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Additive L-Theory

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Abstract. The cobordism groups of quadratic Poincaré complexes in an additive category with involution $\mathbb{A}$ are identified with the Wall L-groups of quadratic forms and formations in $\mathbb{A}$, generalizing earlier work for modules over a ring with involution by avoiding kernels and cokernels.

Key words. Additive category, L-theory, surgery obstruction.

0. Introduction

The quadratic L-groups $L_*(\mathbb{A})$ of an additive category with involution $\mathbb{A}$ can be defined both as the Witt groups of forms and formations in $\mathbb{A}$ and as the cobordism groups of quadratic Poincaré complexes in $\mathbb{A}$. The main result of this paper is that these two definitions agree, allowing quadratic L-theory to be defined unambiguously in situations where it is inopportune to deal with kernels and cokernels.

Wall [17] defined the n-dimensional quadratic L-group $L_n(A)$ of a ring with involution A for $n(\text{mod } 4)$, with $L_{2i}(A)$ the Witt group of nonsingular $(-)^i$-quadratic forms on based f.g. free A-modules, and $L_{2i+1}(A)$ the stable automorphism group of such forms, requiring isomorphisms to be simple in the sense of Whitehead torsion. In Ranicki [8], the automorphisms were replaced by formations, which are forms with an ordered pair of Lagrangians ('subkernels' in [17]). In Section 2, the method of [8] is applied to define the n-dimensional quadratic L-group $L_n(\mathbb{A})$ for any additive category with involution $\mathbb{A}$ and $n(\text{mod } 4)$, with $L_{2i}(\mathbb{A})$ the Witt group of nonsingular $(-)^i$-quadratic forms in $\mathbb{A}$, and $L_{2i+1}(\mathbb{A})$ the Witt group of $(-)^i$ quadratic formations in $\mathbb{A}$. A ring with involution A determines additive categories with involution

$$\mathbb{F}(A) = \{\text{f.g. free } A\text{-modules}\},$$

$$\mathbb{P}(A) = \{\text{f.g. projective } A\text{-modules}\}$$

such that

$$L_*(\mathbb{F}(A)) = L_*^h(A), \qquad L_*(\mathbb{P}(A)) = L_*^p(A),$$

with $L_*^h(A)$ the free L-groups and $L_*^p(A)$ the projective L-groups. The original simple L-groups $L_*^s(A) = L_*(A)$ are expressed in Section 7 as the quadratic L-groups of $\mathbb{B}(A) = \{\text{based f.g. free } A\text{-modules}\}$ with the appropriate torsion considerations.

The quadratic L-groups $L_*^h(A)$ (resp. $L_*^p(A)$) were expressed in Ranicki [11] as the cobordism groups of quadratic Poincaré complexes, that is f.g. free (resp. projective) A-module chain complexes with quadratic Poincaré duality. In Section 3 the method of [11] is applied to define the n-dimensional quadratic L-group $L_n(\mathbb{A})$ for any additive category with involution $\mathbb{A}$ and any $n \geqslant 0$ as the cobordism group of n-dimensional quadratic Poincaré complexes in $\mathbb{A}$. The method of [11] also applies to define the n-dimensional symmetric L-groups $L^*(\mathbb{A})$, but these groups are not 4-periodic in general, and can be expressed as Witt groups only for $n = 0, 1$. The proof in [11] that the quadratic Poincaré cobordism groups are isomorphic to the Witt groups for $\mathbb{F}(A)$ and $\mathbb{P}(A)$ used kernel and cokernel modules, albeit only for split surjections and split surjections. An additive category need not have kernels and cokernels, although it can be embedded in an Abelian category. For example, the kernels and cokernels for $\mathbb{F}(A)$ and $\mathbb{P}(A)$ are defined using the embeddings

$$\mathbb{F}(A) \to \mathbb{P}(A) \to \{A\text{-modules}\}.$$

In principle, it is possible to verify the agreement of the two definitions of $L_*(\mathbb{A})$ for any $\mathbb{A}$ by using an embedding of $\mathbb{A}$ in an Abelian category, exercising due care with the involution. We adopt a more intrinsic approach here, staying inside $\mathbb{A}$ throughout and avoiding the use of kernels and cokernels by stabilization.

An n-dimensional quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ is an n-dimensional chain complex in $\mathbb{A}$

$$C: \cdots \to 0 \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \to$$
$$\cdots \to C_1 \xrightarrow{d} C_0,$$

together with a quadratic structure ψ defining a Poincaré duality chain equivalence

$$(1 + T)\psi_0 : C^{n-*} \to C.$$

(C, ψ) is 'highly connected' if

$$C_j = 0 \quad \text{if} \begin{cases} j \neq i, & \text{for } n = 2i, \\ j \neq i, i+1, & \text{for } n = 2i+1. \end{cases}$$

A $(-)^i$ quadratic form (resp. formation) in $\mathbb{A}$ is essentially the same as a highly connected n-dimensional quadratic Poincaré complex (C, ψ) for $n = 2i$ (resp. $n = 2i + 1$). The Witt group of forms (resp. formations) in $\mathbb{A}$ is essentially the same as the cobordism group of the highly connected quadratic Poincaré complexes in $\mathbb{A}$, as recalled in Section 4. The identification in Ranicki [11] of the cobordism group $L_n(\mathbb{A})$ with the Witt group for $\mathbb{A} = \mathbb{F}(A)$ and $\mathbb{P}(A)$ was by an algebraic mimicry of geometric surgery below the middle dimension. To each n-dimensional quadratic Poincaré complex (C, ψ) there was associated an 'instant surgery obstruction' (C', ψ'), a cobordant highly connected complex such that

$$C'_i = \ker\left(\begin{bmatrix} d & (-)^{n+1}(1+T)\psi_0 \\ 0 & (-)^i d^* \end{bmatrix} : C_i \oplus C^{n-i+1} \to C_{i-1} \oplus C^{n-i+2}\right)$$

$$\text{for } n = 2i \text{ or } 2i+1,$$

$$C'_{i+1} = C_{i+1} \quad \text{for } n = 2i+1.$$

The cobordism class $(C, \psi) \in L_n(\mathbb{A})$ is the Witt class of the form (resp. formation) associated to (C', ψ').

The formula defining the instant surgery obstruction (C', ψ') uses the notion of kernel, and so does not apply in an arbitrary additive category with involution $\mathbb{A}$. In Section 5, we associate to each n-dimensional quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ an 'additive instant surgery obstruction' $(C^\oplus, \psi^\oplus)$, a cobordant highly connected quadratic Poincaré complex such that the cobordism class of (C, ψ) is the Witt class of the form (resp. formation) associated to $(C^\oplus, \psi^\oplus)$, with

$$(C^\oplus)_i = \begin{cases} \sum\limits_{j=-\infty}^{\infty} (C_{i+2j} \oplus C^{i+2j+1}) & \text{for } n = 2i, \\ \sum\limits_{j=0}^{\infty} (C_{i-2j} \oplus C^{i+2j+2}) & \text{for } n = 2i+1, \end{cases}$$

$$(C^\oplus)_{i+1} = \sum_{j=0}^{\infty} (C_{i-2j+1} \oplus C^{i+2j+3}) \quad \text{for } n = 2i+1.$$

The quadratic structure $\psi^\oplus$ and the differential

$$d^\oplus : (C^\oplus)_{i+1} \to (C^\oplus)_i \quad (\text{for } n = 2i+1)$$

depend on a choice of contraction

$$\Gamma : 1 \simeq 0 : C((1+T)\psi_0) \to C((1+T)\psi_0)$$

for the algebraic mapping cone $C((1+T)\psi_0 : C^{n-*} \to C)$. In Section 8 the nonsingular $(-)^i$-quadratic form defined by $((C^\oplus)_i, (\psi^\oplus)_0)$ in the case $n = 2i$ is expressed as a 'stable radical quotient' of the singular $(-)^i$-quadratic form

$$\left(C^i \oplus C_{i+1}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right).$$

The additive instant surgery obstruction $(C^\oplus, \psi^\oplus)$ can be regarded as the L-theoretic analogue of the instant torsion of a contractible finite chain complex C in an additive category $\mathbb{A}$, the isomorphism defined for any chain contraction $\Gamma : 1 \simeq 0 : C \to C$ by

$$d + \Gamma : C_{\text{odd}} = \sum_{j=0}^{\infty} C_{2j+1} \to C_{\text{even}} = \sum_{j=0}^{\infty} C_{2j}.$$

For the additive category $\mathbb{B}(A) = \{\text{based f.g. free } A\text{-modules}\}$ this is the traditional method of defining the Whitehead torsion $\tau(C) = \tau(d + \Gamma) \in \tilde{K}_1(A)$. The additive instant surgery obstruction is also an analogue of the instant finiteness obstruction of Ranicki [13]. See Section 6 (resp. Section 7) for the combination of additive L-theory with the additive K_0- (resp. K_1-) theory of Ranicki [13] (resp. [14]).

The quadratic L-theory of additive categories with involution arises in contexts other than the standard examples $\mathbb{F}(A)$ and $\mathbb{P}(A)$ for a ring with involution A. Here are two of these:

(i) A fibration $F \to E \xrightarrow{p} B$ with the fibre F an m-dimensional geometric Poincaré complex induces geometric transfer maps

$$p^!: L_n(\mathbb{Z}[\pi_1(B)]) \to L_{m+n}(\mathbb{Z}[\pi_1(E)]),$$

with $L_* \equiv L_*^h$ for finite F. In Lück and Ranicki [4], $p^!$ is described algebraically as a composite

$$p^!: L_n(\mathbb{Z}[\pi_1(B)]) \to L_n(\mathbb{D}_m(\mathbb{Z}[\pi_1(E)])) \to L_{m+n}(\mathbb{Z}[\pi_1(E)]),$$

with $\mathbb{D}_m(A)$ defined for any ring with involution A to be the additive category of m-dimensional chain complexes C in $\mathbb{F}(A)$ and chain homotopy classes of chain maps, with the m-duality involution $C \to C^{m-*}$.

(ii) Pedersen and Weibel [6, 7] constructed for any idempotent complete filtered additive category $\mathbb{A}$ a nonconnective delooping $\mathbb{K}(\mathbb{A})$ of algebraic K-theory, with $\pi_*(\mathbb{K}(\mathbb{A})) = K_*(\mathbb{A})$, and expressed the generalized homology groups $H_*(X; \mathbb{K}(\mathbb{A}))$ of a compact polyhedron X as the K-groups $K_{*+1}(\mathbb{P}(\mathbb{C}_{O(X)}(\mathbb{A})))$ of the idempotent completion of the additive category $\mathbb{C}_{O(X)}(\mathbb{A})$ of $O(X)$-graded objects in $\mathbb{A}$, with $O(X)$ the open cone of X. Ferry and Pedersen [2] use the additive L-theory of this paper to obtain an analogous nonconnective delooping $\mathbb{L}^{\langle -\infty \rangle}(\mathbb{A})$ of the lower quadratic L-theory of an idempotent complete filtered additive category with involution $\mathbb{A}$ (Ranicki [9, 15]), and an analogous expression for the generalized homology groups $H_*(X; \mathbb{L}^{\langle -\infty \rangle}(\mathbb{A}))$ as the L-groups $L_{*+1}(\mathbb{P}(\mathbb{C}_{O(X)}(\mathbb{A})))$ of the idempotent completion of $\mathbb{C}_{O(X)}(\mathbb{A})$, at least if $\mathbb{A} = \mathbb{F}(A)$ for a ring with involution A and up to 2-torsion. See Remarks 6.16 and 7.24 for a discussion of such expressions for the lower K- and L-groups in the special case $X = S^i$.

The additive instant surgery obstruction $(C^{\oplus}, \psi^{\oplus})$ has an application to the surgery obstruction of a normal map.

The kernel $\mathbb{Z}[\pi_1(X)]$-modules of an n-dimensional normal map $(f, b): M^n \to X$ are defined by

$$K_j(M) = H_{j+1}(\tilde{f}: \tilde{M} \to \tilde{X})$$

with $\tilde{X}$ the universal cover of X, $\tilde{M} = f^*\tilde{X}$ the pullback cover of M, and $\tilde{f}: \tilde{M} \to \tilde{X}$ a $\pi_1(X)$-equivariant lift of f. The theory of Ranicki [12] associates to (f, b) an

n-dimensional quadratic Poincaré complex in $\mathbb{F}(\mathbb{Z}[\pi_1(X)])$

$$\sigma_*(f, b) = (C, \psi)$$

such that $H_*(C) = K_*(M)$, with $C = C(f^!)$ the algebraic mapping cone of the Umkehr $\mathbb{Z}[\pi_1(X)]$-module chain map

$$f^!: C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} C(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{n-*} \xrightarrow{([M]\cap -)} C(\tilde{M}).$$

The Wall surgery obstruction of (f, b) is the cobordism class

$$\sigma_*(f, b) = (C, \psi) \in L_n(\mathbb{Z}[\pi_1(X)]),$$

vanishing if (and for $n \geqslant 5$ only if) (f, b) is normal bordant to a homotopy equivalence. Here, $L_* = L_*^h$.

By definition, a normal map $(f, b): M^n \to X$ is k-connected if $K_j(M) = 0$ for $j \leqslant k$. Classical surgery theory below the middle dimension shows that for $n \geqslant 5$ every n-dimensional normal map $(f, b): M^n \to X$ is normal bordant to a highly connected normal map $(f', b'): M'^n \to X$, meaning $(i-1)$-connected if $n = 2i$ or $2i + 1$. The standard procedure is to construct a sequence of n-dimensional normal maps

$$(f^{(j)}, b^{(j)}): M^{(j)} \to X \quad (0 \leqslant j \leqslant i)$$

such that

(i) $(f^{(0)}, b^{(0)}) = (f, b): M^{(0)} = M \to X$,
(ii) $(f^{(j)}, b^{(j)})$ is $(j-1)$-connected,
(iii) for $1 \leqslant j \leqslant i$ $(f^{(j)}, b^{(j)})$ is obtained from $(f^{(j-1)}, b^{(j-1)})$ by a sequence of elementary surgeries on a finite set of generators of the $\mathbb{Z}[\pi_1(X)]$-module $K_j(M^{(j)}) = \pi_{j+1}(f^{(j)})$, so that $(f^{(j)}, b^{(j)})$ is normal bordant to $(f^{(j-1)}, b^{(j-1)})$.

The last normal map in the sequence $(f^{(j)}, b^{(j)})$ $(0 \leqslant j \leqslant i)$

$$(f', b') = (f^{(i)}, b^{(i)}): M' = M^{(i)} \to X$$

is highly connected. The procedure involves choices, so (f', b') is only determined up to normal bordism (which can itself be made highly connected by surgery below the middle dimension). The instant surgery obstruction (C', ψ') of the quadratic Poincaré complex $\sigma_*(f, b) = (C, \psi)$ determines one particular set of choices, depending on choices of bases in the chain modules of $C = C(f^!)$. The quadratic Poincaré complex of the corresponding highly connected normal map (f', b') is such that

$$\sigma_*(f', b') = (C', \psi'),$$

realizing the instant surgery obstruction. The additive instant surgery obstruction $(C^{\oplus}, \psi^{\oplus})$ of (C, ψ) is realized by another normal bordant highly connected normal map $(f^{\oplus}, b^{\oplus}): M^{\oplus} \to X$, such that

$$\sigma_*(f^{\oplus}, b^{\oplus}) = (C^{\oplus}, \psi^{\oplus}).$$

Both (C', ψ') and $(C^{\oplus}, \psi^{\oplus})$ represent the surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$, but $(C^{\oplus}, \psi^{\oplus})$ is obtained from (C, ψ) without using kernels and cokernels.

1. Additive Categories with Involution

It is assumed that the reader is already familiar with the definition of an additive category.

DEFINITION 1.1. A sequence of objects and morphisms in an additive category $\mathbb{A}$

$$0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0$$

is *split exact* if there exists a morphism $k: N \to M$ such that

(i) $jk = 1: N \to N$,
(ii) $(i \;\; k): L \oplus N \to M$ is an isomorphism. □

DEFINITION 1.2 An *involution* on an additive category $\mathbb{A}$ is a contravariant functor

$$*: \mathbb{A} \to \mathbb{A}; M \to M^*, \qquad (f: M \to N) \to (f^*: N^* \to M^*)$$

together with a natural equivalence

$$e: \mathrm{id}_{\mathbb{A}} \to **: \mathbb{A} \to \mathbb{A}; M \to (e(M): M \to M^{**})$$

such that for any object M of $\mathbb{A}$

$$e(M^*) = (e(M)^{-1})^*: M^* \to M^{***}.$$ □

EXAMPLE 1.3. Let A be an associative ring with 1 and with an involution, that is a function

$$\bar{\ }: A \to A; a \to \bar{a}$$

such that

$$\overline{(a + b)} = \bar{a} + \bar{b}, \qquad \overline{(ab)} = \bar{b} \cdot \bar{a}, \quad \bar{\bar{a}} = a, \bar{1} = 1 \in A \quad (a, b \in A).$$

Define the *duality* involution on the additive category $\mathbb{P}(A)$ of f.g. projective (left) A-modules

$$*: \mathbb{P}(A) \to \mathbb{P}(A); P \to P^*$$

by

$$P^* = \mathrm{Hom}_A(P, A), A \times P^* \to P^*; (a, f) \to (x \to f(x) \cdot \bar{a})$$
$$f^*: Q^* \to P^*; (g: Q \to A) \to (gf: P \to A) \quad (f \in \mathrm{Hom}_A(P, Q))$$
$$e(P): P \to P^{**}; x \to (f \to \overline{f(x)}).$$

The dual of a f.g. free A-module is f.g. free, so that the involution on $\mathbb{P}(A)$ restricts to an involution on the full subcategory $\mathbb{F}(A)$ of f.g. free A-modules. □

Given an additive category with involution $\mathbb{A}$, we shall use the natural isomorphisms $e(M): M \to M^{**}$ to identify $M^{**} = M$.

DEFINITION 1.4. The *duality* isomorphism

$$T_{M,N}: \mathrm{Hom}_{\mathbb{A}}(M, N^*) \to \mathrm{Hom}_{\mathbb{A}}(N, M^*); \psi \to \psi^*$$

is such that

$$T_{N,M} T_{M,N} = 1: \mathrm{Hom}_{\mathbb{A}}(M, N^*) \to \mathrm{Hom}_{\mathbb{A}}(N, M^*) \to \mathrm{Hom}_{\mathbb{A}}(M, N^*). \qquad \square$$

In particular for $M = N$ we have a duality involution

$$T = T_{M,M}: \mathrm{Hom}_{\mathbb{A}}(M, M^*) \to \mathrm{Hom}_{\mathbb{A}}(M, M^*); \psi \to \psi^*.$$

DEFINITION 1.5. For $\varepsilon = \pm 1$ and M in $\mathbb{A}$ let the generator $T \in \mathbb{Z}_2$ act on $\mathrm{Hom}_A(M, M^*)$ by the *ε-duality involution*

$$T_\varepsilon = \varepsilon T: \mathrm{Hom}_{\mathbb{A}}(M, M^*) \to \mathrm{Hom}_{\mathbb{A}}(M, M^*); \psi \to \varepsilon\psi^*. \qquad \square$$

2. Forms and Formations

The theory of ε-quadratic forms and formations over a ring with involution A developed in Ranicki [8] works just as well in any additive category with involution $\mathbb{A}$, the original case being for $\mathbb{A} = \mathbb{P}(A)$. We only state here the most essential definitions and properties of the extension of the theory to any $\mathbb{A}$.

DEFINITION 2.1. An *ε-quadratic form in $\mathbb{A}$* (M, ψ) is an object M in $\mathbb{A}$ together with an element in the $\mathbb{Z}_2$-homology group

$$\psi \in Q_\varepsilon(M) = \mathrm{coker}(1 - T_\varepsilon: \mathrm{Hom}_{\mathbb{A}}(M, M^*) \to \mathrm{Hom}_{\mathbb{A}}(M, M^*)).$$

The form (M, ψ) is *nonsingular* if the morphism

$$(1 + T_\varepsilon)\psi = \psi + \varepsilon\psi^*: M \to M^*$$

is an isomorphism in $\mathbb{A}$. $\square$

DEFINITION 2.2. A *morphism* of ε-quadratic forms in $\mathbb{A}$

$$f: (M, \psi) \to (M', \psi')$$

is a morphism $f: M \to M'$ in $\mathbb{A}$ such that

$$f^*\psi' f = \psi \in Q_\varepsilon(M). \qquad \square$$

DEFINITION 2.3. A *Lagrangian* L in a nonsingular ε-quadratic form (M, ψ) is a morphism of forms

$$i: (L, 0) \to (M, \psi)$$

such that there is defined a split exact sequence in $\mathbb{A}$

$$0 \to L \xrightarrow{i} M \xrightarrow{i^*(\psi + \varepsilon\psi^*)} L^* \to 0. \qquad \square$$

DEFINITION 2.4. The *hyperbolic ε-quadratic form $H_\varepsilon(L)$* for an object L in $\mathbb{A}$ is the nonsingular ε-quadratic form in $\mathbb{A}$

$$H_\varepsilon(L) = \left(L \oplus L^*, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

with a Lagrangian defined by the morphism of forms

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : (L, 0) \to H_\varepsilon(L). \qquad \square$$

DEFINITION 2.5. Let (M, ψ), (M', ψ') be nonsingular ε-quadratic forms with Lagrangians L, L', respectively. An isomorphism $f: (M, \psi) \to (M', \psi')$ *sends* L to L' if there exists an isomorphism $e \in \mathrm{Hom}_{\mathbb{A}}(L, L')$ such that

$$i'e = fi : L \to M',$$

in which case there is defined an isomorphism of split exact sequences

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{i} & M & \xrightarrow{i^*(\psi + \varepsilon\psi^*)} & L^* & \to & 0 \\ & & {\scriptstyle e}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e^{*-1}} & & \\ 0 & \to & L' & \xrightarrow{i'} & M' & \xrightarrow{i'^*(\psi' + \varepsilon\psi'^*)} & L'^* & \to & 0. \end{array} \qquad \square$$

PROPOSITION 2.6. *An ε-quadratic form (M, ψ) admits a Lagrangian L if and only if it is isomorphic to $H_\varepsilon(L)$.*

Proof. An isomorphism of forms $f: H_\varepsilon(L) \to (M, \psi)$ determines a Lagrangian L of (M, ψ) with

$$i : L \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} L \oplus L^* \xrightarrow{f} M.$$

Conversely, suppose that (M, ψ) has a Lagrangian L, and let $i: L \to M$ be the inclusion. Choose a splitting morphism $j \in \mathrm{Hom}_{\mathbb{A}}(L^*, M)$ for the split exact sequence

$$0 \to L \xrightarrow{i} M \xrightarrow{i^*(\psi + \varepsilon\psi^*)} L^* \to 0,$$

so that

$$i^*(\psi + \varepsilon\psi^*)j = 1 \in \mathrm{Hom}_A(L^*, L^*).$$

For any $k \in \mathrm{Hom}_A(L^*, L)$ there is defined another splitting

$$j' = j + ik : L^* \to M$$

such that

$$\begin{aligned} j'^*\psi j' &= j^*\psi j + k^*i^*\psi ik + k^*i^*\psi j + j^*\psi ik \\ &= j^*\psi j + k \in Q_\varepsilon(L^*). \end{aligned}$$

Thus, there exists a splitting $j: L^* \to M$ which is the inclusion of a Lagrangian, with

$j^*\psi j = 0 \in Q_\varepsilon(L^*)$, in which case

$$(i \quad j): H_\varepsilon(L) \to (M, \psi)$$

is an isomorphism of ε-quadratic forms. □

DEFINITION 2.7. The *Witt group of ε-quadratic forms* $L_\varepsilon(\mathbb{A})$ is the Abelian group with one generator (M, ψ) for each isomorphism class of nonsingular ε-quadratic forms in $\mathbb{A}$ and relations

(i) $(M, \psi) + (M', \psi') = (M \oplus M', \psi \oplus \psi')$
(ii) $H_\varepsilon(L) = 0$. □

PROPOSITION 2.8. *A nonsingular ε-quadratic form in $\mathbb{A}$ (M, ψ) represents* 0 *in the Witt group $L_\varepsilon(\mathbb{A})$ if and only if there exists an isomorphism of forms*

$$(M, \psi) \oplus H_\varepsilon(G) \to H_\varepsilon(F)$$

for some objects F, G in $\mathbb{A}$.

Proof. Immediate from the definition. □

In particular, for any nonsingular ε-quadratic form (M, ψ) the diagonal morphism defines a Lagrangian

$$\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : (M, 0) \to (M \oplus M, \psi \oplus -\psi),$$

so that

$$-(M, \psi) = (M, -\psi) \in L_\varepsilon(\mathbb{A}).$$

DEFINITION 2.9. A *nonsingular ε-quadratic formation in* $\mathbb{A}$ $(M, \psi; F, G)$ is a nonsingular ε-quadratic form (M, ψ) together with an ordered pair of Lagrangians (F, G). □

DEFINITION 2.10. (i) An *isomorphism* of formations in $\mathbb{A}$

$$f: (M, \psi, F, G) \to (M', \psi'; F', G')$$

is an isomorphism of forms $f: (M, \psi) \to (M', \psi')$ sending F to F', and G to G'.

(ii) A *stable isomorphism* of formations in $\mathbb{A}$

$$[f]: (M, \psi; F, G) \to (M', \psi'; F', G')$$

is an isomorphism of formations

$$f: (M, \psi; F, G) \oplus (H_\varepsilon(P); P, P^*) \to (M', \psi'; F', G') \oplus (H_\varepsilon(P'); P', P'^*)$$

for some objects P, P' in $\mathbb{A}$. □

DEFINITION 2.11. The *Witt group of ε-quadratic formations* $M_\varepsilon(\mathbb{A})$ is the ›elian group with one generator $(M, \psi; F, G)$ for each stable isomorphism class of nonsingular

ε-quadratic formations in $\mathbb{A}$ and relations

(i) $(M, \psi; F, G) + (M', \psi'; F', G') = (M \oplus M', \psi \oplus \psi'; F \oplus F', G \oplus G')$,

(ii) $(M, \psi; F, G) + (M, \psi; G, H) = (M, \psi; F, H)$. □

The inverses in $M_\varepsilon(\mathbb{A})$ are given by

$$-(M, \psi; F, G) = (M, \psi; G, F) = (M, -\psi; F, G) \in M_\varepsilon(\mathbb{A}).$$

REMARK 2.12. As in Ranicki [8] define the *boundary* of a $(-\varepsilon)$-quadratic form (P, θ) in $\mathbb{A}$ to be the nonsingular ε-quadratic formation in $\mathbb{A}$

$$\partial(P, \theta) = (H_\varepsilon(P); P, \Gamma_{(P,\theta)})$$

with

$$\Gamma_{(P,\theta)} = \begin{bmatrix} 1 \\ \theta - \varepsilon\theta^* \end{bmatrix} : P \to P \oplus P^*.$$

A nonsingular ε-quadratic formation $(M, \psi; F, G)$ in $\mathbb{A}$ represents 0 in $M_\varepsilon(\mathbb{A})$ if and only if it is stably isomorphic to the boundary $\partial(P, \theta)$ of a $(-\varepsilon)$-quadratic form (P, θ) in $\mathbb{A}$. This is a translation into the language of forms and formations of Proposition 5.2 of Ranicki [11]. □

By the proof of Proposition 2.6, every nonsingular formation is isomorphic to one of the type $(H_\varepsilon(F); F, G)$, as determined by a split exact sequence in $\mathbb{A}$ of the type

$$0 \to G \xrightarrow{\begin{bmatrix} \gamma \\ \mu \end{bmatrix}} F \oplus F^* \xrightarrow{(\varepsilon\mu^* \ \gamma^*)} G^* \to 0$$

with $\gamma \in \mathrm{Hom}_{\mathbb{A}}(G, F)$, $\mu \in \mathrm{Hom}_{\mathbb{A}}(G, F^*)$ such that

$$\gamma^*\mu = \theta - \varepsilon\theta^* : G \to G^*$$

for some $(-\varepsilon)$-quadratic form (G, θ). We write such a formation as $(F, \begin{bmatrix} \gamma \\ \mu \end{bmatrix} G)$.

REMARK 2.13. The condition of 2.12 can be made more precise, making further use of the material in Section 5 of Ranicki [11]. A nonsingular ε-quadratic formation $(F, \begin{bmatrix} \gamma \\ \mu \end{bmatrix} G)$ in $\mathbb{A}$ represents 0 in the Witt group $M_\varepsilon(\mathbb{A})$ if and only if there exist a $(-\varepsilon)$-quadratic form in $\mathbb{A}$ (H, χ) and a morphism $j: F \to H^*$ such that the morphism defined in $\mathbb{A}$ by

$$\begin{bmatrix} \mu^* & \gamma^* j^* \\ j & \chi - \varepsilon\chi^* \end{bmatrix} : F \oplus H \to G^* \oplus H^*$$

is an isomorphism. If there exist such (H, χ) and j then (F, G) is stably isomorphic to the boundary $\partial(F \oplus G \oplus H, \psi)$ of any $(-\varepsilon)$-quadratic form $(F \oplus G \oplus H, \psi)$ such that

$$\psi - \varepsilon\psi^* = \begin{bmatrix} 0 & -\varepsilon\mu & -\varepsilon j^* \\ \mu^* & \gamma^*\mu & \gamma^* j^* \\ j & -\varepsilon j\gamma & \chi - \varepsilon\chi^* \end{bmatrix} : F \oplus G \oplus H \to F^* \oplus G^* \oplus H^*. \qquad \square$$

DEFINITION 2.14. A *nonsingular split ε-quadratic formation* $(F, (\left[\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}\right], \theta)\, G)$ in $\mathbb{A}$ is a nonsingular ε-quadratic formation $(F, \left[\begin{smallmatrix}\gamma\\ \mu\end{smallmatrix}\right] G)$ with a particular choice of form $(G, \theta \in Q_{-\varepsilon}(G))$ such that

$$\gamma^*\mu = \theta - \varepsilon\theta^*: G \to G^*. \qquad \square$$

The theory of split formations developed in Section 2 of Ranicki [11] for $\mathbb{P}(A)$ works just as well in any additive category with involution $\mathbb{A}$. Again, the Witt group of nonsingular split ε-quadratic formations is isomorphic to the Witt group $M_\varepsilon(\mathbb{A})$ of 2.11, so that the choice of 'Hessian' form (G, θ) does not affect the Witt class $(F, G) \in M_\varepsilon(\mathbb{A})$.

3. Quadratic Poincaré Complexes

The theory of ε-quadratic Poincaré complexes over a ring with involution A developed in Ranicki [11] works just as well in any additive category with involution $\mathbb{A}$, the original case being for $\mathbb{A} = \mathbb{P}(A)$. Again, we only state here the most essential definitions and properties of the extension of the theory to any $\mathbb{A}$.

DEFINITION 3.1. The *dual* of a chain complex in $\mathbb{A}$

$$C: \cdots \to C_{r+1} \xrightarrow{d_C} C_r \xrightarrow{d_C} C_{r-1} \to \cdots$$

is the chain complex in $\mathbb{A}$

$$C^*: \cdots \to C^{-r-1} \xrightarrow{d_{C^*}} C^{-r} \xrightarrow{d_{C^*}} C^{-r+1} \to \cdots$$

defined by $d_{C^*} = (d_C)^*$:

$$(C^*)_r = C^{-r} = (C_{-r})^* \to (C^*)_{r-1} = C^{-r+1} = (C_{-r+1})^*. \qquad \square$$

DEFINITION 3.2. The *n-dual* of a chain complex C in $\mathbb{A}$ is the chain complex C^{n-*} in $\mathbb{A}$ defined for any $n \in \mathbb{Z}$ by

$$d_{C^{n-*}} = (-)^r(d_C)^*: (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1}. \qquad \square$$

DEFINITION 3.3. The *algebraic mapping cone* of a chain map $f: C \to D$ in $\mathbb{A}$ is the chain complex $C(f)$ in $\mathbb{A}$ defined by

$$d_{C(f)} = \begin{bmatrix} d_D & (-)^r f \\ 0 & d_C \end{bmatrix}:$$

$$C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2}. \qquad \square$$

REMARKS 3.4. A chain map $f: C \to D$ of finite chain complexes in $\mathbb{A}$ is a chain equivalence if and only if the algebraic mapping cone $C(f)$ is chain contractible. See Proposition 1.1 of Ranicki [13] for an explicit proof. $\square$

DEFINITION 3.5. Given chain complexes C, D in $\mathbb{A}$, let $\mathrm{Hom}_{\mathbb{A}}(C, D)$ be the Abelian group chain complex defined by

$$d_{\mathrm{Hom}_{\mathbb{A}}}(C, D)\colon \mathrm{Hom}_{\mathbb{A}}(C, D)_r = \sum_{q-p=r} \mathrm{Hom}_{\mathbb{A}}(C_p, D_q) \to \mathrm{Hom}_{\mathbb{A}}(C, D)_{r-1};$$
$$f \to d_D f + (-)^q f d_C. \qquad \square$$

There is a natural one-to-one correspondence between chain maps $f\colon C \to D$ and 0-cycles $f' \in \mathrm{Hom}_{\mathbb{A}}(C, D)_0$, with

$$f' = (-)^n f\colon C_n \to D_n \quad (n \in \mathbb{Z}).$$

Similarly for chain homotopies and 1-chains. Thus, $H_0(\mathrm{Hom}_{\mathbb{A}}(C, D))$ is isomorphic to the additive group of chain homotopy classes of chain maps $C \to D$.

Given a chain complex C in $\mathbb{A}$ and $\varepsilon = \pm 1$ use the signed ε-duality isomorphisms

$$T_\varepsilon\colon \mathrm{Hom}_{\mathbb{A}}(C^p, C_q) \to \mathrm{Hom}_{\mathbb{A}}(C^q, C_p); \psi \to (-)^{pq}\varepsilon\psi^*$$

to define an involution

$$T_\varepsilon\colon \mathrm{Hom}_{\mathbb{A}}(C^*, C) \to \mathrm{Hom}_{\mathbb{A}}(C^*, C),$$

so that $\mathrm{Hom}_{\mathbb{A}}(C^*, C)$ is a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex. Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module of $\mathbb{Z}$

$$W\colon \cdots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2].$$

DEFINITION 3.6. The *ε-quadratic Q-groups* of a chain complex C in $\mathbb{A}$ are the $\mathbb{Z}_2$-hyperhomology groups of $\mathrm{Hom}_{\mathbb{A}}(C^*, C)$, the Abelian groups

$$Q_n(C, \varepsilon) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_{\mathbb{A}}(C^*, C)) \quad (n \in \mathbb{Z}). \qquad \square$$

An element $\psi \in Q_n(C, \varepsilon)$ is represented by a collection of morphisms

$$\psi = \{\psi_s \in \mathrm{Hom}_{\mathbb{A}}(C^{n-r-s}, C_r) \mid r \in \mathbb{Z}, s \geqslant 0\}$$

such that

$$d_C\psi_s + (-)^r\psi_s(d_C)^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T_\varepsilon\psi_{s+1}) = 0\colon$$
$$C^{n-r-s-1} \to C_r.$$

DEFINITION 3.7. An *n-dimensional ε-quadratic Poincaré complex in* $\mathbb{A}$ (C, ψ) is an n-dimensional chain complex in $\mathbb{A}$

$$C\colon \cdots \to 0 \to C_n \to \cdots \to C_{r+1} \xrightarrow{d_C} C_r$$
$$\xrightarrow{d_C} C_{r-1} \to \cdots \to C_0 \to 0 \to \cdots$$

together with an element $\psi \in Q_n(C, \varepsilon)$ such that the chain map

$$(1 + T_\varepsilon)\psi_0\colon C^{n-*} \to C$$

is a chain equivalence. $\square$

See Section 3 of Ranicki [11] for the details of the definition of an *$(n + 1)$-dimensional*

ε-quadratic pair $(f: C \to D, (\delta\psi, \psi))$, a *cobordism* $((f\ \ f'): C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ of n-dimensional ε-quadratic Poincaré complexes (C, ψ), (C', ψ'), and of *algebraic surgery*.

DEFINITION 3.8. The *n-dimensional ε-quadratic L-group of* $\mathbb{A}$ $L_n(\mathbb{A}, \varepsilon)$ $(n \geqslant 0)$ is the cobordism group of n-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$. □

For $\varepsilon = 1$ the terminology is abbreviated by

$$1\text{-quadratic} = \text{quadratic}, \quad T_1 = T,$$
$$Q_*(C, 1) = Q_*(C), \qquad L_*(A, 1) = L_*(A).$$

EXAMPLE 3.9. (i) The projective quadratic L-groups of a ring with involution A are the quadratic L-groups of the additive category with involution $\mathbb{P}(A)$ of f.g. projective A-modules

$$L_*^p(A) = L_*(\mathbb{P}(A)).$$

(ii) The free quadratic L-groups of A are the quadratic L-groups of the full subcategory $\mathbb{F}(A) \subseteq \mathbb{P}(A)$ of f.g. free A-modules

$$L_*^h(A) = L_*(\mathbb{F}(A)).$$ □

A chain map $f: C \to D$ in $\mathbb{A}$ induces morphisms in the Q-groups

$$f_\%: Q_n(C, \varepsilon) \to Q_n(D, \varepsilon);\ \psi \to f_\%\psi = \{f\psi_s f^* \mid s \geqslant 0\}$$

which depend only on the chain homotopy class of f (Proposition 1.1 of Ranicki [11]). In particular, a chain equivalence f induces isomorphisms $f_\%: Q_*(C, \varepsilon) \to Q_*(D, \varepsilon)$.

DEFINITION 3.10. A *homotopy equivalence* of n-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$

$$f: (C, \psi) \to (C', \psi')$$

is a chain equivalence $f: C \to C'$ such that

$$f_\%(\psi) = \psi' \in Q_n(C', \varepsilon).$$ □

A homotopy equivalence $f: (C, \psi) \to (C', \psi')$ determines a cobordism $((f\ \ 1): C \oplus C' \to C', (\delta\psi, \psi \oplus -\psi'))$, so that

$$(C, \psi) = (C', \psi') \in L_n(\mathbb{A}).$$

4. High Connections

We establish the 4-periodicity

$$L_n(\mathbb{A}, \varepsilon) = L_{n+4}(\mathbb{A}, \varepsilon) \quad (n \geqslant 0)$$

by defining maps $L_n(\mathbb{A}, \varepsilon) \to L_{n+2}(\mathbb{A}, -\varepsilon)$ and using algebraic surgery below the middle dimension to prove that they are isomorphisms, as in Ranicki [11] for $\mathbb{A} = \mathbb{P}(A)$ but without using kernels and cokernels.

DEFINITION 4.1. An n-dimensional ε-quadratic Poincaré complex in $\mathbb{A}$ (C, ψ) is *highly connected* if it is such that

$$C_r = 0 \quad \text{if} \begin{cases} r \neq i, & \text{for } n = 2i, \\ r \neq i, i+1, & \text{for } n = 2i+1, \end{cases}$$

for that

$$\begin{cases} C\colon \cdots \to 0 \to C_i \to 0 \to \cdots & \text{if } n = 2i, \\ C\colon \cdots \to 0 \to C_{i+1} \xrightarrow{d} C_i \to 0 \to & \text{if } n = 2i+1. \end{cases}$$ □

PROPOSITION 4.2. *There is a natural one-to-one correspondence between the homotopy equivalence classes of highly connected* $\left[{2i \atop 2i+1}\right]$*-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$ and the* $\left[{- \atop stable}\right]$ *isomorphism classes of nonsingular* $\left[{- \atop split}\right]$ $(-)^i\varepsilon$*-quadratic* $\left[{forms \atop formations}\right]$ *in $\mathbb{A}$.*

Proof. A highly connected $2i$-dimensional ε-quadratic Poincaré complex (C, ψ) corresponds to the nonsingular $(-)^i\varepsilon$-quadratic form (C^i, ψ_0). A highly connected $(2i+1)$-dimensional ε-quadratic Poincaré complex (C, ψ) corresponds to the nonsingular split $(-)^i\varepsilon$-quadratic formation $(F, (\left[{\gamma \atop \mu}\right], \theta)G)$ defined by

$$d_C = \mu^*\colon C_{i+1} = F \to C_i = G^*, \quad C_r = 0 \text{ for } r \neq i, i+1,$$

$$\psi_0 = \begin{cases} \varepsilon\gamma\colon C^i = G \to C_{i+1} = F, \\ 0\colon C^{i+1} = F^* \to C_i = G^*, \end{cases}$$

$$\psi_1 = \theta\colon C^i = G \to C_i = G^*.$$

The verification that the homotopy equivalence of highly connected complexes corresponds to the (stable) isomorphism of forms (formations) proceeds as in Section 2 of Ranicki [11]. □

PROPOSITION 4.3. *The cobordism group of highly connected* $\left[{2i \atop 2i+1}\right]$*-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$ is naturally isomorphic to the Witt group* $\left[{L_{(-)^i\varepsilon}(\mathbb{A}) \atop M_{(-)^i\varepsilon}(\mathbb{A})}\right]$ of nonsingular $(-)^i\varepsilon$*-quadratic* $\left[{forms \atop formations}\right]$ *in $\mathbb{A}$.*

Proof. The verification that the cobordism of highly connected complexes corresponds to the Witt relations for forms and formations proceeds as in the case $\mathbb{A} = \mathbb{P}(A)$ considered in [11]. □

For $i = 0$, Proposition 4.3 gives identifications

$$L_0(\mathbb{A}, \varepsilon) = L_\varepsilon(\mathbb{A}), \qquad L_1(\mathbb{A}, \varepsilon) = M_\varepsilon(\mathbb{A}).$$

DEFINITION 4.4. The *suspension* SC of a chain complex C in $\mathbb{A}$ is the chain complex in $\mathbb{A}$ defined by

$$d_{SC} = d_C\colon SC_r = C_{r-1} \to SC_{r-1} = C_{r-2}.$$ □

DEFINITION 4.5. The *skew-suspension* of an n-dimensional ε-quadratic (Poincaré) complex in $\mathbb{A}$ (C,ψ) is the $(n+2)$-dimensional $(-\varepsilon)$-quadratic (Poincaré) complex in $\mathbb{A}$

$$\bar{S}(C,\psi) = (SC, \bar{S}\psi),$$

with $\bar{S}\psi \in Q_{n+2}(SC, -\varepsilon)$ defined by

$$(\bar{S}\psi)_s = (-)^r\psi_s : (SC)^{r+1} = C^r \to (SC)_{n+1-r-s} = C_{n-r-s}. \qquad \square$$

We wish to prove that the i-fold skew-suspension maps

$$\bar{S}^i : L_{(-)^i\varepsilon}(\mathbb{A}) = L_0(\mathbb{A},(-)^i\varepsilon) \to L_{2i}(\mathbb{A},\varepsilon),$$
$$\bar{S}^i : M_{(-)^i\varepsilon}(\mathbb{A}) = L_1(\mathbb{A},(-)^i\varepsilon) \to L_{2i+1}(\mathbb{A},\varepsilon)$$

are isomorphisms for any $i \geqslant 0$.

The highly connected n-dimensional ε-quadratic Poincaré complexes for $n = 2i$ or $2i+1$ are precisely the i-fold skew-suspensions $\bar{S}^i(C,\psi)$ of the $(n-2i)$-dimensional $(-)^i\varepsilon$-quadratic Poincaré complexes (C,ψ).

DEFINITION 4.6. Let $n = 2i$ or $2i+1$. An n-dimensional ε-quadratic Poincaré complex (C,ψ) is *stably highly connected* if $C_r = 0$ for $r \geqslant n-i+1$, so that

$$\begin{cases} C : \cdots \to 0 \to C_i \xrightarrow{d} C_{i-1} \to \cdots \to C_0 & \text{if } n = 2i, \\ C : \cdots \to 0 \to C_{i+1} \xrightarrow{d} C_i \to \cdots \to C_0 & \text{if } n = 2i+1. \end{cases} \qquad \square$$

Highly connected complexes in the sense of 4.1 are stably highly connected.

REMARK 4.7. For a stably high connected n-dimensional ε-quadratic Poincaré complex (C,ψ) in $\mathbb{A}$ a chain contraction

$$\Gamma : 1 \simeq 0 : C((1+T_\varepsilon)\psi_0 : C^{n-*} \to C) \to C((1+T_\varepsilon)\psi_0)$$

includes morphisms $\Gamma \in \mathrm{Hom}_{\mathbb{A}}(C_r, C_{r+1})$ $(r \leqslant i-1)$ such that

$$d\Gamma + \Gamma d = 1 : C_r \to C_r \quad (r \leqslant i-1).$$

If $\mathbb{A}$ is fully embedded in an Abelian category, then (C,ψ) is homotopy equivalent to a highly connected complex (C',ψ') with

$$C'_i = \ker(d : C_i \to C_{i-1}) \quad \text{if } n = 2i \quad \text{or} \quad 2i+1,$$
$$C'_{i+1} = C_{i+1} \quad \text{if } n = 2i+1.$$

In particular, this applies to the full embedding of the additive category $\mathbb{A} = \mathbb{P}(A)$ of f.g. projective A-modules in the Abelian category of all A-modules, for any ring with involution A. $\square$

DEFINITION 4.8. Given any chain complex C in $\mathbb{A}$ and integers $i \leqslant j$, let $C[i,j]$ be the chain complex in $\mathbb{A}$ defined by

$$C[i,j]_r = \begin{cases} C_r & \text{if } i \leqslant r \leqslant j, \\ 0 & \text{otherwise}, \end{cases}$$

$$d_{C[i,j]} = d_C : C[i,j]_r = C_r \to C[i,j]_{r-1} = C_{r-1} \quad (i < r \leqslant j).$$

If $C_{j+1} = 0$, let $1: C \to C[i,j]$ be the chain map defined by the identity maps $1: C_r \to C[i,j]_r$ $(i \leqslant r \leqslant j)$. □

PROPOSITION 4.9. *Let $n = 2i$ or $2i + 1$. For every n-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$, there exists a cobordism $(C \oplus C' \to C^{n-*}[i, n], (\delta\psi, \psi \oplus -\psi'))$ to a stably highly connected complex (C', ψ') with*

$$C'_r = \begin{cases} C_r \oplus C^{n+1-r} & \text{if } r \leqslant i, \\ C_{i+1} & \text{if } r = i+1 \text{ and } n = 2i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in Proposition 5.3 of Ranicki [11] there is defined an $(n + 1)$-dimensional ε-quadratic pair $(1: C \to C[n - i + 1, n], (0, \psi))$. Algebraic surgery on this pair followed by a homotopy equivalence results in a cobordism $(C \oplus C' \to C^{n-*}[i, n], (\delta\psi, \psi \oplus -\psi'))$ to a stably highly-connected complex (C', ψ') with

$$d_{C'} = \begin{bmatrix} d_C & (-)^{n+1}(1 + T_\varepsilon)\psi_0 \\ 0 & (-)^r d_C^* \end{bmatrix}:$$
$$C'_r = C_r \oplus C^{n+1-r} \to C'_{r-1} = C_{r-1} \oplus C^{n+2-r} \quad (r \leqslant i). \qquad \square$$

PROPOSITION 4.10. *For every stably highly connected n-dimensional ε-quadratic Poincaré complex (C', ψ') in $\mathbb{A}$ with $\left[\begin{smallmatrix} n=2i \\ n=2i+1 \end{smallmatrix}\right]$ there exists a $\left[\begin{smallmatrix} \text{cobordism} & (C' \oplus C^{\oplus} \to D, (\delta\psi', \psi' \oplus -\psi^{\oplus})) \\ \text{homotopy equivalence} & (C', \psi') \to (C^{\oplus}, \psi^{\oplus}) \end{smallmatrix}\right]$ to a highly connected n-dimensional ε-quadratic Poincaré complex $(C^{\oplus}, \psi^{\oplus})$ with*

$$\begin{cases} (C^{\oplus})_i = \sum_{j=0}^{\infty} (C'_{i-2j} \oplus C'^{i-2j-1}), \\ (C^{\oplus})_i = \sum_{j=0}^{\infty} C'_{i-2j}, \quad (C^{\oplus})_{i+1} = \sum_{j=0}^{\infty} C'_{i+1-2j}, \end{cases}$$

$$D_r = \sum_{j=0}^{\infty} C'_{i-2j} \quad \text{if } r = i, = 0 \text{ if } r \neq i.$$

Proof. Given a chain contraction Γ^1 of $C((1 + T_\varepsilon)\psi'_0: C'^{n-*} \to C')$ let C'' be the chain complex defined for $n = \left[\begin{smallmatrix} 2i \\ 2i+1 \end{smallmatrix}\right]$ by

$$\begin{cases} d'' = d' + \Gamma': C''_i = \sum_{j=0}^{\infty} C'_{i-2j} \to C''_{i-1} = \sum_{j=0}^{\infty} C'_{i-2j-1}, \\ d'' = d' + \Gamma': C''_{i+1} = \sum_{j=0}^{\infty} C'_{i+1-2j} \to C''_i = \sum_{j=0}^{\infty} C'_{i-2j}, \end{cases}$$

$$C''_r = 0 \text{ for } \begin{cases} r \neq i-1, i \\ r \neq i, i+1 \end{cases}.$$

The chain map $f: C' \to C''$ defined by

$$f: C'_r \to C''_r = \sum_{j=0}^{\infty} C'_{r-2j}; x \to (x, 0, 0, \ldots) \quad \left(r = \begin{cases} i-1, i \\ i, i+1 \end{cases}\right)$$

is a chain equivalence, so that there is defined a homotopy equivalence of n-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$

$$f:(C',\psi')\to(C'',\psi'')$$

with ψ'' defined by

$$\psi''=f_\%(\psi')\in Q_n(C'',\varepsilon).$$

For $n=2i+1$ (C'',ψ'') is highly connected, so that we can set

$$(C^\oplus,\psi^\oplus)=(C'',\psi'').$$

For $n=2i$ (C'',ψ'') is cobordant to a highly connected $2i$-dimensional ε-quadratic Poincaré complex $(C^\oplus,\psi^\oplus)$ by a cobordism

$$((1\,f^\oplus):C''\oplus C^\oplus\to D,(\delta\psi'',\psi''\oplus-\psi^\oplus))$$

with

$$f^\oplus=(1\;\;0):(C^\oplus)_i=C''_i\oplus C''^{i-1}\to C''_i,\quad D=C''[i,i].$$

By the chain homotopy invariance of the Q-groups this cobordism is homotopy equivalent to a cobordism $((C'\oplus C^\oplus\to D,(\delta\psi',\psi'\oplus-\psi^\oplus))$. $\square$

5. The Additive Instant Surgery Obstruction

Given an n-dimensional ε-quadratic Poincaré complex (C,ψ) in $\mathbb{A}$ let (C',ψ') be the cobordant stably highly connected complex given by 4.9, and let $(C^\oplus,\psi^\oplus)$ be the highly connected complex $\left[\begin{smallmatrix}\text{cobordant}\\ \text{homotopy}\end{smallmatrix}\text{ equivalent}\right]$ to (C',ψ') given by 4.10.

DEFINITION 5.1. $(C^\oplus,\psi^\oplus)$ is the *additive instant surgery obstruction* of (C,ψ), such that

$$\left[\begin{array}{ll}(C^\oplus)_i=\sum\limits_{j=-\infty}^{\infty}(C_{i+2j}\oplus C^{i+2j+1}) & \\ (C^\oplus)_i=\sum\limits_{j=0}^{\infty}(C_{i-2j}\oplus C^{i+2j+2}), & (C^\oplus)_{i+1}=\sum\limits_{j=0}^{\infty}(C_{i-2j+1}\oplus C^{i+2j+3})\end{array}\right.$$

for $n=\begin{cases}2i\\ 2i+1\end{cases}$. $\square$

THEOREM 5.2. *The i-fold skew-suspension morphism of Abelian groups*

$$\begin{cases}\bar{S}^i:L_{(-)^i\varepsilon}(\mathbb{A})=L_0(\mathbb{A},(-)^i\varepsilon)\to L_{2i}(\mathbb{A},\varepsilon),\\ \bar{S}^i:M_{(-)^i\varepsilon}(\mathbb{A})=L_1(\mathbb{A},(-)^i\varepsilon)\to L_{2i+1}(\mathbb{A},\varepsilon),\end{cases}$$

is an isomorphism for any $i\geqslant 0$, with the inverse $\bar{S}^{-i}$ sending the cobordism class of a $\left\{\begin{smallmatrix}2i\\ 2i+1\end{smallmatrix}\right\}$-dimensional ε-quadratic Poincaré complex (C,ψ) to the Witt class of the *nonsingular $(-)^i\varepsilon$-quadratic $\left[{}^{form}_{formation}\right]$ associated to the additive instant surgery obstruction $(C^{\oplus}, \psi^{\oplus})$.*

Proof. As for the case $\mathbb{A} = P(A)$ in Proposition 4.4 of Ranicki [11] algebraic surgery below the middle dimension shows that $L_n(\mathbb{A}, \varepsilon)$ for any $\mathbb{A}$ is isomorphic to the cobordism group of stably highly connected complexes. The additive instant surgery obstruction is the extra step required to pass from stably highly connected complexes to highly connected complexes. □

REMARK 5.3. If $\mathbb{A}$ is embedded in an Abelian category, then the complex (C', ψ') used to define $(C^{\oplus}, \psi^{\oplus})$ in 5.1 is homotopy equivalent to the highly-connected complex associated to the nonsingular $(-)^i\varepsilon$-quadratic $\left[{}^{\text{form}}_{\text{formation}}\right]$

$$(M, \theta) = \left(\operatorname{coker}\left(\begin{bmatrix} d^* & 0 \\ (-)^{i+1}(1+T_\varepsilon)\psi_0 & d \end{bmatrix}: C^{i-1} \oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right)$$

$$\left(F, \begin{bmatrix} \gamma \\ \mu \end{bmatrix} G\right) = \left(C_{i+1}, \begin{bmatrix} [(1+T_\varepsilon)\psi_0\, d] \\ [\varepsilon d^*\, 0] \end{bmatrix} \operatorname{coker}\left(\begin{bmatrix} -d^*\, 0 \\ (-)^i(1+T_\varepsilon)\psi_0 d \end{bmatrix}: C^{i-1} \oplus C_{i+3} \to C^i \oplus C_{i+2}\right)\right).$$

This is the instant surgery obstruction of Ranicki [11]. □

6. Class

We apply the additive K_0-theory of Ranicki [13] to L-theory.

DEFINITION 6.1. The *class group* $K_0(\mathbb{A})$ of an additive category $\mathbb{A}$ is the Abelian group with one generator $[A]$ for each isomorphism class of objects A in $\mathbb{A}$, subject to the relations

$$[A \oplus B] = [A] + [B].$$

DEFINITION 6.2. The *class* of a finite chain complex C in $\mathbb{A}$ is the chain homotopy invariant

$$[C] = [C_{\text{even}}] - [C_{\text{odd}}] \in K_0(\mathbb{A}),$$

with

$$C_{\text{even}} = \sum_{j=0}^{\infty} C_{2j}, \qquad C_{\text{odd}} = \sum_{j=0}^{\infty} C_{2j+1}.$$

(It is assumed that $C_r = 0$ for $r < 0$.)

An involution $*: \mathbb{A} \to \mathbb{A}$ of the additive category $\mathbb{A}$ induces an involution of the class group

$$*: K_0(\mathbb{A}) \to K_0(\mathbb{A}); [A] \to [A]^* = [A^*],$$

such that for any finite chain complex C and $n \in \mathbb{Z}$

$$[C^{n-*}] = (-)^n [C]^* \in K_0(\mathbb{A}).$$

DEFINITION 6.3. The *class* of an n-dimensional ε-quadratic complex (C, ψ) in $\mathbb{A}$ is the class of the underlying chain complex

$$[C, \psi] = [C] \in K_0(\mathbb{A}). \qquad \square$$

PROPOSITION 6.4. (i) *The class of an n-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ is such that*

$$[C, \psi]^* = (-)^n [C, \psi] \in K_0(\mathbb{A}).$$

(ii) *For any cobordism $((f\ f'): C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ of n-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$*

$$[C, \psi] + [C', \psi'] = [D] + (-)^n [D]^* \in K_0(\mathbb{A}),$$

and also

$$[C, \psi] - [C', \psi'] = [C(f')] + (-)^n [C(f')]^* \in K_0(\mathbb{A})$$

with

$$[C(f')] = [D] - [C'] \in K_0(\mathbb{A}).$$

Proof: (i) C is chain equivalent to C^{n-*}.

(ii) The algebraic mapping cone $C(f\ f')$ is chain equivalent to D^{n+1-*}. Also, $C(f)$ is chain equivalent to $C(f')^{n+1-*}$. $\square$

PROPOSITION 6.5. *For any n-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ with additive instant surgery obstruction $(C^{\oplus}, \psi^{\oplus})$ the classes are such that*

$$[C, \psi] - [C^{\oplus}, \psi^{\oplus}] = [P] + (-)^n [P]^* \in K_0(\mathbb{A}),$$

with

$$[P] = \begin{cases} \sum\limits_{j=-\infty}^{\infty} (-)^{i+2j+1} [C_{i+2j+1}] & \text{if } n = 2i, \\ \sum\limits_{j=i+2}^{\infty} (-)^j [C_j] & \text{if } n = 2i+1. \end{cases}$$

Proof. By 4.9, there is a cobordism $(C \oplus C' \to C^{n-*}[i, n], (\delta\psi, \psi \oplus -\psi'))$. In the case $n = 2i$, Proposition 4.10 gives a cobordism $(C' \oplus C^{\oplus} \to D, (\delta\psi', \psi' \oplus -\psi^{\oplus}))$ with

$$D_i = \sum_{j=0}^{\infty} C'_{i-2j} = \sum_{j=0}^{\infty} (C_{i-2j} \oplus C^{i+2j+1}), \qquad D_r = 0 \quad \text{for } r \neq i,$$

so that

$$\begin{aligned}[C]-[C^{\oplus}] &= ([C]+[C'])-([C']+[C^{\oplus}])\\ &= [C^{2i-*}[i,2i]]+[C^{2i-*}[i,2i]]^*-[D]-[D]^*\\ &= [P]+[P]^*\in K_0(\mathbb{A}).\end{aligned}$$

In the case $n=2i+1$, Proposition 4.10 gives a homotopy equivalence $(C',\psi')\to(C^{\oplus},\psi^{\oplus})$, so that

$$\begin{aligned}&[C]-[C^{\oplus}]\\ &= ([C^{2i+1-*}[i,2i+1]]-[C^{\oplus}])-([C^{2i+1-*}[i,2i+1]]-[C^{\oplus}])^*\\ &= [P]-[P]^*\in K_0(\mathbb{A}).\end{aligned}$$ □

Next, we generalize the Rothenberg exact sequence of Ranicki [8, 10]

$$\begin{aligned}&\cdots\to L_n^Y(A)\to L_n^X(A)\to \hat{H}^n(\mathbb{Z}_2;X/Y)\\ &\to L_{n-1}^Y(A)\to\cdots(Y\subseteq X\subseteq \tilde{K}_0(A))\end{aligned}$$

to the quadratic L-theory of any additive category with involution $\mathbb{A}$.

DEFINITION 6.6. Let $\mathbb{A}$ be an additive category with involution $*:\mathbb{A}\to\mathbb{A}$. A subgroup $X\subseteq K_0(\mathbb{A})$ is *$*$-invariant* if $x^*\in X$ for all $x\in X$. □

DEFINITION 6.7. The *intermediate Witt group of ε-quadratic forms* $L_\varepsilon^X(\mathbb{A})$ is defined for a $*$-invariant subgroup $X\subseteq K_0(\mathbb{A})$ to be the Abelian group with one generator for each formal difference $(M,\psi)-(M',\psi')$ of isomorphism classes of nonsingular ε-quadratic forms in $\mathbb{A}$ such that

$$[M]-[M']\in X\subseteq K_0(\mathbb{A}),$$

subject to the relations

(i) $(M,\psi)-(M,\psi)=0$,
(ii) $((M,\psi)-(M',\psi'))+((N,\theta)-(N',\theta'))$
$= (M\oplus N,\psi\oplus\theta)-(M'\oplus N',\psi'\oplus\theta')$,
(iii) $H_\varepsilon(L)-H_\varepsilon(L')=0\quad$ if $[L]-[L']\in X\subseteq K_0(\mathbb{A})$.

In particular,

$$L_\varepsilon^{K_0(\mathbb{A})}(\mathbb{A})=L_\varepsilon(\mathbb{A}).$$

DEFINITION 6.8. The *intermediate Witt group of* ε-quadratic formations $M_\varepsilon^X(\mathbb{A})$ is defined for a $*$-invariant subgroup $X\subseteq K_0(\mathbb{A})$ to be the Abelian group with one generator for each stable isomorphism class of nonsingular ε-quadratic formations $(M,\psi;F,G)$ in $\mathbb{A}$, such that

$$[G]-[F^*]\in X\subseteq K_0(\mathbb{A}),$$

subject to the relations

(i) $(M, \psi; F, G) + (M', \psi'; F', G') = (M \oplus M', \psi \oplus \psi', F \oplus F', G \oplus G')$
(ii) $(M, \psi; F, G) + (M, \psi; G, H) = (M, \psi; F, H)$ if $[F], [G], [H] \in X$. □

In particular,

$$M_\varepsilon^{K_0(\mathbb{A})}(\mathbb{A}) = M_\varepsilon(\mathbb{A}).$$

DEFINITION 6.9. The *intermediate ε-quadratic L-groups* $L_n^X(\mathbb{A}, \varepsilon)$ $(n > 0)$ are defined for a $*$-invariant subgroup $X \subseteq K_0(\mathbb{A})$ to be the cobordism groups of n-dimensional ε-quadratic Poincaré complexes (C, ψ) in $\mathbb{A}$ with class $[C, \psi] \in X \subseteq K_0(\mathbb{A})$, and with the corresponding condition on the cobordisms. For $n = 0$ define

$$L_0^X(\mathbb{A}, \varepsilon) = L_\varepsilon^X(\mathbb{A}).$$ □

In particular,

$$L_n^{K_0(\mathbb{A})}(\mathbb{A}, \varepsilon) = L_n(\mathbb{A}, \varepsilon) \quad (n \geqslant 0).$$

THEOREM 6.10. *For any $*$-invariant subgroup $X \subseteq K_0(\mathbb{A})$ there are natural identifications of the intermediate cobordism groups and the intermediate Witt groups*

$$L_n^X(\mathbb{A}, \varepsilon) = \begin{cases} L_{(-)^i\varepsilon}^X(\mathbb{A}) & \text{if } n = 2i, \\ M_{(-)^i\varepsilon}^X(\mathbb{A}) & \text{if } n = 2i+1. \end{cases}$$

Proof. This is just the intermediate version of Theorem 5.2 and is proved similarly. The intermediate i-fold skew-suspension maps

$$\begin{cases} \bar{S}^i \colon L_{(-)^i\varepsilon}^X(\mathbb{A}) = L_0^X(\mathbb{A}, (-)^i\varepsilon) \to L_{2i}^X(\mathbb{A}, \varepsilon), \\ \bar{S}^i \colon M_{(-)^i\varepsilon}^X(\mathbb{A}) = L_1^X(\mathbb{A}, (-)^i\varepsilon) \to L_{2i+1}^X(\mathbb{A}, \varepsilon), \end{cases}$$

are isomorphisms, using the additive instant surgery obstruction to define the inverses

$$\bar{S}^{-1} \colon L_n^X(\mathbb{A}, \varepsilon) \to L_{n-2i}^X(\mathbb{A}, \varepsilon);$$

$$(C, \psi) \to \begin{cases} (C^\oplus, \psi^\oplus) \oplus H_{(-)^i\varepsilon}(P) & \text{if } n = 2i, \\ (C^\oplus, \psi^\oplus) \oplus (H_{(-)^i\varepsilon}(P); P, P) & \text{if } n = 2i+1, \end{cases}$$

with P as in 6.5. □

THEOREM 6.11. *The intermediate ε-quadratic L-groups of $*$-invariant subgroups $Y \subseteq X \subseteq K_0(\mathbb{A})$ are related by the Rothenberg exact sequence*

$$\cdots \to L_n^Y(\mathbb{A}, \varepsilon) \to L_n^X(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; X/Y) \to L_{n-1}^Y(\mathbb{A}, \varepsilon) \to \cdots$$
$$\to L_{n-1}^Y(\mathbb{A}, \varepsilon) \to \cdots$$

with the Tate $\mathbb{Z}_2$-cohomology groups defined by

$$\hat{H}^n(\mathbb{Z}_2; X/Y) = \{x \in X/Y \mid x^* = (-)^n x\}/\{y + (-)^n y^* \mid y \in X/Y\}$$

and

$$L^X_n(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; X/Y); (C, \psi) \to [C, \psi] = [C].$$

Proof. This can be proved using either forms and formations as in Ranicki [8, 10], or using quadratic Poincaré complexes as in Ranicki [11]. The proofs in the original case $\mathbb{A} = \mathbb{P}(A)$ did not use kernels and cokernels, and so generalize directly to any additive category with involution $\mathbb{A}$. □

For $\varepsilon = 1$ we write

$$L^X_*(\mathbb{A}, 1) = L^X_*(\mathbb{A}).$$

DEFINITION 6.12. The *idempotent completion* $\mathbb{P}(\mathbb{A})$ of an additive category $\mathbb{A}$ is the additive category with objects (A, p) defined by objects A of $\mathbb{A}$ together with a projection $p = p^2: A \to A$. A morphism in $\mathbb{P}(\mathbb{A})$

$$f: (A, p) \to (B, q)$$

is a morphism $f: A \to B$ in $\mathbb{A}$ such that

$$qfp = f: A \to B. \qquad \square$$

The full embedding

$$\mathbb{A} \to \mathbb{P}(\mathbb{A}); A \to (A, 1)$$

is used to identify $\mathbb{A}$ with a full subcategory of $\mathbb{P}(\mathbb{A})$.

DEFINITION 6.13. The *reduced class group* of $\mathbb{P}(\mathbb{A})$ is defined by

$$\tilde{K}_0(\mathbb{P}(\mathbb{A})) = \operatorname{coker}(K_0(\mathbb{A}) \to K_0(\mathbb{P}(\mathbb{A}))). \qquad \square$$

We refer to Ranicki [13] for the algebraic theory of finiteness obstruction in an arbitrary additive category $\mathbb{A}$. In particular, it was proved there that a finite chain complex C in $\mathbb{P}(\mathbb{A})$ is chain equivalent to a finite chain complex in $\mathbb{A}$ if and only if $[C] = 0 \in \tilde{K}_0(\mathbb{P}(\mathbb{A}))$.

An involution $*: \mathbb{A} \to \mathbb{A}$ extends to an involution

$$*: \mathbb{P}(\mathbb{A}) \to \mathbb{P}(\mathbb{A}); (A, p) \to (A^*, p^*).$$

EXAMPLE 6.14. For a ring A and $\mathbb{A} = \mathbb{F}(A) = \{\text{f.g. free } A\text{-modules}\}$ the additive functor

$$\begin{aligned}\mathbb{P}(\mathbb{A}) &\to \mathbb{P}(A)\\ &= \{\text{f.g. projective } A\text{-modules}\}; (A, p) \to \operatorname{im}(p)\end{aligned}$$

is an equivalence of additive categories. The class group of $\mathbb{P}(\mathbb{A})$ is the usual class group of A

$$K_0(\mathbb{P}(\mathbb{A})) = K_0(\mathbb{P}(A)) = K_0(A),$$

and, similarly, for the reduced class groups

$$\tilde{K}_0(\mathbb{P}(\mathbb{A})) = \tilde{K}_0(\mathbb{P}(A)) = \tilde{K}_0(A).$$

For a ring with involution A and an object (F, p) in $\mathbb{P}(\mathbb{F}(A))$ there is a natural isomorphism of f.g. projective A-modules

$$\mathrm{im}(p^*) \to \mathrm{im}(p)^*; \; f \to (x \to f(x)),$$

so that $\mathbb{P}(\mathbb{F}(A)) \to \mathbb{P}(A)$ is an equivalence of additive categories with involution. The inclusion $\mathbb{A} = \mathbb{F}(A) \to \mathbb{P}(\mathbb{A}) \approx \mathbb{P}(A)$ induces the natural maps

$$L_*(\mathbb{F}(A)) = L_*^h(A) \to L_*(\mathbb{P}(A)) = L_*^p(A). \qquad \square$$

PROPOSITION 6.15. *The quadratic L-groups of an additive category with involution* $\mathbb{A}$ *and the idempotent completion* $\mathbb{P}(\mathbb{A})$ *are related by the Rothenberg exact sequence*

$$\cdots \to L_n(\mathbb{A}) \to L_n(\mathbb{P}(\mathbb{A})) \to \hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{P}(\mathbb{A}))) \to L_{n-1}(\mathbb{A}) \to \cdots.$$

Proof. This is a special case of the exact sequence of 6.11, with $\mathbb{A}$ replaced by $\mathbb{P}(\mathbb{A})$ and

$$Y = \mathrm{im}(K_0(\mathbb{A}) \to K_0(\mathbb{P}(\mathbb{A}))) \subseteq X = K_0(\mathbb{P}(\mathbb{A}))$$

such that

$$X/Y = \tilde{K}_0(\mathbb{P}(\mathbb{A})), \qquad L_*^Y(\mathbb{P}(\mathbb{A})) = L_*(\mathbb{A}). \qquad \square$$

REMARK 6.16. For any filtered additive category $\mathbb{A}$ let $\mathbb{C}_1(\mathbb{A})$ be the filtered additive category defined by Pedersen and Weibel [6], with objects $\mathbb{Z}$-graded formal direct sums $\Sigma_{j=-\infty}^{\infty} A(j)$ of objects $A(j)$ in $\mathbb{A}$, and with bounded morphisms. Define

$$\mathbb{C}_0(\mathbb{A}) = \mathbb{A}, \qquad \mathbb{P}_0(\mathbb{A}) = \mathbb{P}(\mathbb{A}),$$
$$\mathbb{C}_i(\mathbb{A}) = \mathbb{C}_1(\mathbb{C}_{i-1}(\mathbb{A})), \qquad \mathbb{P}_i(\mathbb{A}) = \mathbb{P}(\mathbb{C}_i(\mathbb{A})) \quad (i \geqslant 1)$$

and for a ring A write

$$\mathbb{C}_i(\mathbb{F}(A)) = \mathbb{C}_i(A), \qquad \mathbb{P}_i(\mathbb{F}(A)) = \mathbb{P}_i(A) \quad (i \geqslant 0).$$

It was proved in [6] that for $i \geqslant 0$

$$K_1(\mathbb{C}_{i+1}(\mathbb{A})) = K_0(\mathbb{P}_i(\mathbb{A})) = K_{-i}(\mathbb{P}_0(\mathbb{A})) \quad (= K_{-i}(\mathbb{A}) \text{ for } i \geqslant 1)$$

with $K_{-*}(\mathbb{A})$ the lower K-groups. For a ring A and $\mathbb{A} = \mathbb{F}(A)$ this is the earlier result of Pedersen [5] that

$$K_1(\mathbb{C}_{i+1}(A)) = K_0(\mathbb{P}_i(A)) = K_{-i}(A) \quad (i \geqslant 0)$$

with $K_{-i}(A)$ the lower K-groups of Bass [1], designed to fit into split exact sequences

$$0 \to K_{-i+1}(A) \to K_{-i+1}(A[z]) \oplus K_{-i+1}(A[z^{-1}])$$
$$\to K_{-i+1}(A[z, z^{-1}]) \to K_{-i}(A) \to 0 \quad (i \geqslant 0).$$

The additive L-theory of this paper is used in Ranicki [15] to prove the analogous result in L-theory, which we now state. Extend an involution $*\colon \mathbb{A} \to \mathbb{A}$ to an involution $*\colon \mathbb{C}_1(\mathbb{A}) \to \mathbb{C}_1(\mathbb{A})$ by

$$A^*(j) = A(j)^* \quad (j \in \mathbb{Z}),$$

and similarly for $\mathbb{C}_i(\mathbb{A})$, $\mathbb{P}_i(\mathbb{A})$. The L-theoretic analogue is that for any additive category with involution $\mathbb{A}$

$$L_{n+1}(\mathbb{C}_{i+1}(\mathbb{A})) = L_n(\mathbb{P}_i(\mathbb{A})) = L_{n-i}^{\langle -i \rangle}(\mathbb{A}) \quad (i \geqslant 0),$$

with $L_*^{\langle -* \rangle}(\mathbb{A})$ lower L-groups. In particular, for a ring with involution A

$$L_{n+1}(\mathbb{C}_{i+1}(A)) = L_n(\mathbb{P}_i(A)) = L_{n-i}^{\langle -i \rangle}(A) \quad (i \geqslant 0),$$

with $L_*^{\langle -i \rangle}(A) = L_*^{K_{-i}(A)}(A)$ the lower L-groups of Ranicki [9], designed to be such that

$$L_{n+1}^{\langle -i+1 \rangle}(A[z, z^{-1}]) = L_{n+1}^{\langle -i+1 \rangle}(A) \oplus L_n^{\langle -i \rangle}(A) \quad (i \geqslant 0),$$
$$L_n^{\langle 1 \rangle}(A) = L_n^h(A), \qquad L_n^{\langle 0 \rangle}(A) = L_n^p(A)$$

with the involution on A extended to $A[z, z^{-1}]$ by $\bar{z} = z^{-1}$. The additive L-theory Rothenberg exact sequences given by 6.15

$$\cdots \to L_n(\mathbb{C}_i(A)) \to L_n(\mathbb{P}_i(A)) \to \hat{H}^n(\mathbb{Z}_2; \tilde{K}_0(\mathbb{P}_i(A)))$$
$$\to L_{n-1}(\mathbb{C}_i(A)) \to \cdots$$

coincide with the L-theory Rothenberg exact sequences already established in [9]

$$\cdots \to L_{n-i}^{\langle -i+1 \rangle}(A) \to L_{n-i}^{\langle -i \rangle}(A) \to \hat{H}^{n-i}(\mathbb{Z}_2; \tilde{K}_{-i}(A))$$
$$\to L_{n-i-1}^{\langle -i+1 \rangle}(A) \to \cdots. \qquad \square$$

7. Torsion

We apply the additive K_1-theory of Ranicki [14] to L-theory.

DEFINITION 7.1. The *torsion group* $K_1(\mathbb{A})$ of an additive category $\mathbb{A}$ is the Abelian group with one generator $\tau(f)$ for each automorphism $f\colon A \to A$ in $\mathbb{A}$, subject to the relations

(i) $\tau(gf\colon A \to A \to A) = \tau(f) + \tau(g)$,
(ii) $\tau(ifi^{-1}\colon A' \to A \to A \to A') = \tau(f)$ for any isomorphism $i\colon A \to A'$,
(iii) $\tau(f \oplus f'\colon A \oplus A' \to A \oplus A') = \tau(f) + \tau(f')$. $\square$

EXAMPLE 7.2. The torsion group of a ring A is the torsion group of the additive category $\mathbb{P}(A) = \{$f.g. projective A-modules$\}$ and also of the additive category $\mathbb{F}(A) = \{$f.g. free A-modules$\}$

$$K_1(\mathbb{P}(A)) = K_1(\mathbb{F}(A)) = K_1(A). \qquad \square$$

DEFINITION 7.3 [14]. A *stable isomorphism* between objects M, N in an additive category $\mathbb{A}$

$$[f]: M \to N$$

is an equivalence class of isomorphisms $f: M \oplus X \to N \oplus X$ in $\mathbb{A}$ under the equivalence relation

$$(f: M \oplus X \to N \oplus X) \sim (g: M \oplus Y \to N \oplus Y)$$

if

$$\tau((g^{-1} \oplus 1_X)(f \oplus 1_Y): M \oplus X \oplus Y \to M \oplus X \oplus Y) = 0 \in K_1(\mathbb{A}). \qquad \square$$

The objects M, N in $\mathbb{A}$ are stably isomorphic if and only if

$$[M] = [N] \in K_0(\mathbb{A}).$$

DEFINITION 7.4 [14]. A *canonical stable structure* $[\phi]$ on an additive category $\mathbb{A}$ is a collection of stable isomorphisms $\{[\phi_{M,N}]: M \to N\}$, one for each ordered pair $\{M, N\}$ of stably isomorphic objects in $\mathbb{A}$, such that

(i) $[\phi_{M,M}] = [1_M]: M \to M$,
(ii) $[\phi_{M,P}] = [\phi_{N,P}][\phi_{M,N}]: M \to N \to P$,
(iii) $[\phi_{M \oplus M', N \oplus N'}] = [\phi_{M,N}] \oplus [\phi_{M',N'}]: M \oplus M' \to N \oplus N'$. $\square$

EXAMPLE 7.5. If A is a ring such that f.g. free A-modules have well-defined rank, then the additive category $\mathbb{B}(A) = \{$based f.g. free A-modules$\}$ has the canonical (un)stable structure $[\phi]$ with $\phi_{M,N}: M \to N$ the isomorphism sending the base of M to the base of N. $\square$

EXAMPLE 7.6. For any filtered additive category $\mathbb{A}$ there is defined a canonical stable structure $[\phi]$ on $\mathbb{C}_1(\mathbb{A})$, with every object M in $\mathbb{C}_1(\mathbb{A})$ stably isomorphic to 0. See Section 5 of [14] for details. Pedersen and Weibel [6] show that

$$K_0(\mathbb{C}_1(\mathbb{A})) = 0, \qquad K_1(\mathbb{C}_1(\mathbb{A})) = K_0(\mathbb{P}(\mathbb{A})). \qquad \square$$

PROPOSITION 7.7 [14]. *A stable canonical structure $[\phi]$ on an additive category $\mathbb{A}$ determines a torsion function*

$$\tau: \{\textit{isomorphisms in } \mathbb{A}\} \to K_1(\mathbb{A});$$
$$(f: M \to N) \to \tau(f) = \tau([\phi_{N,M}]f: M \to M) = \tau(f[\phi_{N,M}]: N \to N)$$

such that

(i) $\tau(gf: M \to N \to P) = \tau(f: M \to N) + \tau(g: N \to P)$,
(ii) $\tau(f \oplus f': M \oplus M' \to N \oplus N') = \tau(f: M \to N) + \tau(f': M' \to N')$. $\square$

DEFINITION 7.8. The *reduced torsion group* $\tilde{K}_1(\mathbb{A})$ of an additive category $\mathbb{A}$ with respect to a canonical stable structure $[\phi]$ is the quotient of $K_1(\mathbb{A})$ by the subgroup generated by the *sign* elements

$$\varepsilon(M,N) = \tau\left(\begin{bmatrix} 0 & 1_N \\ 1_M & 0 \end{bmatrix} : M \oplus N \to N \oplus M\right) \in K_1(\mathbb{A})$$

for all objects M, N in $\mathbb{A}$

$$\tilde{K}_1(\mathbb{A}) = K_1(\mathbb{A})/\{\varepsilon(M,N)\}. \qquad \square$$

EXAMPLE 7.9. For a ring A the reduced torsion group of $\mathbb{B}(A)$ with respect to the canonical stable structure given by 7.5. is just the usual reduced torsion group of A

$$\tilde{K}_1(\mathbb{B}(A)) = \tilde{K}_1(A) = \operatorname{coker}(K_1(\mathbb{Z}) \to K_1(A)),$$

with

$$\varepsilon(A^m, A^n) = \tau((-1)^{mn} : A \to A) \in \operatorname{im}(K_1(\mathbb{Z}) \to K_1(A)) \subseteq K_1(A). \qquad \square$$

EXAMPLE 7.10. The reduced torsion group of $\mathbb{C}_1(\mathbb{A})$ with respect to the canonical stable structure on $\mathbb{C}_1(\mathbb{A})$ given by 7.6 is isomorphic to the absolute torsion group

$$\tilde{K}_1(\mathbb{C}_1(\mathbb{A})) = K_1(\mathbb{C}_1(\mathbb{A}))/\{\varepsilon(M,N)\} = K_0(\mathbb{P}(\mathbb{A})),$$

with $\varepsilon(M,N) = 0 \in K_1(\mathbb{C}_1(\mathbb{A}))$ for all objects M, N in $\mathbb{C}_1(\mathbb{A})$ by virtue of $K_0(\mathbb{C}_1(\mathbb{A})) = 0$.

DEFINITION 7.11. Let $\mathbb{A}$ be an additive category with a canonical stable structure $[\phi]$.

(i) The *torsion* of a contractible finite chain complex C in $\mathbb{A}$ is defined by

$$\tau(C) = \tau(d + \Gamma : C_{\mathrm{odd}} \to C_{\mathrm{even}}) \in K_1(\mathbb{A})$$

using any chain contraction $\Gamma : 1 \simeq 0 : C \to C$.

(ii) The *reduced torsion* of a chain equivalence $f : C \to D$ of finite chain complexes in $\mathbb{A}$ is defined by

$$\tau(f) = \tau(C(f)) \in \tilde{K}_1(\mathbb{A}). \qquad \square$$

DEFINITION 7.12. A finite chain complex C in $\mathbb{A}$ is *round* if $[C] = 0 \in K_0(\mathbb{A})$, i.e. if C_{odd} is stably isomorphic to C_{even}. $\square$

We refer to Ranicki [14] for the definition and properties of the *torsion* $\tau(f) \in K_1(\mathbb{A})$ of a chain equivalence $f : C \to D$ of round finite chain complexes in $\mathbb{A}$ with respect to a canonical stable structure $[\phi]$ on $\mathbb{A}$, as the sum of $\tau(C(f))$ and a certain element in the sign subgroup $\{\varepsilon(M,N)\} \subseteq K_1(\mathbb{A})$.

We now bring an involution into play.

DEFINITION 7.13. A canonical stable structure $[\phi]$ on an additive category $\mathbb{A}$ is *compatible* with the involution $* : \mathbb{A} \to \mathbb{A}$ if the duals of the canonical stable isomorphisms $\phi_{M,N} : M \oplus X \to N \oplus X$ are again canonical

$$[(\phi_{M,N})^*] = [\phi_{N^*, M^*}] : N^* \to M^*. \qquad \square$$

EXAMPLE 7.14. For a ring with involution A the canonical stable structure on $\mathbb{B}(A)$

given by 7.5 is compatible with the duality involution $*\colon \mathbb{B}(A) \to \mathbb{B}(A)$. □

EXAMPLE 7.15. For a filtered additive category with involution $\mathbb{A}$ the canonical stable structure on $\mathbb{C}_1(\mathbb{A})$ given by 7.6 is compatible with the duality involution $*\colon \mathbb{C}_1(\mathbb{A}) \to \mathbb{C}_1(\mathbb{A})$. □

For the rest of Section 7 we assume that we are dealing with an additive category with involution $\mathbb{A}$ which is equipped with a particular choice of compatible canonical stable structure $[\phi]$. The involution defined on $K_1(\mathbb{A})$ by

$$*\colon K_1(\mathbb{A}) \to K_1(\mathbb{A}); \tau(f\colon M \to M) \to \tau(f)^* = \tau(f^*\colon M^* \to M^*)$$

is then such that

PROPOSITION 7.16. (i) *For any isomorphism* $f\colon M \to N$ *in* $\mathbb{A}$

$$\tau(f\colon M \to N)^* = \tau(f^*\colon N^* \to M^*) \in K_1(\mathbb{A}).$$

(ii) *For any chain equivalence* $f\colon C \to D$ *of finite chain complexes in* $\mathbb{A}$

$$\tau(f^{n-*}\colon D^{n-*} \to C^{n-*}) = (-)^n \tau(f\colon C \to D)^* \in \tilde{K}_1(\mathbb{A}).$$

(iii) *For any chain equivalence* $f\colon C \to D$ *of round finite chain complexes in* $\mathbb{A}$

$$\tau(f^{n-*}\colon D^{n-*} \to C^{n-*}) = (-)^n \tau(f\colon C \to D)^* \in K_1(\mathbb{A}).$$ □

DEFINITION 7.17. An n-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ is *round* if C is a round finite chain complex in $\mathbb{A}$, that is if $[C, \psi] = 0 \in K_0(A)$. □

DEFINITION 7.18. The *round n-dimensional ε-quadratic L-groups* $L^r_*(\mathbb{A}, \varepsilon)$ are the intermediate ε-quadratic L-groups (6.9) defined by

$$L^r_n(\mathbb{A}, \varepsilon) = L_n^{\{0\} \subseteq K_0(\mathbb{A})}(\mathbb{A}, \varepsilon) \quad (n \geqslant 0).$$ □

By a special case of 6.11, there is defined an exact sequence

$$\cdots \to L^r_n(\mathbb{A}, \varepsilon) \to L_n(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; K_0(\mathbb{A}))$$
$$\to L^r_{n-1}(\mathbb{A}, \varepsilon) \to \cdots.$$

For $n > 0$, $L^r_n(\mathbb{A}, \varepsilon)$ is the cobordism group of round finite n-dimensional ε-quadratic Poincaré complexes in $\mathbb{A}$.

EXAMPLE 7.19. For a ring with involution A and $\mathbb{A} = \mathbb{F}(A)$, the round quadratic L-groups

$$L^r_*(A) = L^r_*(\mathbb{F}(A))$$

are the round L-groups of Hambleton, Ranicki and Taylor [3]. □

DEFINITION 7.20. (i) The *reduced torsion* of an n-dimensoinal ε-quadratic Poincaré compelx (C, ψ) in $\mathbb{A}$ is

$$\tau(C, \psi) = \tau((1 + T_\varepsilon)\psi_0\colon C^{n-*} \to C) \in \tilde{K}_1(\mathbb{A}).$$

(ii) The *torsion* of a round n-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ is

$$\tau(C, \psi) = \tau((1 + T_\varepsilon)\psi_0 : C^{n-*} \to C) \in K_1(\mathbb{A}). \qquad \square$$

In both cases of 7.20, the torsion is such that

$$\tau(C, \psi)^* = (-)^n \tau(C, \psi).$$

DEFINITION 7.21. (i) Given a $*$-invariant subgroup $X \subseteq \tilde{K}_1(\mathbb{A})$ define the *intermediate n-dimensional ε-quadratic L-group* $L_n^X(\mathbb{A}, \varepsilon)$ $(n \geqslant 0)$ to be the cobordism group of n-dimensional ε-quadratic Poincaré complexes (C, ψ) in $\mathbb{A}$ such that

$$\tau(C, \psi) \in X \subseteq \tilde{K}_1(\mathbb{A}).$$

(ii) Given a $*$-invariant subgroup $X \subseteq K_1(\mathbb{A})$ define the *intermediate round n-dimensional ε-quadratic L-group* $L_n^{rX}(\mathbb{A}, \varepsilon)$ $(n \geqslant 1)$ to be the cobordism group of round n-dimensional ε-quadratic Poincaré complexes (C, ψ) in $\mathbb{A}$ such that

$$\tau(C, \psi) \in X \subseteq K_1(\mathbb{A}).$$

For $n = 0$ set

$$L_0^{rX}(\mathbb{A}, \varepsilon) = L_\varepsilon^{rX}(\mathbb{A}),$$

the Witt group of formal differences $(M, \psi) - (M', \psi')$ of nonsingular ε-quadratic forms in $\mathbb{A}$ such that

$$[M] - [M'] = 0 \in K_0(\mathbb{A}),$$
$$\tau(M, \psi) - \tau(M', \psi') \in X \subseteq K_1(\mathbb{A}). \qquad \square$$

As for the intermediate L-groups in Section 6 it is possible to use the additive instant additive surgery obstruction to prove that the intermediate L-groups $L_n^X(\mathbb{A}, \varepsilon)$, $L_n^{rX}(\mathbb{A}, \varepsilon)$ $(n \geqslant 0)$ are isomorphic to the corresponding Witt groups of forms and formations with prescribed torsion.

THEOREM 7.22. (i) *For any $*$-invariant subgroups $Y \subseteq X \subseteq \tilde{K}_1(\mathbb{A})$, there is defined a Rothenberg exact sequence*

$$\cdots \to L_n^Y(\mathbb{A}, \varepsilon) \to L_n^X(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; X/Y)$$
$$\to L_{n-1}^Y(\mathbb{A}, \varepsilon) \to \cdots,$$

with

$$L_n^X(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; X/Y); (C, \psi) \to \tau(C, \psi).$$

(ii) *For any $*$-invariant subgroups $Y \subseteq X \subseteq K_1(\mathbb{A})$ there is defined a Rothenberg exact sequence*

$$\cdots \to L_n^{rY}(\mathbb{A}, \varepsilon) \to L_n^{rX}(\mathbb{A}, \varepsilon) \to \hat{H}^n(\mathbb{Z}_2; X/Y)$$
$$\to L_{n-1}^{rY}(\mathbb{A}, \varepsilon) \to \cdots.$$

EXAMPLE 7.23. Given a group π and a morphism $w: \pi \to \mathbb{Z}_2 = \{\pm 1\}$, let $A = \mathbb{Z}[\pi]$ be the group ring with the w-twisted involution

$$^{-}: \mathbb{Z}[\pi] \to \mathbb{Z}[\pi];\ g \to \bar{g} = w(g)g^{-1} \quad (g \in \pi).$$

Let $X = \mathrm{im}(\pi \to \tilde{K}_1(\mathbb{Z}[\pi]))$ be the $*$-invariant subgroup with quotient the Whitehead group

$$\tilde{K}_1(\mathbb{Z}[\pi])/X = \mathrm{Wh}(\pi).$$

The original surgery obstruction groups of Wall [17] are the intermediate quadratic L-groups

$$L_*(\pi, w) = L_*^s(\mathbb{Z}[\pi]) = L_*^X(\mathbb{Z}[\pi]).$$

The original Rothenberg exact sequence (Shaneson [16]) is

$$\begin{aligned} \cdots &\to L_n^s(\mathbb{Z}[\pi]) \to L_n^h(\mathbb{Z}[\pi]) \to \hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(\pi)) \\ &\to L_{n-1}^s(\mathbb{Z}[\pi]) \to \cdots . \end{aligned}$$

□

REMARK 7.24. Continuing the discussion in 6.16, it is also proved in Ranicki [15] that for a filtered additive category with involution $\mathbb{A}$

$$L_{n+1}^X(\mathbb{C}_{i+1}(\mathbb{A})) = L_n(\mathbb{C}_i(\mathbb{A})) = L_{n-i}^{\langle -i+1 \rangle}(\mathbb{A}) \quad (i \geqslant 0)$$

with

$$\begin{aligned} X &= \mathrm{im}(K_0(\mathbb{C}_i(\mathbb{A})) \to K_0(\mathbb{P}_i(\mathbb{A}))) \quad (=\{0\} \text{ for } i > 0) \\ &\subseteq K_0(\mathbb{P}_i(\mathbb{A})) = K_1(\mathbb{C}_{i+1}(\mathbb{A})) = \tilde{K}_1(\mathbb{C}_{i+1}(\mathbb{A})) \end{aligned}$$

such that

$$\tilde{K}_1(\mathbb{C}_{i+1}(\mathbb{A}))/X = \tilde{K}_0(\mathbb{P}_i(\mathbb{A})) \quad (=K_{-i}(\mathbb{A}) \text{ for } i > 0).$$

For a ring with involution A the generalized Rothenberg exact sequence for intermediate L-groups

$$\begin{aligned} \cdots &\to L_{n+1}^X(\mathbb{C}_{i+1}(A)) \to L_{n+1}(\mathbb{C}_{i+1}(A)) \\ &\to \hat{H}^{n+1}(\mathbb{Z}_2; \tilde{K}_1(\mathbb{C}_{i+1}(A))/X) \to L_n^X(\mathbb{C}_{i+1}(A)) \to \cdots \end{aligned}$$

coincides with the exact sequence

$$\begin{aligned} \cdots &\to L_{n-i}^{\langle -i+1 \rangle}(A) \to L_{n-i}^{\langle -i \rangle}(A) \\ &\to \hat{H}^{n-i}(\mathbb{Z}_2; \tilde{K}_{-i}(A)) \to L_{n-i-1}^{\langle -i+1 \rangle}(A) \to \cdots \end{aligned}$$

which was expressed in 6.16 as a generalized Rothenberg exact sequence for intermediate L-groups. □

8. The Stable Radical

We interpret the additive instant surgery obstruction $(C^{\oplus}, \psi^{\oplus})$ of a $2i$-dimensional ε-quadratic Poincaré complex (C, ψ) as a 'stable radical quotient' of the $(-)^i\varepsilon$-quadratic form

$$\left(C^i \oplus C_{i+1}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right),$$

for any additive category with involution $\mathbb{A}$. We start by recalling the radical of a form in the additive category $\mathbb{A} = \mathbb{P}(A) = \{\text{f.g. projective } A\text{-modules}\}$.

DEFINITION 8.1. For a ring with involution A the *radical* of an ε-quadratic form (M, ψ) in $\mathbb{P}(A)$ is the submodule

$$M^{\perp} = \ker(\psi + \varepsilon\psi^*: M \to M^*) \subseteq M. \qquad \square$$

If the inclusion $j \in \mathrm{Hom}_A(M^{\perp}, M)$ is a split injection which defines a morphism of ε-quadratic forms

$$j: (M^{\perp}, 0) \to (M, \psi)$$

and the sequence

$$0 \to M^{\perp} \xrightarrow{j} M \xrightarrow{\psi + \varepsilon\psi^*} M^* \xrightarrow{j^*} (M^{\perp})^* \to 0$$

is exact there is defined a quotient nonsingular ε-quadratic form in $\mathbb{P}(A)$

$$(M, \psi)/M^{\perp} = (M/M^{\perp}, [\psi]).$$

Let $\mathbb{A}$ now be an additive category with involution.

DEFINITION 8.2. A *stable radical* of an ε-quadratic form (M, ψ) in $\mathbb{A}$ is a morphism of ε-quadratic forms

$$j: (L, 0) \to (M, \psi)$$

together with a contractible chain complex in $\mathbb{A}$ of the type

$$M^{\perp}: 0 \to L \xrightarrow{\begin{bmatrix} j \\ k \end{bmatrix}} M \oplus N \xrightarrow{\begin{bmatrix} \psi + \varepsilon\psi^* & 0 \\ 0 & 0 \end{bmatrix}} M^* \oplus N^*$$
$$\xrightarrow{(j^* \; k^*)} L^* \to 0. \qquad \square$$

It is not claimed that every ε-quadratic form admits a stable radical, or that stable radicals are unique.

DEFINITION 8.3. A *stable radical quotient* of an ε-quadratic form (M, ψ) is the nonsingular ε-quadratic form in $\mathbb{A}$ associated to a stable radical $M^{\perp}$ by

$$(M, \psi)/M^{\perp} = \left(M \oplus N \oplus L^*, \begin{bmatrix} \psi & 0 & g^* \\ 0 & 0 & h^* \\ 0 & 0 & 0 \end{bmatrix}\right)$$

with $(g\ h): M \oplus N \to L$ any splitting map for $\begin{bmatrix} j \\ k \end{bmatrix}: L \to M \oplus N$, such that

$$gj + hk = 1: L \to L.$$

REMARK 8.4. A nonsingular form (M, ψ) has a stable radical with $L = N = 0$, such that the quotient is $(M, \psi)/M^{\perp} = (M, \psi)$. □

REMARK 8.5. If $\mathbb{A}$ is fully embedded in an Abelian category and (M, ψ) is an ε-quadratic form in $\mathbb{A}$ such that the inclusion

$$j: M^{\perp} = \ker(\psi + \varepsilon\psi^*: M \to M^*) \to M$$

is a split injection in $\mathbb{A}$ defining a morphism of forms $j: (M^{\perp}, 0) \to (M, \psi)$, and such that the morphism

$$\operatorname{coker}(\psi + \varepsilon\psi^*) \to \ker(\psi + \varepsilon\psi^*)^*; f \to (x \to f(x))$$

is an isomorphism in $\mathbb{A}$, then (M, ψ) has a stable radical with

$$L = \ker(\psi + \varepsilon\psi^*), \quad N = 0.$$

Moreover, there is induced a nonsingular ε-quadratic form $(\operatorname{coker}(j), [\psi])$ in $\mathbb{A}$ such that up to isomorphism

$$(M, \psi)/M^{\perp} = (\operatorname{coker}(j, [\psi]) \oplus H_{\varepsilon}(L).$$

In the terminology of Ranicki [8], L is a 'sub-Lagrangian' of $(M, \psi)/M^{\perp}$. □

PROPOSITION 8.6. *The nonsingular $(-)^i\varepsilon$-quadratic form of the additive instant obstruction $(C^{\oplus}, \psi^{\oplus})$ of a $2i$-dimensional ε-quadratic Poincaré complex (C, ψ) in $\mathbb{A}$ is a stable radical quotient*

$$((C^{\oplus})^i, (\psi^{\oplus})_0) = \left(C^i \oplus C_{i+1}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right) \Big/ (C^i \oplus C_{i+1})^{\perp}.$$

Proof. Let (C', ψ') be the stably highly connected $2i$-dimensional ε-quadratic Poincaré complex cobordant to (C, ψ) given by 4.9, with

$$C': \cdots \to 0 \to C_i \oplus C^{i+1} \to C_{i-1} \oplus C^{i+2} \to \cdots.$$

Let (C'', ψ'') be the homotopy equivalent complex constructed in the proof of 4.10, with

$$C''_i = \sum_{j=0}^{\infty} C'_{i-2j} = \sum_{j=0}^{\infty} (C_{i-2j} \oplus C^{i+2j+1}),$$

$$C''_{i-1} = \sum_{j=0}^{\infty} C'_{i-2j-1} = \sum_{j=0}^{\infty} (C_{i-2j-1} \oplus C^{i+2j+2}),$$

$$C''_r = 0 \text{ for } r \neq i-1, i.$$

Define the $(-)^i\varepsilon$-quadratic form in $\mathbb{A}$

$$(M,\theta) = \left(C^i \oplus C_{i+1}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right).$$

The algebraic mapping cone of the chain equivalence

$$(1 + T_\varepsilon)\psi_0'': C''^{2i-*} \to C''$$

is a contractible chain complex

$$C((1 + T)\psi_0'') = M^\perp : 0 \to L \xrightarrow{\begin{bmatrix} j \\ k \end{bmatrix}} M \oplus N \xrightarrow{\begin{bmatrix} \theta + (-)^i\varepsilon\theta^* & 0 \\ 0 & 0 \end{bmatrix}} M^* \oplus N^*$$
$$\xrightarrow{(j^*\ k^*)} L^* \to 0$$

defining a stable radical $M^\perp$ of (M, θ), with

$$L = C''^{i-1} = \sum_{j=0}^{\infty} (C^{i-2j-1} \oplus C_{i+2j+2}),$$

$$N = C''^i/C'^i = \sum_{j=1}^{\infty} (C^{i-2j} \oplus C_{i+2j+1}).$$

The stable radical quotient $(M, \theta)/M^\perp$ corresponds to the highly connected $2i$-dimensional ε-quadratic Poincaré complex $(C^\oplus, \psi^\oplus)$ cobordant to (C', ψ') given by 4.10.

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