MOD 2 INDICES

3 ATIYAH - REES INVARIANT (1976)

1 ATIYAH (1971)

2 ROKHLIN (1972)

RIEMANN

R-R $\chi(X,L) = \dim H^0(X,L) - \dim H^1(X,L)$

$= c_1(L) - (g-1)$ (TOP. INV) "PRIMARY

DUALITY $H^1(X,L)^* \cong H^0(X, L^* \otimes K)$

$\chi(X,L) = 0$

IF $L^* \otimes K \cong 1$ $(L^2 \cong K)$

$[c_1(K) = 2g-2]$

BUT $h^0(X,L) = \dim H^0(X,L)$ MOD 2

(1) IS TOP. INVARIANT "SECONDARY"

$L^2 \cong K \longleftrightarrow$ SPIN-STRUCTURE ON X

$h^0(X,L)$ MOD 2 (ONLY) SPIN-COBORDISM INVARIANT

② ROKHLIN

a) M spin 4-MANIFOLD

$$\frac{\operatorname{sign}(M)}{8} \equiv 0 \quad \text{MOD } 2$$

b) $W \subset M$ dim 2, ORIENTED, DUAL TO ω_2

$$\frac{\operatorname{Sign}(M) - W^2}{8} \equiv \operatorname{Arf}(W, M) \quad \text{MOD } 2$$

NOTE $\operatorname{Arf}(W, M) = h^0(W, L)$ L INDUCED SPIN-STRUCTURE

"ADJUNCTION FORMULA"

COMPLEX CASE M, W COMPLEX $[W] = K_M \otimes \mathcal{D}^{-2}$

$(\mathcal{D}_W^2 = K_W$: Spin structure)

$W = -c_1 - 2d$

$\boxed{\text{SIGNATURE FORMULA}}$

$$\operatorname{sign}(M) - W^2 = \frac{c_1^2 - 2c_2}{3} - (c_1 + 2d)^2$$

$$= -\frac{2}{3}(c_1^2 + c_2) - 4c_1 d - 4d^2$$

$\boxed{\text{HRR}}$

$$-8\chi(M, \mathcal{D}) = -8\, e^d \tau(M)[M]$$

$$= -8\left(1 + d + \frac{d^2}{2}\right)\left(1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12}\right)$$

$$= -\frac{2}{3}(c_1^2 + c_2) - 4c_1 d - 4d^2$$

$$\boxed{\frac{\operatorname{Sign} M - W^2}{8} = -\chi(M, \mathcal{D})}$$

SHEAF THEORY

$$0 \to K_M \otimes \mathcal{D}^{-1} \xrightarrow{s^*} \mathcal{D} \to \mathcal{D}|_W \to 0$$

[s^* multiplication by section s of $\mathcal{L} = K_M \otimes \mathcal{D}^{-2}$ defining W]

COHOMOLOGY SEQUENCE + SERRE DUALITY

$$H^q(\;, L)^* \cong H^{n-q}(\;, L^* \otimes K)$$

$$\Rightarrow \quad \chi(M, \mathcal{D}) \equiv h^0(W, \mathcal{D}_W) \quad \text{mod } 2$$

ROKHLIN FOR COMPLEX M

ROKHLIN FOR ALL M . USE

$\Rightarrow$ (i) BOTH SIDES $Spin^c$ - COBORDISM INV

(ii) $U(n) \to Spin^c(n) \Rightarrow$ ISOMORPHISM IN COBORDISM UP TO dim 4

INDEX THEORY

M Spin manifold dim 4 . Dirac operator D

$$\text{index } D = -\frac{p_1}{24} = -\frac{\text{sign}(M)}{8}$$

IS EVEN BECAUSE SPINORS in dim 4

ARE Quaternionic

EXPLAINS ORIGINAL ROKHLIN THEOREM (R)

INDEX THEORY ~ REAL K-THEORY

~ HOMOTOPY OF O

BOTT PERIODICITY FOR O

$$Z \quad Z_2 \quad Z_2 \quad 0 \quad Z \quad 0 \quad 0 \quad 0 \quad Z$$

$\left(\frac{O}{SO}\right)$ Spin

FOR symplectic group SP SHIFT BY 4

$$Z \quad 0 \quad 0 \quad 0 \quad Z \quad Z_2 \quad Z_2 \quad 0 \quad Z$$

$\Rightarrow$ "INDEX" OF DIRAC OPERATOR

INTEGER OR INTEGER MOD 2

$\dim M = 8k+1$ $\quad \dim_{\mathbb{R}} H \mod 2$

$= 8k+2$ $\quad \dim_{\mathbb{C}} H \mod 2$

(H = HARMONIC SPINORS)

$\underline{k=0}$ $\dim M = 1$ M CIRCLE

2 spin structures

H = EVEN / ODD FUNCTIONS ON S^1

$\underline{k=1}$ $\dim M = 2$ RIEMANN SURFACE

spin structure $L^2 \cong K$ $h^0(M,L) \bmod 2$

is "index"

M COMPLEX E HOLOMORPHIC VECTOR BUNDLE

$\chi(M,E) = \sum_q (-1)^q \dim H^q(M,E)$ is $\underline{\text{INDEX}}$

(of $\bar\partial$-COMPLEX of E, or of D_E^c DIRAC^c

operator of E)

$\underline{\text{SERRE DUALITY}}$

$$H^q(M,E)^* \cong H^{n-q}(M, E^* \otimes K)$$

$(n = \dim_{\mathbb{C}} M)$

IF $E^* \otimes K \cong E$ + $\underline{n \text{ odd}}$

$\Rightarrow \chi(M,E) = 0$

BUT SEMI-CHARACTERISTIC

$$R(M,E) = \sum_q \dim H^{2q}(M,E) \ \underline{\bmod 2}$$

IS INVARIANT (UNDER DEFORMATION)

<u>if</u> $\exists$ NON-DEGENERATE PAIRING

$$E \otimes E \to K$$

WHICH IS

<u>SYMMETRIC</u> IF $\dim_{\mathbb{C}} M = 1 \mod 4$

<u>SKEW-SYMMETRIC</u> IF $\dim_{\mathbb{C}} M = 3 \mod 4$

<u>EXAMPLES</u>

1. $\dim_{\mathbb{C}} M = 1$ SPIN-STRUCTURE $E = L$
$L^2 \cong K$

2. $M = \mathbb{C}P_3$ E rank 2 $c_1(E)$ EVEN

$$R(M, E) = \dim H^0 + \dim H^2 \mod 2$$

3. <u>ATIYAH-REES INVARIANT</u>

RELATED TO $\pi_5 U(2) = \pi_5(Sp(1)) = \mathbb{Z}_2$

USED TO PROVE ANY TOP. rank 2 VECTOR BUNDLE ON $\mathbb{C}P_3$ HAS HOLOMORPHIC STRUCTURE.

HIGHER DIMENSIONS

SHEAF THEORY PROOF OF ROKHLIN FOR COMPLEX M EXTENDS TO HIGHER DIMS

$\dim_{\mathbb{C}} M = 4k+2$

$W \subset M \quad \dim_{\mathbb{C}} W = 4k+1$

$[W] \cong K_M^{-1} \otimes \mathcal{D}^2$

$\left(\mathcal{D}_W^2 \cong K_W \right)$

$$\chi(M, \mathcal{D}) \equiv h(W, \mathcal{D}) \mod 2$$

FOR GENERAL M?

1) REFORMULATE IN KO-THEORY

2) PROVE BY KO-THEORY

3) **PROVE BY ANALYSIS - INDEX THEORY**

A) CUT OUT TUBULAR NBD N OF W IN M
APPLY INDEX DIRAC WITH APS BDY. COND
INVOLVES η-INVARIANT

B) SHRINK CIRCLE FIBRE OF N
(ADIABATIC LIMIT : WITTEN,
BISMUT)

PUT TOGETHER BY GILKEY

EXAMPLE $\quad M = \mathbb{C}P_2$

1. $W = \mathbb{C}P_1 \qquad \mathcal{D} = \mathcal{O}(1)$

$\operatorname{sign} M = 1 \quad , \quad W^2 = 1 \qquad h^0(W, \mathcal{D}) = 0$

2. $W =$ cubic curve $\quad g = 1$

$\mathcal{D} = \mathcal{O}$

$\operatorname{sign} M = 1 \qquad W^2 = 9 \qquad h^0(W, \mathcal{D}) = 1$

3. NON-ORIENTABLE

$W = \mathbb{R}P_2 \qquad\qquad W^2 = -1$

$$\frac{\operatorname{Sign} M - W^2}{8} = \frac{1}{4} \quad \text{NOT INTEGER!}$$

ROKHLIN FOR NON-ORIENTABLE W

{REPRESENTING $\omega_2 \in H^2(M, Z_2)$}

1. $Arf(W, M)$ CAN BE DEFINED BY HOMOLOGY
   $\in$ [INTEGERS MOD 8]

2. W INHERITS PIN^- - STRUCTURE
   $(\omega_1^2 + \omega_2 = 0)$

3. PIN^- COBORDISM GROUP dim 4
   $\cong Z_8$

4. INDEX THEORY USES η - INVARIANT
   (FOR EVEN-dim. "SECONDARY" TOP. INV)

5. NOTE $\widetilde{KO}(RP^2) \cong Z_4$

6. WEIPING - ZHANG PROVES
   HIGHER-DIM VERSION OF
   NON-ORIENTABLE ROKHLIN
   INDEX $\in \widetilde{KO}(RP^{8k+2}) = Z_{2^{4k+2}}$

References for the 2012 Bristol lecture
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