ L_\varepsilon(A,S)\\ \tilde{L}_\varepsilon(A,S)\end{cases}$) if and only if it has 0 boundary (resp. is isomorphic to the boundary $\begin{cases}\partial(M,\lambda;F,G)\\ \partial(M,\lambda;F,G)\\ \partial(M,\lambda,\mu;F,G)\\ \partial(F,G)\end{cases}$ of an $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) $\begin{cases}(M,\lambda;F,G)\\ (M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\\ (F,G)\end{cases}$)) .

<u>Proposition 13.13</u> Let $\begin{cases}(C,\varphi\in Q^2(C,-\varepsilon))\\(C,\psi\in Q_2(C,-\varepsilon))\end{cases}$ be an S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré complex over A, and let $\begin{cases}(M,\lambda;F,G)\\(F,G)\end{cases}$ be an associated non-singular $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\\text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S).

i) The S-acyclic cobordism class $\begin{cases}(C,\varphi)\in L^1(A,S,\varepsilon)\\(C,\psi)\in L_1(A,S,\varepsilon)\end{cases}$ depends only on the stable equivalence class of $\begin{cases}(M,\lambda;F,G)\\(F,G)\end{cases}$.

ii) $\begin{cases}(C,\varphi) = 0\in L^1(A,S,\varepsilon)\\(C,\psi) = 0\in L_1(A,S,\varepsilon)\end{cases}$ if and only if $\begin{cases}(M,\lambda;F,G)\\(F,G)\end{cases}$ is stably equivalent to the boundary $\begin{cases}\partial(K,\alpha;I,J)\\\partial(K,\beta;I,J)\end{cases}$ of a non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ formation over A $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ such that $\begin{cases}S^{-1}(K,\alpha;I,J) = 0\in M^{\varepsilon}_S(S^{-1}A) = L^1_S(S^{-1}A,\varepsilon)\\S^{-1}(K,\beta;I,J) = 0\in M^S_{\varepsilon}(S^{-1}A) = L^S_1(S^{-1}A,\varepsilon)\end{cases}$.

$\begin{cases}\text{If } \ker(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{\varepsilon}}\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0\\\text{For all } A,S,\varepsilon\end{cases}$ it is possible to choose $\begin{cases}(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ to be an $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ S-formation over A (i.e. such that $S^{-1}K = S^{-1}I\oplus S^{-1}J$), so that $\begin{cases}(C,\varphi) = 0\in L^1(A,S,\varepsilon)\\(C,\psi) = 0\in L_1(A,S,\varepsilon)\end{cases}$ if and only if $\begin{cases}(M,\lambda;F,G)\\(F,G)\end{cases}$ is stably equivalent to the boundary $\begin{cases}\partial(N,\theta)\\\partial(N,\theta,\mu)\end{cases}$ of a $\begin{cases}(-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ linking form over (A,S) $\begin{cases}(N,\theta)\\(N,\theta,\mu)\end{cases}$.

<u>Proof</u>: i) Immediate from Proposition 13.12 $\begin{cases}\text{ii)}\\\text{iii)}\end{cases}$.

ii) By the S-acyclic counterpart of Proposition 5.4 iii) an S-acyclic 2-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré complex over A $\begin{cases}(C,\varphi\in Q^2(C,-\varepsilon))\\(C,\psi\in Q_2(C,-\varepsilon))\end{cases}$ represents 0 in $\begin{cases}L^1(A,S,\varepsilon)\\L_1(A,S,\varepsilon)\end{cases}$ if and only if it is homotopy equivalent to the boundary $\begin{cases}\partial(D,\eta)\\\partial(D,\xi)\end{cases}$ of a connected S-acyclic 3-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D,\eta\in Q^3(D,-\varepsilon))\\(D,\xi\in Q_3(D,-\varepsilon))\end{cases}$ with D a f.g. projective A-module chain complex

$$D : \ldots\longrightarrow 0\longrightarrow D_3\xrightarrow{d}D_2\xrightarrow{d}D_1\xrightarrow{d}D_0\longrightarrow 0\longrightarrow\ldots .$$

Let $\begin{cases}(C,\varphi) = \partial(D,\eta)\\(C,\psi) = \partial(D,\xi)\end{cases}$ be the boundary of such a complex $\begin{cases}(D,\eta)\\(D,\xi)\end{cases}$ as above. Let $\begin{cases}(D',\eta'\in Q^3(D',-\varepsilon))\\(D',\xi'\in Q_3(D',-\varepsilon))\end{cases}$ be the connected 3-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ complex over A obtained from $\begin{cases}(D,\eta)\\(D,\xi)\end{cases}$ by surgery on the connected 4-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\(-\varepsilon)\text{-quadratic}\end{cases}$ pair over A $\begin{cases}(f:D\longrightarrow E,(0,\eta)\in Q^4(f,-\varepsilon))\\(f:D\longrightarrow E,(0,\xi)\in Q_4(f,-\varepsilon))\end{cases}$ defined by

$$f = 1 : D_3\longrightarrow E_3 = D_3\ ,\quad E_r = 0\ (r\neq 3)\ .$$

Then $\begin{cases}(D',\eta') = \bar{S}(D'',\eta'')\\(D',\xi') = \bar{S}(D'',\xi'')\end{cases}$ is the skew-suspension of a 1-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D'',\eta''\in Q^1(D'',\varepsilon))\\(D'',\xi''\in Q_1(D'',\varepsilon))\end{cases}$ such that $\begin{cases}S^{-1}(D'',\eta'')\\S^{-1}(D'',\xi'')\end{cases}$ is Poincaré and null-cobordant over $S^{-1}A$. The homotopy equivalence classes of 1-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\\varepsilon\text{-quadratic}\end{cases}$ complexes over A which become Poincaré and null-cobordant over $S^{-1}A$

are in a natural one-one correspondence with the stable isomorphism classes of non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ formations over A. In particular, the non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over A associated to $\begin{cases}(D'',\eta'')\\ (D'',\xi'')\end{cases}$ is given up to stable isomorphism by

$$\begin{cases}(K,\alpha;I,J) = \left(D_2\oplus D^2, \begin{pmatrix}0 & 1\\ \varepsilon & \eta_1\end{pmatrix}; D_2, \mathrm{im}\left(\begin{pmatrix}d & \eta_0\\ 0 & d^*\end{pmatrix} : D_3\oplus D^1 \longrightarrow D_2\oplus D^2\right)\right)\\ (K,\beta;I,J) = \left(D_2\oplus D^2, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}; D_2, \mathrm{im}\left(\begin{pmatrix}d & (1+T_{-\varepsilon})\xi_0\\ 0 & d^*\end{pmatrix} : D_3\oplus D^1 \longrightarrow D_2\oplus D^2\right)\right)\end{cases}$$

and is such that

$$\begin{cases}S^{-1}(K,\alpha;I,J) = 0 \in M^{\varepsilon}_{S}(S^{-1}A)\\ S^{-1}(K,\beta;I,J) = 0 \in M^{S}_{\varepsilon}(S^{-1}A)\end{cases} .$$

The non-singular $\begin{cases}\text{even }\varepsilon\text{-symmetric}\\ \text{split }\varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) associated to $\begin{cases}(C,\varphi)\\ (C,\psi)\end{cases}$ is the boundary

$$\begin{cases}(M,\lambda;F,G) = \partial(K,\alpha;I,J)\\ (F,G) = \partial(K,\beta;I,J)\end{cases} .$$

Since D is S-acyclic there exist $s\in S$, $g\in \mathrm{Hom}_A(D_2,D_3)$ such that

$$gd = s \in \mathrm{Hom}_A(D_3,D_3) .$$

Let $\tilde{f}: D \longrightarrow \tilde{E}$ be the A-module chain map defined by

$$\begin{array}{ccccccccccccccc} D: & \dots \longrightarrow & 0 & \longrightarrow & D_3 & \xrightarrow{d} & D_2 & \xrightarrow{d} & D_1 & \xrightarrow{d} & D_0 & \longrightarrow & 0 & \longrightarrow & \dots\\ \tilde{f}\big\downarrow & & & & \big\downarrow 1 & & \big\downarrow \varepsilon & & \big\downarrow & & \big\downarrow & & & & \\ \tilde{E}: & \dots \longrightarrow & 0 & \longrightarrow & D_3 & \xrightarrow{s} & D_3 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array} ,$$

so that $\begin{cases}\text{if } v^S_0(\eta) = 0 : H^3(D) \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\\ \end{cases}$ then

$$\begin{cases}\tilde{f}^{\%}(\varphi) = 0 \in Q^3(\tilde{E},-\varepsilon)\\ \tilde{f}_{\%}(\psi) = 0 \in Q_3(\tilde{E},-\varepsilon)\end{cases} .$$

The connected S-acyclic 3-dimensional $\begin{cases}\text{even }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(D',\xi'\in Q^3(D',-\varepsilon))\\ (D',\eta'\in Q_3(D',-\varepsilon))\end{cases}$ obtained from $\begin{cases}(D,\xi)\\ (D,\eta)\end{cases}$ by surgery on $\begin{cases}(\tilde{f}:D\longrightarrow\tilde{E},(0,\xi)\in Q^4(f,-\varepsilon)\\ (\tilde{f}:D\longrightarrow\tilde{E},(0,\eta)\in Q_4(f,-\varepsilon)\end{cases}$ is the skew-suspension $\begin{cases}(\tilde{D}',\tilde{\eta}') = \bar{S}(\tilde{D}'',\tilde{\eta}'')\\ (\tilde{D}',\tilde{\xi}') = \bar{S}(\tilde{D}'',\tilde{\xi}'')\end{cases}$ of an S-acyclic 1-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A $\begin{cases}(\tilde{D}'',\tilde{\eta}''\in Q^1(D'',\varepsilon))\\ (\tilde{D}'',\tilde{\xi}''\in Q_1(D'',\varepsilon))\end{cases}$. The $\begin{cases}(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ linking form over (A,S) associated to $\begin{cases}(\tilde{D}'',\tilde{\eta}'')\\ (\tilde{D}'',\tilde{\xi}'')\end{cases}$

$$\begin{cases}(N,\theta) = (H^1(\tilde{D}''),\tilde{\eta}''_0)\\ (N,\theta,\mu) = (H^1(\tilde{D}''),(1+T_\varepsilon)\tilde{\xi}''_0,v^0_S(\tilde{\xi}''))\end{cases}$$

is such that

$$\begin{cases}(M,\lambda;F,G) = \partial(N,\theta)\\ (F,G) = \partial(N,\theta,\mu)\end{cases} .$$

If $\ker(\hat{\delta}:\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\longrightarrow\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0$ then for any S-acyclic 3-dimensional even $(-\varepsilon)$-symmetric complex over A $(D,\eta\in Q^3(D,-\varepsilon))$ we have

$$v_0(\eta) = 0 : H^3(D) \xrightarrow{v^S_0(\eta)} \hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \xrightarrow{\hat{\delta}} \hat{H}^1(\mathbb{Z}_2;A,\varepsilon)$$

and so $v^S_0(\eta) = 0$.

Conversely, given a non-degenerate $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over A $\begin{cases}(K,\alpha;I,J)\\ (K,\beta;I,J)\end{cases}$ such that $\begin{cases}S^{-1}(K,\alpha;I,J) = 0 \in M^{\varepsilon}_{S}(S^{-1}A)\\ S^{-1}(K,\beta;I,J) = 0 \in M^{S}_{\varepsilon}(S^{-1}A)\end{cases}$ we have to show that the S-acyclic 2-dimensional $\begin{cases}\text{even }(-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ Poincaré complex over A

$\begin{cases}(C,\varphi\in Q^2(C,-\varepsilon))\\(C,\psi\in Q_2(C,-\varepsilon))\end{cases}$ associated to the boundary $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation $\begin{cases}\partial(K,\alpha;I,J) = (M,\lambda;F,G)\\ \partial(K,\beta;I,J) = (F,G)\end{cases}$ is an S-acyclic boundary. As I is a lagrangian of $\begin{cases}(K,\alpha)\\(K,\beta)\end{cases}$ we can identify

$$\begin{cases}(K,\alpha) = \left(I\oplus I^*,\begin{pmatrix}0 & 1\\ \varepsilon & \theta\end{pmatrix}\right)\\[2ex] (K,\beta) = \left(I\oplus I^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\end{cases}$$

for some ε-symmetric form (I^*,θ) (by Proposition 1.5). Write the inclusion of J in $K = I\oplus I^*$ as

$$\begin{pmatrix}j\\k\end{pmatrix} : J \longrightarrow I\oplus I^* \quad ,$$

so that in the ε-quadratic case

$$j^*k = \lambda - \varepsilon\lambda^* : J \longrightarrow J^*$$

for some $(-\varepsilon)$-quadratic form (J,λ). Define a 1-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ complex over A which becomes Poincaré over $S^{-1}A$ $\begin{cases}(D,\eta\in Q^1(D,\varepsilon))\\(D,\xi\in Q_1(D,\varepsilon))\end{cases}$ by

$$d = k^* : D_1 = I \longrightarrow D_0 = J^* \quad , \quad D_r = 0 \ (r \neq 0,1)$$

$$\begin{cases}\eta_0 = \begin{cases}\varepsilon j : D^0 = J \longrightarrow D_1 = I\\ j^*+k^*\theta : D^1 = I^* \longrightarrow D_0 = J^*\end{cases} , \quad \eta_1 = \theta : D^1 = I^* \longrightarrow D_1 = I\\[3ex] \xi_0 = \begin{cases}\varepsilon j : D^0 = J \longrightarrow D_1 = I\\ 0 : D^1 = I^* \longrightarrow D_0 = J^*\end{cases} , \quad \xi_1 = -\lambda : D^0 = J \longrightarrow D_0 = J^* \quad .\end{cases}$$

Now

$$\begin{cases}S^{-1}(D,\eta) = S^{-1}(K,\alpha;I,J) = 0 \in L^1_S(S^{-1}A,\varepsilon) = M^\varepsilon_S(S^{-1}A)\\ S^{-1}(D,\xi) = S^{-1}(K,\beta;I,J) = 0 \in L_1^S(S^{-1}A,\varepsilon) = M^S_\varepsilon(S^{-1}A)\end{cases} \quad ,$$

so that there exists a 2-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ pair over A

$\begin{cases}(f:D \longrightarrow \delta D,(\delta\eta,\eta)\in Q^2(f,\varepsilon))\\ (f:D \longrightarrow \delta D,(\delta\xi,\xi)\in Q_2(f,\varepsilon))\end{cases}$ which becomes Poincare over $S^{-1}A$.

Let $\begin{cases}(D',\eta'\in Q^3(D',-\varepsilon))\\(D',\xi'\in Q_3(D',-\varepsilon))\end{cases}$ be the connected S-acyclic 3-dimensional $\begin{cases}\text{even } (-\varepsilon)\text{-symmetric}\\ (-\varepsilon)\text{-quadratic}\end{cases}$ complex over A obtained from the skew-suspension $\begin{cases}\bar{S}(D,\eta)\\ \bar{S}(D,\xi)\end{cases}$ by surgery on the skew-suspension $\begin{cases}\bar{S}(f:D \longrightarrow \delta D,(\delta\eta,\eta))\\ \bar{S}(f:D \longrightarrow \delta D,(\delta\xi,\xi))\end{cases}$.

The boundary $\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) $\begin{cases}\partial(K,\alpha;I,J)\\ \partial(K,\beta;I,J)\end{cases}$ is the linking formation associated to the S-acyclic boundary

$\begin{cases}(C,\varphi\in Q^2(C,-\varepsilon)) = \partial(D',\eta')\\ (C,\psi\in Q_2(C,-\varepsilon)) = \partial(D',\xi')\end{cases}$ $\left(= \begin{cases}\partial\bar{S}(D,\eta)\\ \partial\bar{S}(D,\xi)\end{cases}\right.$, up to homotopy equivalence).

[]

Define the Witt group of $\left\{\begin{array}{l}\underline{\varepsilon\text{-symmetric}}\\ \underline{\text{even } \varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\\ \underline{\text{split } \varepsilon\text{-quadratic}}\end{array}\right.$ linking formations

over (A,S) $\left\{\begin{array}{l} M^{\varepsilon}(A,S)\\ M\langle v_0\rangle^{\varepsilon}(A,S)\\ M_{\varepsilon}(A,S)\\ \widetilde{M}_{\varepsilon}(A,S)\end{array}\right.$ to be the abelian group with one generator for

each isomorphism class of non-singular $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking

formations over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\\ (F,G)\end{array}\right.$ subject to the relations:

in the (even) ε-symmetric case

$(M,\lambda;F,G) + (M',\lambda';F',G') = (M\oplus M',\lambda\oplus\lambda';F\oplus F',G\oplus G')$

$(M,\lambda;F,G) + (M,\lambda;G,H) = (M,\lambda;F,H)$

$(M,\lambda;F,G) = (L^{\perp}/L,\lambda^{\perp}/\lambda;F\cap L^{\perp},G/L)$ if L is a sublagrangian of $(M,\lambda;F,G)$

$(M,\lambda;F,G) = (L^{\perp}/L,\lambda^{\perp}/\lambda;F/L,G/L)$ if L is a sublagrangian of (M,λ) such that $L\subseteq F\cap G$,

similarly in the ε-quadratic case,

in the split ε-quadratic case

$(F,G) + (F',G') = (F\oplus F',G\oplus G')$

$(F,G) = (F\cap L^{\perp},G/L)$ if L is a sublagrangian of (F,G)

$\partial(M,\lambda,\mu) = 0$ if (M,λ,μ) is a $(-\varepsilon)$-quadratic linking form over (A,S).

In particular, stably equivalent linking formations represent the same element in the Witt group. There are defined forgetful maps

$$M\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow M^{\varepsilon}(A,S)\ ;\ (M,\lambda;F,G)\longmapsto(M,\lambda;F,G)$$
$$M_{\varepsilon}(A,S)\longrightarrow M\langle v_0\rangle^{\varepsilon}(A,S)\ ;\ (M,\lambda,\mu;F,G)\longmapsto(M,\lambda;F,G)$$
$$\widetilde{M}_{\varepsilon}(A,S)\longrightarrow M_{\varepsilon}(A,S)\ ;\ (F,G)\longmapsto(H_{\varepsilon}(F);F,G)\ .$$

In order to verify that $\widetilde{M}_{\varepsilon}(A,S)\longrightarrow M_{\varepsilon}(A,S)$ is well-defined we have to show that $\partial(M,\lambda) = 0\in M_{\varepsilon}(A,S)$ for any $(-\varepsilon)$-quadratic linking form over (A,S) (M,λ,μ). For any non-singular ε-quadratic linking formation over (A,S) of the type $(H_{\varepsilon}(F);F,G)$ we have

$$(H_{\varepsilon}(F);F,G) = (H_{\varepsilon}(F);F,F^{\wedge})\oplus(H_{\varepsilon}(F);F^{\wedge},G) = (H_{\varepsilon}(F);F^{\wedge},G)\in M_{\varepsilon}(A,S)\ ,$$

so that for any even $(-\varepsilon)$-symmetric linking form over (A,S) (M,λ)

$$\partial(M,\lambda) = (H_{\varepsilon}(M);M,\Gamma_{(M,\lambda)}) = (H_{\varepsilon}(M);M^{\wedge},\Gamma_{(M,\lambda)}) = 0\in M_{\varepsilon}(A,S)\ .$$

<u>Proposition 13.14</u> A non-singular $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking formation

over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\\ (F,G)\end{array}\right.$ represents 0 in the Witt group $\left\{\begin{array}{l}M\langle v_0\rangle^{\varepsilon}(A,S)\\ M_{\varepsilon}(A,S)\\ \widetilde{M}_{\varepsilon}(A,S)\end{array}\right.$

if and only if it is stably equivalent to the boundary $\left\{\begin{array}{l}\partial(K,\alpha;I,J)\\ \partial(K,\alpha;I,J)\\ \partial(K,\beta;I,J)\end{array}\right.$ of a

non-degenerate $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ formation over A $\left\{\begin{array}{l}(K,\alpha;I,J)\\ (K,\alpha;I,J)\\ (K,\beta;I,J)\end{array}\right.$ such that

$\left\{\begin{array}{l}S^{-1}(K,\alpha;I,J) = 0\in M^{\varepsilon}_{S}(S^{-1}A)\\ S^{-1}(K,\alpha;I,J) = 0\in M\langle v_0\rangle^{\varepsilon}_{S}(S^{-1}A).\\ S^{-1}(K,\beta;I,J) = 0\in M^{S}_{\varepsilon}(S^{-1}A)\end{array}\right.$ $\left\{\begin{array}{l}\text{If } \ker(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon)\xrightarrow{\hat{S}}\hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0\\ \text{For all } A,S,\varepsilon\\ \text{For all } A,S,\varepsilon\end{array}\right.$

it is possible to choose $\begin{cases}(K,\alpha;I,J)\\(K,\alpha;I,J)\\(K,\beta;I,J)\end{cases}$ to be an S-formation (i.e. such that $S^{-1}K = S^{-1}I\oplus S^{-1}J$), so that $\begin{cases}(M,\lambda;F,G) = 0\in M\langle v_0\rangle^{\varepsilon}(A,S)\\(M,\lambda,\mu;F,G) = 0\in M_{\varepsilon}(A,S)\\(F,G) = 0\in \widetilde{M}_{\varepsilon}(A,S)\end{cases}$ if and only if $\begin{cases}(M,\lambda;F,G)\\(M,\lambda,\mu;F,G)\\(F,G)\end{cases}$ is stably equivalent to the boundary $\begin{cases}\partial(N,\theta)\\\partial(N,\theta)\\\partial(N,\theta,\psi)\end{cases}$ of an $\begin{cases}(-\varepsilon)\text{-symmetric}\\\text{even }(-\varepsilon)\text{-symmetric linking form over }(A,S)\\(-\varepsilon)\text{-quadratic}\end{cases}$ $\begin{cases}(N,\theta)\\(N,\theta)\\(N,\theta,\psi)\end{cases}$.

(We thus have an S-acyclic analogue of the "Bruhat-Tits decomposition" of the unitary groups, cf. Proposition 12.2).

Proof: It is convenient to introduce the following terminology, associating to non-singular ε-symmetric forms over A $(Q,\varphi),(Q',\varphi')$ and an isomorphism of the induced non-singular ε-symmetric forms over $S^{-1}A$

$$f : S^{-1}(Q,\varphi)\longrightarrow S^{-1}(Q',\varphi')$$

an element $[(Q,\varphi),f,(Q',\varphi')]\in M\langle v_0\rangle^{\varepsilon}(A,S)$ in the Witt group of even ε-symmetric linking formations over (A,S).

Let $u\in \mathrm{Hom}_A(P,Q)$ be an S-isomorphism of f.g. projective A-modules such that $fu\in \mathrm{Hom}_{S^{-1}A}(S^{-1}P,S^{-1}Q')$ is induced from an A-module S-isomorphism $fu\in \mathrm{Hom}_A(P,Q')$, and define an A-module morphism

$$\theta = u^*\varphi u : P\longrightarrow P^* .$$

The S-isomorphisms of non-degenerate ε-symmetric forms over A

$$u : (P,\theta)\longrightarrow(Q,\varphi)\ ,\ fu : (P,\theta)\longrightarrow(Q',\varphi')$$

correspond by Proposition 13.8 i) to lagrangians

$$F = \mathrm{coker}(u:P\longrightarrow Q)\ ,\ G = \mathrm{coker}(fu:P\longrightarrow Q')$$

of the boundary even ε-symmetric linking form over (A,S)

$$(M,\lambda) = \partial(P,\theta) .$$

Set

$$[(Q,\varphi),f,(Q',\varphi')] = (M,\lambda;F,G)\in M\langle v_0\rangle^{\varepsilon}(A,S) .$$

<u>Lemma 1</u> The Witt class $[(Q,\varphi),f,(Q',\varphi')]\in M\langle v_0\rangle^{\varepsilon}(A,S)$ is independent of the choice of S-isomorphism $u\in \mathrm{Hom}_A(P,Q)$.

<u>Proof</u>: If $\tilde{u}\in \mathrm{Hom}_A(\tilde{P},Q)$ is another choice of S-isomorphism there exist a f.g. projective A-module $\tilde{\tilde{P}}$ and S-isomorphisms $v\in \mathrm{Hom}_A(\tilde{\tilde{P}},P)$, $\tilde{v}\in \mathrm{Hom}_A(\tilde{\tilde{P}},\tilde{P})$ such that

$$uv = \tilde{u}\tilde{v}\in \mathrm{Hom}_A(\tilde{\tilde{P}},Q) .$$

Therefore it is sufficient to consider the effect of replacing $u\in \mathrm{Hom}_A(P,Q)$ by $\tilde{u} = uv\in \mathrm{Hom}_A(\tilde{P},Q)$ for some S-isomorphism $v\in \mathrm{Hom}_A(\tilde{P},P)$. The non-singular even ε-symmetric linking formation over (A,S)

$$(M,\lambda;F,G) = (\partial(P,\theta);\mathrm{coker}(u:P\longrightarrow Q),\mathrm{coker}(fu:P\longrightarrow Q'))$$

is replaced by

$$(\tilde{M},\tilde{\lambda};\tilde{F},\tilde{G}) = (\partial(\tilde{P},\tilde{\theta});\mathrm{coker}(\tilde{u}:\tilde{P}\longrightarrow Q),\mathrm{coker}(f\tilde{u}:\tilde{P}\longrightarrow Q')) ,$$

with

$$\tilde{\theta} = \tilde{u}^*\varphi\tilde{u} = v^*\theta v\in \mathrm{Hom}_A(\tilde{P},\tilde{P}^*) .$$

Let $H = \mathrm{coker}(v:\tilde{P}\longrightarrow P)$ be the sublagrangian of $(\tilde{M},\tilde{\lambda}) = \partial(\tilde{P},\tilde{\theta})$ associated by Proposition 13.8 i) to the S-isomorphism of non-degenerate ε-symmetric forms over A

$$v : (\tilde{P},\tilde{\theta})\longrightarrow(P,\theta) .$$

Then $H\subseteq \tilde{F}\cap\tilde{G}$ and there is defined an isomorphism of non-singular even ε-symmetric linking formation over (A,S)

$$(M,\lambda;F,G)\longrightarrow(H^{\perp}/H,\tilde{\lambda}^{\perp}/\tilde{\lambda};\tilde{F}/H,\tilde{G}/H) ,$$

so that

$$(M,\lambda;F,G) = (H^{\perp}/H,\lambda^{\perp}/\lambda;\tilde{F}/H,\tilde{G}/H) = (\tilde{M},\tilde{\lambda};\tilde{F},\tilde{G})\in M\langle v_0\rangle^{\varepsilon}(A,S) .$$

[]

Lemma 2 Given non-singular ε-symmetric forms over A $(Q,\varphi),(Q',\varphi'),(Q'',\varphi'')$ and isomorphisms of the induced ε-symmetric forms over $S^{-1}A$

$$f : S^{-1}(Q,\varphi) \longrightarrow S^{-1}(Q',\varphi') \ , \ f' : S^{-1}(Q',\varphi') \longrightarrow S^{-1}(Q'',\varphi'')$$

there is a "Whitehead lemma" identity

$$[(Q,\varphi),f,(Q',\varphi')] \oplus [(Q',\varphi'),f',(Q'',\varphi'')] = [(Q,\varphi),f'f,(Q'',\varphi'')] \in M\langle v_0\rangle^{\varepsilon}(A,S) \ .$$

Proof: There exists an S-isomorphism of non-degenerate ε-symmetric forms over A

$$u : (P,\theta) \longrightarrow (Q,\varphi)$$

such that both $fu \in \mathrm{Hom}_{S^{-1}A}(S^{-1}P,S^{-1}Q')$ and $f'fu \in \mathrm{Hom}_{S^{-1}A}(S^{-1}P,S^{-1}Q'')$ are induced from A-module S-isomorphisms $fu \in \mathrm{Hom}_A(P,Q')$, $f'fu \in \mathrm{Hom}_A(P,Q'')$. Let F,G,H be the lagrangians of $\partial(P,\theta) = (M,\lambda)$ associated by Proposition 13.9 i) to the S-isomorphisms of non-degenerate ε-symmetric forms over A

$$u : (P,\theta) \longrightarrow (Q,\varphi) \ , \ fu : (P,\theta) \longrightarrow (Q',\varphi') \ , \ f'fu : (P,\theta) \longrightarrow (Q'',\varphi'') \ .$$

Then

$$\begin{aligned}[(Q,\varphi),f,(Q',\varphi')] &\oplus [(Q',\varphi'),f',(Q'',\varphi'')] \\ &= (M,\lambda;F,G)\oplus(M,\lambda;G,H) = (M,\lambda;F,H) \\ &= [(Q,\varphi),f'f,(Q'',\varphi'')] \in M\langle v_0\rangle^{\varepsilon}(A,S) \ .\end{aligned}$$

[]

Lemma 3 Given metabolic ε-symmetric forms over A $(Q,\varphi),(Q',\varphi')$ with lagrangians L,L' let

$$f : S^{-1}(Q,\varphi) \longrightarrow S^{-1}(Q',\varphi')$$

be an isomorphism of the induced metabolic forms over $S^{-1}A$. If

$$f(S^{-1}L) = S^{-1}L' \subseteq S^{-1}Q'$$

then

$$[(Q,\varphi),f,(Q',\varphi')] = 0 \in M\langle v_0\rangle^{\varepsilon}(A,S) \ .$$

Proof: Choose a direct complement to L in Q, so that

$$(Q,\varphi) = (L\oplus L^*,\begin{pmatrix}0 & 1\\ \varepsilon & \alpha\end{pmatrix} \in Q^{\varepsilon}(L\oplus L^*))$$

for some ε-symmetric form over A $(L^*,\alpha\in Q^{\varepsilon}(L^*))$. Similarly, choosing a direct complement to L' in Q' we have

$$(Q',\varphi') = (L'\oplus L'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \alpha'\end{pmatrix} \in Q^{\varepsilon}(L'\oplus L'^*))$$

for some $(L'^*,\alpha'\in Q^{\varepsilon}(L'^*))$. The isomorphism $f\in \mathrm{Hom}_{S^{-1}A}(S^{-1}Q,S^{-1}Q')$ can be expressed as

$$f = \begin{pmatrix}\frac{g}{s} & \frac{k}{s}\\ 0 & \frac{g'}{s}\end{pmatrix} : S^{-1}Q = S^{-1}L\oplus S^{-1}L^* \longrightarrow S^{-1}Q' = S^{-1}L'\oplus S^{-1}L'^*$$

for some $s\in S$, $g\in \mathrm{Hom}_A(L,L')$, $g'\in \mathrm{Hom}_A(L^*,L'^*)$, $k\in \mathrm{Hom}_A(L^*,L')$. The S-isomorphism of non-degenerate ε-symmetric forms over A

$$u = \begin{pmatrix}s & 0\\ 0 & s\end{pmatrix} : (P,\theta) = (L\oplus L^*,\begin{pmatrix}0 & \bar{s}s\\ \varepsilon\bar{s}s & \bar{s}\alpha s\end{pmatrix}) \longrightarrow (Q,\varphi) = (L\oplus L^*,\begin{pmatrix}0 & 1\\ \varepsilon & \alpha\end{pmatrix})$$

can be used to define a non-singular ε-symmetric linking formation over (A,S)

$$(M,\lambda;F,G) = (\partial(P,\theta);\mathrm{coker}(u:P\longrightarrow Q),\mathrm{coker}(fu:P\longrightarrow Q'))$$

such that

$$[(Q,\varphi),f,(Q',\varphi')] = (M,\lambda;F,G) \in M\langle v_0\rangle^{\varepsilon}(A,S) \ .$$

The S-isomorphisms of non-degenerate ε-symmetric forms over A

$$h = \begin{pmatrix}\bar{s}s & 0\\ 0 & 1\end{pmatrix} : (P,\theta) = (L\oplus L^*,\begin{pmatrix}0 & \bar{s}s\\ \varepsilon\bar{s}s & \bar{s}\alpha s\end{pmatrix}) \longrightarrow (L\oplus L^*,\begin{pmatrix}0 & 1\\ \varepsilon & \bar{s}\alpha s\end{pmatrix})$$

$$i = \begin{pmatrix}s & 0\\ 0 & 1\end{pmatrix} : (P,\theta) = (L\oplus L^*,\begin{pmatrix}0 & \bar{s}s\\ \varepsilon\bar{s}s & \bar{s}\alpha s\end{pmatrix}) \longrightarrow (L\oplus L^*,\begin{pmatrix}0 & s\\ \varepsilon\bar{s} & \bar{s}\alpha s\end{pmatrix})$$

$$j = \begin{pmatrix}g & 0\\ 0 & 1\end{pmatrix} : (P,\theta) = (L\oplus L^*,\begin{pmatrix}0 & \bar{s}s\\ \varepsilon\bar{s}s & \bar{s}\alpha s\end{pmatrix}) \longrightarrow (L'\oplus L^*,\begin{pmatrix}0 & h\\ \varepsilon h^* & \bar{s}\alpha s\end{pmatrix})$$

correspond by Proposition 13.9 i) to a lagrangian

$$H = \mathrm{coker}(h : L\oplus L^* \longrightarrow L\oplus L^*)$$

and sublagrangians

$$I = \mathrm{coker}(i : L\oplus L^* \longrightarrow L\oplus L^*)$$

$$J = \mathrm{coker}(j : L\oplus L^* \longrightarrow L'\oplus L^*)$$

of the non-singular even ε-symmetric linking form over (A,S) $(M,\lambda) = \partial(P,\theta)$. Now $I \subseteq F\cap H$, $J \subseteq G\cap H$ and the even ε-symmetric linking formations over (A,S) $(I^{\perp}/I,\lambda^{\perp}/\lambda;F/I,H/I)$, $(J^{\perp}/J,\lambda^{\perp}/\lambda;H/J,G/J)$ are stably equivalent to 0, so that

$$\begin{aligned}[(Q,\varphi),f,(Q',\varphi')] &= (M,\lambda;F,G) = (M,\lambda;F,H)\oplus(M,\lambda;H,G) \\ &= (I^{\perp}/I,\lambda^{\perp}/\lambda;F/I,H/I)\oplus(J^{\perp}/J,\lambda^{\perp}/\lambda;H/J,G/J) = 0 \in M\langle v_0\rangle^{\varepsilon}(A,S)\ .\end{aligned}$$

[]

The boundary $\partial(K,\alpha;I,J)$ of a non-degenerate ε-symmetric formation over A $(K,\alpha;I,J)$ may be described as follows. Choose a direct complement to the lagrangian I in (K,α), so that

$$(K,\alpha) = (I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix}\in Q^{\varepsilon}(I\oplus I^*))$$

for some ε-symmetric form over A $(I^*,\theta\in Q^{\varepsilon}(I^*))$. The inclusion of the S-lagrangian

$$\begin{pmatrix} j \\ k \end{pmatrix} : (J,0) \longrightarrow (I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix})$$

extends to an S-isomorphism of non-degenerate ε-symmetric forms over A

$$\begin{pmatrix} j & \widetilde{j} \\ k & \widetilde{k} \end{pmatrix} : (J\oplus J^*,\begin{pmatrix} 0 & \bar{s} \\ \varepsilon s & \varphi \end{pmatrix}) \longrightarrow (I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix})$$

for some $s\in S$, $\varphi\in Q^{\varepsilon}(J^*)$ (by Proposition 1.5, applied to the induced forms over $S^{-1}A$). Define an isomorphism of metabolic ε-symmetric forms over $S^{-1}A$

$$f = \begin{pmatrix} \frac{j}{s} & \widetilde{j} \\ \frac{k}{s} & \widetilde{k} \end{pmatrix} : S^{-1}(J\oplus J^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix}) \longrightarrow S^{-1}(I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix})\ .$$

Then

$$(K,\alpha;I,J) = [(J\oplus J^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix}),f,(I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix})] \in M\langle v_0\rangle^{\varepsilon}(A,S).$$

(To verify that this is the linking formation associated to the non-singular ε-symmetric S-form over A $(K,\alpha;J)$ used to define $\partial(K,\alpha;I,J)$ use the S-isomorphism

$$u = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} : P = J\oplus J^* \longrightarrow J\oplus J^*$$

in the construction of $[(Q,\varphi),f,(Q',\varphi')]$).

Lemma 4 Let $(K,\alpha;I,J)$, $(K',\alpha';I',J')$ be non-degenerate ε-symmetric formations over A such that there exists an isomorphism of the induced non-singular ε-symmetric formations over $S^{-1}A$

$$g : S^{-1}(K,\alpha;I,J) \longrightarrow S^{-1}(K',\alpha';I',J')\ .$$

Then

$$\partial(K,\alpha;I,J) = \partial(K',\alpha';I',J') \in M\langle v_0\rangle^{\varepsilon}(A,S)\ .$$

Proof: As above, let

$$f : S^{-1}(J\oplus J^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix}) \longrightarrow S^{-1}(I\oplus I^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta \end{pmatrix}) = S^{-1}(K,\alpha)$$

$$f' : S^{-1}(J'\oplus J'^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi' \end{pmatrix}) \longrightarrow S^{-1}(I'\oplus I'^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \theta' \end{pmatrix}) = S^{-1}(K',\alpha')$$

be isomorphisms of metabolic ε-symmetric forms over $S^{-1}A$ extending the inclusions of the lagrangians $S^{-1}(J,0)\longrightarrow S^{-1}(K,\alpha)$, $S^{-1}(J',0)\longrightarrow S^{-1}(K',\alpha')$.

The isomorphisms of metabolic ε-symmetric forms over $S^{-1}A$

$$g : S^{-1}(K,\alpha) \longrightarrow S^{-1}(K',\alpha')$$

$$h = f'^{-1}gf : S^{-1}(J\oplus J^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix}) \longrightarrow S^{-1}(J'\oplus J'^*,\begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi' \end{pmatrix})$$

are such that

$$g(S^{-1}I) = S^{-1}I' \subseteq S^{-1}K'\ ,\ h(S^{-1}J) = S^{-1}J' \subseteq S^{-1}J'\oplus S^{-1}J'^*\ .$$

Applying Lemma 3, we have

$$[(K,\alpha),g,(K',\alpha')] = [(J\oplus J^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi\end{pmatrix}),h,(J'\oplus J'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi'\end{pmatrix})]$$

$$= 0 \in M\langle v_0\rangle^{\varepsilon}(A,S)\ .$$

Applying Lemma 3, we have

$$\partial(K,\alpha;I,J) = [(J\oplus J^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi\end{pmatrix}),f,(I\oplus I^*,\begin{pmatrix}0 & 1\\ \varepsilon & \theta\end{pmatrix})]$$

$$= [(J\oplus J^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi\end{pmatrix}),h,(J'\oplus J'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi'\end{pmatrix})]$$

$$\oplus[(J'\oplus J'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi'\end{pmatrix}),f',(I'\oplus I'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \theta'\end{pmatrix})]\oplus[(K',\alpha'),g^{-1},(K,\alpha)]$$

$$= [(J'\oplus J'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \varphi'\end{pmatrix}),f',(I'\oplus I'^*,\begin{pmatrix}0 & 1\\ \varepsilon & \theta'\end{pmatrix})]$$

$$= \partial(K',\alpha';I',J') \in M\langle v_0\rangle^{\varepsilon}(A,S)\ .$$

[]

<u>Lemma 5</u> If $(K,\alpha;I,J)$ is a non-degenerate ε-symmetric formation over A such that $S^{-1}(K,\alpha;I,J) = 0 \in M_S^{\varepsilon}(S^{-1}A)$ then

$$\partial(K,\alpha;I,J) = 0 \in M\langle v_0\rangle^{\varepsilon}(A,S)\ .$$

<u>Proof</u>: Let $(D,\eta\in Q^1(D,\varepsilon))$ be a 1-dimensional ε-symmetric complex over A associated to $(K,\alpha;I,J)$, with

$$d = k^* : D_1 = I \longrightarrow D_0 = J^*\quad ,\quad D_r = 0\ (r \neq 0,1)$$

$$\eta_0 = \begin{cases}\varepsilon j : D^0 = J \longrightarrow D_1 = I\\ j^* + k^*\theta : D^1 = I^* \longrightarrow D_0 = J^*\end{cases}\ ,\quad \eta_1 = \theta : D^1 = I^* \longrightarrow D_1 = I$$

for some ε-symmetric form over A (I^*,θ) such that the inclusion of the S-lagrangian $(J,0)\longrightarrow(K,\alpha)$ extends to an S-isomorphism of non-degenerate ε-symmetric forms over A

$$\begin{pmatrix} j & \tilde{j}\\ k & \tilde{k}\end{pmatrix} : (J\oplus J^*,\begin{pmatrix}0 & \bar{s}\\ \varepsilon s & \varphi\end{pmatrix})\longrightarrow(K,\alpha) = (I\oplus I^*,\begin{pmatrix}0 & 1\\ \varepsilon & \theta\end{pmatrix})\quad (s\in S)\ .$$

Then $S^{-1}(D,\eta)$ is Poincaré, and

$$S^{-1}(D,\eta) = S^{-1}(K,\alpha;I,J) = 0 \in L_S^1(S^{-1}A,\varepsilon) = M_S^{\varepsilon}(S^{-1}A)$$

so that there exists a 2-dimensional ε-symmetric pair over A $(f:D\longrightarrow\delta D,(\delta\eta,\eta)\in Q^2(f,\varepsilon))$ which becomes Poincaré over $S^{-1}A$, with δD a f.g. projective A-module chain complex

$$\delta D : \ldots\longrightarrow 0\longrightarrow\delta D_2\xrightarrow{\ d\ }\delta D_1\xrightarrow{\ d\ }\delta D_0\longrightarrow 0\longrightarrow\ldots\ .$$

Define an A-module chain map $g:\delta D\longrightarrow\delta D'$ by

$$\begin{array}{ccccccccccccc}
\delta D: & \ldots\longrightarrow & 0 & \longrightarrow & \delta D_2 & \xrightarrow{d} & \delta D_1 & \xrightarrow{d} & \delta D_0 & \longrightarrow & 0 & \longrightarrow & \ldots\\
g\big\downarrow & & & & \big\downarrow 1 & & \big\downarrow 1 & & \big\downarrow & & & & \\
\delta D': & \ldots\longrightarrow & 0 & \longrightarrow & \delta D_2 & \xrightarrow{d} & \delta D_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots\ ,
\end{array}$$

and let $(D',\eta'\in Q^1(D',\varepsilon))$ be the 1-dimensional ε-symmetric complex over A obtained from (D,η) by surgery on the 2-dimensional ε-symmetric pair over A $(gf:D\longrightarrow\delta D',(g,1)^{\%}(\delta\eta,\eta)\in Q^2(gf,\varepsilon))$. (Strictly speaking, the algebraic surgery of §7 was only defined for connected complexes and connected pairs and neither (D,η) nor $(gf:D\longrightarrow\delta D',(g,1)^{\%}(\delta\eta,\eta))$ need be connected. However, the formulae of algebraic surgery are still applicable, $S^{-1}(D,\eta)$ is Poincaré and $S^{-1}(gf:D\longrightarrow\delta D',(g,1)^{\%}(\delta\eta,\eta))$ is connected, so that (D',η') becomes Poincaré over $S^{-1}A$). The non-degenerate ε-symmetric formation over A associated to (D',η')

$$(K',\alpha';I',J') = (D_1'\oplus D'^1,\begin{pmatrix}0 & 1\\ \varepsilon & \eta_1'\end{pmatrix};D_1',\mathrm{im}\begin{pmatrix}\bar{\varepsilon}\eta_0'\\ d'^*\end{pmatrix}:D'^0\longrightarrow D_1'\oplus D'^1))$$

has boundary even ε-symmetric linking formation over (A,S) $\partial(K',\alpha';I',J')$ stably equivalent to $\partial(K,\alpha;I,J)$ (by Propositions 7.1 i), 13.12 ii), since $\partial(K,\alpha;I,J)$ agrees with the S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A $\partial\bar{S}(D,\eta)$). Define a 2-dimensional ε-symmetric pair over A $(g':D'\longrightarrow\delta D',(0,\eta')\in Q^2(g',\varepsilon))$ using the A-module chain map $g':D'\longrightarrow\delta D'$ given by

$$\begin{array}{ccccccccc}
D' : \cdots \longrightarrow & 0 & \longrightarrow & D_1 \oplus \delta D_2 \oplus \delta D^1 & \xrightarrow{d' = \begin{pmatrix} d & 0 & \eta_0 f^* \\ -f & d & -\delta\eta_0 \\ 0 & 0 & -d^* \end{pmatrix}} & D_0 \oplus \delta D_1 \oplus \delta D^2 & \longrightarrow & 0 & \longrightarrow \cdots \\
\Big\downarrow g' & \Big\downarrow & & \Big\downarrow & & \Big\downarrow g' = (f \quad d \quad -\delta\eta_0) & & & \\
\delta D' : \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \delta D_0 & \longrightarrow & 0 & \longrightarrow \cdots .
\end{array}$$

There exist $i \in \mathrm{Hom}_A(D_1', \delta D^0)$, $s' \in S$ such that

$$i\eta_0' g'^* = s' : \delta D^0 \longrightarrow \delta D^0 .$$

and such that the A-module chain map $h : D' \longrightarrow D''$ given by

$$\begin{array}{ccccccccc}
D' : \cdots \longrightarrow & 0 & \longrightarrow & D_1' & \xrightarrow{d'} & D_0' & \longrightarrow & 0 & \longrightarrow \cdots \\
\Big\downarrow h & & & \Big\downarrow i & & \Big\downarrow g' & & & \\
D'' : \cdots \longrightarrow & 0 & \longrightarrow & \delta D^0 & \xrightarrow{0} & \delta D_0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

becomes a homotopy equivalence of 1-dimensional ε-symmetric Poincaré complexes over $S^{-1}A$

$$h : S^{-1}(D', \eta') \longrightarrow S^{-1}(D'', \eta'')$$

where $\eta'' = h^{\%}(\eta') \in Q^1(D'', \varepsilon)$. The 1-dimensional ε-symmetric complex over A $(D'', \eta'' \in Q^1(D'', \varepsilon))$ has associated non-degenerate ε-symmetric formation over A

$$(K'', \alpha''; I'', J'') = (\delta D^0 \oplus \delta D_0, \begin{pmatrix} 0 & 1 \\ \varepsilon & i\eta_1' i^* \end{pmatrix}; \delta D^0, \mathrm{im}(\begin{pmatrix} \bar{\varepsilon} s' \\ 0 \end{pmatrix} : \delta D^0 \longrightarrow \delta D^0 \oplus \delta D_0))$$

such that

$$S^{-1}I'' = S^{-1}J'' \subseteq S^{-1}K'' .$$

It follows from Proposition 1.7 that there is defined an isomorphism of non-singular ε-symmetric formations over $S^{-1}A$

$$S^{-1}(K', \alpha'; I', J') \oplus S^{-1}(H^{\varepsilon}(J''^*); J''^*, J'')$$
$$\longrightarrow S^{-1}(K'', \alpha''; I'', J'') \oplus S^{-1}(H^{\varepsilon}(J'^*); J'^*, J') .$$

Applying Lemma 4, we have

$$\partial(K, \alpha; I, J) = \partial(K', \alpha'; I', J') = \partial(K'', \alpha''; I'', J'')$$
$$= \partial(K'', \alpha''; I'', I'') = 0 \in M\langle v_0 \rangle^{\varepsilon}(A, S) .$$

[]

It follows from Proposition 13.13 ii) and Lemma 5 that the correspondence of Proposition 13.12 ii)

(S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complexes over A (C, φ))

$\longleftrightarrow$ (non-singular even ε-symmetric linking formations over (A,S) $(M, \lambda; F, G)$)

can be used to define an abelian group morphism

$$L^1(A, S, \varepsilon) \longrightarrow M\langle v_0 \rangle^{\varepsilon}(A, S) \ ; \ (C, \varphi) \longmapsto (M, \lambda; F, G) .$$

We shall prove that this is in fact an isomorphism, so that applying Proposition 13.13 ii) again it will follow that a non-singular ε-symmetric linking formation over (A,S) $(M, \lambda; F, G)$ representing 0 in $M\langle v_0 \rangle^{\varepsilon}(A, S)$ is stably equivalent to the boundary $\partial(K, \alpha; I, J)$ of a non-degenerate ε-symmetric formation over A $(K, \alpha; I, J)$ such that $S^{-1}(K, \alpha; I, J) = 0 \in M_S^{\varepsilon}(S^{-1}A)$. In order to verify that the correspondence of Proposition 13.11 also defines an abelian group morphism

$$M\langle v_0 \rangle^{\varepsilon}(A, S) \longrightarrow L^1(A, S, \varepsilon) \ ; \ (M, \lambda; F, G) \longmapsto (C, \varphi)$$

we have to show that the S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A associated to the non-singular even ε-symmetric linking formation over (A,S) $\left\{ \begin{array}{l} (M, \lambda; F, G) \oplus (M, \lambda; G, H) \\ (M, \lambda; F, G) \\ (M, \lambda; F, G) \end{array} \right.$ is S-acyclic cobordant to the complex associated to $\left\{ \begin{array}{l} (M, \lambda; F, H) \\ (L^{\perp}/L, \lambda^{\perp}/\lambda; F \cap L^{\perp}, G/L) \\ (L^{\perp}/L, \lambda^{\perp}/\lambda; F/L, G/L) \end{array} \right.$, for any non-singular even ε-symmetric linking form over (A,S) (M, λ) and lagrangians F, G together with a $\left\{ \begin{array}{l} \text{lagrangian } H \text{ of } (M, \lambda) \\ \text{sublagrangian } L \text{ of } (M, \lambda; F, G) \\ \text{sublagrangian } L \text{ of } (M, \lambda) \text{ such that } L \subseteq F \cap G. \end{array} \right.$ We shall consider the three cases seperately.

Recall from the proof of Proposition 13.12 ii) that the S-acyclic complex (C,φ) associated to the linking formation $(M,\lambda;F,G)$ is the union

$$(C,\varphi) = (\delta D\cup_D \delta D', -\delta\eta\cup_\eta\delta\eta' \in Q^2(\delta D\cup_D\delta D',-\varepsilon))$$

of the S-acyclic null-cobordisms $(f:D\longrightarrow \delta D,(\delta\eta,\eta)\in Q^2(f,-\varepsilon))$, $(f':D\longrightarrow \delta D',(\delta\eta',\eta)\in Q^2(f',-\varepsilon))$ associated to the lagrangians F and G by Proposition 13.8 ii), with $(D,\eta\in Q^1(D,-\varepsilon))$ the S-acyclic complex associated to the linking form (M,λ) by Proposition 13.3. Let $(f'':D\longrightarrow\delta D'',(\delta\eta'',\eta)\in Q^2(f'',-\varepsilon))$ be the S-acyclic null-cobordism of (D,η) associated to the lagrangian H of (M,λ), so that the S-acyclic complexes associated to the linking formations $(M,\lambda;G,H)$ and $(M,\lambda;F,H)$ are the unions

$$(C',\varphi') = (\delta D'\cup_D\delta D'', -\delta\eta'\cup_\eta\delta\eta''\in Q^2(\delta D'\cup_D\delta D'',-\varepsilon))$$
$$(C'',\varphi'') = (\delta D\cup_D\delta D'', -\delta\eta\cup_\eta\delta\eta''\in Q^2(\delta D\cup_D\delta D'',-\varepsilon)) \ .$$

Now (C'',φ'') is homotopy equivalent to the S-acyclic complex obtained from $(C,\varphi)\oplus(C',\varphi')$ by surgery on the connected S-acyclic 3-dimensional $(-\varepsilon)$-symmetric pair over A $((g\ g'):C\oplus C'\longrightarrow\delta D',(0,\varphi\oplus-\varphi')\in Q^3((g\ g'),-\varepsilon))$, where

$$g = (0\quad 0\quad 1) : C_r = \delta D_r\oplus D_{r-1}\oplus\delta D'_r \longrightarrow \delta D'_r$$
$$g' = (1\quad 0\quad 0) : C'_r = \delta D'_r\oplus D_{r-1}\oplus\delta D''_r \longrightarrow \delta D'_r \quad .$$

It follows from the S-acyclic counterpart of Proposition 7.1 that

$$(C,\varphi)\oplus(C',\varphi') = (C'',\varphi'')\in L^1(A,S,\varepsilon) \ .$$

The non-singular ε-symmetric linking formations $(M,\lambda;F,G)$ and $(L^\perp/L,\lambda^\perp/\lambda;F\cap L^\perp,G/L)$ are stably equivalent, so that the associated S-acyclic complexes are homotopy equivalent (by Proposition 13.12 ii)) and hence represent the same element of $L^1(A,S,\varepsilon)$.

Given a non-singular even ε-symmetric linking formation over (A,S) $(M,\lambda;F,G)$ let $(K,\alpha;L)$ be a non-singular ε-symmetric S-form over A associated to $(M,\lambda;F,G)$ by Proposition 13.11 i). Let $j\in\mathrm{Hom}_A(L,K)$ be the inclusion, and let $k\in\mathrm{Hom}_A(L^*,K)$, $s\in S$ be such that

$$j^*\alpha k = s\in\mathrm{Hom}_A(L^*,L^*) \ .$$

Given a sublagrangian H of (M,λ) such that $H\subseteq F\cap G$ there exist a f.g. projective A-module L' and A-module morphisms $u\in\mathrm{Hom}_A(L,L')$, $v\in\mathrm{Hom}_A(L',L)$, $j'\in\mathrm{Hom}_A(L',K)$ such that the inclusions $H\longrightarrow F$, $H\longrightarrow G$ have resolutions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L^* & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}} & L'\oplus L^* & \longrightarrow & H & \longrightarrow & 0 \\
 & & \Big\downarrow 1 & \begin{pmatrix} \bar{s} & 0 \\ 0 & 1\end{pmatrix} & \Big\downarrow \begin{pmatrix} v & 0 \\ 0 & 1\end{pmatrix} & & \Big\downarrow & & \\
0 & \longrightarrow & L\oplus L^* & \longrightarrow & L\oplus L^* & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L\oplus L^* & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}} & L'\oplus L^* & \longrightarrow & H & \longrightarrow & 0 \\
 & & \Big\downarrow 1 & & \Big\downarrow (j'\ k) & & \Big\downarrow & & \\
0 & \longrightarrow & L\oplus L^* & \xrightarrow{(j\ k)} & K & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Let $(C,\varphi\in Q^2(C,-\varepsilon))$ be the S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A in normal form associated to the S-form $(K,\alpha;L)$ (as in the proof of Proposition 13.12 ii)), and define an A-module chain map $f:C\longrightarrow D$ by

$$\begin{array}{ccccccccccccc}
C: & \dots & \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{\alpha^* j} & K^* & \xrightarrow{j^*} & L^* & \longrightarrow & 0 & \longrightarrow \dots \\
f\Big\downarrow & & & & & \Big\downarrow & & \Big\downarrow j'^* & & \Big\downarrow 1 & & & \\
D: & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & L'^* & \xrightarrow{u^*} & L^* & \longrightarrow & 0 & \longrightarrow \dots \ .
\end{array}$$

Let $(C',\varphi'\in Q^2(C',-\varepsilon))$ be the S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A obtained from (C,φ) by surgery on the connected S-acyclic 3-dimensional even $(-\varepsilon)$-symmetric pair over A $(f:C\longrightarrow D,(0,\varphi)\in Q^3(f,-\varepsilon))$. Let $(C'',\varphi''\in Q^2(C'',-\varepsilon))$ be the S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A in normal form associated by Proposition 13.12 ii) to the non-singular ε-symmetric S-form over A $(K,\alpha;\mathrm{im}(j':L'\longrightarrow K))$, which corresponds by Proposition 13.12 i) to the non-singular even ε-symmetric linking formation over (A,S) $(H^\perp/H,\lambda^\perp/\lambda;F/H,G/H)$. The A-module chain equivalence $h: C' \longrightarrow C''$ given by

$$\begin{array}{ccccccccccccccc}
C' : & \cdots \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{\begin{pmatrix}\varepsilon\\ u\end{pmatrix}} & L \oplus L' & \xrightarrow{(\alpha^* j\ \ -\alpha j')} & K^* & \xrightarrow{\begin{pmatrix} j \\ -j'^*\end{pmatrix}} & L^* \oplus L'^* & \xrightarrow{(1\ \ u^*)} & L^* & \longrightarrow & 0 \longrightarrow \cdots \\
\downarrow h & & \downarrow & & \downarrow & & \downarrow (u\ \ -\varepsilon) & & \downarrow 1 & & \downarrow (0\ \ -1) & & \downarrow & & \\
C'' : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & L' & \xrightarrow{\alpha^* j'} & K^* & \xrightarrow{j'^*} & L'^* & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots
\end{array}$$

defines a homotopy equivalence of S-acyclic 2-dimensional even $(-\varepsilon)$-symmetric Poincaré complexes over A

$$h : (C',\varphi') \longrightarrow (C'',\varphi'') \ .$$

It follows that

$$(C,\varphi) = (C',\varphi') = (C'',\varphi'') \in L^1(A,S,\varepsilon) \ ,$$

verifying that the S-acyclic complexes associated to the linking formations $(M,\lambda;F,G)$ and $(H^\perp/H,\lambda^\perp/\lambda;F/H,G/H)$ are S-acyclic cobordant.

This completes the identification

$$L^1(A,S,\varepsilon) = M\langle v_0\rangle^\varepsilon(A,S) \ .$$

The verification that a non-singular $\left\{\begin{array}{l}\varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M,\lambda,\mu;F,G)\\ (F,G)\end{array}\right.$ represents 0 in the Witt group $\left\{\begin{array}{l}M_\varepsilon(A,S)\\ \widetilde{M}_\varepsilon(A,S)\end{array}\right.$ if and only if it is stably equivalent to the boundary $\left\{\begin{array}{l}\partial(K,\alpha;I,J)\\ \partial(K,\beta;I,J)\end{array}\right.$ of a non-degenerate $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ formation over A $\left\{\begin{array}{l}(K,\alpha;I,J)\\ (K,\beta;I,J)\end{array}\right.$ such that $\left\{\begin{array}{l}S^{-1}(K,\alpha;I,J) = 0 \in M\langle v_0\rangle^\varepsilon_S(S^{-1}A)\\ S^{-1}(K,\beta;I,J) = 0 \in M^S_\varepsilon(S^{-1}A)\end{array}\right.$ proceeds by analogy with the case of even ε-symmetric linking formations.

It remains to prove that $\left\{\begin{array}{l}\text{if } \ker(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \xrightarrow{\hat{\delta}} \hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0\\ \text{for all } A,S,\varepsilon\\ \text{for all } A,S,\varepsilon\end{array}\right.$ a non-singular $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking formation over (A,S) $\left\{\begin{array}{l}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\\ (F,G)\end{array}\right.$ represents 0 in the Witt group $\left\{\begin{array}{l}M\langle v_0\rangle^\varepsilon(A,S)\\ M_\varepsilon(A,S)\\ \widetilde{M}_\varepsilon(A,S)\end{array}\right.$ if and only if it is stably equivalent to the boundary $\left\{\begin{array}{l}\partial(N,\theta)\\ \partial(N,\theta)\\ \partial(N,\theta,\psi)\end{array}\right.$ of an $\left\{\begin{array}{l}(-\varepsilon)\text{-symmetric}\\ \text{even } (-\varepsilon)\text{-symmetric linking}\\ (-\varepsilon)\text{-quadratic}\end{array}\right.$ form over (A,S) $\left\{\begin{array}{l}(N,\theta)\\ (N,\theta)\\ (N,\theta,\psi)\end{array}\right.$. For $\left\{\begin{array}{l}\text{even } \varepsilon\text{-symmetric}\\ \text{split } \varepsilon\text{-quadratic}\end{array}\right.$ linking formations this follows from Proposition 13.13 ii). (The projection of Proposition 13.12 iii)

(S-acyclic 2-dimensional $(-\varepsilon)$-quadratic Poincaré complexes over A (C,ψ))

$\longrightarrow$(non-singular split ε-quadratic linking formations over (A,S) (F,G))

can thus be used to define an isomorphism of abelian groups

$$L_1(A,S,\varepsilon) \longrightarrow \widetilde{M}_\varepsilon(A,S) \ ; \ (C,\psi) \longmapsto (F,G)).$$

The method of proof of Proposition 13.12 ii) is readily modified to prove the corresponding result for ε-quadratic linking formations.

[]

A non-singular $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over $S^{-1}A$ $\begin{cases}(Q,\varphi;F,G)\\ (Q,\psi;F,G)\end{cases}$

such that $[G]-[F^*]\in S=\mathrm{im}(\widetilde{K}_0(A)\longrightarrow \widetilde{K}_0(S^{-1}A))$ is stably isomorphic to

$\begin{cases}S^{-1}(K,\alpha;I,J)\\ S^{-1}(K,\beta;I,J)\end{cases}$ for some non-degenerate $\begin{cases}\text{(even) } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ formation over A

$\begin{cases}(K,\alpha;I,J)\\ (K,\beta;I,J)\end{cases}$. It follows from Proposition 13.13 that the boundary operation

$$\partial : \text{(non-degenerate formations over } A)\longrightarrow \text{(linking formations over } (A,S))$$

can be used to define abelian group morphisms

$$\partial : M_S^{\varepsilon}(S^{-1}A)\longrightarrow M\langle v_0\rangle^{\varepsilon}(A,S)\ ;\ S^{-1}(K,\ ;I,J)\longmapsto \partial(K,\ ;I,J)$$

$$\partial : M\langle v_0\rangle_S^{\varepsilon}(S^{-1}A)\longrightarrow M_{\varepsilon}(A,S)\ ;\ S^{-1}(K,\alpha;I,J)\longmapsto \partial(K,\alpha;I,J)$$

$$\partial : M_{\varepsilon}^S(S^{-1}A)\longrightarrow \widetilde{M}_{\varepsilon}(A,S)\ ;\ S^{-1}(K,\beta;I,J)\longmapsto \partial(K,\beta;I,J)\quad .$$

There is also defined a morphism

$$\partial : M_S^{\varepsilon}(S^{-1}A)\longrightarrow M^{\varepsilon}(A,S)\ ;\ S^{-1}(K,\alpha;I,J)\longmapsto \partial(K,\alpha;I,J)\ ,$$

namely the composite

$$M_S^{\varepsilon}(S^{-1}A)\xrightarrow{\ \partial\ } M\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow M^{\varepsilon}(A,S)\quad .$$

The correspondence of Proposition 13.11 i) associates to a non-singular

$\begin{cases}\text{even } \varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\\ \text{split } \varepsilon\text{-quadratic}\end{cases}$ linking formation over (A,S) $\begin{cases}(M,\lambda;F,G)\\ (M,\lambda,\mu;F,G)\\ (F,G)\end{cases}$ a stable

isomorphism class of non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \text{even } \varepsilon\text{-symmetric } S\text{-forms over } A\\ \varepsilon\text{-quadratic}\end{cases}$ $\begin{cases}(K,\alpha;L)\\ (K,\alpha;L).\\ (K,\beta;L)\end{cases}$

It follows from Proposition 13.13 that there are defined abelian group morphisms

$$M\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow L^{\varepsilon}(A)\ ;\ (M,\lambda;F,G)\longmapsto (K,\alpha)$$

$$M_{\varepsilon}(A,S)\longrightarrow L\langle v_0\rangle^{\varepsilon}(A)\ ;\ (M,\lambda,\mu;F,G)\longmapsto (K,\alpha)$$

$$\widetilde{M}_{\varepsilon}(A,S)\longrightarrow L_{\varepsilon}(A,S)\ ;\ (F,G)\longmapsto (K,\beta)\quad .$$

Define abelian groups $\begin{cases}L^{2k+1}(A,S,\varepsilon)\\ L_{2k+1}(A,S,\varepsilon)\end{cases}$ $(k\leqslant -1)$ by

$$L_{2k+1}(A,S,\varepsilon)=L_{2k+2i+1}(A,S,(-)^i\varepsilon)\quad (k+i\geqslant 0)$$

(extending the periodicity of Proposition 13.1 ii))

$$L^{-1}(A,S,\varepsilon)=M_{-\varepsilon}(A,S)$$

$$L^{2k+1}(A,S,\varepsilon)=L_{2k+1}(A,S,\varepsilon)\quad (k\leqslant -2)\ .$$

<u>Proposition 13.15</u> i) The localization exact sequence of algebraic Poincaré cobordism groups

$$L^{2k+1}(A,(-)^k\varepsilon)\longrightarrow L_S^{2k+1}(S^{-1}A,(-)^k\varepsilon)\longrightarrow L^{2k+1}(A,S,(-)^k\varepsilon)$$
$$\longrightarrow L^{2k}(A,(-)^k\varepsilon)\longrightarrow L_S^{2k}(S^{-1}A,(-)^k\varepsilon)\qquad (*)_{2k+1}$$

is naturally isomorphic for $\begin{cases}k=0\\ k=-1\\ k\leqslant -2\end{cases}$ to a localization exact sequence of

Witt groups

$$\begin{cases}M^{\varepsilon}(A)\longrightarrow M_S^{\varepsilon}(S^{-1}A)\longrightarrow M\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow L^{\varepsilon}(A)\longrightarrow L_S^{\varepsilon}(S^{-1}A)\\ M\langle v_0\rangle^{\varepsilon}(A)\longrightarrow M\langle v_0\rangle_S^{\varepsilon}(S^{-1}A)\longrightarrow M_{\varepsilon}(A,S)\longrightarrow L\langle v_0\rangle^{\varepsilon}(A)\longrightarrow L\langle v_0\rangle_S^{\varepsilon}(S^{-1}A)\\ M_{\varepsilon}(A)\longrightarrow M_{\varepsilon}^S(S^{-1}A)\longrightarrow \widetilde{M}_{\varepsilon}(A,S)\longrightarrow L_{\varepsilon}(A)\longrightarrow L_{\varepsilon}^S(S^{-1}A)\quad .\end{cases}$$

The long exact sequence of L-groups

$$\ldots\longrightarrow L^n(A,\varepsilon)\longrightarrow L_S^n(S^{-1}A,\varepsilon)\longrightarrow L^n(A,S,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon)\longrightarrow\ \ldots\quad (n\in\mathbb{Z}$$

is naturally isomorphic in the range $n\leqslant 2$ to the long exact sequence of

Witt groups

$$\ldots\longrightarrow L^2(A,S,\varepsilon)\longrightarrow M^{\varepsilon}(A)\longrightarrow M_S^{\varepsilon}(S^{-1}A)\longrightarrow M\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow L^{\varepsilon}(A)\longrightarrow L_S^{\varepsilon}(S^{-1}A)$$
$$\longrightarrow L\langle v_0\rangle^{\varepsilon}(A,S)\longrightarrow M\langle v_0\rangle^{-\varepsilon}(A)\longrightarrow M\langle v_0\rangle_S^{-\varepsilon}(S^{-1}A)\longrightarrow M_{-\varepsilon}(A,S)\longrightarrow L\langle v_0\rangle^{-\varepsilon}(A)$$
$$\longrightarrow L\langle v_0\rangle_S^{-\varepsilon}(S^{-1}A)\longrightarrow L_{-\varepsilon}(A,S)\longrightarrow M_{\varepsilon}(A)\longrightarrow M_{\varepsilon}^S(S^{-1}A)\longrightarrow \widetilde{M}_{\varepsilon}(A,S)\longrightarrow L_{\varepsilon}(A)$$
$$\longrightarrow L_{\varepsilon}^S(S^{-1}A)\longrightarrow \widetilde{L}_{\varepsilon}(A,S)\longrightarrow M_{-\varepsilon}(A)\longrightarrow M_{-\varepsilon}^S(S^{-1}A)\longrightarrow \widetilde{M}_{-\varepsilon}(A,S)\longrightarrow\ \ldots\quad .$$

ii) There are defined natural abelian group morphisms

$$M^{\varepsilon}(A,S) \longrightarrow L^{2k+1}(A,S,(-)^k\varepsilon) \quad (k \geqslant 1)$$

for all A,S,ε. If $\begin{cases} (A,S) \text{ is 1-dimensional} \\ \ker(\hat{S}:\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)) = 0 \end{cases}$ then for $\begin{cases} k \geqslant 1 \\ k = 1 \end{cases}$

these are isomorphisms.

iii) If (A,S) is 0-dimensional

$$M^{\varepsilon}(A,S) = M\langle v_0\rangle^{\varepsilon}(A,S) = M_{\varepsilon}(A,S) = 0 \ ,$$

and there are defined localization exact sequences of Witt groups

$$0 \longrightarrow L^{\varepsilon}(A) \longrightarrow L^{\varepsilon}_S(S^{-1}A) \xrightarrow{\partial} L^{\varepsilon}(A,S) \longrightarrow M^{-\varepsilon}(A) \longrightarrow M^{-\varepsilon}_S(S^{-1}A) \longrightarrow 0$$

$$0 \longrightarrow L^{\varepsilon}(A) \longrightarrow L^{\varepsilon}_S(S^{-1}A) \to L\langle v_0\rangle^{\varepsilon}(A,S) \to M\langle v_0\rangle^{-\varepsilon}(A) \to M\langle v_0\rangle^{-\varepsilon}_S(S^{-1}A) \longrightarrow 0$$

$$0 \longrightarrow L\langle v_0\rangle^{\varepsilon}(A) \longrightarrow L\langle v_0\rangle^{\varepsilon}_S(S^{-1}A) \longrightarrow L_{\varepsilon}(A,S) \longrightarrow M_{-\varepsilon}(A) \longrightarrow M^S_{-\varepsilon}(S^{-1}A) \longrightarrow \widetilde{M}_{-\varepsilon}(A,S)$$

$$\longrightarrow L_{-\varepsilon}(A) \longrightarrow L^S_{-\varepsilon}(S^{-1}A) \longrightarrow \widetilde{L}_{-\varepsilon}(A,S) \longrightarrow M_{\varepsilon}(A) \longrightarrow M^S_{\varepsilon}(S^{-1}A) \longrightarrow \widetilde{M}_{\varepsilon}(A,S) \longrightarrow \dots \ .$$

iv) If $\begin{cases} \operatorname{im}(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \longrightarrow \hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0 \\ \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon) \text{ is an isomorphism} \\ \hat{H}^1(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A,\varepsilon) \text{ is an isomorphism} \end{cases}$ then the forgetful map

identifies the Witt groups

$$\begin{cases} M\langle v_0\rangle^{\varepsilon}(A,S) = M^{\varepsilon}(A,S) \\ M_{\varepsilon}(A,S) = M\langle v_0\rangle^{\varepsilon}(A,S) \\ \widetilde{M}_{\varepsilon}(A,S) = M_{\varepsilon}(A,S) \ . \end{cases}$$

In particular, if $1/2 \in A$ then

$$\widetilde{M}_{\varepsilon}(A,S) = M_{\varepsilon}(A,S) = M\langle v_0\rangle^{\varepsilon}(A,S) = M^{\varepsilon}(A,S) \ .$$

Proof: i) We have already verified (in the course of the proof of Proposition 13.14) that there are natural identifications

$$L^1(A,S,\varepsilon) = M\langle v_0\rangle^{\varepsilon}(A,S)$$

$$L_1(A,S,\varepsilon) = \widetilde{M}_{\varepsilon}(A,S) \ .$$

The sequence of Witt groups

$$\begin{cases} M^{\varepsilon}(A) \longrightarrow M^{\varepsilon}_S(S^{-1}A) \xrightarrow{\partial} M\langle v_0\rangle^{\varepsilon}(A,S) \longrightarrow L^{\varepsilon}(A) \longrightarrow L^{\varepsilon}_S(S^{-1}A) \\ M_{\varepsilon}(A) \longrightarrow M^S_{\varepsilon}(S^{-1}A) \xrightarrow{\partial} \widetilde{M}_{\varepsilon}(A,S) \longrightarrow L_{\varepsilon}(A) \longrightarrow L^S_{\varepsilon}(S^{-1}A) \end{cases}$$

can thus be identified with the exact sequence of L-groups $\begin{cases} (*)_1 \\ (*)_{-3} \end{cases}$. The exactness of this sequence of Witt groups can also be established directly, using Proposition 13.14. The direct method applies also to the verification of the exactness of $(*)_{-1}$

$$M\langle v_0\rangle^{\varepsilon}(A) \longrightarrow M\langle v_0\rangle^{\varepsilon}_S(S^{-1}A) \xrightarrow{\partial} M_{\varepsilon}(A,S) \longrightarrow L\langle v_0\rangle^{\varepsilon}(A) \longrightarrow L\langle v_0\rangle^{\varepsilon}_S(S^{-1}A) \ .$$

ii) Define abelian group morphisms

$$M^{\varepsilon}(A,S) \longrightarrow L^{2k+1}(A,S,(-)^k\varepsilon) \ ; \ (M,\lambda;F,G) \longmapsto \bar{S}^k(C,\varphi) \quad (k \geqslant 1)$$

by sending a non-singular ε-symmetric linking formation over (A,S) $(M,\lambda;F,G)$ to the k-fold skew-suspension $\bar{S}^k(C,\varphi)$ of an S-acyclic 2-dimensional $(-\varepsilon)$-symmetric Poincaré complex over A $(C,\varphi \in Q^2(C,-\varepsilon))$ associated to $(M,\lambda;F,G)$ by Proposition 13.12 ii). The S-acyclic cobordism class $\bar{S}^k(C,\varphi) \in L^{2k+1}(A,S,(-)^k\varepsilon)$ depends only on the stable equivalence class of $(M,\lambda;F,G)$ (proved exactly as in Proposition 13.13 i)), vanishing if $(M,\lambda;F,G) = 0 \in M^{\varepsilon}(A,S)$ (proved as in Proposition 13.14), so that the morphisms are well-defined.

If $\begin{cases} (A,S) \text{ is 1-dimensional} \\ \ker(\hat{S}:\hat{H}^1(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon)) = 0 \end{cases}$ then for $\begin{cases} k \geqslant 1 \\ k = 1 \end{cases}$ $L^{2k+1}(A,S,(-)^k\varepsilon)$

is the S-acyclic cobordism group of S-acyclic 2-dimensional $(-\varepsilon)$-symmetric Poincaré complexes over A (by Proposition $\begin{cases} 13.2 \\ 13.3 \text{ i)} \end{cases}$), so that the morphisms are onto. If $(M,\lambda;F,G) \in \ker(M^{\varepsilon}(A,S) \longrightarrow L^{2k+1}(A,S,(-)^k\varepsilon))$ then (C,φ) is homotopy equivalent to the boundary $\partial(D,\eta)$ of a connected S-acyclic 3-dimensional $(-\varepsilon)$-symmetric complex over A $(D,\eta \in Q^3(D,-\varepsilon))$, and the proof of Proposition 13.14 generalizes to show that $(M,\lambda;F,G) = 0 \in M^{\varepsilon}(A,S)$, so that the morphisms are also one-one.

iii) If (A,S) is 0-dimensional

$$L^3(A,S,-\varepsilon) = L^1(A,S,\varepsilon) = 0$$

by Proposition 13.2, so that

$$M^{\varepsilon}(A,S) = M\langle v_0\rangle^{\varepsilon}(A,S) = 0 \ .$$

The proof of Proposition 13.2 generalizes immediately to show that

$$L^{-1}(A,S,-\varepsilon) = M_{\varepsilon}(A,S) = 0 \ .$$

iv) If $\begin{cases} \mathrm{im}(\hat{H}^0(\mathbb{Z}_2;S^{-1}A/A,\varepsilon) \xrightarrow{\hat{\delta}} \hat{H}^1(\mathbb{Z}_2;A,\varepsilon)) = 0 \\ \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon) \text{ is an isomorphism} \end{cases}$ we have an identification of categories

$$\begin{cases} (\text{even } \varepsilon\text{-symmetric linking forms over } (A,S)) \\ \qquad = (\varepsilon\text{-symmetric linking forms over } (A,S)) \\ (\varepsilon\text{-quadratic linking forms over } (A,S)) \\ \qquad = (\text{even } \varepsilon\text{-symmetric linking forms over } (A,S)) \end{cases}$$

by Proposition 13.5 ii). We can thus identify the categories

$$\begin{cases} (\text{even } \varepsilon\text{-symmetric linking formations over } (A,S)) \\ \qquad = (\varepsilon\text{-symmetric linking formations over } (A,S)) \\ (\varepsilon\text{-quadratic linking formations over } (A,S)) \\ \qquad = (\text{even } \varepsilon\text{-symmetric linking formations over } (A,S)), \end{cases}$$

and also the Witt groups

$$\begin{cases} M\langle v_0\rangle^{\varepsilon}(A,S) = M^{\varepsilon}(A,S) \\ M_{\varepsilon}(A,S) = M\langle v_0\rangle^{\varepsilon}(A,S) \ . \end{cases}$$

If $\hat{H}^1(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^1(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is an isomorphism Proposition 13.5 ii) gives identifications of categories

(split ε-quadratic linking forms over (A,S))

$\qquad = (\varepsilon$-quadratic linking forms over $(A,S))$

($(-\varepsilon)$-quadratic linking forms over (A,S))

$\qquad = ($even $(-\varepsilon)$-symmetric linking forms over $(A,S))$,

so that it is possible to identify the stable equivalence classes

(split ε-quadratic linking formations over (A,S))

$\qquad = (\varepsilon$-quadratic linking formations over $(A,S))$

and also the Witt groups

$$\tilde{M}_{\varepsilon}(A,S) = M_{\varepsilon}(A,S) \ .$$

[]

In Proposition 13.3 ii) we showed that if $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is an isomorphism then

$$L^n(A,S,-\varepsilon) = L^{n+2}(A,S,\varepsilon) \quad (n \geqslant 0) \ .$$

We shall now extend this to all $n \in \mathbb{Z}$, obtaining an S-acyclic analogue of Proposition 9.4 (if $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon) = 0$ then $L^n(A,-\varepsilon) = L^{n+2}(A,\varepsilon)$ $(n \in \mathbb{Z})$).

<u>Proposition 13.16</u> If $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \longrightarrow \hat{H}^0(\mathbb{Z}_2;S^{-1}A,\varepsilon)$ is an isomorphism then the skew-suspension maps

$$\bar{S} : L^n(A,S,-\varepsilon) \longrightarrow L^{n+2}(A,S,\varepsilon) \quad (n \in \mathbb{Z})$$

are isomorphisms.

<u>Proof</u>: Immediate from Propositions 13.3 ii), 13.15 iv).

[]

The localization exact sequence is natural, in the following sense:

<u>Proposition 13.17</u> Given a morphism of rings with involution $f:A \longrightarrow B$ such that $f(S) \subset T$ for some multiplicative subsets $S \subset A$, $T \subset B$ there is defined a morphism of exact sequences of abelian groups

$$\begin{array}{ccccccccc} \dots \longrightarrow & L^n(A,\varepsilon) & \longrightarrow & L^n_S(S^{-1}A,\varepsilon) & \longrightarrow & L^n(A,S,\varepsilon) & \longrightarrow & L^{n-1}(A,\varepsilon) & \longrightarrow \dots \\ & \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & \\ \dots \longrightarrow & L^n(B,\varepsilon) & \longrightarrow & L^n_T(T^{-1}B,\varepsilon) & \longrightarrow & L^n(B,T,\varepsilon) & \longrightarrow & L^{n-1}(B,\varepsilon) & \longrightarrow \dots \end{array} \quad (n \in \mathbb{Z}) \ .$$

Similarly for L_*.

<u>Proof</u>: If C is an S-acyclic n-dimensional A-module chain complex then $B \otimes_A C$ is a T-acyclic n-dimensional B-module chain complex, since

$$T^{-1}(B \otimes_A C) = T^{-1}B \otimes_{S^{-1}A} S^{-1}C \ .$$

[]

Given a central indeterminate x over a ring A there is defined a multiplicative subset $S = \{x^k \mid k \geqslant 0\}$ of the polynomial extension $A[x]$, such that

$$S^{-1}A[x] = A[x,x^{-1}] .$$

The "fundamental theorem of algebraic K-theory", the split exact sequence

$$0 \longrightarrow K_n(A) \longrightarrow K_n(A[x]) \oplus K_n(A[x^{-1}]) \longrightarrow K_n(A[x,x^{-1}]) \longrightarrow K_{n-1}(A) \longrightarrow 0 \qquad (n \in \mathbb{Z})$$

may be obtained from the localization exact sequence

$$\ldots \longrightarrow K_n(A[x]) \longrightarrow K_n(A[x,x^{-1}]) \longrightarrow K_n(A[x],S) \longrightarrow K_{n-1}(A[x]) \longrightarrow \ldots ,$$

using the equivalence of exact categories

(h.d.1 S-torsion $A[x]$-modules M)

= (f.g. projective A-modules with a nilpotent endomorphism $x: M \longrightarrow M$)

(Bass [1] for $n \leq 1$, Quillen [1] for $n \geqslant 2$). Similarly, we could use the L-theoretic localization exact sequence of Proposition 13.1

$$\ldots \longrightarrow L^n(A[x],\varepsilon) \longrightarrow L^n_S(A[x,x^{-1}],\varepsilon) \longrightarrow L^n(A[x],S,\varepsilon) \longrightarrow L^{n-1}(A[x],\varepsilon) \longrightarrow \ldots$$

to obtain analogous results for the L-theory (in the range $n \leqslant 1$) of the polynomial extensions $A[x], A[x,x^{-1}]$ of a ring with involution A, where the involution is extended by

$$\bar{x} = x .$$

We have already studied the quadratic L-theory of such polynomial extensions in Ranicki [4], so that in fact we can reverse the process and deduce information about the localization exact sequence

$$\ldots \longrightarrow L_n(A[x],\varepsilon) \longrightarrow L^S_n(A[x,x^{-1}],\varepsilon) \longrightarrow L_n(A[x],S,\varepsilon) \longrightarrow L_{n-1}(A[x],\varepsilon) \longrightarrow \ldots$$

It follows from Ranicki [4] that this breaks up into split exact sequences

$$0 \longrightarrow L_n(A[x],\varepsilon) \longrightarrow L^S_n(A[x,x^{-1}],\varepsilon) \longrightarrow L_n(A[x],S,\varepsilon) \longrightarrow 0 ,$$

and that there are defined natural isomorphisms

$$L_n(A[x],S,\varepsilon) \longrightarrow L^K_n(A[x^{-1}],\varepsilon) \qquad (n \in \mathbb{Z},\ K = \mathrm{im}(\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(A[x^{-1}])))$$

We thus have a commutative diagram of Witt groups

$$\begin{array}{ccc} L_\varepsilon(A) & \longrightarrow & L\langle v_0\rangle^\varepsilon(A) \\ \downarrow & & \downarrow \\ \widetilde{L}_\varepsilon(A[x],S,\varepsilon) & \longrightarrow & L_\varepsilon(A[x],S,\varepsilon) \end{array}$$

in which $L_\varepsilon(A) = L_0(A,\varepsilon) \longrightarrow \widetilde{L}_\varepsilon(A[x],S,\varepsilon) = L_0(A[x],S,\varepsilon)$ is a split injection. ($L\langle v_0\rangle^\varepsilon(A) = L^{-2}(A,-\varepsilon) \longrightarrow L_\varepsilon(A[x],S,\varepsilon) = L^{-2}(A[x],S,-\varepsilon)$ is also a split injection). In words, the extra structure afforded an ε-quadratic linking form over $(A[x],S)$ (M,λ,μ) by a choice of split ε-quadratic linking form over $(A[x],S)$ (M,λ,ν) such that

$$\nu : M \xrightarrow{\mu} Q_\varepsilon(A[x,x^{-1}]/A[x]) \xrightarrow{p} Q_\varepsilon(A[x],S)$$

corresponds to the structure of an ε-quadratic refinement of an even ε-symmetric form over A. In particular, the extra structure detects the Arf invariant:

<u>Proposition 13.18</u> Let $c = (M,\lambda,\nu)$ be the non-singular split skew-quadratic linking form over $(\mathbb{Z}[x], S)$ $(S = \{x^k \mid k \geqslant 0\})$ defined by

$$M = \mathbb{Z} \oplus \mathbb{Z} ,\ xM = 0 ,\ \lambda : M \times M \longrightarrow \mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x] ;\ ((m,n),(m',n')) \longmapsto x^{-1}(mn' - m'n)$$

$$\nu : M \longrightarrow Q_{-1}(\mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x]) = \mathbb{Z}[x,x^{-1}]/\mathbb{Z}[x] + 2\mathbb{Z}[x,x^{-1}] ;$$
$$(m,n) \longmapsto x^{-1}(m^2 + n^2 + mn) .$$

Then $c \neq 0 \in \ker(\widetilde{L}_\varepsilon(\mathbb{Z}[x],S) \longrightarrow L_\varepsilon(\mathbb{Z}[x],S)) = \mathbb{Z}_2$ $(\varepsilon = -1)$.

<u>Proof</u>: The element $c \in \widetilde{L}_\varepsilon(\mathbb{Z}[x],S) = L_2(\mathbb{Z}[x],S)$ is the image of the generator $c \in L_2(\mathbb{Z}) = \mathbb{Z}_2$ under the natural injection $L_2(\mathbb{Z}) = \mathbb{Z}_2 \longrightarrow \widetilde{L}_\varepsilon(\mathbb{Z}[x],S)$.

[]

The main result of Ranicki [4] is a split exact sequence

$$0 \longrightarrow V_n(A,\varepsilon) \longrightarrow V_n(A[x],\varepsilon) \oplus V_n(A[x^{-1}],\varepsilon) \longrightarrow V_n(A[x,x^{-1}],\varepsilon) \longrightarrow U_n(A,\varepsilon) \longrightarrow 0 .$$

The same arguments apply to also give:

<u>Proposition 13.19</u> There is a natural split exact sequence for $n \leqslant 1$

$$0 \longrightarrow V^n(A,\varepsilon) \longrightarrow V^n(A[x],\varepsilon) \oplus V^n(A[x^{-1}],\varepsilon) \longrightarrow V^n(A[x,x^{-1}],\varepsilon) \longrightarrow U^n(A,\varepsilon) \longrightarrow 0 .$$

[]

It may be conjectured that the result of Proposition 13.19 holds for all $n \in \mathbb{Z}$.

There are also intermediate versions of the localization exact sequence, for the intermediate L-groups of §12.

Given a $*$-invariant subgroup $X\subseteq\widetilde{K}_0(A)$ let $L^n_X(A,S,\varepsilon)$ $(n\in\mathbb{Z})$ be the L-groups defined exactly as $L^n(A,S,\varepsilon)$ but using only S-acyclic algebraic Poincaré complexes over A (C,φ) such that $[C]\in X\subseteq\widetilde{K}_0(A)$. In particular,

$$L^n_{\widetilde{K}_0(A)}(A,S,\varepsilon) = L^n(A,S,\varepsilon) .$$

Given $*$-invariant subgroups $X\subseteq\widetilde{K}_1(A)$, $Y\subseteq\widetilde{K}_1(S^{-1}A)$ such that $S^{-1}X\subseteq Y$ let $L^n_{X,Y}(A,S,\varepsilon)$ $(n\in\mathbb{Z})$ be the L-groups defined exactly as $L^n(A,S,\varepsilon)$ but using only S-acyclic algebraic Poincaré complexes over A (C,φ) such that C is based, $\tau(\varphi_0:C^{n-*}\longrightarrow C)\in X\subseteq\widetilde{K}_1(A)$, $\tau(S^{-1}C)\in Y\subseteq\widetilde{K}_1(S^{-1}A)$. In particular,

$$L^n_{\widetilde{K}_1(A),\widetilde{K}_1(S^{-1}A)}(A,S,\varepsilon) = L^n_{\{0\}\subseteq K_0(A)}(A,S,\varepsilon) .$$

<u>Proposition 13.20</u> i) Given a $*$-invariant subgroup $X\subseteq\widetilde{K}_0(A)$ there is defined an exact sequence of abelian groups

$$\dots\longrightarrow L^n_X(A,\varepsilon)\longrightarrow L^n_{S^{-1}X}(S^{-1}A,\varepsilon)\longrightarrow L^n_X(A,S,\varepsilon)\longrightarrow L^{n-1}_X(A,\varepsilon)\longrightarrow\dots\quad(n\in\mathbb{Z}).$$

ii) Given $*$-invariant subgroups $X\subseteq\widetilde{K}_1(A)$, $Y\subseteq\widetilde{K}_1(S^{-1}A)$ such that $S^{-1}X\subseteq Y$ there is defined an exact sequence of abelian groups

$$\dots\longrightarrow L^n_X(A,\varepsilon)\longrightarrow L^n_Y(S^{-1}A,\varepsilon)\longrightarrow L^n_{X,Y}(A,S,\varepsilon)\longrightarrow L^{n-1}_X(A,\varepsilon)\longrightarrow\dots\quad(n\in\mathbb{Z}) .$$

[]

As a particular case of Proposition 11.5 ii) we have:

<u>Proposition 13.21</u> Let A,R be rings with involution such that A is an R-module, and let $S\subset A$ be a multiplicative subset. The symmetric L-groups $L^*(R)$ act on the localization exact sequence

$$\begin{cases}\dots\longrightarrow L^n(A,\varepsilon)\longrightarrow L^n_S(S^{-1}A,\varepsilon)\longrightarrow L^n(A,S,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon)\longrightarrow\dots\\ \dots\longrightarrow L_n(A,\varepsilon)\longrightarrow L^S_n(S^{-1}A,\varepsilon)\longrightarrow L_n(A,S,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\longrightarrow\dots\end{cases}\quad(n\in\mathbb{Z}) .$$

The element $(R,1)\in L^0(R)$ acts by the identity.

[]

In our applications of the localization exact sequence we shall make much use of the following criterion for a morphism of rings with involution and multiplicative subsets

$$f : (A,S)\longrightarrow(B,T)$$

to induce isomorphisms of L-groups

$$\begin{cases}f : L^n(A,S,\varepsilon)\longrightarrow L^n(B,T,\varepsilon)\\ f : L_n(A,S,\varepsilon)\longrightarrow L_n(B,T,\varepsilon)\end{cases}\quad(n\in\mathbb{Z}) .$$

Define a partial ordering on S by

$$s\leq s' \text{ if there exists } t\in S \text{ such that } s' = st\in S.$$

Define also a direct system of abelian groups $\{A/sA\,|\,s\in S\}$ with structure maps

$$A/sA\longrightarrow A/stA ;\ x\longmapsto tx .$$

The abelian group morphisms

$$A/sA\longrightarrow S^{-1}A/A ;\ a\longmapsto \frac{a}{s}$$

allow the identification

$$\varinjlim_{s\in S} A/sA = S^{-1}A/A .$$

The involution

$$- : S^{-1}A/A\longrightarrow S^{-1}A/A ;\ \frac{a}{s}\longmapsto\frac{\bar{a}}{\bar{s}}$$

is identified with the involution

$$- : \varinjlim_{s\in S} A/sA\longrightarrow\varinjlim_{s\in S} A/sA ;\ \{a_s\in A/sA\,|\,s\in S\}\longmapsto\{\bar{a}_{\bar{s}}\in A/sA\,|\,s\in S\} .$$

A morphism of rings with involution and multiplicative subsets

$$f : (A,S)\longrightarrow(B,T)$$

is <u>cartesian</u> if $f(S) = T$ and if for every $s\in S$ the map

$$f : A/sA\longrightarrow B/tB ;\ x\longmapsto f(x)\quad(t = f(s)\in T)$$

is an isomorphism of abelian groups, in which case there is induced an isomorphism of abelian groups with involution

$$f : \varinjlim_{s\in S} A/sA = S^{-1}A/A\longrightarrow\varinjlim_{t\in T} B/tB = T^{-1}B/B ;\ x\longmapsto f(x) .$$

Cartesian morphisms were introduced by Karoubi [2] (Appendix 1) who proved that such a morphism induces an isomorphism of exact categories

$$f : (\text{h.d. 1 } S\text{-torsion } A\text{-modules}) \longrightarrow (\text{h.d. 1 } T\text{-torsion } B\text{-modules}) \ ;$$

$$M \longmapsto B\otimes_A M \ (= M \text{ as an } A\text{-module}) \ ,$$

and hence induces isomorphisms in hermitian K-theory. (The nomenclature reflects the cartesian property of the commutative square

$$\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ f\big\downarrow & & \big\downarrow f \\ B & \longrightarrow & T^{-1}B \end{array} ,$$

that the sequence

$$0 \longrightarrow A \longrightarrow S^{-1}A\oplus B \longrightarrow T^{-1}B \longrightarrow 0$$

is exact. We shall discuss the L-theory of cartesian squares in §16 below).

<u>Proposition 13.22</u> A cartesian morphism of rings with involution and multiplicative subsets

$$f : (A,S) \longrightarrow (B,T)$$

induces isomorphisms of L-groups

$$\left\{\begin{array}{l} f : L^n(A,S,\varepsilon) \longrightarrow L^n(B,T,\varepsilon) \\ f : L_n(A,S,\varepsilon) \longrightarrow L_n(B,T,\varepsilon) \end{array}\right. \quad (n\in\mathbb{Z}) \ .$$

<u>Proof</u>: Define an <u>n-dimensional (A,S)-module chain complex</u> to be a chain complex

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots$$

of h.d. 1 S-torsion A-modules.

<u>Lemma</u> There is a natural identification of chain homotopy classes

(S-acyclic (n+1)-dimensional A-module chain complexes)

= (n-dimensional (A,S)-module chain complexes) $(n\geqslant 0)$.

<u>Proof</u>: Given an n-dimensional (A,S)-module chain complex C write a f.g. projective A-module resolution of C_r $(0\leqslant r\leqslant n)$ as

$$0 \longrightarrow P_r \xrightarrow{f} Q_r \longrightarrow C_r \longrightarrow 0 \ ,$$

and resolve $d\in \mathrm{Hom}_A(C_r,C_{r-1})$ $(1\leqslant r\leqslant n)$ by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_r & \xrightarrow{f} & Q_r & \longrightarrow & C_r & \longrightarrow & 0 \\ & & g\big\downarrow & & \big\downarrow h & & \big\downarrow d & & \\ 0 & \longrightarrow & P_{r-1} & \xrightarrow{f} & Q_{r-1} & \longrightarrow & C_{r-1} & \longrightarrow & 0 \end{array} .$$

As $d^2 = 0$ there exist chain homotopies $k\in \mathrm{Hom}_A(Q_r,P_{r-2})$ $(2\leqslant r\leqslant n)$ such that

$$g^2 = kf\in \mathrm{Hom}_A(P_r,P_{r-2}) \ , \quad h^2 = fk\in \mathrm{Hom}_A(Q_r,Q_{r-2}) \ .$$

Call a collection such as (P,Q,f,g,h,k) a <u>resolution</u> of C. A resolution of C determines an S-acyclic (n+1)-dimensional A-module chain complex D, with

$$d_D = \begin{pmatrix} g & (-)^r k \\ f & (-)^r h \end{pmatrix} : D_r = P_{r-1}\oplus Q_r \longrightarrow D_{r-1} = P_{r-2}\oplus Q_{r-1} \quad (1\leqslant r\leqslant n+2) \ ,$$

such that $H_*(D) = H_*(C)$. A chain equivalence of (A,S)-module chain complexes $C \longrightarrow C'$ determines a chain equivalence of S-acyclic A-module chain complexes $D \longrightarrow D'$.

Conversely, given an S-acyclic (n+1)-dimensional A-module chain complex D it is possible to define an n-dimensional (A,S)-module chain complex C with a resolution which determines D (up to chain equivalence), as follows. As $S^{-1}D$ is a chain contractible $S^{-1}A$-module chain complex there exist $s\in S$, $e\in \mathrm{Hom}_A(D_r,D_{r+1})$ $(0\leqslant r\leqslant n)$ such that

$$de + ed = s : D_r \longrightarrow D_r \quad (0\leqslant r\leqslant n+1) .$$

Define a resolution (P,Q,f,g,h,k) by

$$f = \left\{\begin{array}{l} \begin{pmatrix} d & 0 & 0 & . \\ e & d & 0 & . \\ 0 & e & d & . \\ . & . & . & . \end{pmatrix} : P_0 = D_1\oplus D_3\oplus D_5\oplus\ldots \longrightarrow Q_0 = D_0\oplus D_2\oplus D_4\oplus\ldots \\ \begin{pmatrix} s & 0 & 0 & . \\ e^2 & s & 0 & . \\ 0 & e^2 & s & . \\ . & . & . & . \end{pmatrix} : P_r = D_{r+1}\oplus D_{r+3}\oplus D_{r+5}\oplus\ldots \longrightarrow Q_r = D_{r+1}\oplus D_{r+3}\oplus D_{r+5}\oplus\ldots \quad (r\geqslant 1) \end{array}\right.$$

$$g = \begin{pmatrix} d & 0 & 0 & . \\ e & d & 0 & . \\ 0 & e & d & . \\ . & . & . & . \end{pmatrix} : P_r = D_{r+1}\oplus D_{r+3}\oplus D_{r+5}\oplus\ldots \longrightarrow P_{r-1} = D_r\oplus D_{r+2}\oplus D_{r+4}\oplus\ldots \quad (r\geqslant 0)$$

$$h = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} : Q_1 = D_2 \oplus D_4 \oplus D_6 \oplus \dots \longrightarrow Q_0 = D_0 \oplus D_2 \oplus D_4 \oplus \dots \\ \begin{pmatrix} d & 0 & 0 & \cdot \\ e & d & 0 & \cdot \\ 0 & e & d & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} : Q_r = D_{r+1} \oplus D_{r+3} \oplus D_{r+5} \oplus \dots \longrightarrow Q_{r-1} = D_r \oplus D_{r+2} \oplus D_{r+4} \oplus \dots \quad (r \geq 1) \end{cases}$$

$$k = \begin{pmatrix} 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} : Q_r = D_{r+1} \oplus D_{r+3} \oplus D_{r+5} \oplus \dots \longrightarrow P_{r-2} = D_{r-1} \oplus D_{r+1} \oplus D_{r+3} \oplus \dots \quad (r \geq 2) .$$

This is a resolution of an n-dimensional (A,S)-module chain complex C such that $H_*(C) = H_*(D)$.

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Applying the Lemma twice we obtain the following identifications of chain homotopy classes:

(S-acyclic (n+1)-dimensional A-module chain complexes)

= (n-dimensional (A,S)-module chain complexes)

= (n-dimensional (B,T)-module chain complexes)

= (T-acyclic (n+1)-dimensional B-module chain complexes) .

Moreover, if D is an S-acyclic (n+1)-dimensional A-module chain complex then $B \otimes_A D$ is a T-acyclic (n+1)-dimensional B-module chain complex such that

$$\begin{cases} Q^{n+1}(D,\varepsilon) = Q^{n+1}(B \otimes_A D,\varepsilon) \\ Q_{n+1}(D,\varepsilon) = Q_{n+1}(B \otimes_A D,\varepsilon) \end{cases} .$$

It is now immediate that we can identify

$$\begin{cases} L^n(A,S,\varepsilon) = L^n(B,T,\varepsilon) \\ L_n(A,S,\varepsilon) = L_n(B,T,\varepsilon) \end{cases} \quad (n \in \mathbb{Z}) .$$

[]

Given a ring with involution A and a multiplicative subset $S \subset A$ define the <u>S-adic completion of A</u> to be the inverse limit

$$\hat{A} = \varprojlim_{s \in S} A/sA$$

of the inverse system of rings $\{A/sA \mid s \in S\}$ with structure maps the projections

$$A/stA \longrightarrow A/sA \quad (s,t \in S) .$$

The completion $\hat{A}$ is a ring with involution

$$\bar{\ } : \hat{A} \longrightarrow \hat{A} \ ; \ \{a_s \in A/sA \mid s \in S\} \longmapsto \{\bar{a}_{\bar{s}} \in A/sA \mid s \in S\} .$$

The inclusion

$$f : A \longrightarrow \hat{A} \ ; \ a \longmapsto \{a \in A/sA \mid s \in S\}$$

is a morphism of rings with involution, such that $\hat{S} = f(S) \subset \hat{A}$ is a multiplicative subset.

<u>Proposition 13.23</u> The inclusion $f:(A,S) \longrightarrow (\hat{A},\hat{S})$ is a cartesian morphism of rings with involution, and so induces isomorphisms

$$\begin{cases} f : L^n(A,S,\varepsilon) \longrightarrow L^n(\hat{A},\hat{S},\varepsilon) \\ f : L_n(A,S,\varepsilon) \longrightarrow L_n(\hat{A},\hat{S},\varepsilon) \end{cases} \quad (n \in \mathbb{Z}) .$$

<u>Proof</u>: Immediate from Proposition 13.22.

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The profinite completion $\hat{\mathbb{Z}} = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$ of $\mathbb{Z}$ is the infinite product

$$\hat{\mathbb{Z}} = \prod_p \hat{\mathbb{Z}}_p$$

over all the primes p of the p-adic completions $\hat{\mathbb{Z}}_p = \varprojlim_k \mathbb{Z}/p^k\mathbb{Z}$, with symmetric Witt group

$$L^0(\hat{\mathbb{Z}}) = \prod_p L^0(\hat{\mathbb{Z}}_p) ,$$

where

$$L^0(\hat{\mathbb{Z}}_p) = \begin{cases} \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 1 \pmod 4 \\ \mathbb{Z}_4 & \text{if } p \equiv 3 \pmod 4 \end{cases} .$$

A ring with involution A is <u>torsion-free</u> if the additive group of A is torsion-free, in which case $S = \mathbb{Z}-\{0\} \subset A$ is a multiplicative subset, such that $S^{-1}A = \mathbb{Q} \otimes_{\mathbb{Z}} A$, $\hat{A} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A$.

A ring with involution A is <u>p^∞-torsion-free</u> if $p1 \in A$ is a non-zero-divisor, in which case $S = \{p^k | k \geqslant 0\} \subset A$ is a multiplicative subset such that
$S^{-1}A = A[\frac{1}{p}] = \mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} A$, $\hat{A} = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} A$.

<u>Proposition 13.24</u> i) If A is a torsion-free ring with involution then the natural maps

$$\begin{cases} L^n(A,\varepsilon) \longrightarrow L^n_S(S^{-1}A,\varepsilon) \\ L_n(A,\varepsilon) \longrightarrow L^S_n(S^{-1}A,\varepsilon) \end{cases} \quad (n \in \mathbb{Z},\ S = \mathbb{Z}-\{0\} \subset A)$$

are isomorphisms modulo 8-torsion.

ii) If A is a p^∞-torsion-free ring with involution then the natural maps

$$\begin{cases} L^n(A,\varepsilon) \longrightarrow L^n_S(S^{-1}A,\varepsilon) \\ L_n(A,\varepsilon) \longrightarrow L^S_n(S^{-1}A,\varepsilon) \end{cases} \quad (n \in \mathbb{Z},\ S = \{p^k | k \geqslant 0\} \subset A)$$

are isomorphisms modulo 8- (resp. 2-, 4-) torsion, according as $p = 2$ (resp. $\equiv 1 \pmod 4$, $\equiv 3 \pmod 4$).

<u>Proof</u>: i) By Proposition 13.23 $L^n(A,S,\varepsilon) = L^n(\hat{A},\hat{S},\varepsilon)$, and by Proposition 13.21 $L^n(\hat{A},\hat{S},\varepsilon)$ is an $L^0(\hat{\mathbb{Z}})$-module, since $\hat{A}$ is a $\hat{\mathbb{Z}}$-module.

ii) As for i), but using the $\hat{\mathbb{Z}}_p$-module structure of $\hat{A}$.

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In particular, the result of Proposition 13.24 i) applies to the integral group ring $A = \mathbb{Z}[\pi]$ of any group π, with $S^{-1}A = \mathbb{Q}[\pi]$ the rational group ring.

Results similar to those of Proposition 13.24 were first obtained by Karoubi [2].

We shall now investigate the general properties of the L-groups $\begin{cases} L^n(A,S,\varepsilon) \\ L_n(A,S,\varepsilon) \end{cases}$ $(n \in \mathbb{Z})$ in the case when the ring with involution A is an algebra over a Dedekind ring R and $S = R-\{0\}$. An S-torsion A-module has a canonical decomposition as a direct sum of $\mathcal{P}$-primary S-torsion A-modules, with $\mathcal{P}$ ranging over all the non-zero prime ideals of R, and there is a corresponding direct sum decomposition for the L-groups.

Given a multiplicative subset $S \subset A$ in a ring with involution A we shall say that the pair (A,S) is a <u>Dedekind algebra</u> if $R = S \cup \{0\}$ is a Dedekind ring with respect to the ring operations inherited from A. The localization $S^{-1}A = F \otimes_R A$ is the induced algebra over the quotient field $F = S^{-1}R$. For example, a torsion-free ring with involution A is the same as a Dedekind algebra $(A,\mathbb{Z}-\{0\})$. A Dedekind ring with involution R is the same as a Dedekind algebra $(R,R-\{0\})$. In dealing with Dedekind algebras (A,S) and the prime ideals $\mathcal{P}$ of R we shall always exclude the case $\mathcal{P} = 0$.

Let (A,S) be a Dedekind algebra.

The <u>annihilator</u> of an S-torsion A-module M is the ideal of R defined by

$$\mathrm{ann}(M) = \{s \in R | sM = 0\} \triangleleft R \ .$$

Like all ideals of R this has a unique expression as a product of powers of distinct prime ideals $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$

$$\mathrm{ann}(M) = \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_r^{k_r} \quad (k_i \geqslant 1) \ .$$

If M is such that the natural map $M \longrightarrow M^{\wedge\wedge}$ is an isomorphism (e.g. if M is h.d.1) then

$$\mathrm{ann}(M^\wedge) = \overline{\mathrm{ann}(M)} \triangleleft R \ .$$

An S-torsion A-module M is <u>$\mathcal{P}$-primary</u> for some prime ideal $\mathcal{P}$ of R if

$$\mathrm{ann}(M) = \mathcal{P}^k$$

for some $k \geqslant 1$. An S-acyclic A-module chain complex C (resp. an S-acyclic algebraic Poincaré complex over A (C,φ)) is <u>$\mathcal{P}$-primary</u> if each of the homology S-torsion A-modules $H_r(C)$ is $\mathcal{P}$-primary.

Define the localization of A at $\mathcal{P}$ for some prime ideal $\mathcal{P}$ of R to be the ring obtained by inverting R-$\mathcal{P}$

$$A_{\mathcal{P}} = (R-\mathcal{P})^{-1}A \ .$$

If $\overline{\mathcal{P}} = \mathcal{P}$ there is defined an involution

$$\overline{} : A_{\mathcal{P}} \longrightarrow A_{\mathcal{P}} \ ; \ \frac{a}{r} \longmapsto \frac{\bar{a}}{\bar{r}} \quad (a\in A,\ r\in R-\mathcal{P}) \ .$$

An h.d. 1 S-torsion A-module M induces an h.d.1 $\mathcal{P}$-primary S-torsion A-module

$$M_{\mathcal{P}} = A_{\mathcal{P}}\otimes_A M \ .$$

If $\mathrm{ann}(M) = \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_r^{k_r}$ it is possible to identify

$$M_{\mathcal{P}} = \begin{cases} \mathcal{P}_1^{k_1}\mathcal{P}_2^{k_2}\ldots\mathcal{P}_{i-1}^{k_{i-1}}\mathcal{P}_{i+1}^{k_{i+1}}\ldots\mathcal{P}_r^{k_r}M & \text{if } \mathcal{P}=\mathcal{P}_i \text{ for some } i,\ 1\leqslant i\leqslant r \\ 0 & \text{if } \mathcal{P}\notin\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_r\} \end{cases}$$

so that

$$M = \bigoplus_{i=1}^{r} M_{\mathcal{P}_i} \ , \quad (M^\wedge)_{\mathcal{P}} = (M_{\overline{\mathcal{P}}})^\wedge \ , \quad \mathrm{Hom}_A(M,M') = \bigoplus_{\mathcal{P}}\mathrm{Hom}_A(M_{\mathcal{P}},M'_{\mathcal{P}}) \ .$$

We thus have a canonical identification of exact categories

(h.d. 1 S-torsion A-modules) = $\bigoplus_{\mathcal{P}}$(h.d. 1 $\mathcal{P}$-primary S-torsion A-modules) ,

with $\mathcal{P}$ ranging over the (non-zero) prime ideals of R. The S-duality functor $M\longmapsto M^\wedge$ sends the $\mathcal{P}$-primary component to the $\overline{\mathcal{P}}$-primary component. Applying the Lemma appearing in the proof of Proposition 13.22 we have also the following identifications of chain homotopy classes

(S-acyclic (n+1)-dimensional A-module chain complexes)

= (n-dimensional (A,S)-module chain complexes)

= $\bigoplus_{\mathcal{P}}$(n-dimensional $\mathcal{P}$-primary (A,S)-module chain complexes)

= $\bigoplus_{\mathcal{P}}$($\mathcal{P}$-primary S-acyclic (n+1)-dimensional A-module chain complexes).

For each prime ideal $\mathcal{P}$ of R such that $\overline{\mathcal{P}}=\mathcal{P}$ define the L-groups $\begin{cases} L^n(A,\mathcal{P}^\infty,\varepsilon) \\ L_n(A,\mathcal{P}^\infty,\varepsilon)\end{cases}$ $(n\in\mathbb{Z})$ in the same way as $\begin{cases} L^n(A,S,\varepsilon) \\ L_n(A,S,\varepsilon)\end{cases}$ but using only $\mathcal{P}$-primary S-acyclic A-module chain complexes. Define a multiplicative subset

$$S_{\mathcal{P}} = \{\tfrac{s}{1}\in A \mid s\in S\}\subset A_{\mathcal{P}} \ .$$

There is a natural identification

(h.d. 1 $\mathcal{P}$-primary S-torsion A-modules)

= (h.d. 1 $S_{\mathcal{P}}$-torsion $A_{\mathcal{P}}$-modules) ,

allowing the identifications

$$\begin{cases} L^n(A,\mathcal{P}^\infty,\varepsilon) = L^n(A_{\mathcal{P}},S_{\mathcal{P}},\varepsilon) \\ L_n(A,\mathcal{P}^\infty,\varepsilon) = L_n(A_{\mathcal{P}},S_{\mathcal{P}},\varepsilon) \end{cases} \quad (n\in\mathbb{Z}) \ .$$

If $\mathcal{P} = \pi R$ is a principal prime ideal of R, with generator $\pi\in\mathcal{P}$, then $\bar{\pi} = \pi u\in\mathcal{P}$ for some unit $u\in R$ such that $u\bar{u} = 1\in R$ and there is defined a multiplicative subset

$$S_\pi = \{\pi^j u^k \mid j\geqslant 0, k\in\mathbb{Z}\}\subset A$$

such that there are identifications

(h.d. 1 $\mathcal{P}$-primary S-torsion A-modules)

= (h.d. 1 S_π-torsion A-modules)

$$\begin{cases} L^n(A,\mathcal{P}^\infty,\varepsilon) = L^n(A,S_\pi,\varepsilon) \\ L_n(A,\mathcal{P}^\infty,\varepsilon) = L_n(A,S_\pi,\varepsilon) \end{cases} \quad (n\in\mathbb{Z}) \ .$$

Proposition 13.25 The L-groups of a Dedekind algebra (A,S) have a canonical direct sum decomposition

$$\begin{cases} L^n(A,S,\varepsilon) = \bigoplus_{\mathcal{P}} L^n(A,\mathcal{P}^\infty,\varepsilon) \\ L_n(A,S,\varepsilon) = \bigoplus_{\mathcal{P}} L_n(A,\mathcal{P}^\infty,\varepsilon) \end{cases} \quad (n\in\mathbb{Z})$$

with $\mathcal{P}$ ranging over all the prime ideals of R such that $\overline{\mathcal{P}}=\mathcal{P}$.

The localization exact sequence of (A,S) can thus be expressed as

$$\begin{cases} \ldots\longrightarrow L^n(A,\varepsilon)\longrightarrow L^n_S(S^{-1}A,\varepsilon)\longrightarrow\bigoplus_{\mathcal{P}}L^n(A,\mathcal{P}^\infty,\varepsilon)\longrightarrow L^{n-1}(A,\varepsilon)\longrightarrow\ldots \\ \ldots\longrightarrow L_n(A,\varepsilon)\longrightarrow L^S_n(S^{-1}A,\varepsilon)\longrightarrow\bigoplus_{\mathcal{P}}L_n(A,\mathcal{P}^\infty,\varepsilon)\longrightarrow L_{n-1}(A,\varepsilon)\longrightarrow\ldots \end{cases} \quad (n\in\mathbb{Z}) \ .$$

Proof: An S-acyclic finite-dimensional A-module chain complex C is chain equivalent to a direct sum $\bigoplus_{\mathcal{P}} C_{\mathcal{P}}$ of $\mathcal{P}$-primary S-acyclic finite-dimensional A-module chain complexes $C_{\mathcal{P}}$. Expressing the spectrum of prime ideals of R as a disjoint union

$$\mathrm{spec}(R) = \{\mathcal{P}\} \cup \{Q\} \cup \{\overline{Q}\} \quad (\overline{\mathcal{P}} = \mathcal{P})$$

we have

$$Q^n(C,\varepsilon) = \bigoplus_{\mathcal{P}} Q^n(C_{\mathcal{P}},\varepsilon) \oplus \bigoplus_{Q} Q^n(C_Q \oplus C_{\overline{Q}},\varepsilon) \quad , \quad Q^n(C_Q,\varepsilon) = 0 \ .$$

Thus an S-acyclic ε-symmetric Poincaré complex over A (C,φ) is homotopy equivalent to the direct sum $\bigoplus_{\mathcal{P}}(C_{\mathcal{P}},\varphi_{\mathcal{P}}) \oplus \bigoplus_{Q}(C_Q \oplus C_{\overline{Q}},\varphi_{Q\oplus\overline{Q}})$ with each $(C_{\mathcal{P}},\varphi_{\mathcal{P}})$ a $\mathcal{P}$-primary S-acyclic ε-symmetric Poincaré complex over A and each $(C_Q \oplus C_{\overline{Q}},\varphi_{Q\oplus\overline{Q}})$ a null-cobordant complex (- with a canonical null-cobordism, as defined by $((1\ 0):C_Q \oplus C_{\overline{Q}} \longrightarrow C_Q,(0,\varphi_{Q\oplus\overline{Q}})))$. It is immediate that

$$L^n(A,S,\varepsilon) = \bigoplus_{\mathcal{P}} L^n(A,\mathcal{P}^\infty,\varepsilon) \ .$$

Similarly for the ε-quadratic case.

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We now specialize to the case of a Dedekind algebra (A,S) such that $A = R = S \cup 0$ is a Dedekind ring, with $S^{-1}A = S^{-1}R = F$ the quotient field. The symmetric Witt groups of Dedekind rings R have been studied by Milnor and Husemoller [1], Durfee [1] and Barge, Lannes, Latour and Vogel [1]. (There is an extensive literature in the case when R is the ring of integers in an algebraic number field F, e.g. Landherr [1], Fröhlich [1], Wall [7], Lannes [1]). In particular, there is the original exact sequence of Milnor

$$0 \longrightarrow L^0(R) \longrightarrow L^0(F) \longrightarrow \bigoplus_{\mathcal{P}} L^0(R/\mathcal{P})$$

with the identity involution on R, $\mathcal{P}$ running over all the maximal ideals of R. This can be deduced from Proposition 13.25 by means of an L-theoretic analogue of the devissage argument of algebraic K-theory (Bass [1] for n = 1, Quillen [1] for $n \geqslant 2$) used to prove

$$K_n(\text{h.d.1 S-torsion R-modules}) = \bigoplus_{\mathcal{P}} K_n(\text{h.d.1 }\mathcal{P}\text{-primary S-torsion R-modules})$$

and hence to establish the localization exact sequence

$$\ldots \longrightarrow K_n(R) \longrightarrow K_n(F) \longrightarrow \bigoplus_{\mathcal{P}} K_{n-1}(R/\mathcal{P}) \longrightarrow K_{n-1}(R) \longrightarrow \ \ldots \ .$$

Given a finite-dimensional R-module chain complex C write

$\begin{cases} T_r(C) \\ F_r(C) = H_r(C)/T_r(C) \end{cases}$ (resp. $\begin{cases} T^r(C) \\ F^r(C) = H^r(C)/T^r(C) \end{cases}$) for the $\begin{cases} \text{S-torsion submodule} \\ \text{S-torsion-free quotient} \end{cases}$ of $H_r(C)$ (resp. $H^r(C)$) $(r \in \mathbb{Z})$, which is $\begin{cases} \text{an h.d.1 S-torsion} \\ \text{a f.g. projective} \end{cases}$ R-module. The universal coefficient theorem gives natural R-module isomorphisms

$$T_r(C) \longrightarrow T^{r+1}(C)\hat{} = \mathrm{Hom}_R(T^{r+1}(C),F/R) \ ; \ x \longmapsto (f \longmapsto \overline{\tfrac{1}{s}f(y)})$$
$$(x \in C_r, y \in C_{r+1}, s \in S, sx = dy, f \in C^{r+1})$$

$$F_r(C) \longrightarrow F^r(C)^* = \mathrm{Hom}_R(F^r(C),R) \ ; \ x \longmapsto (f \longmapsto \overline{f(x)}) \quad (x \in C_r, f \in C^r) \ .$$

<u>Proposition 13.26</u> The L-groups of a Dedekind ring R and the quotient field $F = S^{-1}R$ $(S = R-\{0\})$ are such that

i) The skew-suspension maps

$$\overline{S} : L^n(R,\varepsilon) \longrightarrow L^{n+2}(R,-\varepsilon) \quad (n \geqslant 0)$$
$$\overline{S} : L^n(R,S,\varepsilon) \longrightarrow L^{n+2}(R,S,-\varepsilon) \quad (n \geqslant 1)$$

are isomorphisms.

ii) $M^\varepsilon(F) = 0$, $M\langle v_0\rangle^\varepsilon(F) = 0$, $M_\varepsilon(F) = 0$

$M^\varepsilon(R,S) = 0$, $M\langle v_0\rangle^\varepsilon(R,S) = 0$, $M_\varepsilon(R,S) = 0$.

iii) There are defined exact sequences

$$0 \longrightarrow L^\varepsilon(R) \longrightarrow L^\varepsilon(F) \xrightarrow{\partial} L^\varepsilon(R,S) \longrightarrow M^{-\varepsilon}(R) \longrightarrow 0$$
$$0 \longrightarrow L^\varepsilon(R) \longrightarrow L^\varepsilon(F) \xrightarrow{\partial} L\langle v_0\rangle^\varepsilon(R,S) \longrightarrow M\langle v_0\rangle^{-\varepsilon}(R) \longrightarrow 0$$
$$0 \longrightarrow L\langle v_0\rangle^\varepsilon(R) \longrightarrow L\langle v_0\rangle^\varepsilon(F) \xrightarrow{\partial} L_\varepsilon(R,S) \longrightarrow M_{-\varepsilon}(R) \longrightarrow 0$$
$$0 \longrightarrow \widetilde{M}_\varepsilon(R,S) \longrightarrow L_\varepsilon(R) \longrightarrow L_\varepsilon(F) \xrightarrow{\partial} \widetilde{L}_\varepsilon(R,S) \longrightarrow M_{-\varepsilon}(R) \longrightarrow 0 \quad .$$

In particular, there are natural identifications

$$M^{\varepsilon}(R) = \frac{(\text{non-singular } (-\varepsilon)\text{-symmetric linking forms over } (R,S))}{(\text{boundaries of non-degenerate } (-\varepsilon)\text{-symmetric forms over } R)+(\text{metabolics})}$$

$$M\langle v_0\rangle^{\varepsilon}(R) = \frac{(\text{non-singular even } (-\varepsilon)\text{-symmetric linking forms over } (R,S))}{(\text{boundaries of non-degenerate } (-\varepsilon)\text{-symmetric forms over } R)}$$

$$M_{\varepsilon}(R) = \frac{(\text{non-singular } (-\varepsilon)\text{-quadratic linking forms over } (R,S))}{(\text{boundaries of non-degenerate even } (-\varepsilon)\text{-symmetric forms over } R)}$$

$$= \frac{(\text{non-singular split } (-\varepsilon)\text{-quadratic linking forms over } (R,S))}{(\text{boundaries of non-degenerate } (-\varepsilon)\text{-quadratic forms over } R)} \ .$$

iv) For $n = 2i$ (resp. $n = 2i+1$) the isomorphism

$$\begin{cases} \bar{S}^{-i} : L^{n}(R,\varepsilon) \longrightarrow L^{n-2i}(R,(-)^{i}\varepsilon) \\ \bar{S}^{-i} : L_{n}(R,\varepsilon) \longrightarrow L_{n-2i}(R,(-)^{i}\varepsilon) \end{cases} \quad (i \geqslant 0)$$

sends the cobordism class of an n-dimensional $\begin{cases}\varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over R

$\begin{cases}(C,\varphi\in Q^{n}(C,\varepsilon)) \\ (C,\psi\in Q_{n}(C,\varepsilon))\end{cases}$ to the class in $\begin{cases}L^{0}(R,(-)^{i}\varepsilon) = L^{(-)^{i}\varepsilon}(R) \\ L_{0}(R,(-)^{i}\varepsilon) = L_{(-)^{i}\varepsilon}(R)\end{cases}$ (resp.

$\begin{cases}L^{1}(R,(-)^{i}\varepsilon) = M^{(-)^{i}\varepsilon}(R) \\ L_{1}(R,(-)^{i}\varepsilon) = M_{(-)^{i}\varepsilon}(R)\end{cases}$) of the non-singular $\begin{cases}(-)^{i}\varepsilon\text{-symmetric} \\ (-)^{i}\varepsilon\text{-quadratic}\end{cases}$ form over R

$\begin{cases}(F^{i}(C),\varphi_0\in \mathrm{Hom}_R(F^{i}(C),F^{i}(C)^{*})) \\ (F^{i}(C),(1+T_{\varepsilon})\psi_0,v^{i}(\psi):F^{i}(C)\longrightarrow Q_{(-)^{i}\varepsilon}(R))\end{cases}$ (resp. of the non-singular

$\begin{cases}(-)^{i+1}\varepsilon\text{-symmetric} \\ (-)^{i+1}\varepsilon\text{-quadratic}\end{cases}$ linking form over (R,S)

$\begin{cases}(T^{i+1}(C),\varphi_0\in \mathrm{Hom}_R(T^{i+1}(C),T^{i+1}(C)^{\wedge})) \\ (T^{i+1}(C),(1+T_{\varepsilon})\psi_0,pv_S^{i}(\psi):T^{i+1}(C)\longrightarrow Q_{(-)^{i+1}\varepsilon}(R,S))\end{cases}$).

v) There are natural direct sum decompositions

$$L^{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} L^{\varepsilon}(R,\mathcal{P}^{\infty}) = \bigoplus_{\mathcal{P}} L^{\varepsilon}(R/\mathcal{P})$$

$$L\langle v_0\rangle^{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} L\langle v_0\rangle^{\varepsilon}(R,\mathcal{P}^{\infty})$$

$$L_{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} L_{\varepsilon}(R,\mathcal{P}^{\infty}) \ , \quad \widetilde{L}_{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} \widetilde{L}_{\varepsilon}(R,\mathcal{P}^{\infty}) \ , \quad \widetilde{M}_{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} \widetilde{M}_{\varepsilon}(R,\mathcal{P}^{\infty})$$

with $\mathcal{P}$ running over all the non-zero prime ideals of R such that $\bar{\mathcal{P}} = \mathcal{P}$.

<u>Proof</u>: i) - iv) Immediate from Propositions 7.4, 12.4, 13.10, 13.15 since a Dedekind ring R is 1-dimensional (= noetherian of global dimension 1) and the quotient field F is 0-dimensional (= semi-simple).

v) The direct sum decompositions of the type $L^{\varepsilon}(R,S) = \bigoplus_{\mathcal{P}} L^{\varepsilon}(R,\mathcal{P}^{\infty})$ are immediate from Proposition 13.25. It remains to identify $L^{\varepsilon}(R,\mathcal{P}^{\infty}) = L^{\varepsilon}(R/\mathcal{P})$.

Let $L^{\varepsilon}(R,\mathcal{P}^{k})$ $(k \geqslant 1)$ be the Witt group of non-singular ε-symmetric linking forms over (R,S) (M,λ) such that M is $\mathcal{P}$-primary with annihilator

$$\mathrm{ann}(M) = \mathcal{P}^{j} \ (j \leqslant k).$$

The natural maps

$$L^{\varepsilon}(R,\mathcal{P}^{k}) \longrightarrow L^{\varepsilon}(R,\mathcal{P}^{k+1}) \ ; \ (M,\lambda) \longmapsto (M,\lambda) \qquad (k \geqslant 1)$$

are isomorphisms, with inverses

$$L^{\varepsilon}(R,\mathcal{P}^{k+1}) \longrightarrow L^{\varepsilon}(R,\mathcal{P}^{k}) \ ; \ (M,\lambda) \longmapsto (L^{\perp}/L,\lambda^{\perp}/\lambda) \quad (L = \mathcal{P}^{k}M \subset M) \ .$$

We can thus identify

$$L^{\varepsilon}(R/\mathcal{P}) \equiv L^{\varepsilon}(R,\mathcal{P}) = \varinjlim_{k} L^{\varepsilon}(R,\mathcal{P}^{k}) \equiv L^{\varepsilon}(R,\mathcal{P}^{\infty}) \ .$$

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(The argument used to identify $L^{\varepsilon}(R/\mathcal{P}) = L^{\varepsilon}(R,\mathcal{P}^{\infty})$ breaks down in the ε-quadratic case. Given a non-singular ε-quadratic linking form over (R,S) (M,λ,μ) such that $\mathrm{ann}(M) = \mathcal{P}^{k+1}$ $(k \geqslant 1)$ it need not be the case that $L = \mathcal{P}^{k}M \subset M$ is a sublagrangian. For example, consider the non-singular quadratic linking form over $(\mathbb{Z},\mathbb{Z}-\{0\})$ defined by

$$M = \mathbb{Z}/4\mathbb{Z} \ , \ \lambda : M \times M \longrightarrow \mathbb{Q}/\mathbb{Z} \ ; \ (m,n) \longmapsto \tfrac{1}{4}mn$$

$$\mu : M \longrightarrow Q_{+1}(\mathbb{Z},\mathbb{Z}-\{0\}) = \mathbb{Q}/2\mathbb{Z} \ ; \ m \longmapsto \tfrac{1}{4}m^{2} \ .$$

Then $L = 2\mathbb{Z}/4\mathbb{Z}$ is a sublagrangian of the symmetrization (M,λ) but not of (M,λ,μ), since $\mu(2) = 1 \neq 0 \in \mathbb{Q}/2\mathbb{Z}$. In fact, (M,λ,μ) is a generator of $\ker((1+T):L_0(\mathbb{Z},\mathbb{Z}-\{0\}) \longrightarrow L^{0}(\mathbb{Z},\mathbb{Z}-\{0\})) = \mathbb{Z}/8\mathbb{Z})$.

In particular, we have that $\mathbb{Z}$ is a Dedekind ring, with quotient field $\mathbb{Q}$.

<u>Proposition 13.27</u> The symmetric and quadratic L-groups of $\mathbb{Z}$ are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ 0 \\ 0 \end{cases}, \quad L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}_2 \\ 0 \end{cases} \text{ if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 \qquad (n \geqslant 0) .$$

The invariants are given by

$$L^{4k}(\mathbb{Z}) \longrightarrow \mathbb{Z} \ ;$$
$$(C, \varphi \in Q^{4k}(C)) \longmapsto (\text{signature of } (F^{2k}(C), \varphi_0))$$
$$L^{4k+1}(\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \ ;$$
$$(C, \varphi \in Q^{4k+1}(C)) \longmapsto (\text{deRham invariant of } (T^{2k+1}(C), \varphi_0))$$
$$L_{4k}(\mathbb{Z}) \longrightarrow \mathbb{Z} \ ;$$
$$(C, \psi \in Q_{4k}(C)) \longmapsto \tfrac{1}{8}(\text{signature of } (F^{2k}(C), (1+T)\psi_0))$$
$$L_{4k+2}(\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \ ;$$
$$(C, \psi \in Q_{4k+1}(C)) \longmapsto (\text{Arf invariant of } (F^{2k+1}(C), \psi_0)) .$$

The hyperquadratic L-groups of $\mathbb{Z}$ are given by

$$\hat{L}^n(\mathbb{Z}) = \begin{cases} \mathbb{Z}_8 \\ \mathbb{Z}_2 \\ 0 \\ \mathbb{Z}_2 \end{cases} \text{ if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 \qquad (n \geqslant 0) .$$

<u>Proof</u>: Proposition 13.26 reduces the computation in the $\begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ dimensional case to the stable classification of non-singular $\begin{Bmatrix}\mathbb{Z}\text{-}\\\mathbb{Q}/\mathbb{Z}\text{-}\end{Bmatrix}$ valued forms on $\begin{Bmatrix}\text{f.g. free}\\\text{finite}\end{Bmatrix}$ abelian groups, for which we refer to $\begin{Bmatrix}\text{Milnor and Husemoller [1], Arf [1]}\\\text{deRham [1], Wall [1]}\end{Bmatrix}$.

The generator of $\begin{Bmatrix}L^0(\mathbb{Z})\\L_0(\mathbb{Z})\end{Bmatrix}$ is the non-singular $\begin{Bmatrix}\text{symmetric}\\\text{quadratic}\end{Bmatrix}$ form over $\mathbb{Z}$ $\begin{Bmatrix}(\mathbb{Z}, 1 \in Q^{+1}(\mathbb{Z}))\\(\mathbb{Z}^8, E_8 \in Q_{+1}(\mathbb{Z}))\end{Bmatrix}$ of signature $\begin{Bmatrix}1 \in \mathbb{Z}\\8 \in \mathbb{Z}\end{Bmatrix}$. The generator of $\begin{Bmatrix}L^1(\mathbb{Z}) = L^2(\mathbb{Z}, \mathbb{Z}-\{0\})\\L_2(\mathbb{Z}) = L_3(\mathbb{Z}, \mathbb{Z}-\{0\})\end{Bmatrix}$ is the non-singular $\begin{Bmatrix}\text{symmetric formation}\\\text{skew-quadratic form}\end{Bmatrix}$ over $\mathbb{Z}$

$$\begin{Bmatrix} d = \left(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix} \in Q^{+1}(\mathbb{Z}\oplus\mathbb{Z}); \begin{pmatrix}1\\0\end{pmatrix}\mathbb{Z}, \begin{pmatrix}1\\-2\end{pmatrix}\mathbb{Z}\right) \\ c = \left(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \in Q_{+1}(\mathbb{Z}\oplus\mathbb{Z})\right) \end{Bmatrix} \text{ of } \begin{Bmatrix}\text{deRham}\\\text{Arf}\end{Bmatrix} \text{ invariant } 1 \in \mathbb{Z}_2,$$

corresponding to the non-singular $\begin{Bmatrix}\text{skew-symmetric linking form over } (\mathbb{Z}, \mathbb{Z}-\{0\})\\\text{split skew-quadratic linking formation}\end{Bmatrix}$

$$\begin{Bmatrix} d = (\mathbb{Z}/2\mathbb{Z}, \lambda : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}; (m,n) \longmapsto \tfrac{1}{2}mn) \\ c = \left(\mathbb{Z}/4\mathbb{Z}, \left(\begin{pmatrix}(2\ 0)\\(0\ 2)\end{pmatrix}, \theta\right)\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\right) \ , \ \theta : \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \longrightarrow Q_{+1}(\mathbb{Z}, \mathbb{Z}-\{0\}) = \mathbb{Q}/2\mathbb{Z} ; \ (m,n) \longmapsto m^2 + n^2 \end{Bmatrix} .$$

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§14. Laurent extensions

Given a sequence of algebraic K-functors

$$F_n : (\text{rings}) \longrightarrow (\text{abelian groups}) \quad (n \in \mathbb{Z})$$

it is traditional to aim at splitting theorems of the type

$$F_n(A[z,z^{-1}]) = F_n(A) \oplus F_{n-1}(A) \oplus ? \qquad (n \in \mathbb{Z})$$

with $A[z,z^{-1}]$ the Laurent extension of A, by analogy with

$$K_1(A[z,z^{-1}]) = K_1(A) \oplus K_0(A) \oplus Nil_+(A) \oplus Nil_-(A) \ .$$

We shall now investigate such splitting theorems for the algebraic L-functors

$$L^n, L_n : (\text{rings with involution}) \longrightarrow (\text{abelian groups}) \quad (n \in \mathbb{Z}) \ .$$

The Laurent extension $A[z,z^{-1}]$ of a ring with involution A is the ring of finite polynomials $\sum_j a_j z^j$ $(a_j \in A, j \in \mathbb{Z})$ in a central invertible indeterminate z, with involution by

$$\bar{\ } : A[z,z^{-1}] \longrightarrow A[z,z^{-1}] \ ; \ \sum_j a_j z^j \longmapsto \sum_j \bar{a}_j z^{-j} \ .$$

In particular, for a group ring $\mathbb{Z}[\pi]$ with the w-twisted involution (for some group morphism $w:\pi \longrightarrow \mathbb{Z}_2$) there is a natural identification

$$\mathbb{Z}[\pi][z,z^{-1}] = \mathbb{Z}[\pi \times \mathbb{Z}] \ ,$$

giving $\mathbb{Z}[\pi\times\mathbb{Z}]$ the $(w\times 1)$-twisted involution, $w\times 1:\pi\times\mathbb{Z} \longrightarrow \mathbb{Z}_2; (g,z^j) \longmapsto w(g)$.

Proposition 14.1 There is a natural direct sum decomposition of the ε-quadratic L-groups of the Laurent extension $A[z,z^{-1}]$ of a ring with involution A

$$V_n(A[z,z^{-1}],\varepsilon) = V_n(A,\varepsilon) \oplus U_{n-1}(A,\varepsilon) \qquad (n \in \mathbb{Z}) \ .$$

Proof: The splitting theorem for surgery obstruction groups

$$L_n^s(\pi\times\mathbb{Z}) = L_n^s(\pi) \oplus L_{n-1}^h(\pi)$$

was first obtained geometrically by Shaneson [1] (and implicitly by Wall [5]), by expressing the surgery obstruction $\sigma_*(f,b) \in L_n^s(\pi_1(X\times S^1))$ of an n-dimensional normal bundle map of the type $(f,b):M \longrightarrow X\times S^1$ as the direct sum of two terms

$$\sigma_*(f,b) = \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(X\times S^1)]} \sigma_*(f,b) \oplus \sigma_*(g,c) \in L_n^s(\pi_1(X)) \oplus L_{n-1}^h(\pi_1(X)) \ ,$$

with $\sigma_*(g,c)$ the surgery obstruction of the $(n-1)$-dimensional normal bundle map $(g,c) : N = f^{-1}(X\times \text{regular pt.}) \longrightarrow X$ constructed by transversality.

The splitting theorem was then obtained algebraically by Novikov [2] (modulo 2-torsion, with $1/2 \in A$) and Ranicki [2] (for any A), with isomorphisms

$$(\bar{e}\ \bar{B}) : V_n(A) \oplus U_{n-1}(A) \longrightarrow V_n(A[z,z^{-1}]) \quad (n (\text{mod } 4)) \ .$$

Here $\bar{e}$ is the map induced by the split injection of rings with involution

$$\bar{e} : A \longrightarrow A[z,z^{-1}] \ ; \ a \longmapsto a \ .$$

The corresponding isomorphisms for the intermediate quadratic L-groups

$$(\bar{e}\ \bar{B}) : V_n^X(A) \oplus U_n^Y(A) \longrightarrow V_n^{\bar{e}(X)\oplus\bar{B}(Y)}(A[z,z^{-1}]) \quad (X \subseteq \widetilde{K}_1(A),\ Y \subseteq \widetilde{K}_0(A))$$

were defined in Ranicki [3]. The methods of Ranicki [2],[3] generalize immediately from $\varepsilon = 1 \in A$ to any central unit $\varepsilon \in A$ such that $\varepsilon\bar{\varepsilon} = 1 \in A$.

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(There is a generalization of the decomposition of Proposition 14.1 to twisted Laurent extensions $A_\alpha[z,z^{-1}]$ ($az = z\alpha(a)$ for some automorphism $\alpha : A \longrightarrow A; a \mapsto \alpha(a)$, $\bar{z} = z^{-1}$), involving an exact sequence - cf. Cappell [1] and Ranicki [3].)

The symmetric signature $\sigma^*(S^1) \in L^1(\mathbb{Z}[z,z^{-1}])$ of the circle S^1 is represented by the 1-dimensional symmetric complex $(C, \varphi \in Q^1(C))$ over $\mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z,z^{-1}]$ defined by

$$C_r = \begin{cases} \mathbb{Z}[z,z^{-1}] & r = 0,1 \\ 0 & r \neq 0,1 \end{cases} , \ d = 1-z : C_1 \longrightarrow C_0 \ , \ \varphi_0 = \begin{cases} 1 : C^1 \longrightarrow C_0 \\ z^{-1} : C^0 \longrightarrow C_1 \end{cases} , \ \varphi_1 = 1 : C^1 \longrightarrow C_1 ,$$

corresponding to the non-singular symmetric formation over $\mathbb{Z}[z,z^{-1}]$

$$\sigma^*(S^1) = (\mathbb{Z}[z,z^{-1}] \oplus \mathbb{Z}[z,z^{-1}], \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}[z,z^{-1}], \begin{pmatrix} 1 \\ z-1 \end{pmatrix}\mathbb{Z}[z,z^{-1}]) \ .$$

(Moreover, $\sigma^*(S^1)$ is isomorphic to the symmetric formation $(\mathbb{Z}[z,z^{-1}] \oplus \mathbb{Z}[z,z^{-1}], 1 \oplus -1; \Delta, (z\oplus 1)\Delta)$ associated to the automorphism $z : (\mathbb{Z}[z,z^{-1}],1) \longrightarrow (\mathbb{Z}[z,z^{-1}],1)$, with $\Delta = \{(x,x) \mid x \in \mathbb{Z}[z,z^{-1}]\}$. It is thus seen to correspond to the generator $\tau(z : \mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}[z,z^{-1}]) \in \widetilde{K}_1(\mathbb{Z}[z,z^{-1}]) = \mathbb{Z}$. The connection of the symmetric L-groups with the orthogonal groups of automorphisms of symmetric forms will be discussed further in §15 below).

The split injections of Proposition 14.1 are precisely the products

$$\bar{B} = \sigma^*(S^1) \otimes - : U_n(A,\varepsilon) \longrightarrow V_{n+1}(A[z,z^{-1}],\varepsilon) \quad (n \in \mathbb{Z}) \ .$$

Define similarly products in the ε-symmetric L-groups

$$\bar{B} = \sigma^*(S^1) \otimes - : U^n(A,\varepsilon) \longrightarrow V^{n+1}(A[z,z^{-1}],\varepsilon) \quad (n \in \mathbb{Z}) \ .$$

<u>Conjecture 14.2</u> The map

$$(\bar{e}\ \bar{B})\ :\ V^n(A,\varepsilon)\oplus U^{n-1}(A,\varepsilon) \longrightarrow V^n(A[z,z^{-1}],\varepsilon)$$

is an isomorphism for all $A,\varepsilon,n\in\mathbb{Z}$.

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Proposition 14.1 verifies the conjecture in the ε-quadratic range $n\leqslant -3$. In Proposition 14.6 below we shall establish the conjecture in the even ε-symmetric range $-2\leqslant n\leqslant -1$, and show that $(\bar{e}\ \bar{B})$ is at least a split injection for $0\leqslant n\leqslant 1$. If $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon) = 0$ Proposition 9.4 gives the corresponding results for $n\leqslant 3$.

For any ring with involution A let us write

$$\begin{cases} \bar{L}^n(A) = L^n(A,-1) \\ \bar{L}_n(A) = L_n(A,-1)\ (= L_{n+2}(A)) \end{cases} \quad (n\in\mathbb{Z}) .$$

As usual, questions of periodicity are closely related to Laurent extensions. Combined with the computation of $L^*(\mathbb{Z})$ (Proposition 13.27) Conjecture 14.2 would give

$$L^2(\mathbb{Z}[\mathbb{Z}^2]) = L^2(\mathbb{Z})\oplus L^1(\mathbb{Z})\oplus L^1(\mathbb{Z})\oplus L^0(\mathbb{Z}) = 0\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z} .$$

On the other hand, Propositions 9.7 , 14.1 give that

$$\bar{L}^0(\mathbb{Z}[\mathbb{Z}^2]) = \bar{L}^0(\mathbb{Z})\oplus L_1(\mathbb{Z})\oplus L_1(\mathbb{Z})\oplus L_0(\mathbb{Z}) = 0\oplus 0\oplus 0\oplus 8\mathbb{Z} .$$

This suggests that the skew-suspension map $\bar{S}:\bar{L}^0(\mathbb{Z}[\mathbb{Z}^2]) \longrightarrow L^2(\mathbb{Z}[\mathbb{Z}^2])$ is not onto, and hence that the symmetric L-groups are not 4-periodic in general, $L^n \neq L^{n+4}$. We can prove that much, even without the conjecture:

<u>Proposition 14.3</u> The (k+1)-fold skew-suspension map

$$\bar{S}^{k+1}\ :\ \bar{L}^0(\mathbb{Z}[\mathbb{Z}^2]) \longrightarrow L^{2k+2}(\mathbb{Z}[\mathbb{Z}^2],(-)^k) \quad (k\geqslant 0)$$

is not onto, with $\bar{S}^k\sigma^*(S^1\times S^1)\notin \operatorname{im}(\bar{S}^{k+1})$.

<u>Proof</u>: Consider first the case $k = 0$. The products of Proposition 11.1 fit into a commutative diagram

$$\begin{array}{ccc}
\bar{L}_0(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}_0(\mathbb{Z}) & \xrightarrow{1\otimes(1+\bar{T})} & \bar{L}_0(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}^0(\mathbb{Z}) \\
\downarrow{\scriptstyle (1+\bar{T})\otimes 1} & & \downarrow{\scriptstyle \otimes} \\
\bar{L}^0(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}_0(\mathbb{Z}) & \xrightarrow{\otimes} & L_0(\mathbb{Z}[\mathbb{Z}^2]) \\
\downarrow{\scriptstyle \bar{S}\otimes 1} & & \downarrow{\scriptstyle \bar{S}} \\
L^2(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}_0(\mathbb{Z}) & \xrightarrow{\otimes} & \bar{L}_2(\mathbb{Z}[\mathbb{Z}^2])
\end{array}$$

The skew-symmetrization map $(1+\bar{T}):\bar{L}_0(\mathbb{Z}[\mathbb{Z}^2])\longrightarrow\bar{L}^0(\mathbb{Z}[\mathbb{Z}^2])$ is onto, by Proposition 9.3 . Thus if $\sigma^*(S^1\times S^1)\in\operatorname{im}(\bar{S}:\bar{L}^0(\mathbb{Z}[\mathbb{Z}^2])\longrightarrow L^2(\mathbb{Z}[\mathbb{Z}^2]))$ there exists an element $x\in\bar{L}_0(\mathbb{Z}[\mathbb{Z}^2])$ such that

$$\sigma^*(S^1\times S^1) = \bar{S}(1+\bar{T})(x)\in L^2(\mathbb{Z}[\mathbb{Z}^2]) .$$

The Arf invariant element $c\in\bar{L}_0(\mathbb{Z}) = \mathbb{Z}_2$ is such that

$$\sigma^*(S^1\times S^1)\otimes c = \bar{B}^2(c) \neq 0\in\bar{L}_2(\mathbb{Z}[\mathbb{Z}^2]) ,$$

by Proposition 14.1. The above diagram gives

$$\sigma^*(S^1\times S^1)\otimes c = \bar{S}(x\otimes(1+\bar{T})c) = 0\in\bar{L}_2(\mathbb{Z}[\mathbb{Z}^2]) ,$$

since $(1+\bar{T})(c) = 0\in\bar{L}^0(\mathbb{Z}) = 0$. It follows from this contradiction that there is no such $x\in\bar{L}_0(\mathbb{Z}[\mathbb{Z}^2])$, and hence that $\sigma^*(S^1\times S^1)\notin\operatorname{im}(\bar{S})$. For general $k\geqslant 0$ observe that $\bar{S}:L^2(\mathbb{Z}[\mathbb{Z}^2])\longrightarrow\bar{L}^4(\mathbb{Z}[\mathbb{Z}^2])$ is an isomorphism by Proposition 9.5 , and that $\bar{S}^j:\bar{L}^4(\mathbb{Z}[\mathbb{Z}^2])\longrightarrow L^{4+2j}(\mathbb{Z}[\mathbb{Z}^2],(-)^{j+1})$ $(j\geqslant 1)$ is an isomorphism by Proposition 7.4, since $\mathbb{Z}[\mathbb{Z}^2]$ is 3-dimensional.

[]

Let $d\in L^1(\mathbb{Z}) = \mathbb{Z}_2$ be the generator (= the deRham element).

<u>Proposition 14.4</u> The (k+1)-fold skew-suspension map

$$\bar{S}^{k+1}\ :\ \bar{L}^0(\mathbb{Z}[\mathbb{Z}]) = 0 \longrightarrow L^{2k+2}(\mathbb{Z}[\mathbb{Z}],(-)^k) \quad (k\geqslant 0)$$

is not onto, with $\bar{S}^k\bar{B}(d)\notin\operatorname{im}(\bar{S}^{k+1}) = 0$.

<u>Proof</u>: Let $\mathbb{Z}[\mathbb{Z}^-]$ be the group ring $\mathbb{Z}[\mathbb{Z}]$ with involution $\bar{z} = -z^{-1}$. It is proved in Morgan [1] that the product

$$d\otimes-\ :\ L_3(\mathbb{Z}[\mathbb{Z}^-]) = \mathbb{Z}_2 \longrightarrow L_4(\mathbb{Z}[\mathbb{Z}^-]) = \mathbb{Z}_2$$

is an isomorphism. It follows that the product

$$\bar{S}^k\bar{B}(d)\otimes-\ :\ L_3(\mathbb{Z}[\mathbb{Z}^-]) = \mathbb{Z}_2 \longrightarrow L_{2k+5}(\mathbb{Z}[\mathbb{Z}^-\times\mathbb{Z}],(-)^k) = \mathbb{Z}_2$$

is also an isomorphism, and hence that $\bar{S}^k\bar{B}(d)\neq 0\in L^{2k+2}(\mathbb{Z}[\mathbb{Z}],(-)^k)$.
(Conjecture 14.2 would give that $\bar{S}^k\bar{B}(d)\in L^{2k+2}(\mathbb{Z}[\mathbb{Z}],(-)^k) = \mathbb{Z}_2$ is the generator.)

[]

The symmetrization functor embeds the category of quadratic forms over a group ring $\mathbb{Z}[\pi]$ with the untwisted involution (= category of even symmetric forms over $\mathbb{Z}[\pi]$) in the category of symmetric forms over $\mathbb{Z}[\pi]$. Nevertheless, it need not be the case that the symmetrization map in the Witt groups $(1+T):L_0(\mathbb{Z}[\pi]) \longrightarrow L^0(\mathbb{Z}[\pi])$ is one-one, as shown by the following example. (In Proposition 14.8 below it will be proved that $L_0(\mathbb{Z}[\pi])$ is isomorphic to a direct summand of $L^0(\mathbb{Z}[\pi\times\mathbb{Z}^4])$).

Proposition 14.5 The symmetrization map

$$1+T\ :\ L_0(\mathbb{Z}[\mathbb{Z}^2]) \longrightarrow L^0(\mathbb{Z}[\mathbb{Z}^2])$$

is not one-one, with $\bar{B}^2(c) \neq 0 \in \ker(1+T)$.

Proof: There is defined a commutative diagram

$$\begin{array}{ccccc}
L^2(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}_0(\mathbb{Z}) & \xrightarrow{\ \otimes\ } & \bar{L}_2(\mathbb{Z}[\mathbb{Z}^2]) & \xleftarrow{\ \bar{S}\ } & L_0(\mathbb{Z}[\mathbb{Z}^2]) \\
\downarrow {\scriptstyle 1\otimes(1+\bar{T})} & & \downarrow {\scriptstyle 1+\bar{T}} & & \downarrow {\scriptstyle 1+T} \\
L^2(\mathbb{Z}[\mathbb{Z}^2])\otimes_{\mathbb{Z}}\bar{L}^0(\mathbb{Z}) & \xrightarrow{\ \otimes\ } & \bar{L}^2(\mathbb{Z}[\mathbb{Z}^2]) & \xleftarrow{\ \bar{S}\ } & L^0(\mathbb{Z}[\mathbb{Z}^2])
\end{array}$$

such that the skew-suspensions $\bar{S}$ are isomorphisms (by Propositions 7.3, 9.5), with $\bar{L}^0(\mathbb{Z}) = 0$. The generator $c \in \bar{L}_0(\mathbb{Z}) = \mathbb{Z}_2$ is such that

$$(1+T)\bar{S}^{-1}\bar{B}^2(c) = \bar{S}^{-1}(1+\bar{T})\bar{B}^2(c) = \bar{S}^{-1}\bar{B}^2(1+\bar{T})(c) = 0 \in L^0(\mathbb{Z}[\mathbb{Z}^2])\ .$$

[]

Proposition 14.1 gives

$$L_0(\mathbb{Z}[\mathbb{Z}^2]) = L_0(\mathbb{Z})\oplus\bar{L}_1(\mathbb{Z})\oplus\bar{L}_1(\mathbb{Z})\oplus\bar{L}_0(\mathbb{Z}) = 8\mathbb{Z}\oplus 0\oplus 0\oplus\mathbb{Z}_2\ ,$$

and Conjecture 14.2 would give

$$L^0(\mathbb{Z}[\mathbb{Z}^2]) = L^0(\mathbb{Z})\oplus L^{-1}(\mathbb{Z})\oplus L^{-1}(\mathbb{Z})\oplus L^{-2}(\mathbb{Z}) = \mathbb{Z}\oplus 0\oplus 0\oplus 0\ .$$

At any rate, Proposition 14.6 will show that $L^0(\mathbb{Z}[\mathbb{Z}^2]) = \mathbb{Z}\oplus 0\oplus 0\oplus 0\oplus ?$.

Proposition 14.6 There are defined natural direct sum decompositions of ε-symmetric L-groups

$$V^n(A[z,z^{-1}],\varepsilon) = V^n(A,\varepsilon)\oplus U^{n-1}(A,\varepsilon) \quad (n \leqslant -1)$$

$$V^0(A[z,z^{-1}],\varepsilon) = V^0(A,\varepsilon)\oplus U^{-1}(A,\varepsilon)\oplus T^0(A,\varepsilon)$$

$$V^1(A[z,z^{-1}],\varepsilon) = V^1(A,\varepsilon)\oplus U^0(A,\varepsilon)\oplus T^1(A,\varepsilon)$$

with T^0, T^1 such that

$$T^0(A[z,z^{-1}],\varepsilon) = T^0(A,\varepsilon)$$

$$T^1(A[z,z^{-1}],\varepsilon) = T^1(A,\varepsilon)\oplus T^0(A,\varepsilon)\ .$$

Proof: For $n \leqslant -3$ these are the decompositions of the ε-quadratic L-groups of Proposition 14.1 . We shall now define maps

$$B\ :\ V^n(A[z,z^{-1}],\varepsilon) \longrightarrow U^{n-1}(A,\varepsilon) \quad (-2 \leqslant n \leqslant 1)$$

such that $B\bar{B} = 1$, $B\bar{\varepsilon} = 0$, treating seperately the cases n odd and n even.

Given an ε-symmetric form over $A[z,z^{-1}]$ (L,λ) with $L = A[z,z^{-1}]\otimes_A L_0$ for some f.g. projective A-module L_0 there is defined an isomorphism of ε-symmetric metabolic forms over $A[z,z^{-1}]$

$$\begin{pmatrix} 1 & 0 \\ \bar{\varepsilon}\sum_{j=1}\lambda_j z^j & 1 \end{pmatrix} : H^{\varepsilon}(L,\lambda) = (L\oplus L^*, \begin{pmatrix} \lambda & \varepsilon \\ 1 & 0 \end{pmatrix})$$

$$\longrightarrow A[z,z^{-1}]\otimes_A H^{\varepsilon}(L_0,\lambda_0) = (L\oplus L^*, \begin{pmatrix} \lambda_0 & \varepsilon \\ 1 & 0 \end{pmatrix})$$

$$(\lambda = \sum_{j=-\infty}^{\infty}\lambda_j z^j\ ,\ \lambda_j \in \mathrm{Hom}_A(L_0,L_0^*),\ \lambda_j^* = \lambda_{-j})\ .$$

Given a non-singular ε-symmetric form over $A[z,z^{-1}]$ (Q,φ) and a lagrangian of the type $F = A[z,z^{-1}]\otimes_A F_0$ it is thus possible to choose a direct complement $F' = A[z,z^{-1}]\otimes_A F_0'$ to F in Q such that $\varphi| : F' \longrightarrow F^* = \mathrm{Hom}_{A[z,z^{-1}]}(F,A[z,z^{-1}])$ is an isomorphism sending F_0' to $F_0^* = \mathrm{Hom}_A(F_0,A)$ and $\varphi(F_0')(F_0') \subseteq A \subseteq A[z,z^{-1}]$.

Define

$$B\ :\ V^1(A[z,z^{-1}],\varepsilon) \longrightarrow U^0(A,\varepsilon)\ ;\ (Q,\varphi;F,G) \longmapsto (B_N(F_0\oplus F_0^*, G_0\oplus G_0^*), [\varphi]_0)$$

and verify that $B\bar{B} = 1$, $B\bar{\varepsilon} = 0$ exactly as in the $\pm$quadratic case (§2 of Ranicki [2]). Here $B_N(F_0\oplus F_0^*, G_0\oplus G_0^*) = z^N(F_0\oplus F_0^*)^- \cap (G_0\oplus G_0^*)^+$ for $N \geqslant 0$ so large that $z^N(F_0\oplus F_0^*)^+ \subseteq (G_0\oplus G_0^*)^+$, using direct complements F^*, G^* to F, G in Q chosen as above, with $M^+ = \sum_{j=0}^{\infty} z^j M$, $M^- = \sum_{j=-\infty}^{-1} z^j M \subseteq A[z,z^{-1}]\otimes_A M$ for any A-module M

and $[\varphi]_0(x)(y) = a_0 \in A$ if $\varphi(x)(y) = \sum_{j=-\infty}^{\infty} a_j z^j \in A[z,z^{-1}]$ ($a_j \in A$, $x,y \in Q$).
If $(Q,\varphi;F,G)$ is an even $(-\varepsilon)$-symmetric formation over $A[z,z^{-1}]$ then $B(Q,\varphi;F,G)$ is an even $(-\varepsilon)$-symmetric form over A, and we have also a map

$$B : V^{-1}(A[z,z^{-1}],\varepsilon) \longrightarrow U^{-2}(A,\varepsilon)$$

such that $B\bar{B} = 1$, $B\bar{e} = 0$.

Given a non-singular ε-symmetric form over $A[z,z^{-1}]$ (Q,φ) with $Q = A[z,z^{-1}]\otimes_A Q_0$ for some f.g. projective A-module Q_0 define a non-singular even $(-\varepsilon)$-symmetric formation over A

$$B(Q,\varphi) = (H^{-\varepsilon}(\sum_{j=0}^{N-1} z^j Q_0);\sum_{j=0}^{N-1} z^j Q_0,\{(z^N(1-\nu)z^{-N}x,\nu\varphi(x)) \in \sum_{j=0}^{N-1} z^j Q_0 \oplus \sum_{j=0}^{N-1} z^j Q_0^* \mid x \in \varphi^{-1}(z^N Q_0^{*-}) \cap Q_0^+\} = B_N(Q_0,\varphi))$$

for $N \geqslant 0$ so large that $\varphi(Q_0) \subseteq \sum_{j=-N}^{N} Q_0^*$, $\varphi^{-1}(Q_0^*) \subseteq \sum_{j=-N}^{N} z^j Q_0$, with $\nu = (1\ \ 0) : Q = Q_0^+ \oplus Q_0^- \longrightarrow Q_0^+$. Define

$$B : V^0(A[z,z^{-1}],\varepsilon) \longrightarrow U^{-1}(A,\varepsilon)\ ;\ (Q,\varphi) \longmapsto B(Q,\varphi)$$

and verify that $B\bar{B} = 1$, $B\bar{e} = 0$ exactly as in the $\pm$quadratic case (§3 of Ranicki [2]). If (Q,φ) is an even $(-\varepsilon)$-symmetric form over $A[z,z^{-1}]$ then $B(Q,\varphi) = (H_\varepsilon(\sum_{j=0}^{N-1} z^j Q_0);\sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0,\varphi))$ is an ε-quadratic formation over A, and we also have a map

$$B : V^{-2}(A[z,z^{-1}],\varepsilon) \longrightarrow U^{-3}(A,\varepsilon) = U_1(A,\varepsilon)\ ;\ (Q,\varphi) \longmapsto B(Q,\varphi)$$

such that $B\bar{B} = 1$, $B\bar{e} = 0$.

The exactness of the sequences

$$0 \longrightarrow V^{-1}(A,\varepsilon) \xrightarrow{\bar{e}} V^{-1}(A[z,z^{-1}],\varepsilon) \xrightarrow{B} U^{-2}(A,\varepsilon) \longrightarrow 0$$

$$0 \longrightarrow V^{-2}(A,\varepsilon) \xrightarrow{\bar{e}} V^{-2}(A[z,z^{-1}],\varepsilon) \xrightarrow{B} U^{-3}(A,\varepsilon) \longrightarrow 0$$

may be verified precisely as in §§2,3 of Ranicki [2].

[]

Conjecture 14.2 would imply that the groups T^0,T^1 vanish, that is

$$T^0(A,\varepsilon) = 0\ ,\ T^1(A,\varepsilon) = 0$$

for any A,ε. At any rate we have:

<u>Proposition 14.7</u> If A,ε are such that $\hat{H}^0(\mathbb{Z}_2;A,\varepsilon) = 0$ then

$$T^0(A,\varepsilon) = 0\ ,\ T^1(A,\varepsilon) = 0\ .$$

<u>Proof</u>: $\hat{H}^0(\mathbb{Z}_2;A[z,z^{-1}],\varepsilon) = \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) = 0$, so that by Propositions 9.4, 14.6

$$V^1(A[z,z^{-1}],\varepsilon) = V^{-1}(A[z,z^{-1}],-\varepsilon) = V^{-1}(A,-\varepsilon)\oplus U^{-2}(A,-\varepsilon) = V^1(A,\varepsilon)\oplus U^0(A,\varepsilon)$$

$$V^0(A[z,z^{-1}],\varepsilon) = V^{-2}(A[z,z^{-1}],-\varepsilon) = V^{-2}(A,-\varepsilon)\oplus U^{-3}(A,-\varepsilon) = V^0(A,\varepsilon)\oplus U^{-1}(A,\varepsilon)\ .$$

[]

The decomposition of Proposition 14.6 shows that $\ker(\bar{S}:L^0(A,\varepsilon) \longrightarrow L^2(A,-\varepsilon))$ may be quite large in general (albeit 8-torsion, by Proposition 11.4), since there is defined a commutative diagram

$$\begin{array}{ccc} U^{-1}(A,\varepsilon) & \xrightarrow{\bar{B}} & V^0(A[z,z^{-1}],\varepsilon) \\ \downarrow{\scriptstyle 1+T_{-\varepsilon}} & & \downarrow{\scriptstyle \bar{S}} \\ U^1(A,-\varepsilon) & \xrightarrow{\bar{B}} & V^2(A[z,z^{-1}],-\varepsilon) \end{array}$$

with both the maps $\bar{B}$ split injections. Thus $\ker(\bar{S}:V^0(A[z,z^{-1}],\varepsilon)\to V^2(A[z,z^{-1}],-\varepsilon))$ is at least as large as $\ker((1+T_{-\varepsilon}):U^{-1}(A,\varepsilon) \longrightarrow U^1(A,-\varepsilon))$, which may even be infinitely generated (cf. Proposition 14.9 ii)).

<u>Proposition 14.8</u> For the group ring $\mathbb{Z}[\pi]$ of a group π with the untwisted involution there are natural identifications

$$L^1(\mathbb{Z}[\pi\times\mathbb{Z}]) = L^1(\mathbb{Z}[\pi])\oplus L^0(\mathbb{Z}[\pi])\oplus ?\ ,\ L^0(\mathbb{Z}[\pi\times\mathbb{Z}]) = L^0(\mathbb{Z}[\pi])\oplus \bar{L}^1(\mathbb{Z}[\pi])\oplus ?$$

$$\bar{L}^1(\mathbb{Z}[\pi\times\mathbb{Z}]) = \bar{L}^1(\mathbb{Z}[\pi])\oplus \bar{L}^0(\mathbb{Z}[\pi])\ ,\ \bar{L}^0(\mathbb{Z}[\pi\times\mathbb{Z}]) = \bar{L}^0(\mathbb{Z}[\pi])\oplus L_1(\mathbb{Z}[\pi])$$

$$L^{-1}(\mathbb{Z}[\pi]) = \bar{L}^1(\mathbb{Z}[\pi])\ ,\ L^{-2}(\mathbb{Z}[\pi]) = \bar{L}^0(\mathbb{Z}[\pi])$$

$$\bar{L}^{-1}(\mathbb{Z}[\pi]) = L_1(\mathbb{Z}[\pi])\ ,\ \bar{L}^{-2}(\mathbb{Z}[\pi]) = L_0(\mathbb{Z}[\pi])$$

<u>Proof</u>: Immediate from Propositions 9.5, 14.7, neglecting the difference between U and V.

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<u>Proposition 14.9</u> i) The (k+1)-fold skew-suspension map

$$\bar{S}^{k+1} : \bar{L}^0(\mathbb{Z}[\mathbb{Z}^4]) \longrightarrow L^{2k+2}(\mathbb{Z}[\mathbb{Z}^4],(-)^k)\quad (k \geqslant 0)$$

is not one-one, with $\bar{B}^4(c) \neq 0 \in \ker(\bar{S}^{k+1})$.

ii) The skew-suspension map

$$\bar{S} : \bar{L}^0(\mathbb{Z}[(\mathbb{Z}_2*\mathbb{Z})\times\mathbb{Z}^4]) \longrightarrow L^2(\mathbb{Z}[(\mathbb{Z}_2*\mathbb{Z})\times\mathbb{Z}^4])$$

is not one-one, and $\ker(\bar{S})$ is infinitely generated.

<u>Proof</u>: i) is immediate from Propositions 7.4, 14.8 and $(1+\bar{T})(c) = 0 \in \bar{L}^0(\mathbb{Z}) = 0$.

ii) Cappell [2] has shown that $\ker((1+\bar{T}):\bar{L}_0(\mathbb{Z}[\mathbb{Z}_2*\mathbb{Z}]) \longrightarrow \bar{L}^0(\mathbb{Z}[\mathbb{Z}_2*\mathbb{Z}]))$ is infinitely generated. Now apply Proposition 14.8 again.

[]

(We can apply the decompositions of Proposition 14.8 to express the surgery obstruction of an oriented normal bundle map in terms of the symmetric signature of an associated highly-connected degree 1 map, subsuming the quadratic information in a geometric construction. Let then $(f,b):M \longrightarrow X$ be an n-dimensional normal bundle map with X oriented ($w(X) = 1$) and $n = \begin{cases} 2i \\ 2i+1 \end{cases}$. Let $(f',b'):M'^{n+k} \longrightarrow X \times T^k$ be the highly-connected $(n+k)$-dimensional normal bundle map obtained from $(f\times 1, b\times 1):M\times T^k \longrightarrow X\times T^k$ ($T^k = S^1\times S^1\times\ldots\times S^1$, k times) by framed surgery below the middle dimension, with $k = \begin{cases} 2 \\ 1 \end{cases}$ if $i\equiv 0(\bmod 2)$, $k = \begin{cases} 4 \\ 3 \end{cases}$ if $i\equiv 1(\bmod 2)$.

Set $j = \frac{1}{2}(n+k)$, $\pi = \pi_1(X)$. The split injection $\overline{B}^k : L_n(\mathbb{Z}[\pi]) \longrightarrow \overline{L}^0(\mathbb{Z}[\pi\times\mathbb{Z}^k])$ sends the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi])$ to the symmetric signature

$$\overline{B}^k\sigma_*(f,b) = \sigma^j(f') \in \overline{L}^0(\mathbb{Z}[\pi\times\mathbb{Z}^k]) ,$$

with skew-suspension

$$\overline{S}\sigma^j(f') = \overline{B}^k(1+T)\sigma_*(f,b) \in L^2(\mathbb{Z}[\pi\times\mathbb{Z}^k]) \qquad ((1+T)\sigma_*(f,b) \in \begin{cases} L^0(\mathbb{Z}[\pi],(-)^i) \\ L^1(\mathbb{Z}[\pi],(-)^i) \end{cases})$$

and j-fold skew-suspension

$$\overline{S}^j\sigma^j(f') = \overline{B}^k(\sigma^*(M) - \sigma^*(X)) \in L^{n+k}(\mathbb{Z}[\pi\times\mathbb{Z}^k]) .$$

(All this holds for unoriented normal bundle maps also, provided that we set $k = \begin{cases} 6 \\ 5 \end{cases}$ if $i \equiv 0(\bmod 2)$). In particular, consider the 2-dimensional normal bundle map $(f,b):T^2 \longrightarrow S^2$ defined by the exotic framing b on T^2, with Arf invariant $\sigma_*(f,b) = 1 \in L_2(\mathbb{Z}) = \mathbb{Z}_2$. Let $(f',b'):M^6 \longrightarrow S^2\times T^4$ be the 2-connected 6-dimensional normal bundle map obtained from $(f\times 1, b\times 1):T^2\times T^4 \longrightarrow S^2\times T^4$ by framed surgery below the middle dimension. Then the Arf invariant is detected by $\sigma^3(f') = \overline{B}^4(c) \neq 0 \in \ker(\overline{S}:\overline{L}^0(\mathbb{Z}[\mathbb{Z}^4]) \longrightarrow L^2(\mathbb{Z}[\mathbb{Z}^4]))$, the element constructed in Proposition 14.9 i). Let $(f,b_0):T^2 \longrightarrow S^2$ be the normal map defined by the standard framing b_0 on T^2, with $\sigma_*(f,b_0) = 0 \in L_2(\mathbb{Z})$. We can also write the element as $\sigma^3(f') = (1+\overline{T})\Omega^3(\sigma_*(f\times 1, b\times 1) \oplus -\sigma_*(f\times 1, b_0\times 1)) \in \ker(\overline{S}^3:\overline{L}^0(\mathbb{Z}[\mathbb{Z}^4]) \to L^6(\mathbb{Z}[\mathbb{Z}^4]))$ so that this failure of periodicity in L^* is one of the type discussed following Proposition 7.3).

The realization theorems of §§5,6 of Wall [5] show that for a finitely presented group π every element $x \in L_n(\mathbb{Z}[\pi])$ is the quadratic signature $x = \sigma_*(f,b)$ of an $(i-1)$-connected normal map $(f,b):M \longrightarrow X$ of n-dimensional geometric Poincaré complexes with $\pi_1(X) = \pi$, if $n = 2i$ or $2i+1$, for $n \geqslant 5$. It is not known which elements $x \in L^n(\mathbb{Z}[\pi])$ can be realized as the symmetric signature $x = \sigma^*(X)$ of an n-dimensional geometric Poincaré complex X with $\pi_1(X) = \pi$, or as the symmetric signature $x = \sigma^*(f)$ of a degree 1 map $f:M \longrightarrow X$ of n-dimensional geometric Poincaré complexes with $\pi_1(X) = \pi$. In Proposition 10.5 we gave an example to show that not every element $x \in \hat{L}^n(\mathbb{Z}[\pi])$ can be realized as the hyperquadratic signature $x = \hat{\sigma}^*(X)$ of an n-dimensional normal space X with $\pi_1(X) = \pi$. In Proposition 18.8 we shall show that for $i \neq 2,4,8$ the symmetric signature $\sigma^*(f)$ of an $(i-1)$-connected n-dimensional degree 1 map $f:M \longrightarrow X$ ($n = 2i$ or $2i+1$) is such that $\sigma^*(f) \in \mathrm{im}((1+T):L_n(\mathbb{Z}[\pi]) \longrightarrow L^n(\mathbb{Z}[\pi]))$, $\pi = \pi_1(X)$. All this suggests that not every element of $\mathrm{coker}((1+T):L_n(\mathbb{Z}[\pi]) \longrightarrow L^n(\mathbb{Z}[\pi]))$ is geometrically realizable. An unlikely source of such non-realizable elements are the failures of Conjecture 14.2:

Proposition 14.10 The geometric Poincaré bordism groups are such that

$$\Omega_n^P(X\times S^1) = \Omega_n^P(X) \oplus \Omega_{n-1}^P(X) \quad (n \geqslant 5)$$

for any space X. Geometrically realizable elements of $L^n(\mathbb{Z}[\pi\times\mathbb{Z}])$ lie in $\mathrm{im}((\overline{e}\ \overline{B}):L^n(\mathbb{Z}[\pi]) \oplus L^{n-1}(\mathbb{Z}[\pi]) \longrightarrow L^n(\mathbb{Z}[\pi\times\mathbb{Z}]))$, for any group π.

Proof: (Here, we are neglecting the difference between U and V, and between finite and infinite geometric Poincaré complexes). Applying the 5-lemma to the morphism of exact sequences

$$\begin{array}{ccccccccc}
\ldots \to L_n(\mathbb{Z}[\pi]) \oplus L_{n-1}(\mathbb{Z}[\pi]) & \to & \Omega_n^P(X) \oplus \Omega_{n-1}^P(X) & \to & \Omega_n^N(X) \oplus \Omega_{n-1}^N(X) & \to & L_{n-1}(\mathbb{Z}[\pi]) \oplus L_{n-2}(\mathbb{Z}[\pi]) \to \ldots \\
\downarrow (\overline{e}\ \overline{B}) & & \downarrow & & \downarrow & & \downarrow (\overline{e}\ \overline{B}) \\
\ldots \longrightarrow L_n(\mathbb{Z}[\pi\times\mathbb{Z}]) & \longrightarrow & \Omega_n^P(X\times S^1) & \longrightarrow & \Omega_n^N(X\times S^1) & \longrightarrow & L_{n-1}(\mathbb{Z}[\pi\times\mathbb{Z}]) \to \ldots
\end{array} \quad (\pi = \pi_1(X))$$

we have that $\Omega_n^P(X) \oplus \Omega_{n-1}^P(X) \to \Omega_n^P(X\times S^1)$ is an isomorphism for $n \geqslant 5$, with $\Omega_n^N(X) = H_n(X;\underline{MSG})$ the normal space bordism groups (as in Proposition 10.4). Thus no element of $\mathrm{coker}((\overline{e}\ \overline{B}):L^n(\mathbb{Z}[\pi]) \oplus L^{n-1}(\mathbb{Z}[\pi]) \longrightarrow L^n(\mathbb{Z}[\pi\times\mathbb{Z}]))$ is geometrically realizable as the symmetric signature $\sigma^*(M)$ of an n-dimensional geometric Poincaré complex M with respect to a covering $M \longrightarrow K(\pi\times\mathbb{Z},1)$.

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§15. Classifying spaces

Let X be an n-dimensional geometric Poincaré complex with a topological normal bundle structure $(\nu_X : X \longrightarrow BTOP(k), \rho_X \in \pi_{n+k}(T(\nu_X)))$, e.g. a topological n-manifold. Given a topological normal bundle map

$$(f,b) : (M, \nu_M, \rho_M) \longrightarrow (X, \nu'_X, \rho'_X)$$

there is defined an equivalence of Spivak normal structures $c : (\nu_X, \rho_X) \longrightarrow (\nu'_X, \rho'_X)$, giving a fibre homotopy trivialization of $\nu_X - \nu'_X : X \longrightarrow BTOP$. Conversely, given a fibre homotopy trivialized topological bundle $\xi : X \longrightarrow BTOP$ make the homotopy equivalence $h : E(\xi) \longrightarrow X \times \mathbb{R}^\infty$ transverse regular at the zero section $X \times 0 \subset X \times \mathbb{R}^\infty$ $(n \geqslant 5)$ to obtain a topological normal bundle map

$$(f,b) : (M, \nu_M, \rho_M) \longrightarrow (X, \nu_X - \xi, \rho'_X)$$

with $M = h^{-1}(X \times 0) \subset E(\xi)$. It was first observed by Milnor [3] (for $X = S^n$, with normal vector bundle maps) and then by Sullivan [1] (for any X, with PL normal bundle maps) that the normal bundle bordism classes over a fixed range X are in a natural one-one correspondence with fibre homotopy trivialized bundles over X. The extension to the topological category was carried out by Kirby and Siebenmann [1] (cf. §17B of Wall [5]). Our aim here is to give a purely algebraic description of the surgery obstruction map

$$\theta : [X_+, G/TOP] \longrightarrow L_n(\mathbb{Z}[\pi_1(X)]) .$$

In particular, we shall construct algebraically an Ω-spectrum of simplicial sets $\underline{\mathbb{L}}_0(\mathbb{Z})$ such that the 0th term $\mathbb{L}_0(\mathbb{Z})$ is canonically homotopy equivalent to $L_0(\mathbb{Z}) \times G/TOP$. For any connected space X there is defined a natural map

$$\sigma_* : H_n(X; \underline{\mathbb{L}}_0(\mathbb{Z})) \longrightarrow L_n(\mathbb{Z}[\pi_1(X)])$$

which is an isomorphism for $X = pt.$. An n-dimensional geometric Poincaré complex X with a topological normal bundle structure satisfies Poincaré duality with $\underline{\mathbb{L}}_0(\mathbb{Z})$-coefficients, $H^r(X; \underline{\mathbb{L}}_0(\mathbb{Z})) = H_{n-r}(X; \underline{\mathbb{L}}_0(\mathbb{Z}))$, such that the surgery obstruction map is the composite

$$\theta : [X_+, G/TOP] \hookrightarrow [X_+, \mathbb{L}_0(\mathbb{Z})] = H^0(X; \underline{\mathbb{L}}_0(\mathbb{Z})) = H_n(X; \underline{\mathbb{L}}_0(\mathbb{Z})) \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(X)]) .$$

An <u>(n+1)-ad of chain complexes</u> $\Gamma = \{C, f\}$ is a directed system of chain complexes

$$C = \{C(\alpha) \mid \alpha \subseteq I_n = \{1,2,\ldots,n\}\}$$

(including $\alpha = \emptyset$) and chain maps

$$f = \{f(\alpha,\beta) : C(\beta) \longrightarrow C(\alpha) \mid \beta \subseteq \alpha \subseteq I_n\}$$

such that

$$f(\alpha,\alpha) = 1 \ , \ f(\alpha,\beta)f(\beta,\gamma) = f(\alpha,\gamma) \qquad (\gamma \subseteq \beta \subseteq \alpha) .$$

The <u>homology</u> groups of a chain complex (n+1)-ad Γ are the homology groups

$$H_*(\Gamma) = H_*(C(\Gamma))$$

of the chain complex $C(\Gamma)$ defined by

$$d_{C(\Gamma)} : C(\Gamma)_r = \sum_{\alpha \subseteq I_n} C(\alpha)_{r-n+|\alpha|} \longrightarrow C(\Gamma)_{r-1} ;$$

$$\sum_{\alpha \subseteq I_n} x(\alpha) \longmapsto \sum_{\alpha \subseteq I_n} \Big(d_{C(\alpha)}x(\alpha) + \sum_{i \in \alpha} (-)^{r+|\{j \in \alpha \mid j \leqslant i\}|} f(\alpha, \alpha-\{i\})x(\alpha-\{i\})\Big) .$$

In particular, a 1-ad Γ is just a chain complex $C(\emptyset)$, with $C(\Gamma) = C(\emptyset)$, and a 2-ad Γ is a chain map $f = f(\{1\},\emptyset) : C(\emptyset) \longrightarrow C(\{1\})$, with $C(\Gamma) = C(f)$ the algebraic mapping cone of f.

<u>Proposition 15.1</u> The chain complexes of a CW (n+1)-ad K (in the sense of §0 of Wall [5]) define a chain complex (n+1)-ad $C(K)$ such that the homology groups of $C(K)$ are just those of K.

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Given a chain complex (n+1)-ad $\Gamma = \{C, f\}$ define for each $i \in I_n$ chain complex n-ads $\partial_i\Gamma$, $\delta_i\Gamma$ and for each $i \in I_{n+1}$ an (n+2)-ad $\sigma_i\Gamma$ by

$$\partial_i C(\alpha) = C(\partial_i\alpha) \quad \delta_i C(\alpha) = C(\delta_i\alpha) \quad \sigma_i C(\alpha) = C(\sigma_i\alpha)$$

$$\partial_i f(\alpha,\beta) = f(\partial_i\alpha, \partial_i\beta) \quad \delta_i f(\alpha,\beta) = f(\delta_i\alpha, \delta_i\beta) \quad \sigma_i f(\alpha,\beta) = f(\sigma_i\alpha, \sigma_i\beta)$$

where

$$\partial_i : I_{n-1} \longrightarrow I_n ; \ j \longmapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{cases} , \ \delta_i(\alpha) = \partial_i\alpha \cup \{i\} \quad (i \in I_n, \alpha \subseteq I_{n-1})$$

$$\sigma_i(\alpha) = \partial_i^{-1}(\alpha) \quad (i \in I_{n+1}, \alpha \subseteq I_n) .$$

The abelian group morphisms

$$C(\delta_i\Gamma)_r \longrightarrow C(\Gamma)_r \ ; \ \sum_{\beta\subseteq I_{n-1}} x(\delta_i\beta) \longmapsto \sum_{\beta\subseteq I_{n-1}} (-)^{|\{j\in\beta\,|\,j\geqslant i\}|} x(\delta_i\beta)$$

$$\partial_i : C(\Gamma)_r \longrightarrow SC(\partial_i\Gamma)_r = C(\partial_i\Gamma)_{r-1} \ ; \ \sum_{\alpha\subseteq I_n} x(\alpha) \longmapsto \sum_{\beta\subseteq I_{n-1}} x(\partial_i\beta)$$

define a split short exact sequence of chain complexes and chain maps

$$0 \longrightarrow C(\delta_i\Gamma) \longrightarrow C(\Gamma) \xrightarrow{\partial_i} SC(\partial_i\Gamma) \longrightarrow 0$$

inducing a long exact sequence in the homology groups

$$\ldots \longrightarrow H_r(\partial_i\Gamma) \longrightarrow H_r(\delta_i\Gamma) \longrightarrow H_r(\Gamma) \xrightarrow{\partial_i} H_{r-1}(\partial_i\Gamma) \longrightarrow H_{r-1}(\partial_i\Gamma) \longrightarrow \ldots \ .$$

Define a chain map $\sigma_i : C(\Gamma) \longrightarrow \Omega C(\sigma_i\Gamma)$ by

$$\sigma_i : C(\Gamma)_r \longrightarrow \Omega C(\sigma_i\Gamma)_r = C(\sigma_i\Gamma)_{r+1} \ ; \ \sum_{\alpha\subseteq I_n} x(\alpha) \longmapsto \sum_{\beta\subsetneq I_{n+1}} x(\sigma_i\beta) \ .$$

The abelian group morphisms

$$\partial_i : H_r(\Gamma) \longrightarrow H_{r-1}(\partial_i\Gamma) \ , \ \sigma_i : H_r(\Gamma) \longrightarrow H_{r+1}(\sigma_i\Gamma)$$

satisfy the usual simplicial relations for face and degeneracy maps.

For any chain complex $(n+1)$-ad $\Gamma = \{C,f\}$ we shall denote $C(I_n)$ by $C(|\Gamma|)$, writing $H_*(C(|\Gamma|))$ as $H_*(|\Gamma|)$.

Given an $(n+1)$-ad of A-module chain complexes $\Gamma = \{C,f\}$ define an $(n+1)$-ad of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\Gamma^t\otimes_A\Gamma = \{C^t\otimes_A C, f^t\otimes_A f\}$$

by

$$(C^t\otimes_A C)(\alpha) = C(\alpha)^t\otimes_A C(\alpha) \ , \ (f^t\otimes_A f)(\alpha,\beta) = f(\alpha,\beta)^t\otimes_A f(\alpha,\beta) \ ,$$

with $T\in\mathbb{Z}_2$ acting by the ε-transposition involution T_ε as before.

We thus have abelian group chain complex $(n+1)$-ads $\begin{cases} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\Gamma^t\otimes_A\Gamma) \\ W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\Gamma^t\otimes_A\Gamma) \end{cases}$

and $\begin{cases} \mathbb{Z}_2\text{-hypercohomology} \\ \mathbb{Z}_2\text{-hyperhomology} \end{cases}$ groups

$$\begin{cases} Q^m(\Gamma,\varepsilon) = H_m(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\Gamma^t\otimes_A\Gamma)) \\ Q_m(\Gamma,\varepsilon) = H_m(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\Gamma^t\otimes_A\Gamma)) \ , \end{cases}$$

to be denoted $\begin{cases} Q^m(\Gamma) \\ Q_m(\Gamma) \end{cases}$ for $\varepsilon = 1\in A$. There are defined natural maps

$$Q_m(\Gamma,\varepsilon) \xrightarrow{(1+T_\varepsilon)} Q^m(\Gamma,\varepsilon) \longrightarrow H_m(C(|\Gamma|)^t\otimes_A C(\Gamma))$$

(generalizing the cases $n = 0,1$ considered in §1 above), and there are natural identifications

$$\partial_i(\Gamma^t\otimes_A\Gamma) = \partial_i\Gamma^t\otimes_A\partial_i\Gamma \ , \ \delta_i(\Gamma^t\otimes_A\Gamma) = \delta_i\Gamma^t\otimes_A\delta_i\Gamma \ , \ \sigma_i(\Gamma^t\otimes_A\Gamma) = \sigma_i\Gamma^t\otimes_A\sigma_i\Gamma$$

giving exact sequences

$$\begin{cases} \ldots \longrightarrow Q^m(\partial_i\Gamma,\varepsilon) \longrightarrow Q^m(\delta_i\Gamma,\varepsilon) \longrightarrow Q^m(\Gamma,\varepsilon) \xrightarrow{\partial_i} Q^{m-1}(\partial_i\Gamma,\varepsilon) \longrightarrow \ldots \\ \ldots \longrightarrow Q_m(\partial_i\Gamma,\varepsilon) \longrightarrow Q_m(\delta_i\Gamma,\varepsilon) \longrightarrow Q_m(\Gamma,\varepsilon) \xrightarrow{\partial_i} Q_{m-1}(\partial_i\Gamma,\varepsilon) \longrightarrow \ldots \end{cases}$$

and also morphisms

$$\begin{cases} \sigma_i : Q^m(\Gamma,\varepsilon) \longrightarrow Q^{m+1}(\sigma_i\Gamma,\varepsilon) \\ \sigma_i : Q_m(\Gamma,\varepsilon) \longrightarrow Q_{m+1}(\sigma_i\Gamma,\varepsilon) \ . \end{cases}$$

An <u>m-dimensional</u> $\begin{cases} \underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}} \end{cases}$ <u>(n+1)-ad over A</u> $\begin{cases} (C,\varphi) \\ (C,\psi) \end{cases}$ is an $(n+1)$-ad C of finite-dimensional A-module chain complexes such that

$$\dim C(\alpha) = m - n + |\alpha| \quad (\alpha\subseteq I_n,\ m\in\mathbb{Z},\ C(\alpha) = 0 \text{ if } \dim C(\alpha) < 0)$$

together with a $\begin{cases} \mathbb{Z}_2\text{-hypercohomology} \\ \mathbb{Z}_2\text{-hyperhomology} \end{cases}$ class $\begin{cases} \varphi\in Q^m(C,\varepsilon) \\ \psi\in Q_m(C,\varepsilon) \end{cases}$. Such an $(n+1)$-ad is <u>Poincaré</u> if for each of the 2^n subsets $\alpha\subseteq I_n$ the element of $H_{m-|\alpha|}(C(|\partial_\alpha C|)^t\otimes_A C(\partial_\alpha C))$ determined by $\begin{cases} \partial_\alpha\varphi\in Q^{m-|\alpha|}(\partial_\alpha C,\varepsilon) \\ \partial_\alpha\psi\in Q_{m-|\alpha|}(\partial_\alpha C,\varepsilon) \end{cases}$ induces A-module isomorphisms

$$H^r(|\partial_\alpha C|) \longrightarrow H_{m-|\alpha|-r}(\partial_\alpha C) \quad (0\leqslant r\leqslant m-|\alpha|)$$

via the slant product

$$\backslash : H^r(|\partial_\alpha C|)\otimes_{\mathbb{Z}} H_{m-|\alpha|}(C(|\partial_\alpha C|)^t\otimes_A C(\partial_\alpha C)) \longrightarrow H_{m-|\alpha|-r}(\partial_\alpha C) \ ,$$

where

$$\partial_\alpha = \begin{cases} \partial_{i_p}\partial_{i_{p-1}}\ldots\partial_{i_1} & \text{if } \alpha = \{i_1,i_2,\ldots,i_p\}\subseteq I_n \ , \ 1\leqslant i_1 < i_2 < \ldots < i_p \leqslant n \\ \text{identity} & \text{if } \alpha = \emptyset \ . \end{cases}$$

In particular, an $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré 1-ad (resp. 2-ad, 3-ad) is the same as an $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré complex (resp. pair, triad).

All the results on geometric spaces of §§1 - 11 have $(n+1)$-ad versions, some of which we shall now state.

A CW $(n+1)$-ad $X = \{X(\alpha)\,|\,\alpha \subseteq I_n\}$ is <u>m-dimensional</u> $(m \in \mathbb{Z}, n \geqslant 0)$ if each $X(\alpha)$ $(\alpha \subseteq I_n)$ is a finitely-dominated CW complex of dimension

$$\dim X(\alpha) = m - n + |\alpha| \ ,$$

with $X(\alpha) = \emptyset$ if $m - n + |\alpha| < 0$. (Warning: our convention concerning the dimension of CW $(n+1)$-ads, $\dim X = \dim |X|$, differs from that of §0 of Wall [5], $\dim X = \dim |X| - n$). If $|\widetilde{X}|$ is a covering of $|X| = X(I_n)$ with group of covering translations π then $C(\widetilde{X}) = \{C(\widetilde{X(\alpha)})\,|\,\alpha \subseteq I_n\}$ is an m-dimensional $\mathbb{Z}[\pi]$-module chain complex $(n+1)$-ad.

An <u>m-dimensional geometric Poincaré $(n+1)$-ad</u> X $(m \in \mathbb{Z}, n \geqslant 0)$ is an m-dimensional CW $(n+1)$-ad $\{X(\alpha)\,|\,\alpha \subseteq I_n\}$ together with an orientation map $w : \pi = \pi_1(|X|) \longrightarrow \mathbb{Z}_2$ and a fundamental class $[X] \in H^{\pi}_m(\widetilde{X}; {}^w\mathbb{Z})$ inducing Poincaré duality $\mathbb{Z}[\pi]$-module isomorphisms

$$[X] \cap - \ : \ {}^wH^r_{\pi}(|\widetilde{X}|) \longrightarrow H_{m-r}(\widetilde{X}) \ ,$$

and such that each $\partial_i X$ $(1 \leqslant i \leqslant n)$ is an $(m-1)$-dimensional geometric Poincaré n-ad with compatible orientation map and fundamental class. Then each CW pair $(X(\alpha), \partial X(\alpha) = \bigcup_{\beta \subsetneq \alpha} X(\beta))$ $(\alpha \subseteq I_n)$ is a $(m - n + |\alpha|)$-dimensional geometric Poincaré pair, so that X is a $(m+n)$-dimensional geometric Poincaré $(n+1)$-ad in the sense of §0 of Wall [5].

<u>Proposition 15.2</u> An m-dimensional geometric Poincaré $(n+1)$-ad X $(m \in \mathbb{Z}, n \geqslant 0)$ determines in a natural way an m-dimensional symmetric Poincaré $(n+1)$-ad over $\mathbb{Z}[\pi]$

$$\sigma^*(X) = (C(\widetilde{X}), \Phi \in Q^m(C(\widetilde{X}))) \qquad (\pi = \pi_1(|X|)) \ .$$

A degree 1 map of m-dimensional geometric Poincaré $(n+1)$-ads $f : M \longrightarrow X$ has a kernel m-dimensional symmetric Poincaré $(n+1)$-ad over $\mathbb{Z}[\pi]$, the <u>symmetric kernel</u> $\sigma^*(f)$, such that up to homotopy equivalence of $(n+1)$-ads

$$\sigma^*(M) = \sigma^*(f) \oplus \sigma^*(X) \ .$$

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Given a spherical fibration $p : |X| \longrightarrow BG(k)$ over the total space of a CW $(n+1)$-ad X write the restrictions of p to $X(\alpha)$ $(\alpha \subseteq I_n)$ as

$$p(\alpha) \ : \ X(\alpha) \hookrightarrow |X| \xrightarrow{\ p\ } BG(k) \ ,$$

so that the Thom spaces define a CW $(n+1)$-ad $T(\alpha) = \{T(p(\alpha))\,|\,\alpha \subseteq I_n\}$.

Regard the standard n-simplex Δ^n as an $(n+2)$-ad with $\Delta^n(\alpha)$ the face of Δ^n spanned by the vertices $\alpha \subseteq I_{n+1}$. For $N \geqslant 0$ let $\Delta^n \times D^N$ be the $(n+2)$-ad defined by

$$(\Delta^n \times D^N)(\alpha) = \Delta^n(\alpha) \times D^N \cup \Delta^n \times S^{N-1} \qquad (\alpha \subseteq I_{n+1}) \ .$$

Write $\Delta^{-1} \times D^N$ for the 1-ad (= space) defined by $(\Delta^{-1} \times D^N)(\emptyset) = S^{N-1}$.

A geometric Poincaré $(n+1)$-ad X can be embedded as an $(n+1)$-ad in $\Delta^{n-1} \times D^{k+m+1}$ for sufficiently large $k \geqslant 0$, at least for finite X, with a closed regular neighbourhood E such that each $E(\alpha) = E \cap (\Delta^{n-1} \times D^{k+m+1})(\alpha)$ is a regular neighbourhood of $X(\alpha)$ in $(\Delta^{n-1} \times D^{k+m+1})(\alpha)$. The inclusions $\partial E(\alpha) \subset E(\alpha)$ define $(k-1)$-spherical fibrations

$$S^{k-1} \longrightarrow \partial E(\alpha) \xrightarrow{\ \nu_{X(\alpha)}\ } E(\alpha) = X(\alpha) \ ,$$

and the collapsing map of $(n+1)$-ads

$$\rho_X \ : \ \Delta^{n-1} \times D^{k+m+1} \longrightarrow \Delta^{n-1} \times D^{k+m+1} / \overline{\Delta^{n-1} \times D^{k+m+1} - E} = E/\partial E = T(\nu_X)$$

is of degree 1. The composite

$$\alpha_X \ : \ \Delta^{n-1} \times D^{k+m+1} \xrightarrow{\ \rho_X\ } T(\nu_X) \xrightarrow{\ \Delta\ } \widetilde{X}_+ \wedge_{\pi} T\pi(\nu_X)$$

is the prototype of an $S\pi$-duality map between $(n+1)$-ads of π-spaces.

An <u>m-dimensional normal $(n+1)$-ad</u> (X, ν_X, ρ_X) is an m-dimensional CW $(n+1)$-ad $X = \{X(\alpha)\,|\,\alpha \subseteq I_n\}$ together with a $(k-1)$-spherical fibration $\nu_X : |X| \longrightarrow BG(k)$ and a map of $(n+1)$-ads $\rho_X : \Delta^{n-1} \times D^{k+m+1} \longrightarrow T(\nu_X)$. Call $[X] = h(\rho_X) \cap U_{\nu_X} \in H^{\pi}_{m+n}(\widetilde{X}; {}^w\mathbb{Z})$ $(w = w_1(\nu_X) : \pi = \pi_1(|X|) \longrightarrow \mathbb{Z}_2)$ the <u>fundamental class</u>, and $\alpha_X : \Delta^{n-1} \times D^{k+m+1} \xrightarrow{\ \rho_X\ } T(\nu_X) \xrightarrow{\ \Delta\ } \widetilde{X}_+ \wedge_{\pi} T\pi(\nu_X)$ the <u>fundamental map</u>.

<u>Proposition 15.3</u> i) An m-dimensional geometric Poincaré $(n+1)$-ad X carries a canonical equivalence class of normal structures (ν_X, ρ_X) with a fundamental $S\pi$-duality map α_X.

ii) A normal map $(f, b) : M \longrightarrow X$ of m-dimensional geometric Poincaré $(n+1)$-ads determines in a natural way an m-dimensional quadratic Poincaré $(n+1)$-ad over $\mathbb{Z}[\pi_1(|X|)]$, the <u>quadratic kernel</u> $\sigma_*(f, b)$, such that

$$(1+T)\sigma_*(f, b) = \sigma^*(f) \ .$$

iii) An m-dimensional normal (n+1)-ad X determines in a natural way a pair

$\hat{\sigma}^*(X)$ = ((m-1)-dimensional quadratic Poincaré n-ad over $\mathbb{Z}[\pi_1(|X|)]$ x,

m-dimensional symmetric Poincaré (n+1)-ad y over $\mathbb{Z}[\pi_1(|X|)]$ y)

such that $\partial_{n+2}y = (1+T)x$, $\partial_{n+1}\partial_n\dots\partial_1 y = 0$. If X is Poincaré then $\hat{\sigma}^*(X) = (0,\sigma^*(X))$.

[]

We now define our classifying spaces, as simplicial sets of algebraic Poincaré (n+1)-ads. Let $\begin{cases}\mathbb{L}^m(A,\varepsilon)\\ \mathbb{L}_m(A,\varepsilon)\end{cases}$ $(m\in\mathbb{Z})$ be the simplicial monoid with n-simplexes the (m+n)-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré (n+2)-ads over A $\begin{cases}(C,\varphi\in Q^{m+n}(C,\varepsilon))\\ (C,\psi\in Q_{m+n}(C,\varepsilon))\end{cases}$ such that $C(\emptyset) = 0$, with the direct sum $\oplus$ as monoid operation and 0 as the base simplex in each dimension. The face operations are given by

$$\begin{cases}\partial_i(C,\varphi\in Q^{m+n}(C,\varepsilon)) = (\partial_i C,\partial_i\varphi\in Q^{m+n-1}(\partial_i C,\varepsilon))\\ \partial_i(C,\psi\in Q_{m+n}(C,\varepsilon)) = (\partial_i C,\partial_i\psi\in Q_{m+n-1}(\partial_i C,\varepsilon))\end{cases}\quad(1\leqslant i\leqslant n+1)$$

and the degeneracies by

$$\begin{cases}\sigma_i(C,\varphi\in Q^{m+n}(C,\varepsilon)) = (\sigma_i C,\sigma_i\varphi\in Q^{m+n+1}(\sigma_i C,\varepsilon))\\ \sigma_i(C,\psi\in Q_{m+n}(C,\varepsilon)) = (\sigma_i C,\sigma_i\psi\in Q_{m+n+1}(\sigma_i C,\varepsilon))\end{cases}\quad(1\leqslant i\leqslant n+1)\ .$$

(In Proposition 15.5 below we shall identify $\mathbb{L}_0(\mathbb{Z},1) = \mathbb{Z}\times G/TOP$).

The mapping fibre of the ε-symmetrization map

$$1+T_\varepsilon : \mathbb{L}_m(A,\varepsilon)\longrightarrow\mathbb{L}^m(A,\varepsilon)\ ;\ (C,\psi)\longmapsto(C,(1+T_\varepsilon)\psi)$$

will be denoted by $\hat{\mathbb{L}}^{m+1}(A,\varepsilon)$ $(m\in\mathbb{Z})$. An n-simplex of $\hat{\mathbb{L}}^{m+1}(A,\varepsilon)$ is thus a pair

(an n-simplex $(C,\psi\in Q_{m+n}(C,\varepsilon))$ of $\mathbb{L}_m(A,\varepsilon)$,

an (n+1)-simplex $(D,\nu\in Q^{m+n+1}(D,\varepsilon))$ of $\mathbb{L}^m(A,\varepsilon)$)

such that $\partial_{n+2}(D,\nu) = (1+T_\varepsilon)(C,\psi)$, $\partial_{n+1}\partial_n\dots\partial_1(D,\nu) = 0$. For $\varepsilon = 1\in A$ we shall write

$$\begin{cases}\mathbb{L}^m(A,1) = \mathbb{L}^m(A)\\ \mathbb{L}_m(A,1) = \mathbb{L}_m(A)\ .\\ \hat{\mathbb{L}}^m(A,1) = \hat{\mathbb{L}}^m(A)\end{cases}$$

Proposition 15.4 The simplicial monoids $\begin{cases}\mathbb{L}^m(A,\varepsilon)\\ \mathbb{L}_m(A,\varepsilon)\\ \hat{\mathbb{L}}^m(A,\varepsilon)\end{cases}$ $(m\in\mathbb{Z})$ satisfy the Kan extension condition, and are such that

$$\begin{cases}\mathbb{L}^{m+1}(A,\varepsilon) = \Omega\mathbb{L}^m(A,\varepsilon)\\ \mathbb{L}_{m+1}(A,\varepsilon) = \Omega\mathbb{L}_m(A,\varepsilon)\\ \hat{\mathbb{L}}^{m+1}(A,\varepsilon) = \Omega\hat{\mathbb{L}}^m(A,\varepsilon)\end{cases}$$

up to canonical isomorphism of H-spaces. There is defined a long fibration sequence

$$\dots\longrightarrow\hat{\mathbb{L}}^{m+1}(A,\varepsilon)\xrightarrow{H}\mathbb{L}_m(A,\varepsilon)\xrightarrow{1+T_\varepsilon}\mathbb{L}^m(A,\varepsilon)\xrightarrow{J}\hat{\mathbb{L}}^m(A,\varepsilon)\longrightarrow\dots\ ,$$

and the homotopy groups are given by

$$\pi_n(\mathbb{L}^m(A,\varepsilon)) = \begin{cases}L^{m+n}(A,\varepsilon)\\ 0\end{cases},\ \pi_n(\mathbb{L}_m(A,\varepsilon)) = \begin{cases}L_{m+n}(A,\varepsilon)\\ 0\end{cases}\ \text{if}\begin{cases}m+n\geqslant 0\\ m+n\leqslant -1\end{cases},$$

$$\pi_n(\hat{\mathbb{L}}^m(A,\varepsilon)) = \begin{cases}\hat{L}^{m+n}(A,\varepsilon)\\ \operatorname{coker}((1+T_\varepsilon):L_0(A,\varepsilon)\longrightarrow L^0(A,\varepsilon))\\ 0\end{cases}\ \text{if}\begin{cases}m+n\geqslant 1\\ m+n=0\\ m+n\leqslant -1\end{cases}.$$

Proof: A 0-simplex of $\begin{cases}\mathbb{L}^m(A,\varepsilon)\\ \mathbb{L}_m(A,\varepsilon)\end{cases}$ is an m-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré complex over A, and a 1-simplex is a cobordism of its faces. Thus for $m<0$ $\begin{cases}\mathbb{L}^m(A,\varepsilon)\\ \mathbb{L}_m(A,\varepsilon)\end{cases}$ is connected, while for $m\geqslant 0$ the set of path-components is the cobordism group

$$\begin{cases}\pi_0(\mathbb{L}^m(A,\varepsilon)) = L^m(A,\varepsilon)\\ \pi_0(\mathbb{L}_m(A,\varepsilon)) = L_m(A,\varepsilon)\ .\end{cases}$$

The verification of the Kan extension condition requires a generalization to algebraic Poincaré n-ads of the union construction for cobordisms (= 3-ads) used in the proof of Proposition 5.2. An n-simplex of $\Omega\mathbb{L}^m(A,\varepsilon)$ (= mapping fibre of $0:0\longrightarrow\mathbb{L}^m(A,\varepsilon)$) is an (n+1)-simplex x of $\mathbb{L}^m(A,\varepsilon)$ such that $\partial_{n+2}x = 0$, $\partial_{n+1}\partial_n\dots\partial_1 x = 0$, which is the same as an n-simplex of $\mathbb{L}^{m+1}(A,\varepsilon)$ up to labelling. This identifies $\Omega\mathbb{L}^m(A,\varepsilon) = \mathbb{L}^{m+1}(A,\varepsilon)$ as simplicial sets. The H-space structure on $\Omega\mathbb{L}^m(A,\varepsilon)$ agrees with the monoid operation $\oplus$ in $\mathbb{L}^{m+1}(A,\varepsilon)$, for if x,x' are (n+1)-simplexes of $\mathbb{L}^m(A,\varepsilon)$ such that $\partial_{n+2}x = 0$, $\partial_{n+2}x' = 0$, $\partial_{n+1}\partial_n\dots\partial_1 x = 0$, $\partial_{n+1}\partial_n\dots\partial_1 x' = 0$ then $x'' = \sigma_1 x\oplus\sigma_2 x'$ is an (n+2)-simplex of $\mathbb{L}^m(A,\varepsilon)$ such that

$$\partial_i x'' = \begin{cases} x & i = 1 \\ x\otimes x' & i = 2 \\ x' & i = 3 \\ 0 & i \geq 4 \end{cases} .$$

The higher homotopy groups of $\mathbb{L}^m(A,\varepsilon)$ are thus given by

$$\pi_n(\mathbb{L}^m(A,\varepsilon)) = \pi_0(\Omega^n\mathbb{L}^m(A,\varepsilon)) = \pi_0(\mathbb{L}^{m+n}(A,\varepsilon)) = \begin{cases} L^{m+n}(A,\varepsilon) \\ 0 \end{cases} \text{if} \begin{cases} m+n \geq 0 \\ m+n < 0 \end{cases} .$$

Similarly for $\mathbb{L}_m(A,\varepsilon)$, $\hat{\mathbb{L}}^m(A,\varepsilon)$.

[]

Product with the generator $\bar{S} = 1 \in L^2(\mathbb{Z},-1) = \mathbb{Z}$ defines a skew-suspension map in the $\pm\varepsilon$-quadratic L-spaces

$$\bar{S} : \mathbb{L}_m(A,\varepsilon) \longrightarrow \mathbb{L}_{m+2}(A,-\varepsilon)$$

inducing the skew-suspension isomorphisms in the homotopy groups

$$\bar{S} : \pi_n(\mathbb{L}_m(A,\varepsilon)) = L_{m+n}(A,\varepsilon) \longrightarrow \pi_n(\mathbb{L}_{m+2}(A,-\varepsilon)) = L_{m+n+2}(A,-\varepsilon) \quad (m+n \geq 0)$$

(cf. Proposition 7.3). Thus $\bar{S}:\mathbb{L}_m(A,\varepsilon) \longrightarrow \mathbb{L}_{m+2}(A,-\varepsilon)$ is a homotopy equivalence for $m \geq 0$, by the simplicial Whitehead theorem. The double skew-suspension map

$$\bar{S}^2 : \mathbb{L}_m(A,\varepsilon) \longrightarrow \mathbb{L}_{m+4}(A,\varepsilon)$$

is a homotopy equivalence for $m \geq 0$ (a fortiori), allowing the identifications

$$\mathbb{L}_m(A,\varepsilon) = \begin{cases} \Omega^{m-4j}\mathbb{L}_0(A,\varepsilon) \\ \Omega^{m+4j}(\mathbb{L}_0(A,\varepsilon)(4j,4j+1,\ldots,\infty)) \end{cases} \text{if} \begin{cases} m \geq 0,\ 4j \leq m \\ m \leq 0,\ 4j \geq -m \end{cases}$$

with $\mathbb{L}(4j,4j+1,\ldots,\infty)$ the universal $(4j-1)$-connected space over $\mathbb{L}$. Similarly for the ε-symmetric and ε-hyperquadratic $\mathbb{L}$-spaces $\mathbb{L}^m(A,\varepsilon)$, $\hat{\mathbb{L}}^m(A,\varepsilon)$ of a ~~0-hereditary~~ 1-dimensional ring A (cf. Proposition 7.4). The deformation retract of $\mathbb{L}_m(A)$ $(m \geq 0)$ consisting of the highly-connected quadratic Poincaré $(n+1)$-ads was obtained in Ranicki [5] by a direct construction involving $\pm$quadratic forms and formations. The space $\mathbb{L}^0(A)$ has also been constructed by Mishchenko and Solov'ev [1].(cf. §10 of Mishchenko [3]).

Classifying spaces for normal bundle maps (e.g. G/PL) were constructed geometrically by Sullivan [1] and Casson in the simply-connected case, and then by Quinn [1] in general (cf. Rourke [1], §17A of Wall [5]). Given a space K and a group morphism $w:\pi_1(K) \longrightarrow \mathbb{Z}_2$ define Δ-sets (= simplicial sets without degeneracies) $\mathbb{L}_m(K)$ $(m\in\mathbb{Z})$ with n-simplexes the $(m+n)$-dimensional topological normal bundle maps of $(n+4)$-ads $(f,b):M \longrightarrow X$ with a reference map $|X| \longrightarrow K$, such that $\partial_{n+3}f$, $\partial_{n+2}f$, $\partial_{n+1}\partial_n\ldots\partial_1 f$ are homotopy equivalences and $M(\emptyset) = X(\emptyset) = \emptyset$ - we have modified the definition of the spaces $\mathbb{L}_m(K)$ appearing in Quinn [1] to ensure the characteristic properties $\Omega\mathbb{L}_m(K) = \mathbb{L}_{m+1}(K)$, $\pi_n(\mathbb{L}_m(K)) = L_{m+n}(\mathbb{Z}[\pi_1(K)])$. In particular, $\mathbb{L}_0(pt.) = L_0(\mathbb{Z})\times G/TOP$.

Proposition 15.5 The quadratic kernel construction defines homotopy equivalences of Kan Δ-sets

$$\sigma_* : \mathbb{L}_m(K) \longrightarrow \mathbb{L}_m(\mathbb{Z}[\pi_1(K)]) ; \ (f,b) \longmapsto \sigma_*(f,b) .$$

In particular, there is defined a homotopy equivalence

$$\sigma_* : \mathbb{L}_0(pt.) = L_0(\mathbb{Z})\times G/TOP \longrightarrow \mathbb{L}_0(\mathbb{Z}) .$$

[]

Quinn [1] also defines an "assembly" map

$$A : [X_+,\mathbb{L}_m(K)] \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(K\times X)])$$

for any triangulated topological n-manifold X, which sends a simplicial map $f:X \longrightarrow \mathbb{L}_m(K)$ to the quadratic kernel induced by a reference map $Y \longrightarrow K\times X$ from the topological normal bundle map of 1-ads $(f,b):M \longrightarrow Y$ obtained by glueing together the normal maps classified by $f|_{\Delta^k\subset X}$. In particular, the surgery obstruction map is the composite

$$\theta : [X_+,G/TOP] \hookrightarrow [X_+,\mathbb{Z}\times G/TOP] = [X_+,\mathbb{L}_0(\mathbb{Z})] \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) ,$$

which we wish to describe algebraically. In the first instance we define maps $\sigma^*:MSPL(k) \longrightarrow \mathbb{L}^{-k}(\mathbb{Z})$ inducing the symmetric signatures on PL cobordism $\sigma^*:\Omega_n^{PL} = \varinjlim_k \pi_{n+k}(MSPL(k)) \longrightarrow L^n(\mathbb{Z})$, and which behave well with respect to the multiplicative structure in the Thom spectrum MSPL - the multiplicative structures in the algebraic L-spaces are defined as follows.

The smash product of based simplicial sets X,Y is the based simplicial set $X\wedge Y$ defined by

$$(X\wedge Y)^{(n)} = X^{(n)}\times Y^{(n)}/X^{(n)}\times 0\cup 0\times Y^{(n)}$$

$$\partial_i(x,y) = (\partial_i x,\partial_i y)\ ,\ \sigma_i(x,y) = (\sigma_i x,\sigma_i y)\quad (x\in X^{(n)},\ y\in Y^{(n)})\ ,$$

and there is defined a natural pairing in the homotopy groups

$$\pi_m(X)\otimes_{\mathbb{Z}}\pi_n(Y) \longrightarrow \pi_{m+n}(X\wedge Y)\ .$$

<u>Proposition 15.6</u> The L-group products of §11 are induced by simplicial maps

$$\begin{aligned}
\otimes &: \mathbb{L}^m(A,\varepsilon)\wedge\mathbb{L}^n(B,\eta)\longrightarrow \mathbb{L}^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\\
\otimes &: \mathbb{L}^m(A,\varepsilon)\wedge\mathbb{L}_n(B,\eta)\longrightarrow \mathbb{L}_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\qquad (m,n\in\mathbb{Z})\\
\otimes &: \mathbb{L}_m(A,\varepsilon)\wedge\mathbb{L}_n(B,\eta)\longrightarrow \mathbb{L}_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\\
\otimes &: \hat{\mathbb{L}}^m(A,\varepsilon)\wedge\hat{\mathbb{L}}^n(B,\eta)\longrightarrow \hat{\mathbb{L}}^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)\ .
\end{aligned}$$

<u>Proof</u>: Unfortunately, this requires bisimplicial sets.

Define a <u>chain complex $(m+1,n+1)$-ad</u> $\Gamma = \{C,f\}$ to be a directed system of chain complexes

$$C = \{C(\alpha,\beta)\,|\,\alpha\subseteq I_m,\beta\subseteq I_n\}$$

and chain maps

$$f = \{f(\alpha,\alpha';\beta,\beta'):C(\alpha,\beta)\longrightarrow C(\alpha',\beta')\,|\,\alpha\subseteq\alpha'\subseteq I_m,\beta\subseteq\beta'\subseteq I_n\}\ .$$

The <u>homology</u> groups of Γ are defined to be

$$H_*(\Gamma) = H_*(C(\Gamma))$$

with $C(\Gamma)$ the chain complex given by

$$d_{C(\Gamma)} : C(\Gamma)_r = \sum_{\alpha\subseteq I_m,\beta\subseteq I_n} C(\alpha,\beta)_{r-m-n+|\alpha|+|\beta|}\longrightarrow C(\Gamma)_{r-1}\ ;$$

$$\sum_{\alpha,\beta} x(\alpha,\beta)\longrightarrow \sum_{\alpha,\beta}\Big(d_{C(\alpha,\beta)}x(\alpha,\beta) + \sum_{i\in\alpha}\pm f(\alpha,\alpha-\{i\};\beta,\beta)x(\alpha-\{i\},\beta) + \sum_{j\in\beta}\pm f(\alpha,\alpha;\beta,\beta-\{j\})x(\alpha,\beta-\{j\})\Big)\ .$$

In particular, the tensor product of a chain complex $(m+1)$-ad $\Gamma = \{C,f\}$ and a chain complex $(n+1)$-ad $\Delta = \{D,g\}$ is a chain complex $(m+1,n+1)$-ad

$$\Gamma\otimes\Delta = \{C\otimes D,f\otimes g\}$$

with

$$(C\otimes D)(\alpha,\beta) = C(\alpha)\otimes D(\beta)\ ,\ (f\otimes g)(\alpha,\alpha';\beta,\beta') = f(\alpha,\alpha')\otimes g(\beta,\beta')\ .$$

A k-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré $(m+1,n+1)$-ad over A $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ is an $(m+1,n+1)$-ad $\{C,f\}$ of finite-dimensional A-module chain complexes such that

$$\dim C(\alpha,\beta) = k-m-n+|\alpha|+|\beta|\qquad (\alpha\subseteq I_m,\beta\subseteq I_n,k\in\mathbb{Z})$$

together with a $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\end{cases}$ class

$$\begin{cases}\varphi\in Q^k(C,\varepsilon) = H_k(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,C^t\otimes_A C))\\ \psi\in Q_k(C,\varepsilon) = H_k(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C^t\otimes_A C))\end{cases}$$

giving rise to a total of 2^{m+n} Poincaré duality isomorphisms of the type

$$H^r(|\partial_{\alpha,\beta}C|)\longrightarrow H_{k-|\alpha|-|\beta|-r-1}(\partial_{\alpha,\beta}C)\quad (r\in\mathbb{Z},\alpha\subseteq I_m,\beta\subseteq I_n)\ .$$

Here $\partial_{\alpha,\beta}C$ is the $(m-|\alpha|+1,n-|\beta|+1)$-ad with $\partial_{\alpha,\beta}C(\lambda,\mu) = C(\partial_\alpha\lambda,\partial_\beta\mu)$, $|\partial_{\alpha,\beta}C| = \partial_{\alpha,\beta}C(I_{m-|\alpha|},I_{n-|\beta|}) = C(I_m\backslash\alpha,I_n\backslash\beta)$, and $C^t\otimes_A C$ is the $(m+1,n+1)$-ad of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes with $(C^t\otimes_A C)(\alpha,\beta) = C(\alpha,\beta)^t\otimes_A C(\alpha,\beta)$.

Let $\begin{cases}\mathbb{L}^j(A,\varepsilon)^{**}\\ \mathbb{L}_j(A,\varepsilon)^{**}\end{cases}$ be the bisimplicial set with (m,n)-simplexes the $(j+m+n)$-dimensional $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ Poincaré $(m+2,n+2)$-ads over A $\begin{cases}(C,\varphi)\\(C,\psi)\end{cases}$ with $C(\alpha,\emptyset) = C(\emptyset,\beta) = 0$ $(\alpha\subseteq I_{m+1},\beta\subseteq I_{n+1})$. Let $\begin{cases}\mathbb{L}^j(A,\varepsilon)^*\\ \mathbb{L}_j(A,\varepsilon)^*\end{cases}$ be the diagonal simplicial set of $\begin{cases}\mathbb{L}^j(A,\varepsilon)^{**}\\ \mathbb{L}_j(A,\varepsilon)^{**}\end{cases}$, with $\begin{cases}\mathbb{L}^j(A,\varepsilon)^{(n)} = \mathbb{L}^j(A,\varepsilon)^{(n,n)}\\ \mathbb{L}_j(A,\varepsilon)^{(n)} = \mathbb{L}_j(A,\varepsilon)^{(n,n)}\end{cases}$. The face and degeneracy maps of $\begin{cases}\mathbb{L}^j(A,\varepsilon)^{**}\\ \mathbb{L}_j(A,\varepsilon)^{**}\end{cases}$ are homotopy equivalences $\begin{cases}\mathbb{L}^j(A,\varepsilon)^{(n,*)}\rightleftarrows\mathbb{L}^j(A,\varepsilon)^{(n-1,*)}\\ \mathbb{L}_j(A,\varepsilon)^{(n,*)}\rightleftarrows\mathbb{L}_j(A,\varepsilon)^{(n-1,*)}\end{cases}$ in each degree, so that $\begin{cases}\mathbb{L}^j(A,\varepsilon)^*\\ \mathbb{L}_j(A,\varepsilon)^*\end{cases}$ is homotopy equivalent to

$$\begin{cases}\mathbb{L}^j(A,\varepsilon)^{(0,*)} = \mathbb{L}^j(A,\varepsilon)\\ \mathbb{L}_j(A,\varepsilon)^{(0,*)} = \mathbb{L}_j(A,\varepsilon)\ .\end{cases}$$

The tensor product of a $(j+m)$-dimensional ε-symmetric Poincaré $(m+2)$-ad (C,φ) over A and a $(k+n)$-dimensional η-symmetric Poincaré $(n+2)$-ad (D,ν) over B is a $(j+k+m+n)$-dimensional $\varepsilon\otimes\eta$-symmetric Poincaré $(m+2,n+2)$-ad over $A\otimes_{\mathbb{Z}}B$

$$(C,\varphi)\otimes_{\mathbb{Z}}(D,\nu) = (C\otimes_{\mathbb{Z}}D,\varphi\otimes\nu)\ ,$$

thus defining a bisimplicial map

$$\otimes : \mathbb{L}^j(A,\varepsilon)\times\mathbb{L}^k(B,\eta)/\mathbb{L}^j(A,\varepsilon)\times 0\cup 0\times\mathbb{L}^k(B,\eta) \longrightarrow \mathbb{L}^{j+k}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)^{**}\ .$$

Restriction to the diagonal defines a simplicial map

$$\otimes : \mathbb{L}^j(A,\varepsilon)\wedge\mathbb{L}^k(B,\eta) \longrightarrow \mathbb{L}^{j+k}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)^{*} = \mathbb{L}^{j+k}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)$$

inducing the tensor product in the symmetric L-groups. Similarly for the other products.

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Define an <u>oriented $(G(k),PL(k))$-bundle</u> over a CW pair (X,Y) to be an oriented $G(k)$-bundle (= $(k-1)$-spherical fibration) $\alpha : X \longrightarrow BSG(k)$ over X together with an oriented $PL(k)$-bundle $\beta : Y \longrightarrow BSPL(k)$ over Y such that the restriction of α to Y is $\alpha|_Y : Y \xrightarrow{\beta} BSPL(k) \xrightarrow{J} BSG(k)$. Such objects were first systematically considered in the context of surgery by Sullivan [1], although the case $(X,Y) = (D^N,S^{N-1})$ goes back to Milnor [3]. Oriented $(G(k),PL(k))$-bundles over (X,Y) are classified by the relative homotopy classes of maps of pairs $(\alpha,\beta) : (X,Y) \longrightarrow (BSG(k),BSPL(k))$. The <u>Thom space</u> of (α,β) is

$$T(\alpha,\beta) = T(\alpha)/T(\beta)$$

so that there is a cofibration sequence $T(\beta)\longrightarrow T(\alpha)\longrightarrow T(\alpha,\beta)\longrightarrow \Sigma T(\beta)\longrightarrow \Sigma T(\alpha)\longrightarrow \dots$.

Given $\begin{cases}(\alpha,\beta) : (X,Y)\longrightarrow(BSG(k),BSPL(k))\\ \beta : Y\longrightarrow BSPL(k)\\ \alpha : X\longrightarrow BSG(k)\end{cases}$ let $\begin{cases}\Omega^{N,PL}_{n+1}(X,Y;\alpha,\beta)\\ \Omega^{PL}_n(Y,\beta)\\ \Omega^N_n(X,\alpha)\end{cases}$ be the bordism groups

of maps $\begin{cases}(f,g) : (M,N)\to(X,Y)\\ g : N\longrightarrow Y\\ f : M\longrightarrow X\end{cases}$ from $\begin{cases}\text{finite } (n+1)\text{-dimensional normal pairs } (M,N) \text{ with}\\ \text{compact } n\text{-dimensional PL-manifolds } N\\ \text{finite } n\text{-dimensional normal spaces } M\end{cases}$

$\begin{cases}\text{PL-manifold boundary } N\\ -\\ -\end{cases}$ such that $\begin{cases}(\nu_M,\nu_N) = (f^*\alpha,g^*\beta) : (M,N)\longrightarrow(BSG,BSPL)\\ \nu_N = g^*\beta : N\longrightarrow BSPL\\ \nu_M = f^*\alpha : M\longrightarrow BSG\end{cases}$

The standard transversality argument proves that the Pontrjagin-Thom maps

$$\begin{cases}\Omega^{N,PL}_{n+1}(X,Y;\alpha,\beta)\longrightarrow\pi^S_{n+k+1}(T(\alpha,\beta))\ ;\\ \qquad((f,g):(M^{n+1},N^n)\longrightarrow(X,Y))\longmapsto(S^{n+k+1}\xrightarrow{\rho_{M,N}}T(\nu_M,\nu_N)\xrightarrow{T(f,g)}T(\alpha,\beta))\\ \Omega^{PL}_n(Y,\beta)\longrightarrow\pi^S_{n+k}(T(\beta))\ ;\ (g:N^n\longrightarrow Y)\longmapsto(S^{n+k}\xrightarrow{\rho_N}T(\nu_N)\xrightarrow{T(g)}T(\beta))\\ \Omega^N_n(X,\alpha)\longrightarrow\pi^S_{n+k}(T(\alpha))\ ;\ (f:M^n\longrightarrow X)\longmapsto(S^{n+k}\xrightarrow{\rho_M}T(\nu_M)\xrightarrow{T(f)}T(\alpha))\end{cases}$$

are isomorphisms. There is defined a long exact sequence

$$\dots\longrightarrow\Omega^N_{n+1}(X,\alpha)\longrightarrow\Omega^{N,PL}_{n+1}(X,Y;\alpha,\beta)\longrightarrow\Omega^{PL}_n(Y,\beta)\longrightarrow\Omega^N_n(X,\alpha)\longrightarrow\dots\ .$$

The bordism groups considered at the end of §10 are given by

$$\begin{cases}\Omega^{N,PL}_{n+1}(X) = \varinjlim_k\Omega^{N,PL}_{n+1}(X\times BSG(k),X\times BSPL(k);\alpha,\beta) = \varinjlim_k\pi_{n+k+1}(X_+\wedge MSG(k)/MSPL(k))\\ \Omega^{PL}_n(X) = \varinjlim_k\Omega^{PL}_n(X\times BSPL(k),\beta) = \varinjlim_k\pi_{n+k}(X_+\wedge MSPL(k))\\ \Omega^N_n(X) = \varinjlim_k\Omega^N_n(X\times BSG(k),\alpha) = \varinjlim_k\pi_{n+k}(X_+\wedge MSG(k))\end{cases}$$

with (α,β) = projection : $X\times(BSG(k),BSPL(k))\longrightarrow(BSG(k),BSPL(k))$.

<u>Proposition 15.7</u> Let $\begin{cases}(\alpha,\beta) : (X,Y)\longrightarrow(BSG(k),BSPL(k))\\ \beta : Y\longrightarrow BSPL(k)\\ \alpha : X\longrightarrow BSG(k)\end{cases}$ be an oriented

$\begin{cases}(G(k),PL(k))\text{-}\\ PL(k)\text{-}\\ G(k)\text{-}\end{cases}$ bundle over a connected CW $\begin{cases}\text{pair } (X,Y)\\ \text{complex } Y\\ \text{complex } X\end{cases}$, and let $\begin{cases}\pi_1(X)\longrightarrow\pi\\ \pi_1(Y)\longrightarrow\pi\\ \pi_1(X)\longrightarrow\pi\end{cases}$

be a group morphism. The $\begin{cases}\text{quadratic}\\ \text{symmetric}\\ \text{hyperquadratic}\end{cases}$ signatures

$$\begin{cases}\sigma_* : \Omega^{N,PL}_{n+1}(X,Y;\alpha,\beta) = \pi^S_{n+k+1}(T(\alpha,\beta))\longrightarrow L_n(\mathbb{Z}[\pi])\\ \sigma^* : \Omega^{PL}_n(Y,\beta) = \pi^S_{n+k}(T(\beta))\longrightarrow L^n(\mathbb{Z}[\pi])\\ \hat{\sigma}^* : \Omega^N_n(X,\alpha) = \pi^S_{n+k}(T(\alpha))\longrightarrow \hat{L}^n(\mathbb{Z}[\pi])\end{cases}$$

are induced by natural simplicial maps

$$\begin{cases}\sigma_* : T(\alpha,\beta)\longrightarrow\mathbb{L}_{-k-1}(\mathbb{Z}[\pi])\\ \sigma^* : T(\beta)\longrightarrow\mathbb{L}^{-k}(\mathbb{Z}[\pi])\\ \hat{\sigma}^* : T(\alpha)\longrightarrow\hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi])\end{cases}$$

which fit into a commutative diagram

$$\begin{array}{ccccc} T(\beta) & \longrightarrow & T(\alpha) & \longrightarrow & T(\alpha,\beta) \\ \downarrow \sigma^* & & \downarrow \hat{\sigma}^* & & \downarrow \sigma_* \\ \mathbb{L}^{-k}(\mathbb{Z}[\pi]) & \xrightarrow{J} & \hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi]) & \xrightarrow{H} & \mathbb{L}_{-k-1}(\mathbb{Z}[\pi]) \end{array}$$

The simplicial $\left\{\begin{array}{l}\text{quadratic}\\ \text{symmetric}\\ \text{hyperquadratic}\end{array}\right.$ signature map on the Thom space of the cartesian product

$\left\{\begin{array}{l}(\alpha\times\alpha',\beta\times\beta'):(X\times Y',Y\times Y')\longrightarrow(BSG(k+k'),BSPL(k+k'))\\ \beta\times\beta':Y\times Y'\longrightarrow BSPL(k+k')\\ \alpha\times\alpha':X\times X'\longrightarrow BSG(k+k')\end{array}\right.$ with $\left\{\begin{array}{l}\beta':Y'\longrightarrow BSPL(k')\\ \beta':Y'\longrightarrow BSPL(k')\\ \alpha':X'\longrightarrow BSG(k'),\end{array}\right.$

is such that there are defined commutative diagrams

$$\begin{array}{ccc} T(\alpha,\beta)\wedge T(\beta') & \xrightarrow{\sigma_*\wedge\sigma^*} & \mathbb{L}_{-k-1}(\mathbb{Z}[\pi])\wedge\mathbb{L}^{-k'}(\mathbb{Z}[\pi']) \\ \downarrow\otimes & & \downarrow\otimes \\ T(\alpha\times\beta',\beta\times\beta') & \xrightarrow{\sigma_*} & \mathbb{L}_{-k-k'-1}(\mathbb{Z}[\pi\times\pi']) \end{array}$$

$$\begin{array}{ccc} T(\beta)\wedge T(\beta') & \xrightarrow{\sigma^*\wedge\sigma^*} & \mathbb{L}^{-k}(\mathbb{Z}[\pi])\wedge\mathbb{L}^{-k'}(\mathbb{Z}[\pi']) \\ \downarrow\otimes & & \downarrow\otimes \\ T(\beta\times\beta') & \xrightarrow{\sigma^*} & \mathbb{L}^{-k-k'}(\mathbb{Z}[\pi\times\pi']) \end{array}$$

$$\begin{array}{ccc} T(\alpha)\wedge T(\alpha') & \xrightarrow{\hat{\sigma}^*\wedge\hat{\sigma}^*} & \hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi])\wedge\hat{\mathbb{L}}^{-k'}(\mathbb{Z}[\pi']) \\ \downarrow\otimes & & \downarrow\otimes \\ T(\alpha\times\alpha') & \xrightarrow{\hat{\sigma}^*} & \hat{\mathbb{L}}^{-k-k'}(\mathbb{Z}[\pi\times\pi']) \end{array},$$

the maps between the Thom spaces being the natural homeomorphisms.

Proof: Use transversality in the $\left\{\begin{array}{l}\text{normal}\\ \text{PL}\end{array}\right.$ category to consider the Thom space

$\left\{\begin{array}{l}T(\alpha)\\ T(\beta)\end{array}\right.$ of $\left\{\begin{array}{l}\alpha:X\longrightarrow BSG(k)\\ \beta:Y\longrightarrow BSPL(k)\end{array}\right.$ as the simplicial set of singular simplexes $\left\{\begin{array}{l}f:\Delta^n\longrightarrow T(\alpha)\\ f:\Delta^n\longrightarrow T(\beta)\end{array}\right.$

transverse regular at the zero section $\left\{\begin{array}{l}X\subset T(\alpha)\\ Y\subset T(\beta)\end{array}\right.$, in the sense that

$\left\{\begin{array}{l}M^{n-k}=f^{-1}(X)\subset\Delta^n\\ N^{n-k}=f^{-1}(Y)\subset\Delta^n\end{array}\right.$ is an oriented $-(k+1)$-dimensional $\left\{\begin{array}{l}\text{normal space}\\ \text{PL manifold}\end{array}\right.$ $(n+2)$-ad

with normal $\left\{\begin{array}{l}G(k)\text{-}\\ PL(k)\text{-}\end{array}\right.$ bundle the pullback $\left\{\begin{array}{l}\nu_M=g^*(\alpha):M\longrightarrow BSG(k)\\ \nu_N=g^*(\beta):N\longrightarrow BSPL(k)\end{array}\right.$ along the

restriction $\left\{\begin{array}{l}g=f|:M\longrightarrow X\\ g=f|:N\longrightarrow Y\end{array}\right.$. The construction of Proposition $\left\{\begin{array}{l}15.3\text{ iii)}\\ 15.2\end{array}\right.$ can now be used to define a simplicial map

$$\left\{\begin{array}{l}\hat{\sigma}^*:T(\alpha)\longrightarrow\hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi]);\ (f:\Delta^n\longrightarrow T(\alpha))\longmapsto\hat{\sigma}^*(M)\\ \sigma^*:T(\beta)\longrightarrow\mathbb{L}^{-k}(\mathbb{Z}[\pi]);\ (f:\Delta^n\longrightarrow T(\beta))\longmapsto\sigma^*(N)\end{array}\right..$$

The Thom space products agree with the L-space products because the natural homeomorphism $\left\{\begin{array}{l}\otimes:T(\alpha)\wedge T(\alpha')\longrightarrow T(\alpha\wedge\alpha')\\ \otimes:T(\beta)\wedge T(\beta')\longrightarrow T(\beta\wedge\beta')\end{array}\right.$ can be expressed in terms of our simplicial model using the bisimplicial methods of Proposition 15.6.

[]

(Given an oriented $G(k)$-bundle $\xi:X\longrightarrow BSG(k)$ let $W(\xi)$ be the simplicial set of singular simplexes $f:\Delta^n\longrightarrow T(\xi)$ "Poincaré transverse" to $X\subset T(\xi)$, i.e. such that $M^{n-k}=f^{-1}(X)\subset\Delta^n$ is an $(n-k)$-dimensional geometric Poincaré $(n+2)$-ad with Spivak normal fibration $\nu_M:M\xrightarrow{f|}X\xrightarrow{\xi}BSG(k)$, exactly as in Levitt-Morgan [1]. The method of Proposition 15.7 also gives simplicial maps σ_*,σ^* to fit into a commutative diagram

$$\begin{array}{ccccc} F(\xi) & \longrightarrow & W(\xi) & \longrightarrow & T(\xi) \\ \downarrow\sigma_* & & \downarrow\sigma^* & & \downarrow\hat{\sigma}^* \\ \mathbb{L}_{-k}(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{1+T} & \mathbb{L}^{-k}(\mathbb{Z}[\pi_1(X)]) & \xrightarrow{J} & \hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi_1(X)]) \end{array}$$

with $F(\xi)$ the homotopy-theoretic fibre of the forgetful map $W(\xi)\longrightarrow T(\xi)$.

We can interpret Remark 4.6 of Levitt [2] as stating that the map $\sigma_*:F(\xi)\longrightarrow\mathbb{L}_{-k}(\mathbb{Z}[\pi_1(X)])$ induces isomorphisms $\sigma_*:\pi_{n+k}(F(\xi))\longrightarrow L_n(\mathbb{Z}[\pi_1(X)])$ for $n\geqslant 4, k\geqslant 3$ - at least for $\pi_1(X)=\{1\}$ but in principle for all $\pi_1(X)$.

The map $\sigma^*:W(\xi)\longrightarrow\mathbb{L}^{-k}(\mathbb{Z}[\pi_1(X)])$ induces the symmetric signatures in the geometric Poincaré cobordism groups

$$\sigma^*:\Omega_n^P(X,\xi)=\varinjlim_j\pi_{n+j+k}(W(\xi\oplus\varepsilon^j))\longrightarrow L^n(\mathbb{Z}[\pi_1(X)]).$$

The Levitt-Jones-Quinn exact sequence appearing in Proposition 10.4

$$\ldots\longrightarrow L_n(\mathbb{Z}[\pi_1(X)])\longrightarrow\Omega_n^P(X)\longrightarrow H_n(X;\underline{MSG})\longrightarrow L_{n-1}(\mathbb{Z}[\pi_1(X)])\longrightarrow\ldots$$

is the stable homotopy exact sequence associated to the fibration $F(\xi)\longrightarrow W(\xi)\longrightarrow T(\xi)$ in the special case ξ = projection : $X\times BSG(k)\longrightarrow BSG(k)$.

Given an oriented $TOP(k)$-bundle $\xi:X\longrightarrow BSTOP(k)$ let $\Omega_n^{TOP}(X,\xi)$ be the topological bordism group of maps $f:M^n \longrightarrow X$ from compact topological n-manifolds M with stable normal TOP-bundle $\nu_M = f^*\xi : M \longrightarrow BSTOP$. The Pontrjagin-Thom map

$$\Omega_n^{TOP}(X,\xi) \longrightarrow \pi_{n+k}^S(T(\xi)) \;;\; (f:M^n \longrightarrow X) \longmapsto (S^{n+k} \xrightarrow{f_M} T(\nu_M) \xrightarrow{T(f)} T(\xi))$$

is not in general an isomorphism, owing to the absence of TOP transversality in dimension 4. For the same reason, it is not possible to use the method of Proposition 15.7 to define directly a simplicial map $\sigma^*: T(\xi) \longrightarrow \mathbb{L}^{-k}(\mathbb{Z}[\pi_1(X)])$ such that the symmetric signature map factorizes as

$$\sigma^* : \Omega_n^{TOP}(X,\xi) \longrightarrow \pi_{n+k}^S(T(\xi)) \xrightarrow{\sigma^*} L^n(\mathbb{Z}[\pi_1(X)]) .$$

However, the trick of raising dimensions by crossing with $\mathbb{CP}^2$ can be used to construct a section $T(\xi) \longrightarrow W(\xi)$ of $W(\xi) \longrightarrow T(\xi)$ as in Theorem A of Levitt-Morgan [1] and Theorem 4.5 of Brumfiel-Morgan [1], at least for $\pi_1(X) = \{1\}$ but in principle for all $\pi_1(X)$. (Indeed, liftings $\tilde{\xi}: X \to BSTOP(k)$ of $\xi: X \longrightarrow BSG(k)$ are shown there to correspond to such sections $T(\xi) \longrightarrow W(\xi)$). The composite

$$\sigma^* : T(\xi) \longrightarrow W(\xi) \xrightarrow{\sigma^*} \mathbb{L}^{-k}(\mathbb{Z}[\pi_1(X)])$$

is then a simplicial symmetric signature map. Given an oriented $(G(k),TOP(k))$-bundle $(\alpha,\beta):(X,Y) \longrightarrow (BSG(k),BSTOP(k))$ there is likewise defined a simplicial map

$$\sigma_* : T(\alpha,\beta) \longrightarrow \mathbb{L}_{-k-1}(\mathbb{Z}[\pi_1(X)])$$

such that the quadratic signature on the (N,TOP)-bordism groups is given by the composite

$$\sigma_* : \Omega_{n+1}^{N,TOP}(X;Y;\alpha,\beta) \xrightarrow{\text{Pontrjagin-Thom}} \pi_{n+k+1}^S(T(\alpha,\beta)) \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(X)]) .$$

For $m \in \mathbb{Z}$ let $\begin{cases} \underline{\mathbb{L}}^m(A,\varepsilon) \\ \underline{\mathbb{L}}_m(A,\varepsilon) \\ \hat{\underline{\mathbb{L}}}^m(A,\varepsilon) \end{cases}$ be the simplicial Ω-spectrum whose rth term is $\begin{cases} \mathbb{L}^{m-r}(A,\varepsilon) \\ \mathbb{L}_{m-r}(A,\varepsilon) \quad (r \in \mathbb{Z}) \\ \hat{\mathbb{L}}^{m-r}(A,\varepsilon) \end{cases}$ with structure maps $\begin{cases} \mathbb{L}^{m-r}(A,\varepsilon) \longrightarrow \Omega\,\mathbb{L}^{m-r-1}(A,\varepsilon) \\ \mathbb{L}_{m-r}(A,\varepsilon) \longrightarrow \Omega\,\mathbb{L}_{m-r-1}(A,\varepsilon) \\ \hat{\mathbb{L}}^{m-r}(A,\varepsilon) \longrightarrow \Omega\,\hat{\mathbb{L}}^{m-r-1}(A,\varepsilon) \end{cases}$ the canonical isomorphisms of Proposition 15.4. The products of Proposition 15.6 define pairings of spectra in the sense of G.W.Whitehead [1]

$$\otimes : \underline{\mathbb{L}}^m(A,\varepsilon) \wedge \underline{\mathbb{L}}^n(B,\eta) \longrightarrow \underline{\mathbb{L}}^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)$$
$$\otimes : \underline{\mathbb{L}}^m(A,\varepsilon) \wedge \underline{\mathbb{L}}_n(B,\eta) \longrightarrow \underline{\mathbb{L}}_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)$$
$$\otimes : \underline{\mathbb{L}}_m(A,\varepsilon) \wedge \underline{\mathbb{L}}_n(B,\eta) \longrightarrow \underline{\mathbb{L}}_{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta)$$
$$\otimes : \hat{\underline{\mathbb{L}}}^m(A,\varepsilon) \wedge \hat{\underline{\mathbb{L}}}^n(B,\eta) \longrightarrow \hat{\underline{\mathbb{L}}}^{m+n}(A\otimes_{\mathbb{Z}}B,\varepsilon\otimes\eta) .$$

In particular, for a commutative ring A we have connective commutative ring spectra $\underline{\mathbb{L}}^0(A) = \underline{\mathbb{L}}^0(A,1)$ and $\hat{\underline{\mathbb{L}}}^0(A) = \hat{\underline{\mathbb{L}}}^0(A,1)$, and also a connective $\underline{\mathbb{L}}^0(A)$-algebra spectrum $\underline{\mathbb{L}}_0(A) = \underline{\mathbb{L}}_0(A,1)$, using $A\otimes_{\mathbb{Z}}A \longrightarrow A; a\otimes b \longmapsto ab$ to convert the external products into internal products.

For H = G,PL let <u>MSH</u> be the Thom spectrum whose kth term is the Thom space MSH(k) of the universal oriented H(k)-bundle over BSH(k). The products

$$\otimes : MSH(j) \wedge MSH(k) \longrightarrow MSH(j+k)$$

induced by the Whitney sum of bundles

$$\oplus : BSH(j) \times BSH(k) \longrightarrow BSH(j+k)$$

can be expressed in terms of the simplicial model of Proposition 15.7 (involving singular simplexes transverse regular at the zero section) using bisimplicial sets as in Proposition 15.6. These products make <u>MSH</u> into a ring spectrum.
Let <u>MSG</u>/<u>MSPL</u> be the cofibre of <u>MSPL</u> $\longrightarrow$ <u>MSG</u>, that is the spectrum whose kth term is the cofibre MSG(k)/MSPL(k) of the natural map $MSPL(k) \longrightarrow MSG(k)$. The induced products

$$\otimes : MSPL(j) \wedge MSG(k)/MSPL(k) \longrightarrow MSG(j+k)/MSPL(j+k)$$

make <u>MSG</u>/<u>MSPL</u> into an <u>MSPL</u>-module spectrum.

Proposition 15.8 i) The simplicial $\left\{\begin{array}{l}\text{quadratic}\\ \text{symmetric}\\ \text{hyperquadratic}\end{array}\right.$ signature maps

$$\left\{\begin{array}{l}\sigma_* : MSG(k)/MSPL(k) \longrightarrow \mathbb{L}_{-k-1}(\mathbb{Z})\\ \sigma^* : MSPL(k) \longrightarrow \mathbb{L}^{-k}(\mathbb{Z})\\ \hat{\sigma}^* : MSG(k) \longrightarrow \hat{\mathbb{L}}^{-k}(\mathbb{Z})\end{array}\right.$$

define a natural transformation of cofibrations of spectra

$$\begin{array}{ccccc}\underline{MSPL} & \longrightarrow & \underline{MSG} & \longrightarrow & \underline{MSG/MSPL}\\ \sigma^*\big\downarrow & & \hat{\sigma}^*\big\downarrow & & \sigma_*\big\downarrow\\ \underline{\mathbb{L}}^0(\mathbb{Z}) & \xrightarrow{J} & \underline{\hat{\mathbb{L}}}^0(\mathbb{Z}) & \xrightarrow{H} & \underline{\mathbb{L}}_{-1}(\mathbb{Z})\end{array}$$

ii) For any connected space X and any ring with involution A there is defined a natural transformation of long exact sequences of abelian groups

$$\begin{array}{ccccccccc}\ldots\longrightarrow & H_n(X;\underline{\mathbb{L}}_0(A,\varepsilon)) & \xrightarrow{1+T} & H_n(X;\underline{\mathbb{L}}^0(A,\varepsilon)) & \xrightarrow{J} & H_n(X;\underline{\hat{\mathbb{L}}}^0(A,\varepsilon)) & \xrightarrow{H} & H_{n-1}(X;\underline{\mathbb{L}}_0(A,\varepsilon)) & \longrightarrow\ldots\\ & \sigma_*\big\downarrow & & \sigma^*\big\downarrow & & \hat{\sigma}^*\big\downarrow & & \sigma_*\big\downarrow & \\ \ldots\longrightarrow & L_n(A[\pi_1(X)],\varepsilon) & \xrightarrow{1+T} & L^n(A[\pi_1(X)],\varepsilon) & \xrightarrow{J} & \hat{L}^n(A[\pi_1(X)],\varepsilon) & \xrightarrow{H} & L_{n-1}(A[\pi_1(X)],\varepsilon) & \ldots\end{array}$$

which is the identity for X = pt..

iii) The $\left\{\begin{array}{l}\text{quadratic}\\ \text{symmetric}\\ \text{hyperquadratic}\end{array}\right.$ signature map on $\left\{\begin{array}{l}(N,PL)\text{-}\\ PL\text{-}\\ N\text{-}\end{array}\right.$ bordism is the composite

$$\left\{\begin{array}{l}\sigma_*:\Omega^{N,PL}_{n+1}(X) = H_{n+1}(X;\underline{MSG}/\underline{MSPL}) \xrightarrow{\sigma_*} H_n(X;\underline{\mathbb{L}}_0(\mathbb{Z})) \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(X)])\\ \sigma^*:\Omega^{PL}_n(X) = H_n(X;\underline{MSPL}) \xrightarrow{\sigma^*} H_n(X;\underline{\mathbb{L}}^0(\mathbb{Z})) \xrightarrow{\sigma^*} L^n(\mathbb{Z}[\pi_1(X)])\\ \hat{\sigma}^*:\Omega^N_n(X) = H_n(X;\underline{MSG}) \xrightarrow{\hat{\sigma}^*} H_n(X;\underline{\hat{\mathbb{L}}}^0(\mathbb{Z})) \xrightarrow{\hat{\sigma}^*} \hat{L}^n(\mathbb{Z}[\pi_1(X)])\end{array}\right.$$

<u>Proof</u>: i) Apply the construction of Proposition 15.7 to the universal Thom spaces.

ii) Let $\sigma^*:X_+ = T(0) \longrightarrow \mathbb{L}^0(\mathbb{Z}[\pi_1(X)])$ be the simplicial symmetric signature map associated to $0:X \longrightarrow BSPL(0)$, which is the composite

$$X_+ \longrightarrow K(\pi_1(X),1)_+ \xrightarrow{\sigma^*} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)])$$

with $X \longrightarrow K(\pi_1(X),1)$ the map killing $\pi_r(X)$ $(r \geqslant 2)$ and $\sigma^*:K(\pi,1)_+ \longrightarrow \mathbb{L}^0(\mathbb{Z}[\pi])$ defined as follows $(\pi = \pi_1(X))$.

The classifying space $K(\pi,1)$ of a (discrete) group π has a standard simplicial model, with n-skeleton $K(\pi,1)^{(n)} = \pi\times\pi\times\ldots\times\pi$ (n times). Let Z be the infinite cyclic group written multiplicatively with generator $z \in Z$, so that $K(Z,1) = S^1$ and $\mathbb{Z}[Z] = \mathbb{Z}[z,z^{-1}]$ $(\bar{z} = z^{-1})$. The simplicial symmetric signature map $\sigma^*:K(Z,1)_+ \longrightarrow \mathbb{L}^0(\mathbb{Z}[Z])$ is defined on the 1-skeleton by

$$\sigma^* : K(Z,1)^{(1)} = Z \longrightarrow \mathbb{L}^0(\mathbb{Z}[Z])^{(1)} ;$$

$$z^n \longmapsto \sigma^*(z^n) = (\mathbb{Z}[Z]\oplus\mathbb{Z}[Z],1\oplus-1;\{(x,x)\,|\,x\in\mathbb{Z}[Z]\},\{(x,z^n x)\,|\,x\in\mathbb{Z}[Z]\}) ,$$

regarding non-singular symmetric formations over $\mathbb{Z}[Z]$ as the 1-simplexes of $\mathbb{L}^0(\mathbb{Z}[Z])$ with 0 faces. In particular, $\sigma^*(z) = \sigma^*(S^1) \in L^1(\mathbb{Z}[Z])$. For $g \in \pi$ define a group morphism

$$g : Z \longrightarrow \pi ;\ z^n \longrightarrow g^n ,$$

and consider the commutative diagram

$$\begin{array}{ccc}K(Z,1)_+ & \xrightarrow{\sigma^*} & \mathbb{L}^0(\mathbb{Z}[Z])\\ g\big\downarrow & & \big\downarrow g\\ K(\pi,1)_+ & \xrightarrow{\sigma^*} & \mathbb{L}^0(\mathbb{Z}[\pi])\end{array}$$

to obtain that

$$\sigma^* : K(\pi,1)^{(1)} = \pi \longrightarrow \mathbb{L}^0(\mathbb{Z}[\pi])^{(1)}$$

sends $g \in \pi$ to the non-singular symmetric formation over $\mathbb{Z}[\pi]$

$$\sigma^*(g) = (\mathbb{Z}[\pi]\oplus\mathbb{Z}[\pi],1\oplus-1;\{(x,x)\,|\,x\in\mathbb{Z}[\pi]\},\{(x,gx)\,|\,x\in\mathbb{Z}[\pi]\}).$$

The extension of $\sigma^*:K(\pi,1)^{(1)} \longrightarrow \mathbb{L}^0(\mathbb{Z}[\pi])^{(1)}$ to the 2-skeletons is related to the sum formula

$$(M,\varphi;F,G)\oplus(M,\varphi;G,H) = (M,\varphi;F,H) \in L^1(A)$$

of Proposition 7.7, which has a canonical proof on the representative level (cf. the identity of Lemma 3.3 of Ranicki [3]). Given $g,h \in \pi$ there is defined an isomorphism of non-singular symmetric formations over $\mathbb{Z}[\pi]$

$$1\oplus g : \sigma^*(h) \longrightarrow (\mathbb{Z}[\pi]\oplus\mathbb{Z}[\pi],1\oplus-1;\{(x,gx)\,|\,x\in\mathbb{Z}[\pi]\},\{(x,ghx)\,|\,x\in\mathbb{Z}[\pi]\}) ,$$

so that applying the sum formula

$$\sigma^*(g)\oplus\sigma^*(h) = \sigma^*(gh) \in L^1(\mathbb{Z}[\pi]) .$$

There is a canonical choice of 2-simplex $\sigma^*(g,h) \in \mathbb{L}^0(\mathbb{Z}[\pi])^{(2)}$ such that

$$\partial_1\sigma^*(g,h) = \sigma^*(g) \ ,\quad \partial_2\sigma^*(g,h) = \sigma^*(gh) \ ,\quad \partial_3\sigma^*(g,h) = \sigma^*(h) .$$

The composites

$$\begin{cases} \sigma_* : X_+ \wedge \mathbb{L}_{-k}(A,\varepsilon) \xrightarrow{\sigma^* \wedge 1} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \mathbb{L}_{-k}(A,\varepsilon) \xrightarrow{\otimes} \mathbb{L}_{-k}(A[\pi_1(X)],\varepsilon) \\ \sigma^* : X_+ \wedge \mathbb{L}^{-k}(A,\varepsilon) \xrightarrow{\sigma^* \wedge 1} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \mathbb{L}^{-k}(A,\varepsilon) \xrightarrow{\otimes} \mathbb{L}^{-k}(A[\pi_1(X)],\varepsilon) \\ \hat{\sigma}^* : X_+ \wedge \hat{\mathbb{L}}^{-k}(A,\varepsilon) \xrightarrow{\sigma^* \wedge 1} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \hat{\mathbb{L}}^{-k}(A,\varepsilon) \xrightarrow{\otimes} \hat{\mathbb{L}}^{-k}(A[\pi_1(X)],\varepsilon) \end{cases}$$

define a natural transformation of cofibration sequences of spectra

$$\begin{array}{ccccc} X_+ \wedge \mathbb{L}_0(A,\varepsilon) & \xrightarrow{1+T} & X_+ \wedge \mathbb{L}^0(A,\varepsilon) & \xrightarrow{J} & X_+ \wedge \hat{\mathbb{L}}^0(A,\varepsilon) \\ \sigma_* \downarrow & & \sigma^* \downarrow & & \hat{\sigma}^* \downarrow \\ \mathbb{L}_0(A[\pi_1(X)],\varepsilon) & \xrightarrow{1+T} & \mathbb{L}^0(A[\pi_1(X)],\varepsilon) & \xrightarrow{J} & \hat{\mathbb{L}}^0(A[\pi_1(X)],\varepsilon) \end{array}$$

inducing natural maps in the homotopy groups

$$\begin{cases} \sigma_* : H_n(X;\mathbb{L}_0(A,\varepsilon)) = \varinjlim_k \pi_{n+k}(X_+ \wedge \mathbb{L}_{-k}(A,\varepsilon)) \longrightarrow L_n(A[\pi_1(X)],\varepsilon) \\ \sigma^* : H_n(X;\mathbb{L}^0(A,\varepsilon)) = \varinjlim_k \pi_{n+k}(X_+ \wedge \mathbb{L}^{-k}(A,\varepsilon)) \longrightarrow L^n(A[\pi_1(X)],\varepsilon) \\ \hat{\sigma}^* : H_n(X;\hat{\mathbb{L}}^0(A,\varepsilon)) = \varinjlim_k \pi_{n+k}(X_+ \wedge \hat{\mathbb{L}}^{-k}(A,\varepsilon)) \longrightarrow \hat{L}^n(A[\pi_1(X)],\varepsilon) \end{cases}.$$

iii) Proposition 15.8 gives that the signature maps on the bordism groups

$$\begin{cases} \sigma_* : \Omega^{N,PL}_{n+1}(X) = H_{n+1}(X;\underline{MSG/MSPL}) \longrightarrow L_n(\mathbb{Z}[\pi_1(X)]) \\ \sigma^* : \Omega^{PL}_n(X) = H_n(X;\underline{MSPL}) \longrightarrow L^n(\mathbb{Z}[\pi_1(X)]) \\ \hat{\sigma}^* : \Omega^{N}_n(X) = H_n(X;\underline{MSG}) \longrightarrow \hat{L}^n(\mathbb{Z}[\pi_1(X)]) \end{cases}$$

are induced by the simplicial maps

$$\begin{cases} \sigma_* : X_+ \wedge MSG(k)/MSPL(k) \xrightarrow{\sigma^* \wedge \sigma_*} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \mathbb{L}_{-k-1}(\mathbb{Z}) \xrightarrow{\otimes} \mathbb{L}_{-k-1}(\mathbb{Z}[\pi_1(X)]) \\ \sigma^* : X_+ \wedge MSPL(k) \xrightarrow{\sigma^* \wedge \sigma^*} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \mathbb{L}^{-k}(\mathbb{Z}) \xrightarrow{\otimes} \mathbb{L}^{-k}(\mathbb{Z}[\pi_1(X)]) \\ \hat{\sigma}^* : X_+ \wedge MSG(k) \xrightarrow{\sigma^* \wedge \hat{\sigma}^*} \mathbb{L}^0(\mathbb{Z}[\pi_1(X)]) \wedge \hat{\mathbb{L}}^{-k}(\mathbb{Z}) \xrightarrow{\otimes} \hat{\mathbb{L}}^{-k}(\mathbb{Z}[\pi_1(X)]) \end{cases}.$$

[]

Identifying $\mathbb{L}_0(\mathbb{Z}) = L_0(\mathbb{Z}) \times G/TOP$ as in Proposition 15.5 we have the fibre square of Sullivan [1]

$$\begin{array}{ccc} \mathbb{L}_0(\mathbb{Z}) & \longrightarrow & \prod_{i=0}^{\infty} K(\mathbb{Z}[1/\mathrm{odd}],4i) \times K(\mathbb{Z}_2,4i+2) \\ \Delta \downarrow & & \downarrow \\ (\mathbb{Z} \times BO) \otimes \mathbb{Z}[1/2] & \xrightarrow{ph} & \prod_{i=0}^{\infty} K(\mathbb{Q},4i) \end{array}$$

with ph the Pontrjagin character and

$$ph.\Delta : \mathbb{L}_0(\mathbb{Z}) = L_0(\mathbb{Z}) \times G/TOP \xrightarrow{\text{forget}} \mathbb{Z} \times BTOP \xrightarrow{1/L\text{-genus}} \prod_{i=0}^{\infty} K(\mathbb{Q},4i) .$$

It would be interesting to see this directly in terms of $\mathbb{L}_0(\mathbb{Z})$. At any rate we can identify

$$\mathbb{L}_0(\mathbb{Z}) \otimes \mathbb{Z}[1/2] = \mathbb{L}^0(\mathbb{Z}) \otimes \mathbb{Z}[1/2] = \underline{bo} \otimes \mathbb{Z}[1/2] ,$$

where $\underline{bo}$ is the connective real K-theory Ω-spectrum with kth term $\Omega^{8j-k}(\mathbb{Z} \times BO)(8j,8j+1,\ldots,\infty)$ $(8j \geq k \geq 0)$. The invariant $\Psi(M^n)$ defined for n-dimensional manifolds M in §17H of Wall [5] is just

$$\Psi(M^n) = \sigma^*(M) \otimes E_8 \in L_n(\mathbb{Z}[\pi_1(M)]) \qquad (E_8 = 1 \in L_0(\mathbb{Z}) = \mathbb{Z})$$

(by the surgery product formula of Proposition 11.2 applied to $M \times Q^8 \longrightarrow M \times S^8$), so that the related maps defined there

$$1'_\pi : KO_n(K(\pi,1)) \otimes \mathbb{Z}[1/2] \longrightarrow L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Z}[1/2] = L^n(\mathbb{Z}[\pi]) \otimes \mathbb{Z}[1/2]$$

$$1_\pi : \sum_{i=0}^{\infty} H_{n-4i}(K(\pi,1);\mathbb{Q}) \longrightarrow L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Q} = L^n(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

may be obtained from

$$\sigma^* : H_n(K(\pi,1);\mathbb{L}^0(\mathbb{Z})) \longrightarrow L^n(\mathbb{Z}[\pi])$$

by applying $-\otimes\mathbb{Z}[1/2]$ and $-\otimes\mathbb{Q}$ respectively. Furthermore, the map

$$1(\pi) : KO_n(K(\pi,1)) \otimes \mathbb{Z}[1/2] \longrightarrow L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Z}[1/2]$$

defined in §3.3.1 of Loday [1] is likewise the map obtained from

$$\sigma^* : H_n(K(\pi,1);\mathbb{L}^0(\mathbb{Z})) \longrightarrow L^n(\mathbb{Z}[\pi])$$

by applying $-\otimes\mathbb{Z}[1/2]$, since its construction uses the map $1(\pi) : K(\pi,1)_+ \longrightarrow \mathbb{L}^0(\mathbb{Z}[1/2][\pi])$ sending $g \in \pi$ to the non-singular symmetric formation over $\mathbb{Z}[1/2][\pi]$

$$1(\pi)(g) = (\mathbb{Z}[1/2][\pi] \oplus \mathbb{Z}[1/2][\pi]^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \mathbb{Z}[1/2][\pi],$$

$$\mathrm{im}(\begin{pmatrix} \frac{1}{2}(1+g) \\ 1-g \end{pmatrix} : \mathbb{Z}[1/2][\pi] \longrightarrow \mathbb{Z}[1/2][\pi] \oplus \mathbb{Z}[1/2][\pi]^*))$$

isomorphic to

$$\sigma^*(g) \otimes \mathbb{Z}[1/2] = (\mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi], \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \mathrm{im}(\begin{pmatrix} 1 \\ 1 \end{pmatrix} : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi]),$$

$$\mathrm{im}(\begin{pmatrix} 1 \\ g \end{pmatrix} : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi])) \otimes \mathbb{Z}[1/2]$$

(by $\begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix} : \sigma^*(g) \otimes \mathbb{Z}[1/2] \longrightarrow 1(\pi)(g)$). In principle, we have proved Conjecture 3.3.5 of Loday [1] that $1'_\pi = 1(\pi)$. The map

$$\sigma_* : H_n(X;\mathbb{L}_0(\mathbb{Z}[\pi])) \longrightarrow L_n(\mathbb{Z}[\pi \times \pi_1(X)])$$

agrees with the geometrically defined "assembly map" A_π of Quinn [1],[2].

Let $\underline{R} = \{R_j, \Sigma R_j \to R_{j+1}, \otimes : R_j \wedge R_k \to R_{j+k}, 1 : S^k \to R_k \mid j,k \geq 0\}$ be a commutative ring spectrum. An $\underline{R}$-orientation for a spherical fibration $\nu : X \to BG(k)$ is a Thom class $U \in \dot{H}^k(T(\nu);\underline{R}) = \varinjlim_j [\Sigma^j T(\nu), R_{j+k}]$ which restricts to a $H^*(pt;\underline{R})$-module generator $i^*U \in \dot{H}^k(T(i^*\nu);\underline{R})$ for each point $i : \{pt\} \to X$ ($T(i^*\nu) = S^k$). In particular, $\varepsilon^k : X \to BG(k)$ ($k \geq 0$) has the canonical $\underline{R}$-orientation $1 \in \dot{H}^k(\Sigma^k X_+;\underline{R})$ represented by the composite

$$\Sigma^k X_+ \xrightarrow{\text{collapse}} S^k \xrightarrow{1} R_k \quad .$$

There exists a classifying space $B\underline{R}G$ for $\underline{R}$-oriented spherical fibrations over CW complexes X, with $[X_+, B\underline{R}G]$ the set of $\underline{R}$-orientation preserving stable fibre homotopy classes of pairs

$$(\text{spherical fibration } \nu : X \to BG(k), \underline{R}\text{-orientation } U \in \dot{H}^k(T(\nu);\underline{R})) \quad .$$

The Whitney sum $\oplus$ defines an abelian group structure on $[X_+, B\underline{R}G]$, giving $B\underline{R}G$ an H-space structure. Given $(\nu, U) \in [X_+, B\underline{R}G]$ for a finitely-dominated CW complex X and an $\underline{R}$-module spectrum $\underline{S}$ we have that the Atiyah-Hirzebruch spectral sequence converges

$$E^2_{p,q} = H_p(X; H_q(pt.;\underline{S})) \Longrightarrow H_*(X;\underline{S}) \quad ,$$

giving rise to the usual Thom isomorphisms in $\underline{S}$-homology

$$U \cap - : \dot{H}_{r+k}(T(\nu);\underline{S}) \longrightarrow H_r(X;\underline{S}) \qquad (r \in \mathbb{Z})$$

and similarly for $\underline{S}$-cohomology. An $\underline{R}$-orientation for an n-dimensional geometric Poincaré complex X is an $\underline{R}$-orientation for the Spivak normal fibration ν_X appearing in a given normal structure $(\nu_X : X \to BG(k), \rho_X \in \pi_{n+k}(T(\nu_X)))$, in which case the fundamental S-duality isomorphism

$$\alpha_X \backslash - : \dot{H}^k(T(\nu_X);\underline{R}) \longrightarrow H_n(X;\underline{R})$$

sends the Thom class U_{ν_X} to a fundamental $\underline{R}$-homology class $[X] = \alpha_X \backslash U_{\nu_X} \in H_n(X;\underline{R})$ and there are defined $\underline{S}$-(co)homology Poincaré duality isomorphisms

$$[X] \cap - : H^r(X;\underline{S}) \xrightarrow{(\alpha_X \backslash -)^{-1}} \dot{H}_{n+k-r}(T(\nu_X);\underline{S}) \xrightarrow{U_{\nu_X} \cap -} H_{n-r}(X;\underline{S})$$

for any $\underline{R}$-module spectrum $\underline{S}$. Note that a map of ring spectra $\underline{R} \to \underline{R}'$ sends $\underline{R}$-orientations to $\underline{R}'$-orientations.

An oriented $\begin{Bmatrix} G(k)- \\ PL(k)- \end{Bmatrix}$ bundle $\begin{Bmatrix} \alpha : X \to BSG(k) \\ \beta : Y \to BSPL(k) \end{Bmatrix}$ has a canonical $\begin{Bmatrix} \underline{MSG}- \\ \underline{MSPL}- \end{Bmatrix}$ orientation $\begin{Bmatrix} U_\alpha \in \dot{H}^k(T(\alpha);\underline{MSG}) \\ U_\beta \in \dot{H}^k(T(\beta);\underline{MSPL}) \end{Bmatrix}$, represented by the natural map $\begin{Bmatrix} T(\alpha) \to MSG(k) \\ T(\beta) \to MSPL(k) \end{Bmatrix}$ (cf. Theorem 7.4 of G.W.Whitehead [1]). The Thom isomorphism

$$\begin{cases} U_\alpha \cap - : \dot{\Omega}^N_{n+k}(T(\alpha)) = \dot{H}_{n+k}(T(\alpha);\underline{MSG}) \longrightarrow \Omega^N_n(X) = H_n(X;\underline{MSG}) \\ U_\beta \cap - : \dot{\Omega}^{PL}_{n+k}(T(\beta)) = \dot{H}_{n+k}(T(\beta);\underline{MSPL}) \longrightarrow \Omega^{PL}_n(X) = H_n(X;\underline{MSPL}) \end{cases}$$

sends the singular $\begin{Bmatrix} \text{normal pair} & f : (L^{n+k}, \partial L) \to (T(\alpha), *) \\ \text{PL manifold} & g : (M^{n+k}, \partial M) \to (T(\beta), *) \end{Bmatrix}$ transverse regular in the $\begin{Bmatrix} N- \\ PL- \end{Bmatrix}$ sense at the zero section $\begin{Bmatrix} X \hookrightarrow T(\alpha) \\ Y \hookrightarrow T(\beta) \end{Bmatrix}$ to the $\begin{Bmatrix} N- \\ PL- \end{Bmatrix}$ bordism class $\begin{Bmatrix} f| : U^n = f^{-1}(X) \longrightarrow X \\ g| : V^n = g^{-1}(Y) \longrightarrow Y \end{Bmatrix}$. (The model for this is the reformulation of the smooth transversality results of Thom [2] in terms of $\underline{MSO}$-orientations due to Atiyah[2]).

<u>Proposition 15.9</u> i) An oriented $\begin{Bmatrix} G(k)- \\ PL(k)- \end{Bmatrix}$ bundle $\begin{Bmatrix} \alpha : X \to BSG(k) \\ \beta : Y \to BSPL(k) \end{Bmatrix}$ has a canonical $\begin{Bmatrix} \hat{\underline{L}}^0(\mathbb{Z})- \\ \underline{L}^0(\mathbb{Z})- \end{Bmatrix}$ orientation, with a Thom class $\begin{Bmatrix} U_\alpha \in \dot{H}^k(T(\alpha);\hat{\underline{L}}^0(\mathbb{Z})) \\ U_\beta \in \dot{H}^k(T(\beta);\underline{L}^0(\mathbb{Z})) \end{Bmatrix}$ such that

$$JU_\beta = U_{J\beta} \in \dot{H}^k(T(\beta);\hat{\underline{L}}^0(\mathbb{Z})) \quad .$$

ii) An oriented n-dimensional geometric Poincaré complex X has a canonical $\hat{\underline{L}}^0(\mathbb{Z})$-orientation, with a fundamental class $[\hat{X}] \in H_n(X;\hat{\underline{L}}^0(\mathbb{Z}))$ such that

$$\hat{\sigma}^*([\hat{X}]) = J\sigma^*(X) \in \hat{L}^n(\mathbb{Z}[\pi_1(X)]) \quad .$$

iii) A PL normal structure on an oriented n-dimensional geometric Poincaré complex X

$$(\nu_X : X \longrightarrow BSPL(k), \rho_X \in \pi_{n+k}(T(\nu_X)))$$

determines by PL transversality a PL normal bundle map

$$(f = \rho_X| , b) : M^n = \rho_X^{-1}(X) \longrightarrow X$$

and places an $\underline{L}^0(\mathbb{Z})$-orientation on X, with a fundamental class $[X] \in H_n(X;\underline{L}^0(\mathbb{Z}))$ such that

$$\sigma^*([X]) = \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(X)]) \quad , \quad J[X] = [\hat{X}] \in H_n(X;\hat{\underline{L}}^0(\mathbb{Z}))$$

$$(1+T)\sigma_*(f,b) = \sigma^*([X]) - \sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)]) \quad .$$

Let $(\nu'_X : X \longrightarrow BSPL(k'), \rho'_X \in \pi_{n+k'}(T(\nu'_X)))$ be another PL normal structure on X, classified with respect to (ν_X,ρ_X) by $c : X \longrightarrow G/PL$. The Poincaré duality isomorphism

$$[X]\cap - \; : \; H^0(X;\mathbb{L}_0(\mathbb{Z})) = [X_+,\mathbb{L}_0(\mathbb{Z})] \longrightarrow H_n(X;\mathbb{L}_0(\mathbb{Z}))$$

sends $X_+ \xrightarrow{c} (0\times G/PL)_+ \longrightarrow \mathbb{L}_0(\mathbb{Z}) = L_0(\mathbb{Z})\times G/TOP$ to an element $[c]\in H_n(X;\mathbb{L}_0(\mathbb{Z}))$ such that

$$\sigma_*([c]) = \sigma_*(f',b') - \sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) \;,\quad (1+T)[c] = [X'] - [X] \in H_n(X;\mathbb{L}^0(\mathbb{Z})).$$

Proof: It is sufficient to observe that an oriented $\begin{cases} G(k)- \\ PL(k)- \end{cases}$ bundle $\begin{cases} \alpha : X \longrightarrow BSG(k) \\ \beta : Y \longrightarrow BSPL(k) \end{cases}$ has a canonical $\begin{cases} \hat{\mathbb{L}}^0(\mathbb{Z})- \\ \mathbb{L}^0(\mathbb{Z})- \end{cases}$ orientation $\begin{cases} U_\alpha \in \dot{H}^k(T(\alpha);\hat{\mathbb{L}}^0(\mathbb{Z})) \\ U_\beta \in \dot{H}^k(T(\beta);\mathbb{L}^0(\mathbb{Z})) \end{cases}$, which is represented by

$$\begin{cases} U_\alpha \; : \; T(\alpha) \longrightarrow MSG(k) \xrightarrow{\hat{\sigma}^*} \hat{\mathbb{L}}^{-k}(\mathbb{Z}) \\ U_\beta \; : \; T(\beta) \longrightarrow MSPL(k) \xrightarrow{\sigma^*} \mathbb{L}^{-k}(\mathbb{Z}) \; . \end{cases}$$

[]

Let X be an oriented n-dimensional PL manifold, so that the PL normal bundle maps $(f,b):M\longrightarrow X$ are classified by maps $c:X\longrightarrow G/PL$ (= (G,PL)-bundles over $(*,X)$). The mapping cylinder of a PL normal bundle map is an (N,PL)-bordism element (cf. Proposition 10.4), thus defining a function

$$\varphi \; : \; [X_+,G/PL] \longrightarrow \Omega^{N,PL}_{n+1}(X) \;\; ; \;\; ((f,b):M\longrightarrow X) \longrightarrow ((X,M\cup -X) \longrightarrow X)$$

such that the surgery obstruction is the composite

$$\theta \; : \; [X_+,G/PL] \xrightarrow{\varphi} \Omega^{N,PL}_{n+1}(X) \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(X)]) \; .$$

(Recall from p.110 of Wall [5] the warning that θ is NOT a morphism of abelian groups with respect to the Whitney sum of bundles H-space structure on G/PL).
Let $(\alpha_k,\beta_k) : (*,G(k)/PL(k)) \longrightarrow (BSG(k),BSPL(k))$ be the universal oriented $(G(k),PL(k))$-bundle, with Thom spaces

$$T(\alpha_k) = S^k \;\; , \;\; T(\beta_k) = \Sigma^k(G(k)/PL(k))_+ \;\; , \;\; T(\alpha_k,\beta_k) = \Sigma^{k+1}(G(k)/PL(k))$$

and a map of cofibrations

$$\begin{array}{ccccc} T(\beta_k) & \longrightarrow & T(\alpha_k) & \longrightarrow & T(\alpha_k,\beta_k) \\ \downarrow & & \downarrow & & \downarrow \\ MSPL(k) & \longrightarrow & MSG(k) & \longrightarrow & MSG(k)/MSPL(k) \end{array} \; .$$

The adjoints $G(k)/PL(k) \longrightarrow \Omega^{k+1}(MSG(k)/MSPL(k))$ $(k\geq 0)$ fit together, and define a map

$$\gamma \; : \; G/PL = \varinjlim_k G(k)/PL(k) \longrightarrow \varinjlim_k \Omega^{k+1}(MSG(k)/MSPL(k))$$

such that φ (as defined above) is the composite

$$\varphi \; : \; [X_+,G/PL] \xrightarrow{\gamma} [X_+,\varinjlim_k \Omega^{k+1}(MSG(k)/MSPL(k))] = H^{-1}(X;\underline{MSG}/\underline{MSPL})$$
$$\xrightarrow{[X]\cap -} H_{n+1}(X;\underline{MSG}/\underline{MSPL}) = \Omega^{N,PL}_{n+1}(X) \; .$$

with $[X] = (1:X\longrightarrow X)\in \Omega^{PL}_n(X) = H_n(X;\underline{MSPL})$. The composite

$$G/PL \xrightarrow{\gamma} \varinjlim_k \Omega^{k+1}(MSG(k)/MSPL(k)) \xrightarrow{\sigma_*} \mathbb{L}_0(\mathbb{Z})$$

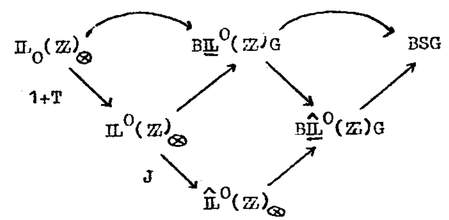
is almost a homotopy equivalence into the zero component of $\mathbb{L}_0(\mathbb{Z})$, inducing the surgery map in the homotopy groups $\pi_n(G/PL)\longrightarrow L_n(\mathbb{Z})$ which is an isomorphism for $n\neq 4$ and maps onto the subgroup of index 2 if $n = 4$. The surgery obstruction map can thus also be factored as

$$\theta \; : \; [X_+,G/PL] \xrightarrow{\sigma_*\gamma} [X_+,\mathbb{L}_0(\mathbb{Z})] = H^0(X;\mathbb{L}_0(\mathbb{Z})) \xrightarrow{[X]\cap -} H_n(X;\mathbb{L}_0(\mathbb{Z})) \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(X)])$$

with $[X]\in H_n(X;\mathbb{L}^0(\mathbb{Z}))$ the canonical L-theoretic orientation of X.

Let $\begin{cases} \sigma^*\in \dot{C}^k(\mathbb{L}^{-k}(\mathbb{Z});\mathbb{Z}) \\ \hat{\sigma}^*\in \dot{C}^k(\hat{\mathbb{L}}^{-k}(\mathbb{Z});\mathbb{Z}_8) \end{cases}$ be the $\begin{cases} \mathbb{Z}- \\ \mathbb{Z}_8- \end{cases}$ valued cochain assigning to each k-simplex of $\begin{cases} \mathbb{L}^{-k}(\mathbb{Z}) \\ \hat{\mathbb{L}}^{-k}(\mathbb{Z}) \end{cases}$ (i.e. a non-singular symmetric form over $\mathbb{Z}$) its signature.

The additivity of the signature (first observed by Milnor [2], and then generalized by Novikov) ensures that the cochains are cocycles, defining cohomology classes $\begin{cases} \sigma^*\in \dot{H}^k(\mathbb{L}^{-k}(\mathbb{Z});\mathbb{Z}) = [\mathbb{L}^{-k}(\mathbb{Z}),K(\mathbb{Z},k)] \\ \hat{\sigma}^*\in \dot{H}^k(\hat{\mathbb{L}}^{-k}(\mathbb{Z});\mathbb{Z}_8) = [\hat{\mathbb{L}}^{-k}(\mathbb{Z}),K(\mathbb{Z}_8,k)] \end{cases}$ and hence a map of ring spectra $\begin{cases} \sigma^* : \mathbb{L}^0(\mathbb{Z}) \longrightarrow \underline{K}(\mathbb{Z},0) \\ \hat{\sigma}^* : \hat{\mathbb{L}}^0(\mathbb{Z}) \longrightarrow \underline{K}(\mathbb{Z}_8,0) \end{cases}$. It follows that every $\begin{cases} \mathbb{L}^0(\mathbb{Z})- \\ \hat{\mathbb{L}}^0(\mathbb{Z})- \end{cases}$ oriented spherical fibration is oriented in the usual sense. The homotopy-theoretic fibre of the forgetful map $\begin{cases} B\mathbb{L}^0(\mathbb{Z})G \longrightarrow BSG \\ B\hat{\mathbb{L}}^0(\mathbb{Z})G \longrightarrow BSG \end{cases}$ classifies the $\begin{cases} \mathbb{L}^0(\mathbb{Z})- \\ \hat{\mathbb{L}}^0(\mathbb{Z})- \end{cases}$ orientations of the trivial oriented spherical fibrations, which for $0:X\longrightarrow BSG(0)$ are the elements $\begin{cases} U \in H^0(X;\underline{\mathbb{L}}^0(\mathbb{Z})) = [X_+,\mathbb{L}^0(\mathbb{Z})] \\ \hat{U} \in H^0(X;\hat{\underline{\mathbb{L}}}^0(\mathbb{Z})) = [X_+,\hat{\mathbb{L}}^0(\mathbb{Z})] \end{cases}$ which send X to the component of $\begin{cases} 1\in\pi_0(\mathbb{L}^0(\mathbb{Z})) = L^0(\mathbb{Z}) = \mathbb{Z} \\ 1\in\pi_0(\hat{\mathbb{L}}^0(\mathbb{Z})) = \hat{L}^0(\mathbb{Z}) = \mathbb{Z}_8 \end{cases}$. It follows that there is defined a commutative braid of fibrations



with $\begin{cases} \mathbb{L}^0(\mathbb{Z})_\otimes \\ \hat{\mathbb{L}}^0(\mathbb{Z})_\otimes \end{cases}$ the path-component in $\begin{cases} \mathbb{L}^0(\mathbb{Z}) \\ \hat{\mathbb{L}}^0(\mathbb{Z}) \end{cases}$ of $\begin{cases} 1\in\pi_0(\mathbb{L}^0(\mathbb{Z})) \\ 1\in\pi_0(\hat{\mathbb{L}}^0(\mathbb{Z})) \end{cases}$ and $\mathbb{L}_0(\mathbb{Z})_\otimes$ the homotopy-theoretic fibre of $J:\mathbb{L}^0(\mathbb{Z})_\otimes \longrightarrow \hat{\mathbb{L}}^0(\mathbb{Z})_\otimes$. Now $\mathbb{L}_0(\mathbb{Z})_\otimes$ can be identified with the zero path-component of $\mathbb{L}_0(\mathbb{Z})$, with the Whitney sum H-space structure inherited from $B\underline{\mathbb{L}}^0(\mathbb{Z})G$ given by

$$\mathbb{L}_0(\mathbb{Z}) \wedge \mathbb{L}_0(\mathbb{Z}) \longrightarrow \mathbb{L}_0(\mathbb{Z}) \; ; \; a\wedge b \longmapsto a\oplus b\oplus(a\otimes b) \; .$$

The L-theoretic orientations of Proposition 15.9 i) define forgetful maps

$$\begin{cases} BSPL \longrightarrow B\underline{\mathbb{L}}^0(\mathbb{Z})G \\ BSG \longrightarrow B\hat{\underline{\mathbb{L}}}^0(\mathbb{Z})G \end{cases}$$

and hence a map of the fibres

$$G/PL \longrightarrow \mathbb{L}_0(\mathbb{Z})_\otimes$$

which can be identified with the almost homotopy equivalence obtained above. In principle, it is also possible to $\underline{\mathbb{L}}^0(\mathbb{Z})$-orient STOP-bundles and improve this to a homotopy equivalence

$$G/TOP \longrightarrow \mathbb{L}_0(\mathbb{Z})_\otimes$$

allowing us to claim:

Proposition 15.10 There is a natural equivalence of categories

(oriented spherical fibrations with an $\underline{\mathbb{L}}^0(\mathbb{Z})$-orientation lifting the canonical $\hat{\underline{\mathbb{L}}}^0(\mathbb{Z})$-orientation)$\longleftrightarrow$(oriented TOP bundles) . []

(Away from 2 this is the result of Sullivan [2] that TOP bundles are the same as $KO\otimes\mathbb{Z}[1/2]$-oriented spherical fibrations).

We shall now define simplicial spectra $\begin{cases} \underline{\mathcal{L}}^*(A,\varepsilon) \\ \underline{\mathcal{L}}_*(A,\varepsilon) \end{cases}$ such that

$$\begin{cases} \pi_n(\underline{\mathcal{L}}^*(A,\varepsilon)) = L^n(A,\varepsilon) \\ \pi_n(\underline{\mathcal{L}}_*(A,\varepsilon)) = L_n(A,\varepsilon) \end{cases} \quad (n\in\mathbb{Z})$$

involving the lower L-groups of §9.

A <u>weakly m-dimensional</u> $\begin{cases} \underline{\varepsilon\text{-symmetric}} \\ \underline{\varepsilon\text{-quadratic}} \end{cases}$ <u>Poincaré (n+1)-ad over A</u> $\begin{cases} (\Gamma,\Phi) \\ (\Gamma,\Psi) \end{cases}$ $(m\in\mathbb{Z}, n\geqslant 0)$ is an A-module chain complex $\Gamma = \{C(\alpha)\,|\,\alpha\subseteq I_n\}$ together with a class $\begin{cases} \Phi \\ \Psi \end{cases}$ such that

i) if $m\geqslant 0$ $\begin{cases} (\Gamma,\Phi\in Q^m(\Gamma,\varepsilon)) \\ (\Gamma,\Psi\in Q_m(\Gamma,\varepsilon)) \end{cases}$ is an m-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré (n+1)-ad over A (as previously defined)

ii) (ε-symmetric case) if $m\leqslant -1$ $\Phi = (\Phi',\Phi'')$ with $\Phi'\in \bar{Q}^{-m}(S^{-m}\Gamma,\varepsilon)$ such that $(S^{-m}\Gamma,\Phi')$ is a $(-m)$-dimensional even ε-symmetric Poincaré (n+1)-ad over A, with

$$v_0(\partial_\alpha\Phi') = 0 : H^{m+n-|\alpha|}(\partial_\alpha\Gamma) \longrightarrow \hat{H}^0(\mathbb{Z}_2;A,\varepsilon) \qquad (\alpha\subseteq I_n) \quad ,$$

and $\Phi''\in Q_{-m}(S^{-m}\Gamma',\varepsilon)$ such that

$$(1+T_\varepsilon)\Phi'' = f^{\%}(\Phi')\in Q^{-m}(S^{-m}\Gamma',\varepsilon) \quad ,$$

where $\Gamma' = \{C'(\alpha)\,|\,\alpha\subseteq I_n\}$ is the A-module chain complex (n+1)-ad defined by

$$C'(\alpha) = \begin{cases} 0 & (m-n+|\alpha| = -1,-2) \\ C(\alpha) & (m-n+|\alpha|\leqslant -3) \end{cases}$$

and $f:\Gamma\longrightarrow\Gamma'$ is the natural projection

ii) (ε-quadratic case) if $m\leqslant -1$ $\Psi\in Q_{-m}(S^{-m}\Gamma,\varepsilon)$ with $(S^{-m}\Gamma,\Phi)$ a $(-m)$-dimensional ε-quadratic Poincaré (n+1)-ad over A.

Define $\begin{cases} \mathcal{L}^m(A,\varepsilon) \\ \mathcal{L}_m(A,\varepsilon) \end{cases}$ $(m\in\mathbb{Z})$ to be the simplicial monoid with n-simplexes the weakly (m+n)-dimensional $\begin{cases} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{cases}$ Poincaré (n+2)-ads over A $\begin{cases} (\Gamma,\Phi) \\ (\Gamma,\Psi) \end{cases}$ such that $C(\emptyset) = 0$, with the direct sum $\oplus$ as monoid operation and 0 as the base simplex in each dimension.

<u>Proposition 15.11</u> The simplicial monoids $\begin{cases}\mathcal{L}^m(A,\varepsilon)\\ \mathcal{L}_m(A,\varepsilon)\end{cases}$ $(m\in\mathbb{Z})$ satisfy the Kan extension condition, and are such that

$$\text{i)}\begin{cases}\pi_n(\mathcal{L}^m(A,\varepsilon)) = L^{m+n}(A,\varepsilon)\\ \pi_n(\mathcal{L}_m(A,\varepsilon)) = L_{m+n}(A,\varepsilon)\end{cases}\quad (n\geqslant 0)$$

$$\text{ii)}\begin{cases}\Omega\mathcal{L}^m(A,\varepsilon) = \mathcal{L}^{m+1}(A,\varepsilon)\\ \Omega\mathcal{L}_m(A,\varepsilon) = \mathcal{L}_{m+1}(A,\varepsilon)\end{cases}$$

$$\text{iii)}\begin{cases}\mathcal{L}^m(A,\varepsilon) = \mathbb{L}^m(A,\varepsilon)\\ \mathcal{L}_m(A,\varepsilon) = \mathbb{L}_m(A,\varepsilon)\end{cases}(m\geqslant 0)\ ,\ \begin{cases}\mathcal{L}^m(A,\varepsilon)(-m,-m+1,\dots,\infty) = \mathbb{L}^m(A,\varepsilon)\\ \mathcal{L}_m(A,\varepsilon)(-m,-m+1,\dots,\infty) = \mathbb{L}_m(A,\varepsilon)\end{cases}(m\leqslant -1)\ .$$

<u>Proof</u>: By analogy with Proposition 15.4.

[]

Thus the simplicial Ω-spectrum $\begin{cases}\underline{\mathcal{L}}^*(A,\varepsilon)\\ \underline{\mathcal{L}}_*(A,\varepsilon)\end{cases}$ with mth term $\begin{cases}\mathcal{L}^{-m}(A,\varepsilon)\\ \mathcal{L}_{-m}(A,\varepsilon)\end{cases}$ $(m\in\mathbb{Z})$

is such that

$$\begin{cases}\pi_n(\underline{\mathcal{L}}^*(A,\varepsilon)) = \pi_{m+n}(\mathcal{L}^{-m}(A,\varepsilon)) = L^n(A,\varepsilon)\\ \pi_n(\underline{\mathcal{L}}_*(A,\varepsilon)) = \pi_{m+n}(\mathcal{L}_{-m}(A,\varepsilon)) = L_n(A,\varepsilon)\end{cases}\quad (n\in\mathbb{Z},\ m+n\geqslant 0)\ .$$

The skew-suspension maps

$$\overline{S} : \mathcal{L}_m(A,\varepsilon) \longrightarrow \mathcal{L}_{m+2}(A,-\varepsilon)\quad (m\in\mathbb{Z})$$

are homotopy equivalences, so that $\mathcal{L}_*(A,\varepsilon)$ is a periodic Ω-spectrum.

The classifying spaces of the discrete orthogonal groups map into our algebraic L-spaces as follows.

Define the <u>orthogonal group</u> $\begin{cases}O^*(M,\varphi)\\ O_*(M,\psi)\end{cases}$ of a non-singular $\begin{cases}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{cases}$ form over A $\begin{cases}(M,\varphi)\\ (M,\psi)\end{cases}$ to be the group of automorphisms $\begin{cases}\alpha:(M,\varphi)\longrightarrow(M,\varphi)\\ \beta:(M,\psi)\longrightarrow(M,\psi)\end{cases}$.

Working as in the proof of Proposition 15.8 it is possible to construct a simplicial map

$$\begin{cases}\sigma^* : BO^*(M,\varphi) \equiv K(O^*(M,\varphi),1) \longrightarrow \mathbb{L}^0(A,\varepsilon)\\ \sigma_* : BO_*(M,\psi) \equiv K(O_*(M,\psi),1) \longrightarrow \mathbb{L}_0(A,\varepsilon)\end{cases}$$

given on the 1-skeletons by

$$\begin{cases}\sigma^*:BO^*(M,\varphi)^{(1)} = O^*(M,\varphi) \longrightarrow \mathbb{L}^0(A,\varepsilon)^{(1)}\ ;\ \alpha\longmapsto(M\oplus M,\varphi\oplus-\varphi;\Delta,(\alpha\oplus 1)\Delta)\\ \sigma_*:BO_*(M,\psi)^{(1)} = O_*(M,\psi) \longrightarrow \mathbb{L}_0(A,\varepsilon)^{(1)}\ ;\ \beta\longmapsto(M\oplus M,\psi\oplus-\psi;\Delta,(\beta\oplus 1)\Delta)\end{cases}$$

with $\Delta = \{(x,x)\in M\oplus M \mid x\in M\}$. (If $\begin{cases}(M,\varphi)\\ (M,\psi)\end{cases}$ is $\begin{cases}\text{metabolic}\\ \text{hyperbolic}\end{cases}$ with lagrangian L then $\begin{cases}\sigma^*(\alpha) = (M,\varphi;L,\alpha(L))\in L^1(A,\varepsilon)\\ \sigma_*(\beta) = (M,\psi;L,\beta(L))\in L_1(A,\varepsilon)\end{cases}$). In particular, the map appearing in Proposition 15.8 $\sigma^*:K(\pi,1)\longrightarrow\mathbb{L}^0(\mathbb{Z}[\pi])$ (for any group π) is the composite

$$\sigma^* : K(\pi,1) \xrightarrow{\ \iota\ } BO^*(\mathbb{Z}[\pi],1) \xrightarrow{\ \sigma^*\ } \mathbb{L}^0(\mathbb{Z}[\pi])\ ,$$

with $(\mathbb{Z}[\pi],1)$ the non-singular symmetric form over $\mathbb{Z}[\pi]$ defined by

$$1 : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi]^*\ ;\ a\longmapsto(b\longmapsto b\overline{a})$$

and ι the map of classifying spaces induced by the group morphism

$$\iota : \pi \longrightarrow O^*(\mathbb{Z}[\pi],1)\ ;\ g\longmapsto(g:(\mathbb{Z}[\pi],1)\longrightarrow(\mathbb{Z}[\pi],1))\quad(\overline{g} = g^{-1})\ .$$

Note also that $\sigma^*:BO^*(\mathbb{Z},1)\longrightarrow\mathbb{L}^0(\mathbb{Z})$ induces an isomorphism

$$\sigma^* : \pi_1(BO^*(\mathbb{Z},1)) = O^*(\mathbb{Z},1) = \mathbb{Z}_2 \longrightarrow L^1(\mathbb{Z}) = \mathbb{Z}_2$$

sending the generator $-1:(\mathbb{Z},1)\longrightarrow(\mathbb{Z},1)$ to the deRham element $d\in L^1(\mathbb{Z})$.

Karoubi [1] defines various L-spaces by applying the +construction of Quillen to the classifying space $B_\varepsilon O(A)$ of the stable ε-quadratic orthogonal group of a ring with involution A ${}_\varepsilon O(A) = \varinjlim_m O_*(H_\varepsilon(A^m))$ (and related spaces), and these L-spaces may be mapped to ours by the map $\sigma_*:B_\varepsilon O(A)^+\longrightarrow\mathbb{L}_0(A,\varepsilon)$ obtained from the maps $\sigma_*:BO_*(H_\varepsilon(A^m))\longrightarrow\mathbb{L}_0(A,\varepsilon)$ $(m\geqslant 1)$. Similarly, there is defined a map $\sigma^*:B^\varepsilon O(A)^+\longrightarrow\mathcal{L}^{-2}(A,-\varepsilon)$ with ${}^\varepsilon O(A) = \varinjlim_m O^*(H^\varepsilon(A^m))$.

§16. Mayer-Vietoris sequences

We shall now seek algebraic L-theoretic analogues of the Mayer-Vietoris exact sequence in the algebraic K-groups of a cartesian square of rings

$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B' & \longrightarrow & A' \end{array}$ with $B \longrightarrow A'$ onto (Milnor [5])

$$K_1(A) \longrightarrow K_1(B) \oplus K_1(B') \longrightarrow K_1(A') \longrightarrow K_0(A) \longrightarrow K_0(B) \oplus K_0(B') \longrightarrow K_0(A') \longrightarrow \dots ,$$

as well as of the Mayer-Vietoris exact sequence

$$\dots \longrightarrow K_n(A) \longrightarrow K_n(\hat{A}) \oplus K_n(S^{-1}A) \longrightarrow K_n(\hat{S}^{-1}\hat{A}) \longrightarrow K_{n-1}(A) \longrightarrow \dots \qquad (n \in \mathbb{Z})$$

of a cartesian localization-completion square $\begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{S}^{-1}\hat{A} \end{array}$ $(\hat{A} = \varprojlim_{s \in S} A/sA)$.

We shall construct the L-groups of commutative diagrams of rings with involution, such as the $\left\{\begin{array}{l} \varepsilon\text{-symmetric} \\ \varepsilon\text{-quadratic} \end{array}\right.$ L-groups $\left\{\begin{array}{l} L^n(\Phi,\varepsilon) \\ L_n(\Phi,\varepsilon) \end{array}\right.$ $(n \in \mathbb{Z})$ of a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \Phi & \downarrow \\ B' & \longrightarrow & A' \end{array}$$

whose vanishing $\left\{\begin{array}{l} L^*(\Phi,\varepsilon) = 0 \\ L_*(\Phi,\varepsilon) = 0 \end{array}\right.$ is the necessary and sufficient condition for there to be excision isomorphisms

$$\left\{\begin{array}{l} L^n(A \longrightarrow B,\varepsilon) \longrightarrow L^n(B' \longrightarrow A',\varepsilon) \ , \ L^n(A \longrightarrow B',\varepsilon) \longrightarrow L^n(B \longrightarrow A',\varepsilon) \\ L_n(A \longrightarrow B,\varepsilon) \longrightarrow L_n(B' \longrightarrow A',\varepsilon) \ , \ L_n(A \longrightarrow B',\varepsilon) \longrightarrow L_n(B \longrightarrow A',\varepsilon) \end{array}\right.$$

and Mayer-Vietoris exact sequences

$$\left\{\begin{array}{l} \dots \longrightarrow L^n(A,\varepsilon) \longrightarrow L^n(B,\varepsilon) \oplus L^n(B',\varepsilon) \longrightarrow L^n(A',\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon) \longrightarrow \dots \\ \dots \longrightarrow L_n(A,\varepsilon) \longrightarrow L_n(B,\varepsilon) \oplus L_n(B',\varepsilon) \longrightarrow L_n(A',\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \dots \end{array}\right.$$

We shall prove that $L_*(\Phi,\varepsilon) = 0$ (using the appropriate intermediate ε-quadratic L-groups) if Φ is a cartesian square with $B \longrightarrow A'$ onto, and also if Φ is a cartesian localization-completion square.

A morphism of rings with involution

$$f : A \longrightarrow B$$

is a function such that

$$f(a+b) = f(a) + f(b) \ , \ f(ab) = f(a)f(b) \ , \ f(1) = 1 \ , \ f(\bar{a}) = \overline{f(a)} \in B \quad (a,b \in A)$$

An (n+1)-ad of rings with involution A $(n \geqslant 0)$ is a directed system of rings with involution

$$\{A(\alpha) \mid \alpha \subseteq I_n = \{1,2,\dots,n\}\}$$

and morphisms

$$\{f(\alpha,\beta) : A(\beta) \longrightarrow A(\alpha) \mid \beta \subseteq \alpha \subseteq I_n\} ,$$

such that

$$f(\alpha,\alpha) = 1, \ f(\alpha,\beta)f(\beta,\gamma) = f(\alpha,\gamma) \qquad (\gamma \subseteq \beta \subseteq \alpha \subseteq I_n) \ .$$

In particular, a 0-ad is just a ring with involution $A(\emptyset)$, a 1-ad is a morphism of rings with involution $f(1,\emptyset) : A(\emptyset) \longrightarrow A(1)$, and a 2-ad is a commutative square of rings with involutions

$$\begin{array}{ccc} A(\emptyset) & \xrightarrow{f(1,\emptyset)} & A(1) \\ {\scriptstyle f(2,\emptyset)}\downarrow & & \downarrow{\scriptstyle f(12,1)} \\ A(2) & \xrightarrow{f(12,2)} & A(12) \end{array}$$

Given an (n+1)-ad of rings with involution A define for each $i \in I_n$ n-ads $\partial_i A$, $\delta_i A$ and for each $i \in I_{n+1}$ an (n+2)-ad $\delta_i A$ by

$$\partial_i A(\alpha) = A(\partial_i \alpha) \ , \ \delta_i A(\alpha) = A(\delta_i \alpha) \ , \ \sigma_i A(\alpha) = A(\sigma_i \alpha)$$

$$\partial_i f(\alpha,\beta) = f(\partial_i \alpha, \partial_i \beta) \ , \ \delta_i f(\alpha,\beta) = f(\delta_i \alpha, \delta_i \beta) \ , \ \sigma_i f(\alpha,\beta) = f(\sigma_i \alpha, \sigma_i \beta)$$

with

$$\partial_i : I_{n-1} \longrightarrow I_n \ ; \ j \longmapsto \left\{\begin{array}{ll} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{array}\right. , \quad \delta_i(\alpha) = \partial_i(\alpha) \cup \{i\} \ (i \in I_n, \alpha \subseteq I_{n-1})$$

$$\sigma_i(\alpha) = \partial_i^{-1}(\alpha) \subseteq I_{n-1} \quad (i \in I_{n+1}, \alpha \subseteq I_n)$$

(exactly as in §15 above, and §0 of Wall [5]).

Our methods serve to give a completely algebraic account of the geometrically defined surgery obstruction groups of (n+1)-ads of Theorem 3.1 of Wall [5].

<u>Proposition 16.1</u> Let A be an (n+1)-ad of rings with involution, and let $\varepsilon=\varepsilon(\emptyset)\in A(\emptyset)$ be a central unit such that $\bar{\varepsilon}=\varepsilon^{-1}\in A(\emptyset)$, and such that each $\varepsilon(\alpha)=f(\alpha,\emptyset)(\varepsilon)\in A(\alpha)$ $(\alpha\subseteq I_n)$ is a central unit. Then there are defined in a natural way the $\left\{\begin{array}{l}\underline{\varepsilon\text{-symmetric}}\\ \underline{\varepsilon\text{-quadratic}}\end{array}\right.$ <u>(n+1)-ad L-groups of A</u> $\left\{\begin{array}{l}L^m(A,\varepsilon)\\ L_m(A,\varepsilon)\end{array}\right.$ $(m\in\mathbb{Z})$ which are invariant under the permutation of factors in A, and come equipped with natural transformations $\left\{\begin{array}{l}\partial_i:L^m(A,\varepsilon)\longrightarrow L^{m-1}(\partial_iA,\varepsilon)\\ \partial_i:L_m(A,\varepsilon)\longrightarrow L_{m-1}(\partial_iA,\varepsilon)\end{array}\right.$ and natural equivalences $\left\{\begin{array}{l}\sigma_i:L^m(A,\varepsilon)\longrightarrow L^{m+1}(\sigma_iA,\varepsilon)\\ \sigma_i:L_m(A,\varepsilon)\longrightarrow L_{m+1}(\sigma_iA,\varepsilon)\end{array}\right.$ such that the inclusions $\partial_iA\xrightarrow{j}\delta_iA$, $\sigma_i\delta_iA\xrightarrow{k}A$ induce an exact sequence

$$\left\{\begin{array}{l}\dots\longrightarrow L^m(\partial_iA,\varepsilon)\xrightarrow{j}L^m(\delta_iA,\varepsilon)\xrightarrow{k\sigma_i}L^m(A,\varepsilon)\xrightarrow{\partial_i}L^{m-1}(\partial_iA,\varepsilon)\longrightarrow\dots\\ \dots\longrightarrow L_m(\partial_iA,\varepsilon)\xrightarrow{j}L_m(\delta_iA,\varepsilon)\xrightarrow{k\sigma_i}L_m(A,\varepsilon)\xrightarrow{\partial_i}L_{m-1}(\partial_iA,\varepsilon)\longrightarrow\dots\end{array}\right.\quad(i\in I_n)\,.$$

<u>Proof</u>: Let $\left\{\begin{array}{l}\underline{\mathcal{L}}^*(A,\varepsilon)\\ \underline{\mathcal{L}}_*(A,\varepsilon)\end{array}\right.$ be the (n+1)-ad of simplicial Ω-spectra defined by

$$\left\{\begin{array}{l}\underline{\mathcal{L}}^*(A,\varepsilon)(\alpha)=\underline{\mathcal{L}}^*(A(\alpha),\varepsilon(\alpha))\\ \underline{\mathcal{L}}_*(A,\varepsilon)(\alpha)=\underline{\mathcal{L}}_*(A(\alpha),\varepsilon(\alpha))\end{array}\right.\quad(\alpha\subseteq I_n)$$

using the construction of §15 for $\left\{\begin{array}{l}\underline{\mathcal{L}}^*\\ \underline{\mathcal{L}}_*\end{array}\right.$ in the 1-ad case, and set

$$\left\{\begin{array}{l}L^m(A,\varepsilon)=\pi_m(\underline{\mathcal{L}}^*(A,\varepsilon))\\ L_m(A,\varepsilon)=\pi_m(\underline{\mathcal{L}}_*(A,\varepsilon))\end{array}\right.\quad(m\in\mathbb{Z})\,.$$

Thus $\left\{\begin{array}{l}L^m(A,\varepsilon)\\ L_m(A,\varepsilon)\end{array}\right.$ $(m\in\mathbb{Z})$ is the cobordism group of collections $\{$weakly $(m+n-|\alpha|)$-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré $(n-|\alpha|+1)$-ad over $A(\alpha)$ $\left\{\begin{array}{l}(\Gamma_\alpha,\Phi_\alpha)\\ (\Gamma_\alpha,\Psi_\alpha)\end{array}\right.|\,\alpha\subseteq I_n\}$ such that $\left\{\begin{array}{l}\partial_i(\Gamma_\alpha,\Phi_\alpha)=A(\alpha)\otimes_{A(\beta)}(\Gamma_\beta,\Phi_\beta)\\ \partial_i(\Gamma_\alpha,\Psi_\alpha)=A(\alpha)\otimes_{A(\beta)}(\Gamma_\beta,\Psi_\beta)\end{array}\right.$ $(i\in\alpha\subseteq I_n,\ \beta=\alpha-\{i\})$.

[]

Let Φ be a commutative square (= 3-ad) of rings with involution

$$\begin{array}{ccc}A&\xrightarrow{\alpha}&B\\ {\scriptstyle\alpha'}\downarrow&\Phi&\downarrow{\scriptstyle\beta}\\ B'&\xrightarrow{\beta'}&A'\end{array}\,.$$

The $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ L-groups of Φ vanish, $\left\{\begin{array}{l}L^n(\Phi,\varepsilon)=0\\ L_n(\Phi,\varepsilon)=0\end{array}\right.$ $(n\in\mathbb{Z})$, if and only if there are excision isomorphisms

$$\left\{\begin{array}{l}(\alpha',\beta):L^n(\alpha,\varepsilon)\longrightarrow L^n(\beta',\varepsilon)\ ,\quad(\alpha,\beta'):L^n(\alpha',\varepsilon)\longrightarrow L^n(\beta,\varepsilon)\\ (\alpha',\beta):L_n(\alpha,\varepsilon)\longrightarrow L_n(\beta',\varepsilon)\ ,\quad(\alpha,\beta'):L_n(\alpha',\varepsilon)\longrightarrow L_n(\beta,\varepsilon)\end{array}\right.\quad(n\in\mathbb{Z})\,.$$

As usual, such excision isomorphisms imply a Mayer-Vietoris exact sequence

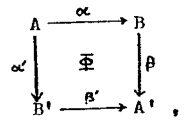
$$\left\{\begin{array}{l}\dots\longrightarrow L^{n+1}(A',\varepsilon)\xrightarrow{\partial}L^n(A,\varepsilon)\xrightarrow{\binom{\alpha}{\alpha'}}L^n(B,\varepsilon)\oplus L^n(B',\varepsilon)\xrightarrow{(\beta\ -\beta')}L^n(A',\varepsilon)\longrightarrow\dots\\ \dots\longrightarrow L_{n+1}(A',\varepsilon)\xrightarrow{\partial}L_n(A,\varepsilon)\xrightarrow{\binom{\alpha}{\alpha'}}L_n(B,\varepsilon)\oplus L_n(B',\varepsilon)\xrightarrow{(\beta\ -\beta')}L_n(A',\varepsilon)\longrightarrow\dots\end{array}\right.\quad(n\in\mathbb{Z})\,.$$

There is an appealing pseudo-geometric interpretation of this condition in terms of the union operation for algebraic Poincaré cobordisms of §5, as follows. Given a weakly (n-1)-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complex over A $\left\{\begin{array}{l}(C,\varphi)\\ (C,\psi)\end{array}\right.$ and null-cobordisms $\left\{\begin{array}{l}(f:B\otimes_AC\longrightarrow D,(\delta\varphi,1\otimes\varphi))\\ (f:B\otimes_AC\longrightarrow D,(\delta\psi,1\otimes\psi))\end{array}\right.$, $\left\{\begin{array}{l}(f':B'\otimes_AC\longrightarrow D',(\delta\varphi',1\otimes\varphi))\\ (f':B'\otimes_AC\longrightarrow D',(\delta\psi',1\otimes\psi))\end{array}\right.$ over B, B' respectively it is possible to glue together $\left\{\begin{array}{l}A'\otimes_B(f:B\otimes_AC\longrightarrow D,(\delta\varphi,1\otimes\varphi))\\ A'\otimes_B(f:B\otimes_AC\longrightarrow D,(\delta\psi,1\otimes\psi))\end{array}\right.$ and $\left\{\begin{array}{l}A'\otimes_{B'}(f':B\otimes_AC\longrightarrow D',(\delta\varphi',1\otimes\varphi))\\ A'\otimes_{B'}(f':B\otimes_AC\longrightarrow D',(\delta\psi',1\otimes\psi))\end{array}\right.$ to define a weakly n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complex over A'

$$\left\{\begin{array}{l}(C',\varphi')=(A'\otimes_BD\cup_{A'\otimes_AC}A'\otimes_{B'}D',1\otimes_B\delta\varphi\cup_{1\otimes_A\varphi}-(1\otimes_{B'}\delta\varphi'))\\ (C',\psi')=(A'\otimes_BD\cup_{A'\otimes_AC}A'\otimes_{B'}D',1\otimes_B\delta\psi\cup_{1\otimes_A\psi}-(1\otimes_{B'}\delta\psi'))\end{array}\right.\,.$$

Then $\left\{\begin{array}{l}L^*(\Phi,\varepsilon)=0\\ L_*(\Phi,\varepsilon)=0\end{array}\right.$ if and only if every element of $\left\{\begin{array}{l}L^n(A',\varepsilon)\\ L_n(A',\varepsilon)\end{array}\right.$ (for every $n\in\mathbb{Z}$) is represented by a weakly n-dimensional $\left\{\begin{array}{l}\varepsilon\text{-symmetric}\\ \varepsilon\text{-quadratic}\end{array}\right.$ Poincaré complex over A' $\left\{\begin{array}{l}(C',\varphi')\\ (C',\psi')\end{array}\right.$ with such a Mayer-Vietoris decomposition, and the decomposition is unique up to cobordism. Actually, we shall find the following criterion for excision more useful.

<u>Proposition 16.2</u> Let Φ be a commutative square of rings with involution



and let $\varepsilon \in A$ be a central unit such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$ and such that the images of ε in A', B, B' (also to be denoted by ε) are central as well.

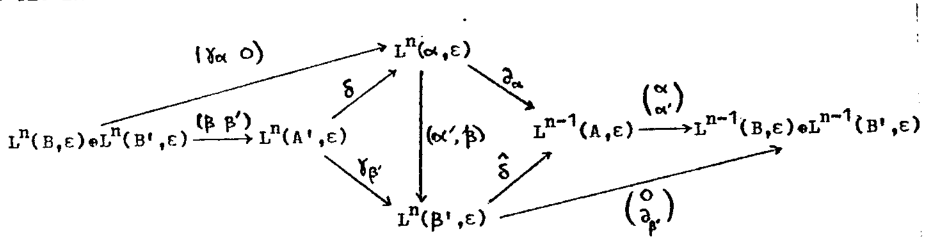
The natural map of ε-symmetric L-groups

$$(\alpha',\beta) \;:\; L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon)$$

is an isomorphism (for some $n \in \mathbb{Z}$) if and only if there exist abelian group morphisms

$$\delta \;:\; L^n(A',\varepsilon) \longrightarrow L^n(\alpha,\varepsilon) \quad , \quad \hat{\delta} : L^n(\beta',\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon)$$

to fit into a commutative diagram



involving the exact sequences

$$L^n(B,\varepsilon) \xrightarrow{\gamma_\alpha} L^n(\alpha,\varepsilon) \xrightarrow{\partial_\alpha} L^{n-1}(A,\varepsilon) \xrightarrow{\alpha} L^{n-1}(B,\varepsilon)$$
$$L^n(B',\varepsilon) \xrightarrow{\beta'} L^n(A',\varepsilon) \xrightarrow{\gamma_{\beta'}} L^n(\beta',\varepsilon) \xrightarrow{\partial_{\beta'}} L^{n-1}(B',\varepsilon) \;.$$

Similarly for the natural map of ε-quadratic L-groups

$$(\alpha',\beta) \;:\; L_n(\alpha,\varepsilon) \longrightarrow L_n(\beta',\varepsilon) \;.$$

<u>Proof</u>: If $(\alpha',\beta) : L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon)$ is an isomorphism define

$$\delta = (\alpha',\beta)^{-1}\gamma_{\beta'} : L^n(A',\varepsilon) \longrightarrow L^n(\alpha,\varepsilon) \;,\; \hat{\delta} = \partial_\alpha(\alpha',\beta)^{-1} : L^n(\beta',\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon) \;.$$

Conversely, given $\delta, \hat{\delta}$ we shall verify that (α',β) is an isomorphism by diagram chasing as follows.

Let $x \in \ker((\alpha',\beta) : L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon))$, so that

$$\partial_\alpha x = \hat{\delta}(\alpha',\beta)x = 0 \in L^{n-1}(A,\varepsilon)$$

and $x \in \ker(\partial_\alpha : L^n(\alpha,\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon)) = \mathrm{im}(\gamma_\alpha : L^n(B,\varepsilon) \longrightarrow L^n(\alpha,\varepsilon))$. Let $y \in L^n(B,\varepsilon)$ be such that

$$\gamma_\alpha(y) = x \in L^n(\alpha,\varepsilon) \;,$$

so that

$$\gamma_{\beta'}\beta(y) = (\alpha',\beta)\gamma_\alpha(y) = (\alpha',\beta)(x) = 0 \in L^n(\beta',\varepsilon)$$

and $\beta(y) \in \ker(\gamma_{\beta'} : L^n(A',\varepsilon) \longrightarrow L^n(\beta',\varepsilon)) = \mathrm{im}(\beta' : L^n(B',\varepsilon) \longrightarrow L^n(A',\varepsilon))$.
Let $z \in L^n(B',\varepsilon)$ be such that

$$\beta(y) = \beta'(z) \in L^n(A',\varepsilon) \;,$$

so that

$$x = \gamma_\alpha(y) = \delta\beta(y) = \delta\beta'(z) = 0 \in L^n(\alpha,\varepsilon)$$

and $(\alpha',\beta) : L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon)$ is one-one.

Let $u \in L^n(\beta',\varepsilon)$, so that

$$\alpha\hat{\delta}(u) = 0 \in L^{n-1}(B,\varepsilon)$$

and $\hat{\delta}(u) \in \ker(\alpha : L^{n-1}(A,\varepsilon) \longrightarrow L^{n-1}(B,\varepsilon)) = \mathrm{im}(\partial_\alpha : L^n(\alpha,\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon))$.
Let $v \in L^n(\alpha,\varepsilon)$ be such that

$$\hat{\delta}(u) = \partial_\alpha(v) \in L^{n-1}(A,\varepsilon) \;,$$

so that

$$(u - (\alpha',\beta)(v)) \in \ker(\hat{\delta} : L^n(\beta',\varepsilon) \rightarrow L^{n-1}(A,\varepsilon)) \subseteq \ker(\partial_{\beta'} : L^n(\beta',\varepsilon) \longrightarrow L^{n-1}(B',\varepsilon))$$
$$= \mathrm{im}(\gamma_{\beta'} : L^n(A',\varepsilon) \longrightarrow L^n(\beta',\varepsilon)).$$

Let $w \in L^n(A',\varepsilon)$ be such that

$$u - (\alpha',\beta)(v) = \gamma_{\beta'}(w) \in L^n(\beta',\varepsilon) \;,$$

so that

$$u = (\alpha',\beta)(v + \delta(w)) \in \mathrm{im}((\alpha',\beta) : L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon))$$

and $(\alpha',\beta) : L^n(\alpha,\varepsilon) \longrightarrow L^n(\beta',\varepsilon)$ is onto.

[]

Proposition 16.3 Let Φ be a cartesian square of rings with involution

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\alpha'}\downarrow & \Phi & \downarrow{\scriptstyle\beta} \\ B' & \xrightarrow{\beta'} & A' \end{array} ,$$

such that the sequence

$$0 \longrightarrow A \xrightarrow{\binom{\alpha}{\alpha'}} B\oplus B' \xrightarrow{(\beta\ -\beta')} A' \longrightarrow 0$$

is exact, and let $X\subseteq \widetilde{K}_m(A), Y\subseteq \widetilde{K}_m(B), Y'\subseteq \widetilde{K}_m(B'), X'\subseteq \widetilde{K}_m(A')$ $(m=0,1)$ be $*$-invariant subgroups such that

$$\ker(\binom{\alpha}{\alpha'}:\widetilde{K}_m(A)\longrightarrow \widetilde{K}_m(B)\oplus\widetilde{K}_m(B'))\subseteq X \ , \ B\otimes_A X\subseteq Y \ , \ B'\otimes_A X'\subseteq Y' \ ,$$
$$A'\otimes_B Y\subseteq X' \ , \ A'\otimes_{B'} Y'\subseteq X'$$

and such that the sequence

$$0\longrightarrow X/\ker(\binom{\alpha}{\alpha'}:\widetilde{K}_m(A)\longrightarrow\widetilde{K}_m(B)\oplus\widetilde{K}_m(B')) \xrightarrow{\binom{\alpha}{\alpha'}} Y\oplus Y' \xrightarrow{(\beta\ -\beta')} X' \longrightarrow 0$$

is exact. Then if

either i) $\beta:B\longrightarrow A'$ (or $\beta':B'\longrightarrow A'$) is onto

or ii) $\alpha':A\longrightarrow B' = S^{-1}A$ and $\beta:B\longrightarrow A' = T^{-1}B$ are localization maps such that $\alpha:(A,S)\longrightarrow(B,T)$ is a cartesian morphism (in the sense of §13)

there are defined excision isomorphisms

$$(\alpha',\beta) : L_n^{Y,X}(\alpha,\varepsilon)\longrightarrow L_n^{X',Y'}(\beta',\varepsilon) \ , \ (\alpha,\beta') : L_n^{Y',X}(\alpha',\varepsilon)\longrightarrow L_n^{X',Y}(\beta,\varepsilon) \ (n\in\mathbb{Z})$$

and a Mayer-Vietoris exact sequence

$$\ldots\longrightarrow L_{n+1}^{X'}(A',\varepsilon)\xrightarrow{\partial} L_n^X(A,\varepsilon)\xrightarrow{\binom{\alpha}{\alpha'}} L_n^Y(B,\varepsilon)\oplus L_n^{Y'}(B',\varepsilon)\xrightarrow{(\beta\ -\beta')} L_n^{X'}(A',\varepsilon)\longrightarrow\ldots \quad (n\in\mathbb{Z}).$$

Proof: It is sufficient to consider only the case

$$X = \widetilde{K}_1(A) \ , \ Y = \widetilde{K}_1(B) \ , \ Y' = \widetilde{K}_1(B') \ , \ X' = \mathrm{im}((\beta\ \beta'):\widetilde{K}_1(B)\oplus\widetilde{K}_1(B')\longrightarrow\widetilde{K}_1(A')).$$

The other cases may be deduced from this by repeated application of the exact sequences of Proposition 12.1, working as in §5 of Ranicki [3].

Given a f.g. projective B-module M, a f.g. projective B'-module M', and an isomorphism of the induced f.g. projective A'-modules

$$f : A'\otimes_B M\longrightarrow A'\otimes_{B'} M'$$

there is defined a f.g. projective A-module

$$(M,f,M') = \{(x,x')\in M\oplus M' \mid f(1\otimes x) = 1\otimes x'\in A'\otimes_{B'} M'\} \ ,$$

with A acting by

$$A\times(M,f,M')\longrightarrow(M,f,M') \ ; \ (a,(x,x'))\longmapsto(\alpha(a)x,\alpha'(a)x') \ .$$

We shall use the natural B-module isomorphism

$$B\otimes_A(M,f,M')\longrightarrow M \ ; \ 1\otimes(x,x')\longmapsto x$$

to identify $B\otimes_A(M,f,M') = M$, and similarly for the natural B'-module isomorphism

$$B'\otimes_A(M,f,M')\longrightarrow M' \ ; \ 1\otimes(x,x')\longmapsto x' \ .$$

There is also defined a natural A-module isomorphism

$$(M^*,f^{*-1},M'^*)\longrightarrow(M,f,M')^* \ ;$$
$$(g,g')\longmapsto((x,x')\longmapsto(g(x),g'(x'))\in A = \ker((\beta\ -\beta'):B\oplus B'\longrightarrow A'))) \ .$$

Morphisms also pull back. Given f.g. projective B-modules M,N, f.g. projective B'-modules M',N' and A'-module isomorphisms $f\in\mathrm{Hom}_{A'}(A'\otimes_B M,A'\otimes_{B'}M')$, $g\in\mathrm{Hom}_{A'}(A'\otimes_B N,A'\otimes_{B'}N')$ there is defined an exact sequence of abelian groups

$$0\longrightarrow\mathrm{Hom}_A((M,f,M'),(N,g,N'))\longrightarrow\mathrm{Hom}_B(M,M')\oplus\mathrm{Hom}_{B'}(N,N')$$
$$\longrightarrow\mathrm{Hom}_{A'}(A'\otimes_B M,A'\otimes_{B'}N')\longrightarrow 0 \ .$$

In particular, for $N = M^*$, $g = f^{*-1}$, $N' = M'^*$ there is an exact sequence

$$0\longrightarrow\mathrm{Hom}_A((M,f,M'),(M,f,M')^*)\longrightarrow\mathrm{Hom}_B(M,M^*)\oplus\mathrm{Hom}_{B'}(M',M'^*)$$
$$\longrightarrow\mathrm{Hom}_{A'}(A'\otimes_B M,A'\otimes_B M^*)\longrightarrow 0$$

so that it is possible to pull back split ε-quadratic forms, as detailed below. (The theory of split ε-quadratic forms was developed in §12. It is also possible to pull back ε-quadratic forms, but the sequence

$$0\longrightarrow Q_\varepsilon((M,f,M'))\longrightarrow Q_\varepsilon(M)\oplus Q_\varepsilon(M')\longrightarrow Q_\varepsilon(A'\otimes_B M)\longrightarrow 0$$

may fail to be exact at $Q_\varepsilon((M,f,M'))$, so that the pullback ε-quadratic form is not uniquely determined. In the following example (due to Pardon)

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_2] & \xrightarrow{e_+} & \mathbb{Z} \\ {\scriptstyle e_-}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{Z} & \xrightarrow{p} & \mathbb{Z}_2 \end{array} \qquad \begin{array}{l} (e_\pm(a+bT) = a\pm b \in \mathbb{Z},\ a,b\in\mathbb{Z} \\ \ p = \text{projection}) \end{array}$$

it is the case that

$$Q_{-1}(\mathbb{Z}[\mathbb{Z}_2]) = \mathbb{Z}[\mathbb{Z}_2]/2\mathbb{Z}[\mathbb{Z}_2] \longrightarrow Q_{-1}(\mathbb{Z})\oplus Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z}_2 \ ;$$
$$[a+bT] \longmapsto ([a+b],[a-b])$$

is not one-one).

Given a split ε-quadratic form over B $(M,\psi\in \mathrm{Hom}_B(M,M^*))$, a split ε-quadratic form over B' $(M',\psi'\in\mathrm{Hom}_{B'}(M',M'^*))$, and an isomorphism of the induced split ε-quadratic forms over A'

$$(f,\chi)\ :\ A'\otimes_B(M,\psi) \longrightarrow A'\otimes_{B'}(M',\psi')$$

define a pullback split ε-quadratic form over A

$$((M,\psi),(f,\chi),(M',\psi'))$$
$$= ((M,f,M'),(\psi+\chi_B-\varepsilon\chi_B^*,\psi'+\chi_{B'}-\varepsilon\chi_{B'}^*):(M,f,M')\longrightarrow(M^*,f^{*-1},M'^*) = (M,f,M')^*),$$

using any of the ε-quadratic forms $(M,\chi_B\in Q_\varepsilon(M)),(M',\chi_{B'}\in Q_\varepsilon(M'))$ such that

$$\chi = 1\otimes\chi_B - f^*(1\otimes\chi_{B'})f \in Q_\varepsilon(A'\otimes_B M) \ .$$

The isomorphism class of the pullback does not depend on the choices of $\chi_B,\chi_{B'}$.

i) Let $\beta:B\longrightarrow A'$ be onto.

Every element of $V_{2i}^{X'}(A',\varepsilon)$ is represented by a non-singular $(-)^i\varepsilon$-quadratic form over A' $(A'^p,\psi'\in Q_{(-)^i\varepsilon}(A'^p))$ (for some $p\geqslant 0$) such that $\tau(\psi'+(-)^i\varepsilon\psi'^*:A'^p\longrightarrow(A'^p)^*)\in X'\subseteq\widetilde{K}_1(A')$, and such that there exists $\psi\in\mathrm{Hom}_B(B^p,(B^p)^*)$ with

$$\psi' = 1\otimes\psi\in Q_{(-)^i\varepsilon}(A'^p) \ .$$

There also exists $\chi\in\mathrm{Hom}_B((B^p)^*,B^p)$ such that

$$(\psi'+(-)^i\varepsilon\psi'^*)^{-1}\psi'(\psi'+(-)^i\varepsilon\psi'^*)^{-1} = 1\otimes\chi\in Q_{(-)^i\varepsilon}((A'^p)^*) \ .$$

The split $(-)^{i-1}\varepsilon$-quadratic formation over A

$$(F,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)G) = (A^p=(B^p,1,B'^p),(\begin{pmatrix}(1-(\chi+(-)^i\varepsilon\chi^*)(\psi+(-)^i\varepsilon\psi^*),0)\\ (\psi+(-)^i\varepsilon\psi^*,1)\end{pmatrix},$$
$$\psi-(\psi+(-)^i\varepsilon\psi^*)^*\chi(\psi+(-)^i\varepsilon\psi^*))(B^p,\psi'+(-)^i\varepsilon\psi'^*,(B'^p)^*))$$

becomes null-cobordant over B, with an isomorphism

$$(1,1,\chi)\ :\ B\otimes_A(F,G) = (B^p,(\begin{pmatrix}1-(\chi+(-)^i\varepsilon\chi^*)(\psi+(-)^i\varepsilon\psi^*)\\ \psi+(-)^i\varepsilon\psi^*\end{pmatrix},$$
$$\psi-(\psi+(-)^i\varepsilon\psi^*)^*\chi(\psi+(-)^i\varepsilon\psi^*))B^p)$$
$$\longrightarrow \partial(B^p,\psi) = (B^p,(\begin{pmatrix}1\\ \psi+(-)^i\varepsilon\psi^*\end{pmatrix},\psi)B^p) \ .$$

This defines an abelian group morphism

$$\delta\ :\ V_{2i}^{X'}(A',\varepsilon)\longrightarrow V_{2i}(\alpha,\varepsilon)\ ;\ (A'^p,\psi')\longmapsto((F,G),(B^p,\psi),(1,1,\chi)) \ .$$

Every element of $V_{2i+1}^{X'}(A',\varepsilon)$ is represented by an automorphism of split hyperbolic $(-)^i\varepsilon$-quadratic forms over A' $(f,\chi):\widetilde{H}_{(-)^i\varepsilon}(A'^p)\to\widetilde{H}_{(-)^i\varepsilon}(A'^p)$ (for some $p\geqslant 0$) such that $\tau(f:A'^p\oplus(A'^p)^*\longrightarrow A'^p\oplus(A'^p)^*)\in X'\subseteq\widetilde{K}_1(A')$. The non-singular split $(-)^i\varepsilon$-quadratic form over A

$$(M,\widetilde{\psi}) = (\widetilde{H}_{(-)^i\varepsilon}(B^p),(f,\chi),\widetilde{H}_{(-)^i\varepsilon}(B'^p))$$

becomes null-cobordant over B, with an isomorphism of $(-)^i\varepsilon$-quadratic forms over A

$$1\ :\ B\otimes_A(M,\psi) = H_{(-)^i\varepsilon}(B^p)\longrightarrow\partial(B^p,0) = H_{(-)^i\varepsilon}(B^p) \ .$$

This defines an abelian group morphism

$$\delta:\ V_{2i+1}^{X'}(A',\varepsilon)\longrightarrow V_{2i+1}(\alpha,\varepsilon)\ ;$$
$$(H_{(-)^i\varepsilon}(A'^p);A'^p,f(A'^p))\longmapsto((M,\psi),(B^p,0),1) \ .$$

Every element of $V_{2i}^{X'}(\beta',\varepsilon)$ is represented by a non-singular split $(-)^{i-1}\varepsilon$-quadratic formation over B' $(B'^p,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)B'^p)$ together with a $(-)^i\varepsilon$-quadratic form over A' $(A'^p,\lambda'\in Q_{(-)^i\varepsilon}(A'^p))$ and an isomorphism of split $(-)^i\varepsilon$-quadratic formations over A'

$$(a,b,c)\ :\ A'\otimes_{B'}(B'^p,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)B'^p)\longrightarrow\partial(A'^p,\lambda')=(A'^p,(\begin{pmatrix}1\\ \lambda'+(-)^i\varepsilon\lambda'^*\end{pmatrix},\lambda')A'^p)\ ,$$

such that $\tau(a:A'^p\longrightarrow A'^p),\ \tau(b:A'^p\longrightarrow A'^p)\in X'\subseteq\widetilde{K}_1(A')$.

There exists $\chi\in Q_{(-)^i\varepsilon}(B^p)$ such that

$$\lambda' = 1\otimes\chi\in Q_{(-)^i\varepsilon}(A'^p)\ .$$

Define a non-singular split $(-)^{i-1}\varepsilon$-quadratic form over A

$$(M,\widetilde{\psi}\in\mathrm{Hom}_A(M,M^*)) = (\widetilde{H}_{(-)^i\varepsilon}(B^p),(\begin{pmatrix}a & a(c+(-)^i\varepsilon c^*)^*\\ 0 & a^{*-1}\end{pmatrix},\begin{pmatrix}0&0\\0&c\end{pmatrix}),\widetilde{H}_{(-)^i\varepsilon}(B'^p))\ .$$

Define lagrangians F,G for the associated $(-)^{i-1}\varepsilon$-quadratic form over A $(M,\psi\in Q_{(-)^{i-1}\varepsilon}(M))$ by

$$F = \mathrm{im}((\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}):A^p=(B^p,1,B'^p)\longrightarrow M)$$

$$G = \mathrm{im}((\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\begin{pmatrix}1\\ \chi+(-)^i\varepsilon\chi^*\end{pmatrix}):(B^p,b,B'^p)\longrightarrow M)\qquad .$$

This defines a morphism of abelian groups

$$\hat{\delta}\ :\ V_{2i}^{X'}(\beta',\varepsilon)\longrightarrow V_{2i-1}(A,\varepsilon)\ ;$$
$$((B'^p,(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta)B'^p),(A'^p,\chi'),(a,b,c))\longmapsto(M,\psi;F,G)\ .$$

Every element of $V_{2i+1}^{X'}(\beta',\varepsilon)$ is represented by a non-singular split $(-)^i\varepsilon$-quadratic form over B' (B'^{2p},ψ') together with an isomorphism of split $(-)^i\varepsilon$-quadratic forms over A'

$$(f,\chi)\ :\ \widetilde{H}_{(-)^i\varepsilon}(A'^p)\longrightarrow A'\otimes_{B'}(B'^{2p},\psi')$$

such that $\tau(f:A'^p\oplus(A'^p)^*\longrightarrow A'^{2p})\in X'\subseteq\widetilde{K}_1(A')$. This defines an abelian group morphism

$$\hat{\delta}\ :\ V_{2i+1}^{X'}(\beta',\varepsilon)\longrightarrow V_{2i}(A,\varepsilon)\ ;$$
$$((B'^{2p},\psi'),(f,\chi))\longmapsto(H_{(-)^i\varepsilon}(B^p),(f,\chi),(B'^{2p},\psi'))\qquad .$$

The abelian group morphisms

$$\delta\ :\ V_n^{X'}(A',\varepsilon)\longrightarrow V_n(\alpha,\varepsilon)\ ,\quad \hat{\delta}\ :\ V_n^{X'}(\beta',\varepsilon)\longrightarrow V_{n-1}(A,\varepsilon)\qquad (n\in\mathbb{Z})$$

satisfy the hypotheses of Proposition 16.2, so that we have excision isomorphisms

$$(\alpha',\beta)\ :\ V_n(\alpha,\varepsilon)\longrightarrow V_n^{X'}(\beta',\varepsilon)\ ,\quad (\alpha,\beta'):V_n(\alpha',\varepsilon)\longrightarrow V_n^{X'}(\beta,\varepsilon)\qquad (n\in\mathbb{Z})$$

and a Mayer-Vietoris exact sequence

$$\ldots\longrightarrow V_n(A,\varepsilon)\longrightarrow V_n(B,\varepsilon)\oplus V_n(B',\varepsilon)\longrightarrow V_n^{X'}(A',\varepsilon)\longrightarrow V_{n-1}(A,\varepsilon)\longrightarrow\ldots\qquad (n\in\mathbb{Z}).$$

ii) Let $\alpha:(A,S)\longrightarrow(B,T)$ be a cartesian morphism of rings with involution and multiplicative subsets, and let Φ be the cartesian square

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & B\\ \downarrow & \Phi & \downarrow\\ S^{-1}A & \longrightarrow & T^{-1}B\end{array}$$

Neither $B\longrightarrow T^{-1}B$ nor $S^{-1}A\longrightarrow T^{-1}B$ is onto (in general), but for every element $x\in T^{-1}B$ there exists $t\in T$ such that $tx\in\mathrm{im}(B\longrightarrow T^{-1}B)$. Thus every ε-quadratic form over $T^{-1}B$ $(M,\psi\in Q_\varepsilon(M))$ with M a f.g. free $T^{-1}B$-module is isomorphic to the form induced over $T^{-1}B$ from an ε-quadratic form over B $(\hat{M},\hat{\psi}\in Q_\varepsilon(\hat{M}))$, via an isomorphism of the type

$$t^{-1}\ :\ (M,\psi)\longrightarrow(M,\bar{t}\psi t) = T^{-1}B\otimes_B(\hat{M},\hat{\psi})\quad (t\in T).$$

We can thus work as in i), and define abelian group morphisms

$$\delta\ :\ V_n^{X'}(S^{-1}A,\varepsilon)\longrightarrow V_n(A\longrightarrow B,\varepsilon)\ ,\ \hat{\delta}:\ V_n^{X'}(S^{-1}A\longrightarrow T^{-1}B,\varepsilon)\longrightarrow V_{n-1}(A,\varepsilon)$$
$$(n\in\mathbb{Z},\ X'=\mathrm{im}(\widetilde{K}_1(S^{-1}A)\oplus\widetilde{K}_1(B)\longrightarrow\widetilde{K}_1(T^{-1}B)))$$

to satisfy the conditions of Proposition 16.2, obtaining excision isomorphisms and a Mayer-Vietoris sequence. Alternatively, we can use the localization exact sequence of Proposition 13.15 and the natural equivalence of categories

(h.d.1 S-torsion A-modules) = (h.d.1 T-torsion B-modules)

(cf. Proposition 13.23). []

Mayer-Vietoris exact sequences such as those of Proposition 16.3 have also been obtained by Wall [9], Karoubi [2], Bak [1].

Let I be a ring with involution, possibly without $1 \in I$. Define the <u>quadratic L-groups of I</u>

$$L_n(I) = \ker(L_n(I^+) \longrightarrow L_n(\mathbb{Z})) \quad (n \in \mathbb{Z})$$

where $I^+ = I \oplus \mathbb{Z}$ is a ring with involution and $1 = (0,1) \in I^+$, and

$$I^+ \longrightarrow \mathbb{Z}\ ;\ (i,z) \longmapsto z\ .$$

If there exists $1 \in I$ this agrees with the previous definition of $L_n(I)$.

Let I be a 2-sided ideal of a ring with involution A (such that $1 \in A$) which is invariant under the involution, $\bar{i} \in I \subseteq A$ $(i \in I)$, so that the quotient A/I is also a ring with involution. Define the <u>quadratic L-groups of (A,I)</u>

$$L_n(A,I) = L^X_{n+1}(A \longrightarrow A/I) \quad (n \in \mathbb{Z})$$

with $A \longrightarrow A/I$ the natural projection and $X = \operatorname{im}(\widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(A/I)) \subseteq \widetilde{K}_0(A/I)$.

There is thus an exact sequence of quadratic L-groups

$$\ldots \longrightarrow L_n(A,I) \longrightarrow L_n(A) \longrightarrow L^X_n(A/I) \longrightarrow L_{n-1}(A,I) \longrightarrow \ldots \quad (n \in \mathbb{Z})\ .$$

By analogy with the excision isomorphisms in algebraic K-theory (Bass [1])

$$K_n(I) = K_{n+1}(I^+ \longrightarrow \mathbb{Z}) \longrightarrow K_n(A,I) = K_{n+1}(A \longrightarrow A/I) \quad (n \leq 0)$$

we have the following excision isomorphisms in algebraic L-theory:

<u>Proposition 16.4</u> If I is a 2-sided ideal of a ring with involution A which is invariant under the involution the natural maps

$$L_n(I) = L_{n+1}(I^+ \longrightarrow \mathbb{Z}) \longrightarrow L_n(A,I) = L^X_{n+1}(A \longrightarrow A/I) \quad (n \in \mathbb{Z})$$

are excision isomorphisms.

<u>Proof</u>: Apply Proposition 16.3 ii) to the cartesian square of rings with involution

$$\begin{array}{ccc} I^+ & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/I \end{array},$$

in which $A \longrightarrow A/I$ is onto.

[]

The ε-symmetric L-groups do not appear to have as good excision and Mayer-Vietoris properties for cartesian squares as the ε-quadratic L-groups. Here are some partial results.

As before, let Φ be a cartesian square of rings with involution

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \alpha'}\downarrow & \Phi & \downarrow{\scriptstyle \beta} \\ B' & \xrightarrow{\beta'} & A' \end{array},$$

and let $X \subseteq \widetilde{K}_m(A), X' \subseteq \widetilde{K}_m(A'), Y \subseteq \widetilde{K}_m(B), Y' \subseteq \widetilde{K}_m(B')$ $(m = 0,1)$ be $*$-invariant subgroups such that there is defined an exact sequence

$$0 \longrightarrow X/\ker(\begin{pmatrix}\alpha\\ \alpha'\end{pmatrix}: \widetilde{K}_m(A) \longrightarrow \widetilde{K}_m(B) \oplus \widetilde{K}_m(B')) \xrightarrow{\begin{pmatrix}\alpha\\ \alpha'\end{pmatrix}} Y \oplus Y' \xrightarrow{(\beta\ -\beta')} X' \longrightarrow 0\ .$$

<u>Proposition 16.5</u> i) If $\beta : B \longrightarrow A'$ is onto there are defined excision isomorphisms

$$(\alpha',\beta) : L^n_{Y,X}(\alpha,\varepsilon) \longrightarrow L^n_{X',Y'}(\beta',\varepsilon)\ ,\ (\alpha,\beta') : L^n_{Y',X}(\alpha',\varepsilon) \longrightarrow L^n_{X',Y}(\beta,\varepsilon) \quad (n \leq -1)$$

and a Mayer-Vietoris exact sequence

$$L^{-1}_X(A,\varepsilon) \xrightarrow{\begin{pmatrix}\alpha\\ \alpha'\end{pmatrix}} L^{-1}_Y(B,\varepsilon) \oplus L^{-1}_{Y'}(B',\varepsilon) \xrightarrow{(\beta\ -\beta')} L^{-1}_{X'}(A',\varepsilon) \xrightarrow{\partial} L^{-2}_X(A,\varepsilon) \xrightarrow{\begin{pmatrix}\alpha\\ \alpha'\end{pmatrix}} L^{-2}_Y(B,\varepsilon) \oplus L^{-2}_{Y'}(B',\varepsilon) \xrightarrow{(\beta\ -\beta')} \ldots\ .$$

ii) If Φ is the cartesian square $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S^{-1}A & \longrightarrow & T^{-1}B \end{array}$ associated to a cartesian morphism $\alpha : (A,S) \longrightarrow (B,T)$ there are defined excision isomorphisms

$$\left.\begin{array}{l} L^n_{Y,X}(A \longrightarrow B,\varepsilon) \longrightarrow L^n_{X',Y'}(S^{-1}A \longrightarrow T^{-1}B,\varepsilon) \\ L^n_{Y',X}(A \longrightarrow S^{-1}A,\varepsilon) \longrightarrow L^n_{X',Y}(B \longrightarrow T^{-1}B,\varepsilon) \end{array}\right. \quad (n \in \mathbb{Z})$$

and a Mayer-Vietoris exact sequence

$$\ldots \longrightarrow L^n_X(A,\varepsilon) \longrightarrow L^n_Y(B,\varepsilon) \oplus L^n_{Y'}(S^{-1}A,\varepsilon) \longrightarrow L^n_{X'}(T^{-1}B,\varepsilon) \longrightarrow L^{n-1}_X(A,\varepsilon) \longrightarrow \ldots\ .$$

<u>Proof</u>: i) By analogy with the proof of Proposition 16.3.

ii) Immediate from Propositions 13.23, 12.1.

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§17 Codimension 2 surgery

Algebraic Poincaré complexes can also be used to construct the homology surgery obstruction groups arising in codimension 2 surgery, such as the knot cobordism groups C_n of Kervaire [1] and the non-simply-connected generalizations of Cappell and Shaneson [1], Matsumoto [1] and Smith [2]. Given morphisms of rings with involution $A \longrightarrow B$, $B \longrightarrow B'$ we shall define cobordism groups $L_n(A;B,B',\varepsilon)$ $(n \geqslant 0)$ of n-dimensional ε-quadratic complexes over A which become Poincaré over B and contractible over B'. For a group ring $A = \mathbb{Z}[\pi]$ the quadratic L-groups $L_n(A;B,B')$ are the obstruction groups to performing surgery to B-homology equivalence on the interior of an n-dimensional normal map of pairs $(f,b):(M,\partial M) \longrightarrow (X,\partial X)$ $(\pi = \pi_1(X))$ such that $f|:\partial M \longrightarrow \partial X$ is a B-homology equivalence and $f:M \longrightarrow X$ is a B'-homology equivalence, in such a way that the trace of the surgeries is also a B'-homology equivalence. We shall only deal with the ε-quadratic L-groups of this type, leaving the ε-symmetric theory for a later occasion.

Let $f:A \longrightarrow B$ be a morphism of rings with involution, and let $\varepsilon_A \in A$, $\varepsilon_B \in B$ be central units such that

$$\bar{\varepsilon}_A = \varepsilon_A^{-1} \in A \ , \ \bar{\varepsilon}_B = \varepsilon_B^{-1} \in B \ , \ f(\varepsilon_A) = \varepsilon_B \in B \ .$$

We shall denote both ε_A and ε_B by ε, as before.

An A-module chain complex C is B-acyclic if there exists a collection of A-module morphisms $\{h \in \mathrm{Hom}_A(C_r,C_{r+1}) | r \in \mathbb{Z}\}$ such that the B-module morphisms

$$1 \otimes_A (dh + hd) : B \otimes_A C_r \longrightarrow B \otimes_A C_r \quad (r \in \mathbb{Z})$$

are isomorphisms, in which case the collection of B-module morphisms

$$\{(1 \otimes_A (dh+hd))^{-1}(1 \otimes_A h) \in \mathrm{Hom}_B(B\otimes_A C_r, B \otimes_A C_{r+1}) | r \in \mathbb{Z}\}$$

is a chain contraction of $B \otimes_A C$.

If $f:A \longrightarrow B$ is locally epic in the sense of Cappell and Shaneson [1] (that is, for any finite collection $\{b_i \in B | 1 \leqslant i \leqslant m\}$ there exists a unit $u \in B$ such that $ub_i \in \mathrm{im}(f:A \longrightarrow B)$ $(1 \leqslant i \leqslant m)$) a finite-dimensional A-module

chain complex C is B-acyclic if and only if $B \otimes_A C$ is a contractible B-module chain complex, or equivalently if $H_*(B \otimes_A C) = 0$. In particular, if $f:A \longrightarrow B$ is onto or if it is a localization map $f:A \longrightarrow B = S^{-1}A$ (as in §13) then it is locally epic.

An n-dimensional ε-quadratic complex over A $(C,\psi \in Q_n(C,\varepsilon))$ is B-acyclic (resp. B-Poincaré) if the A-module chain complex C (resp. $C((1+T_\varepsilon)\psi_0:C^{n-*} \longrightarrow C)$) is B-acyclic, in which case $B \otimes_A (C,\psi)$ is an n-dimensional ε-quadratic contractible (resp. Poincaré) complex over B. Similarly for pairs.

Given morphisms of rings with involution

$$A \longrightarrow B \quad , \quad B \longrightarrow B'$$

define the n-dimensional ε-quadratic L-group of $(A;B,B')$ $L_n(A;B,B',\varepsilon)$ $(n \geqslant 0)$ to be the cobordism group of n-dimensional ε-quadratic B-Poincaré B'-acyclic complexes over A. For example,

$$L_n(A;A,0,\varepsilon) = L_n(A,\varepsilon) \quad (n \geqslant 0).$$

Proposition 17.1 i) The skew-suspension maps

$$\bar{\bar{S}} : L_n(A;B,B',\varepsilon) \longrightarrow L_{n+2}(A;B,B',-\varepsilon) \quad (n \geqslant 1)$$

are isomorphisms. Given morphisms of rings with involution

$$A \longrightarrow B \ , \ B \longrightarrow B' \ , \ B' \longrightarrow B''$$

there is defined an exact sequence

$$\ldots \longrightarrow L_{n+1}(A;B',B'',\varepsilon) \longrightarrow L_n(A;B,B',\varepsilon) \longrightarrow L_n(A;B,B'',\varepsilon) \longrightarrow L_n(A;B',B'',\varepsilon) \longrightarrow \ldots$$
$$\ldots \longrightarrow L_1(A;B',B'',\varepsilon) \ .$$

ii) The skew-suspension map

$$\bar{\bar{S}} : L_0(A;B,0,\varepsilon) \longrightarrow L_2(A;B,0,-\varepsilon)$$

is an isomorphism, and there is defined an exact sequence

$$L_1(A;B,0,\varepsilon) \longrightarrow L_1(A;B',0,\varepsilon) \longrightarrow L_2(A;B,B',-\varepsilon) \longrightarrow L_0(A;B,0,\varepsilon) \longrightarrow L_0(A;B',0,\varepsilon) \ .$$

Proof: i) In the first instance, we show that every element of $L_n(A;B,B',\varepsilon)$ $(n\geqslant 1)$ is represented by a connected complex (C,ψ). Given a representative complex $(C,\psi\in Q_n(C,\varepsilon))$ we may take C to be a f.g. projective chain complex of the type

$$C:\ \dots\longrightarrow 0\longrightarrow C_n\xrightarrow{d}C_{n-1}\longrightarrow\dots\longrightarrow C_1\xrightarrow{d}C_0\longrightarrow 0\longrightarrow\dots\ .$$

The boundary of the skew-suspension $\partial\bar{S}(C,\psi)$ is a B-acyclic $(n+1)$-dimensional $(-\varepsilon)$-quadratic Poincaré complex over A, which we shall denote by $(D,\chi\in Q_{n+1}(D,-\varepsilon))$. Let $\{h\in \mathrm{Hom}_A(D_r,D_{r+1})\,|\,0\leqslant r\leqslant n\}$ be a collection of A-module morphisms such that the B-module morphisms $\{1\otimes_A(dh+hd)\in \mathrm{Hom}_B(B\otimes_A D_r,B\otimes_A D_r)\,|\,0\leqslant r\leqslant n+1\}$ are isomorphisms. Define an A-module chain map $g:D\longrightarrow E$ by

$$\begin{array}{ccccccccccc}
D: & \dots\longrightarrow & 0 & \longrightarrow & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \longrightarrow & \dots \\
g\downarrow & & & & \downarrow 1 & & \downarrow h & & & & \\
E: & \dots\longrightarrow & 0 & \longrightarrow & D_{n+1} & \xrightarrow{hd} & D_{n+1} & \longrightarrow & 0 & \longrightarrow & \dots\ ,
\end{array}$$

and let $(D',\chi'\in Q_{n+1}(D',-\varepsilon))$ be the well-connected B-acyclic $(n+1)$-dimensional $(-\varepsilon)$-quadratic Poincaré complex over A obtained from (D,χ) by B-acyclic surgery on the connected B-acyclic $(n+2)$-dimensional $(-\varepsilon)$-quadratic pair $(g:D\longrightarrow E,(0,\chi)\in Q_{n+2}(g,-\varepsilon))$. By Proposition 7.2 there is a B-acyclic cobordism between (D,χ) and (D',χ'), and there is also a B'-acyclic null-cobordism of (D,χ). The union of these cobordisms is a B'-acyclic null-cobordism of (D',χ'), corresponding by Proposition 5.4 ii) to the skew-suspension of a connected n-dimensional ε-quadratic B-Poincaré B'-acyclic complex over A $(C',\psi'\in Q_n(C',\varepsilon))$ such that

$$(C,\psi)=(C',\psi')\in L_n(A;B,B',\varepsilon)\ .$$

Next, we define an isomorphism inverse to the skew-suspension map

$$\bar{S}:\ L_n(A;B,B',\varepsilon)\longrightarrow L_{n+2}(A;B,B',-\varepsilon)\quad (n\geqslant 1)\ .$$

Given a connected $(n+2)$-dimensional $(-\varepsilon)$-quadratic B-Poincaré B'-acyclic complex over A $(C,\psi\in Q_{n+2}(C,-\varepsilon))$ let $\{h\in \mathrm{Hom}_A(C_r,C_{r+1})\,|\,0\leqslant r\leqslant n+1\}$ induce isomorphisms $\{1\otimes_A(dh+hd)\in \mathrm{Hom}_{B'}(B'\otimes_A C_r,B'\otimes_A C_r)\,|\,0\leqslant r\leqslant n+2\}$, and define an A-module chain map $g:C\longrightarrow D$ by

$$\begin{array}{ccccccccccc}
C: & \dots\longrightarrow & 0 & \longrightarrow & C_{n+2} & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \longrightarrow & \dots \\
g\downarrow & & & & \downarrow 1 & & \downarrow h & & \downarrow & & \\
D: & \dots\longrightarrow & 0 & \longrightarrow & C_{n+2} & \xrightarrow{hd} & C_{n+2} & \longrightarrow & 0 & \longrightarrow & \dots\ .
\end{array}$$

The complex obtained from (C,ψ) by surgery on the pair $(g:C\longrightarrow D,(0,\psi))$ is the skew-suspension $\bar{S}(C',\psi')$ of an n-dimensional ε-quadratic B-Poincaré B'-acyclic complex over A $(C',\psi'\in Q_n(C',\varepsilon))$. Define

$$\bar{S}^{-1}:\ L_{n+2}(A;B,B',-\varepsilon)\longrightarrow L_n(A;B,B',\varepsilon)\ ;\ (C,\psi)\longmapsto(C',\psi')\quad(n\geqslant 1).$$

The relative L-groups L_n $(n\geqslant 1)$ appearing in the exact sequence

$$\dots\longrightarrow L_n(A;B,B',\varepsilon)\longrightarrow L_n(A;B,B'',\varepsilon)\longrightarrow L_n\longrightarrow L_{n-1}(A;B,B',\varepsilon)\longrightarrow\dots$$

are the cobordism groups of n-dimensional ε-quadratic B-Poincaré pairs over A $(g:C\longrightarrow D,(\delta\psi,\psi)\in Q_n(g,\varepsilon))$ such that C is B'-acyclic and D is B''-acyclic. The algebraic Thom complex construction used in the proof of Proposition 5.4 i) gives an n-dimensional ε-quadratic B'-Poincaré B''-acyclic complex over A $(C(g),\delta\psi/\psi\in Q_n(C(g),\varepsilon))$, thus defining an isomorphism

$$L_n\longrightarrow L_n(A;B',B'',\varepsilon)\ ;\ (g:C\longrightarrow D,(\delta\psi,\psi))\longmapsto(C(g),\delta\psi/\psi)\quad(n\geqslant 1).$$

ii) Work as in i), replacing $g:D\longrightarrow E$ in the first paragraph by

$$\begin{array}{ccccccccccc}
D: & \dots\longrightarrow & 0 & \longrightarrow & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \longrightarrow & \dots \\
g\downarrow & & & & \downarrow 1 & & \downarrow & & \downarrow & & \\
E: & \dots\longrightarrow & 0 & \longrightarrow & D_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots\ ,
\end{array}$$

and similarly for $g:C\longrightarrow D$ in the second paragraph.

[]

In the case $B=A$, $B'=S^{-1}A$, $B''=0$ the exact sequence of Proposition 17.1

$$\dots\longrightarrow L_{n+1}(A;S^{-1}A,0,\varepsilon)\longrightarrow L_n(A;A,S^{-1}A,\varepsilon)\longrightarrow L_n(A;A,0,\varepsilon)\longrightarrow L_n(A;S^{-1}A,0,\varepsilon)\longrightarrow\dots\quad(n\geqslant 1)$$

is just the localization exact sequence of Proposition 13.1

$$\dots\longrightarrow L^S_{n+1}(S^{-1}A,\varepsilon)\longrightarrow L_{n+1}(A,S,\varepsilon)\longrightarrow L_n(A,\varepsilon)\longrightarrow L_n(S^{-1}A,\varepsilon)\longrightarrow\dots\ ,$$

as there are natural identifications

$$L_n(A;A,0,\varepsilon)=L_n(A,\varepsilon)\ ,\ L_n(A;S^{-1}A,0,\varepsilon)=L^S_n(S^{-1}A,\varepsilon)\ ,\ L_n(A;A,S^{-1}A,\varepsilon)=L_{n+1}(A,S,\varepsilon)\quad(n\geqslant 1)\ .$$

Given a group π and a morphism $w:\pi \longrightarrow \mathbb{Z}_2$ let the group ring $\mathbb{Z}[\pi]$ have the w-twisted involution. Assume given morphisms of rings with involution

$$\mathbb{Z}[\pi] \longrightarrow B' \ , \ B' \longrightarrow B'' \ .$$

Let $(f,b):(M,\partial M) \longrightarrow (X,\partial X)$ be an n-dimensional normal bundle map of pairs such that $(\pi_1(X),w(X)) = (\pi,w)$, $K_*(M;B'') = 0$, $K_*(\partial M;B') = 0$ (i.e. $f:M \longrightarrow X$ is a B''-homology equivalence and $\partial f:\partial M \longrightarrow \partial X$ is a B'-homology equivalence). The quadratic kernel $\sigma_*(f,b)$ is an n-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi]$ which is B''-acyclic, with a boundary $\sigma_*(\partial f,\partial b)$ which is B'-acyclic. Applying the algebraic Thom complex construction (exactly as in the proof of Proposition 17.1 i), with $B = A = \mathbb{Z}[\pi]$, $\varepsilon = 1$) there is obtained an n-dimensional quadratic B'-Poincaré B''-acyclic complex over $\mathbb{Z}[\pi]$, whose cobordism class is an element $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi];B',B'')$.

<u>Proposition 17.2</u> The cobordism class $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi];B',B'')$ $(n \geqslant 6)$ is the obstruction to making $(f,b):(M,\partial M) \longrightarrow (X,\partial X)$ a B'-homology equivalence of pairs $(f',b'):(M',\partial M) \longrightarrow (X,\partial X)$ by a sequence of surgeries on the interior of M such that the trace $(W;M,M') \longrightarrow (X \times I; X \times 0, X \times 1)$ is a B''-homology equivalence. []

The following special cases of Proposition 17.2 are of particular interest:

i) $B' = \mathbb{Z}[\pi]$, $B'' = 0$. This is the classical surgery obstruction theory of Wall [5] (since $\mathbb{Z}[\pi]$- homology equivalence of spaces with fundamental group π = homotopy equivalence, by Whitehead's theorem), and

$$L_n(\mathbb{Z}[\pi];\mathbb{Z}[\pi],0) = L_n(\mathbb{Z}[\pi]) \quad (n \geqslant 0) \ .$$

ii) $B'' = 0$. This is the homology surgery obstruction theory of Cappell and Shaneson [1], whose Γ-groups here appear as

$$L_n(\mathbb{Z}[\pi];B',0) = \Gamma_n(\mathbb{Z}[\pi] \longrightarrow B') \quad (n \geqslant 0).$$

Indeed, the groups $L_n(\mathbb{Z}[\pi];B',B'')$ (for arbitrary B'') are the relative Γ-groups appearing in the exact sequence

$$\dots \longrightarrow \Gamma_{n+1}(\mathbb{Z}[\pi] \longrightarrow B') \longrightarrow \Gamma_{n+1}(\mathbb{Z}[\pi] \longrightarrow B'') \longrightarrow L_n(\mathbb{Z}[\pi];B',B'') \longrightarrow \Gamma_n(\mathbb{Z}[\pi] \longrightarrow B') \longrightarrow \dots \ .$$

We shall discuss the Γ-groups further below.

iii) $B' = \mathbb{Z}[\pi]$, $B'' = \mathbb{Q}[\pi]$. This is the local surgery obstruction theory studied by Pardon [4]. The groups $L_n(\mathbb{Z}[\pi];\mathbb{Z}[\pi],\mathbb{Q}[\pi]) = L_{n+1}(\mathbb{Z}[\pi],S)$ $(S = \mathbb{Z}-\{0\})$ fit into the localization exact sequence

$$\dots \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow L_n^S(\mathbb{Q}[\pi]) \longrightarrow L_n(\mathbb{Z}[\pi],S) \longrightarrow L_{n-1}(\mathbb{Z}[\pi]) \longrightarrow \dots \ .$$

iv) $B' = \mathbb{Z}[\pi]$. This is the dual surgery obstruction theory studied by Smith [2]. The groups $L_n(\mathbb{Z}[\pi];\mathbb{Z}[\pi],B'')$ fit into the exact sequence relating L- and Γ-groups

$$\dots \longrightarrow L_{n+1}(\mathbb{Z}[\pi]) \longrightarrow \Gamma_{n+1}(\mathbb{Z}[\pi] \longrightarrow B'') \longrightarrow L_n(\mathbb{Z}[\pi];\mathbb{Z}[\pi],B'') \longrightarrow L_n(\mathbb{Z}[\pi]) \longrightarrow \dots \ .$$

We shall now relate the <u>Γ-groups</u> defined by

$$\Gamma_n(f:A \longrightarrow B,\varepsilon) = L_n(A;B,0,\varepsilon) \quad (n \geqslant 0)$$

with the Γ-groups as defined by Cappell and Shaneson [1].

An ε-quadratic form over A $(M,\psi \in Q_\varepsilon(M))$ is <u>B-non-singular</u> if the A-module chain complex

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{\psi+\varepsilon\psi^*} M^* \longrightarrow 0 \longrightarrow \dots$$

is B-acyclic. Then $B \otimes_A (M,\psi)$ is a non-singular ε-quadratic form over B.

A <u>B-lagrangian</u> of a B-non-singular ε-quadratic form over A (M,ψ) is a f.g. projective A-module L which is a submodule of M such that

i) $j^*\psi j = 0 \in Q_\varepsilon(L)$, with $j \in \mathrm{Hom}_A(L,M)$ the inclusion

ii) the A-module chain complex

$$\dots \longrightarrow 0 \longrightarrow L \xrightarrow{j} M \xrightarrow{j^*(\psi+\varepsilon\psi^*)} L^* \longrightarrow 0 \longrightarrow \dots$$

is B-acyclic.

Then $B \otimes_A L$ is a lagrangian of $B \otimes_A (M,\psi)$.

A <u>B-non-singular ε-quadratic formation over A</u> $(M,\psi;F,G)$ is a non-singular ε-quadratic form over A (M,ψ) together with a lagrangian F and a B-lagrangian G. Then $B \otimes_A (M,\psi;F,G)$ is a non-singular ε-quadratic formation over B.

<u>Proposition 17.3</u> i) The even-dimensional Γ-group $\Gamma_{2i}(f,\varepsilon) \equiv L_{2i}(A;B,0,\varepsilon)$ $(i \geq 0)$ is naturally isomorphic to the Witt group of B-non-singular $(-)^i\varepsilon$-quadratic forms over A.

ii) The odd-dimensional Γ-group $\Gamma_{2i+1}(f,\varepsilon) = L_{2i+1}(A;B,0,\varepsilon)$ $(i \geq 0)$ is naturally isomorphic to the subgroup of $L_1^X(B,\varepsilon)$ $(X = \mathrm{im}(\tilde{K}_0(A) \longrightarrow \tilde{K}_0(B)))$ consisting of elements of type $B\otimes_A(M,\psi;F,G)$, with $(M,\psi;F,G)$ a B-non-singular $(-)^i\varepsilon$-quadratic formation over A.

<u>Proof</u>: Use the periodicity $L_n(A;B,0,\varepsilon) = L_{n+2}(A;B,0,-\varepsilon)$ $(n \geq 0)$ given by Proposition 17.1 to reduce to the case $i = 0$, and then proceed as in Propositions 7.6,7.7 (which is the special case A = B).

[]

Thus the Γ-groups $\Gamma_n(f) \equiv \Gamma_n(f,1)$ defined above agree with the Γ-groups defined in Cappell and Shaneson [1].

Define the <u>Δ-groups</u>

$$\Delta_n(f:A \longrightarrow B,\varepsilon) = L_{n+1}(A;A,B,-\varepsilon) \quad (n \geq 1)$$

As mentioned above, these fit into an exact sequence

$$\dots \longrightarrow L_n(A,\varepsilon) \longrightarrow \Gamma_n(f,\varepsilon) \longrightarrow \Delta_n(f,\varepsilon) \longrightarrow L_{n-1}(A,\varepsilon) \longrightarrow \dots .$$

A <u>non-singular ε-quadratic B-form over A</u> $(M,\psi;L)$ is a non-singular ε-quadratic form over A (M,ψ) together with a B-lagrangian L.

A <u>non-singular split ε-quadratic formation over A</u> (F,G) is a non-singular split ε-quadratic formation over A such that the A-module chain complex

$$0 \longrightarrow G \xrightarrow{\ \mu\ } F^* \longrightarrow 0$$

is B-acyclic.

<u>Proposition 17.4</u> The Δ-group $\begin{cases}\Delta_{2i}(f,\varepsilon)\\ \Delta_{2i-1}(f,\varepsilon)\end{cases}$ $(i \geq 1)$ is naturally isomorphic to the Witt group of non-singular $\begin{cases}\text{split}\\ -\end{cases}$ $(-)^{i+1}\varepsilon$-quadratic $\begin{cases}\text{B-formations}\\ \text{B-forms}\end{cases}$ over A.

[]

The identification of Proposition 17.4 in the special case $B = S^{-1}A$ has already been carried out in §13, with

$$\Delta_n(f:A \to S^{-1}A,\varepsilon) = L_n(A,S,\varepsilon).$$

Matsumoto [1] has defined obstruction groups $P_n(\mathcal{E})$ for the problem of finding locally flat spines of codimension 2, which may be expressed as Γ-groups (cf. Matsumoto [2],[3]). Given a group π and a central element $g\in\pi$ let C be the cyclic normal subgroup of π generated by g, and let

$$\mathcal{E} : \{1\} \longrightarrow C \longrightarrow \pi \longrightarrow \pi/C \longrightarrow \{1\}$$

be the associated short exact sequence of groups. The obstruction groups are

$$P_n(\mathcal{E}) = \Gamma_n(\mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi/C],g) ,$$

so that

$P_{2i}(\mathcal{E})$ = the Witt group of $\mathbb{Z}[\pi/C]$-non-singular $(-)^i g$-quadratic forms over $\mathbb{Z}[\pi]$

$P_{2i+1}(\mathcal{E})$ = the subgroup of $L_1^X(\mathbb{Z}[\pi/C],(-)^i)$ consisting of the non-singular $(-)^i$-quadratic formations over $\mathbb{Z}[\pi/C]$ of the type $\mathbb{Z}[\pi/C]\otimes_{\mathbb{Z}[\pi]}(M,\psi;F,G)$ with $(M,\psi;F,G)$ a $\mathbb{Z}[\pi/C]$-non-singular $(-)^i g$-quadratic formation over $\mathbb{Z}[\pi]$ and $X = \mathrm{im}(\tilde{K}_0(\mathbb{Z}[\pi]) \to \tilde{K}_0(\mathbb{Z}[\pi/C]))$.

Kervaire [1] introduced the cobordism groups C_n of knots $S^n \subset S^{n+2}$, and showed that $C_{2k} = 0$ $(k \geq 0)$. Levine [1] gave an algebraic characterization of C_{2k+1} $(k \geq 2)$ as a Witt group of Seifert matrices over $\mathbb{Z}$. The Witt group of Seifert matrices over $\mathbb{Q}$ was computed by Levine [2], and the isomorph of C_{2k+1} in this group has been described by Stoltzfus [1] using a localization exact sequence. We shall now give an algebraic characterization of the groups C_n as the algebraic cobordism groups of (n+1)-dimensional symmetric Poincaré complexes over $\mathbb{Z}$ with the extra structure afforded the Poincaré duality of the chain complex of a Seifert surface $M^{n+1} \subset S^{n+2}$ of a knot $S^n = \partial M \subset S^{n+2}$. In particular, we shall obtain an algebraic interpretation (Proposition 17.16) of the identification

$$C_n = \Gamma_{n+3}(\Phi) \quad (n \geq 4)$$

derived geometrically by Cappell and Shaneson [1], where Φ is the square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\ 1\ } & \mathbb{Z}[\mathbb{Z}] \\ {\scriptstyle 1}\downarrow & \Phi & \downarrow{\scriptstyle e} \\ \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\ e\ } & \mathbb{Z} \end{array}$$

and $e : \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}\,;\ z \longmapsto 1$ is the augmentation map.

The geometric methods of Matsumoto [1] identified

$$C_n = \Gamma_{n+1}(e:\mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z},z) \quad (n \geqslant 4) ,$$

and the algebraic methods of Matsumoto [3] interpreted this in terms of Seifert matrices - we shall give an interpretation in terms of algebraic Poincaré complexes (Propositions 17.13, 17.14).

A locally flat n-knot $S^n \subset S^{n+2}$ admits a Seifert surface, that is a locally flat (n+1)-dimensional dimensional submanifold $M^{n+1} \subset S^{n+2}$ such that $\partial M = S^n \subset S^{n+2}$. Inclusion defines a degree 1 map of (n+1)-dimensional geometric Poincaré pairs which is the identity on the boundaries

$$f : (M,\partial M) \longrightarrow (D^{n+3},S^n) .$$

The transversality used to construct the Seifert surface equips M with a normal vector field in S^{n+2}, so that f is a normal bundle map. The corresponding embedding $M \times \mathbb{R}^1 \subset S^{n+2}$ gives a geometric Umkehr map

$$F : (\Sigma D^{n+3}_+,\Sigma S^n_+) \longrightarrow (\Sigma M_+,\Sigma\partial M_+)$$

involving just a single suspension Σ, so that the associated $\mathbb{Z}_2$-hypercohomology class $\varphi_f \in Q^{n+1}(C)$ $(C = C(f^!))$ is such that

$$\varphi_f \in \ker(S:Q^{n+1}(C) \longrightarrow Q^{n+2}(SC)) \subseteq \ker(J:Q^{n+1}(C) \longrightarrow \hat{Q}^{n+1}(C)) .$$

This suggests developing an abstract theory of chain complexes (over any ring with involution A) with Poincaré duality stemming from elements of the relative groups $Q^{n+2}(S)$ appearing in the long exact sequence

$$\ldots \longrightarrow Q^{n+2}(S) \longrightarrow Q^{n+1}(C,\varepsilon) \xrightarrow{S} Q^{n+2}(SC,\varepsilon) \longrightarrow Q^{n+1}(S) \longrightarrow Q^n(C,\varepsilon) \longrightarrow \ldots .$$

The relevant part of the exact braid of Proposition 1.3 gives identifications

$$Q^{n+2}(S) = Q^{n+1}_{[0,0]}(C,\varepsilon) = H_{n+1}(C^t\otimes_A C) .$$

More precisely:

Proposition 17.5 For any A-module chain complex C there is defined a long exact sequence of abelian groups

$$\ldots \longrightarrow H_n(C^t\otimes_A C) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{S} Q^{n+1}(SC,\varepsilon) \xrightarrow{\partial} H_{n-1}(C^t\otimes_A C) \longrightarrow \ldots$$

where

$$1+T_\varepsilon : H_n(C^t\otimes_A C) \longrightarrow Q^n(C,\varepsilon) ; \psi \longmapsto (1+T_\varepsilon)\psi , ((1+T_\varepsilon)\psi)_s = \begin{cases}(1+T_\varepsilon)\psi & s = 0 \\ 0 & s \geqslant 1\end{cases}$$

$$\partial : Q^{n+1}(SC,\varepsilon) \longrightarrow H_{n-1}(C^t\otimes_A C) ; \psi \longmapsto \partial\psi , (\partial\psi)_s = \begin{cases}\psi_0 & s = 0 \\ 0 & s \geqslant 1 .\end{cases}$$

[]

An n-dimensional ε-symmetric Seifert complex over A (C,ψ) is an n-dimensional A-module chain complex C together with a homology class $\psi \in H_n(C^t\otimes_A C)$ such that slant product with $(1+T_\varepsilon)\psi \in H_n(C^t\otimes_A C)$ induces A-module isomorphisms

$$(1+T_\varepsilon)\psi : H^r(C) \longrightarrow H_{n-r}(C) \quad (0 \leqslant r \leqslant n) .$$

For f.g. projective chain complexes C the identification $C^t\otimes_A C = \mathrm{Hom}_A(C^*,C)$ makes apparent that $\psi \in H_n(C^t\otimes_A C)$ is just a chain homotopy class of chain maps

$$\psi : C^{n-*} \longrightarrow C$$

such that

$$(1+T_\varepsilon)\psi : C^{n-*} \longrightarrow C$$

is a chain equivalence.

A homotopy equivalence of n-dimensional ε-symmetric Seifert complexes over A

$$f : (C,\psi) \longrightarrow (C',\psi')$$

is a chain equivalence

$$f : C \longrightarrow C'$$

such that

$$(f^t\otimes_A f)\psi = \psi' \in H_n(C'^t\otimes_A C') .$$

Working as in §1 we can identify such structures for $n = 0,1$ with Seifert forms and formations, as follows.

A <u>Seifert ε-form over A</u> (M,Ψ) is a f.g. projective A-module M together with an A-module morphism $\Psi \in \mathrm{Hom}_A(M,M^*)$ such that $(1+T_\varepsilon)\Psi \in \mathrm{Hom}_A(M,M^*)$ is an isomorphism. The Seifert ε-forms appearing in the isotopy classification of odd-dimensional simple knots of Levine [3] are the case $\varepsilon = \pm 1 \in A = \mathbb{Z}$.

An <u>isomorphism</u> of Seifert ε-forms over A

$$f : (M,\Psi) \longrightarrow (M',\Psi')$$

is an A-module isomorphism $f \in \mathrm{Hom}_A(M,M')$ such that

$$f^*\Psi' f = \Psi \in \mathrm{Hom}_A(M,M^*) .$$

<u>Proposition 17.6</u> The homotopy equivalence classes of 0-dimensional ε-symmetric Seifert complexes over A are in a natural one-one correspondence with the isomorphism classes of Seifert ε-forms over A.

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A <u>lagrangian</u> of a Seifert ε-form (M,Ψ) is a direct summand L of M such that the inclusion $j \in \mathrm{Hom}_A(L,M)$ fits into a short exact sequence

$$0 \longrightarrow L \xrightarrow{\ j\ } M \xrightarrow{\ j^*(1+T_\varepsilon)\Psi\ } L^* \longrightarrow 0$$

and also

$$j^*\Psi j = 0 \in \mathrm{Hom}_A(L,L^*) .$$

A <u>Seifert ε-formation over A</u> $(M,\Psi;F,G)$ is a Seifert ε-form (M,Ψ) together with an ordered pair of lagrangians F,G. Such objects appear in the isotopy classification of even-dimensional simple knots of Kearton [2] for $\varepsilon = \pm 1 \in A = \mathbb{Z}$. (The actual classification involves also an abelian group Π fitting into an exact sequence

$$0 \longrightarrow (M/F+G)\otimes_{\mathbb{Z}}\mathbb{Z}_2 \xrightarrow{\ i\ } \Pi \xrightarrow{\ h\ } F\cap G \longrightarrow 0 ,$$

together with a bilinear pairing

$$\varphi : \Pi \times \Pi \longrightarrow \mathbb{Z}_2$$

such that

$$\varphi(ia,b) = \Psi(a,hb) ,\ \varphi(b,ia) = \Psi(hb,a) \in \mathbb{Z}_2 \quad (a \in M/F+G ,\ b \in F\cap G)).$$

An <u>isomorphism</u> of Seifert ε-formations

$$f : (M,\Psi;F,G) \longrightarrow (M',\Psi';F',G')$$

is an isomorphism of Seifert ε-forms

$$f : (M,\Psi) \longrightarrow (M',\Psi')$$

such that $f(F) = F'$, $f(G) = G'$. A <u>stable isomorphism</u> of Seifert ε-formations

$$[f] : (M,\Psi;F,G) \longrightarrow (M',\Psi';F',G')$$

is an isomorphism

$$f : (M,\Psi;F,G)\oplus(N,\theta;H,K) \longrightarrow (M',\Psi';F',G')\oplus(N',\theta';H',K')$$

with $(N,\theta;H,K),(N',\theta';H',K')$ such that $N = H\oplus K$, $N' = H'\oplus K'$.

<u>Proposition 17.7</u> The homotopy equivalence classes of 1-dimensional ε-symmetric Seifert complexes over A are in a natural one-one correspondence with the stable isomorphism classes of Seifert ε-formations over A.

<u>Proof</u>: Let $(C,\Psi \in H_1(C^t\otimes_A C))$ be a 1-dimensional ε-symmetric Seifert complex, with C a f.g. projective A-module chain complex

$$C : \ldots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\ d\ } C_0 \longrightarrow 0 \longrightarrow \ldots ,$$

so that Ψ is represented by A-module morphisms

$$\Psi : C^0 \longrightarrow C_1 ,\ \tilde{\Psi} : C^1 \longrightarrow C_0$$

such that

$$d\Psi + \tilde{\Psi}d^* = 0 : C^0 \longrightarrow C_0 ,$$

and such that the chain map

$$\begin{array}{ccccccccccc} C^{1-*} & : & \ldots \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{-d^*} & C^1 & \longrightarrow & 0 & \longrightarrow \ldots \\ \big\downarrow (1+T_\varepsilon)\Psi & & & & & \big\downarrow \Psi+\varepsilon\tilde{\Psi}^* & & \big\downarrow \tilde{\Psi}+\varepsilon\Psi^* & & & \\ C & : & \ldots \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{\ d\ } & C_0 & \longrightarrow & 0 & \longrightarrow \ldots \end{array}$$

is a chain equivalence. The algebraic mapping cone $C((1+T_\varepsilon)\Psi)$ defines a split short exact sequence of f.g. projective A-modules

$$0 \longrightarrow C^0 \xrightarrow{\begin{pmatrix} d^* \\ \Psi+\varepsilon\tilde{\Psi}^* \end{pmatrix}} C^1\oplus C_1 \xrightarrow{(\tilde{\Psi}+\varepsilon\Psi^*\ \ d)} C_0 \longrightarrow 0 .$$

Choosing a splitting map for $(\tilde{\Psi}+\varepsilon\Psi^*\ \ d)$

$$\begin{pmatrix} j \\ k \end{pmatrix} : C_0 \longrightarrow C^1\oplus C_1$$

such that

$$(\tilde{\psi}+\varepsilon\psi^* \quad d)\begin{pmatrix} j \\ k \end{pmatrix} = (\tilde{\psi}+\varepsilon\psi^*)j + dk = 1 : C_0 \longrightarrow C_0$$

we have a Seifert ε-formation over A

$$(M,\psi;F,G) = (C_1 \oplus C^1, \begin{pmatrix} 0 & \varepsilon j\tilde{\psi} \\ 1-\tilde{\psi}^* j^* & k\tilde{\psi} - \varepsilon\tilde{\psi}^* k^* \end{pmatrix}; C_1, \mathrm{im}(\begin{pmatrix} \psi+\varepsilon\tilde{\psi}^* \\ d^* \end{pmatrix} : C^0 \longrightarrow C_1 \oplus C^1)) \quad .$$

Conversely, every Seifert ε-formation is isomorphic to one of this type.

[]

An <u>(n+1)-dimensional ε-symmetric Seifert pair over A</u> $(f:C \longrightarrow D,(\varphi,\psi))$ is a chain map $f:C \longrightarrow D$ from an n-dimensional A-module chain complex C to an (n+1)-dimensional A-module chain complex D together with a relative homology class $(\varphi,\psi) \in H_{n+1}(f^t\otimes_A f: C^t\otimes_A C \longrightarrow D^t\otimes_A D)$ such that slant product with $((1+T_\varepsilon)\varphi,(1+T_\varepsilon)\psi) \in H_{n+1}(f^t\otimes_A f)$ defines A-module isomorphisms

$$((1+T_\varepsilon)\varphi,(1+T_\varepsilon)\psi) : H^r(f) \longrightarrow H_{n+1-r}(D) \qquad (0 \leq r \leq n+1) \quad .$$

The n-dimensional ε-symmetric Seifert complex $(C,\psi \in H_n(C^t\otimes_A C))$ is the <u>boundary</u> of the pair $(f:C \longrightarrow D,(\varphi,\psi))$.

The n-dimensional ε-symmetric Seifert complexes $(C,\psi),(C',\psi')$ are <u>cobordant</u> if $(C,\psi)\oplus(C',-\psi')$ is the boundary of an (n+1)-dimensional ε-symmetric Seifert pair $((f\ f'):C\oplus C' \longrightarrow D,(\varphi,\psi\oplus-\psi'))$.

<u>Proposition 17.8</u> Cobordism is an equivalence relation on n-dimensional ε-symmetric Seifert complexes over A, such that homotopy equivalent Seifert complexes are cobordant. The cobordism classes define the <u>n-dimensional ε-symmetric Seifert group</u> $\Lambda_n(A,\varepsilon)$, with

$$(C,\psi) + (C',\psi') = (C\oplus C',\psi\oplus\psi') \ , \ -(C,\psi) = (C,-\psi) \in \Lambda_n(A,\varepsilon) \quad .$$

[]

Given an n-dimensional ε-symmetric Seifert complex $(C,\psi \in H_n(C^t\otimes_A C))$ there is defined an n-dimensional ε-quadratic Poincaré complex $(C,\hat{\psi} \in Q_n(C,\varepsilon))$ with

$$\hat{\psi}_s = \begin{cases} \psi \in (C^t\otimes_A C)_n & s = 0 \\ 0 \in (C^t\otimes_A C)_{n-s} & s \geq 1 \end{cases} \quad .$$

This defines a forgetful map in the cobordism groups

$$\Lambda_n(A,\varepsilon) \longrightarrow L_n(A,\varepsilon) \ ; \ (C,\psi) \longmapsto (C,\hat{\psi})$$

which is a monomorphism for n odd (cf. Proposition 17.12 below).

Corresponding to Propositions 7.6, 7.7 and 17.6, 17.7 we have

<u>Proposition 17.9</u> The 0-dimensional ε-symmetric Seifert group $\Lambda_0(A,\varepsilon)$ is the group of stable isomorphism classes of Seifert ε-forms over A modulo those admitting lagrangians.

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<u>Proposition 17.10</u> The 1-dimensional ε-symmetric Seifert group $\Lambda_1(A,\varepsilon)$ is the abelian group with one generator for each isomorphism class of Seifert ε-formations over A $(M,\psi;F,G)$ subject to the relations

$$(M,\psi;F,G) + (M',\psi';F,G') = (M\oplus M',\psi\oplus\psi';F\oplus F',G\oplus G')$$

$$(M,\psi;F,G) + (M,\psi;G,H) = (M,\psi;F,H)$$

$$(M,\psi;F,G) = 0 \text{ if } M = F\oplus G \ .$$

[]

The <u>skew-suspension</u> of an n-dimensional ε-symmetric Seifert complex (C,ψ) is the (n+2)-dimensional $(-\varepsilon)$-symmetric Seifert complex $(SC,\bar{S}\psi)$ defined by

$$\bar{S}\psi = \psi \in H_{n+2}(SC^t\otimes_A SC) = H_n(C^t\otimes_A C) \ .$$

By analogy with Proposition 7.3 we have

<u>Proposition 17.11</u> The skew-suspension map in the Seifert groups

$$\bar{S} : \Lambda_n(A,\varepsilon) \longrightarrow \Lambda_{n+2}(A,-\varepsilon) \ ; \ (C,\psi) \longmapsto (SC,\bar{S}\psi)$$

is an isomorphism for all $A,\varepsilon,n \geq 0$.

<u>Proof</u>: We shall perform algebraic surgery on an (n+2)-dimensional $(-\varepsilon)$-symmetric Seifert complex $(C,\psi \in H_{n+2}(C^t\otimes_A C))$ to obtain a cobordant skew-suspension of an n-dimensional ε-symmetric Seifert complex $\Omega(C,\psi)$, thus defining an isomorphism

$$\Omega : \Lambda_{n+2}(A,-\varepsilon) \longrightarrow \Lambda_n(A,\varepsilon) \ ; \ (C,\psi) \longmapsto \Omega(C,\psi)$$

inverse to $\bar{S}:\Lambda_n(A,\varepsilon) \longrightarrow \Lambda_{n+2}(A,-\varepsilon)$. It may be assumed that C is a f.g. projective A-module chain complex of the type

$$C : \ldots \longrightarrow 0 \longrightarrow C_{n+2} \xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots ,$$

so that ψ can be expressed as a chain map

$$\psi : C^{n+2-*} \longrightarrow C$$

such that

$$(1+T_{-\varepsilon})\psi : C^{n+2-*} \longrightarrow C$$

is a chain equivalence. In particular

$$H_0((1+T_{-\varepsilon})\psi : C^{n+2-*} \longrightarrow C) = \operatorname{coker}(((1+T_{-\varepsilon})\psi \;\; d) : C^{n+2} \oplus C_1 \longrightarrow C_0) = 0 ,$$

so that there exists an A-module morphism

$$\begin{pmatrix} j \\ k \end{pmatrix} : C_0 \longrightarrow C^{n+2} \oplus C_1$$

such that

$$((1+T_{-\varepsilon})\psi \;\; d)\begin{pmatrix} j \\ k \end{pmatrix} = (1+T_{-\varepsilon})\psi j + dk = 1 : C_0 \longrightarrow C_0 .$$

Let $(C', \psi' \in H_{n+2}(C'^t \otimes_A C'))$ be the $(n+2)$-dimensional $(-\varepsilon)$-symmetric Seifert complex defined by

$$\begin{array}{ccccccccccc}
C'^{n+2-*} : C^0 & \xrightarrow{\begin{pmatrix} (-)^n d^* \\ -(1+T_{-\varepsilon})\psi^* \end{pmatrix}} & C^1 \oplus C_{n+2} & \xrightarrow{((-)^{n-1} d^* \;\; 0)} & C^2 & \to \cdots \to & C^n & \xrightarrow{\begin{pmatrix} d^* \\ 0 \end{pmatrix}} & C^{n+1} \oplus C^{n+2} & \xrightarrow{(-d^* \;\; (-)^{n+1})} & C^{n+2} \\
\psi' \Big\downarrow \quad \Big\downarrow \psi & & \Big\downarrow \begin{pmatrix} \psi & 0 \\ \psi k^* & (-)^{n+1}\psi j^* \end{pmatrix} & & \Big\downarrow \psi & & \Big\downarrow \psi & & \Big\downarrow \begin{pmatrix} \psi & (-)^n k T_{-\varepsilon}\psi \\ 0 & 1 - j T_{-\varepsilon}\psi \end{pmatrix} & & \Big\downarrow \psi \\
C' : C_{n+2} & \xrightarrow[\begin{pmatrix} d \\ (-)^n \end{pmatrix}]{} & C_{n+1} \oplus C_{n+2} & \xrightarrow[(d \;\; 0)]{} & C_n & \xrightarrow[d]{} \cdots \xrightarrow[d]{} & C_2 & \xrightarrow[\begin{pmatrix} d \\ 0 \end{pmatrix}]{} & C_1 \oplus C^{n+2} & \xrightarrow[(d \;\; (-)^{n+1}(1+T_{-\varepsilon})\psi)]{} & C_0
\end{array}$$

Then (C', ψ') is cobordant to (C, ψ), and it is also the skew-suspension

$$(C', \psi') = \bar{S}\Omega(C, \psi)$$

of an n-dimensional ε-symmetric Seifert complex $\Omega(C, \psi) = (\Omega C', \psi' \in H_n(\Omega C'^t \otimes_A \Omega C'))$.

[]

Proposition 17.11 gives a 4-periodicity

$$\Lambda_n(A, \varepsilon) = \Lambda_{n+4}(A, \varepsilon)$$

via the double skew-suspension map.

Algebraic surgery can also be used to identify the ε-symmetric Λ-groups of A with the εz-quadratic Γ-groups of the augmentation

$$A[z, z^{-1}] \longrightarrow A \; ; \; z \longmapsto 1 \qquad (\bar{z} = z^{-1}) .$$

Proposition 17.12 The natural map

$$\Lambda_n(A, \varepsilon) \longrightarrow \Gamma_n(A[z, z^{-1}] \longrightarrow A, \varepsilon z) \; ; \; (C, \psi) \longmapsto (A[z, z^{-1}] \otimes_A C, 1 \otimes \psi)$$

is an isomorphism of abelian groups.

[]

Given an $(n-1)$-knot $k : S^{n-1} \subset S^{n+1}$ and a choice of Seifert surface $M^n \subset S^{n+1}$ such that $\partial M = k(S^{n-1}) \subset S^{n+1}$ define an n-dimensional symmetric Seifert complex over $\mathbb{Z}$

$$\sigma_*(k, M) = (C, \psi \in H_n(C^t \otimes_{\mathbb{Z}} C))$$

by considering (as above) the kernel $C = C(f^!)$ of the n-dimensional normal bundle map

$$(f, b) : (M, \partial M) \longrightarrow (D^{n+2}, k(S^{n-1}))$$

given by inclusion. (A direct construction for $\sigma_*(k, M)$ is given as follows. Use the map $M \longrightarrow S^{n+1} - M$ defined by the normal field and Alexander duality to define a chain map

$$\psi : C^{n-*} = \dot{C}(M) \longrightarrow \dot{C}(S^{n+1} - M) = C \;).$$

An $(n-1)$-knot $k : S^{n-1} \subset S^{n+1}$ is <u>simple</u> if

$$\pi_r(S^{n+1} - k(S^{n-1})) = \pi_r(S^1) \text{ for } r < i \text{ if } n = \begin{cases} 2i \\ 2i+1 \end{cases} ,$$

in which case there exists an $(i-1)$-connected Seifert surface $M^n \subset S^{n+1}$. The associated Seifert complex $\sigma_*(k, M)$ is then the i-fold skew-suspension

$$\sigma_*(k, M) = \bar{S}^i \sigma_i(k, M)$$

of a $\begin{cases} 0\text{-} \\ 1\text{-} \end{cases}$dimensional $(-)^i$symmetric Seifert complex over $\mathbb{Z}$ $\sigma_i(k, M)$, corresponding to a Seifert $\begin{cases} (-)^i\text{-form} \\ (-)^i\text{-formation} \end{cases}$ $\sigma_i(k, M) = \begin{cases} (H_i(M), \psi \in \operatorname{Hom}_{\mathbb{Z}}(M, M^*)) \\ (N, \psi; F, G) \text{ with } \dot{C}(M) : \ldots \to 0 \to G \to N/F \to 0 \to \ldots \end{cases}$

by Proposition $\begin{cases} 17.6 \\ 17.7 \end{cases}$, exactly as in the classification of simple $(n-1)$-knots due to $\begin{cases} \text{Levine [3]} \\ \text{Kearton [2]} \end{cases}$. (In the case $n = 2i+1$ one has also to consider the homotopy linking

$$\varphi : \pi_{i+1}(M) \times \pi_{i+1}(M) \longrightarrow \mathbb{Z}_2$$

defined as follows. Use the given framing of the normal bundle of the embedding $(M, \partial M) \subset (D^{2i+3}, S^{2i+2})$ to define an S-duality map

$$\alpha : S^{2i+3} = D^{2i+3}/S^{2i+2} \xrightarrow{\text{collapse}} \Sigma^2(M/\partial M) \xrightarrow{\Delta} \Sigma^2(M/\partial M) \wedge M_+ ,$$

giving an S-duality isomorphism

$$\alpha : \pi^S_{i+1}(M/\partial M) = \{S^{i+1}, M/\partial M\} \longrightarrow \pi^i_S(M_+) = \{M_+, S^i\} ,$$

and let φ be the composite

$$\varphi : \pi_{i+1}(M)\times\pi_{i+1}(M) \xrightarrow{\text{proj.}\times\text{proj.}} \pi^S_{i+1}(M_+)\times\pi^S_{i+1}(M/\partial M) \xrightarrow{1\times\alpha} \pi^S_{i+1}(M_+)\times\pi^i_S(M_+) \xrightarrow{\text{evaluation}} \pi^S_1 = \mathbb{Z}_2 .$$

In fact, $\pi_{i+1}(M)$ fits into an exact sequence

$$0 \longrightarrow H_i(M)\otimes_{\mathbb{Z}}\mathbb{Z}_2 \longrightarrow \pi_{i+1}(M) \longrightarrow H_{i+1}(M) \longrightarrow 0$$

and φ agrees with the Seifert $(-)^i$-form $\psi\in\mathrm{Hom}_{\mathbb{Z}}(N,N^*)$.

Kervaire [1] has defined the cobordism of n-knots, with cobordism groups C_n. The normal bundle maps $(f,b):(M,\partial M) \longrightarrow (D^{n+2}, k(S^{n-1}))$ associated to the various of choices of Seifert surface for an $(n-1)$-knot $k:S^{n-1}\subset S^{n+1}$ are normal cobordant, by a relative version of the Seifert surface construction, so that the corresponding Seifert complexes $\sigma_*(k,M)$ are also cobordant.

Proposition 17.13 The natural map

$$\sigma_* : C_{n-1} \longrightarrow \Lambda_n(\mathbb{Z}) ;\ (k:S^{n-1}\subset S^{n+1}) \longmapsto \sigma_*(k,M)$$

is an isomorphism of abelian groups for $n\geq 5$, where $\Lambda_n(\mathbb{Z}) = \Lambda_n(\mathbb{Z},1)$.

Proof: In the case $n = 2i$ we have that

$$\Lambda_{2i}(\mathbb{Z}) = \Lambda_0(\mathbb{Z},(-)^i)$$

by Proposition 17.11, and by Proposition 17.9 that $\Lambda_0(\mathbb{Z},(-)^i)$ is precisely the cobordism group of Seifert $(-)^i$forms over $\mathbb{Z}$ used to characterize C_{2i-1} in Levine [1] for $i\geq 3$.

In the case $n = 2i+1$ we have that $C_{2i} = 0$ by Kervaire [1] and that $\Lambda_{2i+1}(\mathbb{Z}) = \Gamma_{2i+1}(\mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}, z) \subseteq L_{2i+1}(\mathbb{Z}) = 0$ by Proposition 17.12.

[]

Algebraic Seifert complexes serve to clarify the algebraic relationship between the Poincaré duality of a Seifert surface for a knot and the Blanchfield duality of the infinite cyclic cover of the knot complement. Again, we start by developing the abstract theory, over any ring with involution A.

Define the Blanchfield complex associated to an n-dimensional ε-symmetric Seifert complex over A (C,ψ) to be the $(n+1)$-dimensional ε-symmetric Seifert complex over $A[z,z^{-1}]$ $(\bar{z} = z^{-1})$

$$\beta(C,\psi) = (D,\varphi)$$

defined by

$$D_r = A[z,z^{-1}]\otimes_A(C_r\oplus C^{n-r+1})$$

$$d_D = \begin{pmatrix} d_C & (-)^{r-1}(1+T_\varepsilon z)\psi \\ 0 & (-)^{r-1}d_C^* \end{pmatrix} : D_r = A[z,z^{-1}]\otimes_A(C_r\oplus C^{n-r+1}) \longrightarrow D_{r-1} = A[z,z^{-1}]\otimes_A(C_{r-1}\oplus C^{n-r+2})$$

$$\varphi = \begin{pmatrix} 0 & (-)^{r(n-r)}\varepsilon z \\ (-)^{n-r} & 0 \end{pmatrix} : D^{n-r+1} = A[z,z^{-1}]\otimes_A(C^{n-r+1}\oplus C_r) \longrightarrow D_r = A[z,z^{-1}]\otimes_A(C_r\oplus C^{n-r+1}) .$$

We are assuming that C is f.g. projective in order to use the matrix notation. Note that D becomes contractible over A under the augmentation $A[z,z^{-1}] \longrightarrow A$; $z \longmapsto 1$, since $A\otimes_{A[z,z^{-1}]}D = C((1+T_\varepsilon)\psi : C^{n-*} \longrightarrow C)$ is the algebraic mapping cone of a chain equivalence.

Proposition 17.14 $\Lambda_n(A,\varepsilon)$ is isomorphic to the cobordism group of $(n+1)$-dimensional ε-symmetric Seifert complexes over $A[z,z^{-1}]$ which become contractible over A.

Proof: By the Blanchfield complex construction and algebraic surgery.

[]

Call n-dimensional ε-symmetric Seifert complexes over A (C,ψ), (C',ψ') S-equivalent if the associated Blanchfield complexes $\beta(C,\psi)$, $\beta(C',\psi')$ are homotopy equivalent $(n+1)$-dimensional ε-symmetric Seifert complexes over $A[z,z^{-1}]$. S-equivalence is an equivalence relation, such that S-equivalent Seifert complexes are cobordant (by Proposition 17.14).

Given an (n-1)-knot $k:S^{n-1}\subset S^{n+1}$ let $N = S^{n-1}\times D^2\subset S^{n+1}$ be a closed regular neighbourhood of S^{n-1} in S^{n+1}, and let $X = \overline{S^{n+1}\backslash N}$. We shall construct as follows a normal map of (n+1)-dimensional geometric Poincaré pairs

$$f : (X,\partial X) \longrightarrow (D^{n+2}\times S^1, S^{n-1}\times S^1)$$

such that

i) $f|_{\partial X}$ = identity : $\partial X \longrightarrow S^{n-1}\times S^1$

ii) f is an integral homology equivalence, with $f:\pi_1(X)\longrightarrow\pi_1(D^{n+2}\times S^1) = \mathbb{Z}$ the Hurewicz map,

iii) there is given a $\mathbb{Z}$-equivariant geometric Umkehr map

$$F : \Sigma(D^{n+2}\times\mathbb{R}/S^{n-1}\times\mathbb{R}) \longrightarrow \Sigma(\widetilde{X}/\partial\widetilde{X})$$

involving just a single suspension Σ, with $\widetilde{X}$ the infinite cyclic cover of X obtained from the universal cover $D^{n+2}\times\mathbb{R}$ of $D^{n+2}\times S^1$ by pullback along f.

Choose a Seifert surface $M^n\subset S^{n+1}$ for k. Assume that M intersects $\partial N = \partial X = S^{n-1}\times S^1$ transversally, and also that $M\cap N$ is a collar of $\partial M = S^{n-1}$ in M. Then $P^n = M\cap X$ is isomorphic to M, and there is defined a normal map of n-dimensional geometric Poincaré pairs

$$e : (P,\partial P) \longrightarrow (D^{n+2}, S^{n-1})$$

exactly as in the definition of $\sigma_*(k,M)$ above. Let Y be the space obtained from X by cutting along P, so that ∂Y contains two copies of P, P_1 and P_2 say, identified along the common boundary. There is an essentially unique way of extending the corresponding two copies of e to a normal map of (n+1)-dimensional geometric Poincaré triads

$$h : (Y;P_1,P_2,\partial P) \longrightarrow (D^{n+2}\times I; D^{n+2}\times 0, D^{n+2}\times 1, S^{n-1})$$

involving just a single suspension in the Umkehr. Define a $\mathbb{Z}$-equivariant map $\widetilde{f} : \widetilde{X} \longrightarrow D^{n+2}\times\mathbb{R}$ by glueing together a countable number of copies of h end to end. This gives the desired f, a different choice of Seifert surface M giving a homotopic f. The kernel of such a normal map $f : (X,\partial X) \longrightarrow (D^{n+2}\times S^1, S^{n-1}\times S^1)$ is an (n+1)-dimensional symmetric Seifert complex over $\mathbb{Z}[z,z^{-1}]$ $\rho_*(k) = (D,\psi)$ which becomes contractible over $\mathbb{Z}$, whose homotopy type depends only on the isotopy class of the knot k. The kernel

$\mathbb{Z}[z,z^{-1}]$-modules are the usual $\mathbb{Z}[z,z^{-1}]$-torsion knot modules

$$K_*(X) = H_*(D) = \dot{H}_*(\widetilde{X})$$

The duality isomorphisms given by $(1+T)\psi \in H_{n+1}(D^t\otimes_{\mathbb{Z}[z,z^{-1}]}D)$

$$H^r(D) \longrightarrow H_{n+1-r}(D) \quad (0\leq r\leq n+1)$$

are just those established by Blanchfield [1] using linking numbers. This duality has been studied more recently by Levine [4]. The construction of the Blanchfield complex $\beta\sigma_*(k,M)$ is a precise algebraic analogue of the construction of $\rho_*(k)$, replacing the geometric glueing by its algebraic analogue, the union operation of §5.

<u>Proposition 17.15</u> Up to homotopy equivalence of (n+1)-dimensional Seifert complexes over $\mathbb{Z}[z,z^{-1}]$

$$\rho_*(k) = \beta\sigma_*(k,M)$$

for any knot $k:S^{n-1}\subset S^{n+1}$ and Seifert surface $M^n\subset S^{n+1}$. The S-equivalence class of $\sigma_*(k,M)$ is an isotopy invariant of k.

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Proposition 17.15 generalizes the relationship between a Seifert $(-)^i$form $\sigma_*(k,M) = (H_i(M),\varphi\in \mathrm{Hom}_{\mathbb{Z}}(H_i(M),H_i(M)^*))$ of a simple (2i-1)-knot $k:S^{2i-1}\subset S^{2i+1}$ and the $(-)^{i+1}$symmetric Blanchfield duality pairing $(1+T)\psi:H_{i+1}(\widetilde{X})\longrightarrow H^i(\widetilde{X}) = \mathrm{Hom}_{\mathbb{Z}[z,z^{-1}]}(H_{i+1}(\widetilde{X}),F/\mathbb{Z}[z,z^{-1}])$ studied by Trotter [1] and Kearton [1], where $F = (\mathbb{Z}[z,z^{-1}]-\{0\})^{-1}\mathbb{Z}[z,z^{-1}]$ is the quotient field. It is shown there that Seifert ε-forms (over $\mathbb{Z}$) are S-equivalent in the sense of Trotter [1] if and only if they induce isomorphic $(-\varepsilon)$-symmetric Blanchfield duality pairings ($\varepsilon = \pm 1$). Proposition 12.3 of Levine [4] implies that there are natural identifications of categories

($\mathbb{Z}$-acyclic 1-dimensional ε-symmetric Seifert complexes over $\mathbb{Z}[z,z^{-1}]$)

= ($\mathbb{Z}$-acyclic 1-dimensional ε-quadratic Poincaré complexes over $\mathbb{Z}[z,z^{-1}]$)

= ($\mathbb{Z}$-acyclic 1-dimensional ε-symmetric Poincaré complexes over $\mathbb{Z}[z,z^{-1}]$) .

Thus our notion of S-equivalence of 0-dimensional ε-symmetric Seifert complexes over $\mathbb{Z}$ is just S-equivalence in the sense of Trotter [1]. S-equivalence of 1-dimensional ε-symmetric Seifert complexes over $\mathbb{Z}$ is just the T-equivalence of Seifert ε-formations defined by Kearton [2] (provided the homotopy linking is neglected).

Proposition 17.16 Let Φ be the commutative square of rings with involution

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \xrightarrow{1} & \mathbb{Z}[\mathbb{Z}] \\ {\scriptstyle 1}\downarrow & \Phi & \downarrow{\scriptstyle e} \\ \mathbb{Z}[\mathbb{Z}] & \xrightarrow{e} & \mathbb{Z} \end{array}$$

with $e : \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}\ ;\ z \longmapsto 1$ the augmentation map.

The Blanchfield complex construction defines natural isomorphisms of algebraic cobordism groups

$$\beta : \Lambda_n(\mathbb{Z}) = (\text{n-dimensional symmetric Seifert complexes over } \mathbb{Z})$$
$$\longrightarrow \Gamma_{n+2}(\Phi) = \Delta_{n+2}(e) = (\mathbb{Z}\text{-acyclic (n+1)-dimensional quadratic Poincaré complexes over } \mathbb{Z}[\mathbb{Z}])\ ;\ (C,\psi) \longmapsto \beta(C,\psi) \quad (n \geqslant 0)\ .$$

[]

We thus have an algebraic interpretation of the identification

$$C_n = \Gamma_{n+3}(\Phi) \ (= \Lambda_{n+1}(\mathbb{Z})) \quad (n \geqslant 4)$$

of Cappell and Shaneson [1].

Proposition 17.17 Let $S \subset \mathbb{Z}[\mathbb{Z}]$ be the multiplicative subset

$$S = \{p \in \mathbb{Z}[\mathbb{Z}] \,|\, p(1) = 1 \in \mathbb{Z}\}\ .$$

The forgetful maps define isomorphisms of algebraic cobordism groups

$$L_n^S(S^{-1}\mathbb{Z}[\mathbb{Z}]) = \Gamma_n(\mathbb{Z}[\mathbb{Z}] \longrightarrow S^{-1}\mathbb{Z}[\mathbb{Z}])$$
$$= (\text{n-dimensional quadratic } S^{-1}\mathbb{Z}[\mathbb{Z}]\text{-Poincaré complexes over } \mathbb{Z}[\mathbb{Z}])$$
$$\longrightarrow \Gamma_n(e : \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}) = (\text{n-dimensional quadratic } \mathbb{Z}\text{-Poincaré complexes over } \mathbb{Z}[\mathbb{Z}])\ ,$$

$$L_n(\mathbb{Z}[\mathbb{Z}],S) = \Delta_n(\mathbb{Z}[\mathbb{Z}] \longrightarrow S^{-1}\mathbb{Z}[\mathbb{Z}])$$
$$= (S^{-1}\mathbb{Z}[\mathbb{Z}]\text{-acyclic (n-1)-dimensional quadratic Poincaré complexes over } \mathbb{Z}[\mathbb{Z}])$$
$$\longrightarrow \Delta_n(e : \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}) = (\mathbb{Z}\text{-acyclic (n-1)-dimensional quadratic Poincaré complexes over } \mathbb{Z}[\mathbb{Z}]) \quad (n \geqslant 0).$$

Proof: There is a natural identification of categories

(h.d.1 S-torsion $\mathbb{Z}[\mathbb{Z}]$-modules)

= ($S^{-1}\mathbb{Z}[\mathbb{Z}]$-acyclic 1-dimensional $\mathbb{Z}[\mathbb{Z}]$-module chain complexes)

= ($\mathbb{Z}$-acyclic 1-dimensional $\mathbb{Z}[\mathbb{Z}]$-module chain complexes)

[]

We thus have an algebraic interpretation of the identification

$$C_n = L_{n+3}(\mathbb{Z}[\mathbb{Z}],S) \ (= \Delta_{n+3}(e)) \quad (n \geqslant 4)$$

of Smith [1] and Pardon [1].

It follows from Propositions 17.16, 17.17 that the Blanchfield complex construction defines an isomorphism

$$\beta : C_n = \Lambda_{n+1}(\mathbb{Z}) \longrightarrow L_{n+3}(\mathbb{Z}[\mathbb{Z}],S)\ ;\ (C,\psi) \longmapsto \beta(C,\psi) \quad (n \geqslant 0)$$

which fits into a commutative square

$$\begin{array}{ccc} \Lambda_{n+1}(\mathbb{Z}) & \longrightarrow & L_{n+1}(\mathbb{Z}) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \overline{B}} \\ L_{n+3}(\mathbb{Z}[\mathbb{Z}],S) & \longrightarrow & L_{n+2}(\mathbb{Z}[\mathbb{Z}]) \end{array}$$

involving the forgetful maps

$$\Lambda_{n+1}(\mathbb{Z}) \longrightarrow L_{n+1}(\mathbb{Z})\ ,\ L_{n+3}(\mathbb{Z}[\mathbb{Z}],S) \longrightarrow L_{n+2}(\mathbb{Z}[\mathbb{Z}])$$

and the splitting map

$$\overline{B} = \sigma^*(S^1)\otimes - : L_{n+1}(\mathbb{Z}) \longrightarrow L_{n+2}(\mathbb{Z}[\mathbb{Z}])$$

of Ranicki [2] (cf. Proposition 14.1). The commutativity of such a square has a geometric interpretation. Given a knot $k : S^n \subset S^{n+2}$ and a Seifert surface $V^{n+1} \subset S^{n+2}$ let $x = (S^{n+2} - V \times I; V \times 0, V \times 1, \partial V \times I)$ be the relative cobordism used above in the construction of an (n+2)-dimensional normal map

$$(f,b) : (X,\partial X) \longrightarrow (S^1 \times D^{n+3}, S^1 \times S^n) \quad (X = \text{knot complement})$$

with $\mathbb{Z}$-acyclic symmetric Seifert kernel

$$\sigma_*(f,b) = \rho_*(k) = \beta\sigma_*(k,V)\ .$$

Applying the same infinite cyclic glueing construction to the relative cobordism $x' = (V \times I; V \times 0, V \times 1, \partial V \times I)$ results in an (n+2)-dimensional normal map

$$(f',b') : (S^1 \times V, S^1 \times \partial V) \longrightarrow (S^1 \times D^{n+3}, S^1 \times S^n)$$

with symmetric Seifert kernel

$$\sigma_*(f',b') = \sigma^*(S^1)\otimes\sigma_*(k,V)\ .$$

Now $x \cup -x' = S^{n+2} = \partial D^{n+3}$, so that there exists an (n+3)-dimensional normal bordism between (f,b) and (f',b')

$$((g;f,f'),(c;b,b')) : (W^{n+3}; X^{n+2}, S^1 \times V^{n+1}) \longrightarrow S^1 \times D^{n+3} \times (I;0,1)$$

implying that

$$\beta\sigma_*(k,V) = \sigma_*(f,b) = \sigma_*(f',b') = \sigma^*(S^1)\otimes\sigma_*(k,V) \in L_{n+2}(\mathbb{Z}[\mathbb{Z}])$$

As a particular case of the knot module realizability theorem of Levine [4] we have that for any non-singular $(-)^{i+1}$-symmetric linking form over $(\mathbb{Z}[\mathbb{Z}],S)$

$$(M,\lambda : M \longrightarrow M^{\wedge} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(M, S^{-1}\mathbb{Z}[\mathbb{Z}]/\mathbb{Z}[\mathbb{Z}]))$$

there exists a simple knot $k : S^{2i-1} \subset S^{2i+1}$ $(i \geqslant 2)$ such that

$$\rho_*(k) = (M,\lambda) \ .$$

Let $V^{2i} \subset S^{2i+1}$ be a Seifert surface for k, and let $N = H_i(V)/\text{torsion}$ so that $\sigma_*(k,V) = (N, \theta \in \mathrm{Hom}_{\mathbb{Z}}(N,N^*))$ is a Seifert $(-)^i$-form over $\mathbb{Z}$ such that

$$(M,\lambda) = \beta(N,\theta)$$

(by Proposition 17.15). Explicitly, (M,λ) is given in terms of (N,θ) by

$$M = \mathrm{coker}(\theta + (-)^i z\theta^* : N_z \longrightarrow N_z^*)$$

$$\lambda : M \times M \longrightarrow S^{-1}\mathbb{Z}[\mathbb{Z}]/\mathbb{Z}[\mathbb{Z}] \ ; \ ([x],[y]) \longmapsto \frac{(1-z)}{p} x(w)$$

$$(x,y \in N_z^*,\ w \in N_z,\ p \in S,\ py = (\theta + (-)^i z\theta^*)(w) \in N_z^*) \ .$$

Trotter [2] has used localization to give a direct algebraic proof that for each Blanchfield linking form (M,λ) there exists a Seifert form (N,θ), in effect describing the isomorphism

$$\beta^{-1} : L_{2i+2}(\mathbb{Z}[\mathbb{Z}],S) \longrightarrow \Lambda_{2i}(\mathbb{Z}) \ ; \ (M,\lambda) \longmapsto (N,\theta)$$

inverse to β, which fits into a commutative square

$$\begin{array}{ccc} L_{2i+2}(\mathbb{Z}[\mathbb{Z}],S) & \longrightarrow & L_{2i+1}(\mathbb{Z}[\mathbb{Z}]) \\ \beta^{-1}\big\downarrow & & \big\downarrow B \\ \Lambda_{2i}(\mathbb{Z}) & \longrightarrow & L_{2i}(\mathbb{Z}) \end{array}$$

involving the splitting map

$$B : L_{2i+1}(\mathbb{Z}[\mathbb{Z}]) \longrightarrow L_{2i}(\mathbb{Z}) \qquad (B\bar{B} = 1)$$

of Ranicki [2] (cf. Proposition 14.1).

§18. Wu classes

The equivariant S-duality of §3 was developed for the purpose of converting a map of Spivak normal fibrations $b : \nu_M \longrightarrow \nu_X$ over a degree 1 map $f : M \longrightarrow X$ of geometric Poincaré complexes into a quadratic Poincaré complex $\sigma_*(f,b)$ such that $(1+T)\sigma_*(f,b) \oplus \sigma^*(X) = \sigma^*(M)$. In this appendix to §3 we describe the extent to which the symmetric Poincaré complex $\sigma^*(X) = (C(\tilde{X}), \varphi_{\tilde{X}}[X] \in Q^n(C(\tilde{X})))$ associated to an n-dimensional geometric Poincaré complex X reflects properties of the Spivak stable normal fibration $\nu_X : X \longrightarrow BG$. For any stable spherical fibration $p : X \longrightarrow BG$ over any n-dimensional CW complex X we shall define an n-dimensional hyperquadratic complex over $\mathbb{Z}[\pi_1(X)]$ $\hat{\sigma}^*(p) = (C(\tilde{X})^{n-*}, \theta_p \in \hat{Q}^n(C(\tilde{X})^{n-*}))$. The hyperquadratic Wu classes of $\hat{\sigma}^*(p)$ are equivariant analogues of the Wu classes $v_*(p) \in H^*(X;\mathbb{Z}_2)$. For the Spivak stable normal fibration $\nu_X : X \longrightarrow BG$ of an n-dimensional geometric Poincaré complex X it is possible to identify

$$J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$$

(up to homotopy equivalence), giving rise to an equivariant analogue of the Wu formula $v_r(X) = v_r(\nu_X) \in H^r(X;\mathbb{Z}_2)$ relating the diagonal structure of $C(X;\mathbb{Z}_2)$ to ν_X. The quadratic structure in $\sigma_*(f,b)$ expresses the vanishing of the equivariant Wu classes on the kernel of a normal map $(f,b) : M \longrightarrow X$. We shall also develop equivariant analogues of the suspended Wu classes $(\sigma v)_*(h) \in H^{*-1}(X;\mathbb{Z}_2)$ of automorphisms $h : p \longrightarrow p$ of stable spherical fibrations $p : X \longrightarrow BG$ (over any complex X), as required for describing the effect on the quadratic kernel $\sigma_*(f,b)$ of a normal map $(f,b) : M \longrightarrow X$ of a change in the map $b : \nu_M \longrightarrow \nu_X$.

To the symmetric and quadratic constructions of §2

$$\varphi_X : H_*(X) \longrightarrow Q^*(C(X)) \ , \ \psi_F : H_*(X) \longrightarrow Q_*(C(Y)) \qquad (F \in \{X,Y\})$$

we now add a 'hyperquadratic construction'

$$\theta_X : H^*(X) \longrightarrow \hat{Q}^*(C(X)^*) \ .$$

This is defined to be the composite

$$\theta_X : H^*(X) = H_*(Y) \xrightarrow{\varphi_Y} Q^*(C(Y)) = Q^*(C(X)^*) \xrightarrow{J} \hat{Q}^*(C(X)^*)$$

for any S-(or Sπ-)dual Y of X. For example, if $\nu_X : X \longrightarrow BG(k)$ is a Spivak normal fibration for an n-dimensional geometric Poincaré complex X, and $\widetilde{X}$ is an oriented covering of X with data (π,w), then the $S\pi$-duality between $\widetilde{X}_+$ and $T\pi(\nu_X)$ given by Proposition 3.12 expresses the symmetric construction on the fundamental class $[X]\in H_n^{\pi}(\widetilde{X};{}^w\mathbb{Z})$ in terms of the hyperquadratic construction on the Thom class $U_{\nu_X}\in \dot{H}_{\pi}^{k}(T\pi(\nu_X);{}^w\mathbb{Z})$.

Given a group π and an $S\pi$-duality map $\alpha:S^N\longrightarrow X\wedge_{\pi}Y$ between finitely-dominated $CW\pi$-complexes X,Y there is defined a chain equivalence of finite-dimensional $R[\pi]$-module chain complexes

$$\chi_{\alpha}=(\alpha_*[S^N]\backslash -)^{-1}\ :\ \dot{C}(Y;R)\longrightarrow \dot{C}(X;R)^{N-*}$$

for any commutative coefficient ring R, which is obtained by applying $R[\pi]\otimes_{\mathbb{Z}[\pi]}-$ to the $\mathbb{Z}[\pi]$-module chain equivalence $\chi_{\alpha}:\dot{C}(Y)\longrightarrow \dot{C}(X)^{N-*}$ given by Proposition 3.8. Given a group morphism $w:\pi\longrightarrow\mathbb{Z}_2$ endow $R[\pi]$ with the w-twisted involution, and define an $R[\pi]$-module chain map

$$\begin{aligned}\theta_{X,\alpha}\ :\ \mathrm{Hom}_{R[\pi]}(\dot{C}(X;R),{}^wR)&\xrightarrow{\ \chi_{\alpha *}\ }\mathrm{Hom}_{R[\pi]}(\dot{C}(Y;R)^{N-*},{}^wR)=R^t\otimes_{R[\pi]}\dot{C}(Y;R)_{N-*}\\ &\xrightarrow{\ \dot{\varphi}_Y=1\otimes\dot{\Delta}_Y\ }\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\dot{C}(Y;R)^t\otimes_{R[\pi]}\dot{C}(Y;R))_{N-*}\\ &\xrightarrow{\ \chi_{\alpha}^{\%}\ }\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,(\dot{C}(X;R)^{N-*})^t\otimes_{R[\pi]}\dot{C}(X;R)^{N-*})_{N-*}\ .\end{aligned}$$

This induces R-module morphisms in homology

$$\theta_{X,\alpha}\ :\ \dot{H}_{\pi}^{k}(X;{}^wR)\longrightarrow Q^{N-k}(\dot{C}(X;R)^{N-*})$$

such that there is defined commutative diagram of R-module morphisms

$$\begin{array}{ccc}\dot{H}_n^{\pi}(Y;{}^wR)&\xrightarrow{\ \dot{\varphi}_Y\ }&Q^n(\dot{C}(Y;R))\\ \Big\downarrow{\scriptstyle\chi_{\alpha}}&&\Big\downarrow{\scriptstyle\chi_{\alpha}^{\%}}\\ \dot{H}_{\pi}^{N-n}(X;{}^wR)&\xrightarrow{\ \theta_{X,\alpha}\ }&Q^n(\dot{C}(X;R)^{N-*})\end{array}$$

with $\dot{\varphi}_Y$ the symmetric construction of Proposition 2.2. Thus for the $S\pi$-duality map $\alpha_X:S^{n+k}\longrightarrow\widetilde{X}_+\wedge_{\pi}T\pi(\nu_X)$ associated to a normalized n-dimensional geometric Poincaré complex $(X,\nu_X:X\longrightarrow BG(k),\rho_X\in\pi_{n+k}(T(\nu_X)))$ and an oriented covering $\widetilde{X}$ with data (π,w) we have a commutative diagram

$$\begin{array}{ccc}H_n^{\pi}(\widetilde{X};{}^wR)&\xrightarrow{\ \varphi_{\widetilde{X}}\ }&Q^n(C(\widetilde{X};R))\\ \Big\downarrow{\scriptstyle\chi_{\alpha_X *}}&&\Big\downarrow{\scriptstyle\chi_{\alpha_X}^{\%}}\\ \dot{H}_{\pi}^{k}(T\pi(\nu_X);{}^wR)&\xrightarrow{\ \theta_{T\pi(\nu_X),\alpha_X}\ }&Q^n(\dot{C}(T\pi(\nu_X);R)^{n+k-*})\end{array}$$

using the untwisted dual $R[\pi]$-module structure in $\dot{C}(T\pi(\nu_X);R)^{n+k-*}$. Evaluating on the fundamental class $[X]\in H_n^{\pi}(\widetilde{X};{}^wR)$ and using the isomorphism $\chi_{\alpha}^{\%}$ as an identification we can write

$$\varphi_{\widetilde{X}}[X]=\theta_{T\pi(\nu_X),\alpha_X}(U_{\nu_X})\in Q^n(C(\widetilde{X};R))\ ,$$

with $U_{\nu_X}\in\dot{H}_{\pi}^{k}(T\pi(\nu_X);{}^wR)$ the Thom class of ν_X. (We are only using the orientability of $\widetilde{X}$ with coefficients in R here).

We shall now show that for a fixed finitely dominated $CW\pi$-complex X the composite

$$\dot{H}_{\pi}^{k}(X;{}^wR)\xrightarrow{\ \theta_{X,\alpha}\ }Q^{N-k}(\dot{C}(X;R)^{N-*})\xrightarrow{\ J\ }\hat{Q}^{-k}(\dot{C}(X;R)^{-*})$$

is independent of the $S\pi$-duality map $\alpha:S^N\longrightarrow X\wedge_{\pi}Y$, with J as in Proposition 1.2. We have the following <u>hyperquadratic construction</u>.

<u>Proposition 18.1</u> Let π be a group, $w:\pi\longrightarrow\mathbb{Z}_2$ a group morphism, R a commutative ring, and give the group ring $R[\pi]$ the w-twisted involution.

Given a finitely-dominated $CW\pi$-complex X there are defined in a natural way R-module morphisms

$$\theta_X\ :\ \dot{H}_{\pi}^{k}(X;{}^wR)\longrightarrow\hat{Q}^{-k}(\dot{C}(X;R)^{-*})\qquad (k\geqslant 0)$$

with the untwisted dual $R[\pi]$-module structure on $\dot{C}(X;R)^{-*}$, such that

i) if $\alpha:S^N\longrightarrow X\wedge_{\pi}Y$ is an $S\pi$-duality map there is defined a commutative diagram of R-modules

$$\begin{array}{ccc}&&Q^{N-k}(\dot{C}(X;R)^{N-*})\\ &\nearrow^{\theta_{X,\alpha}}&\Big\downarrow{\scriptstyle J}\\ \dot{H}_{\pi}^{k}(X;{}^wR)&\searrow_{\theta_X}&\hat{Q}^{-k}(\dot{C}(X;R)^{-*})\end{array}\quad ,$$

ii) if $f:X\longrightarrow Y$ is a π-map of finitely-dominated CWπ-complexes then there is defined a commutative diagram of R-modules

$$\begin{array}{ccc} \dot{H}^k_\pi(Y;{}^wR) & \xrightarrow{\ \theta_Y\ } & \hat{Q}^{-k}(\dot{C}(Y;R)^{-*}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \hat{f}^{*\%}} \\ \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\ \theta_X\ } & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \end{array}\ ,$$

iii) the construction is invariant under suspension, in that there is defined a commutative diagram of R-modules

$$\begin{array}{ccc} \dot{H}^{k+1}_\pi(\Sigma X;{}^wR) & \xrightarrow{\ \theta_{\Sigma X}\ } & \hat{Q}^{-k-1}(\dot{C}(\Sigma X;R)^{-*}) \\ {\scriptstyle \Sigma^*_X}\downarrow & & \downarrow{\scriptstyle \hat{\Sigma}^{*\%}_X} \\ \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\ \theta_X\ } & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \end{array}$$

in which the vertical maps are the suspension isomorphisms,

iv) if $h:R\longrightarrow S$ is a morphism of commutative rings, with $\dot{H}^{k-1}_\pi(X;{}^wS) = 0$, there is defined a commutative diagram of R-modules

$$\begin{array}{ccc} \dot{H}^k_\pi(X;{}^wR) & \xrightarrow{\ \theta_X\ } & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \dot{H}^k_\pi(X;{}^wS) & \xrightarrow{\ \theta_X\ } & \hat{Q}^{-k}(\dot{C}(X;S)^{-*}) \end{array}$$

in which the vertical maps are the change of rings $h:R[\pi]\longrightarrow S[\pi]$.

Proof: If $\alpha:S^N\longrightarrow X\wedge_\pi Y$ is an Sπ-duality map then so is $\Sigma\alpha:S^{N+1}\longrightarrow X\wedge_\pi \Sigma Y$, and Proposition 2.4 gives a commutative diagram

$$\begin{array}{ccc} & \overset{\theta_{X,\alpha}}{\nearrow} & \hat{Q}^{N-k}(\dot{C}(X;R)^{N-*}) \\ \dot{H}^k_\pi(X;{}^w\mathbb{Z}) & & \downarrow{\scriptstyle S} \\ & \underset{\theta_{X,\Sigma\alpha}}{\searrow} & \hat{Q}^{N-k+1}(\dot{C}(X;R)^{N+1-*}) \end{array}\ .$$

If $\alpha':S^N\longrightarrow X\wedge_\pi Y'$ is another Sπ-duality map let $F\in\{Y,Y'\}_\pi$ be the image of $1\in\{X,X\}_\pi$ under the Sπ-duality isomorphism

$$\{X,X\}_\pi \xrightarrow{(\alpha\backslash -)} \{S^N,Y'\wedge_\pi X\} \xrightarrow{(\alpha\backslash -)^{-1}} \{Y,Y'\}_\pi\ .$$

Applying the quadratic construction of Proposition 2.5 we obtain

$$\dot{\psi}_F : \dot{H}^\pi_{N-k}(Y;{}^wR) \longrightarrow Q_{N-k}(\dot{C}(Y';R))$$

such that

$$F^\%\dot{\varphi}_Y - \dot{\varphi}_{Y'}F_* = (1+T)\dot{\psi}_F : \dot{H}^\pi_{N-k}(Y;{}^wR) \longrightarrow Q^{N-k}(\dot{C}(Y';R))\ .$$

The composite

$$Q_{N-k}(\dot{C}(Y';R)) \xrightarrow{\ 1+T\ } Q^{N-k}(\dot{C}(Y';R)) \xrightarrow{\ J\ } \hat{Q}^{N-k}(\dot{C}(Y';R))$$

is 0 (Proposition 1.2), so that there is defined a commutative diagram

$$\begin{array}{ccccccc} \dot{H}^k_\pi(X;{}^wR) & & & \xrightarrow{\ J\theta_{X,\alpha}\ } & & & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \\ & \overset{\chi_{\alpha*}}{\searrow} & & & & \overset{\hat{\chi}^\%_\alpha}{\nearrow} & \\ & & \dot{H}^\pi_{N-k}(Y;{}^wR) & \xrightarrow{\dot{\varphi}_Y} Q^{N-k}(\dot{C}(Y;R)) \xrightarrow{J} & \hat{Q}^{N-k}(\dot{C}(Y;R)) & & \\ {\scriptstyle 1}\downarrow & & {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle \hat{F}^\%} & & \downarrow{\scriptstyle 1} \\ & & \dot{H}^\pi_{N-k}(Y';{}^wR) & \xrightarrow{\dot{\varphi}_{Y'}} Q^{N-k}(\dot{C}(Y';R)) \xrightarrow{J} & \hat{Q}^{N-k}(\dot{C}(Y';R)) & & \\ & \underset{\chi_{\alpha'*}}{\nearrow} & & & & \underset{\hat{\chi}^\%_{\alpha'}}{\searrow} & \\ \dot{H}^k_\pi(X;{}^wR) & & & \xrightarrow{\ J\theta_{X,\alpha'}\ } & & & \hat{Q}^{-k}(\dot{C}(X;R)^{-*}) \end{array}\ .$$

Thus

$$J\theta_{X,\alpha} = J\theta_{X,\alpha'} : \dot{H}^k_\pi(X;{}^wR) \longrightarrow \hat{Q}^{-k}(\dot{C}(X;R)^{-*})$$

is independent of the Sπ-duality maps involved, and may be written as θ_X. []

Applying the hyperquadratic Wu class operations $\hat{v}_r$ of §1 to the hyperquadratic construction for $\pi = \{1\}$, $R = \mathbb{Z}_2$ we recover the duals of the Steenrod squares.

Proposition 18.2 Let X be a finitely-dominated connected CW complex. The composite

$$\dot{H}^k(X;\mathbb{Z}_2) \xrightarrow{\ \theta_X\ } \hat{Q}^{-k}(\dot{C}(X;\mathbb{Z}_2)^{-*}) \xrightarrow{\ \hat{v}_r\ } \mathrm{Hom}_{\mathbb{Z}_2}(\dot{H}_{k+r}(X;\mathbb{Z}_2),\mathbb{Z}_2)$$

is given by

$$\hat{v}_r(\theta_X(x))(y) = \langle \chi(Sq^r)(x),y\rangle \in \mathbb{Z}_2 \qquad (x\in \dot{H}^k(X;\mathbb{Z}_2),\ y\in \dot{H}_{k+r}(X;\mathbb{Z}_2))$$

with $\chi(Sq^r)$ the image of Sq^r under the canonical anti-automorphism χ of the mod2 Steenrod algebra, as characterized by $\sum_{i+j=r}\chi(Sq^i)Sq^j = \begin{cases} Sq^0 & \text{if } r=0 \\ 0 & \text{if } r\neq 0\ . \end{cases}$

Proof: Apply Proposition 2.3 to an S-dual Y of X, and use the result of Thom [1] that Steenrod squares in Y correspond to the duals of the Steenrod squares in X. []

We shall say that a space X is <u>n-dimensional</u> if it is of the homotopy type of a connected finitely dominated CW complex and the universal cover $\widetilde{X}$ is such that $H^r_\pi(\widetilde{X}) = 0$ for $r > n$, in which case $C(\widetilde{X})$ is an n-dimensional $\mathbb{Z}[\pi]$-module chain complex ($\pi = \pi_1(X)$). In particular, an n-dimensional geometric Poincaré complex is an n-dimensional space.

The hyperquadratic construction associates a hyperquadratic complex to every oriented covering of the base space of a stable spherical fibration over a finite-dimensional space.

<u>Proposition 18.3</u> Given a stable spherical fibration $p:X \longrightarrow BG$ over an n-dimensional space X and an oriented covering $\widetilde{X}$ with data (π,w), and given also a commutative ring R, there is defined in a natural way an n-dimensional hyperquadratic complex over $R[\pi]$ with the w-twisted involution, the <u>Wu complex</u> of p,

$$\hat{\sigma}^*(p) = ({}^wC(\widetilde{X};R)^{n-*}, \theta_{T\pi(p)}(U_p)) \in \hat{Q}^n({}^wC(\widetilde{X};R)^{n-*})$$

depending only on the stable fibre homotopy class of p.

The hyperquadratic Wu classes of $\hat{\sigma}^*(p)$ are the <u>Wu classes</u> of p, $R[\pi]$-module morphisms

$$v_r(p) = \hat{v}_r(\theta_{T\pi(p)}(U_p)) : H_r(\widetilde{X};R) \longrightarrow \hat{H}^r(\mathbb{Z}_2;R[\pi]) \quad (r \geq 0)$$

such that

i) the 0th Wu class is the augmentation map

$$v_0(p) : H_0(\widetilde{X};R) \longrightarrow \hat{H}^0(\mathbb{Z}_2;R[\pi])\,;\ \sum_{g\in\pi} n_g gx \longmapsto \sum_{g\in\pi} n_g \in R/2R = \hat{H}^0(\mathbb{Z}_2;R) \subseteq \hat{H}^0(\mathbb{Z}_2;R[\pi])$$

with $x \in H_0(\widetilde{X};R)$ the geometric $R[\pi]$-module generator defined by any path-component of $\widetilde{X}$,

ii) if $f:M \longrightarrow X$ is a map of n-dimensional spaces with induced cover $\widetilde{M}$ and pullback fibration $f^*p : M \xrightarrow{f} X \xrightarrow{p} BG$ then there is defined a map of Wu complexes

$$\check{f}^* : \hat{\sigma}^*(p) \longrightarrow \hat{\sigma}^*(f^*p)$$

and the Wu classes are such that there is defined a commutative diagram

$$\begin{array}{ccc} H_r(\widetilde{M};R) & \xrightarrow{\tilde{f}_*} & H_r(\widetilde{X};R) \\ & {\scriptstyle v_r(f^*p)} \searrow \quad \swarrow {\scriptstyle v_r(p)} & \\ & \hat{H}^r(\mathbb{Z}_2;R[\pi]) & \end{array}$$

iii) if $p:X \longrightarrow BG$ is stably fibre homotopy trivial then

$$v_r(p) = 0 : H_r(\widetilde{X};R) \longrightarrow \hat{H}^r(\mathbb{Z}_2;R[\pi]) \quad (r > 0)\ ,$$

iv) $\hat{\sigma}^*(p)$ is induced via $R[\pi]\otimes_{R[\pi_1(X)]} -$ from the Wu complex $\hat{\sigma}^*(\tilde{p})$ associated to the universal cover $\widetilde{\widetilde{X}}$, and there is defined a commutative diagram

$$\begin{array}{ccc} H_r(\widetilde{\widetilde{X}};R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;R[\pi_1(X)]) \\ \downarrow & & \downarrow \\ H_r(\widetilde{X};R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;R[\pi]) \end{array}$$

in which the vertical maps are the change of rings $R[\pi_1(X)] \longrightarrow R[\pi]$, with $\pi_1(X) \longrightarrow \pi$ the characteristic map of the covering,

v) if $h:R \longrightarrow S$ is a morphism of commutative rings there is defined a commutative diagram

$$\begin{array}{ccc} H_r(\widetilde{X};R) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;R[\pi]) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_r(\widetilde{X};S) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;S[\pi]) \end{array}$$

in which the vertical maps are the change of rings $h:R[\pi] \longrightarrow S[\pi]$.

<u>Proof</u>: Choose a representative (k-1)-spherical fibration $p:X \longrightarrow BG(k)$, evaluate

$$\theta_{T\pi(p)} : \dot{H}^k_\pi(T\pi(p);{}^wR) \longrightarrow \hat{Q}^{-k}(\dot{C}(T\pi(p);R)^{-*}) = \hat{Q}^n(\dot{C}(T\pi(p);R)^{n+k-*})$$

on the Thom class $U_p \in \dot{H}^k_\pi(T\pi(p);{}^wR)$, and use the Thom equivalence

$$U_p \cap - : \dot{C}(T\pi(p);R) \longrightarrow {}^wS^kC(\widetilde{X};R)$$

to obtain an element $\theta_{T\pi(p)}(U_p) \in \hat{Q}^n({}^wC(\widetilde{X};R)^{n-*})$.

To prove iii), that $v_r(0) = 0$ $(r > 0)$, let Y be a skeleton of $K(\pi,1)$ of dimension $>r$ containing the image of the classifying map $f:X \longrightarrow K(\pi,1)$ of the covering $\widetilde{X}$ (assuming that X is finite, in the first instance), and apply the naturality property ii) to obtain a commutative diagram

$$\begin{array}{ccc} H_r(\widetilde{X};R) & \xrightarrow{\tilde{g}_*} & H_r(\widetilde{Y};R) = 0 \\ & {\scriptstyle v_r(g^*0) = v_r(0)} \searrow \quad \swarrow {\scriptstyle v_r(0)} & \\ & \hat{H}^r(\mathbb{Z}_2;R[\pi]) & \end{array} \qquad (g = f| : X \longrightarrow Y)$$

[]

The <u>mod 2 Stiefel-Whitney classes</u> $w_*(p) \in H^*(X;\mathbb{Z}_2)$ of a spherical fibration $p: X \longrightarrow BG(k)$ are characterized by the property

$$U_p \cap w_j(p) = Sq^j(U_p) \in \dot{H}^{j+k}(T(p);\mathbb{Z}_2)$$

which may be expressed in terms of the symmetric construction and the symmetric Wu classes as

$$w_j(p) : H_j(X;\mathbb{Z}_2) \xrightarrow{(U_p\cap -)^{-1}} \dot{H}_{j+k}(T(p);\mathbb{Z}_2) \xrightarrow{\dot{\phi}_{T(p)}} Q^{j+k}(\dot{C}(T(p);\mathbb{Z}_2))$$

$$\xrightarrow{v_j} \mathrm{Hom}_{\mathbb{Z}_2}(\dot{H}^k(T(p);\mathbb{Z}_2), H^{k-j}(\mathbb{Z}_2;\mathbb{Z}_2)) = \begin{cases} \mathbb{Z}_2 & \text{if } j \leqslant k \\ 0 & \text{if } j > k \end{cases}$$

(cf. Proposition 2.3), with $U_p \in \dot{H}^k(T(p);\mathbb{Z}_2)$ the mod 2 Thom class.

The <u>mod 2 Wu classes</u> $v_*(p) \in H^*(X;\mathbb{Z}_2)$ of a stable spherical fibration $p: X \longrightarrow BG$ over a finite-dimensional space X are defined by

$$v_r(p) = \sum_{i+j=r} \chi(Sq^i) w_j(-p) \in H^r(X;\mathbb{Z}_2) \quad (r \geqslant 0)$$

with $-p: X \longrightarrow BG$ any stable inverse for p. The mod 2 Wu classes are characterized by the property

$$v_r(p)(U_p \cap z) = \langle \chi(Sq^r)(U_p), z \rangle \in \mathbb{Z}_2 \qquad (z \in \dot{H}_{r+k}(T(p);\mathbb{Z}_2)) .$$

<u>Proposition 18.4</u> The mod 2 reductions of the Wu classes of a stable spherical fibration $p: X \longrightarrow BG$ over a finite-dimensional space X with respect to an oriented cover $\widehat{X}$ of X with data (π, w) agree with the mod 2 Wu classes, that is there are defined commutative diagrams

$$\begin{array}{ccc} H_r(\widehat{X}) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}[\pi]) \\ \downarrow & & \downarrow \\ H_r(X;\mathbb{Z}_2) & \xrightarrow{v_r(p)} & \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2 \end{array}$$

in which the vertical maps are the change of rings $\mathbb{Z}[\pi] \longrightarrow \mathbb{Z}_2 ; \sum_{g\in\pi} n_g g \longmapsto \sum_{g\in\pi} n_g$.

<u>Proof</u>: Applying Proposition 18.2 we can express the mod 2 Wu classes of p in terms of the hyperquadratic construction by evaluating the composite

$$\dot{H}^k(T(p);\mathbb{Z}_2) \xrightarrow{\theta_{T(p)}} \hat{Q}^{-k}(\dot{C}(T(p);\mathbb{Z}_2)) \xrightarrow{\hat{v}_r} \mathrm{Hom}_{\mathbb{Z}_2}(\dot{H}_{k+r}(T(p);\mathbb{Z}_2),\mathbb{Z}_2)$$

$$= \mathrm{Hom}_{\mathbb{Z}_2}(H_r(X;\mathbb{Z}_2),\mathbb{Z}_2) = H^r(X;\mathbb{Z}_2)$$

on the mod 2 Thom class $U_p \in \dot{H}^k(T(p);\mathbb{Z}_2)$, so that

$$v_r(p) = \hat{v}_r \theta_{T(p)}(U_p) \in H^r(X;\mathbb{Z}_2) .$$

[]

Define the <u>Hopf invariant</u> function

$$H : \pi_{m+n}(S^m) \longrightarrow H^{m-n-1}(\mathbb{Z}_2;\mathbb{Z},(-)^m) = \begin{cases} \mathbb{Z} & \text{if } m = n+1 \text{ , } n \equiv 1 (\mathrm{mod}\ 2) \\ \mathbb{Z}_2 & \text{if } m > n+1 \text{ , } n \equiv 1 (\mathrm{mod}\ 2) \text{ ;} \\ 0 & \text{otherwise} \end{cases}$$

$$(f: S^{m+n} \longrightarrow S^m) \longmapsto H(f)$$

by applying the symmetric construction to the mapping cone $X = S^m \cup_f e^{m+n+1}$, with

$$\mathbb{Z} = \dot{H}_{m+n+1}(X) \xrightarrow{\dot{\phi}_X} Q^{m+n+1}(\dot{C}(X)) \xrightarrow{v_{n+1}} \mathrm{Hom}_{\mathbb{Z}}(\dot{H}^m(X), H^{m-n-1}(\mathbb{Z}_2,\mathbb{Z},(-)^m)) ;$$

$$1 \longmapsto v_{n+1}(\dot{\phi}_X(1)) = H(f) \qquad (\dot{H}^m(X) = \mathbb{Z}) .$$

Alternatively, apply the quadratic construction to $f: \Sigma^m(S^n) = S^{m+n} \longrightarrow \Sigma^m(S^0) = S^m$,

$$\dot{\psi}_f : \dot{H}_n(S^n) = \mathbb{Z} \longrightarrow Q_n^{[0,m-1]}(\dot{C}(S^0)) \underset{(\text{if } n>0)}{=} H^{m-n-1}(\mathbb{Z}_2;\mathbb{Z},(-)^m) \ ; \ 1 \longmapsto \dot{\psi}_f(1) = H(f).$$

Both these ways agree with the construction of the Hopf invariant due to Steenrod [1], by Propositions 2.2 i), 2.3, 2.6. The morphism defined in §4 is the composite $j: \pi_n(SO(m)) \xrightarrow{J} \pi_n(SG(m)) = \pi_{m+n}(S^m) \xrightarrow{H} Q^{m+n+1}(S^m\mathbb{Z})$. The diagram

$$\begin{array}{ccc} \pi_{m+n}(S^m) & \xrightarrow{H} & H^{m-n-1}(\mathbb{Z}_2;\mathbb{Z},(-)^m) \\ \Sigma \downarrow & & \downarrow S \ (= \text{id. if } m > n+1) \\ \pi_{m+n+1}(S^{m+1}) & \xrightarrow{H} & H^{m-n}(\mathbb{Z}_2;\mathbb{Z},(-)^{m+1}) \end{array}$$

commutes, so that it is possible to define the <u>stable Hopf invariant</u>

$$\hat{H} : \pi_n^S = \varinjlim_m \pi_{m+n}(S^m) \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } n \equiv 1 (\mathrm{mod}\ 2) \\ 0 & \text{if } n \equiv 0 (\mathrm{mod}\ 2) \text{ .} \end{cases}$$

<u>Proposition 18.5</u> The Wu classes of an orientable stable spherical fibration $p: S^m \longrightarrow BG$ over S^m $(m \geqslant 1)$ are given by

$$v_0(p) : H_0(S^m) = \mathbb{Z} \longrightarrow \hat{H}^0(\mathbb{Z}_2;\mathbb{Z}) = \mathbb{Z}_2 \ ; \ z \longmapsto z^2 \equiv z (\mathrm{mod}\ 2)$$

$$v_m(p) : H_m(S^m) = \mathbb{Z} \longrightarrow \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } m \equiv 0 (\mathrm{mod}\ 2) \\ 0 & \text{if } m \equiv 1 (\mathrm{mod}\ 2) \end{cases} ;$$

$$z \longmapsto z^2 (\text{stable Hopf invariant of classifying map } p \in \pi_m(BG) = \pi_{m-1}^S).$$

<u>Proof</u>: Choosing a representative $(k-1)$-spherical fibration $p: S^m \longrightarrow BG(k)$ $(k \gg m)$ we have that the Thom space $T(p)$ is the mapping cone of $p \in \pi_m(BG(k)) = \pi_{m+k-1}(S^k)$

$$T(p) = S^k \cup_p e^{k+m}$$

by Lemma 1 of Milnor [3] (see also Proposition 3.7 of Wall [4]). Now

$$v_m(p) = \sum_{i+j=m} \chi(Sq^i) w_j(-p) = w_m(-p) = w_m(p) \in H^m(S^m;\mathbb{Z}_2) = \mathbb{Z}_2 \text{ ,}$$

and $w_m(p) = \hat{H}(p) \in \mathbb{Z}_2$ by construction.

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The <u>rth Wu class</u> of an n-dimensional geometric Poincaré complex X with respect to an oriented cover $\widetilde{X}$ of X with data (π,w) is the rth Wu class of the associated n-dimensional symmetric Poincaré complex $\sigma^*(X) = (C(\widetilde{X}),\varphi_{\widetilde{X}}[X])$

$$v_r(X) = v_r(\varphi_{\widetilde{X}}[X]) : H_r(\widetilde{X}) \longrightarrow \hat{H}^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r}) \quad (r > 0)\ .$$

The <u>mod 2 Wu classes</u> $v_r(X) \in H^r(X;\mathbb{Z}_2)$ of X are characterized by

$$v_r(X)([X]\cap y) = \langle Sq^r(y),[X]\rangle \in \mathbb{Z}_2 \quad (y \in H^{n-r}(X;\mathbb{Z}_2),\ [X] \in H_n(X;\mathbb{Z}_2)),$$

and Proposition 2.3 gives commutative diagrams relating the two types of Wu class

$$\begin{array}{ccc} H_r(\widetilde{X}) & \xrightarrow{v_r(X)} & \hat{H}^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r}) \\ \downarrow & & \downarrow \\ H_r(X;\mathbb{Z}_2) & \xrightarrow{v_r(X)} & \hat{H}^{n-2r}(\mathbb{Z}_2;\mathbb{Z}_2,(-)^{n-r}) = \begin{cases} \mathbb{Z}_2 & \text{if } 2r \leqslant n \\ 0 & \text{if } 2r > n \end{cases} \end{array}$$

in which the vertical maps are the change of rings $\mathbb{Z}[\pi] \longrightarrow \mathbb{Z}_2;\ \sum_{g\in\pi} n_g g \longmapsto \sum_{g\in\pi} n_g$.
The <u>reduced Wu classes</u> of X are defined by passing to the reduced (Tate) cohomology groups

$$\hat{v}_r(X) : H_r(\widetilde{X}) \xrightarrow{v_r(X)} \hat{H}^{n-2r}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r}) \xrightarrow{\text{reduction}} \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}[\pi])\ .$$

Note that $\hat{v}_r(X) = v_r(X)$ for $n \neq 2r$.

<u>Proposition 18.6</u> If X is an n-dimensional geometric Poincaré complex and $\widetilde{X}$ is an oriented covering with data (π,w) then the Poincaré duality chain equivalence

$$[X]\cap - : {}^wC(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$$

defines a homotopy equivalence of n-dimensional hyperquadratic complexes over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$[X]\cap - : \hat{\sigma}^*(\nu_X) = ({}^wC(\widetilde{X})^{n-*},\theta_{T\pi(\nu_X)}(U_{\nu_X}) \in \hat{Q}^n({}^wC(\widetilde{X})^{n-*}))$$
$$\longrightarrow J\sigma^*(X) = (C(\widetilde{X}),J\varphi_{\widetilde{X}}[X] \in \hat{Q}^n(C(\widetilde{X})))\ .$$

In particular, the reduced Wu classes of X are just the Wu classes of the Spivak normal fibration $\nu_X : X \longrightarrow BG$

$$\hat{v}_r(X) = v_r(\nu_X) : H_r(\widetilde{X}) \longrightarrow \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}[\pi]) \quad (r \geqslant 0).$$

<u>Proof</u>: We have already obtained the identity

$$\varphi_{\widetilde{X}}[X] = \theta_{T\pi(\nu_X),\alpha_X}(U_{\nu_X}) \in Q^n(C(\widetilde{X}))$$

(just before Proposition 18.1). Now apply the J-homomorphism of passing to the suspension limit to remove the dependence on the choice of $S\pi$-duality α_X.

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The identities $J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$, $\hat{v}_r(X) = v_r(\nu_X)$ may be considered as equivariant generalizations of the formulae of Wu [1] and Thom [1] relating the mod 2 Wu classes of a manifold X to the mod 2 Stiefel-Whitney classes of the tangent bundle τ_X, since $v_r(X) = v_r(\nu_X) \in H^r(X;\mathbb{Z}_2)$ can be written as

$$v_r(X) = \sum_{i+j=r} \chi(Sq^i)w_j(\tau_X) \in H^r(X;\mathbb{Z}_2)\ ,$$

or equivalently

$$w_r(\tau_X) = \sum_{i+j=r} Sq^i v_j(X) \in H^r(X;\mathbb{Z}_2)\ .$$

If $T \in \mathbb{Z}_2$ acts on a group ring $\mathbb{Z}[\pi]$ by the w-twisted involution, for some group morphism $w:\pi \longrightarrow \mathbb{Z}_2$, then the direct sum decomposition of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$\mathbb{Z}[\pi] = \mathbb{Z} \oplus \mathbb{Z}[\pi]/\mathbb{Z}$$

gives rise to a direct sum decomposition of $\begin{cases} \mathbb{Z}_2\text{-cohomology} \\ \mathbb{Z}_2\text{-homology} \\ \text{Tate } \mathbb{Z}_2\text{-cohomology} \end{cases}$ groups

which we shall write as

$$\begin{cases} H^r(\mathbb{Z}_2;\mathbb{Z}[\pi],\varepsilon) = H^r(\mathbb{Z}_2;\mathbb{Z},\varepsilon) \oplus H^r(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},\varepsilon) \\ H_r(\mathbb{Z}_2;\mathbb{Z}[\pi],\varepsilon) = H_r(\mathbb{Z}_2;\mathbb{Z},\varepsilon) \oplus H_r(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},\varepsilon) \\ \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}[\pi],\varepsilon) = \hat{H}^r(\mathbb{Z}_2;\mathbb{Z},\varepsilon) \oplus \hat{H}^r(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},\varepsilon) \end{cases}\ ,$$

with $\varepsilon = \pm 1 \in \mathbb{Z}$. We shall call elements of these groups <u>regular</u> if they have a decomposition of the type $(?,0)$. The Wu classes of an orientable spherical fibration $p:X \longrightarrow BG(k)$ with respect to the trivial cover $\widetilde{X} = \pi \times X$ take regular values, $v_m(p):H_m(\widetilde{X}) = \mathbb{Z}[\pi]\otimes_{\mathbb{Z}}H_m(X) \longrightarrow \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi])\ ;\ 1\otimes x \longmapsto (v_m(p)(x),0)$.

A map of geometric Poincaré complexes $f:M \longrightarrow X$ such that $\dim M = m \leqslant \dim X = n$ <u>represents</u> the homology class $x \in H_m(\widetilde{X})$ if $\widetilde{X}$ is an oriented cover of X with data (π,w) such that the composite $\pi_1(M) \xrightarrow{f} \pi_1(X) \longrightarrow \pi$ is trivial, so that $\widetilde{M} = \pi \times M$ is the trivial cover of M and M is oriented (since $w(M):\pi_1(M) \xrightarrow{f} \pi_1(X) \longrightarrow \pi \xrightarrow{w} \mathbb{Z}_2$ is trivial), and if the induced $\mathbb{Z}[\pi]$-module morphism $\tilde{f}:H_m(\widetilde{M}) = \mathbb{Z}[\pi]\otimes_{\mathbb{Z}}H_m(M) = \mathbb{Z}[\pi] \longrightarrow H_m(\widetilde{X})$ sends the generator to $\tilde{f}(1\otimes[M]) = x \in H_m(\widetilde{X})$ for some lift $\tilde{f}:\widetilde{M} \longrightarrow \widetilde{X}$. The lift is non-unique, all such lifts being given by $g\tilde{f} = \tilde{f}g : \widetilde{M} \longrightarrow \widetilde{X}$ $(g \in \pi)$, so that if $x \in H_m(\widetilde{X})$

is representable then so is $gx\in H_m(\widetilde{X})$ $(g\in\pi)$. (Note that for $\pi = \{1\}$ we have the result of Levitt [1] that every homology class $x\in H_m(X)$ is representable in this sense, for any CW complex X). The homology class $x\in H_m(\widetilde{X})$ which are represented by maps $f:S^m \longrightarrow X$ are <u>spherical</u>.

<u>Proposition 18.7</u> The mth reduced Wu class of a geometric Poincaré complex X with respect to an oriented cover $\widetilde{X}$ of X with data (π,w)

$$\hat{v}_m(X) : H_m(\widetilde{X}) \longrightarrow \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi])$$

takes regular values on representable homology classes. If $x\in H_m(\widetilde{X})$ is represented by $f:M\longrightarrow X$ then

$$\hat{v}_m(X)(x) = (v_m(f^*\nu_X)([M]),0)\in \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi]) = \hat{H}^m(\mathbb{Z}_2;\mathbb{Z})\oplus\hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z}) ,$$

and if $M = S^m$ then

$$\hat{v}_m(X)(x) = (\text{stable Hopf invariant of } f^*\nu_X\in \pi_m(BG) = \pi^S_{m-1},0) \in \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi]).$$

<u>Proof</u>: Combining Propositions 18.3 ii), 18.6 we have

$$\begin{aligned}\hat{v}_m(X)(\widetilde{f}(1\otimes[M])) &= v_m(\nu_X)(\widetilde{f}(1\otimes[M]))\\ &= v_m(f^*\nu_X)(1\otimes[M]) = 1\otimes v_m(f^*\nu_X)([M])\in \hat{H}^m(\mathbb{Z}_2;\mathbb{Z}[\pi]) .\end{aligned}$$

For spherical homology classes apply Proposition 18.5 to identify the Wu class with the stable Hopf invariant.

[]

The result of Proposition 18.7 restricts the $\pm$symmetric forms and formations occurring as the symmetric kernels of highly-connected degree 1 maps of geometric Poincaré complexes to be the $\pm$symmetrizations of $\pm$quadratic forms and formations, except in dimensions related to the Hopf invariant 1 dimensions.

<u>Proposition 18.8</u> The symmetric kernel of an (i-1)-connected degree 1 map $f:M\longrightarrow X$ of $\begin{Bmatrix}2i\text{-}\\2i+1\text{-}\end{Bmatrix}$ dimensional geometric Poincaré complexes is a non-singular $(-)^i$-symmetric $\begin{Bmatrix}\text{form}\\\text{formation}\end{Bmatrix}$

$$\sigma^i(f) = \begin{cases}(Q,\varphi)\\(Q,\varphi;F,G)\end{cases}$$

which has a vanishing reduced Wu class for $i \neq 2,4,8$

$$\begin{cases}\hat{v}_i(\varphi) = 0 : Q = K_i(M) = \pi_{i+1}(f) \longrightarrow \hat{H}^i(\mathbb{Z}_2;\mathbb{Z}[\pi]) ; x\longmapsto\varphi(x)(x)\\ \hat{v}_i(\varphi) = 0 : Q/F+G = K_i(M) = \pi_{i+1}(f)\longrightarrow \hat{H}^i(\mathbb{Z}_2;\mathbb{Z}[\pi]) ; x\longmapsto\varphi(x)(x)\end{cases} \quad (\pi=\pi_1(X)) ,$$

so that $\varphi = \psi + (-)^i\psi^* \in \ker(1-T_{(-)^i}:\mathrm{Hom}_{\mathbb{Z}[\pi]}(Q,Q^*)\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\pi]}(Q,Q^*))$ for some $\psi\in \mathrm{coker}(1-T_{(-)^i}:\mathrm{Hom}_{\mathbb{Z}[\pi]}(Q,Q^*)\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\pi]}(Q,Q^*))$. For $i = 2,4,8$ the reduced Wu class only takes regular values in $\hat{H}^i(\mathbb{Z}_2;\mathbb{Z}[\pi]) = \mathbb{Z}_2\oplus\hat{H}^i(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z})$.

<u>Proof</u>: This follows from Proposition 18.7 on applying the result of Adams [1] that the stable Hopf invariant map $\hat{H}:\pi^S_{i-1}\longrightarrow\hat{H}^i(\mathbb{Z}_2;\mathbb{Z})$ is 0 for $i \neq 2,4,8$.

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For $i = 2,4,8$ let M be the (i-1)-connected 2i-dimensional geometric Poincaré complex defined by the complex projective plane $\mathbb{C}P^2 = S^2\cup_\eta e^4$, the quaternion projective plane $\mathbb{H}P^2 = S^4\cup_\eta e^8$ and the Cayley projective plane $\mathbb{O}P^2 = S^8\cup_\eta e^{16}$ respectively, with $\eta\in\pi_{2i-1}(S^i)$ the Hopf invariant 1 elements. The symmetric kernels of the associated degree 1 maps $f:M \longrightarrow S^{2i}$ are all given by the symmetric form over $\mathbb{Z}$

$$\sigma^i(f) = (\mathbb{Z},1)$$

with non-trivial reduced Wu class. Further, crossing with S^1 gives (i-1)-connected (2i+1)-dimensional degree 1 maps $f\times 1:M\times S^1\longrightarrow S^{2i}\times S^1$ such that the symmetric kernels are all given by the symmetric formation over $\mathbb{Z}[z,z^{-1}]$ $(\bar{z} = z^{-1})$

$$\sigma^i(f\times 1) = \sigma^i(f)\otimes\sigma^*(S^1) = (\mathbb{Z}[z,z^{-1}]\oplus\mathbb{Z}[z,z^{-1}]^*,\begin{pmatrix}0&1\\1&1\end{pmatrix};\mathbb{Z}[z,z^{-1}],$$

$$\mathrm{im}(\begin{pmatrix}1\\z-1\end{pmatrix}:\mathbb{Z}[z,z^{-1}]\longrightarrow\mathbb{Z}[z,z^{-1}]\oplus\mathbb{Z}[z,z^{-1}]^*))$$

with non-trivial reduced Wu class. (Another example is furnished by SU(3)/SO(3), which is a 1-connected 5-manifold with non-trivial Wu class = deRham invariant). By contrast with Proposition 18.8 the realization Theorems 5.8,6.5 of Wall [5] show that every $(-)^i$quadratic $\begin{Bmatrix}\text{form}\\\text{formation}\end{Bmatrix}$ over the group ring $\mathbb{Z}[\pi]$ of a finitely presented group π is the quadratic kernel $\sigma_i(f,b)$ of an (i-1)-connected normal map $(f,b):M\longrightarrow X$ of $\begin{Bmatrix}2i\\2i+1\end{Bmatrix}$-dimensional geometric Poincaré complexes, with $\pi=\pi_1(X)$.

The stable automorphisms $h:p \longrightarrow p$ over $1:X \longrightarrow X$ of a stable spherical filtration $p:X \longrightarrow BG$ over a finite-dimensional space X are classified by homotopy classes of maps $h:X \longrightarrow G = \varinjlim_m G(m)$, or equivalently by the relative homotopy classes of maps $h:(X\times D^1, X\times S^0) \longrightarrow BG$ such that $h|_{X\times S^0} = p\cup p : X\times S^0 \longrightarrow BG$.

Define the <u>rth suspended mod 2 Wu class</u> of h, $(\sigma v_r)(h) \in H^{r-1}(X;\mathbb{Z}_2)$, to be the image of the universal rth mod 2 Wu class $v_r \in H^r(BG;\mathbb{Z}_2)$ under the composite

$$H^r(BG;\mathbb{Z}_2) \xrightarrow{h^*} H^r(X\times D^1, X\times S^0;\mathbb{Z}_2) \xrightarrow{(\mathrm{proj.})^{*-1}} \dot{H}^r(\Sigma X_+;\mathbb{Z}_2) \xrightarrow{\Sigma^*} H^{r-1}(X;\mathbb{Z}_2).$$

In terms of the mod 2 Stiefel-Whitney classes this is just

$$(\sigma v_r)(h) = \sum_{i+j=r} \chi(Sq^i)(\Sigma^* w_j(-h)) \in H^{r-1}(X;\mathbb{Z}_2)$$

$$(w_j(-h) \in H^j(X\times D^1, X\times S^0;\mathbb{Z}_2) = \dot{H}^j(\Sigma X_+;\mathbb{Z}_2))$$

<u>Proposition 18.9</u> Let $h:p\to p$ be a stable automorphism over $1:X \longrightarrow X$ of a stable spherical fibration $p:X \longrightarrow BG$ over an n-dimensional space X. Let $\widetilde{X}$ be an oriented cover of X with data (π,w), and let R be a commutative ring.

Then there is defined in a natural way an (n+1)-dimensional hyperquadratic complex over $R[\pi]$ with the w-twisted involution, the <u>suspended Wu complex</u> of (p,h)

$$\hat{\sigma}^*(p,h) = ({}^wC(\widetilde{X};R)^{n-*}, \theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\widetilde{X};R)^{n-*}))$$

depending only on the homotopy class of $h:X \to G$, such that

i) if $\alpha: S^{n+k} \longrightarrow Y \wedge_\pi T\pi(p)$ is an $S\pi$-duality map for some finite-dimensional CWπ-complex Y, and $H \in \{Y,Y\}_\pi$ is the $S\pi$-dual of $T\pi(h) \in \{T\pi(p),T\pi(p)\}_\pi$ then the $\mathbb{Z}[\pi]$-module chain equivalence

$$j: {}^wC(\widetilde{X};R)^{n-*} \xrightarrow{U_p \cap -} \dot{C}(T\pi(p);R)^{n+k-*} \xrightarrow{(\alpha[S^{n+k}]\backslash -)} \dot{C}(Y;R)$$

sends $H\theta_{p,h} \in Q_n({}^wC(\widetilde{X};R)^{n-*})$ to $\dot{\psi}_H(\alpha[S^{n+k}]\backslash U_p) \in Q_n(\dot{C}(Y;R))$, (with H as in Proposition 1.2)

$$\dot{\psi}_H(\alpha[S^{n+k}]\backslash U_p) = j_\% H\theta_{p,h} \in Q_n(\dot{C}(Y;R))$$

ii) if $f:M \longrightarrow X$ is a map of n-dimensional spaces with induced cover $\widetilde{M}$ there is defined a map of the suspended Wu complexes

$$\tilde{f}^* : \hat{\sigma}^*(p,h) \longrightarrow \hat{\sigma}^*(f^*p, f^*h)$$

iii) $\theta_{p,gh} = \theta_{p,g} + \theta_{p,h}$, $\theta_{p,1} = 0$

iv) for $R = \mathbb{Z}_2$, the mod 2 reduction of the rth hyperquadratic Wu class of $\theta_{p,h}$ is the rth suspended mod 2 Wu class of h

$$\hat{v}_r(\theta_{p,h}) = (\sigma v_r)(h) \in \mathrm{Hom}_{\mathbb{Z}_2}(H_{r-1}(X;\mathbb{Z}_2),\mathbb{Z}_2) = H^{r-1}(X;\mathbb{Z}_2).$$

For $X = S^n$ $\hat{v}_{n+1}(\theta_{p,h})$ = (stable Hopf invariant of $h \in \pi_n(G) = \pi_n^S) \in H^n(S^n;\mathbb{Z}_2) = \mathbb{Z}_2$.

Proof: The relative version of the Wu complex construction of Proposition 18.3 applied to $h:(X\times D^1, X\times S^0) \longrightarrow BG$ gives a relative Tate $\mathbb{Z}_2$-hypercohomology class $\theta_{T\pi(h)}(U_h) \in \hat{Q}^{n+1}(i = (\mathrm{inclusion})^*: {}^wC(\widetilde{X}\times D^1;R)^{n-*} \longrightarrow {}^wC(\widetilde{X}\times S^0;R)^{n-*})$. The inclusion ${}^wC(\widetilde{X}\times S^0;R)^{n-*} \longrightarrow C(i) = {}^wC(\widetilde{X};R)^{n-*}$ sends $\theta_{T\pi(h)}(U_h)$ to the required element $\theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\widetilde{X};R)^{n-*})$.

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The hyperquadratic Wu classes of the suspended Wu complex $\hat{\sigma}^*(p,h) = ({}^wC(\widetilde{X};R)^{n-*}, \theta_{p,h} \in \hat{Q}^{n+1}({}^wC(\widetilde{X};R)^{n-*}))$ are the <u>suspended Wu classes</u> of an automorphism $h:p\to p$ of $p:X \longrightarrow BG$, $R[\pi]$-module morphisms

$$(\sigma v_r)(h) = \hat{v}_r(\theta_{p,h}) : H_{r-1}(\widetilde{X};R) \longrightarrow \hat{H}^r(\mathbb{Z}_2;R[\pi]) \quad (r \geqslant 1).$$

We have already related these classes with the suspended mod 2 Wu classes, in Proposition 18.9 iv) above. Note that the quadratic Wu classes of $H\theta_{p,h} \in Q_n({}^wC(\widetilde{X};R)^{n-*})$ are given by

$$v^r(H\theta_{p,h}) = H(\sigma v_{r+1})(h) : H_r(\widetilde{X};R) \longrightarrow H_{2r-n}(\mathbb{Z}_2;R[\pi],(-)^{n-r}) \quad (r \geqslant 0)$$

with $H:\hat{Q}^{n+1}({}^wC(\widetilde{X};R)^{n-*}) \longrightarrow Q_n({}^wC(\widetilde{X};R)^{n-*})$ as defined in Proposition 1.2, and $H:\hat{H}^{r+1}(\mathbb{Z}_2;R[\pi]) \longrightarrow H_{2r-n}(\mathbb{Z}_2;R[\pi],(-)^{n-r})$ the natural map.

<u>Proposition 18.10</u> Let $(f,b):(M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$ be a normal map of normalized n-dimensional geometric Poincaré complexes with quadratic kernel $\sigma_*(f,b) = (C(f^!), \psi = e_\% \psi_F[X] \in Q_n(C(f^!)))$. Given an automorphism $c:\nu_M \longrightarrow \nu_M$ of $\nu_M : M \longrightarrow BG(k)$ define normal maps

$$(f,b'):(M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho'_X) \ , \ (f,b''):(M,\nu_M,\rho'_M) \longrightarrow (X,\nu_X,\rho'_X)$$

by $\rho'_M = T(c)\rho_M \in \pi_{n+k}(T(\nu_M))$, $\rho'_X = T(b)\rho'_M \in \pi_{n+k}(T(\nu_X))$, $b' = bc$, $b'' = b : \nu_M \longrightarrow \nu_X$.

Then the quadratic kernels of (f,b'), (f,b'') are given by

$$\sigma_*(f,b') = \sigma_*(f,b'') = (C(f^!), \psi' = \psi + H\theta \in Q_n(C(f^!)))$$

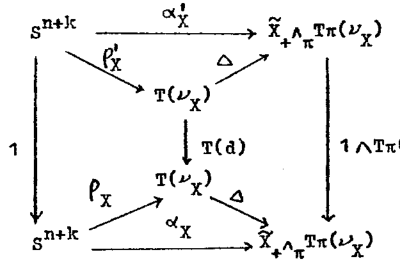
with $\theta = \hat{e}^\% \theta_{\nu_M,c} \in \hat{Q}^{n+1}(C(f^!))$, and the quadratic Wu classes are such that

$$v^r(\psi') - v^r(\psi) = H(\sigma v_{r+1})(c) : K_r(M) \longrightarrow H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi],(-)^{n-r}) \quad (r \geqslant 0).$$

Proof: Let $(1,d):(X,\nu_X,\rho_X^!)\longrightarrow(X,\nu_X,\rho_X)$ be the canonical equivalence of Spivak normal structures given by Proposition 3.12. The fundamental $S\pi$-duality maps

$$\alpha_X = \Delta\rho_X\ ,\ \alpha_X^! = \Delta\rho_X^!\ :\ S^{n+k}\longrightarrow \widetilde{X}_+\wedge_\pi T\pi(\nu_X)$$

are such that there is defined a homotopy commutative diagram



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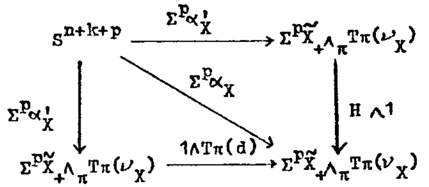
Let $H:\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{X}_+$ be a π-map which is $S\pi$-dual to $T\pi(d):T\pi(\nu_X)\longrightarrow T\pi(\nu_X)$ with respect to $\alpha_X^!$, so that there is defined a homotopy commutative diagram



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Working as in the proof of Theorem 3.5 of Wall [4] we can take H to be

$$H\ :\ \Sigma^p\widetilde{X}_+ = \widetilde{X}_+\wedge S^p\longrightarrow \widetilde{X}_+\wedge S^p\ ;\ \widetilde{x}\wedge s\longmapsto\widetilde{x}\wedge d(x)(s)$$

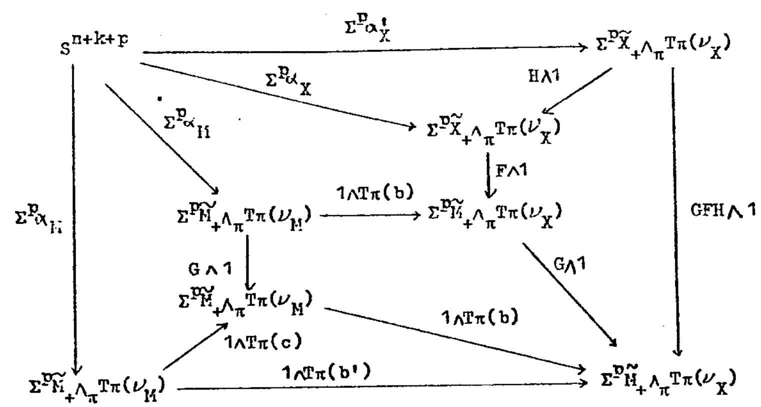
with $d:X\longrightarrow G(p)$ a classifying map for $d:\nu_X\longrightarrow\nu_X$, and similarly for a π-map $G:\Sigma^p\widetilde{M}_+\longrightarrow\Sigma^p\widetilde{M}_+$ $S\pi$-dual to $T\pi(c):T\pi(\nu_M)\longrightarrow T\pi(\nu_M)$ with respect to the fundamental $S\pi$-duality map

$$\alpha_M = \Delta\rho_M\ :\ S^{n+k}\longrightarrow\widetilde{M}_+\wedge_\pi T\pi(\nu_M).$$

By the definition of quadratic kernel we have that

$$\begin{cases}\sigma_*(f,b) = (C(f^!),e_\%\psi_F[X]\in Q_n(C(f^!)))\\ \sigma_*(f,b') = (C(f^!),e_\%\psi_{F'}[X]\in Q_n(C(f^!)))\ ,\end{cases}$$

with $\begin{cases}F:\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+\\ F':\Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+\end{cases}$ a π-map $S\pi$-dual to $\begin{cases}T\pi(b):T\pi(\nu_M)\longrightarrow T\pi(\nu_X)\\ T\pi(b'):T\pi(\nu_M)\longrightarrow T\pi(\nu_X)\end{cases}$ with respect to $\begin{cases}\alpha_M,\alpha_X\\ \alpha_M,\alpha_X^!\end{cases}$. Considering the homotopy commutative diagram



we can identify

$$F' = GFH\ :\ \Sigma^p\widetilde{X}_+\longrightarrow\Sigma^p\widetilde{M}_+\ .$$

Applying the sum formula for the quadratic construction of Proposition 2.5 iii) we obtain

$$\psi_{F'} = \psi_G f_*^! + \psi_F + f_\%^!\psi_H\ :\ H_n^\pi(\widetilde{X};{}^w\mathbb{Z})\longrightarrow Q_n(C(\widetilde{M}))\ ,$$

so that

$$e_\%\psi_{F'}[X] - e_\%\psi_F[X] = e_\%\psi_G[M]\in Q_n(C(f^!))\ .$$

Further, applying the construction of Proposition 18.9 i) to the fundamental $S\pi$-duality map $\alpha_M:S^{n+k}\longrightarrow\widetilde{M}_+\wedge_\pi T\pi(\nu_M)$ we can identify

$$\psi_G[M] = H\theta_{\nu_M,c}\in Q_n(C(\widetilde{M}))$$

with $\theta_{\nu_M,c}\in\hat{Q}^{n+1}({}^wC(\widetilde{M})^{n-*}) = \hat{Q}^{n+1}(C(\widetilde{M}))$, and so

$$e_\%\psi_{F'}[X] - e_\%\psi_F[X] = H(\hat{e}^\%\theta_{\nu_M,c})\in Q_n(C(f^!))\ .$$

Applying the quadratic kernel sum formula of Proposition 3.14 to the composition of normal maps

$$(f,b')\ :\ (M,\nu_M,\rho_M)\xrightarrow{(1,c)}(M,\nu_M,\rho_M^!)\xrightarrow{(f,b'')}(X,\nu_X,\rho_X^!)$$

we have that up to homotopy equivalence

$$\sigma_*(f,b') = \sigma_*(f,b'')\oplus\sigma_*(1,c) = \sigma_*(f,b'')$$

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The mod 2 reduction of the quadratic Wu class identity of Proposition 18.10 in the case $n=2r$ is the formula for the twisting of the Arf form due to Brown [1].

Proposition 18.11 i) Let $(f,b):M \longrightarrow X$, $(f,b'):M \longrightarrow X$ be normal bundle maps with $b' = bc : \nu_M \longrightarrow \nu_X$ for some automorphism $c:\nu_M \longrightarrow \nu_M$ classified by $c:M \longrightarrow SO$. The quadratic kernels

$$\sigma_*(f,b) = (C,\psi) = (C(f^!), e_\% \psi_F[X] \in Q_n(C(f^!)))$$

$$\sigma_*(f,b') = (C,\psi') = (C(f^!), e_\% \psi_{F'}[X] \in Q_n(C(f^!)))$$

are such that if $x \in K_r(M) = H_{r+1}(\widetilde{f})$ is the Hurewicz image of $(h,g) \in \pi_{r+1}(f) = \pi_{r+1}(\widetilde{f})$ with $g:S^r \looparrowright M$ an immersion and $h:D^{r+1} \longrightarrow X$ a null-homotopy of $fg:S^r \longrightarrow X$ then

$$v^r(\psi')(x) - v^r(\psi)(x) = (Hj(cg),0) \in Q_n(S^{n-r}\mathbb{Z}[\pi]) = Q_n(S^{n-r}\mathbb{Z}) \oplus H_{2r-n}(\mathbb{Z}_2;\mathbb{Z}[\pi]/\mathbb{Z},(-)^{n-r})$$

$$(\pi = \pi_1(X))$$

with $Hj:\pi_r(SO) \xrightarrow{j} \hat{Q}^{n+1}(S^{n-r}\mathbb{Z}) \xrightarrow{H} Q_n(S^{n-r}\mathbb{Z})$.

ii) The surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of an $(i-1)$-connected n-dimensional normal bundle map $(f,b):M \longrightarrow X$ for $n = 2i$ or $2i+1$ is independent of the bundle map $b:\nu_M \longrightarrow \nu_X$ for $i \neq 1,3,7$.

<u>Proof</u>: i) The universal cover $\widetilde{X}$ of X induces the trivial cover $\widetilde{S}^r = \pi \times S^r$ of S^r, so that applying Propositions 18.10, 18.9 iv) we have

$$v^r(\psi')(x) - v^r(\psi)(x) = (H\sigma v_{r+1}(c)(x),0) = (Hj(cg),0) \in Q_n(S^{n-r}\mathbb{Z}[\pi]) \text{ ,}$$

since j is the composite $j:\pi_r(SO) \xrightarrow{J} \pi_r(SG) = \pi_r^S \xrightarrow{\text{stable Hopf invariant}} \hat{H}^{r+1}(\mathbb{Z}_2;\mathbb{Z})$ by construction.

ii) Let $\psi,\psi' \in Q_n(C(f^!))$ be the $\mathbb{Z}_2$-hyperhomology classes appearing in the quadratic kernels $\sigma_*(f,b) = (C(f^!),\psi)$, $\sigma_*(f,b') = (C(f^!),\psi')$ of $(i-1)$-connected n-dimensional normal bundle maps $(f,b):M \longrightarrow X$, $(f,b'):M \longrightarrow X$ for $n = 2i$ or $2i+1$, with $b' = bc : \nu_M \longrightarrow \nu_X$ for some automorphism $c:\nu_M \longrightarrow \nu_M$ classified by $c:M \longrightarrow SO$. By the Hurewicz theorem every element $x \in K_i(M) = \pi_{i+1}(f)$ is represented by an immersion $g:S^i \looparrowright M$ together with a null-homotopy $h:D^{i+1} \longrightarrow X$ of $fg:S^i \longrightarrow X$, so that by i)

$$v^i(\psi')(x) - v^i(\psi)(x) = (Hj(cg),0) \in Q_n(S^{n-i}\mathbb{Z}[\pi]) \text{ .}$$

Now $j(cg) = (\text{stable Hopf invariant of } J(cg) \in \pi_i(SG) = \pi_i^S) = 0 \in \hat{H}^{i+1}(\mathbb{Z}_2;\mathbb{Z})$ for $i \neq 1,3,7$ by the result of Adams [1], so that the $(-)^i$quadratic $\begin{cases}\text{forms}\\ \text{formations}\end{cases}$ associated to $\sigma_*(f,b)$, $\sigma_*(f,b')$ are isomorphic by Proposition $\begin{cases}1.5\\ 1.8 \text{ iii)}\end{cases}$.

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For $X = S^{2i}$, $i \equiv 1 \pmod 2$ Proposition 18.11 ii) is the familiar result that the Arf invariant of an $(i-1)$-connected framed $2i$-manifold is independent of the framing for $i \neq 1,3,7$. For $i = 1,3,7$ there exist $(i-1)$-connected $2i$-dimensional normal bundle maps

$$(f,b) : S^i \times S^i \longrightarrow S^{2i}$$

with exotic framings b of $S^i \times S^i$ of Arf invariant $\sigma_*(f,b) = 1 \in L_{2i}(\mathbb{Z}) = \mathbb{Z}_2$, and crossing with S^1 gives $(i-1)$-connected $(2i+1)$-dimensional normal bundle maps

$$(f \times 1, b \times 1) : S^i \times S^i \times S^1 \longrightarrow S^{2i} \times S^1$$

with exotic framings $b \times 1$ of $S^i \times S^i \times S^1$ such that $\sigma_*(f \times 1, b \times 1) = 1 \in L_{2i+1}(\mathbb{Z}[\mathbb{Z}]) = \mathbb{Z}_2$.

The chain homotopy theoretic treatment of the self-intersections of an immersed sphere in §4 will now be generalized to any immersed manifold. Given a map of geometric Poincaré complexes $f:M^m \longrightarrow X^n$ with a Spivak normal fibration $\nu_f:M \longrightarrow BSG(n-m)$ (e.g. an immersion of manifolds) we shall use equivariant S-duality to obtain an $S\pi$-map $F\in\{\tilde{X}_+,T\pi(\nu_f)\}_\pi$ $(\pi = \pi_1(X))$ such that the $\mathbb{Z}_2$-hyperhomology class $\psi(f) = \hat{\psi}_F[X]\in Q_n(S^{n-m}C(\tilde{M}))$ is an obstruction to making (f,ν_f) a Poincaré embedding by a regular homotopy (= homotopy preserving the normal fibrations). In order to do this we shall need the following equivariant generalization of the S-duality of Theorem 3.3 of Atiyah [1].

Proposition 18.12 Let $p:X \longrightarrow BG(j)$, $q:X \longrightarrow BG(k)$ be spherical fibrations over an n-dimensional geometric Poincaré complex X such that the Whitney sum $p\oplus q$ is a Spivak normal fibration for X

$$p\oplus q = \nu_X : X \longrightarrow BG(j+k) ,$$

with a spherical generator $\rho_X\in\pi_{n+j+k}(T(\nu_X))$. If $\tilde{X}$ is a covering of X with group of covering translations π then the composite

$$\alpha_X : S^{n+j+k} \xrightarrow{\rho_X} T(p\oplus q) \xrightarrow{\Delta} T\pi(p)\wedge_\pi T\pi(q)$$

is an $S\pi$-duality map between the Thom π-spaces $T\pi(p),T\pi(q)$.

Proof: Let E be the total space of $p:X\longrightarrow BG(j)$, and let $r:E \xrightarrow{proj.} X \xrightarrow{q} BG(k)$. Then (X,E) is an $(n+j)$-dimensional geometric Poincaré pair, a "Poincaré thickening of X", with a Spivak normal fibration

$$\nu_{(X,E)} = (q,r) : (X,E) \longrightarrow BG(k)$$

and a fundamental $S\pi$-duality map

$$\alpha_X : S^{n+j+k} \xrightarrow{\rho_X} T(p\oplus q) = T(q)/T(r) \xrightarrow{\Delta} (\tilde{X}/\tilde{E})\wedge_\pi T\pi(q) = T\pi(p)\wedge_\pi T\pi(q) .$$

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In particular, for $p = 0 : X \longrightarrow BG(0) = \{pt.\}$ $(E = \emptyset)$, $q = \nu_X : X \longrightarrow BG(k)$ Proposition 18.12 is dealing with the $S\pi$-duality map $\alpha_X:S^{n+k} \longrightarrow \tilde{X}_+\wedge_\pi T\pi(\nu_X)$ of Proposition 3.12.

Let $p:X\longrightarrow BSG(k)$ be an oriented (k-1)-spherical fibration over an n-dimensional geometric Poincaré complex X, and let $\tilde{X}$ be an oriented covering of X with data (π,w). Define the algebraic Thom complex of p to be the (n+k)-dimensional symmetric complex over $\mathbb{Z}[\pi]$ with the w-twisted involution

$$\sigma^*(X,p) = (S^kC(\tilde{X}),\varphi_p[X]\in Q^{n+k}(S^kC(\tilde{X})))$$

with $\varphi_p[X]$ the image of the fundamental class $[X]\in H_n^\pi(\tilde{X};{}^w\mathbb{Z})$ under the composite

$$\varphi_p : H_n^\pi(\tilde{X};{}^w\mathbb{Z}) \xrightarrow{(U_p\cap-)_*^{-1}} \dot{H}^\pi_{n+k}(T\pi(p);{}^w\mathbb{Z}) \xrightarrow{\dot{\varphi}_{T\pi(p)}} Q^{n+k}(\dot{C}(T\pi(p)))$$
$$\xrightarrow{(U_p\cap-)^\%} Q^{n+k}(S^kC(\tilde{X})) ,$$

where $U_p\cap- : \dot{C}(T\pi(p))\longrightarrow S^kC(\tilde{X})$ is the Thom equivalence. Note that $\sigma^*(X,p)$ is obtained by collapsing the boundary $\sigma^*(E)$ in the (n+k)-dimensional symmetric Poincaré pair $\sigma^*(X,E) = (\tilde{p}:C(\tilde{E})\longrightarrow C(\tilde{X}),\varphi_{\tilde{X},\tilde{E}}[X,E]\in Q^{n+k}(\tilde{p}))$ of the associated (n+k)-dimensional geometric Poincaré pair (X,E) with E the total space of p, corresponding to the definition of the Thom π-space $T\pi(p) = \tilde{X}/\tilde{E}$.

Proposition 18.13 The algebraic Thom complex construction enjoys the following properties:

i) $\sigma^*(X,0) = \sigma^*(X)$ $(k=0,\ E = \emptyset)$,

ii) $\sigma^*(X,p\oplus\varepsilon) = S\sigma^*(X,p)$ $(\varepsilon = 0 : X \longrightarrow BSG(1))$

iii) $\varphi_p[X]_0$ is the composite $\mathbb{Z}[\pi]$-module morphism

$$\varphi_p[X]_0 : H^r(S^kC(\tilde{X})) = {}^wH_\pi^{r-k}(\tilde{X}) \xrightarrow{e(p)\cup-} {}^wH_\pi^r(\tilde{X}) \xrightarrow{[X]\cap-} H_{n-r}(\tilde{X}) = H_{n+k-r}(S^kC(\tilde{X}))$$

with $e(p)\in H^k(X) = H_\pi^k(\tilde{X};\mathbb{Z})$ the Euler class of p,

iv) the Poincaré duality chain equivalence

$$[X]\cap- : {}^wC(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})$$

defines a homotopy equivalence of n-dimensional hyperquadratic complexes

$$[X]\cap- : \hat{\sigma}^*(\nu_X - p) = ({}^wC(\tilde{X})^{n-*},\theta_{T\pi(\nu_X-p)}(U_{\nu_X-p})\in\hat{Q}^n({}^wC(\tilde{X})^{n-*}))$$
$$\longrightarrow J\sigma^*(X,p) = (C(\tilde{X}),J\varphi_p[X]\in\hat{Q}^n(C(\tilde{X})))$$

with $\nu_X:X\longrightarrow BG$ the stable Spivak normal fibration of X,

v) the mod 2 reductions of the Wu classes $v_r(\varphi_p[X]) = v_r(\nu_X - p)\in H^r(X;\mathbb{Z}_2)$ are the characteristic classes for the $\mathbb{Z}_2$-module morphism

$$H^{n-r}(X;\mathbb{Z}_2)\longrightarrow H^n(X;\mathbb{Z}_2) = \mathbb{Z}_2 ;\ y\longmapsto \sum_{i+j=r} w_i(p)\cup Sq^j(y) = \langle y\cup v_r(\varphi_p[X]),[X]\rangle .$$

Proof: i) is trivial.

ii) follows from $T\pi(p\oplus\varepsilon) = \Sigma T\pi(p)$ and Proposition 2.4.

iii) The Euler class is by definition $e(p) = i^*(U_p) \in \dot{H}^k_\pi(\tilde{X}_+;\mathbb{Z}) = H^k(X)$, with $i:\tilde{X}_+ \longrightarrow T\pi(p)$ the canonical π-map and $U_p \in \dot{H}^k_\pi(T\pi(p);\mathbb{Z}) = \dot{H}^k(T(p))$ the Thom class, and there is defined a commutative diagram of $\mathbb{Z}[\pi]$-module morphisms

$$\begin{array}{ccc} W_{H_\pi}^{r}(\tilde{X}) & \xrightarrow{\varphi_p[X]_0} & H_{n-k-r}(\tilde{X}) \\ {\scriptstyle U_p\cup -}\downarrow & \searrow{\scriptstyle e(p)\cup -} & \uparrow{\scriptstyle [X]\cap -} \\ W_{\dot{H}_\pi}^{r+k}(T\pi(p)) & \xrightarrow{i^*} & W_{H_\pi}^{r+k}(\tilde{X}) \end{array} .$$

iv) Choose a representative $q:X \longrightarrow BG(j)$ for $\nu_X - p:X \longrightarrow BG$, so that Proposition 18.12 gives an $S\pi$-duality map $\alpha_X:S^{n+j+k} \longrightarrow T\pi(p)\wedge_\pi T\pi(q)$ such that

$$\varphi_p[X] = \theta_{T\pi(q),\alpha_X}(U_q) \in Q^{n+k}(S^kC(\tilde{X})),$$

exactly as in the proof of Proposition 18.6 which is the special case $p = 0, q = \nu_X$.

v) By Proposition 2.3 we have that the mod 2 reduction $v_r(\varphi_p[X]) \in H^r(X;\mathbb{Z}_2)$ is such as to make commutative the following diagram:

$$\begin{array}{ccc} H^{n-r}(X;\mathbb{Z}_2) & \xrightarrow{v_r(\varphi_p[X])\cup -} & H^n(X;\mathbb{Z}_2) = \mathbb{Z}_2 \\ {\scriptstyle U_p\cup -}\downarrow & & \downarrow{\scriptstyle U_p\cup - \,=\, \mathrm{id.}} \\ \dot{H}^{n-r+k}(T(p);\mathbb{Z}_2) & \xrightarrow{Sq^r} & \dot{H}^{n+k}(T(p);\mathbb{Z}_2) = \mathbb{Z}_2 \end{array} .$$

Applying the Cartan formula we have

$$v_r(\varphi_p[X])(y) = Sq^r(U_p \cup y) = \sum_{i+j=r} Sq^i(U_p)\cup Sq^j(y) = \sum_{i+j=r} w_i(p)\cup Sq^j(y) \in \mathbb{Z}_2 \quad (y \in H^{n-r}(X;\mathbb{Z}_2)).$$

[]

(Furthermore, given $p:X^n \longrightarrow BSG(k)$ as above and a fibre homotopy trivialization $F:p\oplus\varepsilon^j \simeq \varepsilon^{j+k}:X \longrightarrow BSG(j+k)$ there is defined a π-homotopy equivalence $F:\Sigma^j T\pi(p) = T\pi(p\oplus\varepsilon^j) \longrightarrow \Sigma^j(\Sigma^k\tilde{X}_+) = T\pi(\varepsilon^{j+k})$ inducing the Thom equivalence $F = U_p \cap - : \dot{C}(T\pi(p)) \longrightarrow S^kC(\tilde{X})$ on the chain level. The quadratic construction gives a $\mathbb{Z}_2$-hyperhomology class $\psi_F[X] \in Q^{[0,j-1]}_{n+k}(S^kC(\tilde{X}))$ (and hence in $Q_{n+k}(S^kC(\tilde{X}))$) such that

$$S^k\varphi_{\tilde{X}}[X] - \varphi_p[X] = (1+T)\psi_F[X] \in Q^{n+k}(S^kC(\tilde{X})).$$

This generalizes to an arbitrary geometric Poincaré complex X the construction of j in Proposition 4.1, where $X = S^n$).

As an unstable complement to Proposition 18.5 we have:

Proposition 18.14 The symmetric Wu classes of the algebraic Thom complex

$$\sigma^*(S^m,p) = (S^kC(S^m),\varphi_p[S^m] \in Q^{m+k}(S^kC(S^m)))$$

of an oriented $(k-1)$-spherical fibration $p:S^m \longrightarrow BSG(k)$ over S^m are given by

$$v_0(\varphi_p[S^m]) : H_0(S^m) = \mathbb{Z} \longrightarrow \hat{H}^{k+m}(\mathbb{Z}_2;\mathbb{Z},(-)^{m+k}) = \mathbb{Z}_2 ; z \longrightarrow z^2 \equiv z (\mathrm{mod}\ 2)$$

$$v_m(\varphi_p[S^m]) : H_m(S^m) = \mathbb{Z} \longrightarrow \hat{H}^{k-m}(\mathbb{Z}_2;\mathbb{Z},(-)^k) = \begin{cases} \mathbb{Z} & \text{if } k = m,\ m \equiv 0 (\mathrm{mod}\ 2) \\ \mathbb{Z}_2 & \text{if } k > m,\ m \equiv 0 (\mathrm{mod}\ 2) ; \\ 0 & \text{otherwise} \end{cases}$$

$$z \longmapsto z^2 (\text{Hopf invariant of } p \in \pi_m(BSG(k)) = \pi_{k+m-1}(S^k)) .$$

Proof: This follows from Propositions 18.5, 18.13 identifying $T(p) = S^k \cup_p e^{k+m}$ as before. (Note that for $k = m = 2r$

$v_m(\varphi_p[S^m]) = $ (Hopf invariant of $p \in \pi_{4r-1}(S^{2r})$) = (Euler number $e(p) \in H^{2r}(S^{2r})) \in \mathbb{Z}$).

[]

A normal fibration (ν_f,b) of fibre dimension k for a map of geometric Poincaré complexes $f:M \longrightarrow X$ is an oriented $(k-1)$-spherical fibration over M

$$\nu_f : M \longrightarrow BSG(k)$$

together with a stable fibre homotopy equivalence over $1:M \longrightarrow M$

$$b : \nu_M \longrightarrow \nu_f \oplus f^*\nu_X$$

for some given normalizations (ν_M,ρ_M), (ν_X,ρ_X). Every map of geometric Poincaré complexes admits normal fibrations for k sufficiently large, and ν_f is unique up to stable fibre homotopy equivalence, by Proposition 3.12. In particular, a normal map of normalized n-dimensional geometric Poincaré complexes $(f,b):(M,\nu_M,\rho_M) \longrightarrow (X,\nu_X,\rho_X)$ is the same as a degree 1 map $f:M \longrightarrow X$ together with a normal fibration (ν_f,b) of fibre dimension 0 such that $T(b)(\rho_M) = \rho_X \in \pi_{n+k}(T(\nu_X))$.

A Poincaré embedding is a map of geometric Poincaré complexes $f:M \longrightarrow X$ such that $m = \dim M \leq n = \dim X$, together with a normal fibration (ν_f,b) of fibre dimension $n-m$ and with a homotopy equivalence

$$X \longrightarrow X_0 \cup_{g\times 1} E\times I \cup_{\nu_f\times 0} M$$

with E the total space of $\nu_f:M \longrightarrow BSG(n-m)$ and $g:E \longrightarrow X_0$ some map.

<u>Proposition 18.15</u> Let $f:M \longrightarrow X$ be a map of geometric Poincaré complexes with a normal fibration (ν_f, b) of fibre dimension $k \geqslant n-m$, $m = \dim M$, $n = \dim X$. Given an oriented cover $\widetilde{X}$ of X with data (π, w) and induced cover $\widetilde{M}$ of M there is defined a <u>self-intersection class</u> $\psi(f) \in Q_{m+k}(S^k C(\widetilde{M}))$ such that

i) $\varphi_{\nu_f}[M] - S^{k+m-n} f^{!\%} \varphi_{\widetilde{X}}[X] = (1+T)\psi(f) \in Q^{m+k}(S^k C(\widetilde{M}))$,

ii) $\psi(f) = 0$ if $k = n-m$ and (ν_f, b) is the normal fibration of a Poincaré embedding.

<u>Proof</u>: Propositions 3.12, 18.12 give $S\pi$-duality maps

$$\alpha_X : S^N \xrightarrow{\rho_X} T(\nu_X) \xrightarrow{\Delta} \widetilde{X}_+ \wedge_\pi T\pi(\nu_X)$$

$$\alpha_f : S^N \xrightarrow{\rho_M} T(\nu_M) \xrightarrow{T(b)} T(\nu_f \oplus f^*\nu_X) \xrightarrow{\Delta} T\pi(\nu_f) \wedge_\pi T\pi(f^*\nu_X) .$$

Let $F \in \{\Sigma^{k+m-n}\widetilde{X}_+, T\pi(\nu_f)\}_\pi$ be the $S\pi$-dual of the π-map $T\pi(f^*\nu_X) \longrightarrow T\pi(\nu_X)$ induced by the fibre map $f^*\nu_X \longrightarrow \nu_X$ over $f:M \longrightarrow X$. Evaluate the quadratic construction

$$\dot{\psi}_F : \dot{H}^\pi_{m+k}(\Sigma^{k+m-n}\widetilde{X}_+ ; {}^w\mathbb{Z}) \longrightarrow Q_{m+k}(\dot{C}(T\pi(\nu_f))) = Q_{m+k}(S^k C(\widetilde{M}))$$

on the $(k+m-n)$-fold suspension of the fundamental class $[X] \in H^\pi_n(\widetilde{X}; {}^w\mathbb{Z})$, and define

$$\psi(f) = \dot{\psi}_F[X] \in Q_{m+k}(S^k C(\widetilde{M})) .$$

Now i) follows from the relation

$$\dot{\varphi}_{T\pi(\nu_f)} F_* - F^{\%} \dot{\varphi}_{\Sigma^{k+m-n}X_+} = (1+T)\dot{\psi}_F : \dot{H}^\pi_{m+k}(\Sigma^{k+m-n}\widetilde{X}_+ ; {}^w\mathbb{Z}) \longrightarrow Q^{m+k}(\dot{C}(T\pi(\nu_f)))$$

given by Proposition 2.5 i), since F induces the Umkehr $f^! : C(\widetilde{X}) \longrightarrow S^{n-m}C(\widetilde{M})$ on chain level. If f is a Poincaré embedding it is possible to represent F by the π-map collapsing the complement $\widetilde{X}_0$

$$F : \widetilde{X}_+ \longrightarrow (\widetilde{X}_0 \cup_{g \times 1} \widetilde{E} \times I \cup_{\nu_f \times 0} \widetilde{M})_+ \xrightarrow{\text{collapse}} (\widetilde{E} \times I \cup_{\nu_f \times 0} \widetilde{M}) / \widetilde{E} \times 1 = T\pi(\nu_f) ,$$

so that $\dot{\psi}_F = 0$ by construction, and $\psi(f) = 0$.

[]

An oriented immersion of manifolds $f:M^m \looparrowright X^n$ (smooth and closed, for simplicity) has a normal bundle $\nu_f : M \longrightarrow BSO(n-m)$ such that $\tau_M \oplus \nu_f = f^*\tau_X : M \longrightarrow BO(n)$. We thus have a normal fibration (ν_f, b) of fibre dimension $(n-m)$, taking for $(\nu_M, \rho_M), (\nu_X, \rho_X)$ the canonical normalizations associated to embeddings of M and X in high-dimensional spheres, so that the given framings of $\tau_M \oplus \nu_M$, $\tau_X \oplus \nu_X$ define a stable bundle isomorphism $b : \nu_M \longrightarrow \nu_f \oplus f^*\nu_X$. If f is an embedding of manifolds then it is a Poincaré embedding as above.

<u>Proposition 18.16</u> The self-intersection class $\psi(f) \in Q_n(S^{n-m}C(\widetilde{M}))$ of an immersion of manifolds $f:M^m \looparrowright X^n$ is such that

i) $\varphi_{\nu_f}[M] - f^{!\%}\varphi_{\widetilde{X}}[X] = (1+T)\psi(f) \in Q^n(S^{n-m}C(\widetilde{M}))$

ii) If $f \times 1 : M^m \looparrowright X^n \times D^p$ can be deformed by a regular homotopy to an embedding $f' : M^m \hookrightarrow \text{interior}(X^n \times D^p)$ then $S^p\psi(f) = 0 \in Q_{n+p}(S^{n-m+p}C(\widetilde{M}))$ and

$$v^r(\psi(f)) = 0 : H_r(\widetilde{M}) \longrightarrow H_{2r-n}(\mathbb{Z}_2; \mathbb{Z}[\pi], (-)^{n-r}) \text{ for } r \geqslant n+p.$$

In particular, if f is already an embedding then $\psi(f) = 0 \in Q_n(S^{n-m}C(\widetilde{M}))$ $(p = 0)$.

iii) If $g : X^n \looparrowright Y^p$ is another immersion of manifolds then $\nu_{gf} = \nu_f \oplus f^*\nu_g$ and

$$\psi(gf) = S^{p-n}\psi(f) + f^!_{\%}\psi(g) \in Q_p(S^{p-m}C(\widetilde{M})) \quad (m \leq n \leq p).$$

<u>Proof</u>: i) is immediate from Proposition 18.15 i).

ii) Let E be a closed tubular neighbourhood of $f'(M)$ in $X \times D^p$, so that the Pontrjagin-Thom construction gives a π-map

$$F : \Sigma^p\widetilde{X}_+ = \widetilde{X} \times D^p / \widetilde{X} \times S^{p-1} \xrightarrow{\text{collapse}} \overline{\widetilde{X} \times D^p / \widetilde{X} \times D^p - \widetilde{E}} = \widetilde{E}/\partial\widetilde{E} = \Sigma^p T\pi(\nu_f)$$

representing $F \in \{\widetilde{X}_+, T\pi(\nu_f)\}_\pi$, so that

$$\psi(f) = \dot{\psi}_F[X] \in \text{im}(Q_n^{[0,p-1]}(S^{n-m}C(\widetilde{M})) \longrightarrow Q_n(S^{n-m}C(\widetilde{M})))$$

$$= \ker(S^p : Q_n(S^{n-m}C(\widetilde{M})) \longrightarrow Q_{n+p}(S^{n-m+p}C(\widetilde{M}))) ,$$

using part of the commutative exact braid of Proposition 1.3.
(Note that $Q_{n+p}(S^{n-m+p}C(\widetilde{M})) = 0$ for dimensional reasons if $n+p \geqslant 2m+1$, corresponding to the general position argument that $f \times 1 : M^m \looparrowright X^n \times D^p$ can be approximated by an embedding in this range).

iii) is an immediate consequence of the sum formula of Proposition 2.5 iii).

[]

The self-intersection of an immersed sphere $g : S^r \looparrowright M^n$ defined in §4 is $\mu(g) = -v^r(\psi(g))(1) \in H_{2r-n}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(M)], (-)^{n-r})$. The relation

$$\lambda(g,g) = (j(\nu_g), 0) + (1+T)\mu(g) \in H^{n-2r}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(M)], (-)^{n-r})$$

of Proposition 4.2 i) can also be obtained by applying the rth symmetric Wu class operation v_r to the relation given by Proposition 18.16 i)

$$g^{!\%}\varphi_M[M] = \varphi_{\nu_g}[S^r] - (1+T)\psi(g) \in Q^n(S^{n-r}C(\widetilde{S}^r)) ,$$

and using Proposition 18.14 to identify

$$v_r(\varphi_{\nu_g}[S^r])(1) = (j(\nu_g), 0) = (\text{Hopf invariant of } J(\nu_g) \in \pi_r(BSG(n-r)) = \pi_{n-1}(S^{n-r}); 0)$$

$$\in H^{n-2r}(\mathbb{Z}_2; \mathbb{Z}[\pi_1(M)], (-)^{n-r}) .$$

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Princeton University

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